

United Arab Emirates
Emirates school establishment
Education Council 3 / domain 5
Almahmood school
12 Advanced
Second semester
Mathematics

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Applications of Differentiation

Learn the notions of local and absolute extrema.

Find the critical points of a given function.

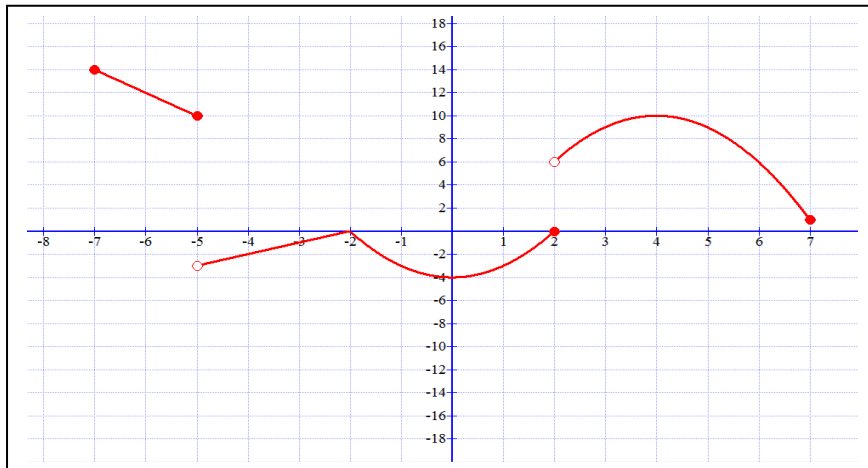
Find the local extrema of a given function.

Find the absolute extrema of a given function.

DEFINITION 3.3

A number c in the domain of a function f is called a **critical number** of f if $f'(c) = 0$ or $f'(c)$ is undefined.

Find Critical Numbers



2) $f(x) = x^2 - 4x + 3$

3) $f(x) = x^3 - 3x^2 + 5$

4) $f(x)=xe^x$

5) $f(x)=\sqrt[3]{x^2-2x}$

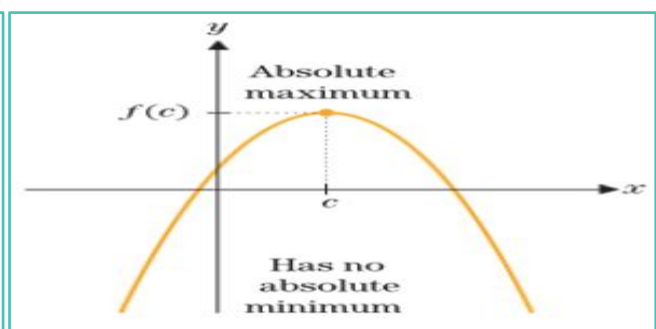
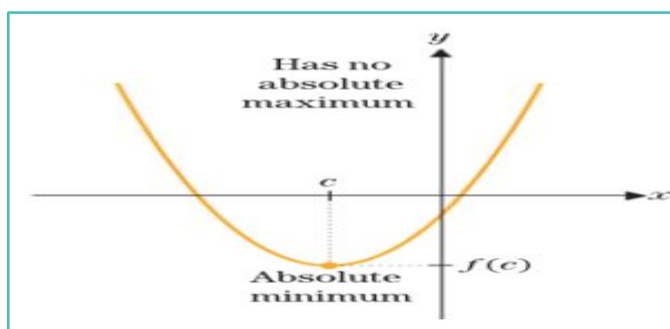
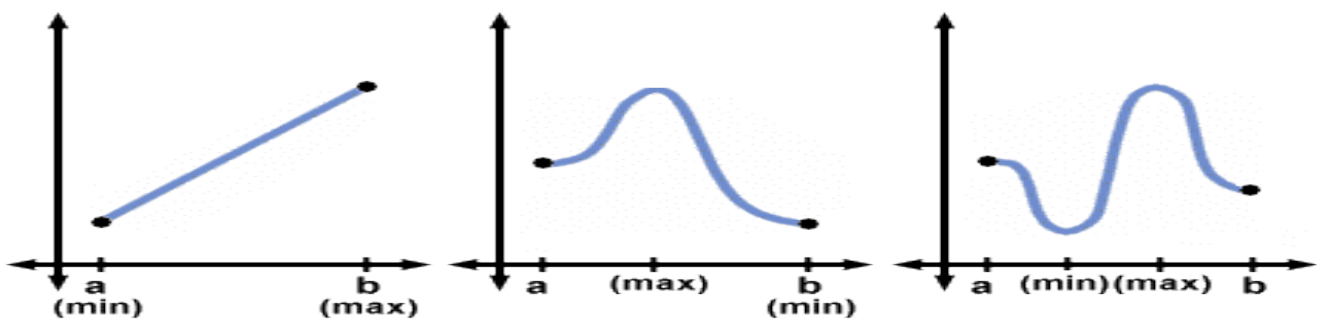
6) $f(x)=x-\frac{1}{x}$

DEFINITION 3.1

For a function f defined on a set S of real numbers and a number $c \in S$,

- (i) $f(c)$ is the **absolute maximum** of f on S if $f(c) \geq f(x)$ for all $x \in S$ and
- (ii) $f(c)$ is the **absolute minimum** of f on S if $f(c) \leq f(x)$ for all $x \in S$.

An absolute maximum or an absolute minimum is referred to as an **absolute extremum**. (The plural form of extremum is **extrema**.)

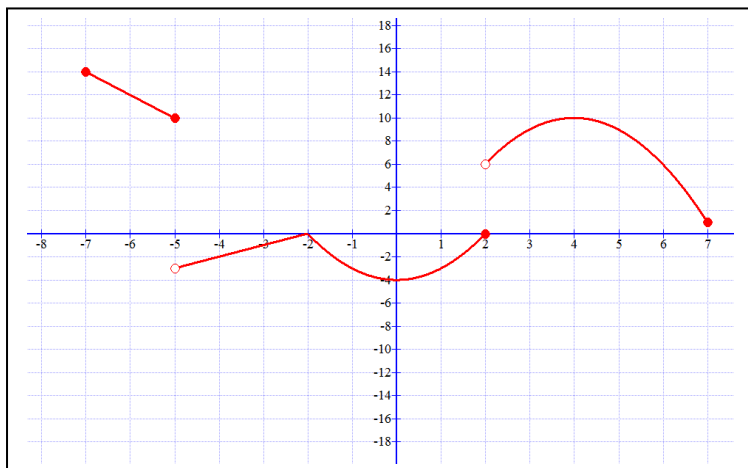
**THEOREM 3.3**

Suppose that f is continuous on the closed interval $[a, b]$. Then, each absolute extremum of f must occur at an endpoint (a or b) or at a critical number.

Finding the absolute extrema of a continuous function on a closed interval

- 1- Find all critical numbers in the interval and compute the function values at these points
- 2- Compute function values at the endpoints
- 3- The largest of these function values is the absolute maximum, and the smallest of these function values is the absolute minimum.

Find the *absolute extrema*



2) $f(x)=x^2-4x+3 \quad x \in [-0, 3]$

3) $f(x)=x^3-3x+5 \quad x \in [-2, 2]$

$$4) f(x) = \sqrt[3]{x^2 - 2x} \quad x \in [-2, 1]$$

$$5) f(x) = |x - 1| \quad x \in [-3, 2]$$

$$6) f(x) = \sin x + \cos x \text{ on interval } [0, 2\pi]$$

increasing and decreasing functions.

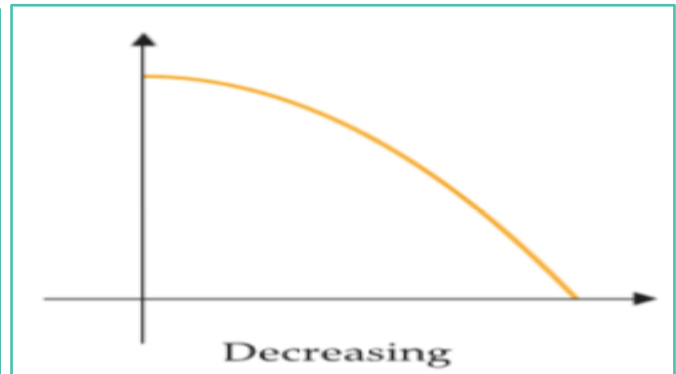
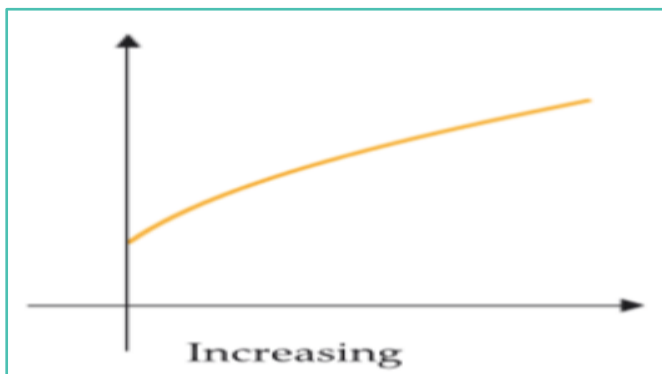
local extrema

- Identify increasing and decreasing functions.
- Find the local extrema of a given function using the First Derivative Test.

