احجز مكانك واستعد للامتحان بثقة كاملة

ملزمة الرياضيات الفصل الثاني

12 متقدم لعام 2026



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انضم للقناة

احصل على شرح الملزمة كاملة

بـ199 درهم فقط

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Chapter (4): Applications of Differentiation

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Differentiation Rules

Function	Derivative	Function	Derivative	Function	Derivative
$y = \csc x$	$\frac{dy}{dx} = -csc\,x\,.cot\,x$	$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x).e^{f(x)}$	$f(x) = constant$	$f'(x) = 0$
$y = \sin^{-1} x$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$	$y = ln\lvert f(x)\rvert$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$	$f(x) = x^n$	$f'(x) = n.x^{n-1}$
$y = \cos^{-1} x$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$	$y = a^{f(x)}$	$\frac{dy}{dx} = f'(x).a^{f(x)}.ln(a)$	$y = f(x).g(x)$	$\frac{dy}{dx} = f'(x).g(x) + f(x).g'(x)$
$y = \tan^{-1} x$	$\frac{dy}{dx} = \frac{1}{1+x^2}$	$y = \sin x$	$\frac{dy}{dx} = \cos x$	$y = \frac{f(x)}{g(x)}$	$\frac{dy}{dx} = \frac{f'(x).g(x) - f(x).g'(x)}{(g(x))^2}$
$y = \cot^{-1} x$	$\frac{dy}{dx} = \frac{-1}{1+x^2}$	$y = \cos x$	$\frac{dy}{dx} = -\sin x$	$y = \frac{c}{f(x)}$	$\frac{dy}{dx} = -\frac{c.f'(x)}{(f(x))^2}$
$y = \sec^{-1} x$	$\frac{dy}{dx} = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$y = \tan x$	$\frac{dy}{dx} = sec^2\,x$	$y = \sqrt{f(x)}$	$\frac{dy}{dx} = \frac{-f'(x)}{2\sqrt{f(x)}}$
$y = \csc^{-1} x$	$\frac{dy}{dx} = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$y = \cot x$	$\frac{dy}{dx} = -csc^2\,x$	$y = f(g(x))$	$\frac{dy}{dx} = f'\big(g(x)\big).g'(x)$
		$y = \sec x$	$\frac{dy}{dx} = sec\,x\,.tan\,x$	$g(x) = f^{-1}(x)$	$g'(x) = \frac{1}{f'\big(g(x)\big)}$

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Chapter (4): Applications of Differentiation

01 First Lesson:

Linear Approximations and Newton's Method



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The Power of the Tangent Line

From Approximating Complex Functions to Solving Unsolvable Equations



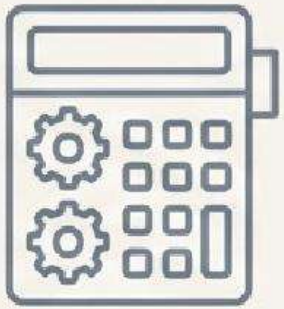
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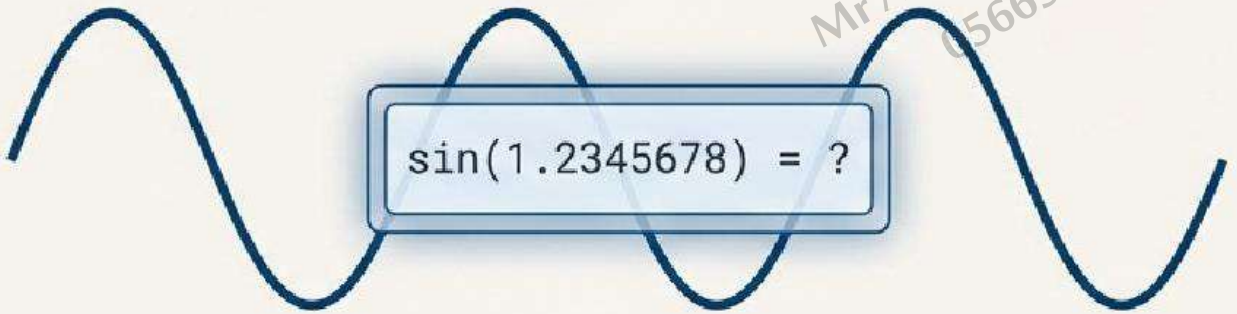
Why Do We Need Approximation?

(The Straightforward)



Some calculations are laborious but direct. For example, 1024×1673. We know **how** to do it, but a calculator is faster.

(The Complex)



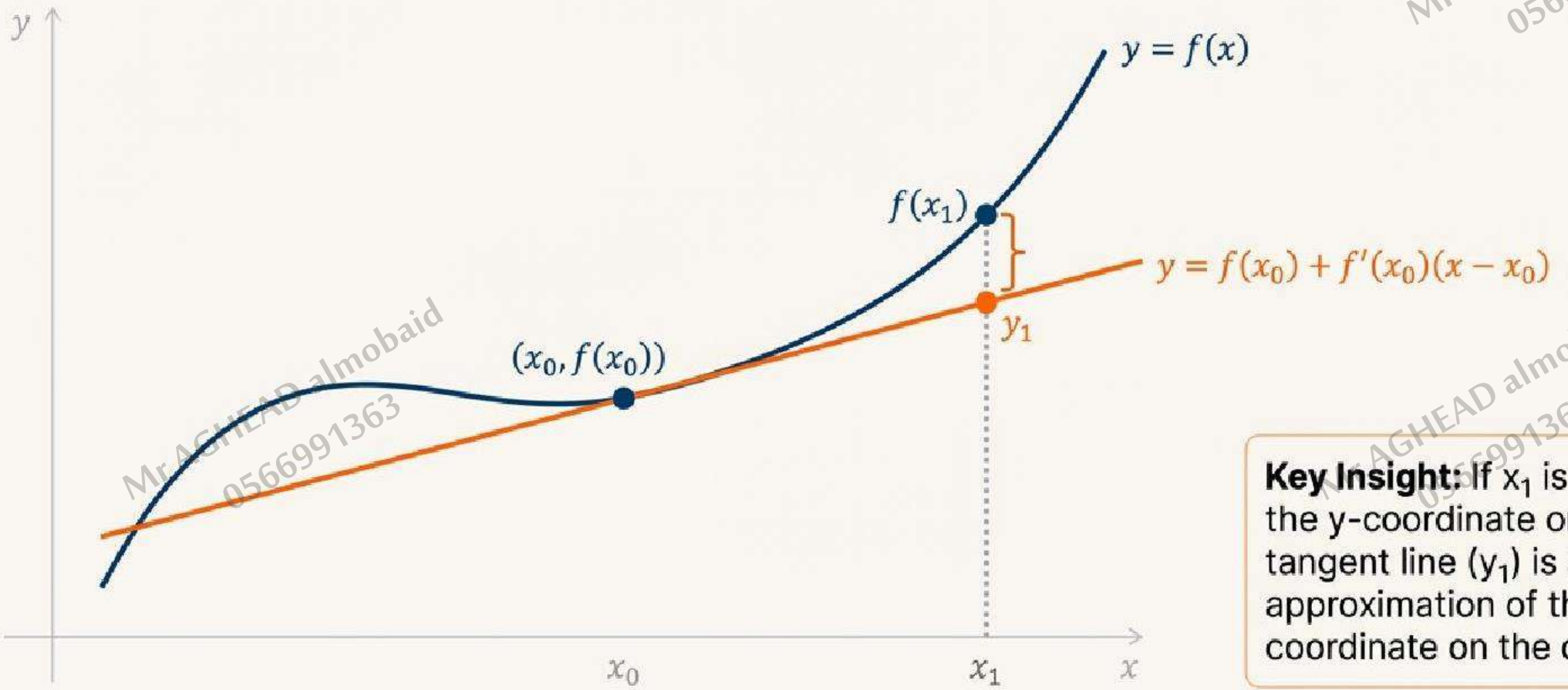
Other calculations are impossible without a specialized tool. How do we compute sin(1.2345678)`? There is no simple arithmetic formula. Your calculator uses built-in programs that generate *approximate** values for functions like sine, cosine, etc.

This chapter introduces two powerful methods, rooted in calculus, that allow us to perform these kinds of approximations ourselves.

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Part I: Using the Tangent Line to Approximate Values

Suppose we need to find $f(x_1)$, but we only know the exact value of the function at a nearby point, $f(x_0)$.



Key Insight: If x_1 is close to x_0, the y-coordinate on the tangent line (y_1) is a very good approximation of the true y-coordinate on the curve ($f(x_1)$).

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Formalizing the Linear Approximation

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Derivation

The equation of the tangent line at $x = x_0$ comes from the point-slope form:

$$m_{\tan} = f'(x_0) = \frac{y - f(x_0)}{x - x_0}$$

Rearranging gives the tangent line equation:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Definition 1.1

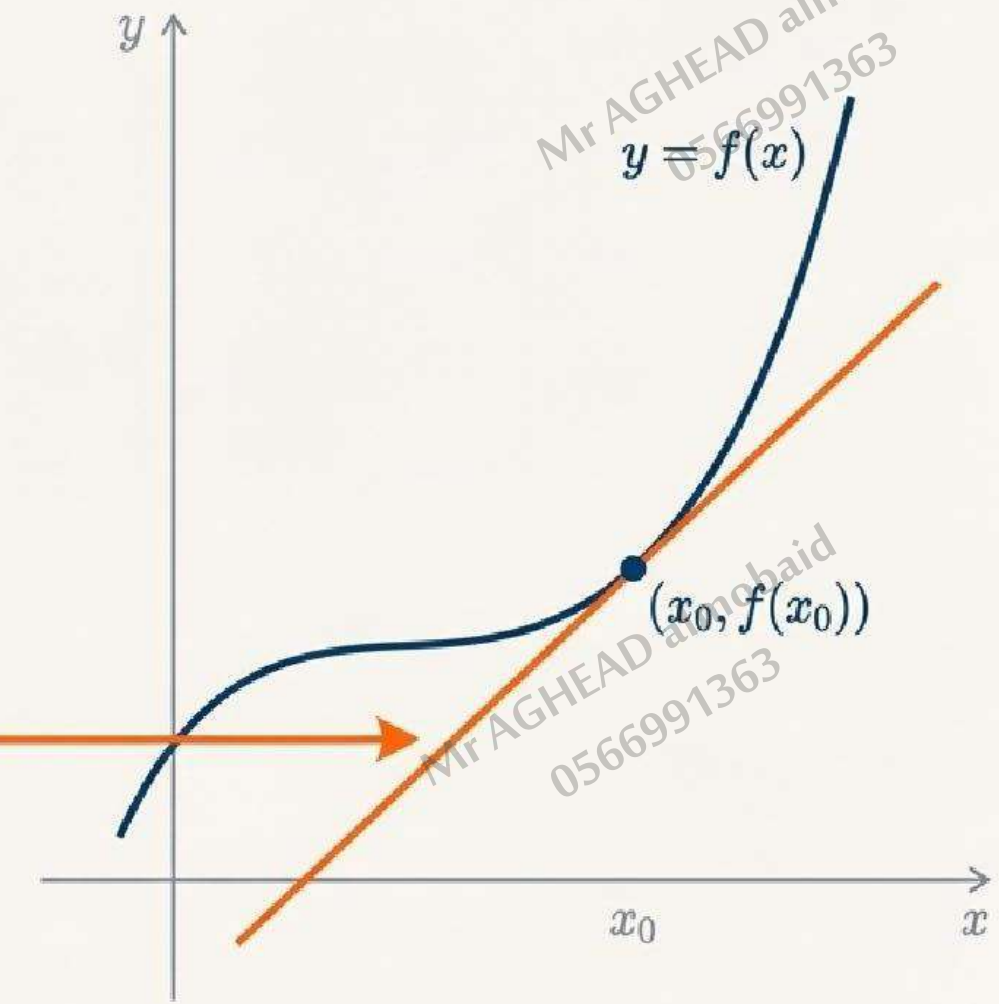
The linear (or tangent line) approximation of $f(x)$ at $x = x_0$ is the function $L(x)$.

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

The Approximation

Therefore, for x near x_0, we have:

$$f(x) \approx L(x)$$



Example: Approximating the Value of cos(1)

We want to approximate $\cos(1)$. We know $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, and $\frac{\pi}{3} \approx 1.047$ is close to 1.

Step 1: Setup

- Function: $f(x) = \cos(x)$
- Point of tangency: $x_0 = \frac{\pi}{3}$
- Derivative: $f'(x) = -\sin(x)$
- Values at x_0: $f\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Step 2: Build the Approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

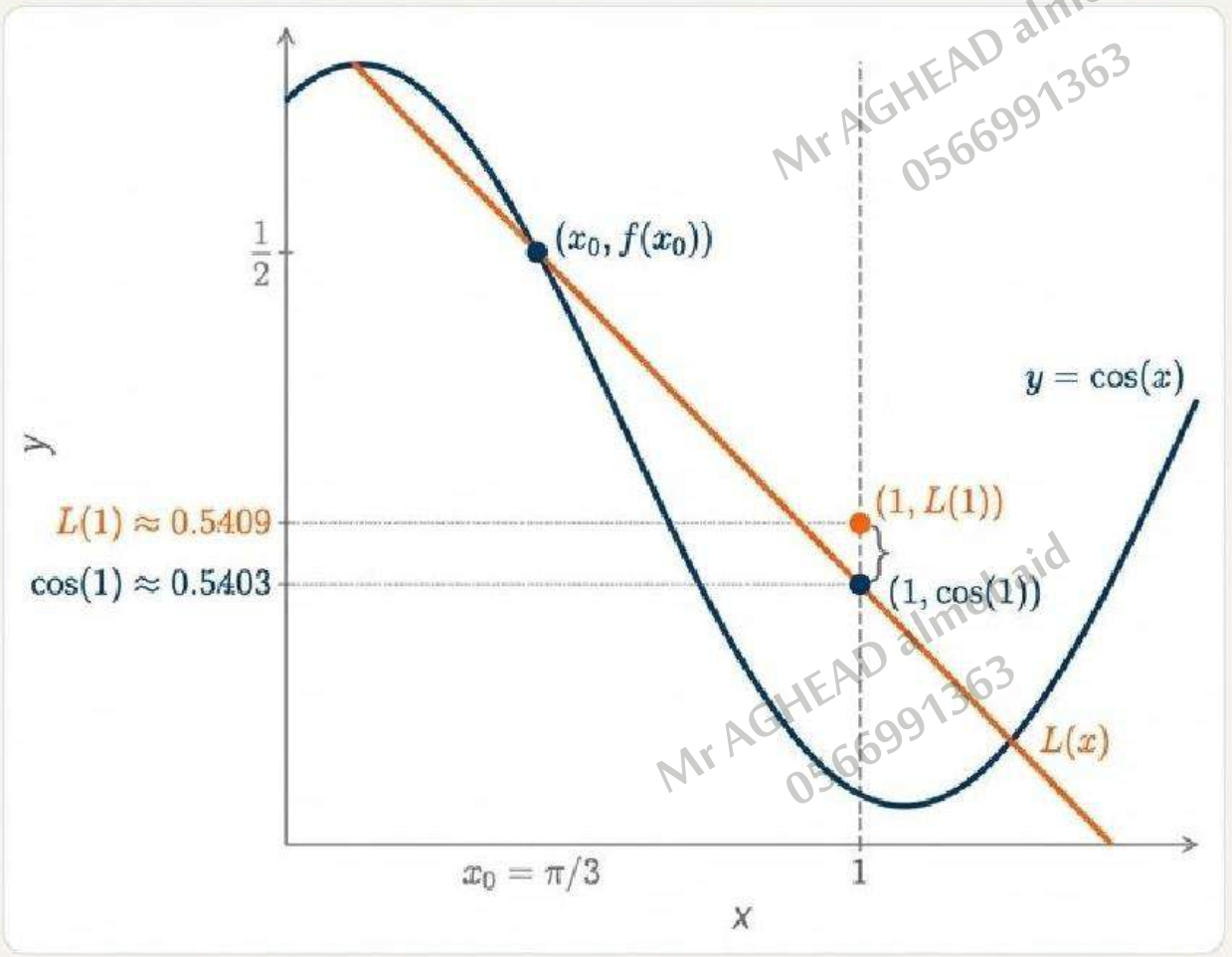
$$L(x) = \frac{1}{2} - \left(\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)\right.$$

Step 3: Calculate the Estimate

$$\cos(1) \approx L(1) = \frac{1}{2} - \left(\frac{\sqrt{3}}{2}\left(1 - \frac{\pi}{3}\right) \approx 0.5409\right.$$

Step 4: Compare with Reality

Approximated Value: 0.5409
Calculator Value: 0.5403
A highly accurate approximation.



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A Famous & Useful Approximation: $\sin(x) \approx x$

Problem

Find the linear approximation of $f(x) = \sin(x)$ for x close to 0.

Setup

Function: $f(x) = \sin(x)$
Point of tangency: $x_0 = 0$
Derivative: $f'(x) = \cos(x)$
Values at x_0: $f(0) = \sin(0) = 0$ and $f'(0) = \cos(0) = 1$

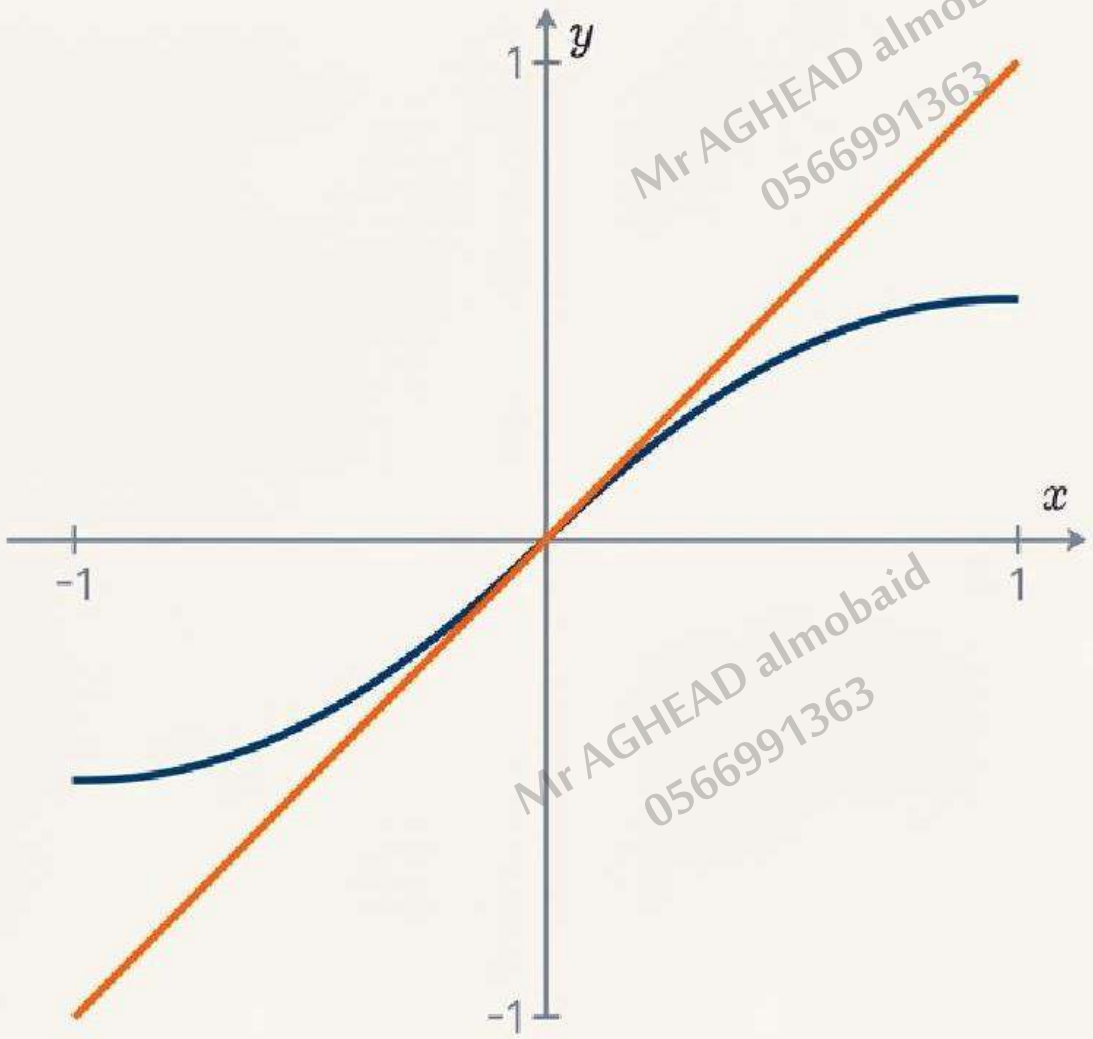
Calculation

$\sin(x) \approx L(x) = f(0) + f'(0)(x - 0)$

$$\sin(x) \approx x$$

Conclusion

For values of x close to 0, $\sin(x)$ is approximately equal to x.
This is a powerful simplification used in physics and engineering.



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Example: Approximating Cube Roots

We'll use linear approximation for $f(x) = \sqrt[3]{x} = x^{1/3}$. The derivative is $f'(x) = \frac{1}{3}x^{-2/3}$.

Case 1: Approximate $\sqrt[3]{8.02}$

Best x_0: The closest number with a known cube root is 8. So, $x_0 = 8$.

$$\sqrt[3]{8.02} \approx f(8) + f'(8)(8.02 - 8) =$$
$$= 2 + \frac{1}{12}(0.02) \approx 2.0016667$$

Calculator Value: 2.0016653 (very close)

Case 2: Approximate $\sqrt[3]{25.2}$

Best x_0: The closest number with a known cube root is 27. So, $x_0 = 27$. Using $x_0 = 8$ would produce a poor result.

$$\sqrt[3]{25.2} \approx f(27) + f'(27)(25.2 - 27) =$$
$$= 3 + \frac{1}{27}(-1.8) \approx 2.9333333$$

Calculator Value: 2.931794 (still a good approximation)

Error in linear approximation (at x_0=8)

x	Error
8.02	1.4×10^{-6}
8.07	1.7×10^{-5}
8.15	7.7×10^{-5}

Notice how the error grows as x moves farther from the point of tangency, 8.

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Part II: Using the Tangent Line to Find Solutions

The New Problem

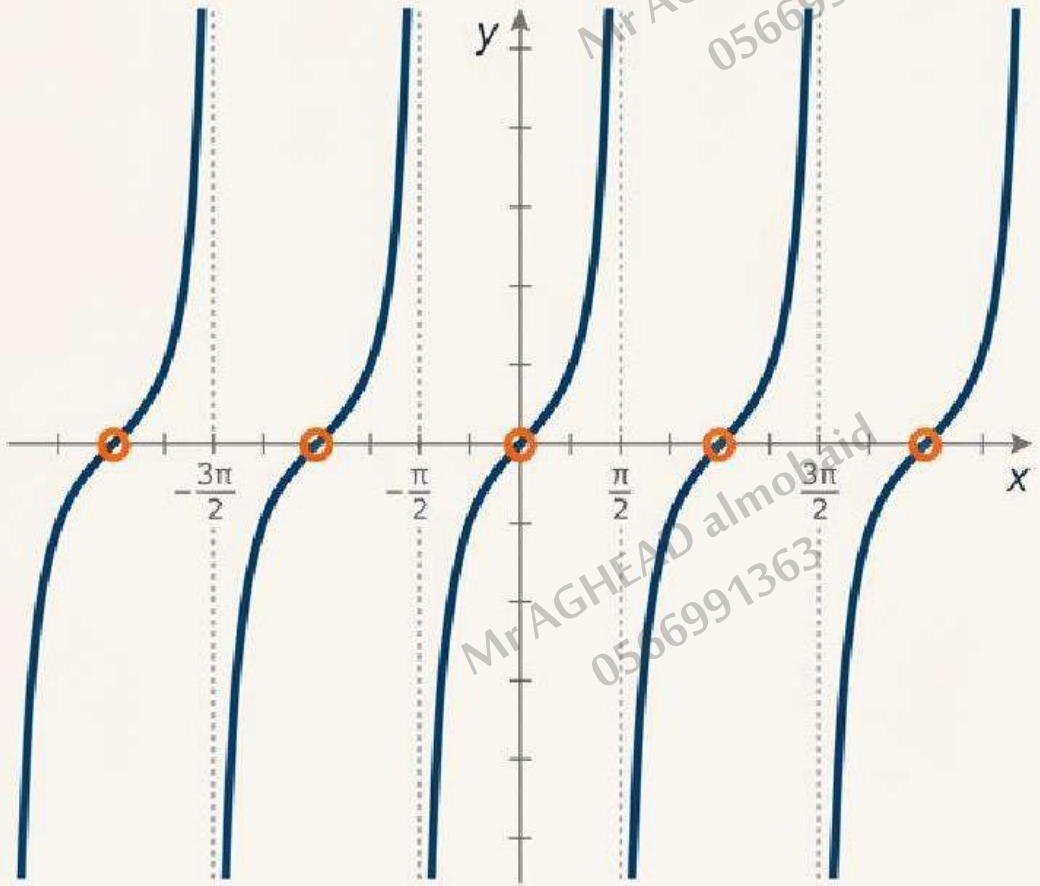
Some equations are easy to solve (e.g., $ax^2 + bx + c = 0$). But how do you find the zeros of a function like $f(x) = \tan(x) - x$?

The Challenge

There are no simple algebraic formulas to find the roots of this equation.

The Goal

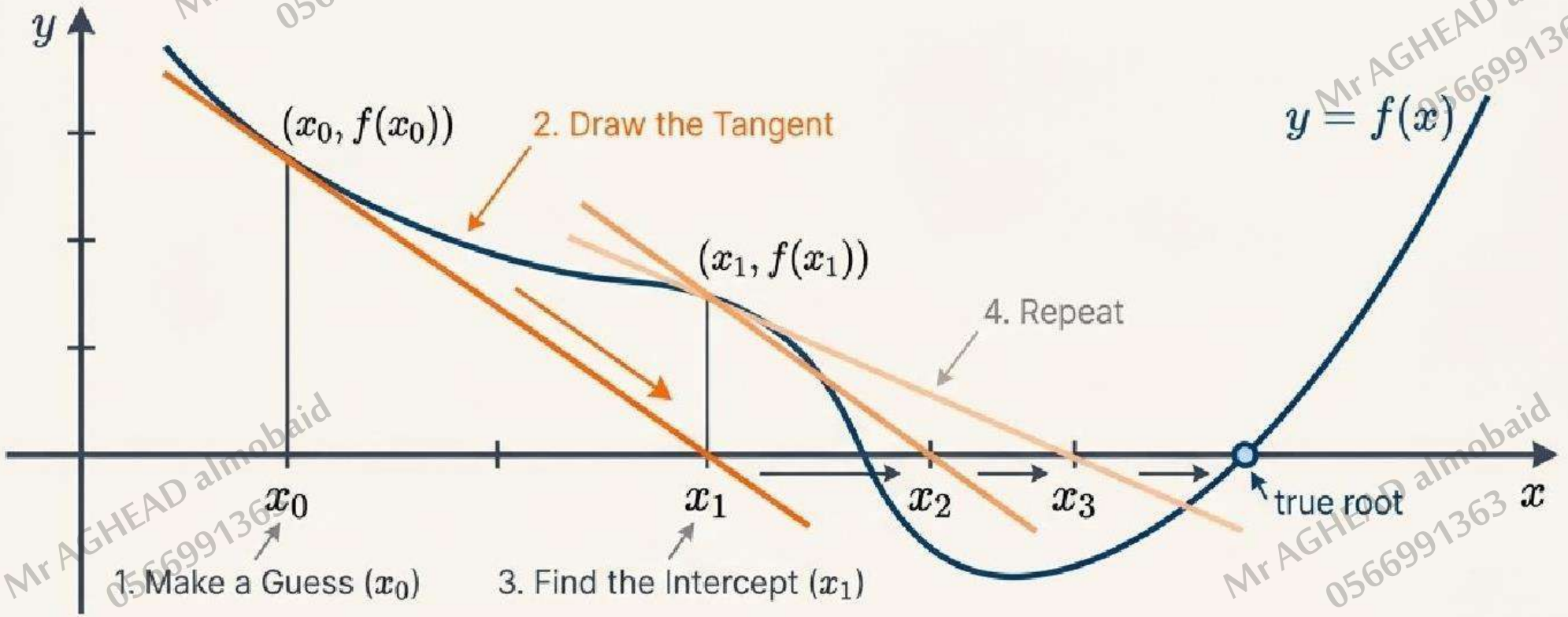
We need a method to find an *approximate* solution to $f(x) = 0$.



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The Iterative Logic of Newton's Method

We can use the tangent line to walk closer and closer to the actual root.



This sequence of approximations, $x_0, x_1, x_2, \ldots$, gets progressively closer to the actual zero of the function.

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Deriving the Formula for Newton's Method

Step 1: The Tangent Line Equation

We start with the equation of the tangent line at a general approximation $x = x_n$:

$$y = f(x_n) + f'(x_n)(x - x_n)$$

Step 2: Find the x-intercept

The next approximation, x_{n+1}, is the x-intercept of this line. We find it by setting $y = 0$ and $x = x_{n+1}$:

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n)$$

Step 3: Solve for the New Approximation

Rearranging the equation to solve for x_{n+1}:

$$f'(x_n)(x_{n+1} - x_n) = -f(x_n)$$

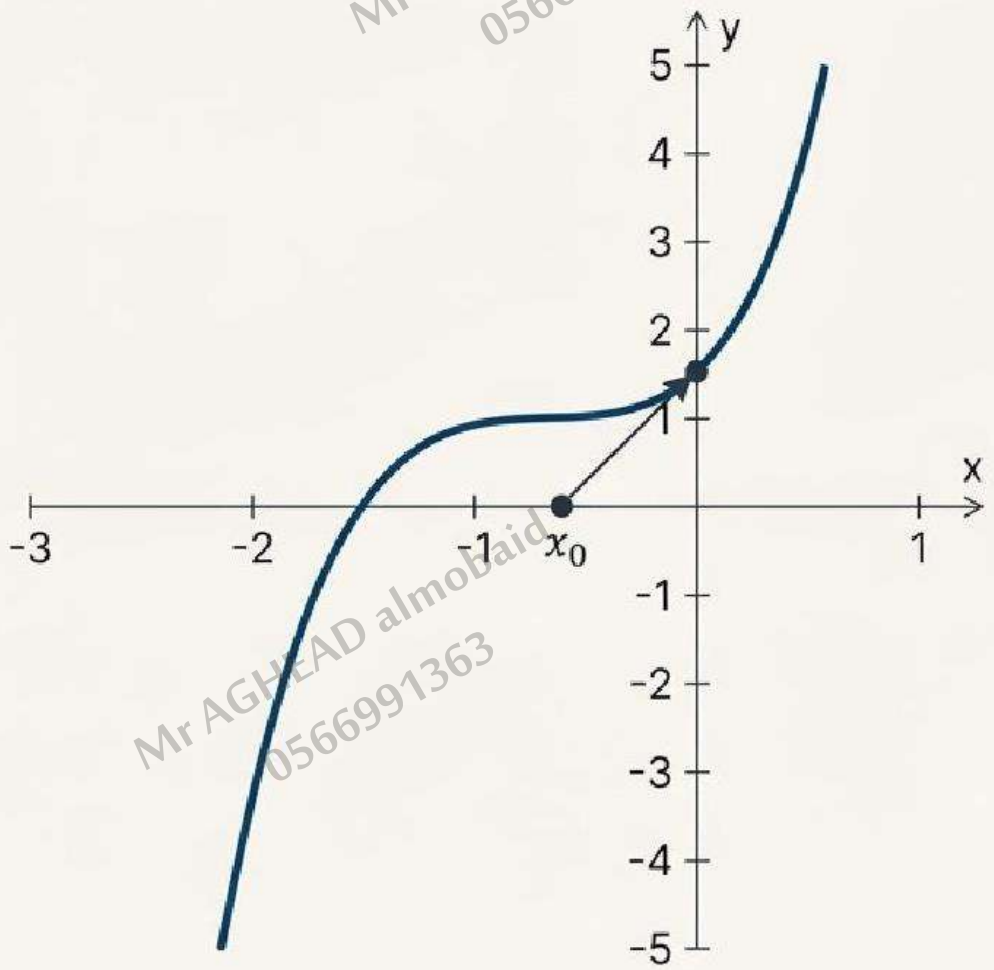
$$x_{n+1} - x_n = \frac{-f(x_n)}{f'(x_n)}$$

Newton's Method Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n = 0, 1, 2, \ldots$$

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Example: Finding a Zero of $f(x) = x^5 - x + 1$



Step 1: Setup and Initial Guess

Function: $f(x) = x^5 - x + 1$

Derivative: $f'(x) = 5x^4 - 1$

From the graph, a root exists between -2 and -1. We choose an initial guess: $x_0 = -1$.

Step 2: Iteration

$x_1 = x_0 - f(x_0)/f'(x_0) = -1 - (1)/4 = -1.25$

$x_2 = x_1 - f(x_1)/f'(x_1) \approx -1.178459$

$x_3 \approx -1.167537$

$x_4 \approx -1.167304$

$x_5 \approx -1.167303978$

$x_6 \approx -1.167303978$

Step 3: Conclusion and Verification

The values stabilize. We stop at x_6.

Check: $f(x_6) \approx 1 \times 10^{-13}$, which is extremely close to 0.

The approximate zero is -1.167303978.

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Example: Using Newton's Method to Find Cube Roots

Approximate the value of $\sqrt[3]{7}$.

Step 1: Reframe as a Root-Finding Problem

- Let $x = \sqrt[3]{7}$.
- Then $x^3 = 7$.
- This is equivalent to finding the root of the equation $x^3 - 7 = 0$.
- So, we apply Newton's Method to the function $f(x) = x^3 - 7$.

Step 2: Setup and Execution

- Function: $f(x) = x^3 - 7$
- Derivative: $f'(x) = 3x^2$
- Initial Guess (since 1^3=1 and 2^3=8, the root is near 2): $x_0 = 2$.

Step 3: Iteration

- $x_1 = 2 - \frac{2^3 - 7}{3 \cdot 2^2} = 2 - \frac{1}{12} \approx 1.916667$
- $x_2 \approx 1.912938$
- $x_3 \approx 1.912931183$
- $x_4 \approx 1.912931183$

Conclusion

$\sqrt[3]{7} \approx 1.912931183$, which compares favorably to a calculator's value.

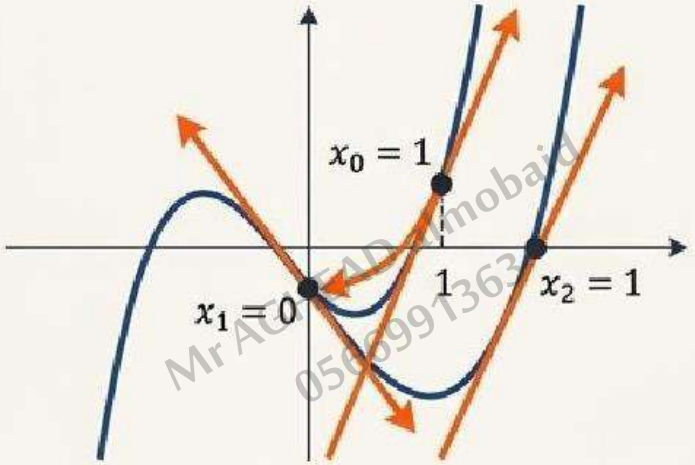
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Caution: When Newton's Method Fails

While powerful, Newtons method is not foolproof. Its success can depend heavily on the initial guess and the shape of the function.

Bad Initial Guess (Cycling)

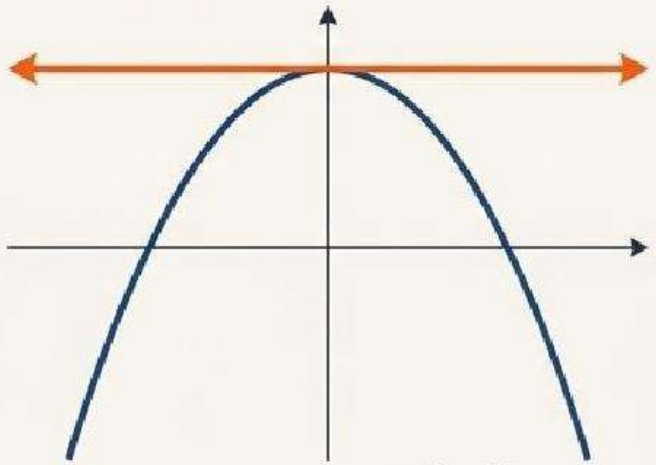
An initial guess of $x_0 = 1$ for $f(x) = x^3 - 3x^2 + x - 1$ causes the method to cycle between $x_1 = 0$ and $x_2 = 1$, never converging.



A better guess, like $x_0 = 2$, quickly converges.

Horizontal Tangent ($f'(x_0) = 0$)

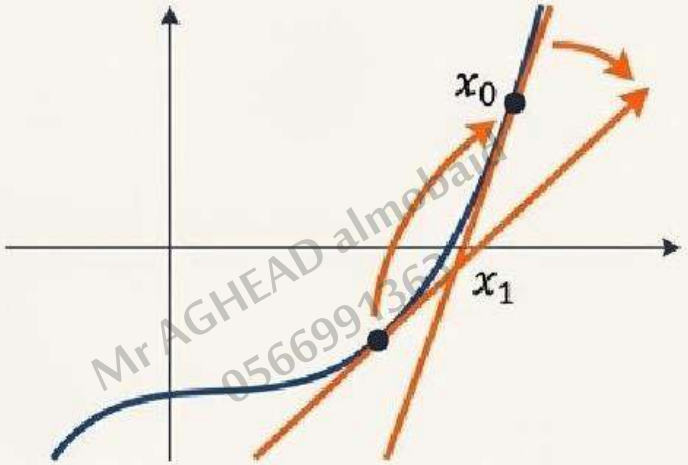
If $f'(x_0) = 0$, the tangent line is horizontal and never intersects the x-axis. The formula involves division by zero and fails.



$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Divergence

A poor initial guess can send the approximations farther and farther *away* from the root.



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A Note on Convergence Speed

Sometimes, even with a good initial guess, Newton's method can converge very slowly.

For $f(x) = \frac{((x-1)^2}{x^2+1}$, the only zero is clearly at $x = 1$.

An initial guess of $x_0 = 0$ *does* converge to the root, but much more slowly than in previous examples.

n	x_n
1	0.5
2	0.70833
3	0.83653
...	...
10	0.9985062
12	0.9996261

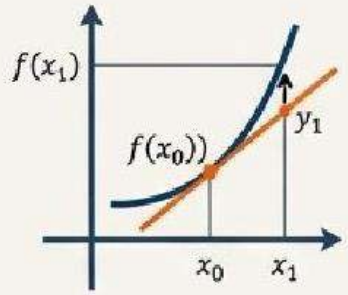
Takeaway: In our $x^5 - x + 1$ example, x_5 was a highly accurate answer. Here, x_{12} is still not as close. Always check if successive values are getting closer and if $f(x_n)$ is approaching zero.

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The Tangent Line: A Fundamental Tool of Calculus

We've explored two powerful applications derived from a single concept—the tangent line.

Linear Approximation

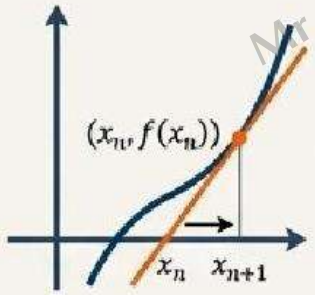


Purpose: To estimate the value of a function $f(x)$ near a known point x_0.

Mechanism: Uses the y-value on the tangent line as an approximation for the y-value on the curve.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Newton's Method



Purpose: To find the roots (or zeros) of an equation $f(x) = 0$.

Mechanism: Uses the x-intercept of the tangent line to iteratively find better approximations of a root.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Approximations are at the heart of calculus. While we may not always find an exact answer, these methods allow us to find answers that are as accurate as we need for any practical purpose.

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Chapter (4): Applications of Differentiation

02 Second Lesson:

Indeterminate Forms and
L'Hôpital's Rule



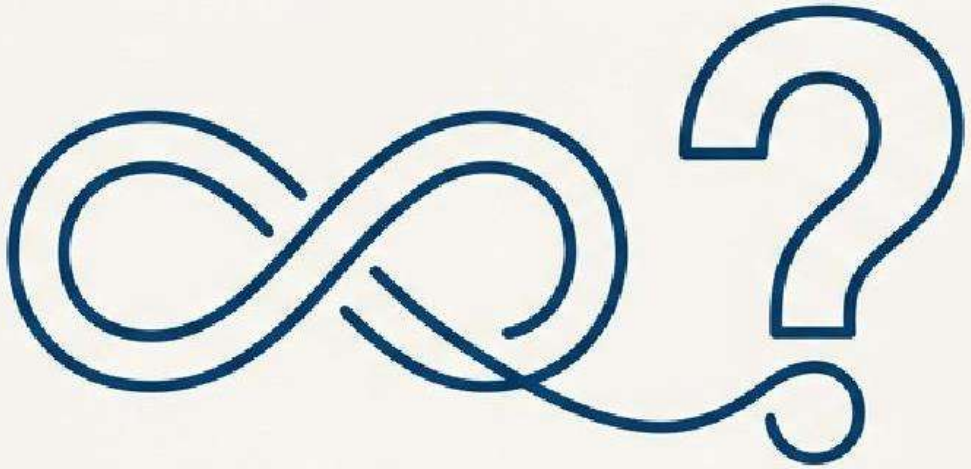
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The Art of the Indeterminate: Mastering L'Hôpital's Rule



A Comprehensive Guide to Solving Ambiguous Limits

Prepared for mr.aghead

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Why '0/0' is a Mathematical Mystery

We often encounter limits where the numerator and denominator both approach 0 or ∞. These are called **Indeterminate Forms**. The term 'indeterminate' means the value cannot be determined from the form alone. Look at these three limits, all of which start as `0/0`:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}$$

becomes

$$\lim_{x\to 1}(x+1)$$

→ 2

$$\lim_{x\to 1}\frac{x-1}{x^2-1}$$

becomes

$$\lim_{x\to 1}\frac{1}{x+1}$$

→ 1/2

$$\lim_{x\to 1}\frac{x-1}{(x-1)^2}$$

becomes

$$\lim_{x\to 1}\frac{1}{x-1}$$

→ Does Not Exist

The expression `0/0` is mathematically meaningless on its own. We need a more powerful tool to uncover the true value of these limits.

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The Tangent Line Tells the Story

If f(c) = 0 and g(c) = 0, then near x=c, the functions behave very much like their tangent lines.

The Logic of Linear Approximation

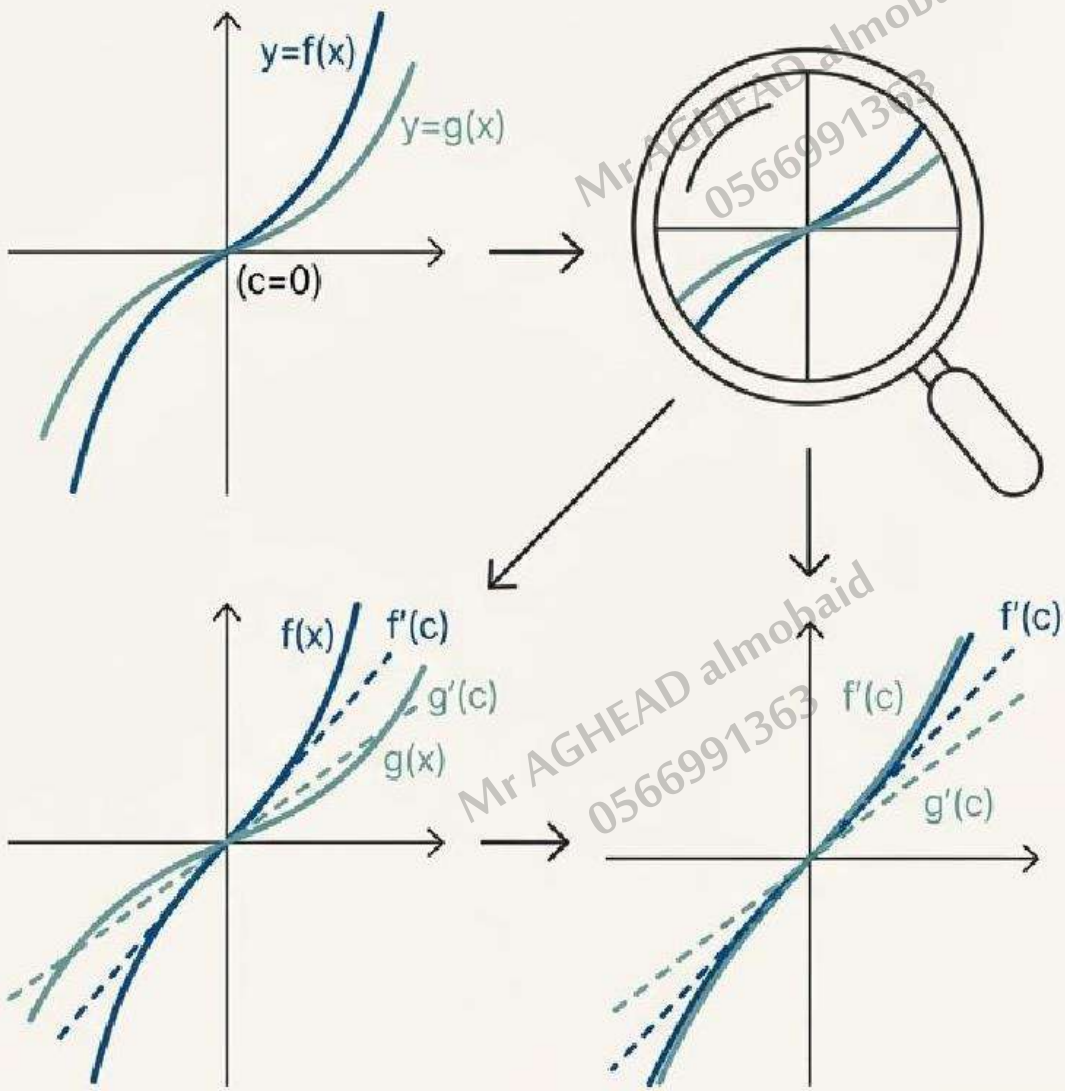
For f(x), the tangent line is $f(x) \approx f(c) + f'(c)(x-c)$.
Since f(c)=0, this simplifies to $f(x) \approx f'(c)(x-c)$.

Similarly for g(x), $g(x) \approx g'(c)(x-c)$.

The Intuitive Leap

Therefore, $\lim f(x)/g(x)$ as $x \to c$ should be approximately $\lim [f'(c)(x-c)] / [g'(c)(x-c)] = f'(c)/g'(c)$.

Key Takeaway: The limit of the ratio of functions is related to the limit of the ratio of their *rates of change* (their derivatives).



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Theorem 2.1: L'Hôpital's Rule

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Formal Statement

Suppose that f and g are differentiable on the interval (a, b), except possibly at c, and that $g'(x) \neq 0$ near c.

Suppose $\lim_{x \to c} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. $\boxed{\frac{0}{0}}$ $\boxed{\frac{\infty}{\infty}}$

And $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists (or is $\pm\infty$).

Then: $$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Important Note (from Remark 2.1)

This rule also applies to one-sided limits ($x \to c^+$, $x \to c^-$) and limits at infinity ($x \to \infty$, $x \to -\infty$).

Core Technique 1: The `0/0` Form in Action

Problem: Evaluate $\lim_{x\to 0} \frac{1-\cos x}{\sin x}$ as (Example 2.1)

Step 1: IDENTIFY The Form

- As $x\to 0$, the numerator $1-\cos(0) \to 1-1=0$.
- As $x\to 0$, the denominator $\sin(0) \to 0$.
- This is the indeterminate form $\frac{0}{0}$.

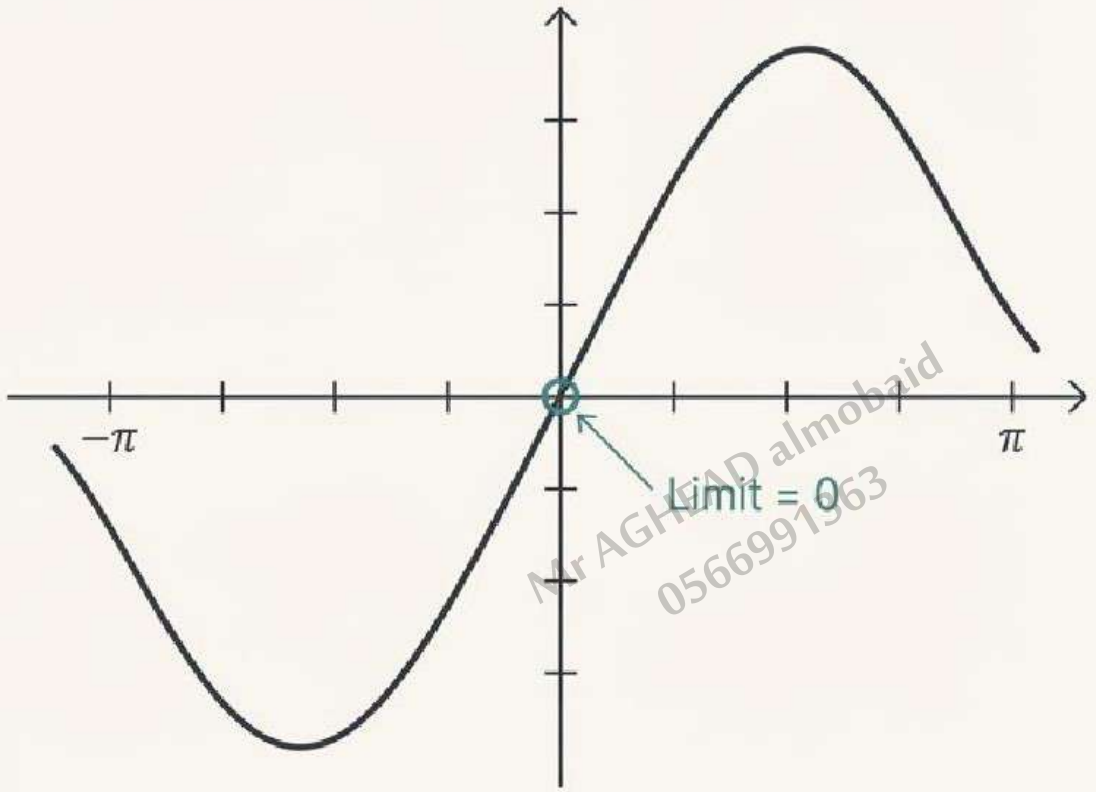
Step 2: APPLY The Rule

- Derivative of numerator: $\frac{d}{dx}(1-\cos x) = \sin x$.
- Derivative of denominator: $\frac{d}{dx}(\sin x) = \cos x$.
- The new limit is $\lim_{x\to 0} \frac{\sin x}{\cos x}$.

Step 3: EVALUATE The New Limit

- Plugging in $x=0$ gives $\frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$.

Visual Confirmation



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Core Technique 2: Tackling the $\frac{\infty}{\infty}$ Form

Problem: Evaluate $\lim_{x\to\infty} \frac{x^2}{e^x}$ as (Example 2.3)

Step 1: IDENTIFY The Form

- As $x\to\infty$, $x^2\to\infty$ and $e^x\to\infty$. This is the $\frac{\infty}{\infty}$ form.

Step 2: APPLY (First Pass)

- $\lim_{x\to\infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{2x}{e^x}$.

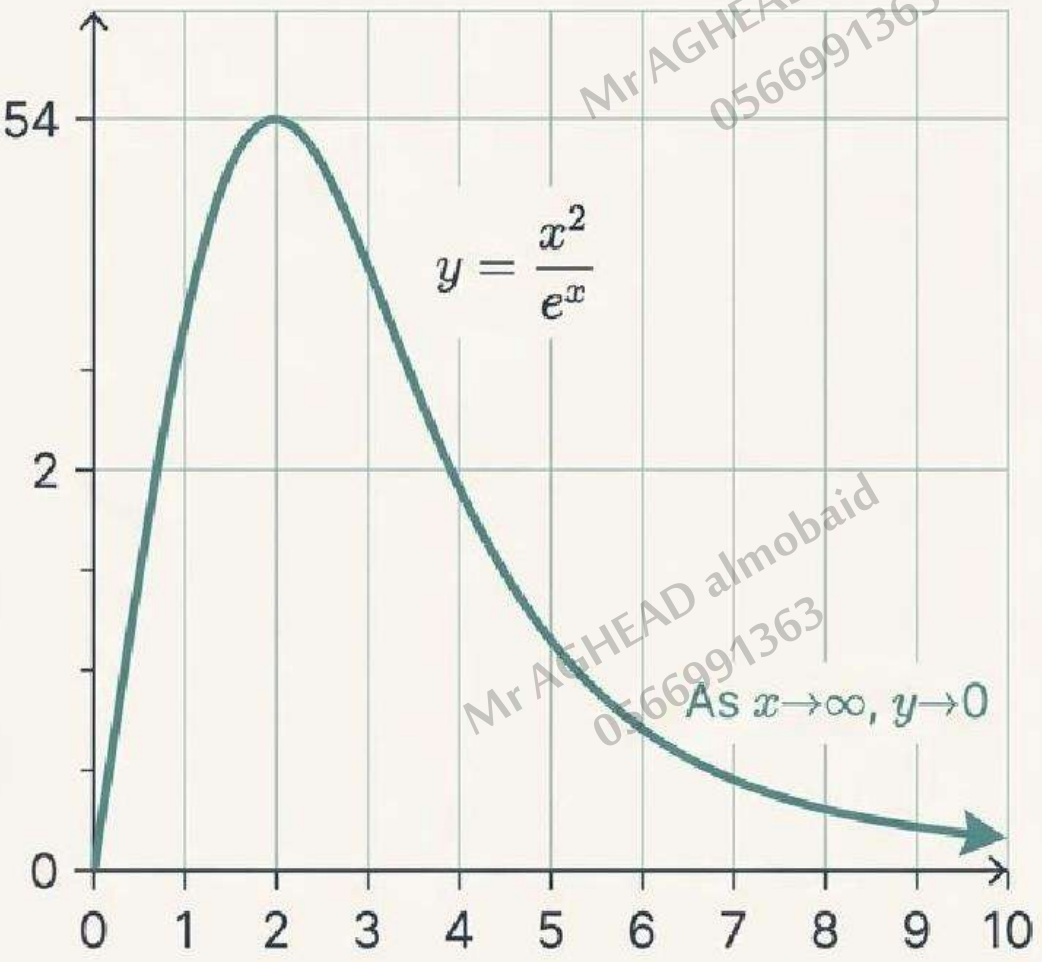
Step 3: RE-IDENTIFY & APPLY AGAIN

- The new limit is *still* an $\frac{\infty}{\infty}$ form. We can apply the rule again.
- $\lim_{x\to\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{2}{e^x}$.

Step 4: EVALUATE The Final Limit

- As $x\to\infty$, $\frac{2}{e^x}\to 0$.

Visual Confirmation



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The Danger Zone: Common Mistakes to Avoid

⚠ **Warning*:** L'Hôpital's Rule is powerful, but it is not universal. Always verify the hypotheses.

Mistake 1: Applying to a Determinate Form (from Example 2.4)

Consider $\lim_{x \to 0} \frac{2x}{e^x}$ as $x \to 0$.

Correctly ✓

The form is $\frac{0}{1}$, which is determinate.

The limit is **0**.

Incorrectly ✗

Applying L'Hôpital's Rule gives

$\lim_{x \to 0} \frac{2}{e^x} = \frac{2}{1} = 2.$ **WRONG.**

Mistake 2: Using the Quotient Rule (from Remark 2.2)

$$\frac{f'(x)}{g'(x)} \neq \frac{f'g - fg')}{g^2}$$

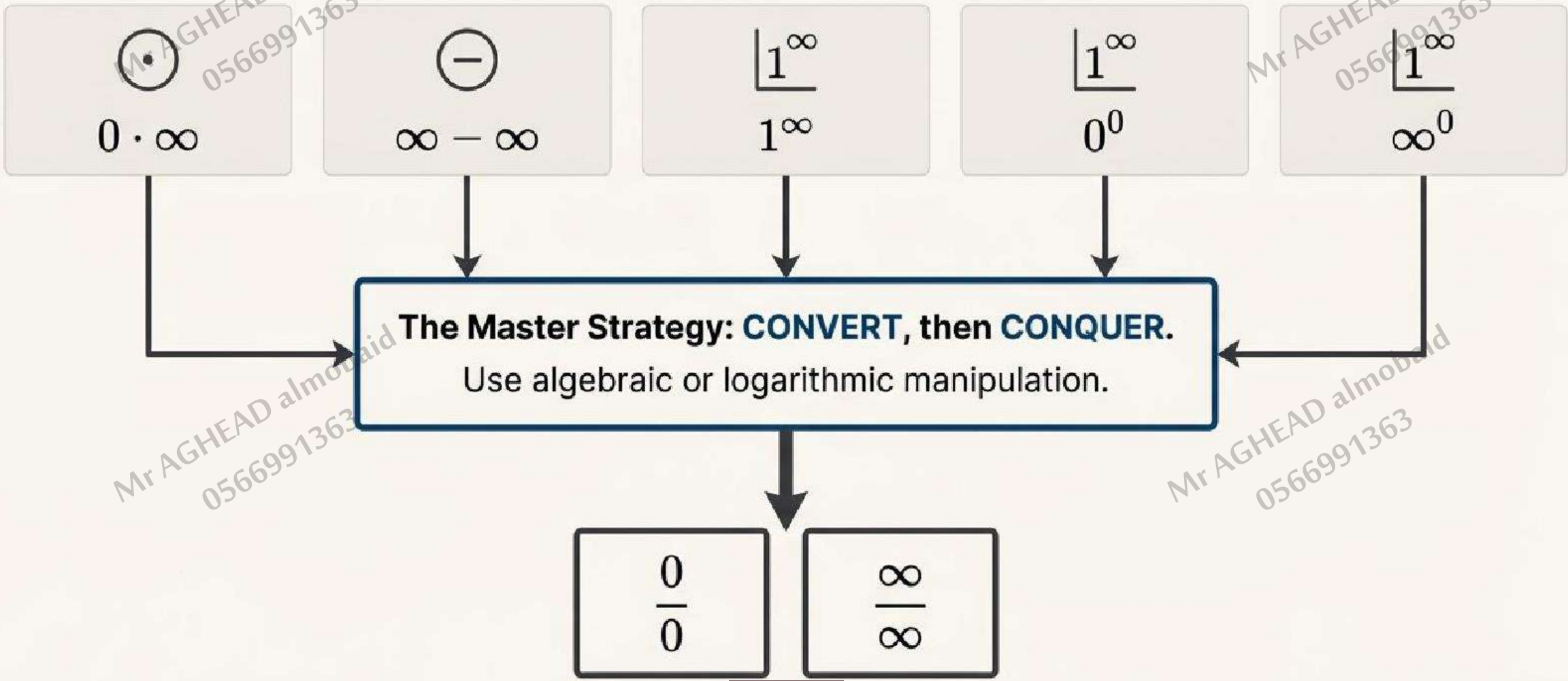
Quotient of the derivatives

Derivative of the quotient

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The Alchemist's Playbook: Other Indeterminate Forms

L'Hôpital's Rule only works directly on $\frac{0}{0}$ and $\frac{\infty}{\infty}$. For other indeterminate forms, our strategy is simple.



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Technique 1: From Products and Differences to Fractions

Form: $0 \cdot \infty$ (Example 2.7)

Problem: $\lim_{x \to \infty} \frac{1}{x} \ln(x)$ as $x \to \infty$.

Technique: Rewrite $f \cdot g$ as $\frac{f}{1/g}$.

Conversion: $\lim_{x \to \infty} \frac{1}{x} \ln(x) = \lim_{x \to \infty} \frac{\ln(x)}{x}$.

Result: This is now an $\frac{\infty}{\infty}$ form, which we can solve. The limit evaluates to **0**.

Form: $\infty - \infty$ (Example 2.6)

Problem: $\lim_{x \to 0} \left[\frac{1}{\ln(x+1)} - \frac{1}{x} \right]$ as $x \to 0$.

Technique: Find a common denominator to combine terms into a single fraction.

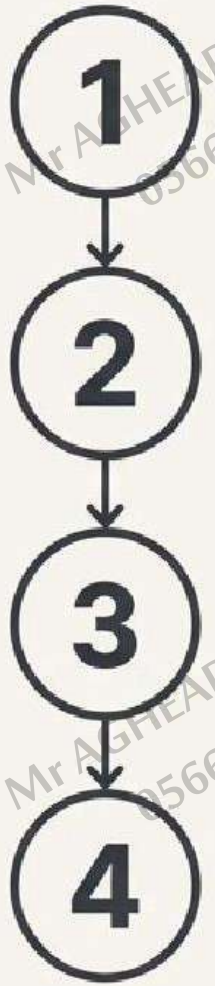
Conversion: $\lim_{x \to 0} \frac{x - \ln(x+1)}{x \ln(x+1)}$.

Result: This is now a 0/0 form. The limit evaluates to **1/2**.

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Technique 2: The Logarithmic Trick for Exponential Forms

Target Forms: 1^∞, 0^0, ∞^0



1 Set Up

Let $y = f(x)^{g(x)}$.

2 $\log$ **Take the Log**

Use log properties to get $\ln(y) = g(x)\ln(f(x))$. This will now be a $0 \cdot \infty$ form. Convert it to a fraction.

3 $\lim_{n\to\infty}$ **Find the Limit of the Log**

Evaluate $L = \lim_{L\to y} \ln(y)$ using L'Hôpital's Rule.

4 e^x **Exponentiate**

e^x The original limit is e^L.

CRITICAL: Do not forget this final step!

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Exponential Forms in Action: Applying the Log Trick

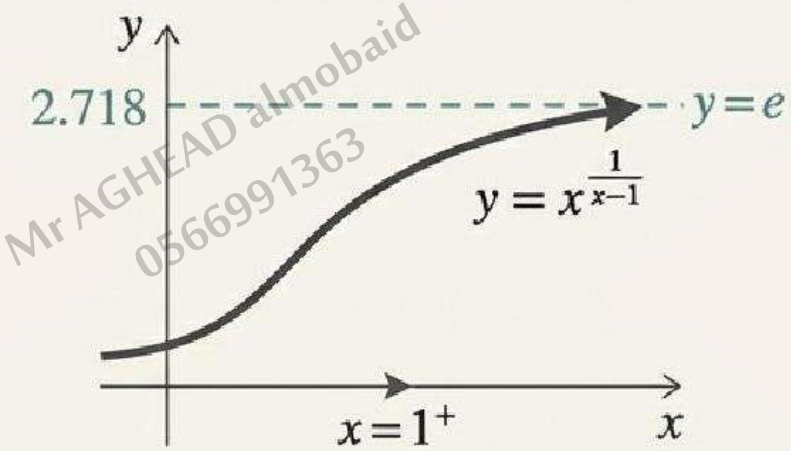
Example 1: The 1^∞ Form (Example 2.8)

Problem: $\lim_{x\to 1^+} x^{1/(x-1)}$ as $x\to 1^+$.

Let $y = x^{1/(x-1)}$. Then $\ln(y) = \frac{\ln(x)}{x-1}$.

$$L = \lim_{x\to 1^+} \ln(y) = \lim_{x\to 1^+} \frac{\frac{1}{x}}{1} = 1.$$

Final Answer: $\lim_{x\to 1^+} y = e^L = e^1 = e$



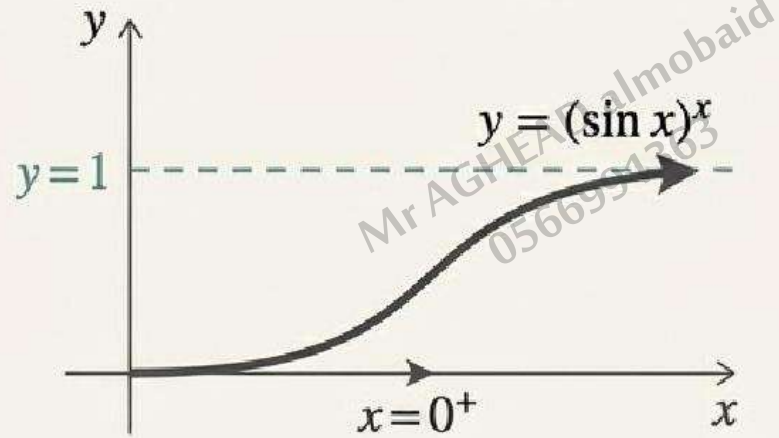
Example 2: The 0^0 Form (Example 2.9)

Problem: $\lim_{x\to 0^+} (\sin x)^x$ as $x\to 0^+$.

Let $y = (\sin x)^x$. Then $\ln(y) = x\ln(\sin x) = \frac{\ln(\sin x)}{1/x}$.

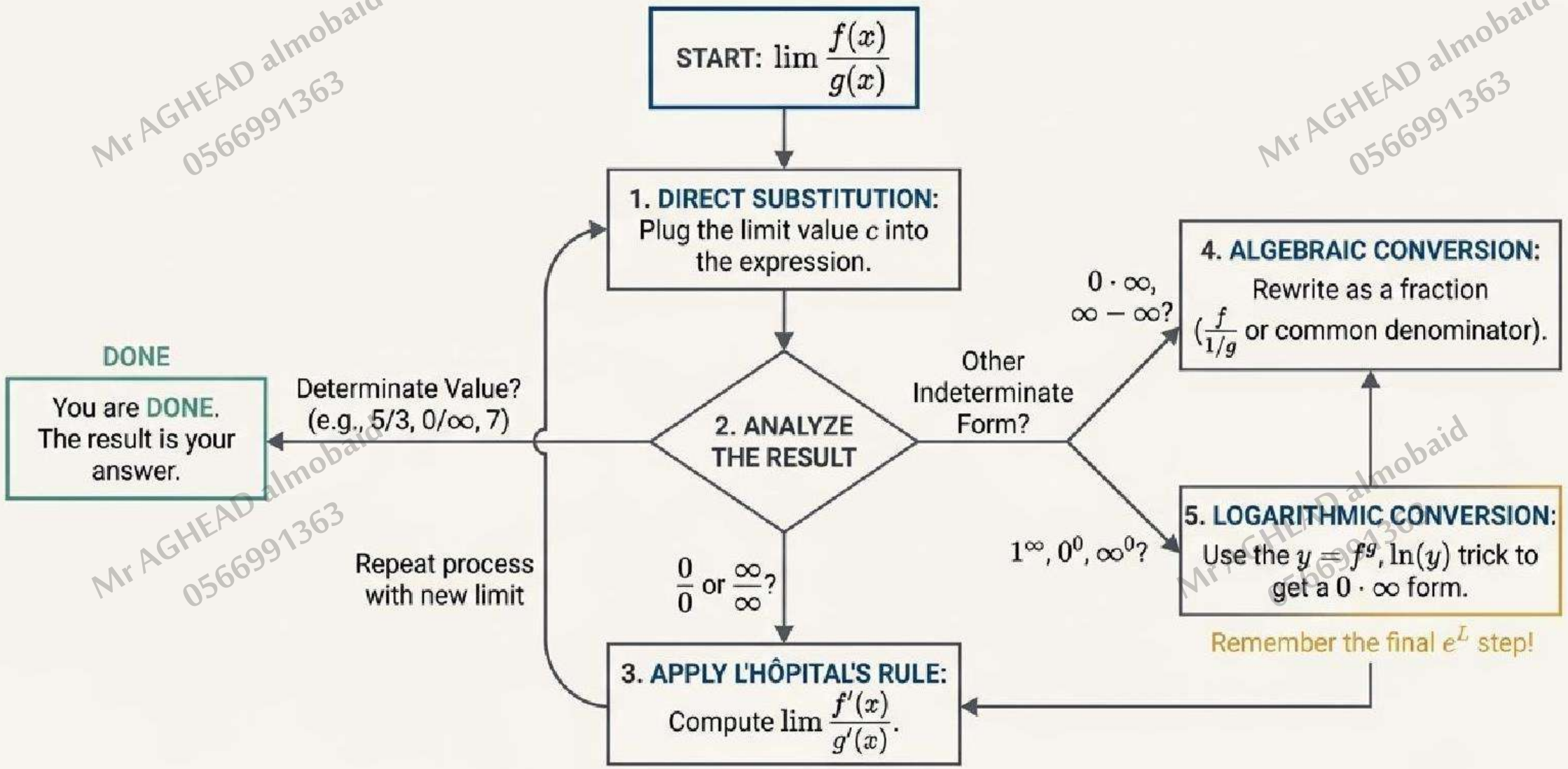
$L = \lim \ln(y) = \cdots = 0$. (Calculation requires two applications).

Final Answer: $\lim_{x\to 0^+} y = e^L = e^0 = 1$



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Your Complete Workflow: The L'Hôpital's Rule Decision Tree



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Practice and Mastery: Sharpen Your Skills

For each limit, first identify the indeterminate form, then choose your strategy and solve.

Problem 1 (Standard `0/0`)

$$\lim_{t \to 0} \frac{e^{2t} - 1}{t}$$

(from Exercise 5)

Problem 2 (Repeated `∞/∞`)

$$\lim_{x \to \infty} \frac{x^3}{e^x}$$

(from Exercise 15)

Problem 3 (Difference `∞ − ∞`)

$$\lim_{x \to 0} \left(\cot x - \frac{1}{x} \right)$$

(from Exercise 18)

Problem 4 (Power `1^∞`)

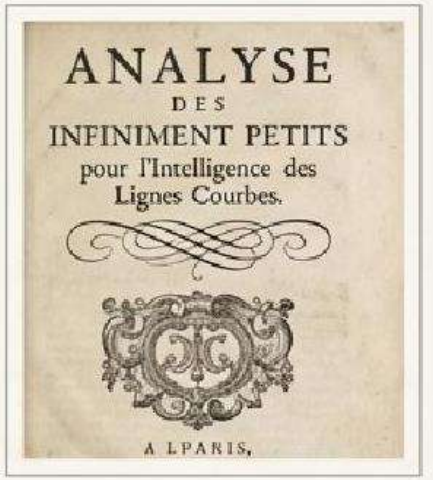
$$\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^x$$

(from Exercise 33, the definition of `e!)

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A Moment in History: Who Was L'Hôpital?





Profile: Guillaume de l'Hôpital (1661–1704) was a French mathematician and member of the nobility.

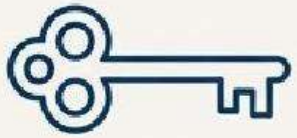
Contribution: He is best known as the author of the first calculus textbook, 'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes'.

The Story Behind the Rule: The rule that bears his name is believed to have been discovered by his brilliant teacher, Johann Bernoulli. As l'Hôpital was Bernoulli's patron and published the result first in his textbook, his name became associated with it.

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You Have Mastered the Art of the Indeterminate

1. **Indeterminate forms are ambiguous**; they require deeper analysis than direct substitution.
2. **L'Hôpital's Rule is a direct tool for 0/0 and ∞/∞**, comparing the functions' rates of change.
3. **All other indeterminate forms are solved through conversion**: use algebra or logarithms to create a 0/0 or ∞/∞ form.
4. **Always verify the form before applying the rule!** Applying it to a determinate form will give the wrong answer.



You now possess a complete and systematic toolkit for tackling the most challenging limits.

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Chapter (4): Applications of Differentiation

03 Third Lesson:

Maximum and Minimum Values



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The Quest for Extrema

A Complete Guide to Finding Maximum and Minimum Values

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For Grade 12 Advanced Mathematics
From core concepts to a complete problem-solving method.

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The Goal: Defining Absolute Extrema

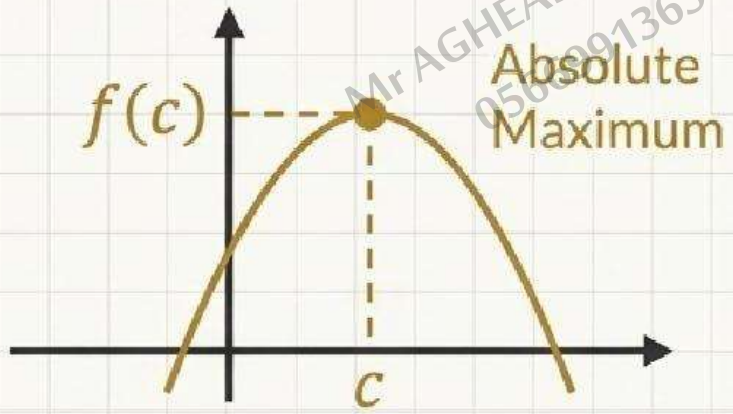
Definition 3.1

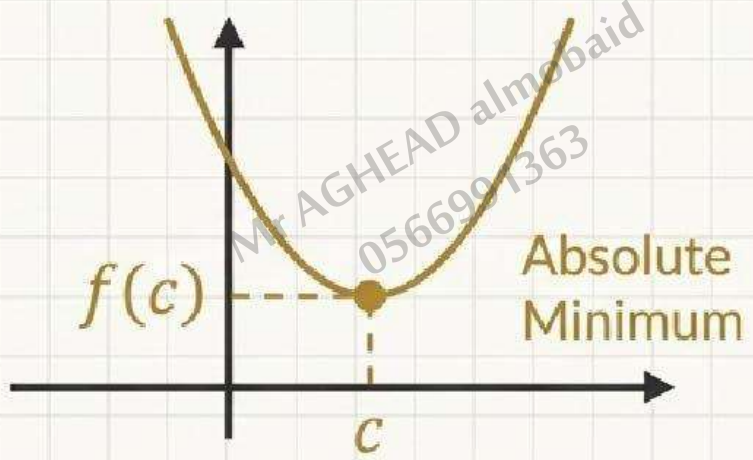
For a function f defined on a set S of real numbers and a number $c \in S$:

(i) $f(c)$ is the **absolute maximum** of f on S if $f(c) \geq f(x)$ for all $x \in S$.

(ii) $f(c)$ is the **absolute minimum** of f on S if $f(c) \leq f(x)$ for all $x \in S$.

An absolute maximum or minimum is called an **absolute extremum**. (Plural: extrema).



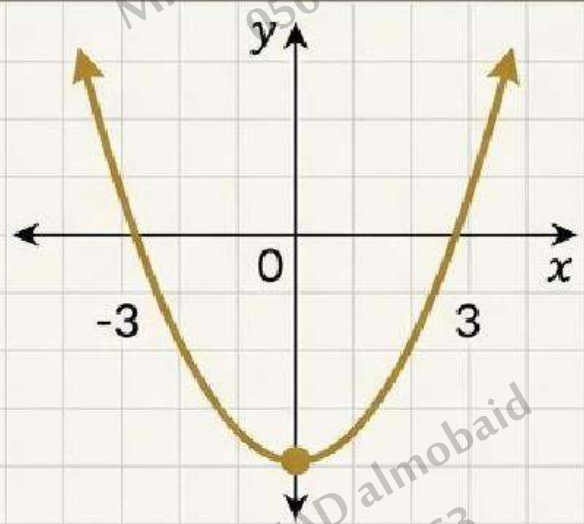
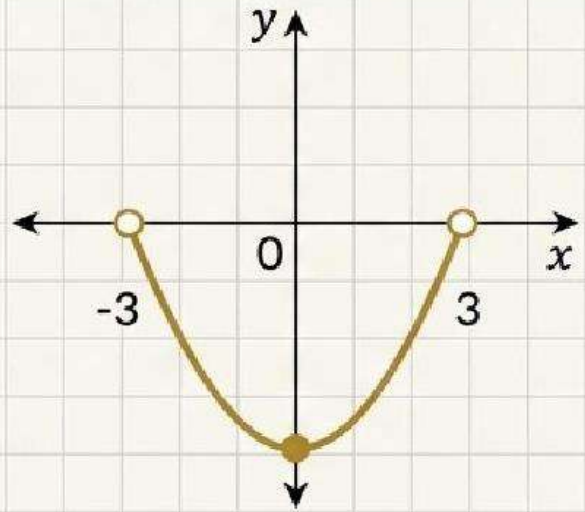
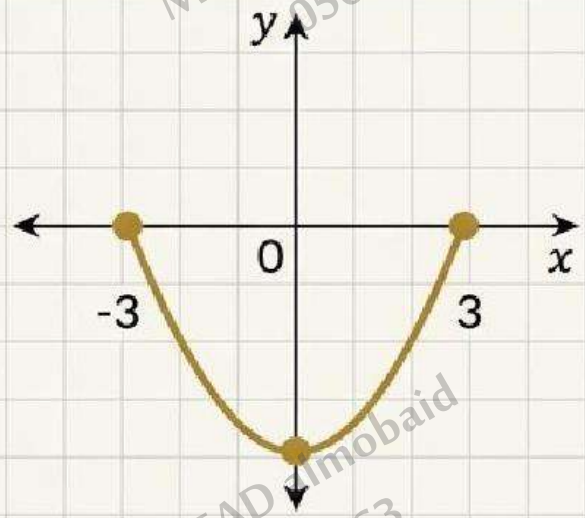


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First Challenge: Do Absolute Extrema Always Exist?

Just because we've defined them, does that mean we can always find them? The answer is no.

Case Study: $f(x) = x^2 - 9$

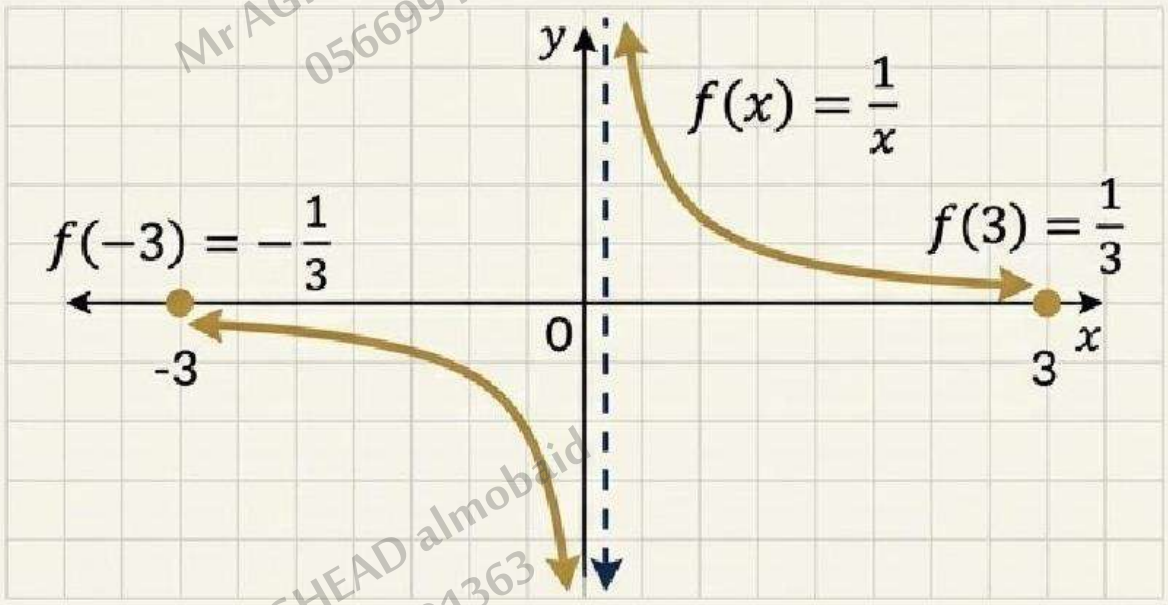
Interval $(-\infty, \infty)$	Open Interval $(-3, 3)$	Closed Interval $[-3, 3]$
		
The function has an absolute minimum of $f(0) = -9$. However, as x approaches ∞ or $-\infty$, $f(x)$ grows without bound. **There is no absolute maximum.**	The function still has an absolute minimum of $f(0) = -9$. The function approaches 0 at the boundaries, but never reaches it. **There is no absolute maximum.**	The function has an absolute minimum of $f(0) = -9$. Now, because the endpoints are included, the function has an absolute maximum of 0, which occurs at $f(-3) = 0$ and $f(3) = 0$.