DEFINITION 4.1

A function f is **increasing** on an interval I if for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) < f(x_2)$ [i.e., $f(x)$ gets larger as x gets larger].

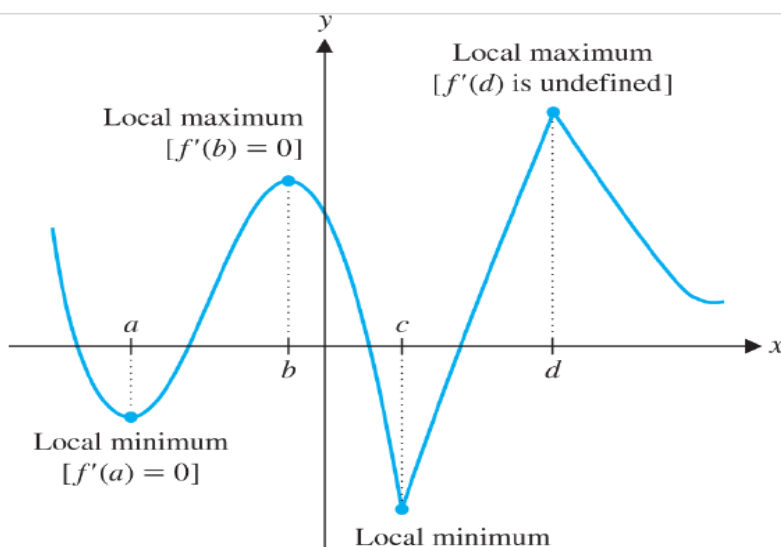
A function f is **decreasing** on the interval I if for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) > f(x_2)$ [i.e., $f(x)$ gets smaller as x gets larger].

**Definition**

(i) $f(c)$ is the local **maximum** of f if $f(c) \geq f(x)$ for all x in some open interval containing c

(ii) $f(c)$ is the local **minimum** of f if $f(c) \leq f(x)$ for all x in some open interval containing c

In either case, we call $f(c)$ a **local extremum** of f .

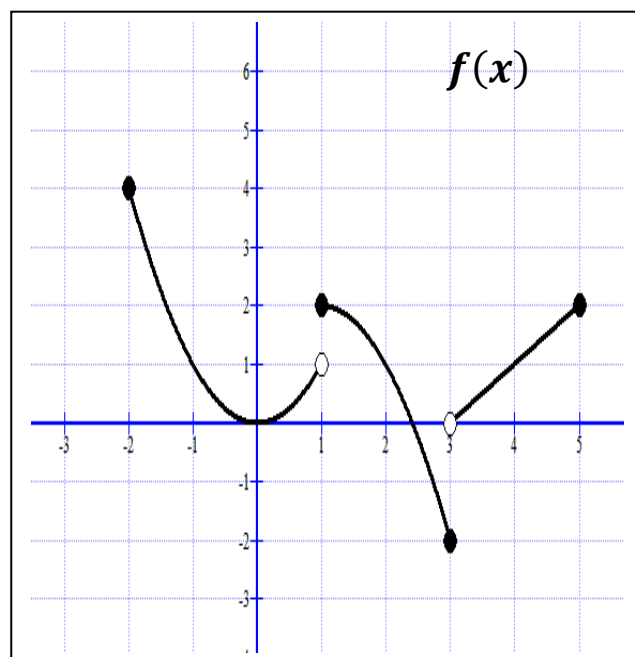


1) $f(x)$ Is increasing on

2) $f(x)$ Is decreasing on...

3) The values of x with f has a local maximum are

4) The values of x with f has a local minimum are



THEOREM 4.1

Suppose that

f is differentiable on an interval I .

- (i) If $f'(x) > 0$ for all $x \in I$, then f is increasing on I .
- (ii) If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I .

Determine intervals of increasing and decreasing and the local extremum

1) $f(x)=2x^3-6x+5$

2) $f(x)=x^3-3x^2+1$

3) $f(x)=\frac{x}{x^2+1}$

4) $f(x) = xe^x$

5) $f(x) = 2\sin x - x + 2 \quad x \in [0, 2\pi]$

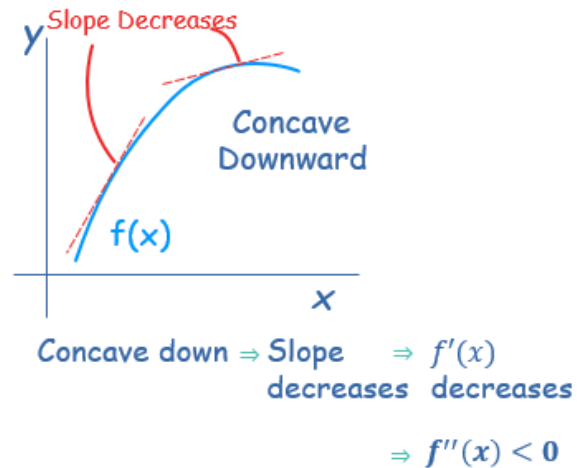
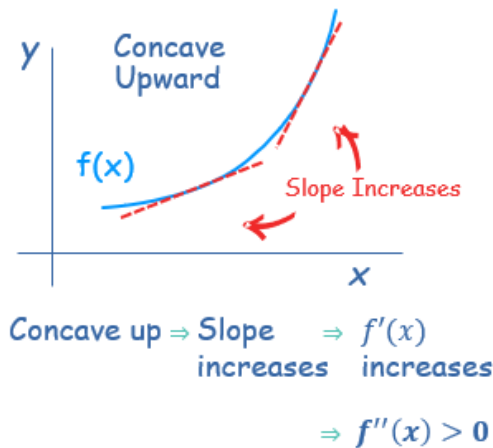
Concavity and the Second Derivative Test

- **Determine the concavity of a function using the first and second derivatives.**
- **Learn the notion of an Inflection Point and find one.**
- **Find the local extrema of a function using the Second Derivative test.**

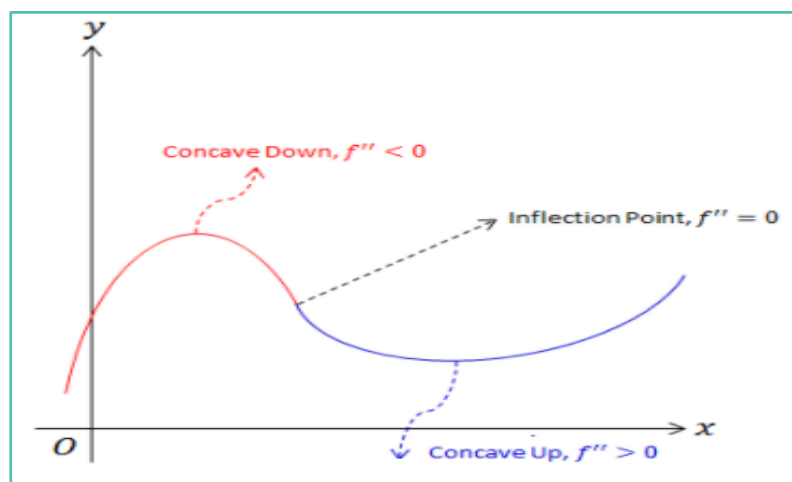
DEFINITION 5.1

For a function f that is differentiable on an interval

- i) The graph of f is concave up on I if f' is increasing on I or
 (ii) Concave down on I if f' is decreasing on I

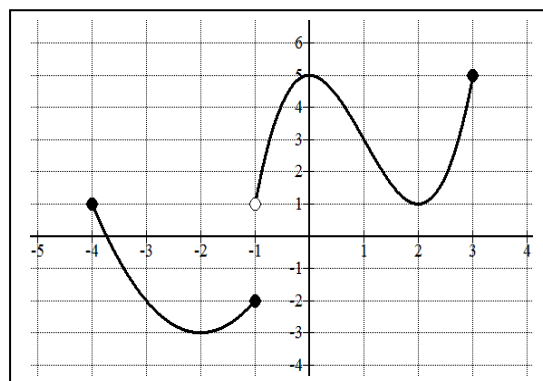
**DEFINITION 5.2**

Suppose that f is continuous on the interval (a, b) and that the graph changes concavity at a point $c \in (a, b)$ (i.e., the graph is concave down on one side of c and concave up on the other). Then, the point $(c, f(c))$ is called an **inflection point** of f .



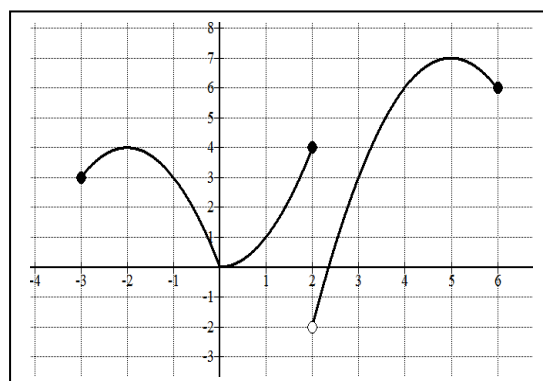
The figure shown is $F(x)$

- 1) $F(x)$ concave up on
- 2) $F(x)$ concave down on
- 3) The inflection points of $F(x)$ is



The figure shown is $g(x)$

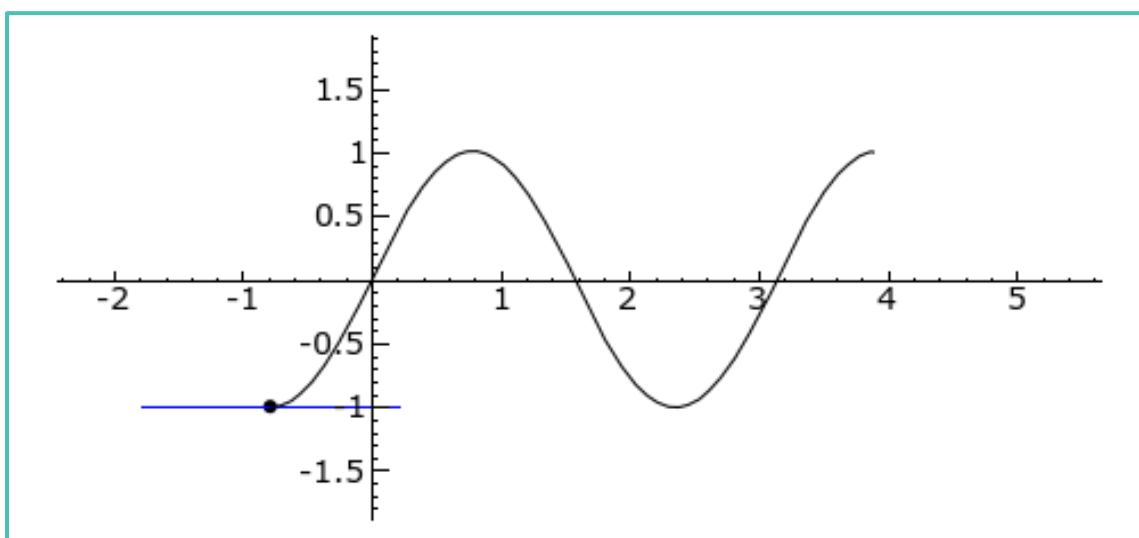
- 1) $g(x)$ concave up on
- 2) $g(x)$ concave down on
- 3) The inflection points of $g(x)$ is



THEOREM 5.1

Suppose that f'' exists on an interval I .

- (i) If $f''(x) > 0$ on I , then the graph of f is concave up on I .
- (ii) If $f''(x) < 0$ on I , then the graph of f is concave down on I .



Determine where the graph of $f(x)$ is concave up and concave down, find any inflection points

1) $F(x)=2x^3-6x^2-9x+10$

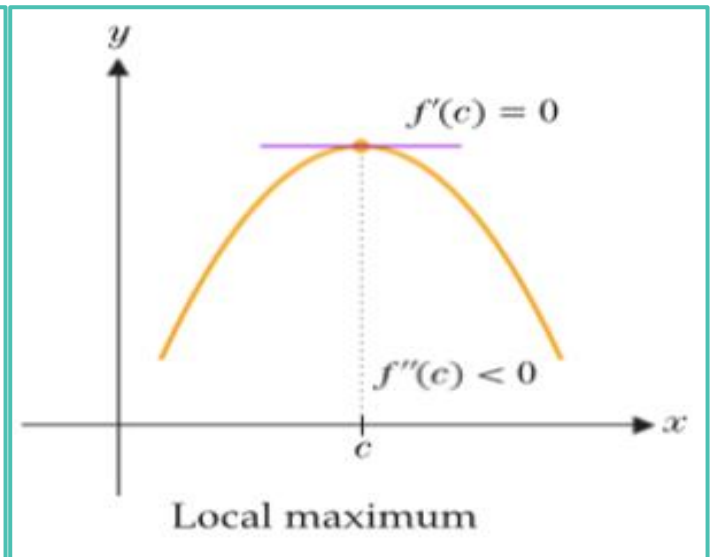
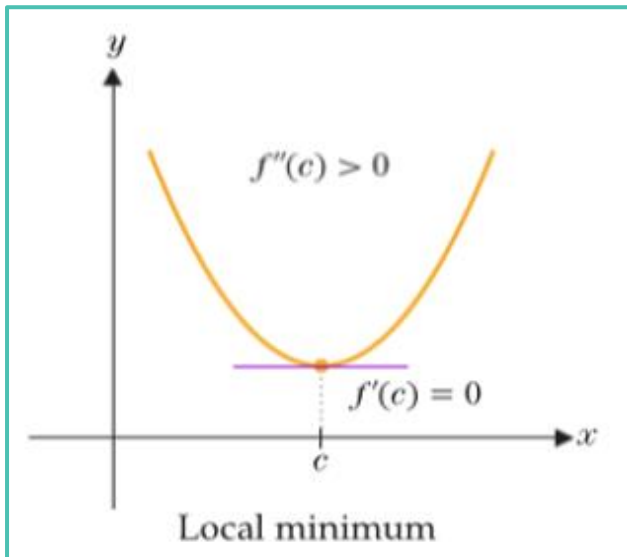
2) $F(x)=x^4-6x^2-x+10$

3) $f(x) = xe^{-4x}$

THEOREM 5.2 (Second Derivative Test)

Suppose that f'' is continuous on the interval (a, b) and $f'(c) = 0$, for some number $c \in (a, b)$.

- (i) If $f''(c) < 0$, then $f(c)$ is a local maximum.
- (ii) If $f''(c) > 0$, then $f(c)$ is a local minimum.

**Second Derivative Test:**

- 1 Find the first derivative and all critical numbers where $f'(c) = 0$.
- 2 Find the second derivative and substitute the critical numbers in the second derivative.
- 3 If:

$f''(c) > 0$	Then, $f(c)$ is a local minimum
$f''(c) < 0$	Then, $f(c)$ is a local maximum
$f''(c) = 0$, or undefined	Then we cannot determine, <u>use first derivative test</u>

Find the extrema points using the second derivative test

1) $f(x) = x^3 - 2x^3 + 5$

2) $f(x) = x^4 - 4x^3 + 4x^2 + 10$

3) If $f''(x) = 2x - 5$, $f(x)$ has acritical points at $x = 2$, $x = 3$

1) $f(x)$ has local maximum value at $x = \dots\dots$

2) $f(x)$ has local minimum value at $x = \dots\dots$

The figure represent $f(x)$ on $[-4, 4]$

1) Critical numbers are ...

2) Intervals of increasing for $f(x)$ are ...

3) Intervals of decreasing $f(x)$ are ...

4) The absolute maximum for $f(x)$ ----

5) The absolute minimum for $f(x)$ ----

6) The values of x where $f(x)$ has a local maximum are

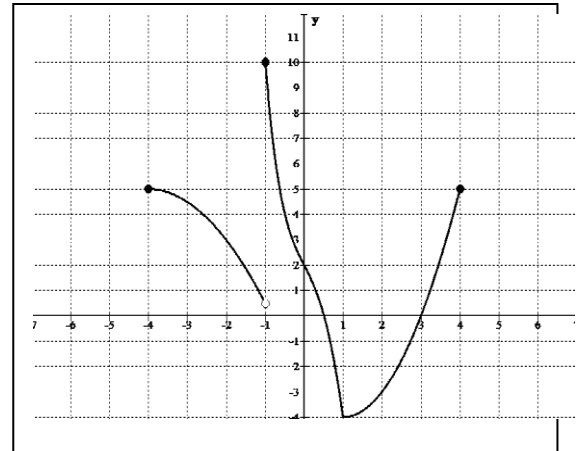
7) The values of x where $f(x)$ has a local minimum are

8) The intervals of concaving up for $f(x)$ are ...