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Second Challenge: The Problem with Breaks

A function's continuity is just as important as its domain.

Case Study: $f(x) = \frac{1}{x}$ on the domain $[-3, 0) \cup (0, 3]$



Supporting Data

Table 1

x	1/x
0.1	10
0.01	100
0.001	1000

Table 2

x	1/x
-0.1	-10
-0.01	-100
-0.001	-1000

- As x approaches 0 from the right, $f(x)$ increases towards $+\infty$.
- As x approaches 0 from the left, $f(x)$ decreases towards $-\infty$.
- Because of the infinite discontinuity at x=0, the function fails to have either an **absolute maximum** or an **absolute minimum** on this domain.

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The Golden Guarantee: The Extreme Value Theorem

Having seen how functions can fail to have extrema, we now introduce the theorem that tells us exactly when they are guaranteed to exist.

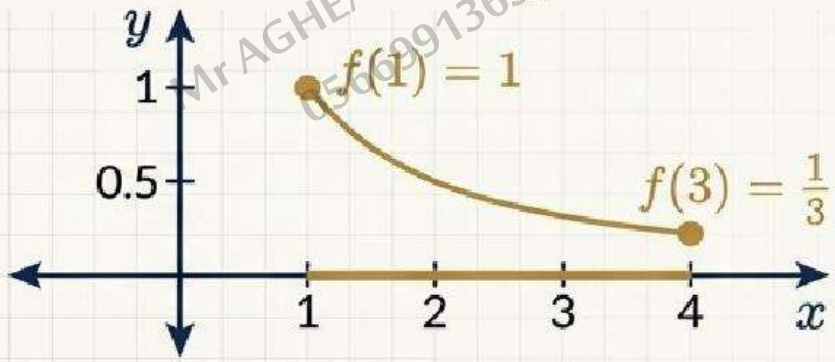
Theorem 3.1 (Extreme Value Theorem)

A **continuous function** f defined on a **closed, bounded interval** $[a, b]$ attains both an absolute maximum and an **absolute minimum** on that interval.

Crucial Conditions

1. The function *must* be **continuous** on $[a, b]$.
2. The interval *must* be **closed** and **bounded**.

Example: $f(x) = \frac{1}{x}$ on the interval $[1, 3]$



On this interval, the function is continuous and the interval is closed. The theorem guarantees extrema exist.

By inspection, the absolute maximum is $f(1) = 1$ and the absolute minimum is $f(3) = \frac{1}{3}$.

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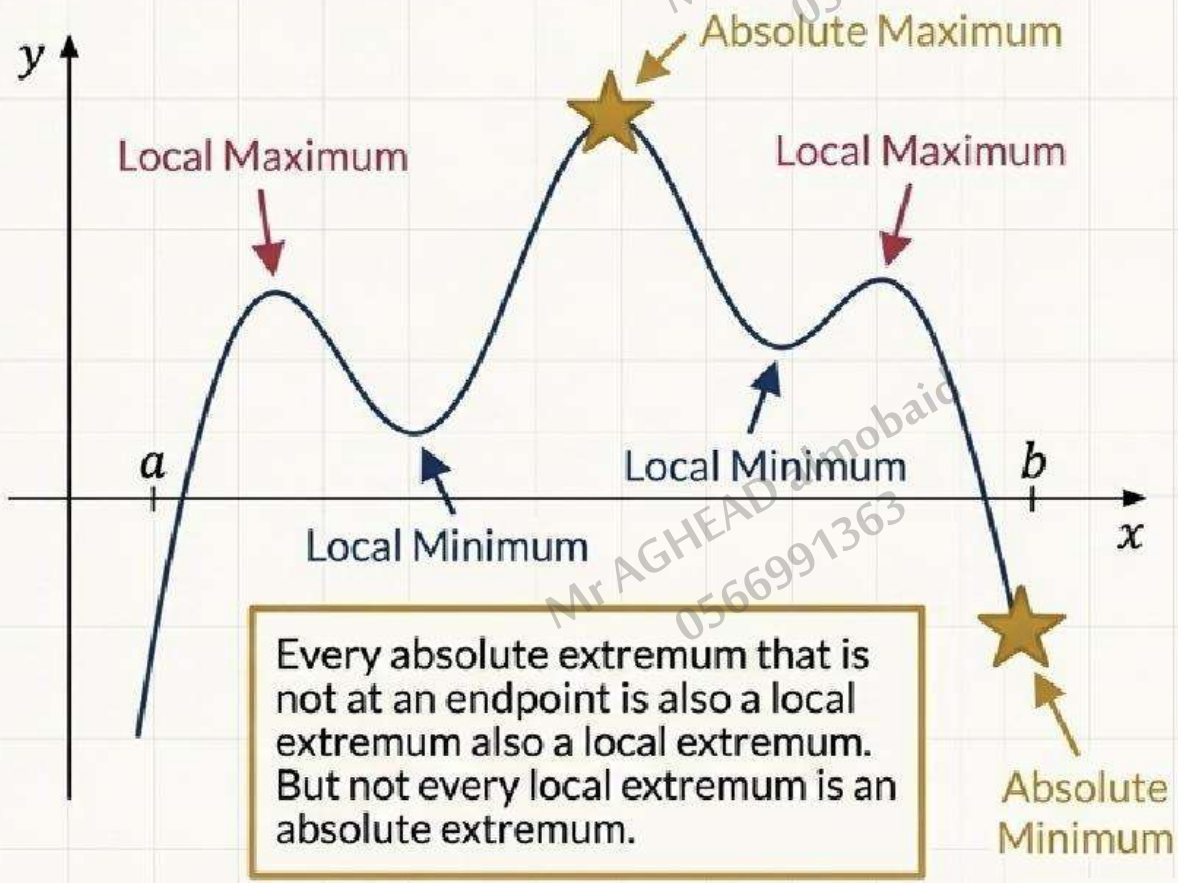
A Closer Look: Finding Extrema within the Interval

Absolute extrema can occur at the endpoints of an interval, but they can also occur at peaks and valleys within the interval. We call these "local" extrema.

Definition 3.2

$f(c)$ is a ***local maximum*** if $f(c) \geq f(x)$ for all x in some open interval containing c.

$f(c)$ is a ***local minimum*** if $f(c) \leq f(x)$ for all x in some open interval containing c.



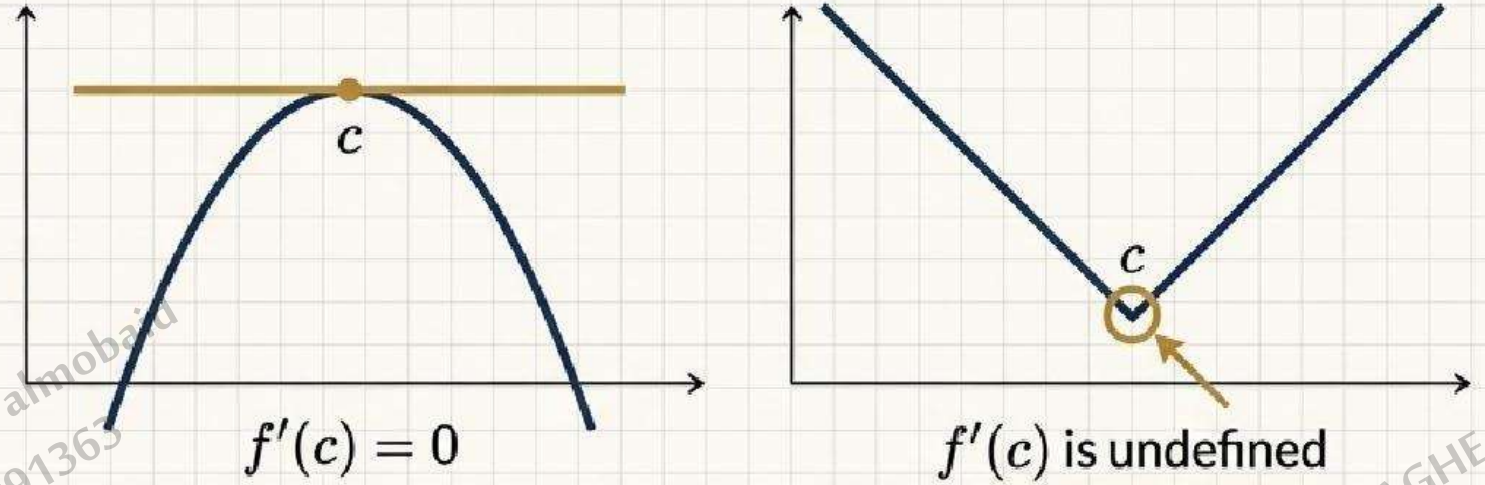
Every absolute extremum that is not at an endpoint is also a local extremum also a local extremum. But not every local extremum is an absolute extremum.

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The Telltale Sign: Where Local Extrema Hide

Key Insight

- At a smooth "hill" or "valley," the tangent line is horizontal. This means the derivative is zero.
- At a sharp "corner" or "cusp," the tangent line is not defined. This means the derivative is undefined.



$f'(c) = 0$ $f'(c)$ is undefined

This observation leads us to a powerful theorem.

Theorem 3.2 (Fermat's Theorem)

If $f(c)$ is a local extremum (local maximum or minimum), then c must be a critical number of f.

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The Suspects: Defining Critical Numbers

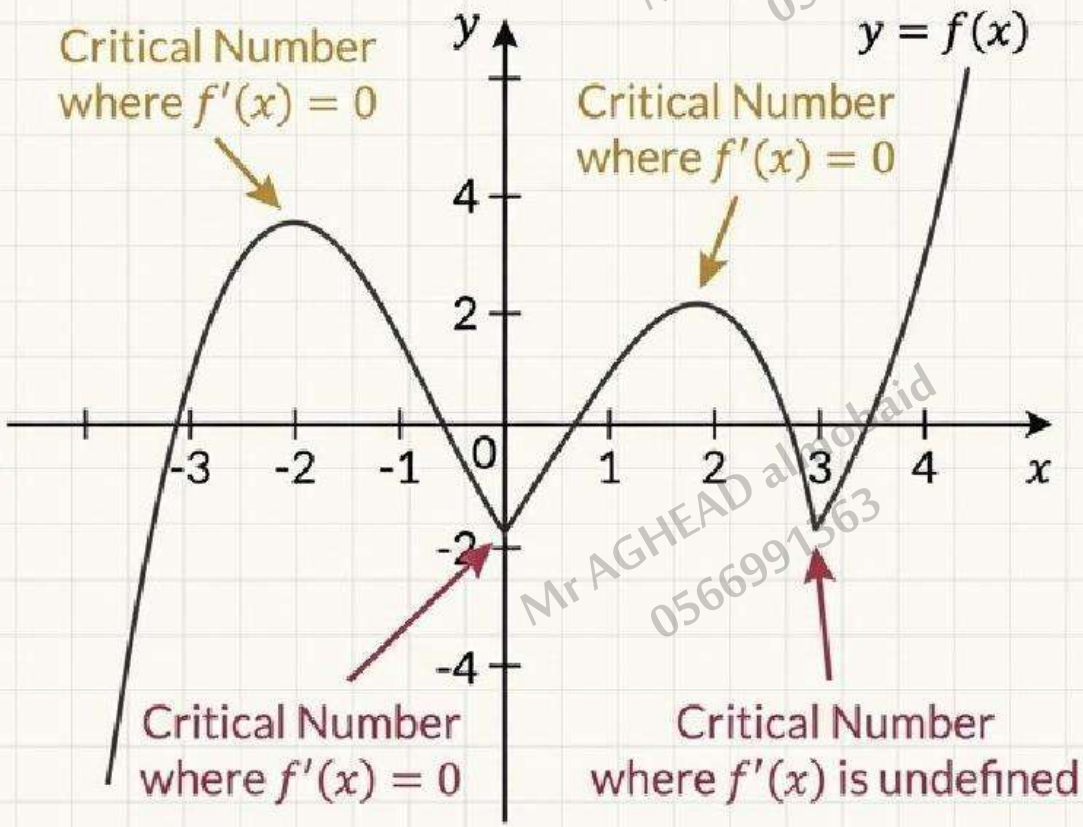
Fermat's Theorem tells us that local extrema can only occur at very specific points in the domain. We give these points a special name.

Definition 3.3

A number c in the domain of a function f is called a **critical number** of f if either:

1. $f'(c) = 0$
2. $f'(c)$ is undefined

The set of critical numbers contains all possible locations for local extrema. To find the highest and lowest points, we no longer have to check every point—we only have to check a few special ones!



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Practice Session: Finding Critical Numbers

For each function, find all critical numbers by finding where $f'(x) = 0$ or is undefined.

Example 1 (Polynomial)

Function: $f(x) = x^3 - 3x^2 + 2$

Derivative: $f'(x) = 3x^2 - 6x = 3x(x-2)$

Analysis: $f'(x) = 0$ when $x = 0$ or $x = 2$.

Critical Numbers: 0, 2

Example 2 (Root Function)

Function: $f(x) = x^2\sqrt{(x+2)}$

Derivative: $f'(x) = 2x\sqrt{(x+2} + \frac{x^2}{2\sqrt{(x+2}}$ (after simplifying)

Analysis: $f'(x) = 0$ when x=0, x=-8/5. $f'(x)$ is undefined when x = -2.

Critical Numbers: -2, -8/5, 0

Example 3 (Rational Function)

Function: $f(x) = \frac{x^2-x+4}{x-1}$

Derivative:

$f'(x) = \frac{(2x-1)(x-1)-(x^2-x+4)]}{(x-1)^2}$

$= \frac{x^2-2x-3}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2}$

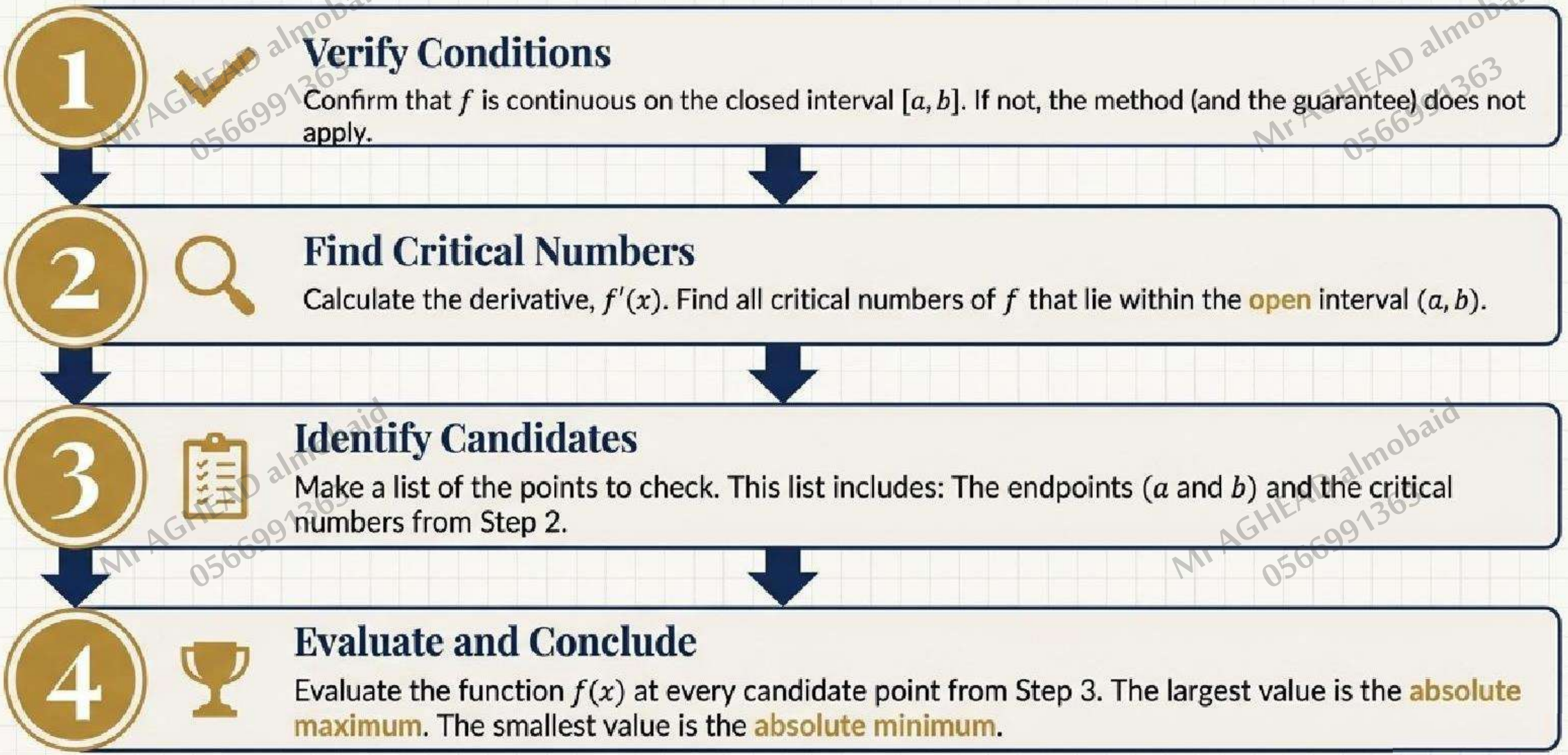
Analysis: $f'(x) = 0$ when $x = 3$ or $x = -1$. $f'(x)$ is undefined at $x = 1$, but 1 is not in the domain of f.

Critical Numbers: -1, 3

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The Master Plan: The Closed Interval Method

We now have a complete, step-by-step method to find the absolute maximum and minimum of a continuous function f on a closed interval $[a, b]$.



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Applying the Method, Example 1: A Polynomial

Find the absolute extrema of $f(x) = x^3 - 6x^2 + 5$ on the interval $[0, 3]$.

Step 1: Conditions

✓ $f(x)$ is a polynomial, so it is continuous everywhere, including on $[0, 3]$. The method applies.

Step 2: Critical Numbers

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Set $f'(x) = 0$: $x = 0, x = 4$.

Only $x = 0$ is a candidate, but as it is an endpoint we consider it in the next step. $x = 4$ is outside the interval $(0, 3)$, so we discard it. There are no critical numbers inside the open interval.

Step 3: Candidates

Endpoints: $x = 0, x = 3$.

Step 4: Evaluate

Candidate (x)	Evaluation $f(x) = x^3 - 6x^2 + 5$	Result
0 (Endpoint)	$f(0) = 0 - 0 + 5$	5
3 (Endpoint)	$f(3) = 27 - 6(9) + 5 = 27 - 54 + 5$	-22

The Absolute Maximum is 5, which occurs at $x = 0$. The Absolute Minimum is -22, which occurs at $x = 3$.

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Applying the Method, Example 2: A Root Function

Find the absolute extrema of $f(x) = x\sqrt{(4 - x^2}$ on the interval $[0, 2]$.

Step 1: Conditions

✓ The function is continuous on its domain, [-2, 2], so it is continuous on [0, 2]. The method applies.

Step 2: Critical Numbers

$$f'(x) = (1)\sqrt{4 - x^2} + x * \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x)$$

$$f'(x) = \frac{[redacted] - x^2}{\sqrt{4 - x^2}} \quad + \sqrt{4 - x^2}$$

$$f'(x) = \frac{4 - x^2 - x^2}{\sqrt{4 - x^2}} = \frac{4 - 2x^2}{\sqrt{4 - x^2}}$$

$f'(x) = 0$ when $4 - 2x^2 = 0 \rightarrow x^2 = 2 \rightarrow x = \sqrt{2}$ (we ignore $-\sqrt{2}$).

$f'(x)$ is undefined when $4 - x^2 = 0 \rightarrow x = 2$ or $x = -2$. $x = 2$ is an endpoint.

Critical number in (0, 2) is $x = \sqrt{2}$.

Step 3: Candidates

Endpoints: $x = 0, x = 2$.
Critical Number: $x = \sqrt{2}$.

Step 4: Evaluate

Candidate (x)	Evaluation $f(x) = x\sqrt{4 - x^2}$	Result
0 (Endpoint)	$f(0) = 0\sqrt{(4)} = 0$	0
$\sqrt{2}$ (Critical)	$f(\sqrt{2}) = \sqrt{2}\sqrt{4 - 2} = \sqrt{2} * \sqrt{2}$	2
2 (Endpoint)	$f(2) = 2\sqrt{(4 - 4} = 0$	0

Conclusion

The Absolute Maximum is 2, which occurs at $x = \sqrt{2}$.
The Absolute Minimum is 0, which occurs at $x = 0$ and $x = 2$.

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Applying the Method, Example 3: A Trigonometric Function

Find the absolute extrema of $f(x) = 2\cos(x) + \sin(2x)$ on the interval $[0, \pi/2]$.

Step 1: Conditions

✓ Trigonometric functions are continuous everywhere. The method applies.

Step 2: Critical Numbers

$f'(x) = -2\sin(x) + 2\cos(2x)$

$f'(x) = -2\sin(x) + 2(1 - 2\sin^2(x)) = -4\sin^2(x) - 2\sin(x) + 2$

Set $f'(x) = 0$: $-2(2\sin^2(x) + \sin(x) - 1) = 0 \rightarrow -2(2\sin(x) - 0)$

$\rightarrow -2(2\sin(x) - 1)(\sin(x) + 1) = 0.$

This gives $\sin(x) = \frac{1}{2}$ or $\sin(x) = -1$.

In the interval $\left(0, \frac{\pi}{2}\right)$, $\sin(x) = \frac{1}{2}$ when $x = \frac{\pi}{6}$.

Critical number in $\left(0, \frac{\pi}{2}\right)$ is $x = \frac{\pi}{6}$.

Step 3: Candidates

Endpoints: $x = 0, x = \frac{\pi}{2}$.

Critical Number: $x = \frac{\pi}{6}$.

Step 3: Candidates

Endpoints: $x = 0, x = \pi/2$.

Critical Number: $x = \pi/6$.

Step 4: Evaluate

Candidate (x)	Evaluation f(x) = 2cos(x) + sin(2x)	Result
0 (Endpoint)	$f(0) = 2(1) + 0 = 2$	2
$\frac{\pi}{6}$ (Critical)	$f\left(\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2}\right) + \sin\left(\frac{\pi}{3}\right) = \sqrt{3} + \frac{\sqrt{3}}{2}$	$\frac{3\sqrt{3}}{2} \approx 2.598$
$\frac{\pi}{2}$ (Endpoint)	$f\left(\frac{\pi}{2}\right) = 2(0) + \sin(\pi) = 0$	0

Conclusion:

The **Absolute Maximum** is $\frac{3\sqrt{3}}{2}$, at $x = \frac{\pi}{6}$.

The **Absolute Minimum** is **0**, at $x = \frac{\pi}{2}$.

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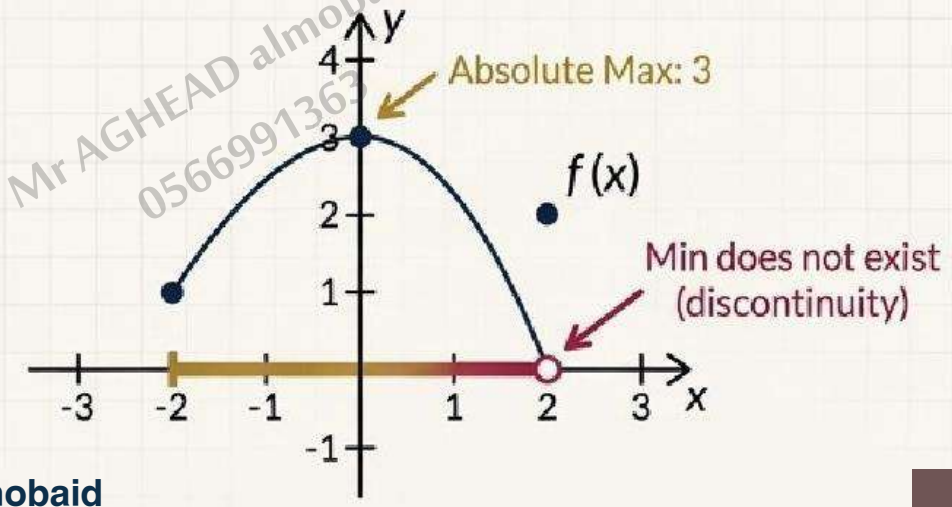
From Algorithm to Intuition: Visualizing the Concepts

Challenge: Sketch a graph of a function that meets the following criteria.

Scenario 1

Goal: Sketch a function $f(x)$ on the closed interval $[-2, 2]$ where the absolute maximum is 3, but the absolute minimum does not exist.

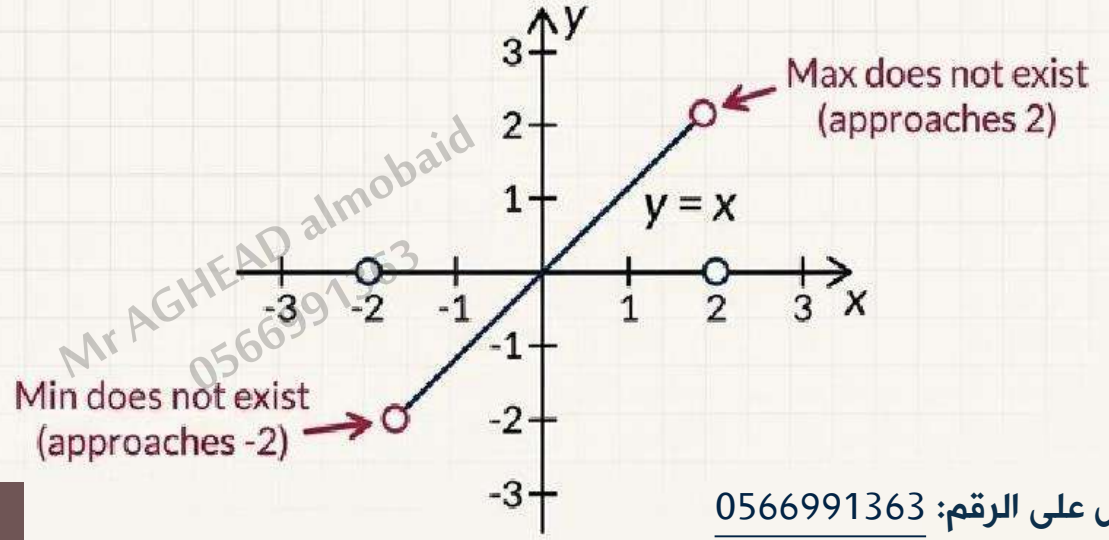
How is this possible? The Extreme Value Theorem requires continuity. The only way for an extremum not to exist on a closed interval is if the function is discontinuous.



Scenario 2

Goal: Sketch a function $f(x)$ on the open interval $(-2, 2)$ where neither the absolute maximum nor the absolute minimum exists.

How is this possible? On an open interval, the function can approach its max/min values at the endpoints without ever reaching them.



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The Power of Extrema: From Theory to Application

The Big Picture



The concepts we've mastered are the foundation of optimization. In business, science, and engineering, the goal is often to find the "best" outcome. This means finding ways to:

- Maximize profit, efficiency, or yield.
- Minimize cost, waste, or risk.

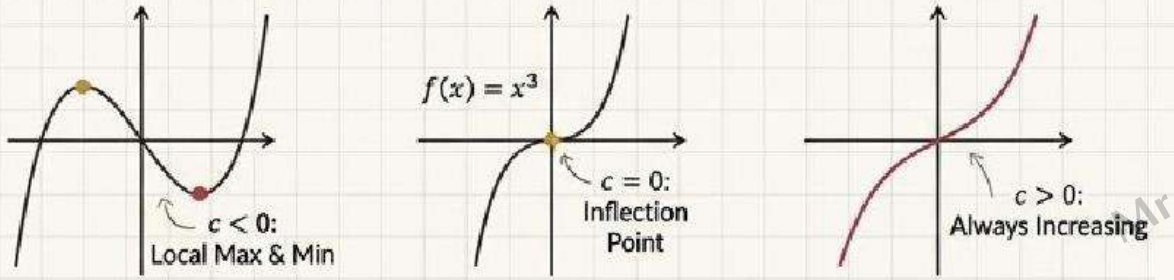
The Closed Interval Method is our first powerful tool for solving these real-world optimization problems.



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A Look Ahead

How do these ideas apply to entire families of functions?
Consider all cubic functions of the form $f(x) = x^3 + cx$.
The number and type of local extrema depend entirely on the value of the constant c.
Understanding critical numbers allows us to describe the behavior of not just one function, but infinite families of them.



The quest for extrema is the quest for the optimal solution.
You now have the map and the tools to find it.

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Chapter (4): Applications of Differentiation

04 Fourth Lesson:
Increasing and Decreasing Functions



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Mastering Function Behavior: An Analyst's Guide to Increasing & Decreasing Functions

تزايد وتناقص الدوال: دليلك لإتقان التحليل

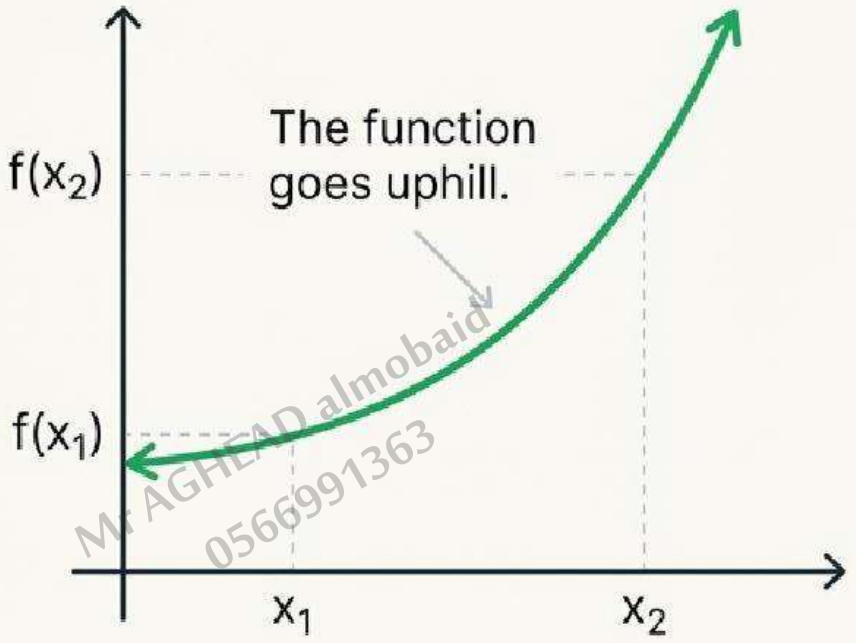
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The Visual Story: What Does a Graph Tell Us?

How can we describe the journey of a function as we move from left to right?

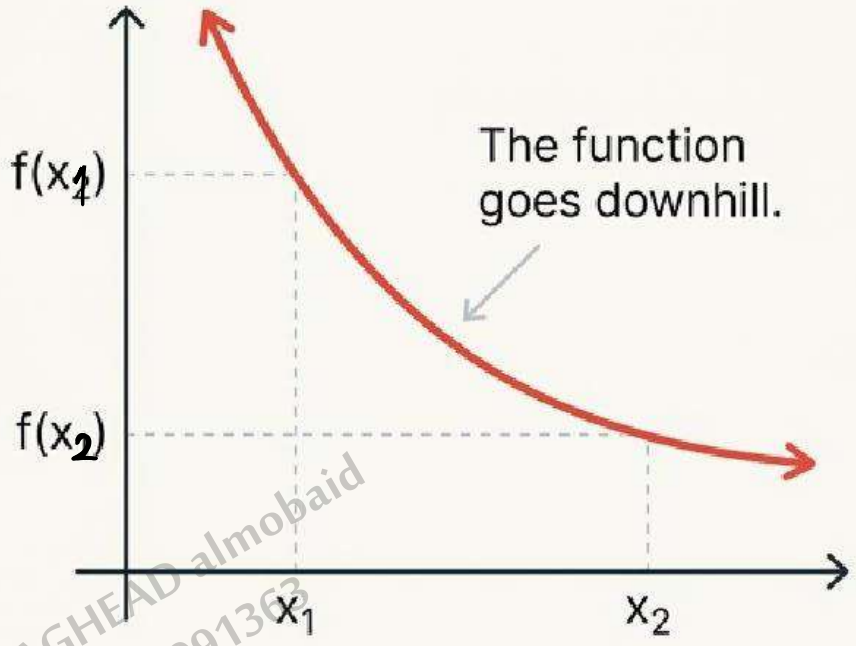
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Increasing (متزايدة)



For any $x_1 < x_2$, it's true that $f(x_1) < f(x_2)$.

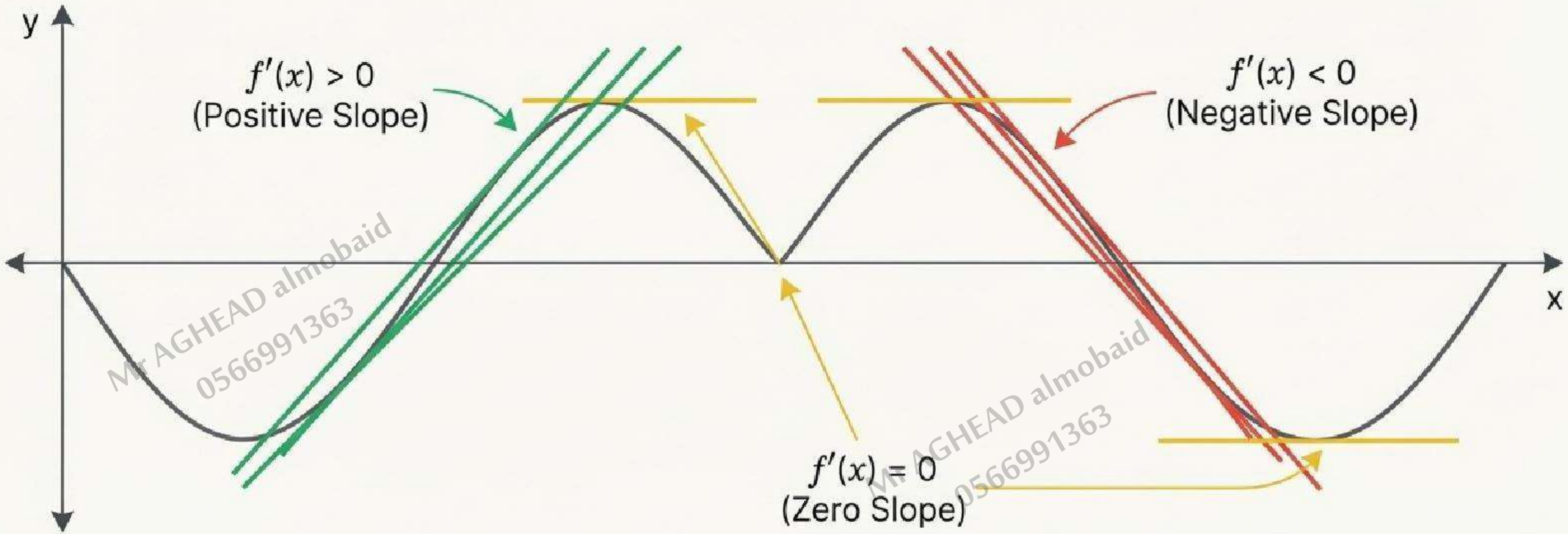
Decreasing (متناقصة)



For any $x_1 < x_2$, it's true that $f(x_1) > f(x_2)$.

The Calculus Connection: The Power of the Derivative (f')

The derivative, $f'(x)$, tells us the slope the **slope** of the **tangent line** at any point. This slope reveals the function's direction.

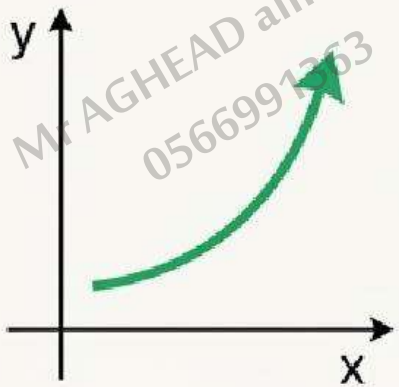


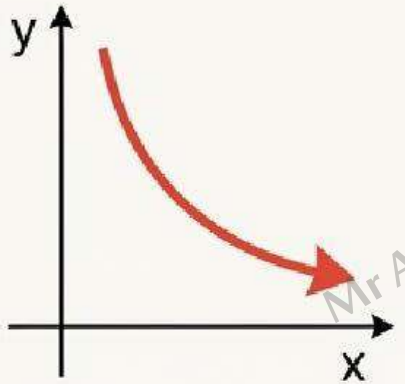
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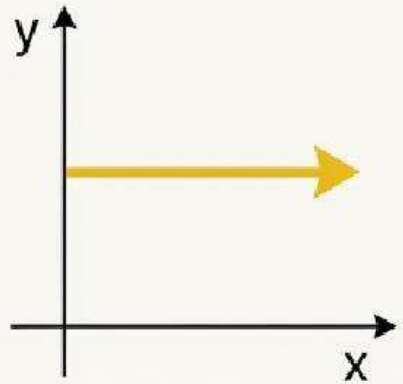
The Core Principle: Connecting $f'(x)$ to Function Behavior

Suppose a function f is differentiable on an interval I.

Column 1	Column 2	Column 3
If $f'(x) > 0$ for all x in I, then f is **increasing** on I.	If $f'(x) < 0$ for all x in I, then f is **decreasing** on I.	If $f'(x) = 0$ for all x in I, then f is **constant** on I.
إذا كانت $f'(x)$ 0 لكل x في الفترة، فإن f تكون متزايدة.	إذا كانت $f'(x)$ 0 لكل x في الفترة، فإن f تكون متناقصة.	إذا كانت $f'(x) =$ 0 لكل x في الفترة، فإن f تكون ثابتة.







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The Analyst's Workflow: A 3-Step Method to Determine Intervals

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1

Find the Critical Numbers (الأعداد الحرجة)

Solve for x where $f'(x) = 0$ or $f'(x)$ is undefined. These are the only points where the function can change direction.

2

Engun for (دراسة السيرجة)

Place the critical numbers on a number line. This divides the line into test intervals. Pick a test $f'(x)$ to determine if the sign is (+) or (-).

Create a Sign Chart (دراسة إشارة المشتقة)

Place the critical numbers on a number line. This divides the line into test intervals. Pick a test value in each interval and plug it into $f'(x)$ to determine

3

State Your Conclusion (تحديد سلوك الدالة)

Use the **Core Principle**. Where the sign chart is (+), the function is **increasing**. Where it is (-), the function is **decreasing**. Write down the intervals.

Worked Example: Analyzing a Polynomial Function

$$f(x) = 2x^3 + 9x^2 - 24x - 10$$

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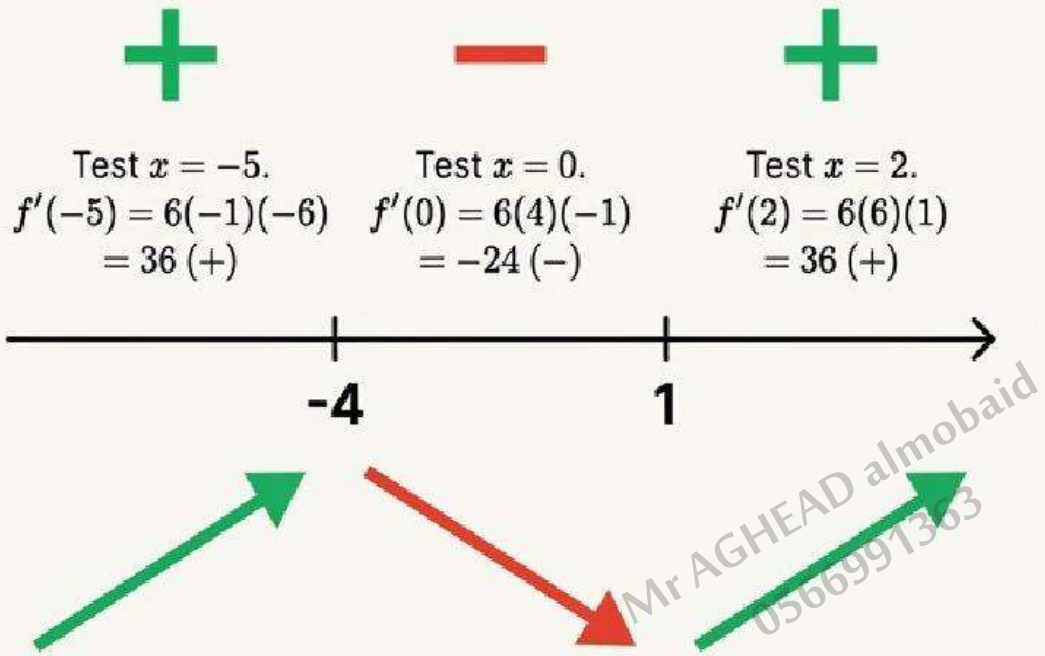
Step 1: Find Critical Numbers

Derivative:
$f'(x) = 6x^2 + 18x - 24$

Factor:
$f'(x) = 6(x^2 + 3x - 4)$
$= 6(x + 4)(x - 1)$

Solve $f'(x) = 0$:
$x = -4$, $x = 1$,
$x = 1$. These are the critical numbers.

Step 2: Create Sign Chart



Step 3: State Conclusion

Increasing (متزايدة) on:
$(-\infty, -4) \cup (1, \infty)$
because $f'(x) > 0$.

Decreasing (متناقصة) on:
$(-4, 1)$ because $f'(x) < 0$.

The First Derivative Test: Classifying Local Extrema

Cairo Bold قيمة صغرى

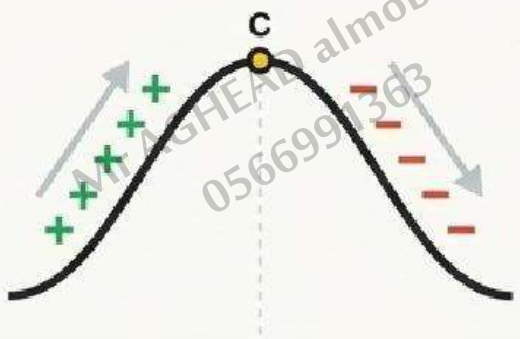
Key Idea: The change in the sign of $f'(x)$ at a critical number c tells us if we have a local maximum, minimum, or neither.

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Local Maximum (قيمة عظمى محلية)

Sign Change at c:
$f'(x)$ changes from + to -

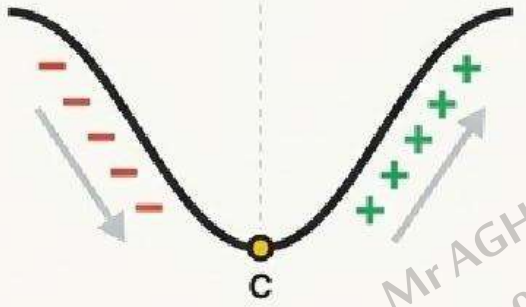
Behavior:
f changes from increasing to decreasing.



Local Minimum (قيمة صغرى محلية)

Sign Change at c:
$f'(x)$ changes from - to +

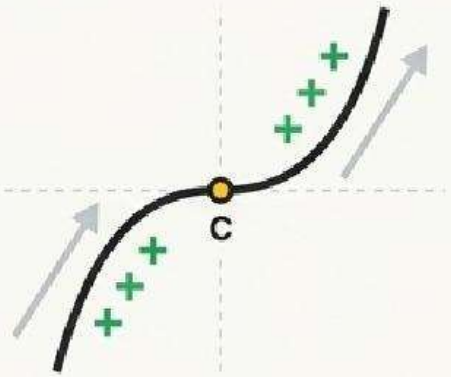
Behavior:
f changes from decreasing to increasing.



Not an Extremum (ليست قيمة قصوى)

Sign Change at c:
No change (e.g., + to + or - to -)

Behavior:
f keeps increasing or keeps decreasing.



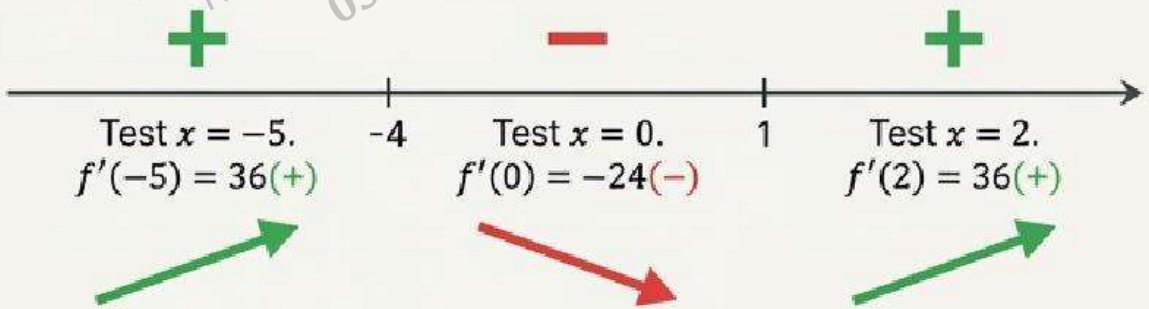
Classifying Extrema: Polynomial Example Revisited

$f(x) = 2x^3 + 9x^2 - 24x - 10$

Analysis

Recall the Sign Chart

The sign chart from slide 6:



Apply the First Derivative Test

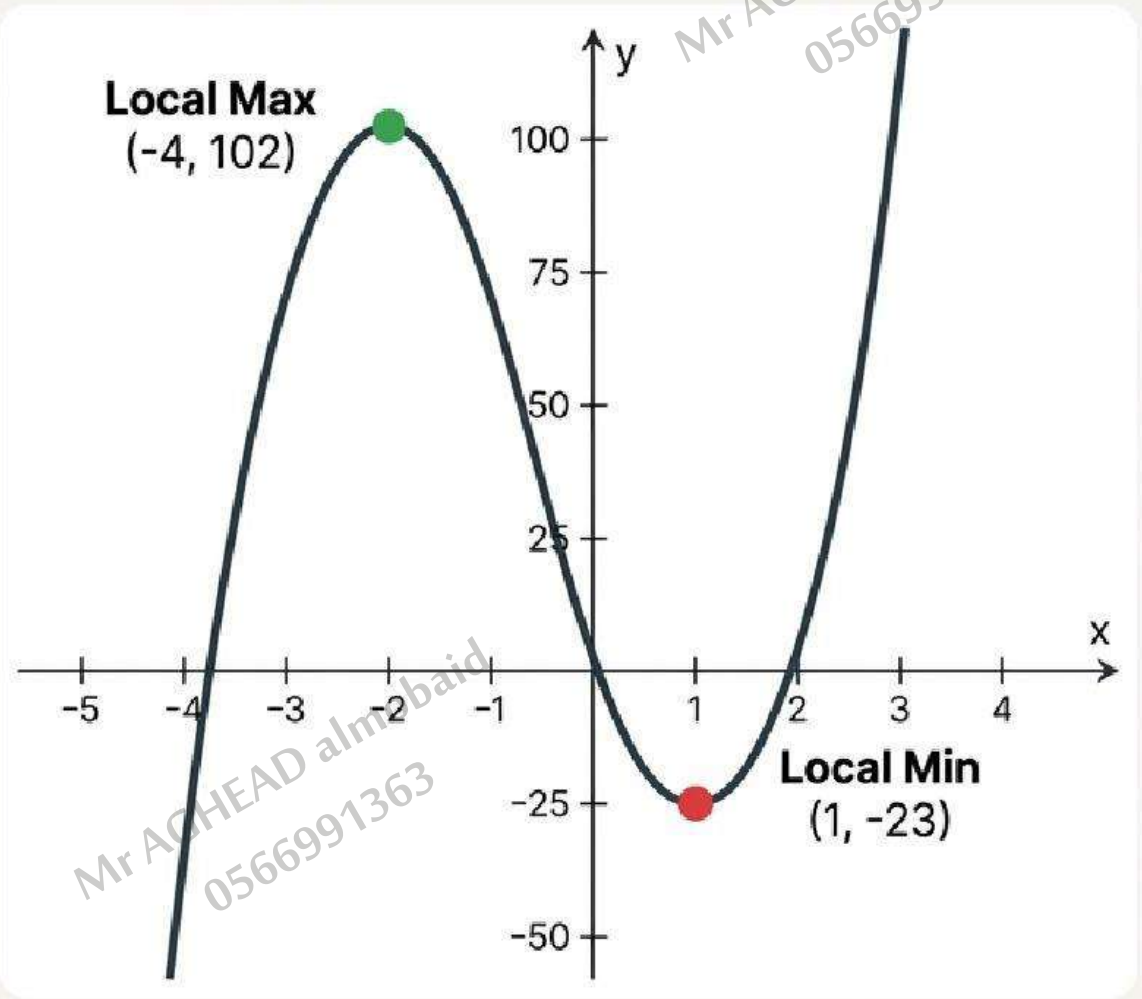
At $x = -4$:

- The sign of f' changes from **(+)** to **(-)**.
- Therefore, there is a **Local Maximum** at $x = -4$.
- The maximum value is $f(-4) = 102$.

At $x = 1$:

- The sign of f' changes from **(-)** to **(+)**.
- Therefore, there is a **Local Minimum** at $x = 1$.
- The minimum value is $f(1) = -23$.

Visualization



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Mastery in Practice: Rational Functions

Function: $f(x) = \frac{x}{x^2 + 1}$

Step 1: Find Critical Numbers

Derivative (Quotient Rule):

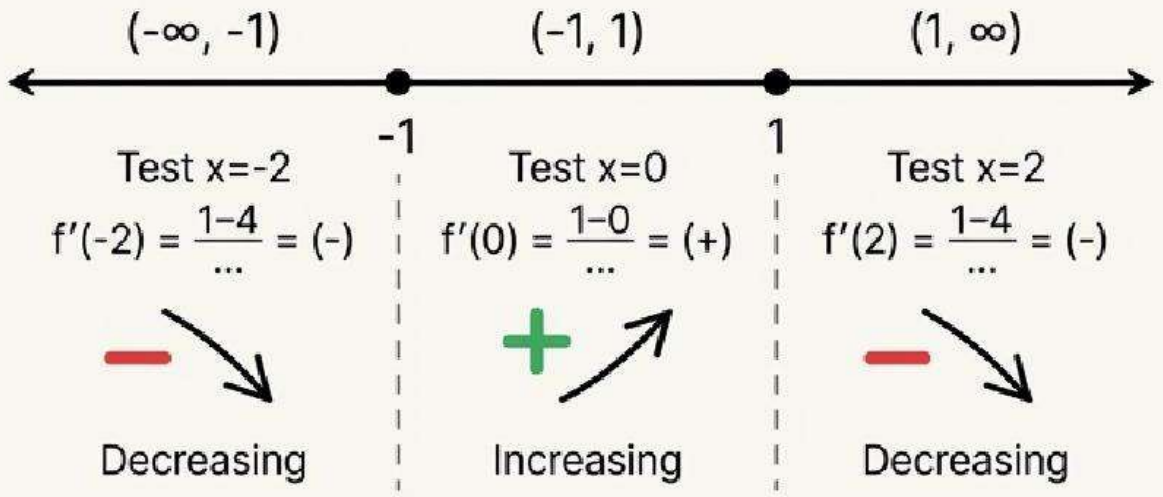
$$f'(x) = \frac{[(1)(x^2 + 1) - (x)(2x)]}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Solve $f'(x) = 0$:

$1 - x^2 = 0 \rightarrow x = 1, x = -1.$

Note: $f'(x)$ is never undefined since the denominator is always positive.

Step 2: Sign Chart



Step 3: Conclusion & Classification

- Increasing on: (-1, 1)
- Decreasing on: (-∞, -1) U (1, ∞)
- Local Minimum at x = -1 (sign changes - to +). Value: f(-1) = -1/2.
- Local Maximum at x = 1 (sign changes + to -). Value: f(1) = 1/2.

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Mastery in Practice: Radicals & Fractional Exponents

$$f(x) = x^{\frac{5}{3}} - 3x^{\frac{2}{3}}$$

Step 1: Find Critical Numbers

Derivative:

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} - 2x^{-\frac{1}{3}} = \frac{5x-6}{3x^{\frac{1}{3}}}$$

Solve $f'(x) = 0$:

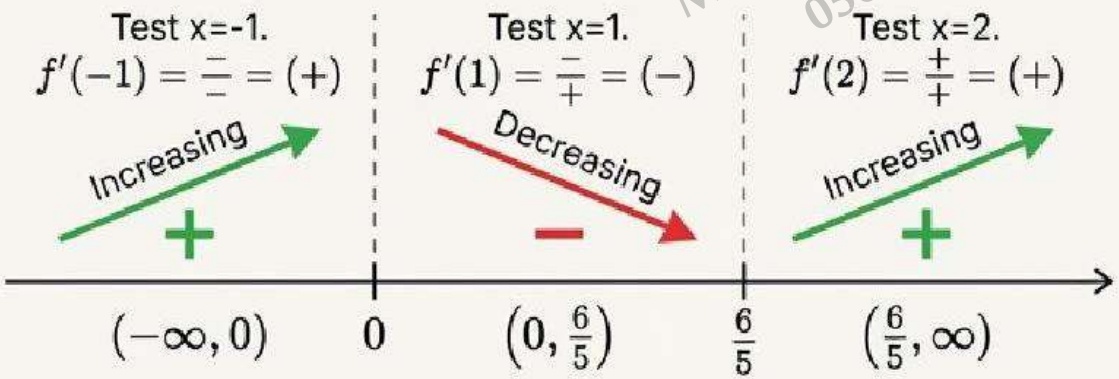
$$5x - 6 = 0 \rightarrow x = \frac{6}{5}$$

Find where $f'(x)$ is undefined:

$$3x^{\frac{1}{3}} = 0 \rightarrow x = 0$$

Critical Numbers are $x = 0$ and $x = \frac{6}{5}$.

Step 2: Sign Chart



Step 3: Conclusion & Classification

Increasing on: $(-\infty, 0) \cup \left(\frac{6}{5}, \infty\right)$

Decreasing on: $\left(0, \frac{6}{5}\right)$

Local Maximum at $x = 0$ (sign changes + to -).
The function has a sharp cusp at this point.

Local Minimum at $x = \frac{6}{5}$ (sign changes - to +).

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Mastery in Practice: Trigonometric & Exponential Functions

Exponential

Function: $f(x) = x^2 e^{-x}$

Derivative: $f'(x) = 2xe^{-x} - x^2e^{-x} = x(2-x)e^{-x}$

Critical Numbers: $x = 0$

$x = 0, x = 2$ (since e^{-x} is never 0).

Sign Chart



Conclusion:

- Increasing on $(0, 2)$ ✅
- Decreasing on $(-\infty, 0) \cup (2, \infty)$ ❌
- Local Min at $x_{min} = 0$, Local Max at $x_{max} = 2$.

Trigonometric

Function: $f(x) = \sin(x) + \cos(x)$ on $[0, 2\pi]$

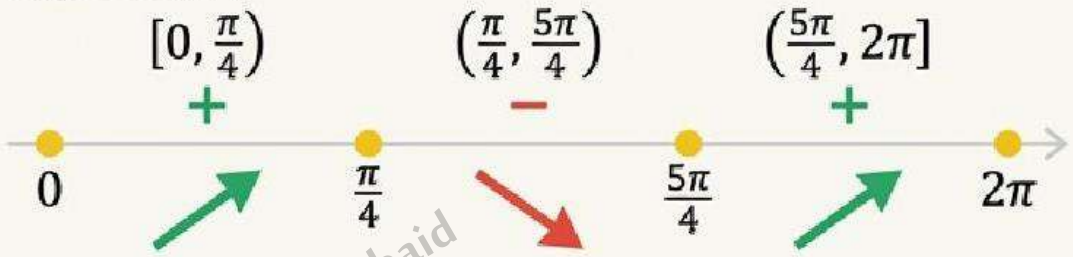
Derivative: $f'(x) = \cos(x) - \sin(x)$

Critical Numbers:

Solve $\cos(x) = \sin(x) \rightarrow \tan(x) = 1$.

In $[0, 2\pi]$, this gives $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$.

Sign Chart



Conclusion:

- Increasing on $[0, \frac{\pi}{4}) \cup (\frac{5\pi}{4}, 2\pi]$ ✅
- Decreasing on $(\frac{\pi}{4}, \frac{5\pi}{4})$ ❌
- Local Max at $x = \frac{\pi}{4}$, Local Min at $x = \frac{5\pi}{4}$.

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Mastery in Practice: Logarithmic & Inverse Trig Functions

Logarithmic

**Function*: $f(x) = \ln(x^2 - 1)$

**Domain*: $x^2 - 1 > 0 \rightarrow x \in (-\infty, -1) \cup (1, \infty)$.

**Derivative*: $f'(x) = \dfrac{2x}{x^2 - 1}$

Critical Numbers

$f'(x) = 0$ at $x = 0$, but 0 is not in the domain.

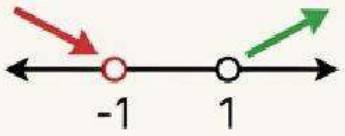
$f'(x)$ is undefined at $x = \pm 1$, also not in the domain.

Conclusion: No critical numbers.

Sign Analysis

For $x > 1$, $f'(x)$ is $\frac{+}{+} = (+)$. ✓

For $x < -1$, $f'(x)$ is $\frac{-}{+} = (-)$. ✗



**Conclusion:

↗ Increasing on $(1, \infty)$ ✓

↘ Decreasing on $(-\infty, -1)$ ✗

No local extrema.

Inverse Trigonometric

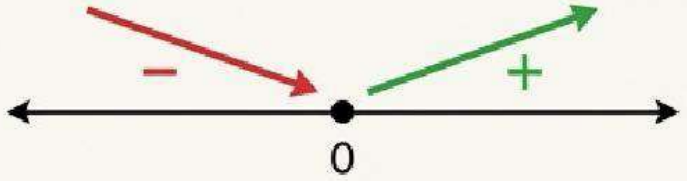
**Function*: $f(x) = \tan^{-1}(x^2)$

**Derivative: $f'(x) = \dfrac{1}{1 + (x^2)^2} \cdot 2x = \dfrac{2x}{1 + x^4}$.

Critical Number

$f'(x) = 0$ when $2x = 0$, so x=0.

**Sign Chart:



**Conclusion:

↗ Increasing on $(0, \infty)$ ✓

↘ Decreasing on $(-\infty, 0)$ ✗

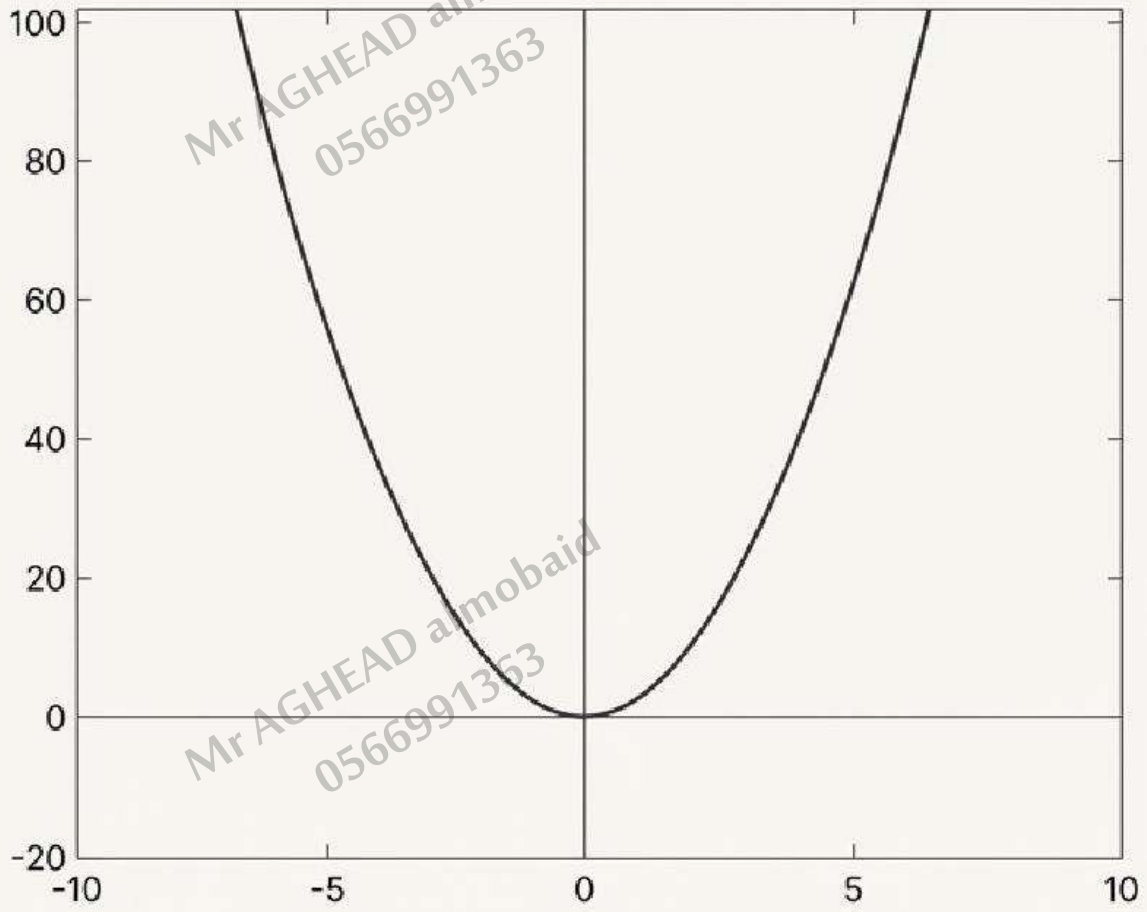
↗ Local Min at $x = 0$. ✓



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Beyond the Calculator: Uncovering Hidden Behavior

The Challenge: Graph $f(x) = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$



A **standard calculator** view might miss important details.

Derivative:

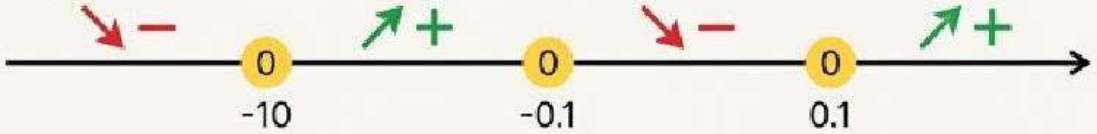
$f'(x) = 12x^3 + 120x^2 - 0.12x - 1.2 = 12(x - 0.1)(x + 0.1)(x + 10)$

Critical Numbers:

$x = -10,\ x = -0.1,\ x = 0.1$

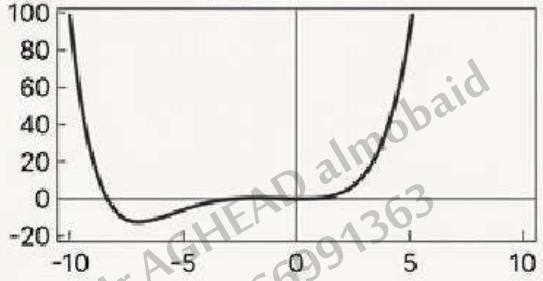
The graph only showed one extremum!

Sign Chart

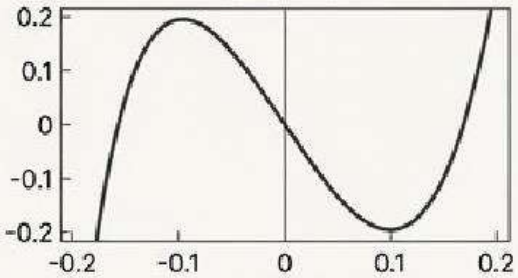


Conclusion

Global View



Local View

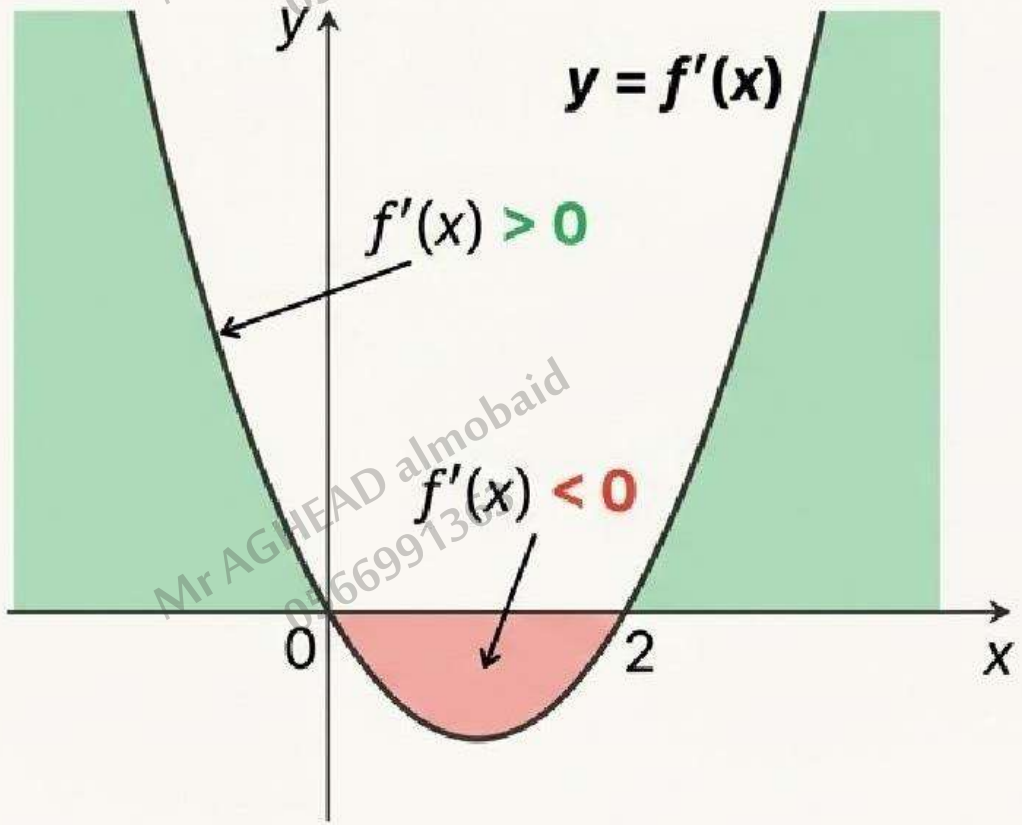


Calculus reveals the complete story. To see all features, we need both a global view for the extremum at x=-10 and a local, zoomed-in view for the behavior near x=0.

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Reading the Derivative's Story: From $f'(x)$ back to $f(x)$

The Task: Given the graph of the derivative, $y = f'(x)$, on what intervals is the original function $f(x)$ increasing or decreasing?



The Key Idea:

- $f(x)$ is **increasing** when $f'(x)$ is **positive** (graph of f' is *above* the x-axis).
- $f(x)$ is **decreasing** when $f'(x)$ is **negative** (graph of f' is *below* the x-axis).

Conclusion for $f(x)$:

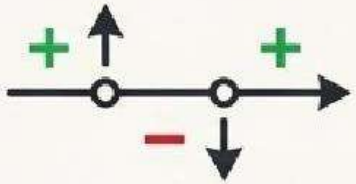
- $f(x)$ is **increasing** on $(-\infty, 0) \cup (2, \infty)$.
- $f(x)$ is **decreasing** on $(0, 2)$.
- Therefore, $f(x)$ has a local maximum at $x=0$ and a local minimum at $x=2$.

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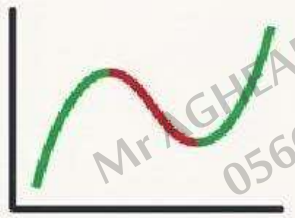
Your Complete Toolkit for Function Analysis

Part 1: The 3-Step Workflow

$f'(x)=0$ Find Critical Numbers



Build Sign Chart



Conclude Intervals

Part 2: The First Derivative Test





With these tools, you can accurately describe the behavior of any differentiable function, revealing insights that graphs alone might hide.

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Chapter (4): Applications of Differentiation

05 Fifth Lesson:
Concavity and the Second Derivative Test



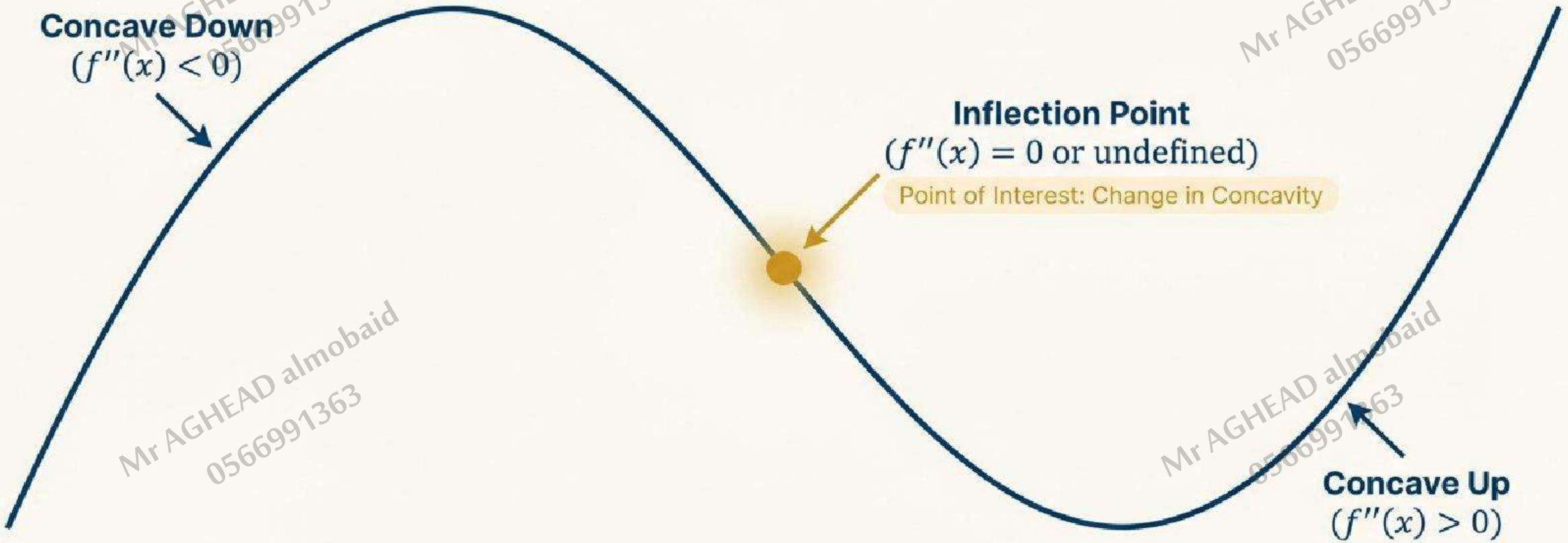
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Mastering the Curve

A Comprehensive Guide to Concavity, Inflection Points, and the Second Derivative Test.



Your complete resource for understanding the shape and behavior of functions.

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Why the First Derivative Isn't the Whole Story

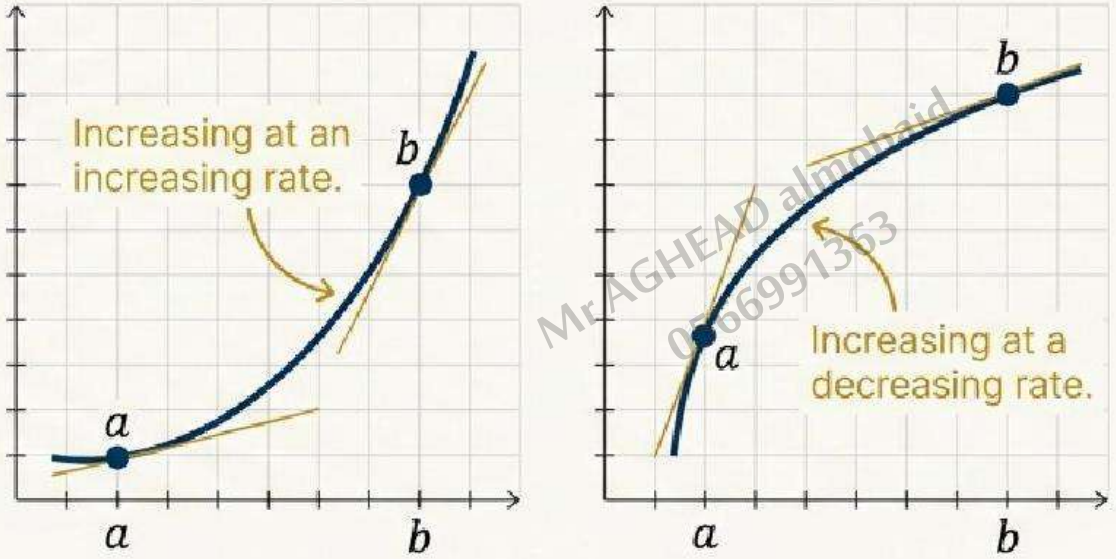
What We Already Know

The first derivative, $f'(x)$, tells us about the direction of a function.

- If $f'(x) > 0$, the function is **increasing**.
- If $f'(x) < 0$, the function is **decreasing**.
- This helps us find local maxima and minima.

But What About the Shape of the Increase?

Both functions below are increasing from point a to b, but they have very different shapes. The first derivative alone can't distinguish between them. We need a new tool to describe this ***curvature***.



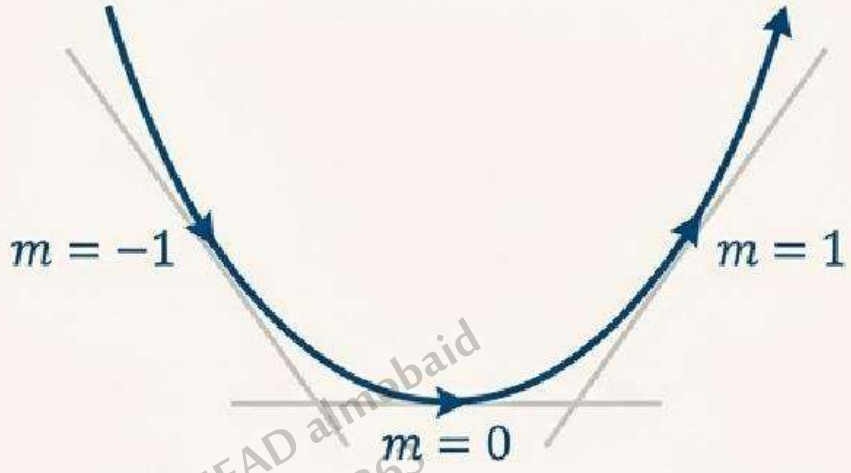
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Introducing Concavity: The Geometry of Curvature

Concavity describes whether a curve bends upwards ("holds water") or downwards ("spills water"). We can understand this by looking at the slopes of the tangent lines.

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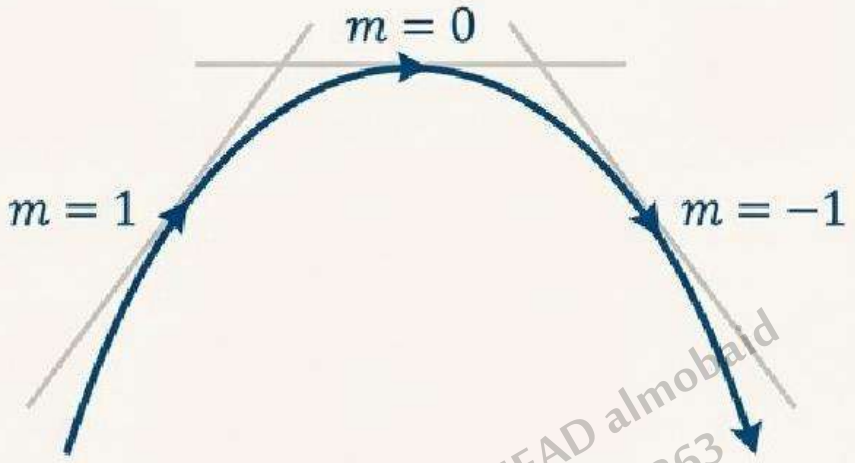
Concave Up (التقعر لأعلى)



The graph of f is **concave up** on an interval I if if it lies **above** all of its tangent lines on I.

The graph of f is **concave up** on I if f' is **increasing** on I.

Concave Down (التقعر للأسفل)



The graph of f is **concave down** on an interval I if it lies **below** all of its tangent lines on I.

The graph of f is **concave down** on I if f' is **decreasing** on I.

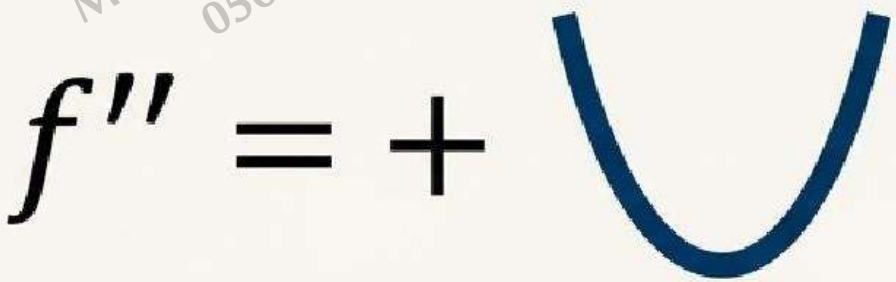
The Second Derivative: Your Tool for Measuring Concavity

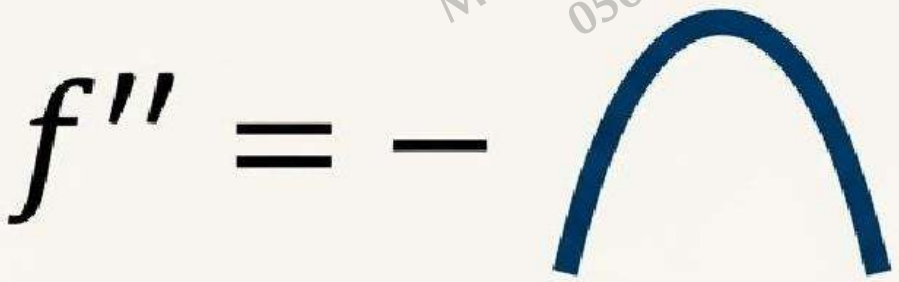
Since concavity depends on whether f' is increasing or decreasing, we can use the derivative of f' (which is f'') to determine the concavity of f. This gives us a direct test.