9) The intervals of concaving down for $f(x)$ are ...

10) The inflection points are

11) The sign of $f'(-2)$. $f''(-2)$ is



The figure represent $f'(x)$

where $f(x)$ is continues

1) Critical numbers for $f(x)$ are ...

2) Intervals of increasing for $f(x)$ are ...

3) Intervals of decreasing for $f(x)$ are ...

4) The values of x where $f(x)$ has a local maximum are

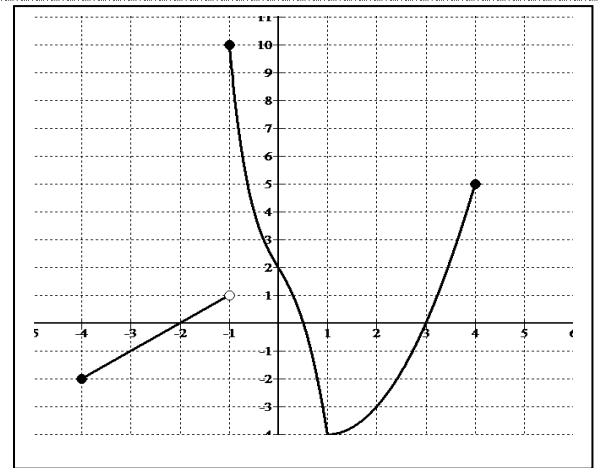
5) The values of x where $f(x)$ has a local minimum are

6) The intervals of concaving up for $f(x)$ are ...

7) The intervals of concaving down for $f(x)$ are ...

8) The inflection points are

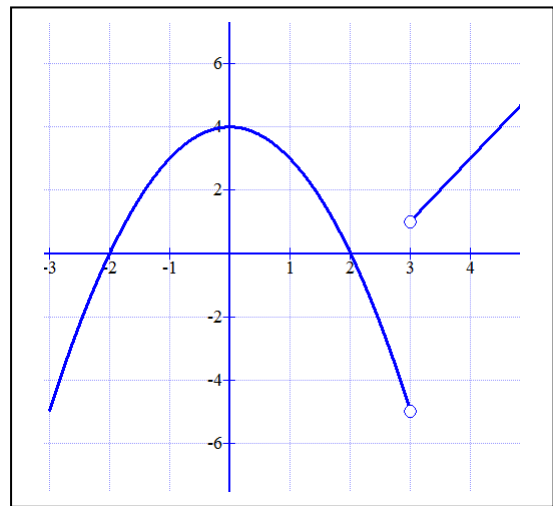
9) The sign of $f(3) - f(2)$ is



The figure represent $f''(x)$ where $f(x)$ is continuous

1) The intervals of concaving up for $f(x)$ are

2) The intervals of concaving down for $f(x)$ are ...



3) The values of x where $f(x)$ has inflection points are ----

4) If $f'(-3) = f'(4) = 0$

$f(x)$ has a local maximum value at $x = \dots\dots$

$f(x)$ has a local minimum value at $x = \dots\dots$

تمرین

If $f(x) = x^3 + bx^2 + kx + h$ has a critical point at $x = 1$, and has inflection point at $(1, 10)$ find b, k, h

Draw the graph of the function

$$f(x)=x^3-3x+5$$

$$2) f(x)= x^4-4x^3+5$$

Draw a graph of $f(x)$ showing all significant features of the function $f(x) = x + \frac{25}{x}$

Draw a graph of $f(x)$ showing all significant features of the function

$$f(x) = (x + 2)^{1/5} + 4$$

sketch a graph of a function with the given properties:

$f(0) = 0, f'(x) > 0$ for $x < -1$ and $-1 < x < 1$, $f'(x) < 0$ for $x > 1$,

$f''(x) > 0$ for $x < -1$, $0 < x < 1$ and $x > 1$

, $f''(x) < 0$ for $-1 < x < 0$

Optimization

Solve mathematical and real-life optimization problems



Guidelines to Solve Optimization Problem

- If there is a picture put it down and label it.
- Determine the variables and how they are related.
- Determine the quantity that needs to be maximized or minimized.
- Write an expression of the amount to be maximized or minimized in term of one variable, by solving for any other variables in term of this one variable.
- Determine the maximum/ minimum allowed values.
- Solve the problem and check your answer.

Two positive numbers whose sum is 12 Find the two numbers when they have the largest product

عدان موجبان مجموعهما 12 اوجد العددين ليكون حاصل ضربهما اكبر ما يمكن

Two positive numbers whose product is 100 Find the two numbers to minimize their sum

عدان موجبان حاصل ضربهما 100 اوجد العددين ليكون مجموعهما اقل ما يمكن

Two positive numbers whose sum is 12 Find the two numbers so that the sum of their squares is as small as possible

Two positive numbers whose sum is 12 . Find the two numbers so that the product of one of them and the square of the other is as large as possible

A rectangle has a perimeter of 20, find its dimensions to maximize its area

For a rectangle whose area is 100 cm^2 , find its dimensions to minimize its perimeter

The sum of the length of its base and height in a right triangle is 10 cm
Find its height so that its area is as large as possible

Right-angled triangle with hypotenuse 10 cm Find its height so that its area is as large as possible

a farmer, wants to fence-in a three-sided area next to river, using of the river as the fourth side of the rectangle. If he has **104** feet of available fencing, what are the dimensions of a maximum area rectangle

A rope whose length is 12 cm is to be divided into two parts to make two squares. Find the length of each square so that the sum of their squares is as small as possible

A circular sector with a perimeter of 12 cm Find its radius so that its area is as large as possible

A cone whose height and radius sum is 10. Find its height and volume as large as possible

A cylinder has an outer area of 600π , find its dimensions to make it as large as possible

The point (x,y) moves in the graph on the graph of the function $f(x)=\sqrt{x}$
Find (x,y) to be as close as possible to the point $(4,0)$.

A factory produces x devices per day, and one device is sold at a price of 200 dirhams. If the cost of producing devices is $x^2 - 10x + 100$, find the number of devices manufactured to achieve the largest possible profit.

A factory produces x devices per day, so if the cost of producing devices is $x^3 - 200x^2 + 5000x$ dirhams, find the number of devices manufactured to achieve the lowest possible average cost

Related Rates

Solve mathematical and real-life problems on related rates. •

Related Rates Problems

Equation relating two or more quantities that are all changing with time.

Use chain rule (with implicit differentiation with respect to time) to determine how different rates (derivatives with respect to time) are related.

STEPS TO SOLVE RELATED RATES PROBLEM



01 - SKETCH

Make a simple sketch, if appropriate.



02 - EQUATION

Set up an equation relating all relevant quantities.



03 - DIFFERENTIATE

Differentiate implicitly both sides of the equation with respect to time t .



04 - SUBSTITUTE

Substitute in values for all known quantities and derivatives.



05 - SOLVE

Solve for the remaining rate.

A rectangle whose length increases at the rate of 2 cm/s and whose width decreases at the rate of 5 cm/s Find the rate of change in its area when its length becomes 10 cm and its width becomes 8 cm

A triangle whose base length is twice its height, if its height increases at a rate of 2 cm/s, find the rate of change in its area when its height becomes 10 cm

A circle whose circumference increases at a rate of 2 cm/s Find the rate of change in its area when its radius becomes 10 cm

A man moves north at a speed of 10m/s and another man moves from the same place eastward at a speed of 15m/s Find the rate of change between the two men when the first is 20m away and the second is 30m away from the intersection point

A ladder of length 10m rests on a vertical wall. If the bottom of the ladder starts to slide away from the wall at a speed of 2m/s when the ladder becomes 8m away from the wall, find

- 1) The speed of the top of the ladder
- 2) The rate of change in the angle measure between the ground and the ladder

A cylinder of ice is melting as its height is twice its radius. If its height changes at a rate of 2 cm/s, find the rate of change in its volume when the height becomes 10cm

An ice cube melts as the length of its edge decreases at a rate of 2 cm/s Find the rate of change in its volume when its side length becomes 10 cm

A water tank in the form of a circular cone with its top down, its height is twice its radius, and water is poured into it at a rate of 2 cm³/s. The velocity of water descending into the tank when its height becomes 10 cm equals 0.02 cm/s

A spherical balloon into which air is pumped at a rate of $2 \text{ cm}^3/\text{s}$ Find the rate of change of its outer surface area when its radius becomes 10 cm

The point (x,y) moves in the graphic plane on the curve of the function $y=\sqrt{x}$ and x was increasing at a rate of $2 \text{ cm}/\text{s}$ Find the rate of change in the distance of the moving point from the point $(4,0)$ when $x=10$

A lamppost with a length of 6 m a man is moving towards the pole at a speed of $2 \text{ m}/\text{s}$. Find the rate of change in the length of the man's shadow

Rates of Change in Economics and Sciences

Solve economical and scientific problems on extrema •

Rates of Change in Economics and Sciences

Suppose that $C(x) = 0.02x^2 + 2x + 4000$ is the **total cost** (in AED) for a company to produce x units of a certain product. Compute the **marginal cost at $x = 100$** and compare this to the **actual cost of producing the 100th unit**.