The Test for Concavity (Theorem 5.1)

Suppose that $f''(x)$ exists on an interval I.

1. If $f''(x) > 0$ on I, then the graph of f is CONCAVE UP on I.
   - *Think: f'' is positive, so f' is increasing, meaning the slopes of the tangent lines are getting steeper.*
2. If $f''(x) < 0$ on I, then the graph of f is CONCAVE DOWN on I.
   - *Think: f'' is negative, so f' is decreasing, meaning the slopes of the tangent lines are getting less steep.*



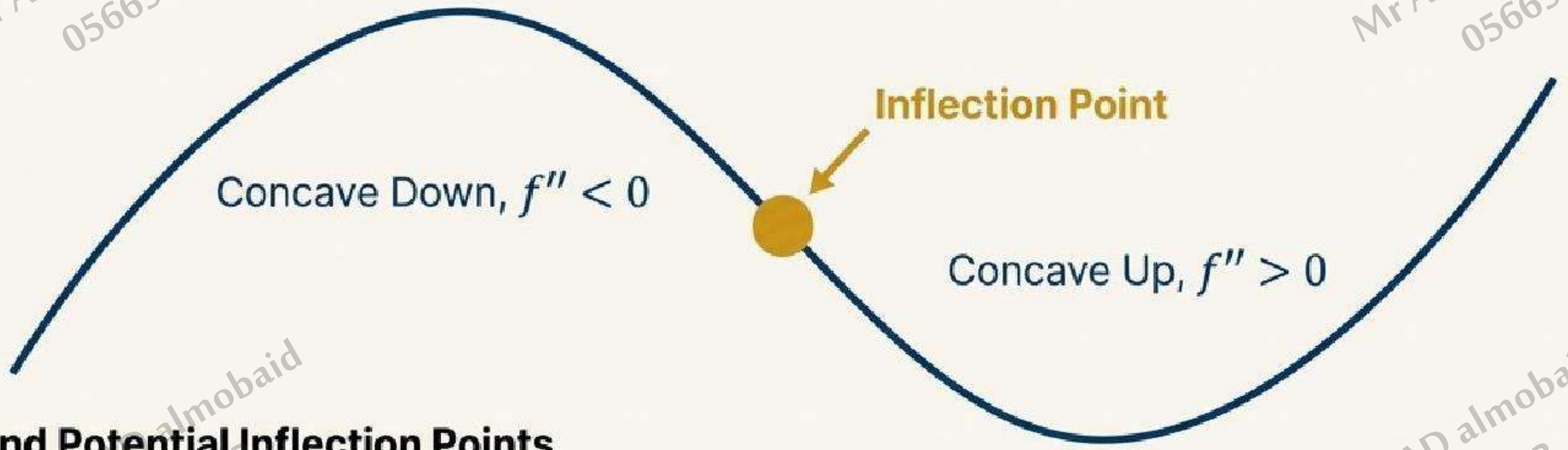


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Inflection Points: Where the Curve Changes its Mind

Inflection Point (نقطة الانعطاف)

An **inflection point** is a point on a continuous curve where the concavity changes (from up to down, or from down to up).



How to Find Potential Inflection Points

If $(c, f(c))$ is an inflection point, then either $f''(c) = 0$ or $f''(c)$ **is undefined.**

- **Important Note:**
  This gives us the *candidates* for inflection points. You must always test the intervals on either side of a candidate point to confirm that the concavity actually changes. We'll see an example where $f''(c)=0$ is NOT an inflection point later.

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The Analytical Method: A Step-by-Step Guide

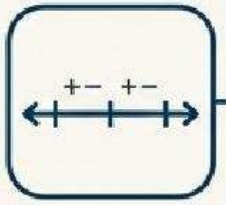
Use this process to determine concavity intervals and find all inflection points for any function $f(x)$.



Step 1: Find the Second Derivative
Calculate $f'(x)$ and then $f''(x)$.



Step 2: Identify Candidate Points
Find all values of x where $f''(x) = 0$. Find all values of x where $f''(x)$ is undefined.
These are your potential inflection points.



Step 3: Create a Number Line and Test Intervals
Place the candidate points from Step 2 on a number line. These points divide the line into intervals.
Choose a test value from within each interval and find the sign (+ or −) of $f''(x)$ at that value.



Step 4: Draw Your Conclusion
If $f''(x) > 0$, the graph is **concave up** in that interval.
If $f''(x) < 0$, the graph is **concave down** in that interval.
If the concavity changes across a candidate point c (and $f(c)$ exists), then $(c, f(c))$ is an **inflection point**.

Worked Example: Analyzing a Polynomial

Find the concavity intervals and inflection points for $f(x) = x^3 - 6x^2$.

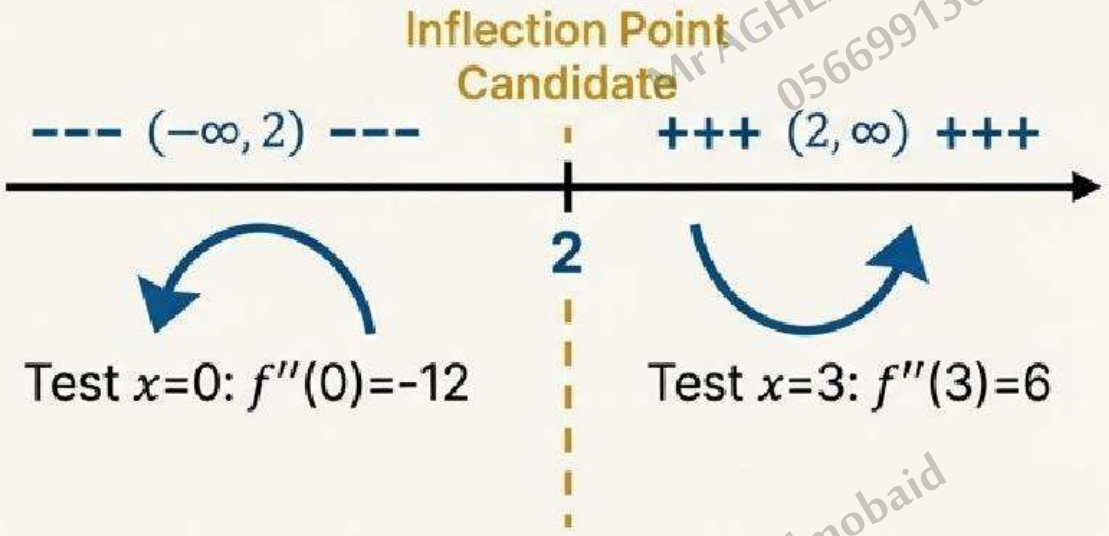
Step 1: Derivatives

$$f'(x) = 3x^2 - 12x$$
$$f''(x) = 6x - 12$$

Step 2: Candidate Points

- Set $f''(x) = 0$:
  - $6x - 12 = 0 \Rightarrow 6x = 12 \Rightarrow x = 2$
- ($f''(x)$ is never undefined).
- Our only candidate is $x = 2$

Step 3: Number Line Test for $f''(x) = 6x - 12$



Step 4: Conclusion

Concave Down: on $(-\infty, 2)$ because $f''(x) < 0$.
Concave Up: on $(2, \infty)$ because $f''(x) > 0$.
Inflection Point: Since concavity changes at x=2, there is an inflection point at $(2, f(2))$.

- $f(2) = (2)^3 - 6(2)^2 = 8 - 24 = -16$.
- The inflection point is **(2, –16)**.

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Example Showcase: Applying the Method to Different Function Types

Rational Function

$$f(x) = x + \frac{1}{x}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$f''(x)$ is never 0.
$f''(x)$ is undefined at $x = 0$.

Concave Down: $(-\infty, 0)$
Concave Up: $(0, \infty)$

Key Takeaway: Concavity changes at $x = 0$, but since the function is not continuous there, it is **not an inflection point**.

Radical Function

$$f(x) = (x+2)^{1/5} + 4$$

$$f'(x) = \frac{1}{5}(x+2)^{-4/5}$$

$$f''(x) = \left(-\frac{4}{25}\right)(x+2)^{-9/5}$$

$f''(x)$ is never 0.
$f''(x)$ is undefined at $x = -2$.

Concave Up: $(-\infty, -2)$
Concave Down: $(-2, \infty)$

Key Takeaway: The concavity changes at a point of vertical tangency, resulting in an inflection point at $x = -2$.

Trigonometric Function

$$f(x) = x^2 + 4\sin(x) \text{ on } [0, 2\pi]$$

$$f'(x) = 2x + 4\cos(x)$$
$$f''(x) = 2 - 4\sin(x)$$

$f''(x) = 0$ when $\sin(x) = \frac{1}{2}$.
This occurs at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

Concave Up: $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$
Concave Down: $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

Key Takeaway: There are two inflection points at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

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A Critical Warning: $f''(c) = 0$ Does Not Guarantee an Inflection Point

Remember, an inflection point requires a **change** in concavity. Simply having a zero in the second derivative isn't enough.

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Analysis

The Classic Counterexample: $f(x) = x^4$

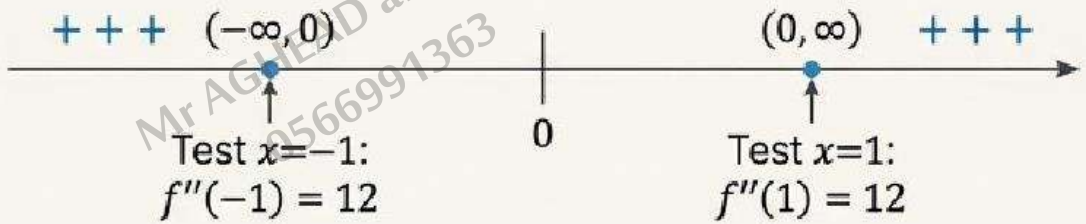
1. **Derivatives:**

$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$

2. **Candidate Point:**

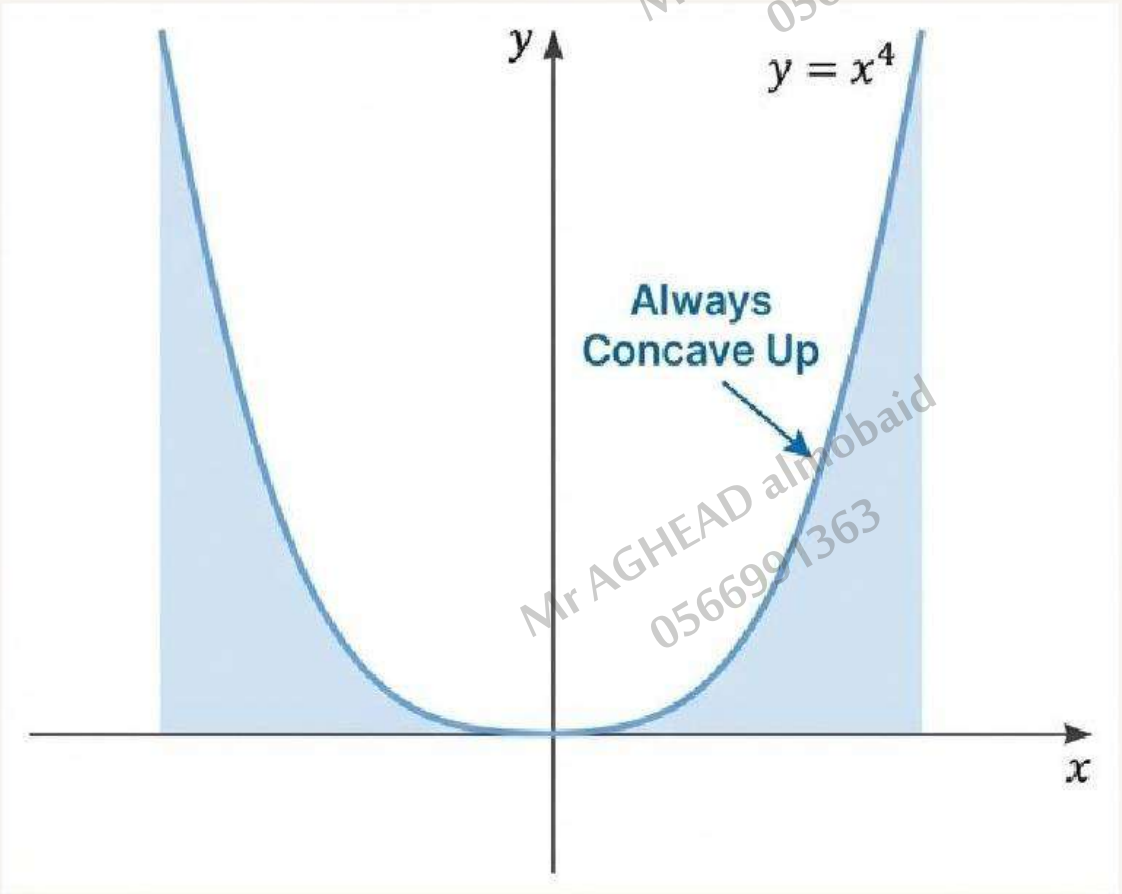
$f''(x) = 0$ when $12x^2 = 0$, so $x = 0$.

3. **Number Line Test for $f''(x) = 12x^2$**



Conclusion: The function is **concave up** on $(-\infty, 0)$ and **concave up** on $(0, \infty)$. Since the concavity **does not change** at $x=0$, there is **NO inflection point**, even though $f''(0)=0$.

Visual



The Second Derivative Test: A Shortcut for Classifying Extrema

We can use concavity to quickly determine if a critical point (where $f'(c) = 0$) is a local maximum or minimum. This is often faster than using a number line for $f'(x)$.

Local Maximum

The Logic (Theorem 5.2)

Suppose $f'(c) = 0$.

1. If $f''(c) < 0$, then f has a LOCAL MAXIMUM at c.

Visual Intuition

The function is flat ($f'(c)=0$) and the graph is concave down (like a hill). This must be a peak.



$f'(c)=0, f''(c)<0$

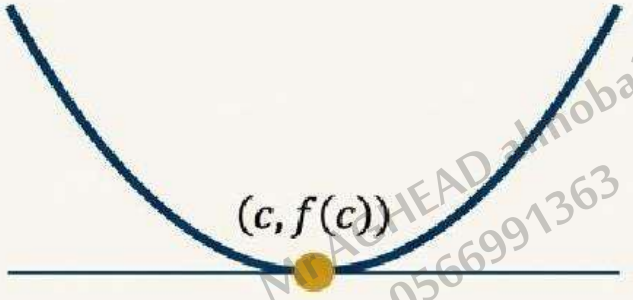
Local Minimum

The Logic (Theorem 5.2)

2. If $f''(c) > 0$, then f has a LOCAL MINIMUM at c.

Visual Intuition

The function is flat ($f'(c)=0$) and the graph is concave up (like a valley). This must be a trough.



$f'(c)=0, f''(c)>0$

Important Caveat:: If $f''(c) = 0$ or is undefined, the test is **inconclusive**. You must go back to the First Derivative Test.

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Applying the Second Derivative Test

Use the Second Derivative Test to find the local extrema of $f(x) = x^4 - 8x^2 + 10$.

Solution

Step 1: Find Critical Numbers

$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2)$

Set $f'(x) = 0$. The critical numbers are $x = -2, x = 0, x = 2$.

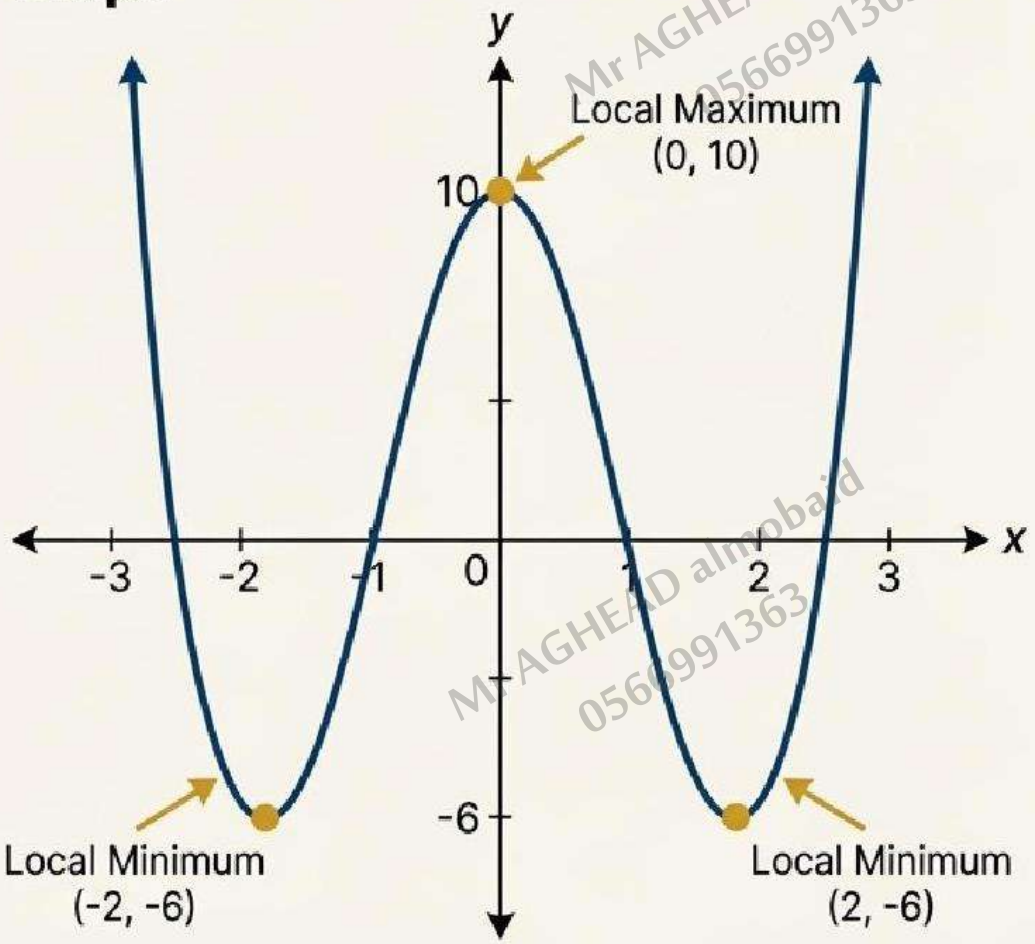
Step 2: Find the Second Derivative

$f''(x) = 12x^2 - 16$

Step 3: Test Each Critical Number in $f''(x)$

- For $x = -2$:
  $f''(-2) = 12(-2)^2 - 16 = 32$.
  Since $f''(-2) > 0$, there is a **local minimum** at $x = -2$.
- For $x = 0$:
  $f''(0) = 12(0)^2 - 16 = -16$.
  Since $f''(0) < 0$, there is a **local maximum** at $x = 0$.
- For $x = 2$:
  $f''(2) = 12(2)^2 - 16 = 48 - 16 = 32$.
  Since $f''(2) > 0$, there is a **local minimum** at $x = 2$.

Graph



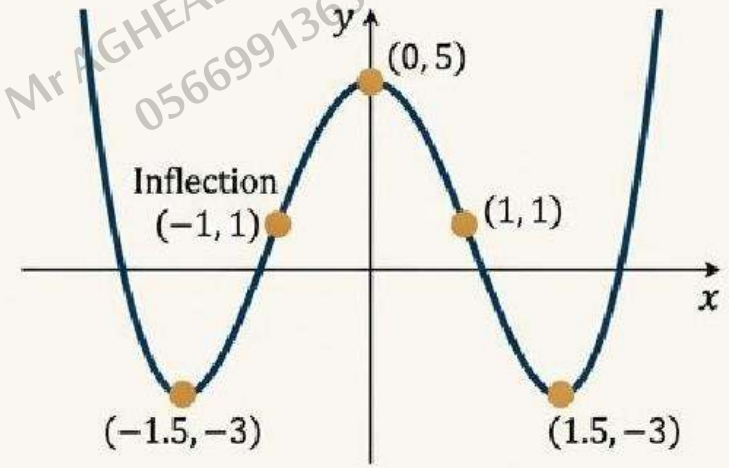
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Reading the Story from the Graph of $f(x)$

You can determine a function's behavior just by looking at its graph. Let's practice identifying key features.

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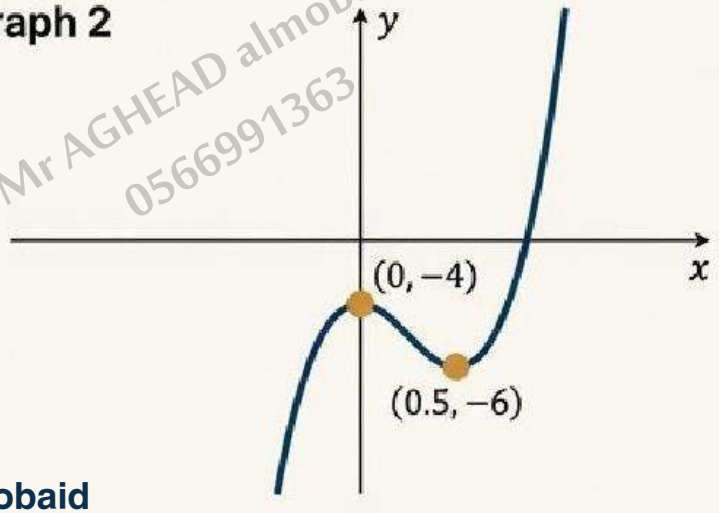
Practice Graph 1



Analysis:

- **Concave Up (لأعلى):** Where the graph opens upwards, like a cup.
  - Interval: $(-\infty, -1) \cup (1, \infty)$
- **Concave Down (للأسفل):** Where the graph opens downwards, like a frown.
  - Interval: $(-1, 1)$
- **Inflection Point (نقطة الانعطاف):** Where the concavity changes.
  - Points: $(-1, 1), (1, 1)$

Practice Graph 2



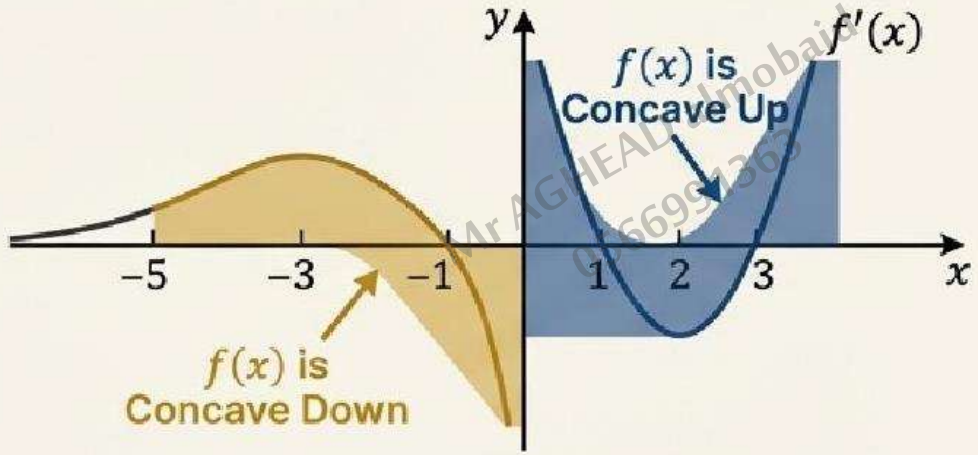
Analysis:

- **Concave Down (للأسفل):** Where the graph opens downwards, like a frown.
  - Interval: $(-\infty, 0.5)$
- **Concave Up (لأعلى):** Where the graph opens upwards, like a cup.
  - Interval: $(0.5, \infty)$
- **Inflection Point (نقطة الانعطاف):** Where the concavity changes.
  - Point: $(0.5, -6)$ approx.

The Relationship Between f, f', and f''

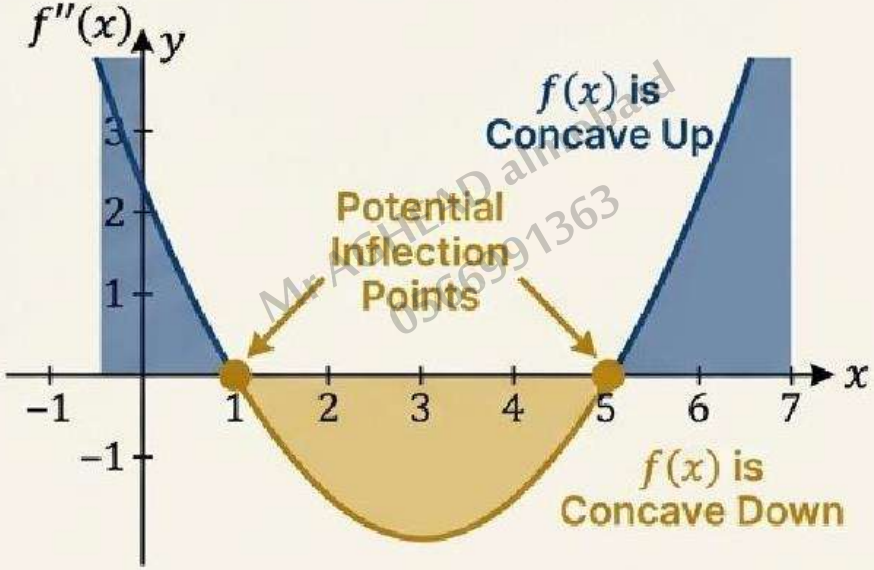
Analyzing the Graph of $f'(x)$

- The behavior of $f'(x)$ tells us about $f(x)$.
- Where $f'(x)$ is **positive** $(f'(x) > 0) \rightarrow f(x)$ is **increasing**.
- Where $f'(x)$ is **increasing** $\rightarrow f'(x)$ has a positive slope $\rightarrow$ $f''(x) > 0 \rightarrow f(x)$ is **concave up**.
- Where $f'(x)$ is **decreasing** $\rightarrow f'(x)$ has a negative slope $\rightarrow$ $f''(x) < 0 \rightarrow f(x)$ is **concave down**.



Analyzing the Graph of $f''(x)$

- The sign of $f''(x)$ tells us directly about the concavity of $f(x)$.
- Where the graph of $f''(x)$ is **above the x-axis** $(f''(x) > 0) \rightarrow$ $f(x)$ is **concave up**.
- Where the graph of $f''(x)$ is **below the x-axis** $(f''(x) < 0) \rightarrow$ $f(x)$ is **concave down**.
- The **x-intercepts** of the $f''(x)$ graph are the potential **inflection points** of $f(x)$.



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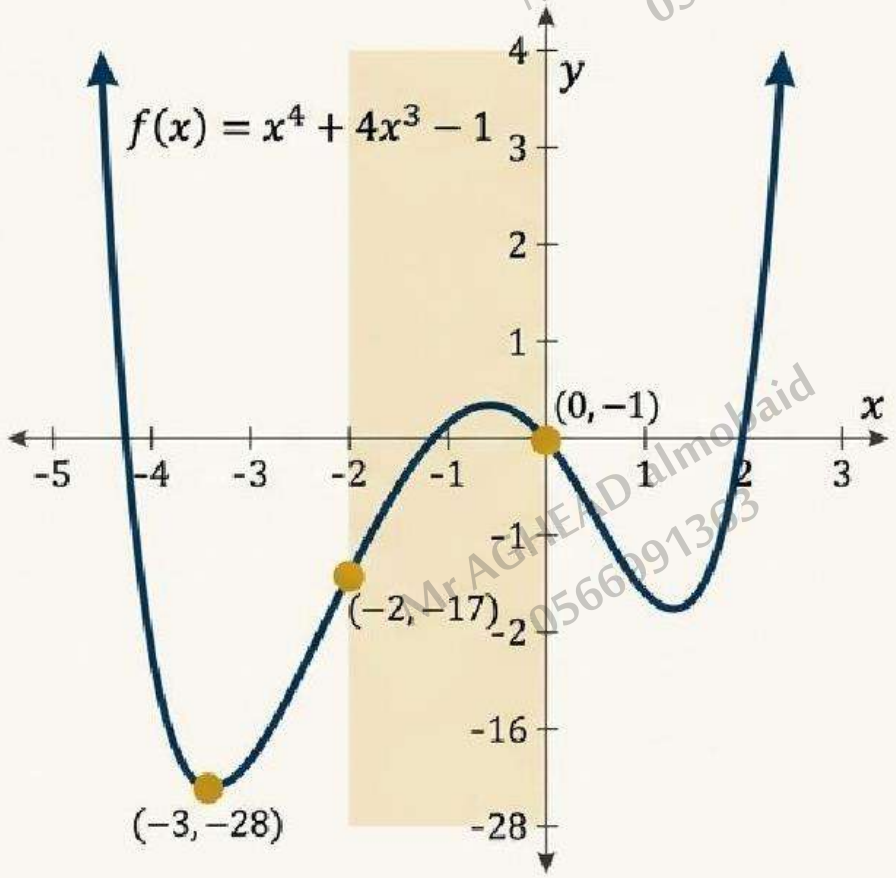
The Grand Synthesis: A Complete Curve Sketching Checklist

Let's combine everything we've learned to analyze a function completely and produce a detailed, accurate graph.
Example: $f(x) = x^4 + 4x^3 - 1$

The Analysis Checklist

1. **Domain:** All real numbers.
2. **First Derivative Analysis**
   - $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$
   - Critical Numbers: $x = 0, x = -3$
   - Decreasing: $(-\infty, -3)$
   - Increasing: $(-3, \infty)$
   - Local Minimum at $x = -3$. No extremum at $x = 0$.
3. **Second Derivative Analysis**
   - $f''(x) = 12x^2 + 24x = 12x(x+2)$
   - Potential Inflection Points: $x = 0, x = -2$
   - Concave Up: $(-\infty, -2) \cup (0, \infty)$
   - Concave Down: $(-2, 0)$
4. **Key Points**
   - Local Min: $(-3, -28)$
   - Inflection Point 1: $(-2, -17)$
   - Inflection Point 2: $(0, -1)$

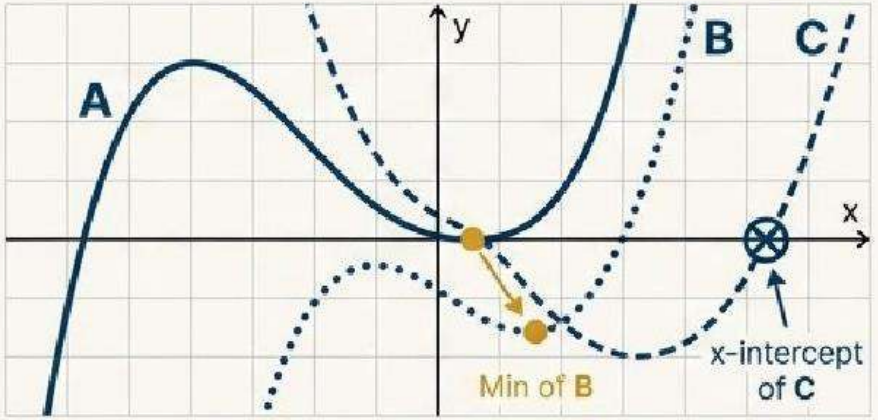
The Final Graph



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Master Challenge: Who is Who?

The graph below shows three functions. One is the original function ***f***, one is the first derivative ***f'***, and one is the second derivative ***f''***. Can you identify them?



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The Logic of Deduction

Look for a curve with a local maximum or minimum. The x-coordinate of that point should be an x-intercept on its derivative's graph.

Clue 1: Find the Extrema

- Look for a curve with a local maximum or minimum. and local at x=3.
- The inflection point of f should correspound to a local extremum on f'.
  - Curve A has an inflection point around x=1.
- This suggests **B is f' and C is f''.**
- This suggests **A is f and B is f'.**

Clue 3: Confirm with Slopes

- Where Curve A is decreasing, Curve B should be negative. This holds true. Where Curve B is decreasing, Curve C aunt C should be negative. This also holds.

The Solution

A = f(x) (The original function)
B = f'(x) (The first derivative)
C = f''(x) (The second derivative)

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Chapter (4): Applications of Differentiation

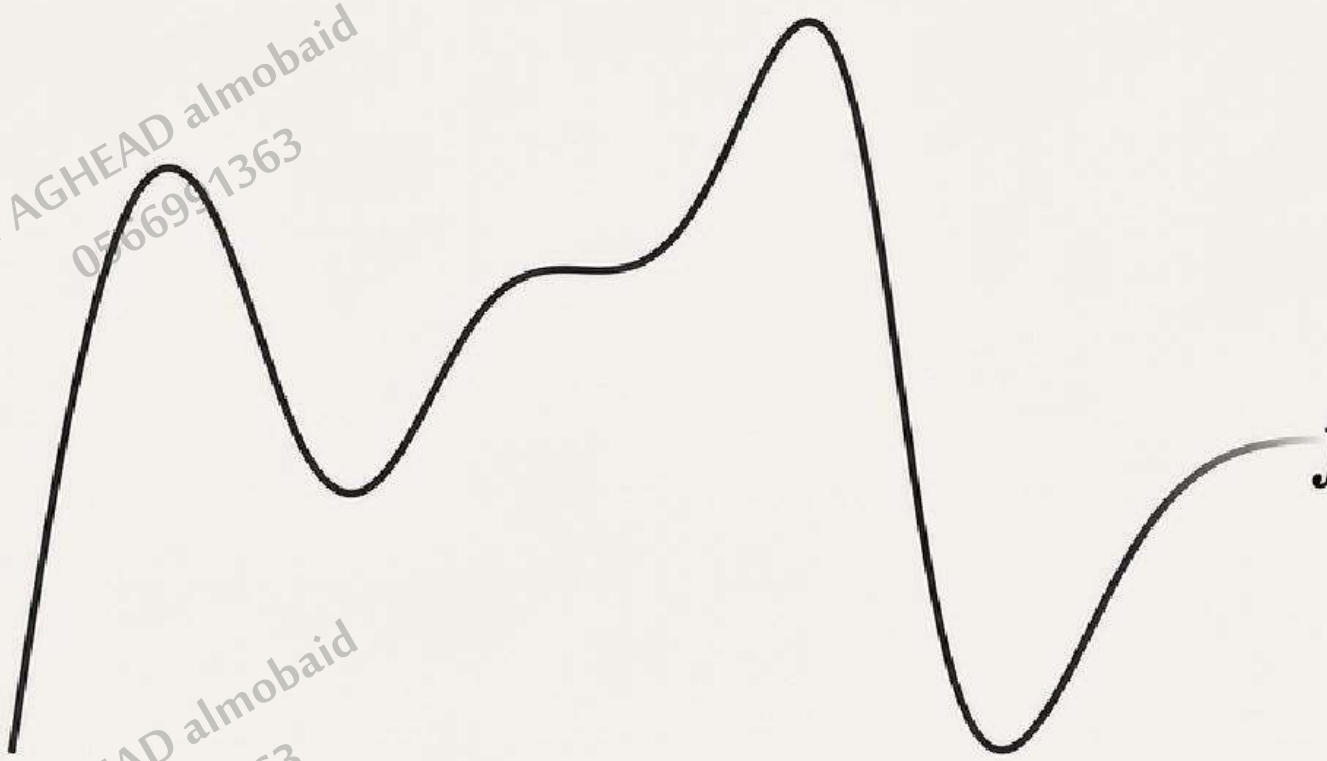
06 Sixth Lesson:

Overview of Curve Sketching



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$$f(x) = \frac{x^3 - 1}{x^2 - 1}$$

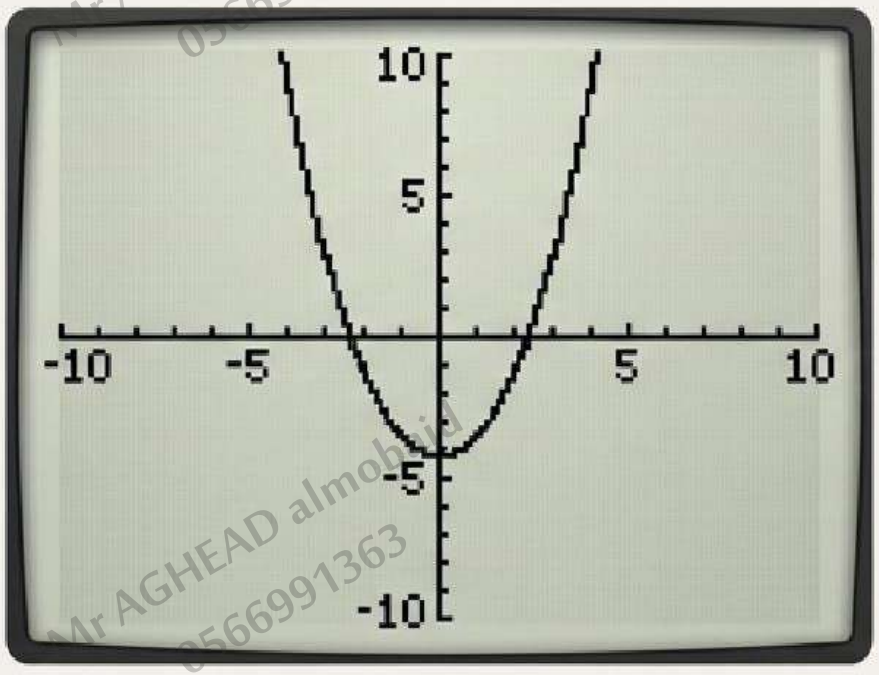
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Sketching the Story of a Function

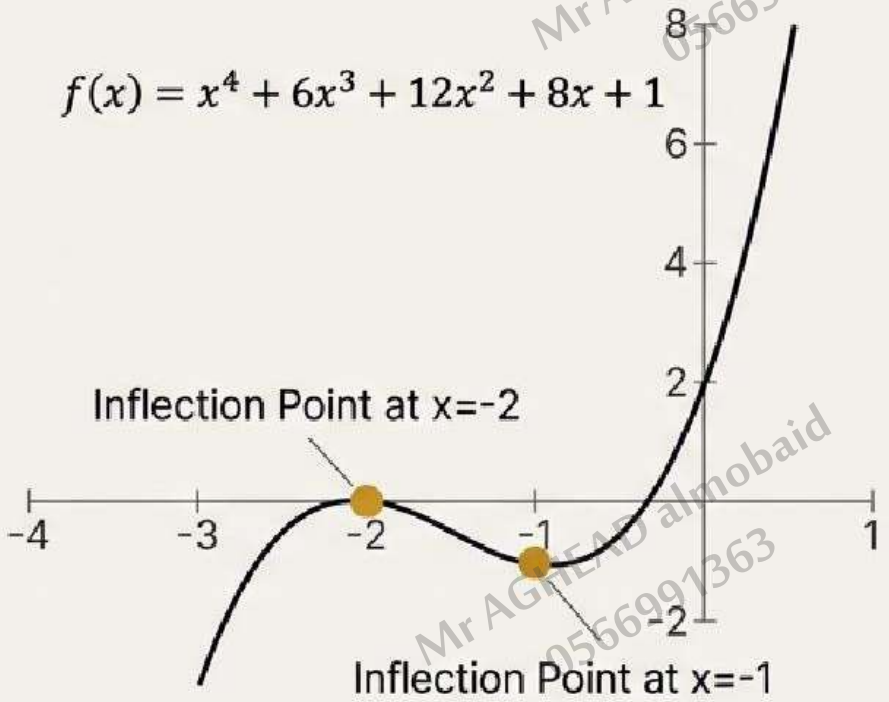
A Comprehensive Calculus Guide to Curve Analysis

Beyond Plotting Points: Seeing the Unseen

The Calculator's View



The Analyst's View



Graphing calculators simply plot points and can easily miss crucial features like local extrema and changes in curvature. Calculus gives us the analytical tools to create a complete and accurate picture, revealing the true story of the function.

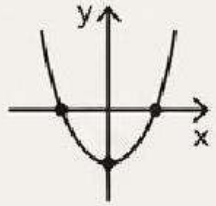
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The Blueprint: Your 7-Step Method for Mastery



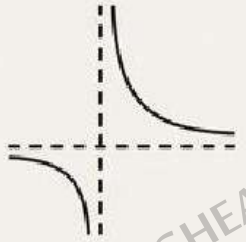
1. Domain

Identify where the function is defined.



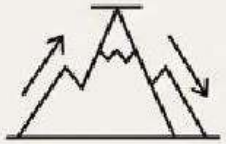
2. Intercepts

Find where the graph crosses the x and y axes.



3. Asymptotes

Locate any vertical, horizontal, or slant lines the graph approaches.



4. First Derivative (f')

Determine intervals of increase/decrease and find local extrema.



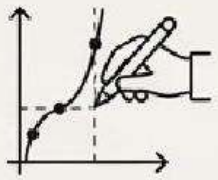
5. Second Derivative (f'')

Determine concavity and find inflection points.



6. Summarize

Collect all key points and intervals in a table or on a combined number line.



7. Sketch

Plot the key points and connect them, following the discovered properties.

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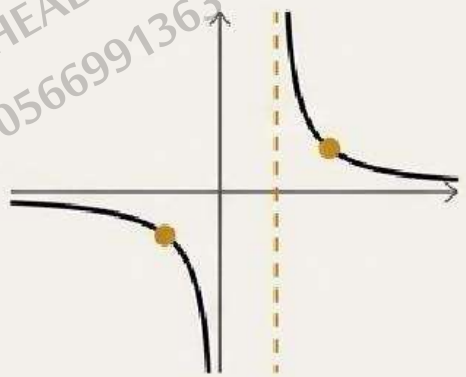
Defining the Boundaries: Vertical & Horizontal Asymptotes

Vertical Asymptotes (x = a)

Concept: Occur where the function approaches $\pm\infty$. For rational functions, this is often at the zeros of the denominator.

Method: Find values a not in the domain. Check that that $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

Example: For $f(x) = \frac{1}{x-2}$, the vertical asymptote is $x = 2$.

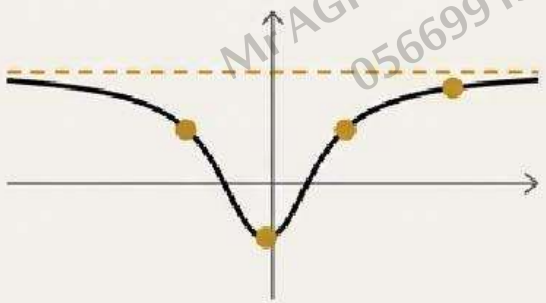


Horizontal Asymptotes (y = L)

Concept: Describe the end behavior of the function as $x \to \pm\infty$.

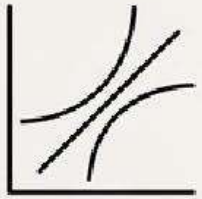
Method: Evaluate $\lim_{x\to\pm\infty} f(x) = L$. For rational functions, compare the degrees of the numerator and denominator.

Example: For $f(x) = \frac{3x^2-1}{x^2+1}$, the horizontal asymptote is $y = 3$ because $\lim_{x\to\infty} f(x) = 3$.



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Beyond the Horizon: Slant (Oblique) Asymptotes



Concept & Method

Concept: A slanted line y = mx + b that the graph approaches. This occurs in rational functions when the degree of the numerator is exactly one greater than the degree of the denominator.

Method: Use polynomial long division to rewrite the function. If f(x) = (mx + b) + remainder/divisor, then the slant asymptote is the linear part, y = mx + b.

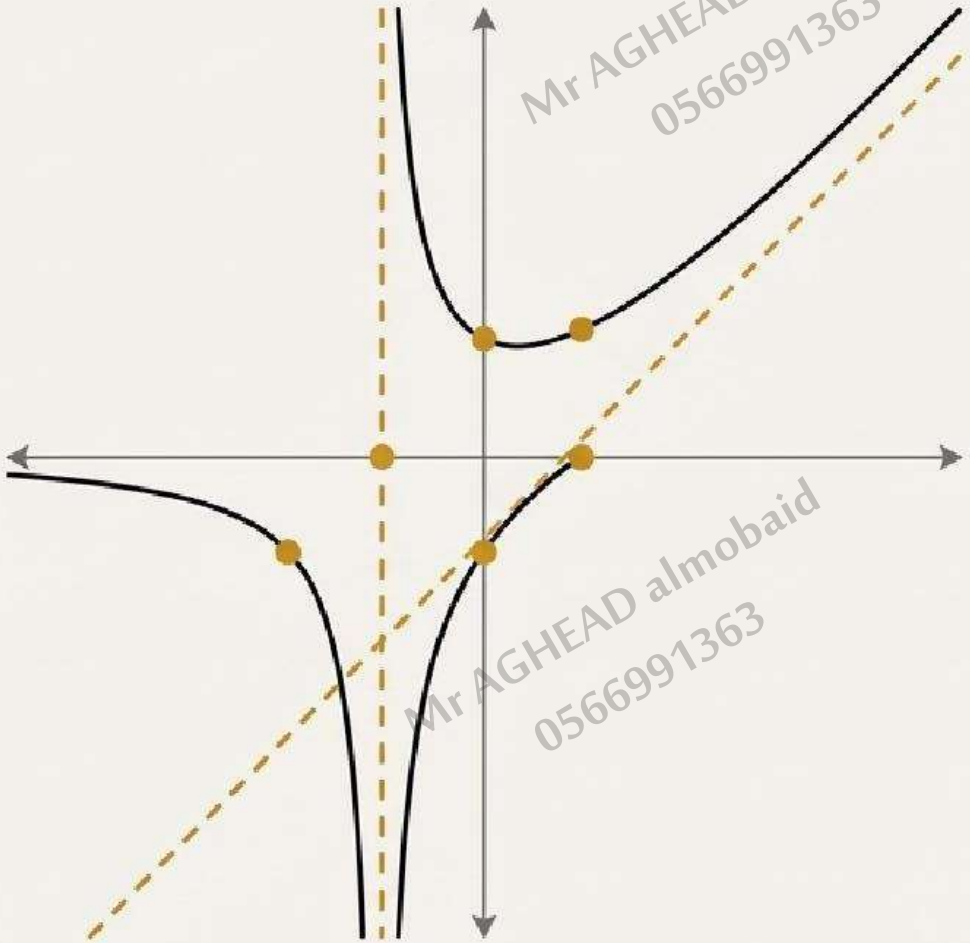
Detailed Example

$f(x) = \frac{x^2-2}{x+1}$:

$$
\begin{array}{r}
x-1 \\
(x+1)\overline{)\,x^2-2} \\
\underline{-x^2+x} \\
-x-2 \\
\underline{--x-1} \\
-1
\end{array}
$$

Result: $f(x) = (x-1) - \frac{1}{x+1}$.

Conclusion: The slant asymptote is $y = x - 1$.



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The Calculus Engine: What Derivatives Reveal

The First Derivative (f') – Direction & Extrema

$f'(x) > 0$ → Function is **increasing** (positive slope).

$f'(x) < 0$ → Function is **decreasing** (negative slope).

$f'(x) = 0$ or DNE → **Critical Points** (potential local max/min).



The Second Derivative (f'') – Curvature & Inflection

$f''(x) > 0$ → **Concave Up** (the curve "holds water").

$f''(x) < 0$ → **Concave Down** (the curve "spills water").

$f''(x) = 0$ or DNE → **Potential Inflection Points** (where concavity changes).



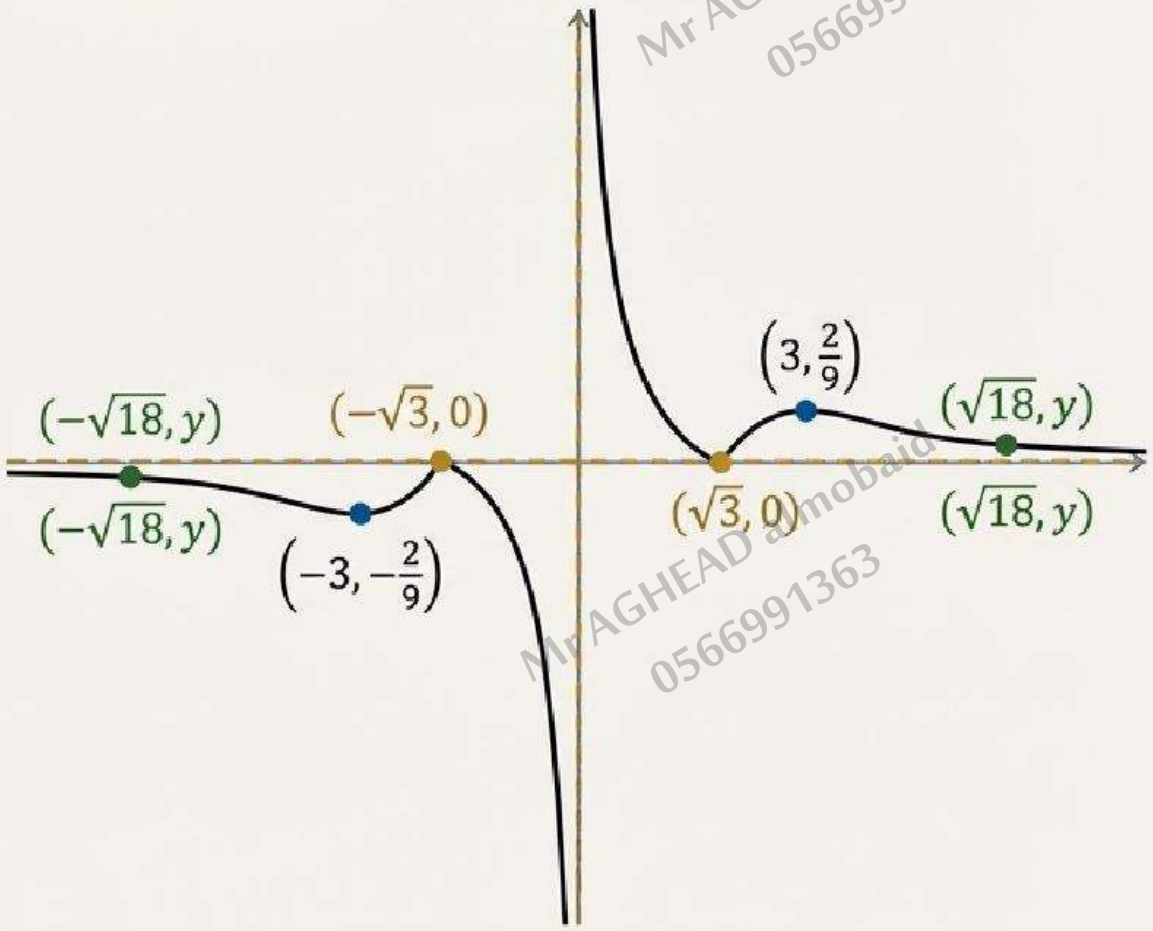
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Case Study: $f(x) = \frac{x^2-3}{x^3}$

The Analysis

1. **Domain:** All real numbers except $x = 0$.
2. **Intercepts:** $x = \pm\sqrt{3}$ (y-intercept does not exist).
3. **Asymptotes:** Vertical Asymptote at x=0, Horizontal Asymptote at y=0.
4. **First Derivative:**
   $f'(x) = \frac{9-x^2}{x^4}$. Critical Points at $x = \pm 3$.
   - **Increasing** on $(-3, 0) \cup (0, 3)$.
   - **Decreasing** on $(-\infty, -3) \cup (3, +\infty)$.
   - **Local Min** at $(-3, -\frac{2}{9})$. **Local Max** at $(3, \frac{2}{9})$.
5. **Second Derivative:**
   $f''(x) = \frac{2(x^2-18)}{x^5}$. Inflection Points at $x = \pm\sqrt{18}$.
   - **Concave Up** on $(-\sqrt{18}, 0) \cup (\sqrt{18}, +\infty)$.
   - **Concave Down** on $(-\infty, -\sqrt{18}) \cup (0, \sqrt{18})$.

The Sketch



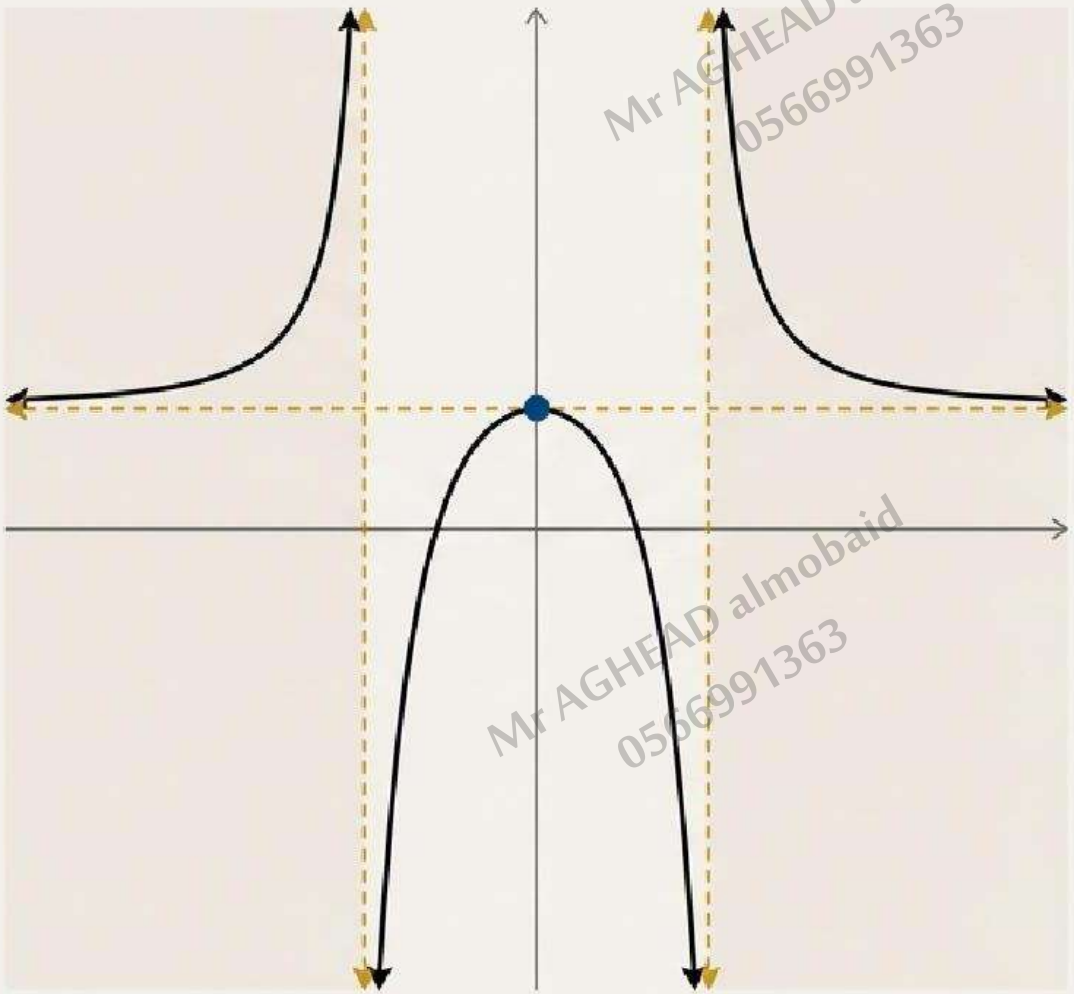
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Case Study: $f(x) = \frac{2x^2}{x^2-1}$

The Analysis

1. **Domain:** All real numbers except x = ±1.
2. **Intercepts:** Origin (0, 0).
3. **Asymptotes**: Vertical Asymptotes at x=1 and x=−1. Horizontal Asymptote at y=2.
4. **First Derivative**: $f'(x) = \frac{-4x}{(x^2-1)^2}$. Critical Point at x=0.
   - Local Maximum at (0, 0).
   - Increasing on $(-\infty, -1) \cup (-1, 0)$.
   - Decreasing on $(0, 1) \cup (1, +\infty)$.
5. **Second Derivative**: $f''(x) = \frac{4(3x^2+1)}{(x^2-1)^3}$. No inflection points.
   - Concave Up on $(-\infty, -1) \cup (1, +\infty)$.
   - Concave Down on $(-1, 1)$.

The Sketch



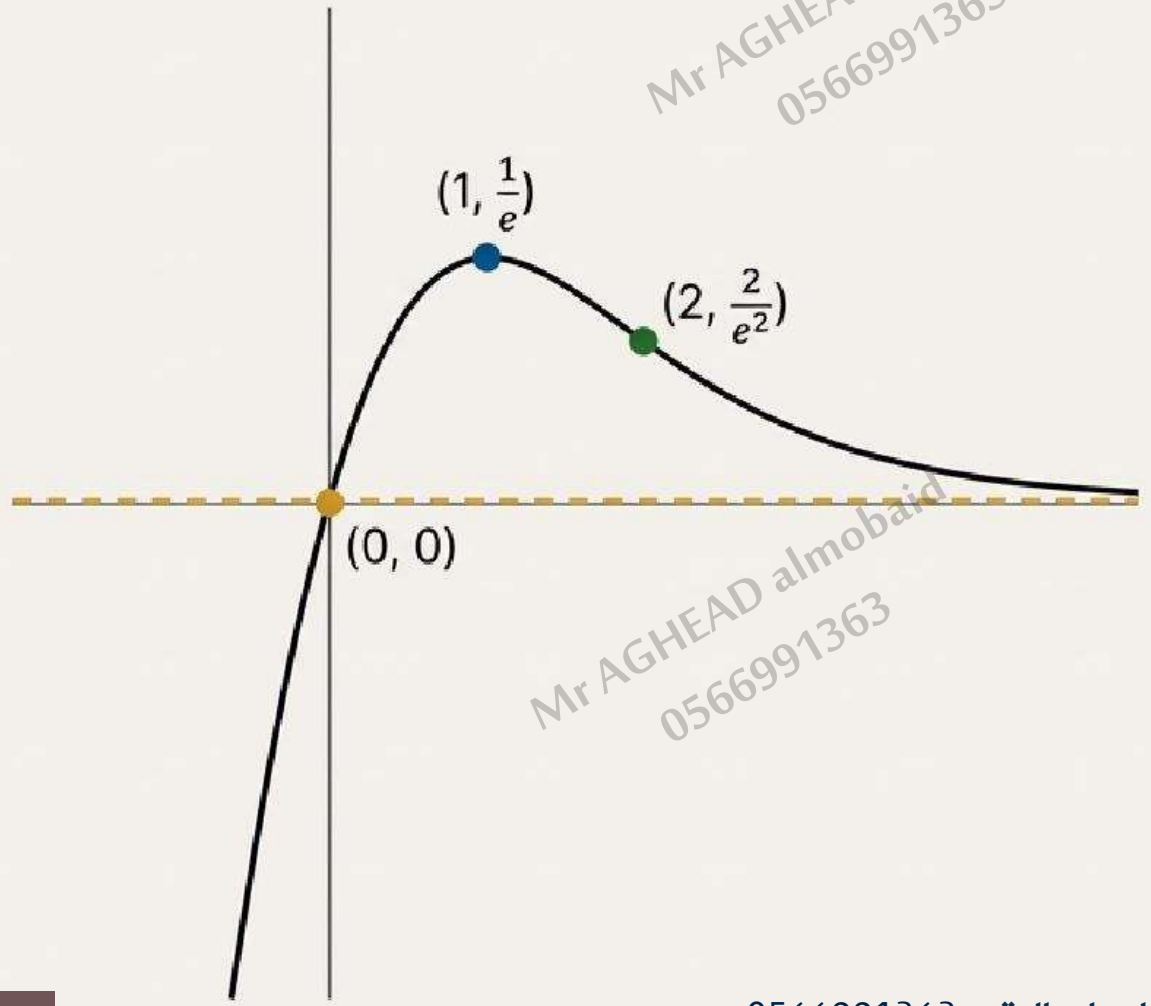
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Case Study: $f(x) = x * e^{-x}$

The Analysis

1. **Domain:** All real numbers.
2. **Intercepts:** Origin (0, 0).
3. **Asymptotes:** Horizontal Asymptote y=0 as $x \to +\infty$. No vertical asymptotes.
4. **First Derivative:** $f'(x) = e^{-x}(1 - x)$.
   Critical Point at x=1.
   - Local Maximum at (1, 1/e).
   - **Increasing** on $(-\infty, 1)$.
   - **Decreasing** on $(1, +\infty)$.
5. **Second Derivative:** $f''(x) = e^{-x}(x - 2)$.
   Inflection Point at x=2.
   - Inflection Point at $(2, 2/e^2)$.
   - **Concave Down** on $(-\infty, 2)$.
   - **Concave Up** on $(2, +\infty)$.

The Sketch



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Case Study: Reconstructing a Graph from Clues

The Given Information

The Task: Sketch a possible graph of a function $f(x)$ that satisfies all the given conditions.

Points

$f(-2) = 8$
$f(0) = 8$
$f(2) = 0$
$f(4) = 0$

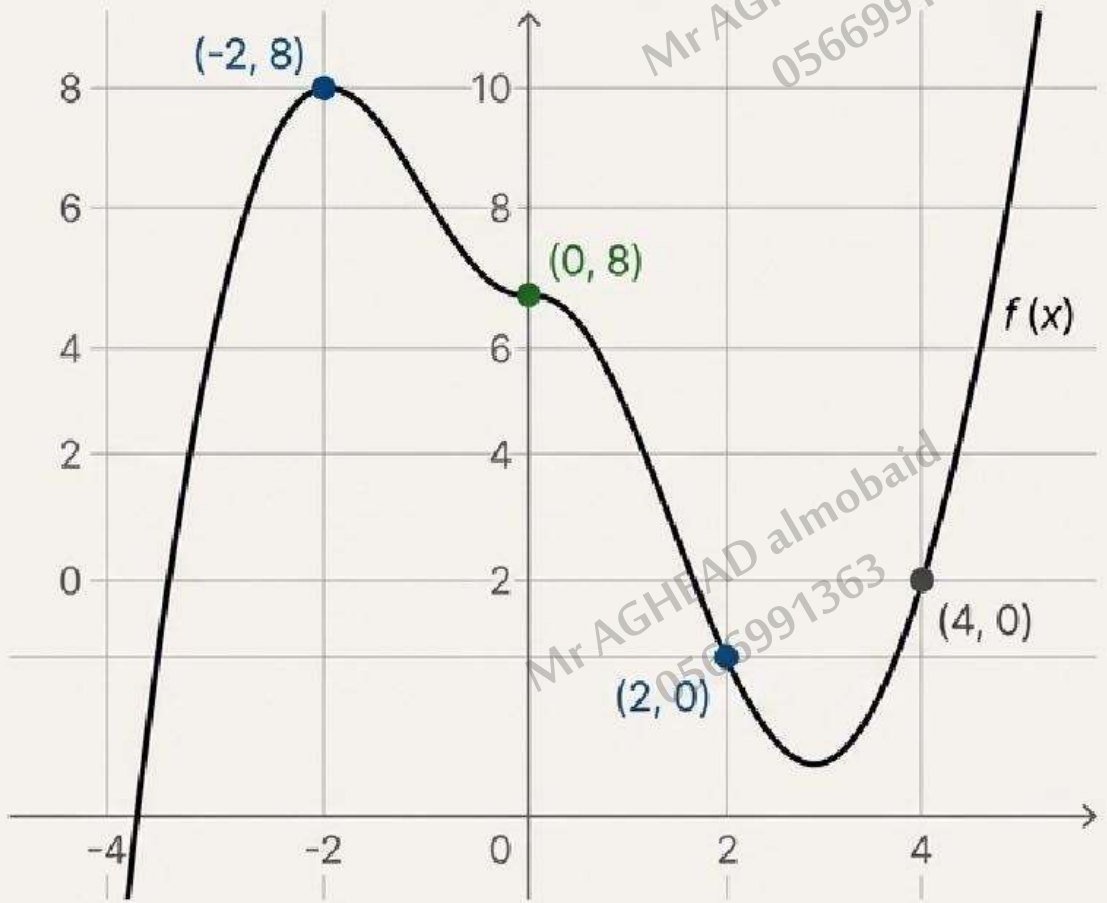
First Derivative Info

$f'(-2) = 0$ and $f'(2) = 0$
Decreasing on $(-2, 2)$
Increasing on $(-\infty, -2) \cup (2, +\infty)$

Second Derivative Info

$f''(0) = 0$
Concave down on $(-\infty, 0)$
Concave up on $(0, +\infty)$

The Sketch



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Special Case: $f(x) = \frac{x^3 - 1}{x^2 - 1}$

The Analysis

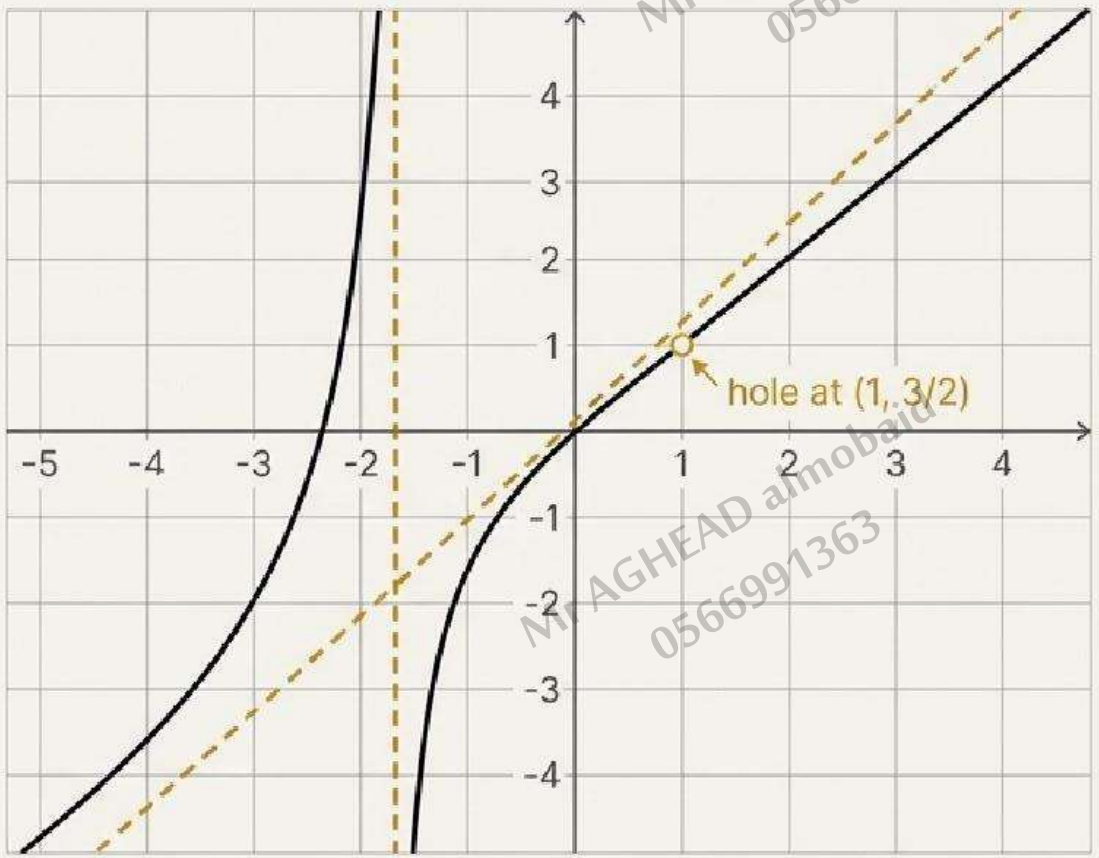
$$f(x) = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1} \text{ for } x \neq 1$$

1. **Discontinuities:** There is a **hole** in the graph at x=1. The y-coordinate is $(1^2+1+1)/(1+1) = 3/2$. **Conclusion:** The hole is at (1, 3/2).

2. **Asymptotes:** Vertical Asymptote at x = -1. **Slant Asymptote:** Long division on the simplified form yields y=x.

3. **Derivatives:** The analysis of f' and f'' for the simplified function determines local extrema and concavity.

The Sketch



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Special Case: $f(x) = (x+2)^{1/5} + 4$

The Analysis

Domain: All real numbers.

f' **First Derivative:** $f'(x) = \frac{1}{5(x+2)^{4/5}}$. Note that $f'(-2)$ is undefined, so x=-2 is a critical point.

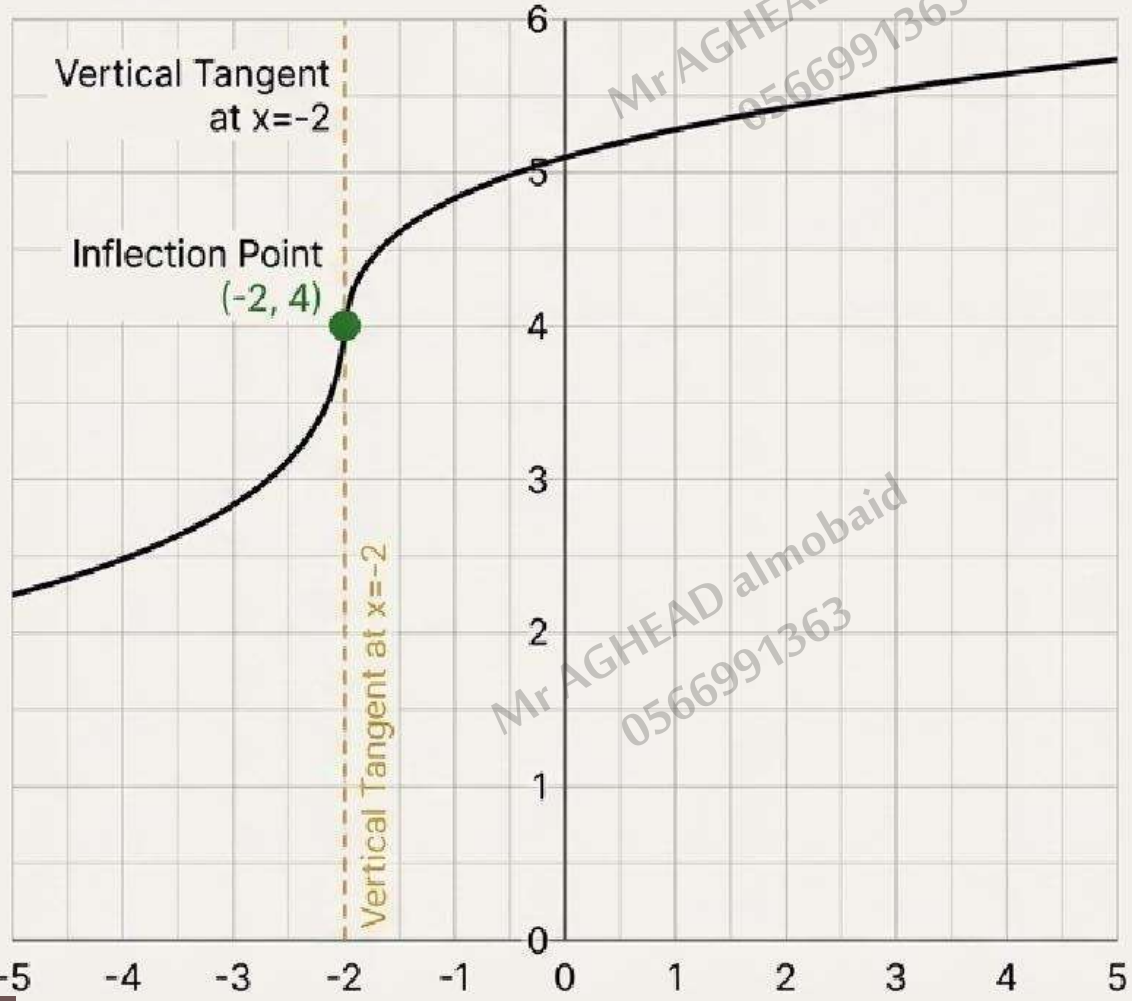
Vertical Tangent: We check the limit: $\lim_{x \to -2} f'(x) = +\infty$. This infinite limit indicates a **vertical tangent line** at x=-2.

f'' **Second Derivative:** $f''(x) = \frac{-4}{25(x+2)^{9/5}}$.

f'' changes sign at x=-2, so (-2, 4) is an inflection point.

Concave Up on $(-\infty, -2)$.
Concave Down on $(-2, +\infty)$.

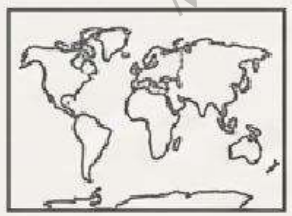
The Sketch



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Your Toolkit for Every Function

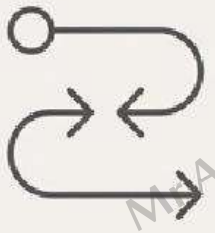
To transform an abstract equation y = f(x) into a complete, accurate graph that tells its story.



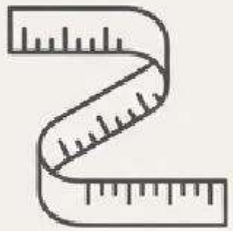
Boundaries: Where does the function exist? (Domain). Where does it go at the edges and breaks? (Asymptotes & End Behavior).



Landmarks: Where does it cross the axes? (Intercepts).



Journey: When is the function rising or falling? (First Derivative). Where are the peaks and valleys? (Local Extrema).



Shape: How is the curve bending? (Second Derivative & Concavity). Where does the bend change? (Inflection Points).

This systematic process demystifies any function. Practice is the key to making it second nature.

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Sharpen Your Skills

Apply the 7-step method to analyze and sketch the following functions. Refer to the examples in your course materials for guidance.

- **Polynomial:** $f(x) = x^4 - 3x^2 + 2$
- **Rational:** $f(x) = \frac{2x}{x^2 - 1}$
- **Transcendental:** $f(x) = x \cdot \ln(x)$
- **Trigonometric:** $f(x) = x + \sin(x)$

Challenge Question

Find a possible equation for a rational function whose graph has vertical asymptotes at $x = 1$ and $x = 2$, and a horizontal asymptote at $y = 3$.

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لا تتردد في التواصل معنا
قم بمسح رمز الـQR

Chapter (4): Applications of Differentiation

07 Seventh Lesson: Optimization



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Mastering Optimization Problems
مسائل القيم المثلى

A Comprehensive Study Guide for 12 Advanced - Term 2

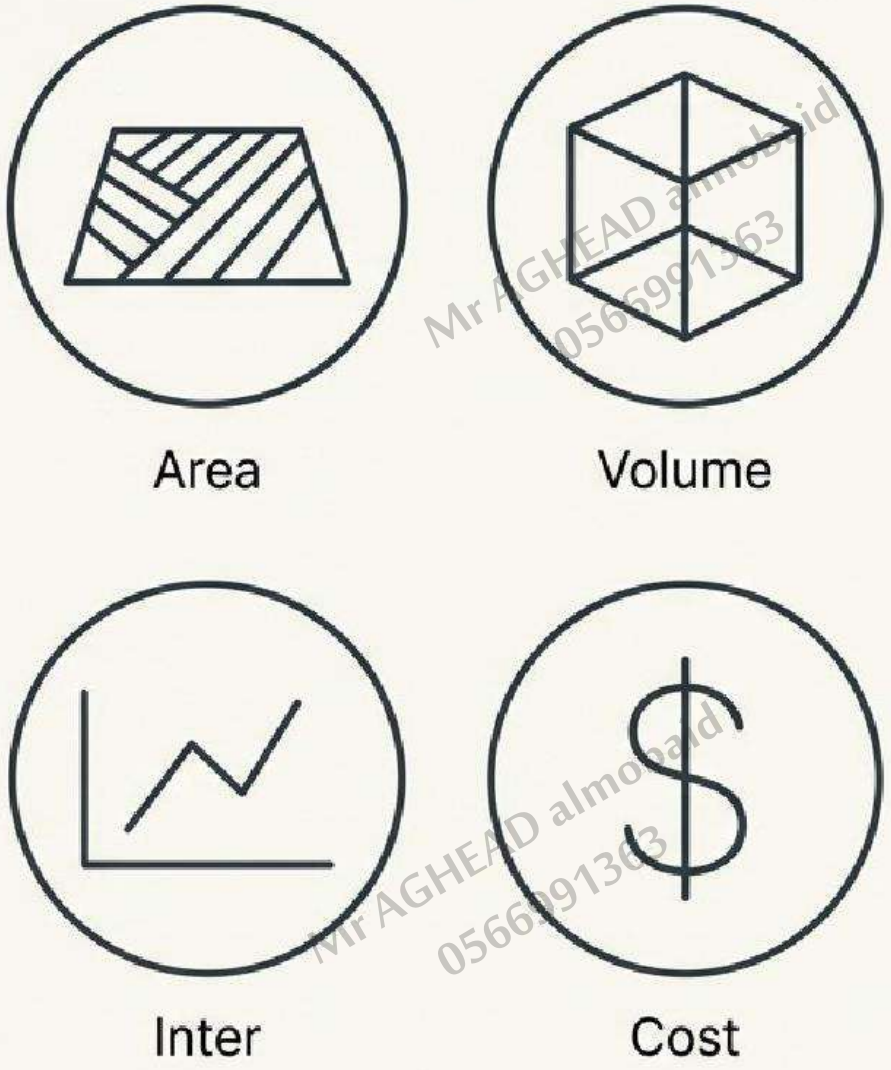
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Finding the Best: An Introduction to Optimization

Optimization is the process of finding the maximum or minimum value of a function under a given set of conditions or constraints. It's about making the most of what you have.

How can we build the largest possible garden with alimited amount of fence? How can a company minimize its packaging material to save costs? How do we find the shortest path between two points?

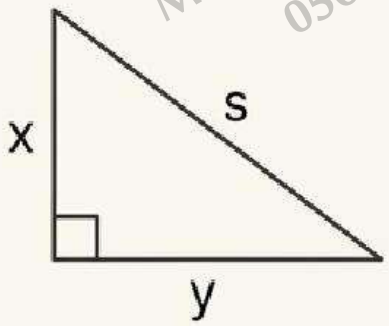


This guide will provide you with a universal strategy to solve these problems and more.

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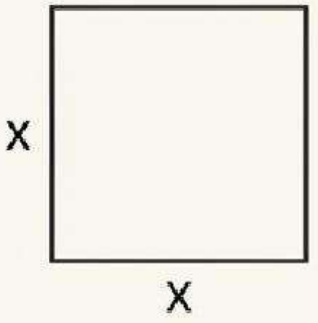
Your Essential Toolkit: Key Formulas | قوانين المساحات والحجوم

Right Triangle | المثلث القائم الزاوية



- Area = (1/2)xy
- Perimeter = x+y+s
- Pythagorean Theorem: $s^2 = x^2 + y^2$

Square | المربع



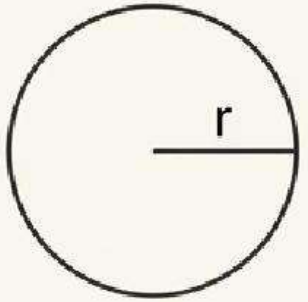
- Area = x^2
- Perimeter = 4x

Rectangle | المستطيل



- Area = xy
- Perimeter = 2(x+y)

Circle | الدائرة



- Area = πr^2
- Circumference = $2\pi r$

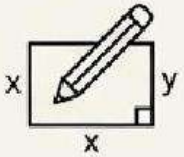
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The Master Key: A 6-Step Strategy for Solving Any Optimization Problem



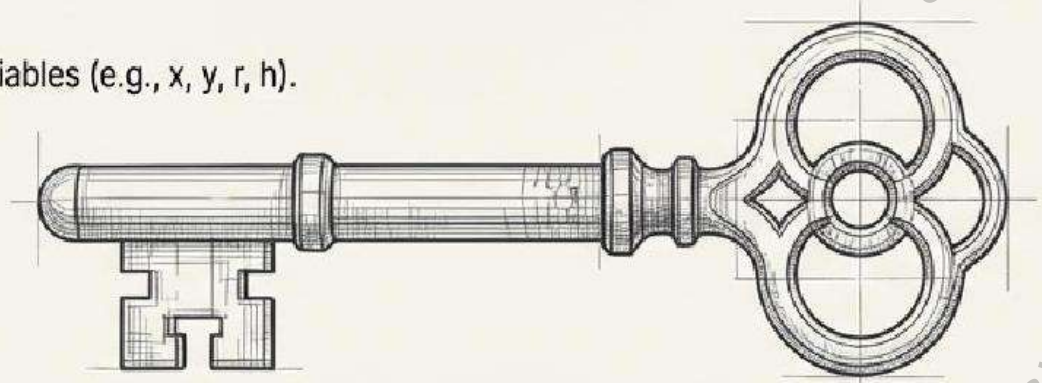
1. Understand & Identify | افهم المسألة

Read the problem carefully. Identify all known quantities and constraints.
Clearly state the objective: what quantity are you trying to maximize or minimize?