If the cost of manufacturing x items is $C(x) = x^3 + 20x^2 + 90x + 15$ (in AED), find the **marginal cost function** and compare the **marginal cost at $x = 50$** with the **actual cost of manufacturing the 50th item**.

The Average Cost

Average Cost per unit of production is equal to the **total cost** of production divided by **number of units** produced.

$$\text{Average Cost} = \frac{\text{Total Cost}}{\text{Number of units (items)}}$$

$$\text{Average Cost} \quad \bar{C}(x) = \frac{C(x)}{x}$$

Companies want to know **level of production that minimize the average cost**.

This can be done by differentiating the average cost function and finding critical numbers (number of units) at which we have minimum value.

Suppose that $C(x) = 0.02x^2 + 2x + 4000$ is the **total cost** (in dollars) for a company to produce x **units** of a certain product. Find the **production level** x that **minimize the average cost**.

Find the production level that will minimize the average cost $C(x) = 10e^{0.02x}$

Elasticity of Demand

In general, a higher price will decrease the demand

This is not always the case

Products  **Elastic** Increase in price will decrease demand and decrease the revenue. Ex: Luxuries
Inelastic Increase in price will not affect demand, it will increase revenue. Ex: Oil

Elasticity of Demand

The response(change) of the demand due to small changes in price

Given: Demand: $x = f(p)$ Price: p Elasticity of Demand at price p : $E = \frac{\text{Relative Change in Demand}}{\text{Relative Change in Price}} = \lim_{\Delta p \rightarrow 0} \frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}}$ Where $\Delta p = (p + h) - p = h$

$$E = \lim_{h \rightarrow 0} \frac{\frac{f(p+h) - f(p)}{f(p)}}{\frac{h}{p}} = \frac{p}{f(p)} \lim_{h \rightarrow 0} \frac{f(p+h) - f(p)}{h}$$

Elasticity of Demand $E = \frac{p}{f(p)} f'(p)$ Prices for which $E < -1$, The demand is elastic
An increase in price will decrease the revenue

Revenue is equal to the number of units sold times the price per unit.

$$\text{Revenue} = R(p) = pf(p)$$

Suppose that $f(p) = 400(20 - p)$ is the demand for an item at price p (in dirhams) with $p < 20$

- Find the elasticity of demand.
- Find the range of prices for which $E < -1$. Compare this to the range of prices for which revenue is a decreasing function of p .

Suppose that $f(p) = 400(20 - p)$ is the demand for an item at price p (in dirhams) with $p < 20$

- Find the elasticity of demand.
- Find the range of prices for which $E < -1$. Compare this to the range of prices for which revenue is a decreasing function of p .

Suppose that

$f(p) = 200(30 - p)$ is the demand

- Find the **elasticity of demand**.
- Find the **range of prices** for which the **demand is elastic** ($E < -1$)

Rate of Chemical Reactions

In Chemistry we study **Rates of Chemical Reactions**.

This gives chemists information about chemical bonds and quantities of reactants and products.

In the chemical reaction:



Concentration of product C is: $[C(t)]$ Moles per liter

Average Reaction rate: $\frac{[C](t_2) - [C](t_1)}{t_2 - t_1}$ (Average Change in Concentration of product C in the interval $[t_1, t_2]$)

Instantaneous Reaction rate: $\lim_{t \rightarrow t_1} \frac{[C](t_2) - [C](t_1)}{t_2 - t_1} = \frac{d[C]}{dt}(t_1)$ (The derivative with respect to time)

In an autocatalytic chemical reaction, the reactants and the product are the same. The reaction continues until some saturation level is reached. From experimental evidence, chemists know that the **rate of reaction is jointly proportional to the amount of the product present and the difference between the saturation level and the amount of the product**. If **initial concentration** of the chemical is **0** and the **saturation level is 1** (100%). The **concentration $x(t)$** of the chemical satisfies the equation :

$$x'(t) = rx(t)[1 - x(t)], \text{ where } r > 0 \text{ is a constant.}$$

Find the concentration of chemical for which $x'(t)$ is **maximum**.

Titration

Acid: Has $pH < 7$

Base: Has $pH > 7$

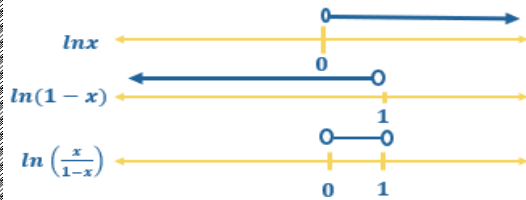
Strong Base + Weak Acid

Titration of a weak acid and a strong base

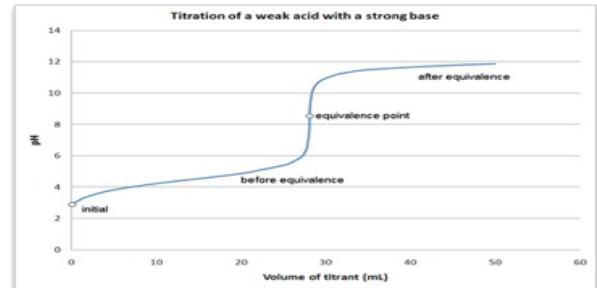
Adding the strong base gradually to the weak acid while monitoring the pH till you reach to an equivalence point which can be used to calculate the concentration of the base

pH is given by: $p(x) = c + \ln\left(\frac{x}{1-x}\right)$ c : Constant

$$p(x) = c + \ln x - \ln(1-x)$$



Domain: $0 < x < 1$ x : Base added = Acid converted



Find the value of x at which the **rate of change of pH** is the **smallest**. Identify the corresponding **point on the titration curve**.

Find the value of x at which the **rate of change of pH** is the **smallest**. Identify the corresponding **point on the titration curve**

In the titration of a weak acid and strong base, the pH is given by $c + \ln\left(\frac{x}{1-x}\right)$ where f is the fraction ($0 < x < 1$) of converted acid. What happens to the rate of change of pH as x approaches 1?

Mass Density

Mass Density is a representation of the amount of **mass** (or the number of particles) of a substance, material or object in relation to the **space it occupies**.

$$\text{Mass Density} = \frac{\text{mass}}{\text{volume}}$$

For **homogeneous materials**
(mass in any portion of the material of a given volume is the same)
The mass density is constant

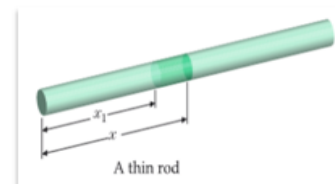
Mass Density for Nonhomogeneous materials

Mass of the first x meter of a thin rod: $f(x)$ Measured in (kilograms)

Total Mass between x and x_1 : $[f(x) - f(x_1)]$

Average Linear Density:
(mass per unit length) $\frac{[f(x) - f(x_1)]}{x - x_1}$

Linear Density:
(Derivative) $\rho(x_1) = \lim_{x \rightarrow x_1} \frac{[f(x) - f(x_1)]}{x - x_1} = f'(x_1)$



Suppose that the **mass** of the first x meter of a thin rod is given by $f(x) = \sqrt{2x}$. Compute the **linear density** at $x = 2$ and $x = 8$ and compare the densities at the two points.

The **mass** of the first x **meters** of a thin rod is given by the function $m(x)$ on the indicated interval. Find the **linear mass density function** for the rod. Briefly **describe the composition** of the rod.

Electrical Current in a Wire

Electric current in a wire, where the charge carriers are electrons, is a measure of the quantity of charge passing any point of the wire per unit of time.