2. Model & Diagram | كوّن نموذجًا رياضيًا

Draw a clear diagram of the situation. Label all relevant parts with variables (e.g., x, y, r, h).

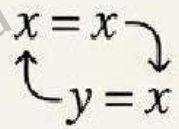


3. Formulate the Primary Equation | ضع متغيرًا واحدًا

$$A = f(x)$$

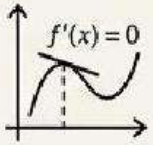
Write the equation for the quantity you want to optimize (e.g., A = xy for area, V = lwh for volume). This is the 'objective function'.

4. Reduce to a Single Variable | أوجد مجال الدالة



Use the problem's constraints to write a 'secondary equation.' Solve this equation for one variable and substitute it into the primary equation. Your objective function should now have only one independent variable. Determine its practical domain.

5. Apply Calculus | حدد النقاط الحرجة



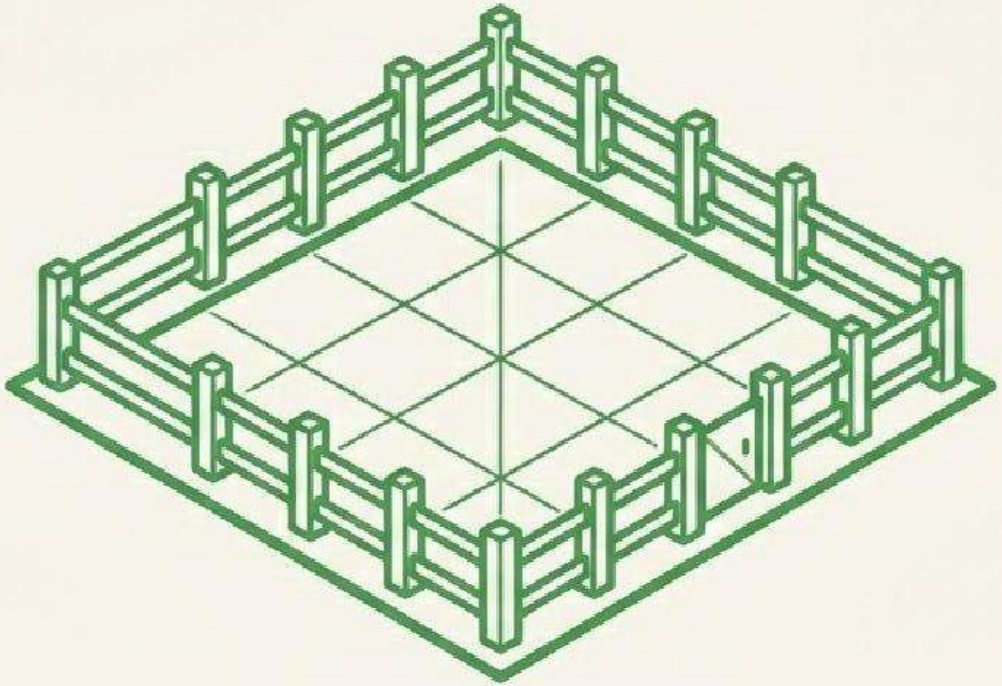
Find the derivative of your single-variable function. Find the critical points by setting the derivative equal to zero or finding where it is undefined.

6. Verify & Interpret | فسّر الحل



Use the First or Second Derivative Test to confirm if the critical point yields a maximum or minimum. Check the endpoints of the domain if applicable. State your final answer clearly, with units, and make sure it's a reasonable solution to the original problem.

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Module 1: Optimizing Area

Inter as I Invv, Arthmic Design, opination Section Place

Maximizing land, minimizing perimeters, and making the most of your space.

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Deep Dive: The Rectangular Garden (Q1)

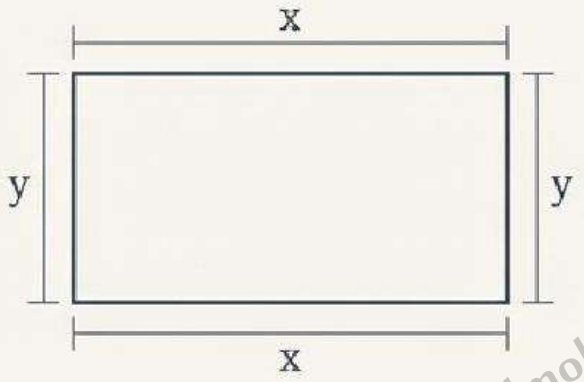
You have 40 linear feet of fencing with which to enclose a rectangular space for a garden. Find the largest area that can be enclosed and the dimensions of the corresponding garden.

لديك 40 قدمًا خطية من السياج لتطويق مساحة مستطيلة لحديقة. أوجد أكبر مساحة يمكن تطويقها وأبعاد الحديقة المقابلة.

Step 1: Understand & Identify | افهم المسألة

- **Given:** Perimeter of fencing = 40 ft. Shape = Rectangle.
  المعطيات: محيط السياج=40قدمًا. الشكل = مستطيل.
- **Objective:** Maximize the Area.
  الهدف: تعظيم المساحة.

Step 2: Model & Diagram | كوّن نموذجًا رياضيًا



Step 3: Primary Equation (Area) | ضع متغيرًا واحدًا

$A = x * y$

- **Substitute:** $A(x) = x * (20 - x) = 20x - x^2$
- **Domain:** The length 'x' must be positive and less than 20. So, $0 < x < 20$.

Step 4: Reduce to a Single Variable | أوجد مجال الدالة

- **Secondary Equation (Perimeter):** $2x + 2y = 40$

$2x + 2y = 40 \Rightarrow x + y = 20 \Rightarrow y = 20 - x$

- **Substitute:** $A(x) = x * (20 - x) = 20x - x^2$

المجال: يجب أن يكون الطول 'س' موجبًا وأقل من 20. إذن، 0 < س < 20.

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Deep Dive: The Rectangular Garden (Solution)

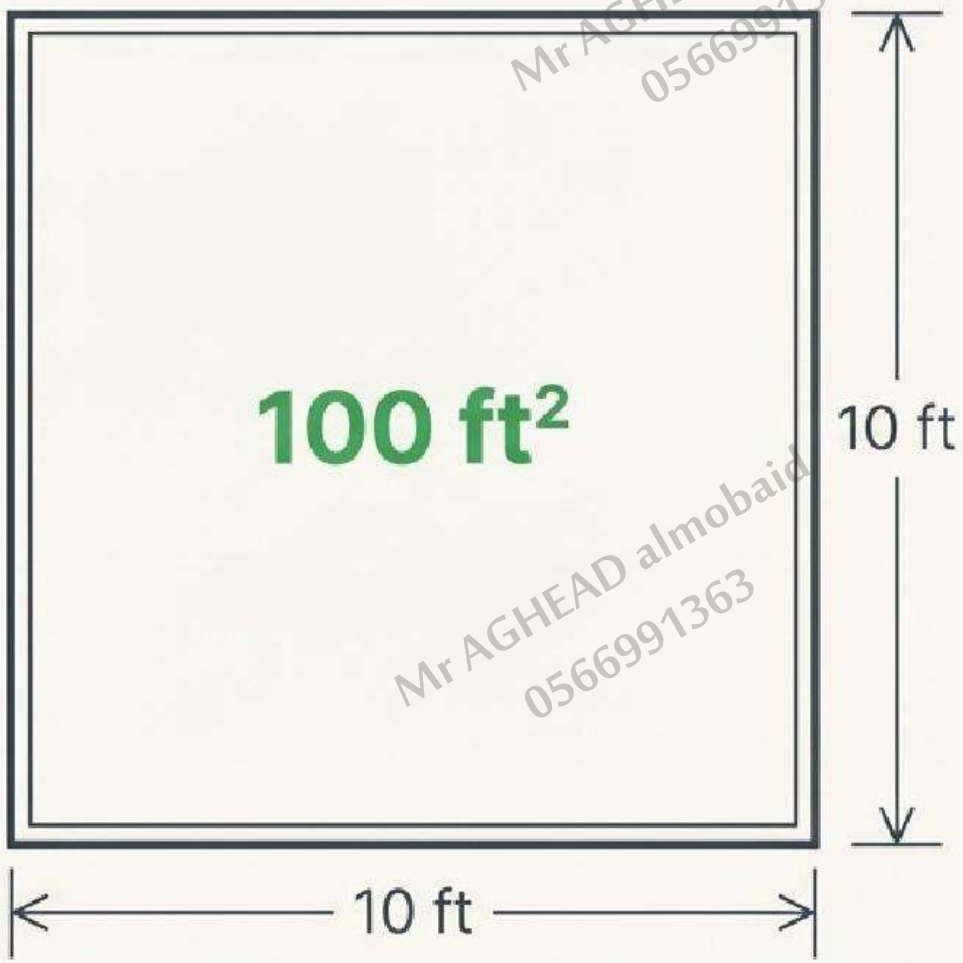
Recap: We need to maximize $A(x) = 20x - x^2$ on the interval $(0, 20)$.

Step 5: Apply Calculus

- **Find Derivative:** A′(x) = 20 - 2x
- **Find Critical Points:** Set A′(x) = 0 => 20 - 2x = 0 => x = 10.

Step 6: Verify & Interpret

- **Verification (Second Derivative Test):** A″(x) = -2. Since A″(10) = -2 < 0, the function has a local maximum at x = 10.
- **Find Dimensions:** If x = 10, then y = 20 - 10 = 10. The garden is a 10 ft by 10 ft square.
- **Final Answer:** The maximum possible area is A = 10 * 10 = 100 sq ft.



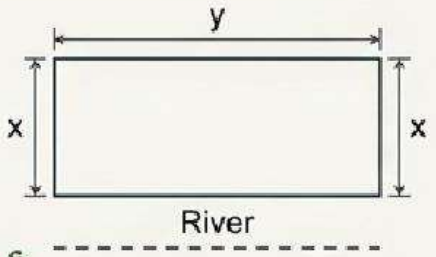
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Practice Your Skills: Area Problems

Use the 6-step strategy to solve the following.

Problem 1 (Q3): River Fence (Maximize Area)

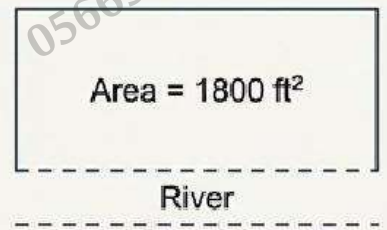
A three-sided fence next to a river with 96 ft of fencing. Find max area.



Answer: Dimensions: 24 ft x 48 ft.
Max Area: 1152 ft².

Problem 2 (Q2): River Fence (Minimize Perimeter)

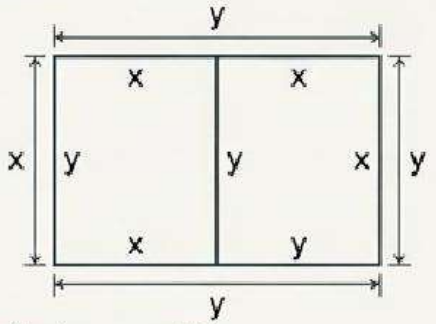
A three-sided fence for an area of 1800 ft². Find min perimeter.



Answer: Dimensions: 30 ft x 60 ft.
Min Perimeter: 120 ft.

Problem 3 (Q4): Two-Pen Corral

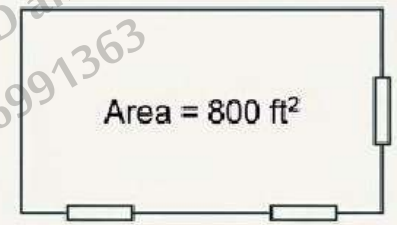
Two identical adjacent rectangular corrals with 120 ft ft of fencing. Find dimensions to max area.



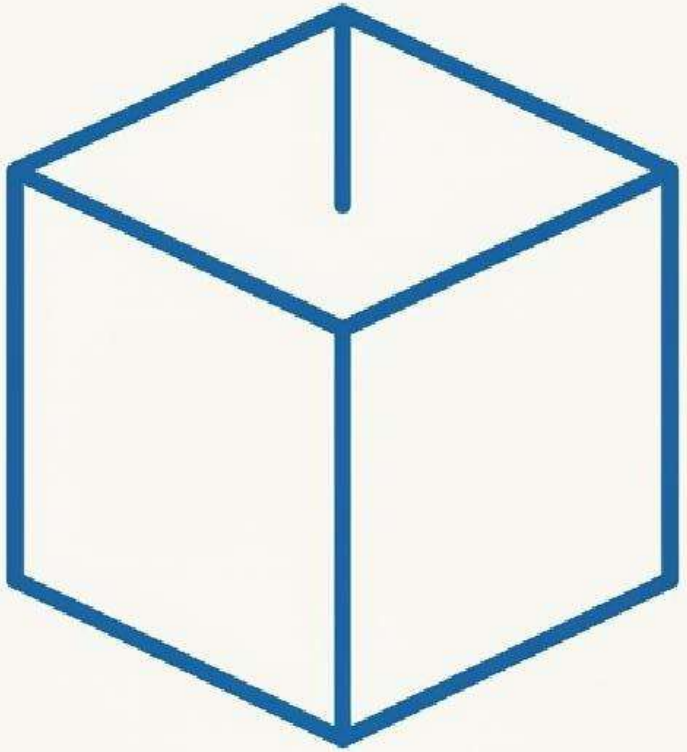
Answer: Dimensions: 20 ft x 30 ft (overall).
Max Area: 600 ft².

Problem 4 (Q5): Showroom Walls

Rectangular room, 800 ft² floor space, walls on 3 sides with door openings. Minimize wall length.



Answer: Dimensions: 20 ft x 40 ft.
Min Wall Length: 64 ft.



Module 2: Optimizing Volume & Surface Area

Building the biggest boxes and the most efficient containers.

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Deep Dive: The Open-Top Box (Q7)

A box with no top is to be built by taking a 10-in by 10-in sheet of cardboard, cutting x-in squares out of each corner and folding up the sides. Find the value of x that maximizes the volume of the box.

Step 1 & 2: Understand, Model, Diagram

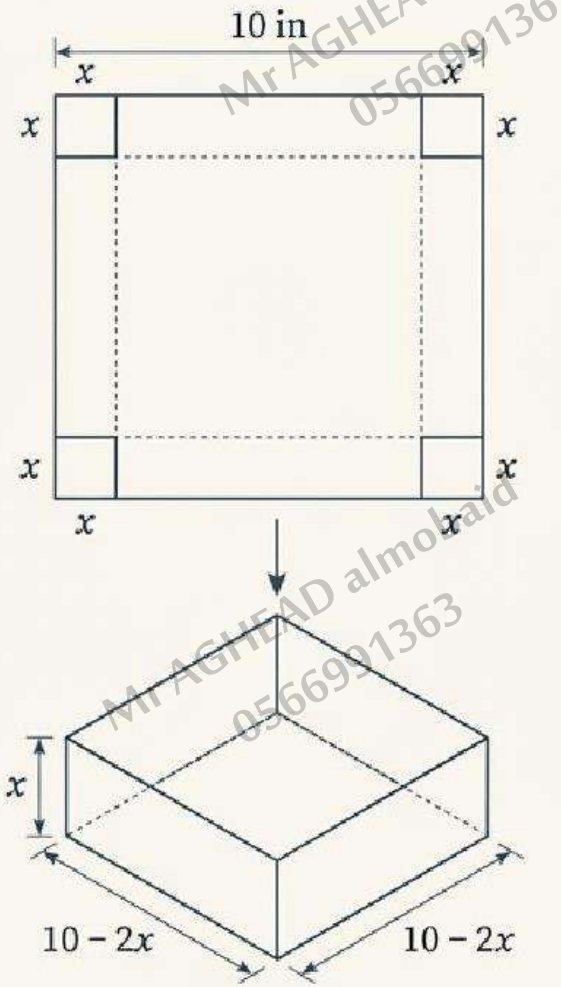
- **Objective:** Maximize Volume.

Step 3: Primary Equation (Volume)

- V = length × width × height

Step 4: Reduce to a Single Variable

- **Dimensions in terms of x:**
  - Height = x
  - Length = 10 − 2x
  - Width = 10 − 2x
- **Substitute:** $V(x) = (10 - 2x)(10 - 2x)(x) = x(10 - 2x)^2$
- **Domain:** We need $x > 0$ and $10 - 2x > 0$, which means $x < 5$. So, $0 < x < 5$.



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Deep Dive: The Open-Top Box (Solution)

Maximize $V(x) = x(10 - 2x)^2 = 100x - 40x^2 + 4x^3$ on $(0, 5)$.

Step 5: Apply Calculus

Find Derivative: $V'(x) = 100 - 80x + 12x^2$

Find Critical Points: Set $V'(x) = 0$.

$$12x^2 - 80x + 100 = 0$$
$$3x^2 - 20x + 25 = 0$$
$$(3x - 5)(x - 5) = 0$$

Solutions: $x = 5/3$ or $x = 5$.

Step 6: Verify & Interpret

Check Domain: The solution $x = 5$ is an endpoint of our domain $(0, 5)$ and results in zero volume, so we discard it. The only critical point inside the domain is $x = 5/3$.

Verification (First Derivative Test):



Final Answer: The volume is maximized when $x = 5/3$ inches.

Note: The prompt asks for the value of x, not the final dimensions or volume, so we answer the specific question asked.

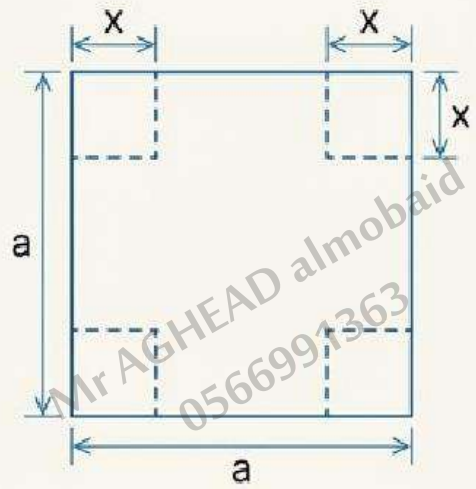
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Practice Your Skills: Volume & Surface Area

Use the 6-step strategy to solve the following.

1. Square Cardboard Box (Q8)

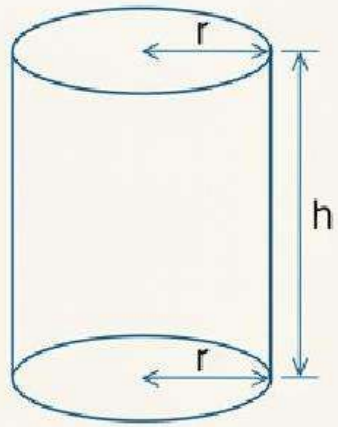
A square sheet of side 'a' is made into an open box. Find the dimensions for maximum volume.



Answer: Cutout size x = a/6.
Dimensions: (2a/3) x (2a/3) x (a/6).

2. Soda Can (Q13)

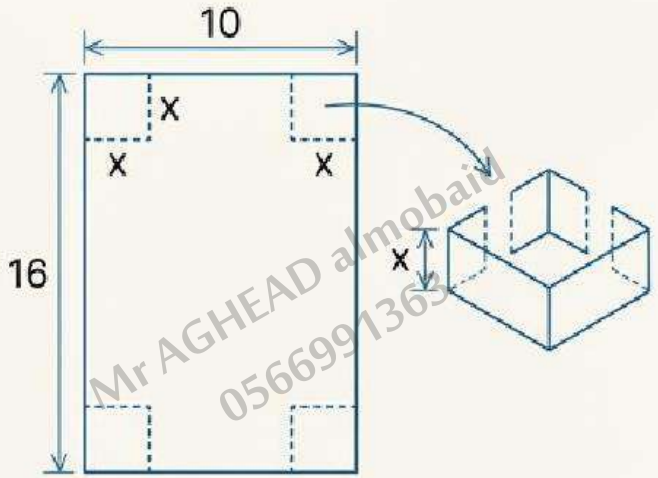
A can must hold 12 fl oz (approx 355 cm^3). Find dimensions (radius r, height h) that minimize the surface area.



Answer: h = 2r.
(The height equals the diameter).

3. Two Boxes (Q9)

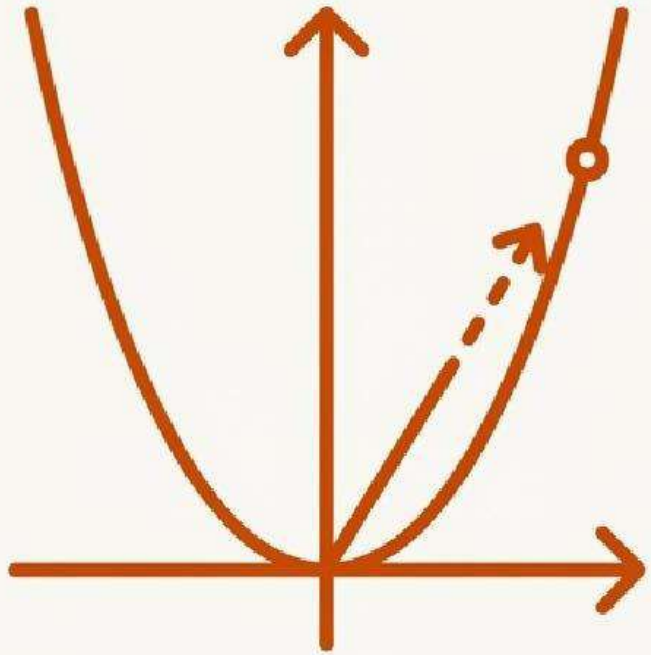
A 10x16 piece of cardboard is used to make two boxes. Find the cutout size 'x' that maximizes the sum of their volumes.



Answer: x = 2.

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Module 3: Optimizing Distance & Cost

Finding the shortest path and the most economical route.

Deep Dive: Closest Point on a Parabola (Q10)

Find the point on the parabola $y = 9 - x^2$ closest to the point (0, 0).

Step 1-3: Understand, Model, Formulate

- **Objective:** Minimize distance from a point (x, y) on the parabola to the origin (0, 0).
- **Primary Equation (Distance Formula):** $d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{(x^2 + y^2}$

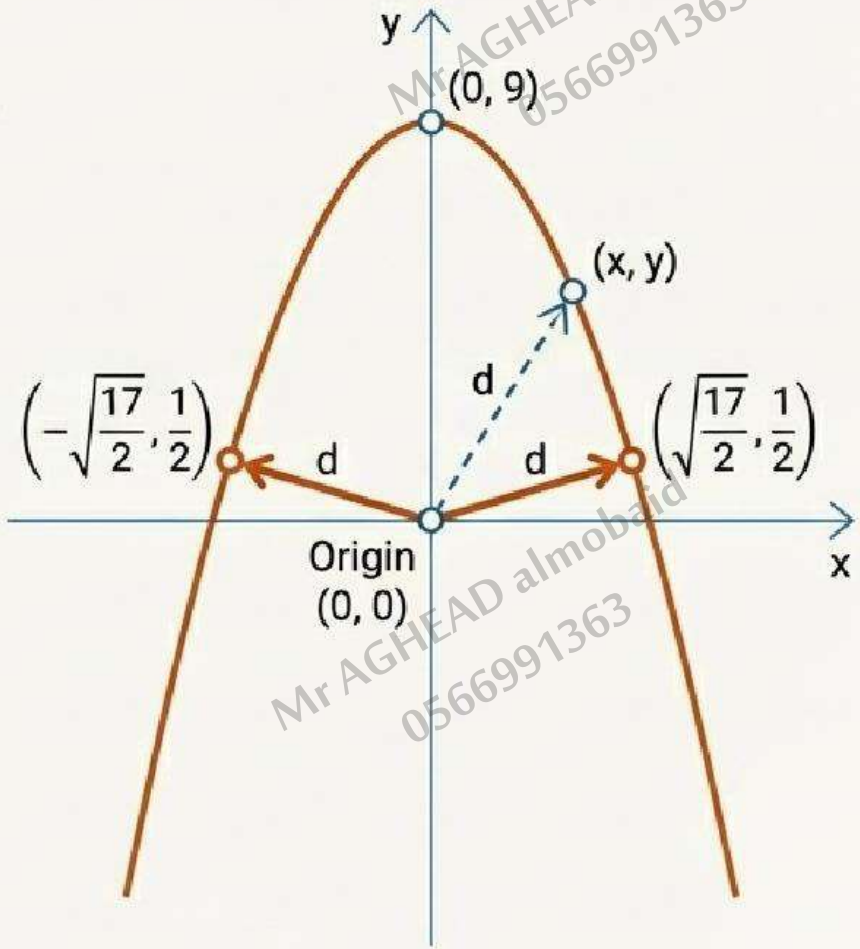
> **Pro-Tip**
> Minimizing distance 'd' is the same as minimizing d^2. This avoids the square root! Let $f = d^2 = x^2 + y^2$.

Step 4: Reduce to a Single Variable

- **Secondary Equation**: The point is on the parabola, so $y = 9 - x^2$.
- **Substitute**: $f(x) = x^2 + (9 - x^2)^2 = x^2 + 81 - 18x^2 + x^4 = x^4 - 17x^2 + 81$.
- **Domain:** x can be any real number.

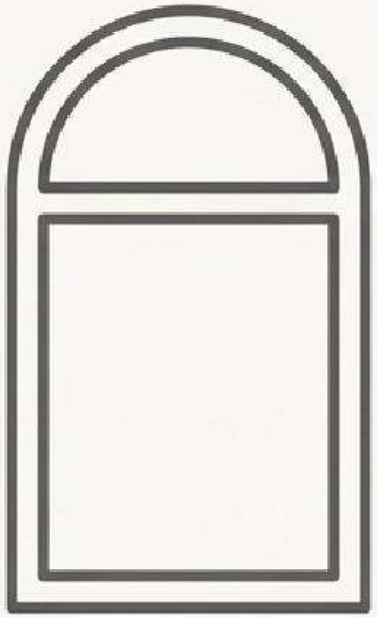
Step 5 & 6: Calculus & Solution

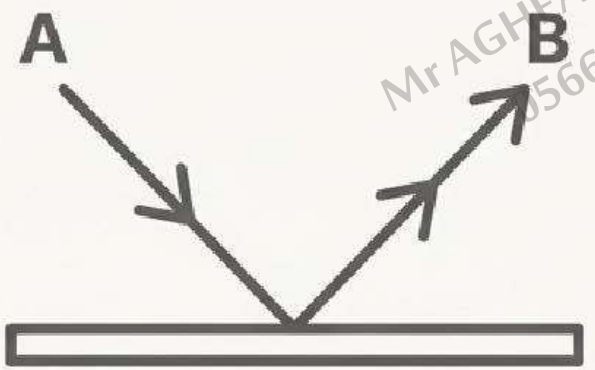
- **Derivative**: $f'(x) = 4x^3 - 34x$
- **Critical Points:** $4x^3 - 34x = 0 \Rightarrow 2x(2x^2 - 17) = 0$. Solutions are $x = 0, x = \pm\sqrt{(17/2}$.
- **Verification & Final Answer:** By testing the points, we find the minimum distance occurs at $x = \pm\sqrt{(17/2}$. The corresponding y-value is $y = 9 - (17/2) = 1/2$.
- **Answer**: The closest points are $(\sqrt{17/2}, 1/2)$ and $(-\sqrt{(17/2}, 1/2)$.



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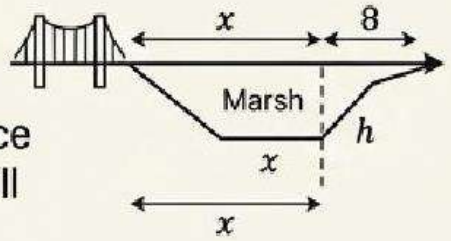
Module 4: Advanced Applications

Putting it all together with complex geometric and real-world scenarios.

Challenge Round: Setups & Solutions

Problem 1 (Q14): Highway Cost

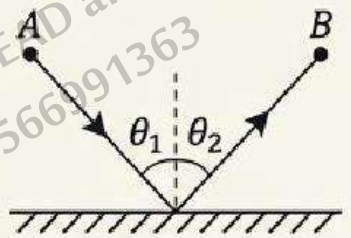
Setup Insight: Define 'x' as the distance east of the bridge where the highway leaves the marsh. Use the Pythagorean theorem to find the distance through the marsh. The cost function will be $C(x) = 10\sqrt{(x^2 + 5^2} + 7(8 - x)$.



Answer: The highway should cross out of the marsh approx. 4.9 miles east of the bridge.

Problem 2 (Q15): Light Reflection

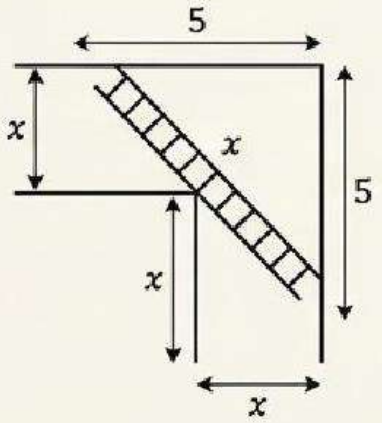
Setup Insight Minimize the total distance traveled by light. Show that when the derivative of the distance function is zero, the condition for equality of angles ($\tan\theta_1 = \tan\theta_2$) is met.



Answer: The point on the mirror is the one for which the angle of incidence equals the angle of reflection.

Problem 3 (Q16): Ladder Around Corner

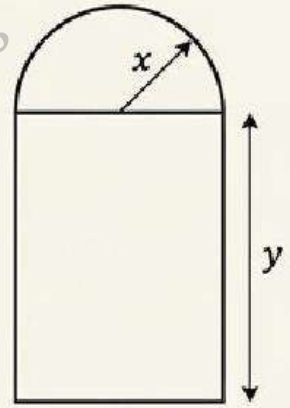
Setup Insight: The problem is equivalent to finding the *minimum* length of a line that touches the inner corner and the two outer walls. Use similar triangles to relate the variables.



Answer: The longest ladder has a length of $\left(4^{2/3} + 5^{2/3}\right)^{3/2} \approx 12.7$ ft.

Problem 4 (Q17): Norman Window

Setup Insight: Perimeter $P = 2x + 2y + \pi x$. Area $A = 2xy + \frac{1}{2}\pi x^2$. Use the perimeter constraint to write Area as a function of x.



Answer: Dimensions are radius $x = \frac{8+\pi}{4+\pi}$ and height $y = \frac{8+\pi}{4+\pi}$. The rectangle's height is equal to the semicircle's radius.

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One Strategy, Many Problems



From fencing a garden to building a highway, the problems we've explored look very different on the surface. But the core solution path is always the same.

True mastery comes not from memorizing individual solutions, but from internalizing the 6-step process.

Understand the goal, model the situation, apply the calculus, and interpret the result. This is the key to optimization.

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لا تتردد في التواصل معنا
قم بمسح رمز الـQR

Chapter (4): Applications of Differentiation

08 Eighth Lesson: Related Rates



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From Theory to Mastery: A Comprehensive Guide to Related Rates

A step-by-step masterclass for calculus students.

Presented by mr.aghead

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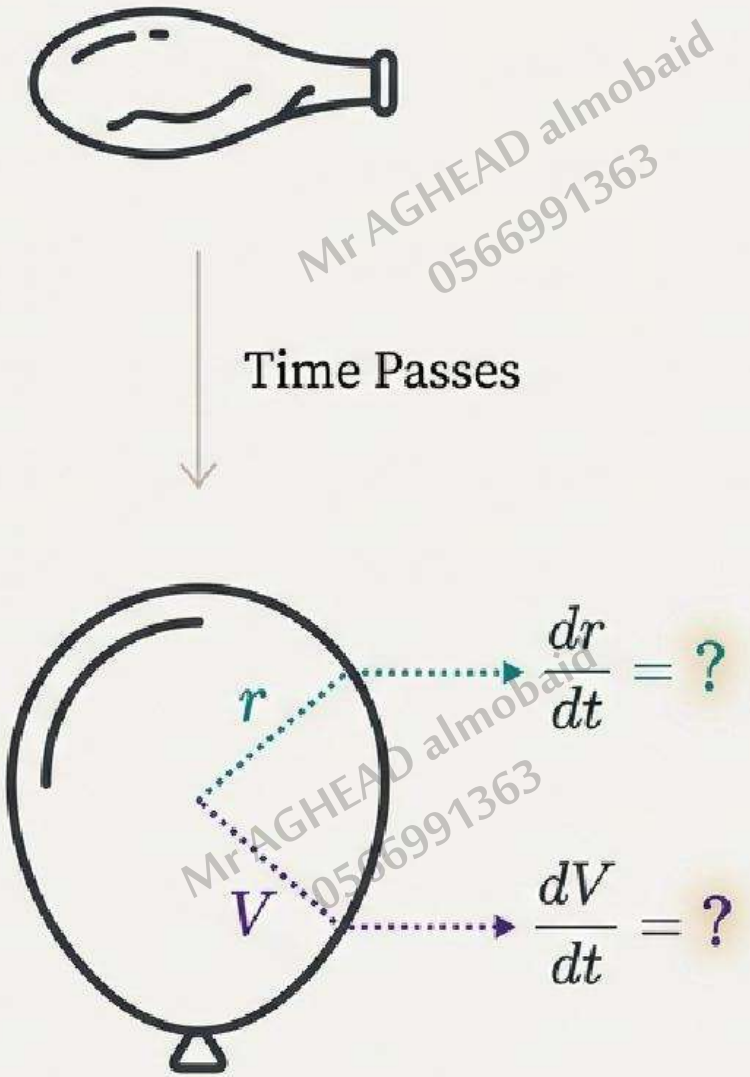
What Are "Related Rates"?

In the real world, quantities rarely change in isolation. When you inflate a balloon, its radius, surface area, and volume all increase simultaneously.
When a car drives away, its distance and the angle you're watching it from both change.

Related Rates is the part of calculus that helps us answer a fundamental question:

If we know how fast one quantity is changing, how fast is a *related* quantity changing?

It's about finding the hidden mathematical connections in a dynamic, interconnected world.



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The Blueprint: A 6-Step Method to Solve Any Problem

Every related rates problem can be solved by following a consistent process. Memorize these steps, and you'll have a reliable tool for any question you face.



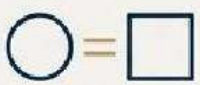
Step 1: Understand & Visualize

Read the problem. Draw a diagram. Label constants and variables.



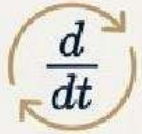
Step 2: Identify & Symbolize

List your "Givens" (known rates, like $\frac{dx}{dt} = 5$) and your "Find" (the rate you need to solve for, like $\frac{dA}{dt} = ?$).



Step 3: Find the Equation

Write the primary equation that relates your variables (e.g., $A = \pi r^2$, $x^2 + y^2 = L^2$).



Step 4: Differentiate

Implicitly differentiate both sides of the equation with respect to time (t). Use the Chain Rule for every variable.



Step 5: Substitute

Plug in all known values for variables and their rates *after* you have differentiated.



Step 6: Solve & Interpret

Solve for the unknown rate. State your answer with the correct units and check that the sign (positive/negative) makes sense.

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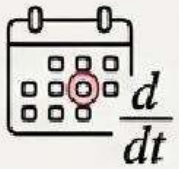
Consultant's Corner: Pro Tips & Common Pitfalls



Key Insight 1: The Sign Matters

A negative rate means the quantity is *decreasing*.

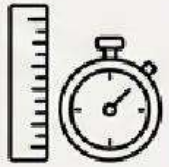
- A ladder sliding *down* a wall: **dy/dt** is negative.
- Cars moving *towards* an intersection: **dx/dt** and **dy/dt** are negative.
- A balloon *leaking* air: **dV/dt** is negative.



Key Insight 2: When to Substitute

This is the most common mistake.

- If a quantity is **constant** (e.g., the length of a ladder), you can substitute its value *before* differentiating.
- If a quantity is **changing with time** (e.g., the height of the ladder on the wall), you must substitute its value *after* differentiating.



Key Insight 3: Keep Units Consistent

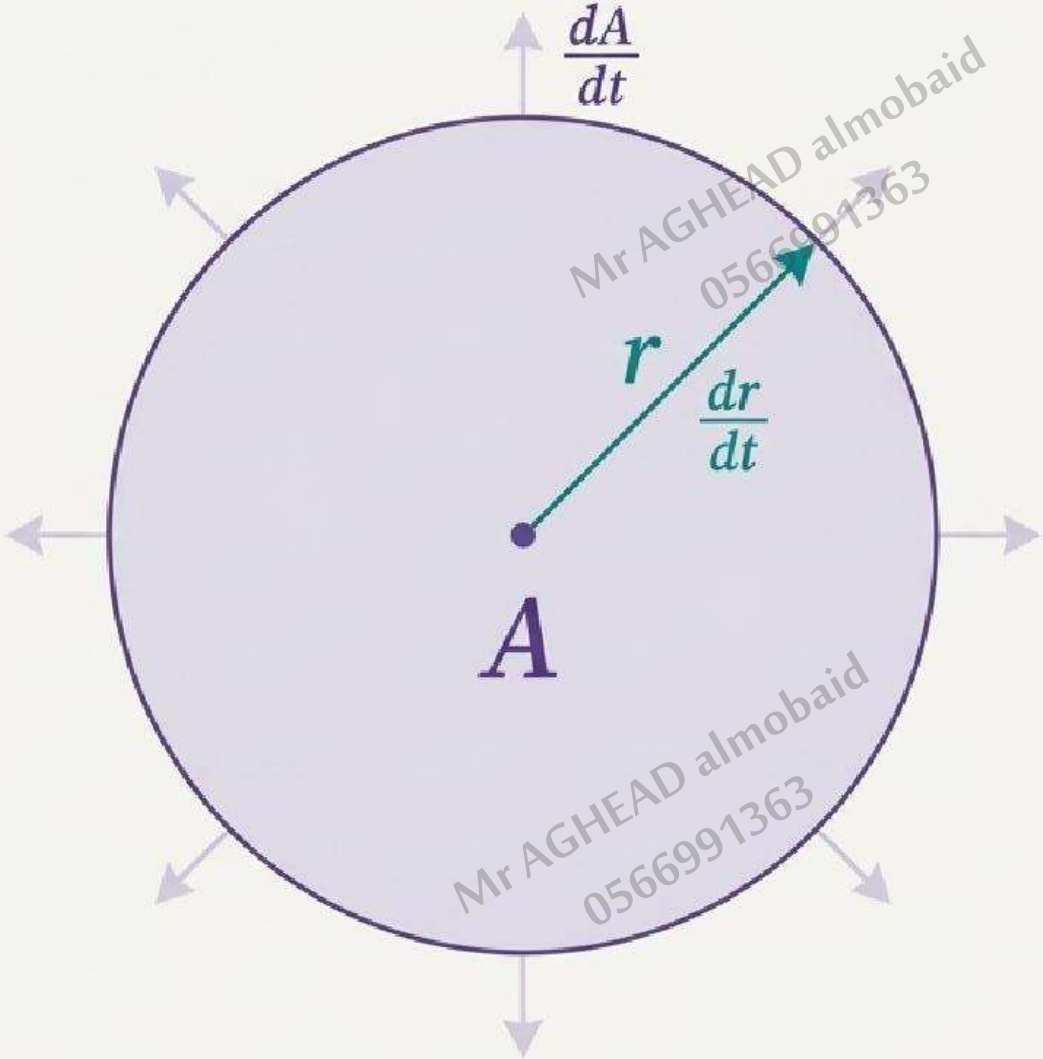
Ensure all your units match before you start. If speed is in ft/sec, all lengths should be in feet.

- Example: An oil spill rate is in gallons/min, but the radius is in feet. You must convert one to match the other.
(Reference: Textbook Example 8.1)

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Example 1: 2D Geometry

An oil spill spreads in a circle. The area of the spill is increasing at a rate of 2000π m²/min. How fast is the radius of the spill increasing at the instant when the radius is 200m?



Given

$\frac{dA}{dt} = 2000\pi$ m²/min

Instantaneous radius $r = 200$ m

Find

$\frac{dr}{dt} = ?$ when $r = 200$ m

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Example 1 Solution: The Expanding Circle

Step 1 & 2: Visualize & Identify (Done on the previous slide)

Step 3: The Equation

The area of a circle is related to its radius by:

$$A = \pi r^2$$

Step 4: Differentiate

Differentiate both sides with respect to time t:

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2) \text{ and } \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} \text{ (Chain Rule)}$$

Step 5: Substitute

Now, substitute the known values: $\frac{dA}{dt} = 2000\pi$ and $r = 200$.

$$2000\pi = 2\pi(200) \cdot \frac{dr}{dt}$$

Step 6: Solve & Interpret

$$2000\pi = 400\pi \cdot \frac{dr}{dt} \text{ and } \frac{dr}{dt} = \frac{2000\pi}{400\pi} = 5$$

Answer: The radius is increasing at a rate of **5 m/min.**

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Example 2: The Pythagorean Theorem

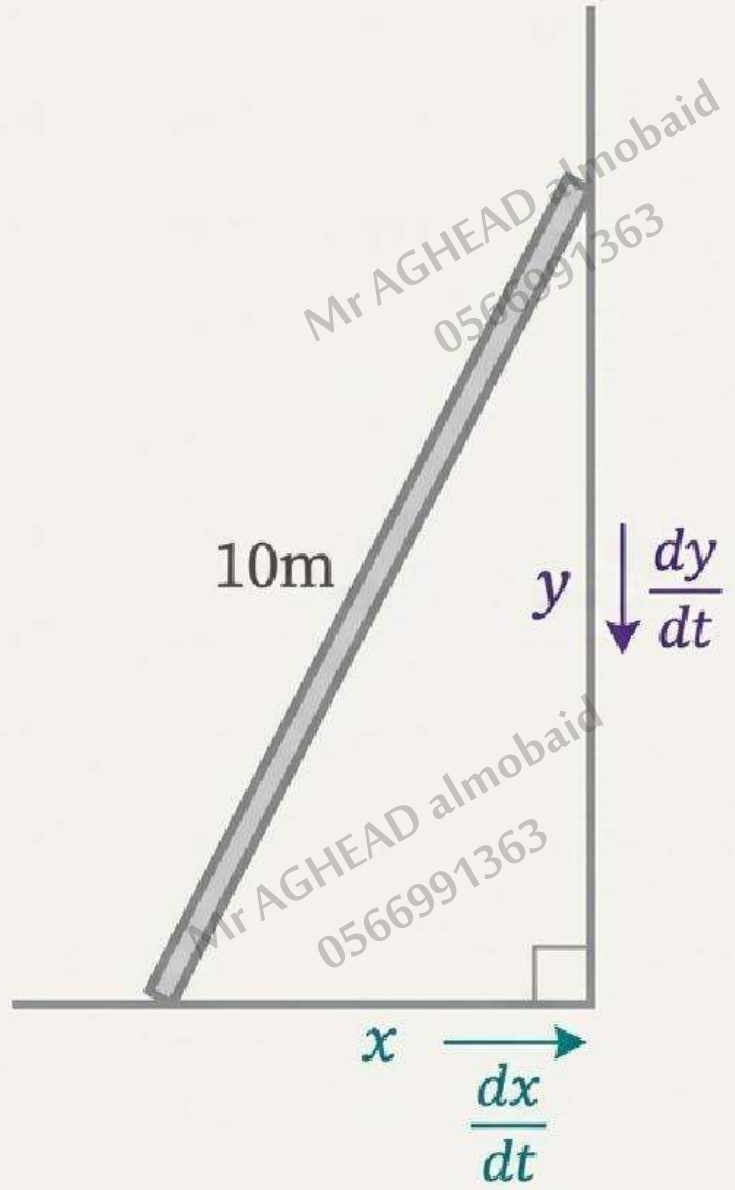
A 10-meter ladder is leaning against a vertical wall. The top of the ladder is sliding down the wall at a constant rate of 3 m/s. How fast is the bottom of the ladder sliding away from the wall when the top of the ladder is 8 meters from the ground?

Given

- Ladder length $L = 10$ m (constant)
- $\frac{dy}{dt} = -3$ m/s (Negative because it's sliding down)
- Instantaneous height $y = 8$ m

Find

- $\frac{dx}{dt} = ?$ when $y = 8$ m



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Example 2 Solution: The Sliding Ladder

Step 3: The Equation: The variables are related by the Pythagorean theorem:

$$x^2 + y^2 = 10^2$$

Step 4: Differentiate: Differentiate both sides with respect to time t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Step 5: Substitute: We need to find x when $y = 8$. Use the primary equation:

$$x^2 + 8^2 = 10^2 \rightarrow x^2 + 64 = 100 \rightarrow x^2 = 36 \rightarrow x = 6\text{ m}$$

Now substitute $x = 6$, $y = 8$, and $\frac{dy}{dt} = -3$:

$$2(6) \cdot \frac{dx}{dt} + 2(8)(-3) = 0$$

Step 6: Solve & Interpret:

$$12 \cdot \frac{dx}{dt} - 48 = 0$$

$$12 \cdot \frac{dx}{dt} = 48$$

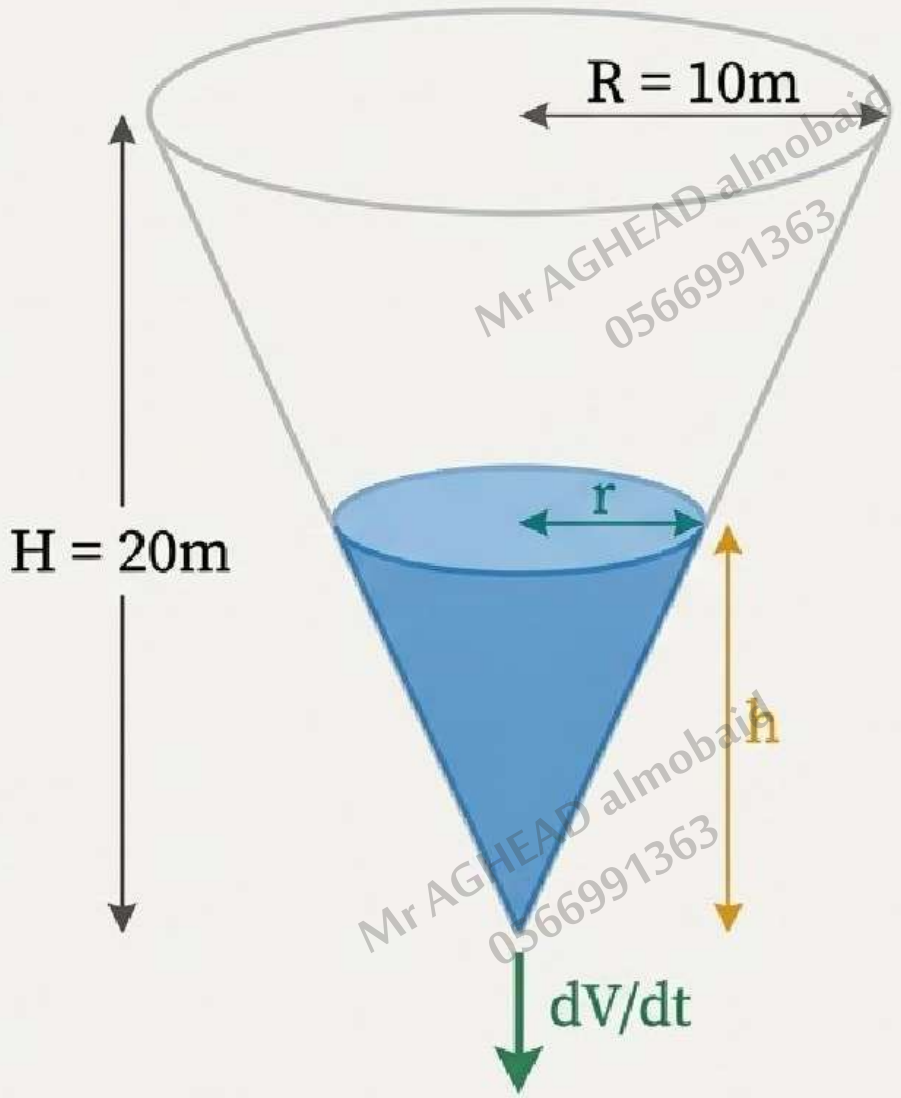
$$dx/dt = 4$$

Answer: The bottom of the ladder is sliding away from the wall at **4 m/s.**

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Example 3: 3D Geometry & Similar Triangles

Water is leaking out of an inverted conical tank at a rate of 10 m^3/min. The tank has a height of 20m and a base radius of 10m. At what rate is the depth of the water in the tank decreasing at the instant the water is 5m deep?



Given

- $dV/dt = -10$ m^3/min (Note: Negative because it's leaking)
- Tank Height $H = 20$ m, Tank Radius $R = 10$ m
- Instantaneous water depth $h = 5$ m

Find

- dh/dt = ? when $h = 5$ m

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Example 3 Solution: The Conical Tank

Step 3: The Equation

Volume of a cone: $V = \frac{1}{3}\pi r^2 h$. This has two variables, r and h. We must eliminate one.

Key Step: Similar Triangles

By similar triangles: $\frac{r}{h} = \frac{R}{H} = \frac{10}{20} = \frac{1}{2}$, so $r = \frac{h}{2}$. Substitute this into the volume equation:

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$

Step 4: Differentiate: Now differentiate the simplified equation:

$$\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \frac{dh}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

Step 5: Substitute: $dV/dt = -10$ and $h = 5$.

$$-10 = \frac{1}{4}\pi(5)^2 \frac{dh}{dt}$$

Step 6: Solve & Interpret:

$$-10 = \frac{25\pi}{4}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-40}{25\pi} = \frac{-8}{5\pi}$$

Answer: The water depth is decreasing at a rate of **$8/(5\pi)$ m/min**.

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Example 4: Trigonometry & Angles

A balloon rises vertically at 75 m/min from a point on the ground. An observer is 150m away from the launch point. Find the rate at which the angle of elevation of the balloon is changing when the balloon is 150m high.

Given

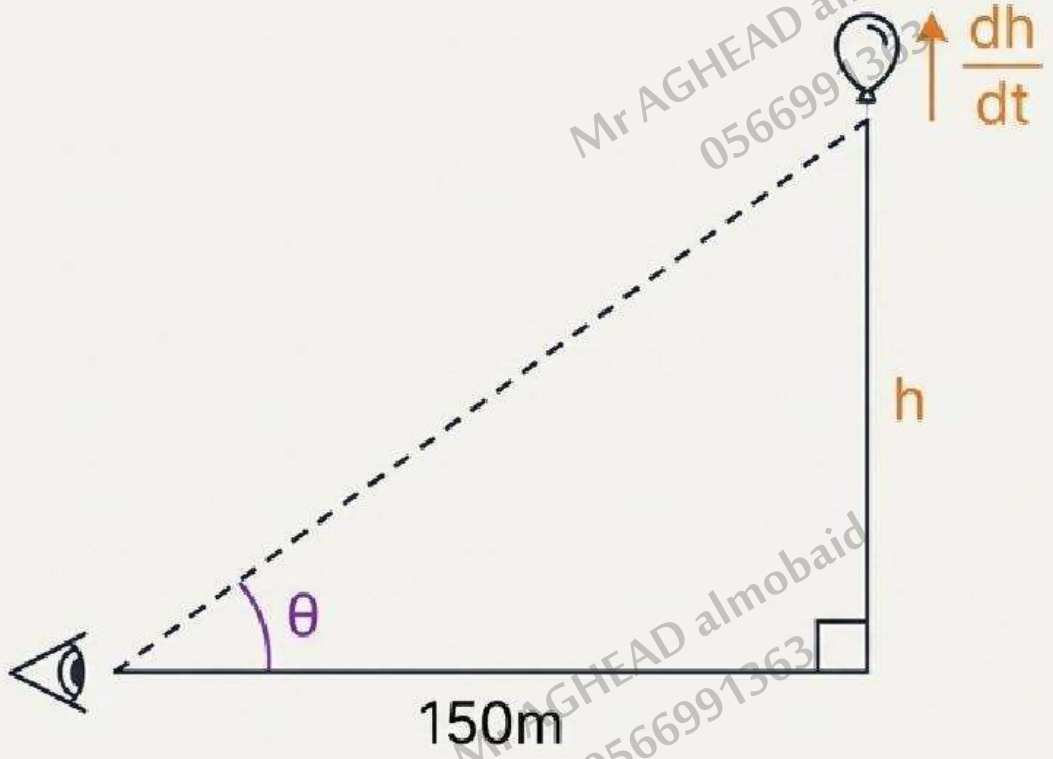
$\frac{dh}{dt}$ = 75 m/min

Observer distance = 150 m (constant)

Instantaneous height h = 150 m

Find

$\frac{d\theta}{dt}$ = ? when h = 150 m



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Example 4 Solution: The Rising Balloon

Step 3: The Equation

The angle θ is related to the height **h** by the tangent function:

$$\tan(\theta) = \frac{h}{150}$$

Step 4: Differentiate

Now differentiate the equation with respect to time **t**:

$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{150} \cdot \frac{dh}{dt}$$

Step 5: Substitute

We know **dh/dt = 75**. We need the value of $\sec^2(\theta)$ when **h = 150.**

When h = 150, the triangle has two sides of 150, so it's an isosceles right triangle, meaning $\theta = \pi/4$ (or 45°). The hypotenuse is $150\sqrt{2}$.

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{150}{150\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

So, $\sec(\theta) = \sqrt{2}$, and $\sec^2(\theta) = 2$.

Substitute into the differentiated equation:

$$2 \cdot \frac{d\theta}{dt} = \frac{1}{150} \cdot 75$$

Step 6: Solve & Interpret

Solve for **dθ/dt**:

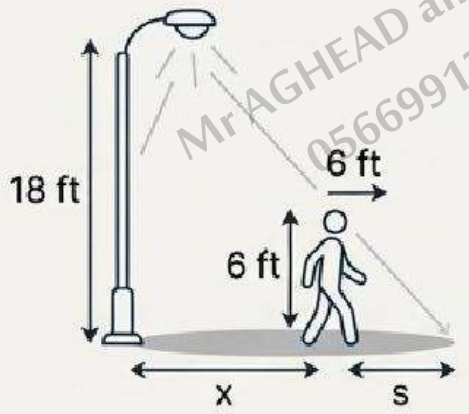
$$2 \cdot \frac{d\theta}{dt} = \frac{1}{2}$$

$$\frac{d\theta}{dt} = \frac{1}{4}$$

Answer: The angle of elevation is changing at **0.25 rad/min.**

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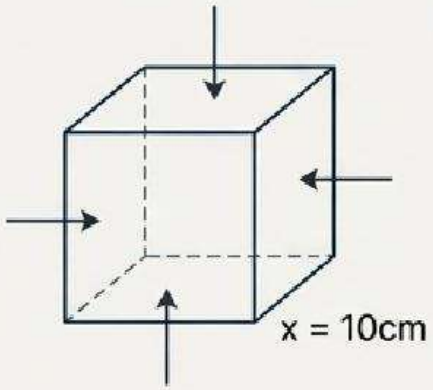
More Applications: A Portfolio of Problems



Shadow Problem

A 6ft man walks away from an 18ft lamppost at 2 ft/s. How fast is his shadow lengthening? (Uses similar triangles).

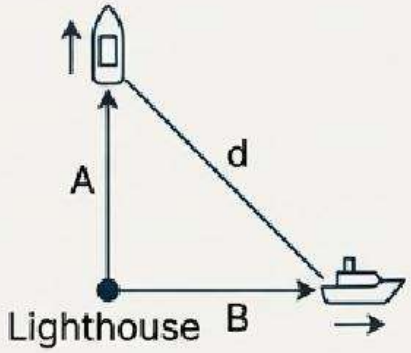
Answer: 1 ft/s



Melting Cube

An ice cube with 10cm sides is melting, with its side length decreasing at 1 cm/h. How fast is the volume changing?

Answer: -300 cm^3/h



Two Ships

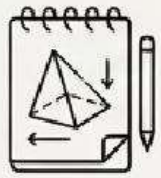
Ship A is south of a point, moving north. Ship B is east, moving east. Find the rate the distance between them is changing.

Answer: (Calculation required, showing a more complex application of Pythagorean theorem)

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Mastering the Method: Your Final Checklist

The 6-Step Blueprint (Recap)



Visualize:
Draw it out.



Identify:
List Givens & Find.

$E=$

Equate:
Find the master formula.

$\frac{d}{dt}$

Differentiate:
Use the chain rule on d/dt.

$V=(\downarrow\frac{}{1})$

Substitute:
Plug in values *after* differentiating.



Solve:
Find the rate and check units/sign.

Top 3 Reminders

Signs matter:
Decreasing = Negative Rate.

Timing is everything:
Substitute changing values last.

Units must match:
Convert first.

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Beyond the Classroom: Seeing the World in Motion

Related Rates are more than just a chapter in a textbook; they are a lens for understanding the world. They reveal the hidden connections between changing systems—in physics, finance, engineering, and beyond.

You haven't just learned to solve a type of problem. You've learned to quantify the intricate dance of change itself.

From: mr.aghead

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Chapter (4): Applications of Differentiation

09 Ninth Lesson:

Rates of Change in Economics and the Sciences



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The Master Key: Applications of the Derivative

Unlocking Rates of Change in Economics and the Sciences



Presented by Mr. Aghead

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How do we pinpoint the moment of maximum opportunity?

Across diverse fields, critical decisions hinge on understanding rates of change.



In Economics:

What is the exact cost of producing just *one more* item?

At what production level is our business most efficient?

What price will maximize our revenue?



In Chemistry:

How fast is a chemical reaction *right now*?

When is the reaction at its peak speed?



In Biology:

What is the current growth rate of a population, and when will it grow the fastest?

These questions seem different, but they all ask the same thing:
How can we measure and optimize change as it happens?

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The Derivative is the Language of Instantaneous Change

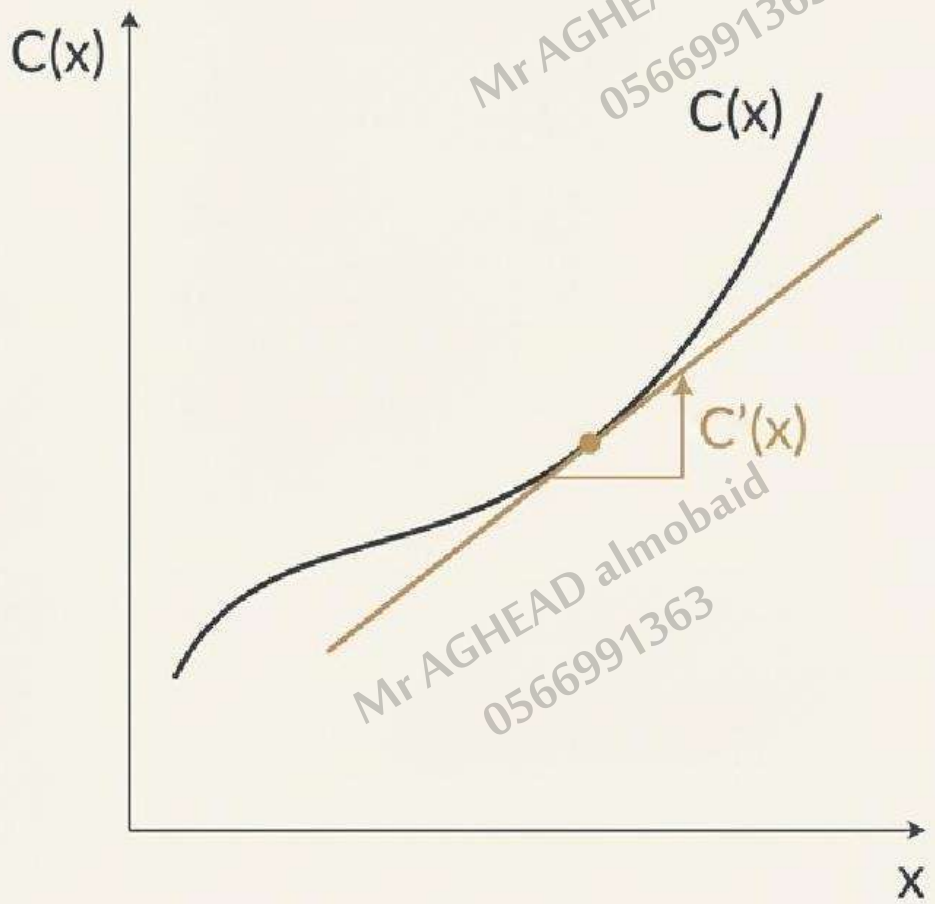
Core Concept

The derivative, $f'(x)$, gives us the instantaneous rate of change of a function. When we see the word 'rate," we should think "derivative."

Key Economic Translations

In business and economics, the derivative has a special name: Marginal.

- **Total Cost Function: C(x)** = Total cost to produce x items.
- **Marginal Cost: C'(x)** or **dC/dx** = The rate of change of cost. It approximates the cost of producing one additional item.
- **Total Revenue Function: R(x)** = Total revenue from selling x items.
- **Marginal Revenue: R'(x)** or **dR/dx** = The rate of change of revenue.
- **Total Profit Function: P(x) = R(x) - C(x)**
- **Marginal Profit: P'(x)** or **dP/dx** = The rate of change of profit.



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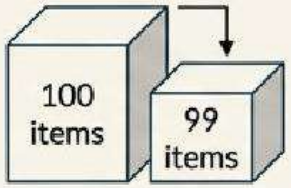
Marginal Cost: A Powerful and Precise Approximation

Core Question: How much does it *really* cost to produce the 100th unit?

1. The Actual Cost

Calculate the total cost of 100 items and subtract the total cost of 99 items.

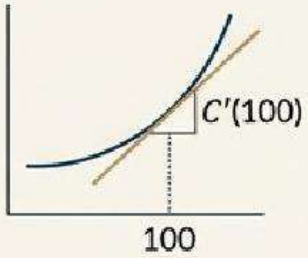
Actual Cost of 100th item = $C(100) - C(99)$



2. The Marginal Cost

Calculate the derivative of the cost function, $C'(x)$, and evaluate it at x=100.

Marginal Cost at x=100 $= C'(100)$



Worked Example: Let the total cost function be $C(x) = 0.02x^2 + 8x + 5000$.

Actual Cost Calculation:

$C(100) = 0.02(100)^2 + 8(100) + 5000 = 6000$

$C(99) = 0.02(99)^2 + 8(99) + 5000 = 5988.02$

$C(100) - C(99) = 11.98$

Marginal Cost Calculation:

$C'(x) = 0.04x + 8$

$C'(100) = 0.04(100) + 8 = 12$

Key Takeaway: The Marginal Cost (12) is extremely close to the Actual Cost (11.98) and is much easier to compute.

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Minimizing Average Cost: Finding the Sweet Spot of Production

Concept, Formula, and Goal

Concept: While managers want to control costs, they are often most interested in minimizing the *average cost per item.*

The Formula: The average cost function $\bar{C}(x)$ is the total cost divided by the number of items produced. $\bar{C}(x) = \frac{C(x)}{x}$.

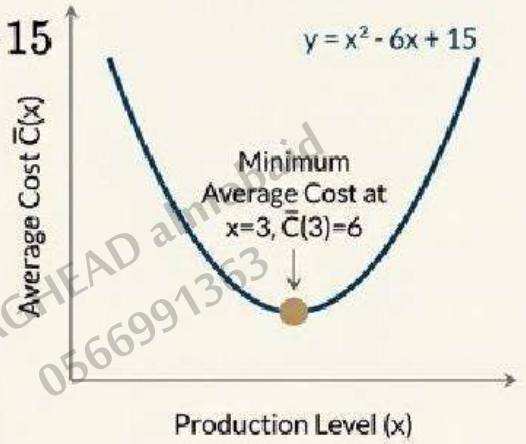
The Goal: To find the production level x that minimizes $\bar{C}(x)$, we find the critical points by setting its derivative to zero: $\bar{C}'(x) = 0$.

Worked Example

Worked Example: For a company with total cost $C(x) = x^3 - 6x^2 + 15x$.

1. Find the Average Cost Function:

$$\bar{C}(x) = \frac{x^3 - 6x^2 + 15x}{x} = x^2 - 6x + 15$$



2. Find the Derivative:

$$\bar{C}'(x) = 2x - 6$$

3. Find the Critical Point:

$$2x - 6 = 0 \Rightarrow x = 3$$

4. Confirm it is a Minimum:

$\bar{C}''(x) = 2$, which is positive.

Therefore, x=3 (e.g., 3000 units) is the production level that minimizes average cost.

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Maximizing Profit: The Golden Rule for Business Success

The Principle: Profit $P(x)$ is maximized when the rate of change of profit is zero, i.e., $P'(x) = 0$.

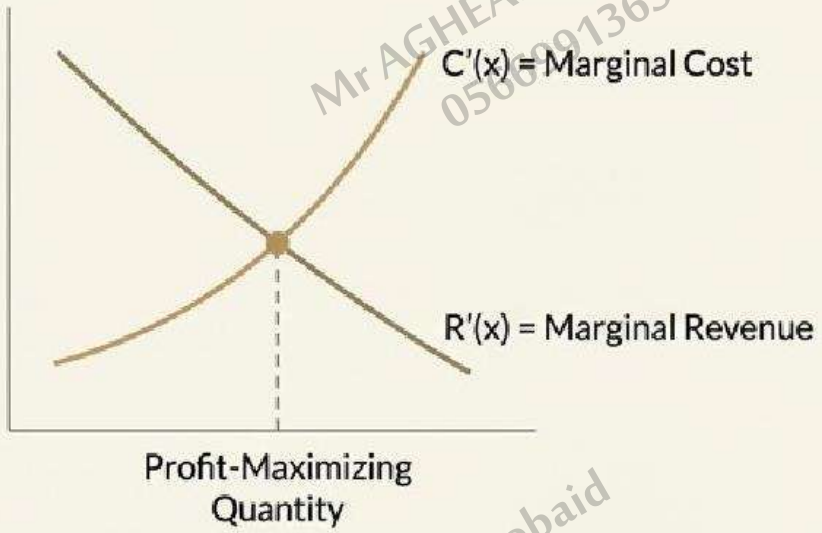
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The Golden Rule

Since $P(x) = R(x) - C(x)$, then $P'(x) = R'(x) - C'(x)$.
Setting this to zero gives us the fundamental condition for maximum profit:

$$R'(x) = C'(x)$$

Marginal Revenue = Marginal Cost



Worked Example: A factory produces an item with a price of 20 per unit.

- Cost Function: $C(x) = 0.002x^2 + 8x + 5000$
- Revenue Function: $R(x) = 20x$

1. **Find Marginal Cost and Revenue:**
   $C'(x) = 0.004x + 8$
   $R'(x) = 20$

2. **Set them equal:**
   $20 = 0.004x + 8$
   $12 = 0.004x \Rightarrow x = 3000$

3. **Conclusion:**
   The factory achieves maximum profit when it produces 3,000 units. The maximum profit is $P(3000) = 13{,}000$.

Elasticity of Demand: A Tool for Strategic Pricing

Concept

Elasticity of Demand (E) measures how sensitive the quantity demanded $x = f(p)$ is to a change in price p.

The Formula

$$E = \frac{p}{f(p)} * f'(p)$$

Interpreting the Results

Elastic Demand

If $E < -1$, demand is elastic. A price increase will lead to a proportionally larger decrease in demand, causing total revenue to fall.

Inelastic Demand

If $-1 < E < 0$, demand is inelastic. A price increase will lead to a proportionally smaller decrease in demand, causing total revenue to rise.

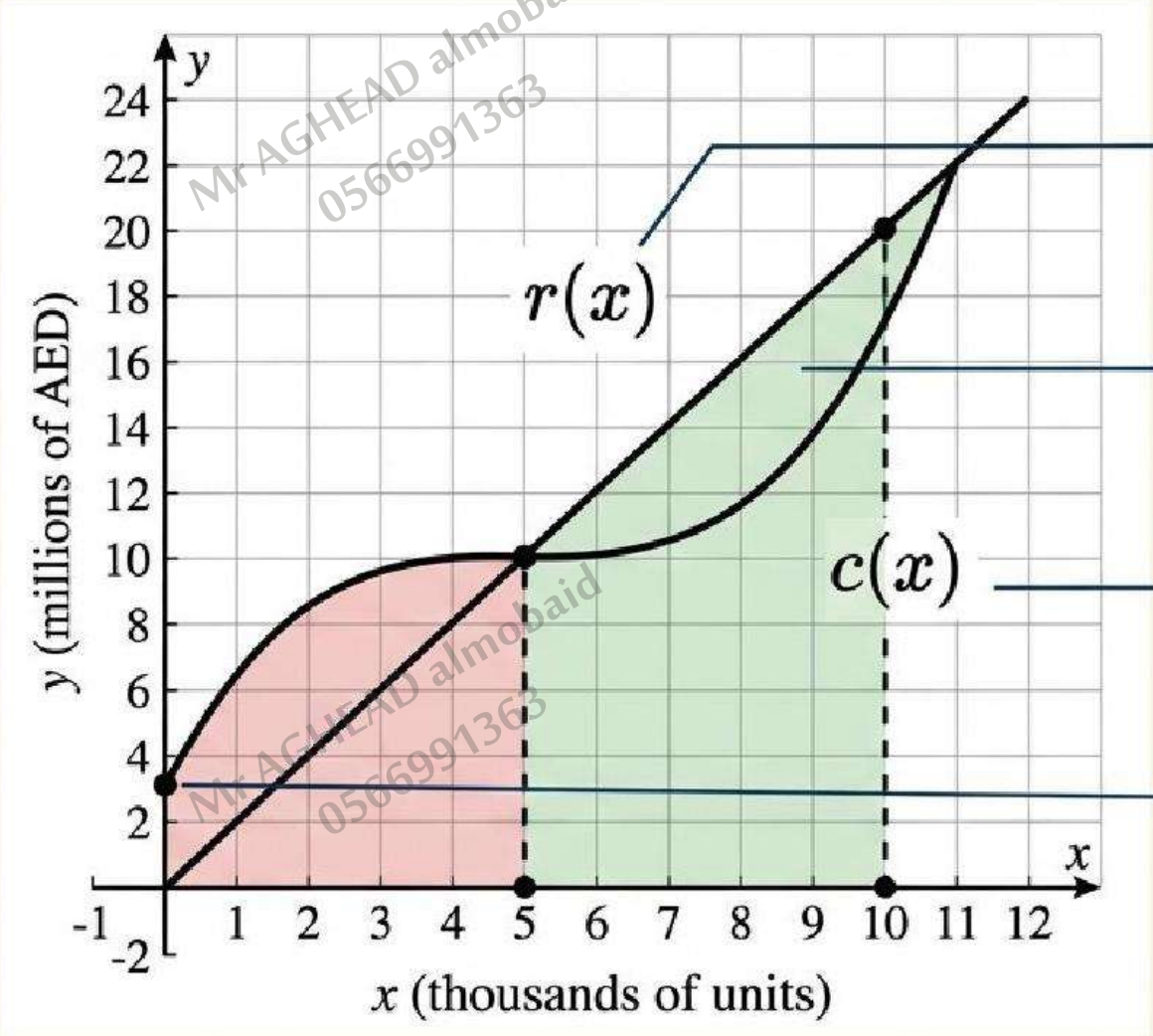
Worked Example

For a product with demand function $f(p) = 200(30 - p)$ for $0 < p < 30$.

1. Find Elasticity E: The derivative is $f'(p) = -200$. $E = \frac{p}{200(30-p))} * (-200) = -\frac{-p}{30-p} = \frac{p}{p-30}$
2. When is demand elastic?: We solve $E < -1$. $\frac{p}{p-30} < -1$. Since $p - 30$ is negative, multiplying by it reverses the inequality: $p > -1 * (p - 30) \Rightarrow p > -p + 30 \Rightarrow 2p > 30 \Rightarrow p > 15$.
3. Conclusion: For prices between 15 and 30, demand is elastic. Increasing the price in this range will decrease total revenue. Revenue increases when the price is between 0 and 15.

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Interpreting Economic Functions from a Graph



Problem & Analysis

Question 1: What is $C(0)$ and what does it represent?
Answer: $C(0) = 3$ (million). This represents the fixed start-up costs of the project.

Question 4: In what interval does the company lose money?
Answer: Loss occurs when $C(x) > R(x)$. This happens in the interval $(0, 5)$.

Question 5: In what interval does the company make a profit?
Answer: Profit occurs when $R(x) > C(x)$. This happens in the interval $(5, 11)$.

Question 6: What is the production level for maximum profit?
Answer: Maximum profit occurs when the vertical distance between $R(x)$ and $C(x)$ is greatest. This is at x=8 (8,000 units), where profit is $R(8) - C(8) = 16 - 12 = 4$ million.

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One Tool, Many Disciplines: The Derivative in the Sciences

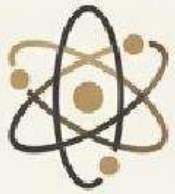


Chemistry – Rate of Reaction

Context: In an autocatalytic reaction, the rate $x'(t)$ is proportional to the amount of product $x(t)$ and the remaining potential (1-x(t)).

Equation: $x'(t) = r * x(t) * [1 - x(t)]$

Key Insight: The reaction rate is maximum when the concentration is exactly half the saturation level (x = 1/2).



Physics – Density & Current

Linear Density: For a rod where $m(x)$ is the mass of the first x meters, the linear density $\rho(x)$ at any point is the derivative.

Equation: $\rho(x) = m'(x)$
Electrical Current: For a wire where $Q(t)$ is the charge at time t, the instantaneous current $I(t)$ is the derivative.

Equation: $I(t) = Q'(t)$



Biology – Population Growth

Context: The logistic model for population growth $p(t)$ assumes the growth rate $p'(t)$ is proportional to the current population and the remaining carrying capacity.

Equation: $p'(t) = r * p(t) * [1 - p(t)]$

Key Insight: The population growth rate is maximum when the population is at half its maximum sustainable level (p = 1/2).

Overarching Takeaway: Notice the identical mathematical models for chemical reactions and population growth. Mathematics reveals the underlying unity in seemingly unrelated natural processes.

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Applied Science: Solving for Rates of Change



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Example 1: Rate of Chemical Reaction

Problem: If a reaction follows $x'(t) = 2x(t)[4 - x(t)]$, find the concentration $x(t)$ where the reaction rate is maximum.

Solution:

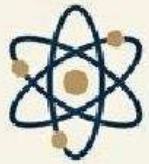
The rate is $f(x) = 8x - 2x^2$.

To find the maximum, find the derivative and set it to zero:

$f'(x) = 8 - 4x$

$8 - 4x = 0 \quad \Rightarrow \quad x = 2$

The rate is maximum when concentration is 2.



Example 2: Linear Density

Problem: The mass of the first x meters of a rod is $m(x) = \sqrt{2x}$. Compute the linear density at x=2 and x=8.

Solution:

Density is $\rho(x) = m'(x) = \frac{1}{2}(2x)^{-1/2} \cdot 2 = \frac{1}{\sqrt{2x}}$.

$\rho(2) = \frac{1}{\sqrt{4}} = \frac{1}{2}$ kg/m.

$\rho(8) = \frac{1}{\sqrt{16}} = \frac{1}{4}$ kg/m.

The rod is denser at the beginning.



Example 3: Population Growth

Problem: A population grows according to $p'(t) = 2p(t)[7 - 2p(t)]$. Find the population level for which the growth rate is maximum.

Solution:

The growth rate is $f(p) = 14p - 4p^2$.

To find the maximum, find the derivative and set it to zero:

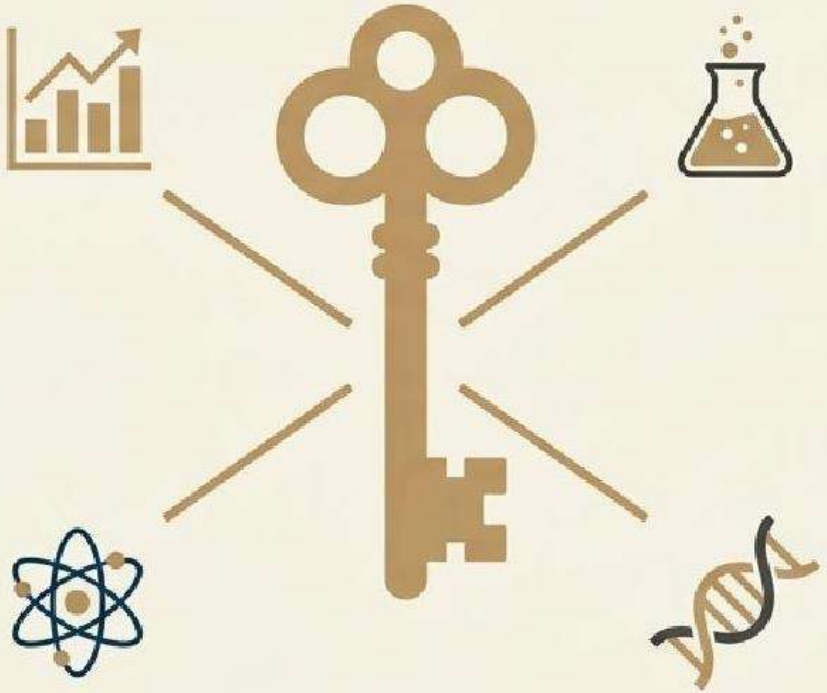
$f'(p) = 14 - 8p$

$14 - 8p = 0 \quad \Rightarrow \quad p = \frac{14}{8} = \frac{7}{4} = 1.75$

The growth rate is maximum at a population level of 1.75.

The Unlocked View: The Power of a Single Idea

Summary: We started with a set of seemingly disconnected problems from business, chemistry, physics, and biology. We discovered that a single mathematical concept—the derivative—was the master key to unlocking them all.



The Key's Power:

- It allows us to move from *average* change to *instantaneous* change.
- It provides the tools for **optimization**: finding the maximum profit, the minimum cost, and the peak reaction rate.
- It reveals the deep, underlying mathematical structures that govern the world around us.

Mastering the derivative is not just about learning a formula; it's about learning to see the world in terms of its dynamic, ever-changing nature.

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تكون هذه الملزمة قد حققت
الفائدة المرجوة ♥

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