Electric Charge in a wire at time t : $Q(t)$ Measured in (coulombs)

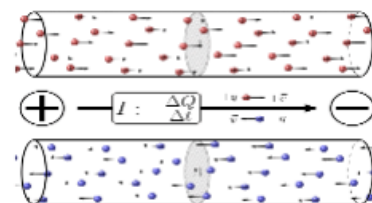
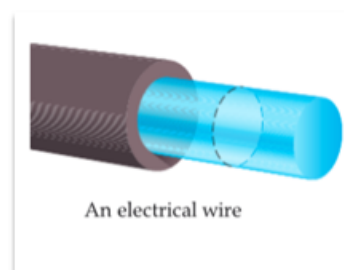
Net Charge passing the cross section in the wire on the interval $[t_1, t_2]$: $Q(t_2) - Q(t_1)$

Average Current on the interval $[t_1, t_2]$: $\frac{Q(t_2) - Q(t_1)}{t_2 - t_1}$
(Charge per Unit time)

Instantaneous Current: $\lim_{t \rightarrow t_1} \frac{Q(t_2) - Q(t_1)}{t_2 - t_1} = Q'(t_1)$
(Derivative of Charge with respect to time)

Instantaneous Current: $I(t) = Q'(t)$

Measured in (amps=coulombs per second)



In the given electric circuit **14-ohm resistor** , a **2-henry inductor**, a **0.05-farad capacitor** and a battery supplying **232 volts of AC** current modeled by the oscillating function **$(323\sin 2t)$** , where t is measured in seconds. Find the **current** in the circuit at any time t .

Suppose that the **charge** in an electrical circuit is $Q(t) = e^{-2t}(\cos 3t - 2\sin 3t)$ coulombs. Find the **current**.

Rate of Population Growth

Logistic Growth is a basic model for population growth.

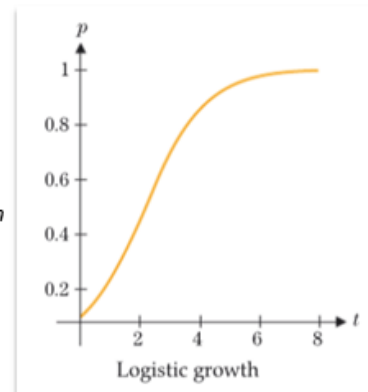
Populations modeled by logistic growth are **constrained** by limitations

Ex: Growth of a colony of bacteria

Population: $p(t)$ is given as a fraction of maximum sustainable population

Rate of Change of Population : $p'(t) = rp(t)[1 - p(t)]$ r : Constant

Rate of change of population is proportional to existing population $p(t)$ and the difference between the maximum population and the existing population $(1 - p(t))$.



$$r = 1, \quad p(0) = 1$$

Suppose that a **population** grows according to the equation $p'(t) = 2p(t)[1 - p(t)]$ (**logistic growth** with $r = 2$). Find the **population** for which the **growth rate** is a maximum. Interpret this point graphically.

Suppose that a **population grows** according to the equation $p'(t) = 2p(t)[1 - p(t)]$ (logistic growth with $r = 2$). Find the **population** for which the **growth rate is a maximum**. Interpret this point graphically.

Antiderivatives

- Find the antiderivative of a given function.
- Understand the notion of indefinite integral as finding an antiderivative.
- Compute straightforward indefinite integrals.

Definition

F is an antiderivatives for $f(x)$ if $F'(x)=f(x)$

Suppose that F and G are both antiderivatives of f on an interval I . Then.

$$G(x) = F(x) + c,$$

For some constant c .

Prove that $F(x)=x^2+\sin x \cos x+5$ is an antiderivatives for $f(x)=2x+\cos 2x$

Prove that $F(x)=e^{2x} \ln x$ is an antiderivatives for $f(x)=e^{2x}(\frac{1}{x} + \ln x^2)$

If $F(x)=(x^2+3)^2$ and $G(x)=x^4+6x^2+15$ are antiderivatives of the function $f(x)$, find the value of the constant for which $F(x)$ differs from $G(x)$

Let F be any antiderivatives of f on an interval I . The **Indefinite Integral** of $f(x)$ (with respect to x) on I , is defined by $\int f(x)dx = F(x) + c$,

Where c is an *arbitrary constant* (the **constant of integration**).

	function	derivative	integration
1	$Y = c$	$\frac{dy}{dx} = 0$	$\int 0 dx = c$
2	$Y = ax$	$\frac{dy}{dx} = ax$	$\int adx = ax + c$
3	$Y = x^n$	$\frac{dy}{dx} = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
5	$Y = \sin x$	$\frac{dy}{dx} = -\cos x$	$\int \cos x dx = \sin x + c$
6	$Y = \cos x$	$\frac{dy}{dx} = -\sin x$	$\int \sin x dx = -\cos x + c$
7	$Y = \tan x$	$\frac{dy}{dx} = \sec^2 x$	$\int \sec^2 x dx = \tan x + c$
8	$Y = \cot x$	$\frac{dy}{dx} = -\csc^2 x$	$\int \csc^2 x dx = -\cot x + c$
9	$Y = \sec x$	$\frac{dy}{dx} = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + c$
11	$Y = \csc x$	$\frac{dy}{dx} = -\csc x \cot x$	$\int \csc x \cot x dx = -\csc x + c$
12	$Y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$	$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$
13	$Y = be^{bx}$	$\frac{dy}{dx} = be^{bx}$	$\int e^{bx} dx = \frac{e^{bx}}{b} + c$
14	$Y = \tan^{-1} x$	$\frac{dy}{dx} = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$
15	$Y = \sin^{-1} x$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
16	$Y = \sec^{-1} x$	$\frac{dy}{dx} = \frac{1}{ x \sqrt{x^2-1}}$	$\int \frac{1}{ x \sqrt{x^2-1}} dx = \sec^{-1} x + c$

THEOREM 1.2 (Power Rule)

$$\int F'(x) dx = F(x) + c$$

$$\int F'(x) dx = F(x) + c$$

$$1) \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$2) \int b dx = bx + c$$

$$1) \int (2x^2 + 5x - 3) dx$$

$$2) \int 5x^3(x^2 + 2x - 3) dx$$

$$3) \int (4x - 2)(3x^2 - 5) dx =$$

$$4) \int \frac{3x^2 + 5x^3 + 7}{x^2} dx =$$

$$5) \int \left(\sqrt{x} + \frac{5}{\sqrt[3]{x}} + \frac{x^2 - 2x}{\sqrt[4]{x^3}} \right) dx$$

$$1) \int \sin x dx = -\cos x + c$$

$$2) \int \cos x dx = \sin x + c$$

$$3) \int \sec^2 x dx = \tan x + c$$

$$4) \int \csc^2 x dx = -\cot x + c$$

$$5) \int \sec x \tan x dx = \sec x + c$$

$$6) \int \csc x \cot x dx = -\csc x + c$$

$$1) \int (2\sin x + 3\cos x + 5) dx$$

تذكر

$$1) \tan x = \frac{\sin x}{\cos x}$$

$$2) \cot x = \frac{\cos x}{\sin x}$$

$$3) \sec x = \frac{1}{\cos x}$$

$$4) \csc x = \frac{1}{\sin x}$$

$$5) \tan^2 x = \sec^2 x - 1$$

$$6) \cot^2 x = \csc^2 x - 1$$

$$7) \sin^2 x = \frac{1}{2} (\cos 2x - 1)$$

$$8) \cos^2 x = \frac{1}{2} (\cos 2x + 1)$$

$$9) \tan^2 x = \csc^2 x - 1$$

$$10) \sin^2 x + \cos^2 x = 1$$

$$11) \sin 2x = 2 \sin x \cos x$$

$$12) \cos 2x = \cos^2 x - \sin^2 x$$

$$2) \int (3\sec^2 x + 5\sec x \tan x + 4\sin x) dx$$

$$3) \int (3\csc^2 x + 5\csc x \cot x + 4\cos x) dx$$

$$4) \int \cot x (\cot x + 3\sin x + 5\csc x + 1) dx$$

$$5) \int \frac{\sin^3 x + 1}{\sin^2 x} dx$$

$$6) \int \frac{\sec^3 x + 5}{\sec x} dx$$

$$7) \int \frac{\tan^3 x + \csc x}{\tan x} dx$$

$$8) \int \frac{\cos x + 1}{\sin^2 x} dx$$

$$1) \int \sin x dx = -\cos x + c$$

$$2) \int \cos x dx = \sin x + c$$

$$3) \int \sec^2 x dx = \tan x + c$$

$$4) \int \csc^2 x dx = -\cot x + c$$

$$5) \int \sec x \tan x dx = \sec x + c$$

$$6) \int \csc x \cot x dx = -\csc x + c$$

تذكر

$$1) \tan x = \frac{\sin x}{\cos x}$$

$$2) \cot x = \frac{\cos x}{\sin x}$$

$$3) \sec x = \frac{1}{\cos x}$$

$$4) \csc x = \frac{1}{\sin x}$$

$$5) \tan^2 x = \sec^2 x - 1$$

$$6) \cot^2 x = \csc^2 x - 1$$

$$7) \sin^2 x = \frac{1}{2} (\cos 2x - 1)$$

$$8) \cos^2 x = \frac{1}{2} (\cos 2x + 1) - 1$$

$$9) \sin^2 x + \cos^2 x = 1$$

$$11) \sin 2x = 2 \sin x \cos x$$

$$12) \cos 2x = \cos^2 x - \sin^2 x$$

$$F'(x) dx = F(x) + c$$

$$1) \int \left(\frac{2x+1}{x^2+x+3} + \frac{1}{x^2+1} + e^{3x} \right) dx$$

$$2) \int \left(\frac{\cos x + 1}{\sin x + x - 2} + \frac{5}{5x+1} + e^{-2x} \right) dx$$

$$3) \int \left(\frac{\sec^2 x}{1 + \tan x} + \frac{1}{\sqrt{1-x^2}} + e^{5x} \right) dx$$

$$4) \int (\cot x + 1)^2 dx$$

$$5) \int \frac{e^{5x} + \sqrt{e^{4x}} + 1}{e^{3x}} dx$$

$$1) \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$$

$$2) \int e^{bx} dx = \frac{e^{bx}}{b} + c$$

$$3) \int \frac{1}{1+x^2} dx = \tan^{-1} x + c$$

$$4) \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$$

$$5) \int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + c$$

Find the function $f(x)$ satisfying a given conditions:

If $f'(x)=3x^2+2x-5$ and $f(1)=10$, find $f(2)$

If the slope of the tangent to the function $f(x)$ is $\frac{5}{x^2+1} + \frac{2x}{x^2+1} + 2e^{2x}$ and $f(0)=10$, find $f(x)$

A particle moves in a straight line where its velocity at instant t is given by the relationship $v(t)=\sin t + \cos t + 5$ and if $s(0)=10$ find $s(t)$

A particle is moving in a straight line where its acceleration at moment t is given by the relationship $a(t)=2t+6$ and $v(0)=10$ Find $v(2)$

1) $\int \cot x \sin x \, dx =$

a) $\sin x + c$ b) $\cos x + c$ c) $\sec x + c$ d) $\csc x + c$

2) $\int \frac{\sin x}{\cos^2 x} \, dx$

a) $\sin x + c$ b) $\cos x + c$ c) $\sec x + c$ d) $\csc x + c$

3) $\int \frac{x+3}{x^2+6x+7} \, dx$

a) $\frac{1}{2} \ln|x^2 + 6x + 7| + c$ b) $\frac{1}{2} \ln|x + 3| + c$

c) $\frac{1}{2} \ln \left| \frac{x^3}{3} + \frac{6x^2}{2} + 7x \right| + c$ d) $\frac{x^3}{3} + \frac{6x^2}{2} + 7x + c$

4) $\int (e^{2x} + \frac{5}{x+1}) \, dx$

a) $2e^{2x} + \frac{1}{5} \ln|x + 1| + c$

b) $\frac{e^{2x}}{2} + 5 \ln|x + 1| + c$

c) $2e^{2x} + \frac{-5}{(x + 1)^2} + c$

d) $e^{x^2} + \frac{5x}{\frac{x^2}{2} + x} + c$

5) $\int (\frac{4}{x^2+1} + \cot x) \, dx$

a) $4 \tan x - \csc^2 x + c$

b) $\frac{1}{4} \tan^{-1} x + \ln|\cos x| + c$

c) $\ln|\cos x(x^2 + 1)| + c$

d) $4 \tan^{-1} x + \ln|\cos x| + c$

Sums and Sigma Notation

Use the sigma notation to compute basic summation.

Sigma Notation

Sigma Notation is a concise and convenient way to represent long sums.

For example,

$$1 + 2 + 3 + 4 + 5 \quad (\text{Sum of the first five whole numbers})$$

$$1 + 4 + 9 + 16 + 25 + 36 \quad (\text{Sum of the squares of first six whole numbers})$$



I'm just a fancy way of saying, "Add everything up!"

There is an obvious pattern for numbers involved

If we take a **sequence** of numbers $a_1, a_2, a_3, \dots, a_n$

then we can write the sum of these numbers (**series**) as $a_1 + a_2 + a_3 + \dots + a_n$

A shorter way of writing this:

Sigma Notation

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \dots + a_n$$

Upper limit (Last value of i) $\rightarrow n$

Greek letter "Sigma" $\rightarrow \sum$

Summation

Index of Summation $\rightarrow i = 1$

General term of the sequence $\rightarrow a_i$

Lower limit (First value of i) $\rightarrow 1$

Find

$$1) \sum_{i=1}^5 (2i + 3) = \dots$$

$$2) \sum_{j=1}^3 \frac{j^2}{j+1} = \dots$$

$$3) \sum_{i=1}^4 3i^3 = \dots$$

Rules of summation notation

$$1) \sum_{i=1}^n c = cn$$

$$2) \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$3) \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$4) \sum_{i=1}^n af(i) = a \sum_{i=1}^n f(i)$$

$$5) \sum_{i=1}^n (f(i) \pm g(i)) = \sum_{i=1}^n f(i) \pm \sum_{i=1}^n g(i)$$

Find

$$1) \sum_{i=1}^{10} (2i + 3)$$

$$2) \sum_{i=1}^{10} (3i^2 + 5)$$

$$3) \sum_{i=4}^{10} (i^2)$$

Write in summation notation :

a) $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{10}$

b) $3^3 + 4^3 + 5^3 + \cdots + 45^3$

c) $2(1)^2 + 2(2)^2 + 2(3)^2 + \cdots + 2(14)^2$

d) $\sqrt{2-1} + \sqrt{3-1} + \sqrt{4-1} + \cdots + \sqrt{15-1}$

Area and Definite of Integral

Estimate the area under a curve on a given interval using rectangles.

Compute the area under a curve using summations and limits.

Understand the notion of a definite integral.

Compute a definite integral using Riemann sums.

Find the area under a curve on a given interval using Riemann sums.

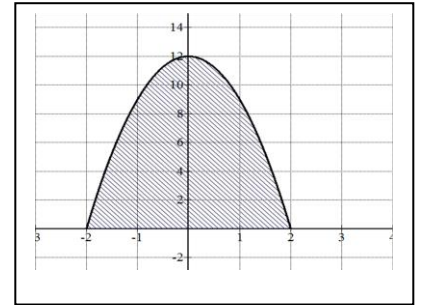
Learn the properties of definite integrals.

Apply the Integral Mean Value Theorem.

Approximate the area under the curve

$f(x) = 12 - 3x^2$ on the interval $[-2, 2]$,

Using 4 rectangles and Left Endpoint



Approximate the area under the curve $f(x) = 8 - x$ on the interval $[0, 8]$, Using 4 rectangles and right Endpoint

Approximate the area under the curve $f(x) = 2x + 1$ on the interval $[0, 10]$, Using 5 rectangles and midpoint

For a function f defined on interval $[a, b]$, if f is continuous on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, the area A under the curve $y = f(x)$ on $[a, b]$ is given by

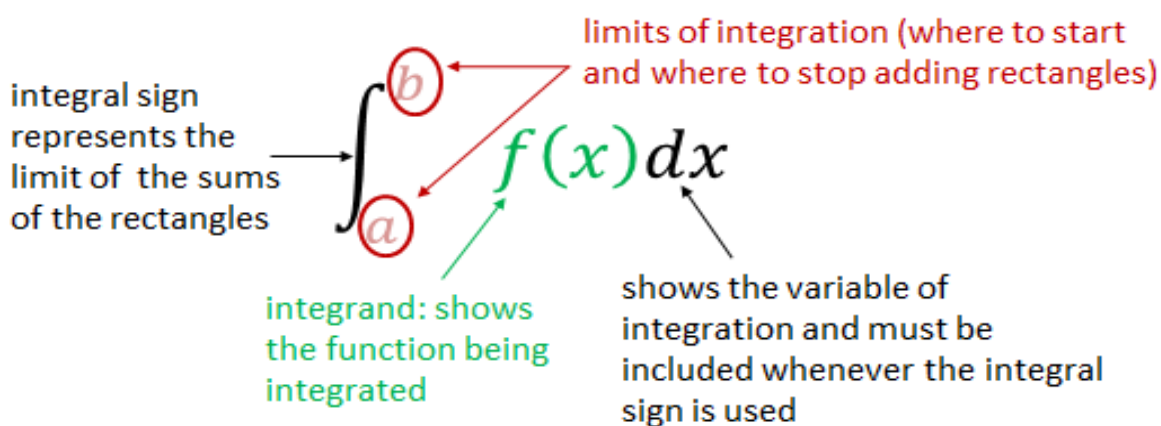
$$A = \lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

Find the area under the curve $y = f(x) = 4 - x^2$ on the interval $[-1, 1]$.

For any function f defined on $[a, b]$, the **definite integral** of f from a to b is

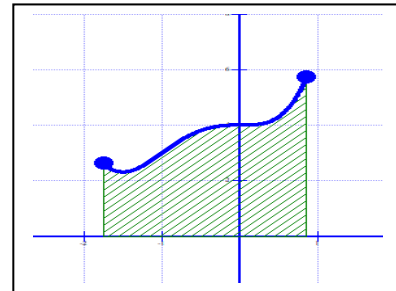
$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x,$$

Whenever the limit exists and is the same for every choice of evaluation points, $c_1, c_2, \dots, c_n$. When the limit exists, we say that f is **integrable** on $[a, b]$.

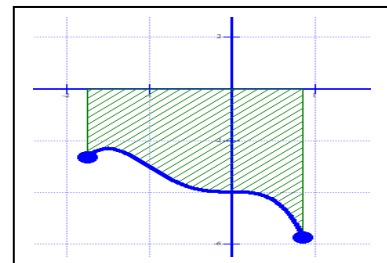


Remarks

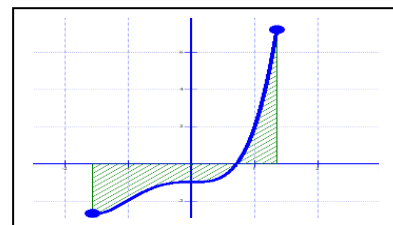
1) If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x) = A$



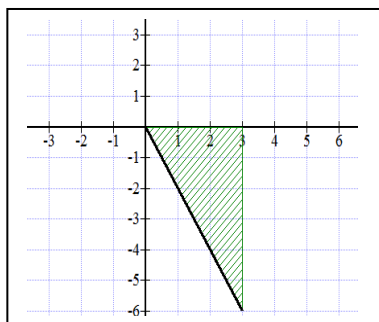
2) If $f(x) \leq 0$ on $[a, b]$ then $\int_a^b f(x) = -A$



3) If $f(x) \leq 0$ on $[a, c]$, $f(x) \geq 0$ on $[c, b]$ then $\int_a^b f(x) = A_2 - A_1$

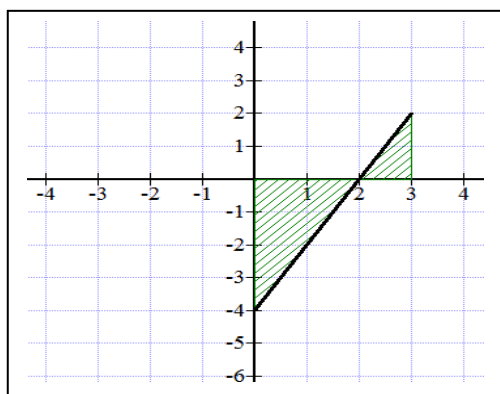


$\int_0^3 -4x \, dx$ The figure represents the function $f(x) = -4x$ find $\int_0^3 -4x \, dx$

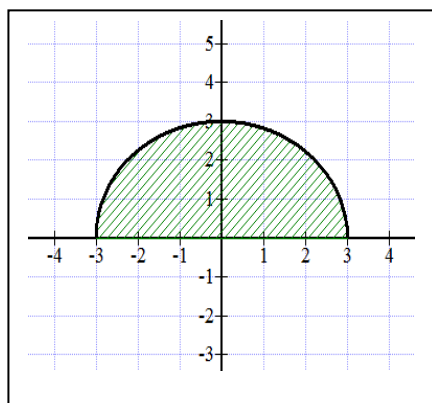


The figure represents the function $f(x) = 2x - 4$

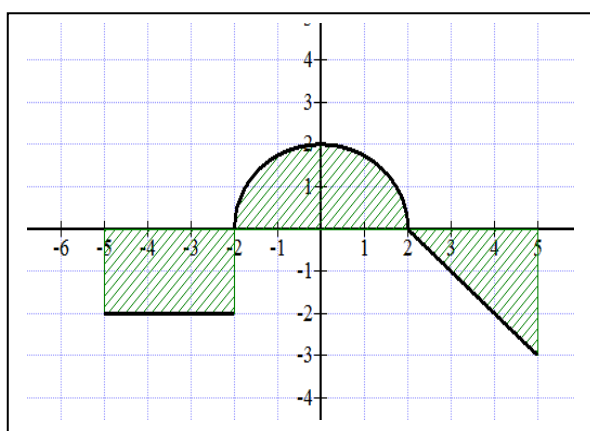
evaluate $\int_0^3 (2x - 4) \, dx$



The figure represents the function $f(x) = \sqrt{9 - x^2}$ find $\int_{-3}^3 \sqrt{9 - x^2} dx$



The figure represents the function $f(x)$ Evaluate $\int_{-5}^5 f(x) dx$



$$\int_a^b f(x)dx = \lim_{x \rightarrow \infty} \sum_1^n f(x_k) \Delta x$$

Use the limit (Riemann) to find

1) $\int_1^3 (2x + 3)dx$

2) $\int_0^2 (3x^2)dx$

properties of the definite integral

$$1) \int_a^a f(x)dx = 0$$

$$2) \int_a^b f(x)dx = - \int_b^a f(x)dx$$

$$3) \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$4) \int_a^b cf(x)dx = c \int_a^b f(x)dx$$

$$5) \int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$4) f(x) > 0 \rightarrow \int_a^b f(x)dx > 0$$

$$5) f(x) > g(x) \rightarrow \int_a^b f(x)dx > \int_a^b g(x)dx$$

Q1

$$\int_1^2 f(x)dx = 2, \int_2^4 f(x)dx = 3, \int_6^4 f(x)dx = -5, \int_4^2 g(x)dx = 6 \text{ find}$$

$$1) \int_2^1 4f(x)dx = \dots \dots \dots$$

$$2) \int_1^4 f(x)dx = \dots \dots \dots$$

$$3) \int_6^1 f(x)dx = \dots \dots \dots$$

$$4) \int_2^4 (4f(x) + g(x))dx = \dots \dots \dots$$

$$5) \int_2^4 4f(x) - 5)dx = \dots \dots \dots$$

$$Q2) \int_1^2 2f(x)dx = 4, \int_2^4 f(x)dx = 3, \int_4^7 \frac{1}{2}f(x)dx = -5 \text{ وجد}$$

$$1) \int_2^1 6f(x)dx = \dots \dots \dots$$

$$2) \int_1^4 f(x)dx = \dots \dots \dots$$

$$3) \int_7^4 6f(x) + 4)dx = \dots \dots \dots$$

Average Value of a Function (Integral Mean Value Theorem)
**Find the value of c that satisfy the conclusion of
the Integral Mean Value Theorem**

Average Value of a Function (Integral)

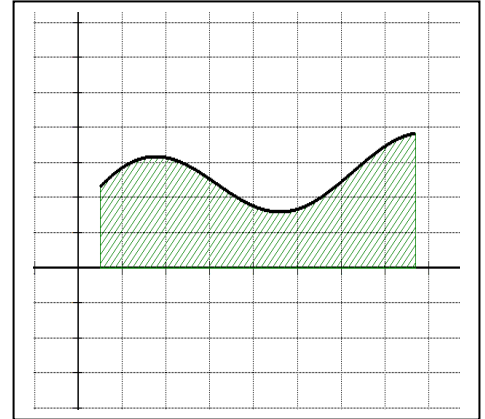
$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx$$

Remark

$$(b-a) f_{ave} = \int_a^b f(x) dx$$

Theorem:

If Average Value of a Function of $f(x)$ over the interval $[a, b]$ is f_{ave} then there exist c on the interval $[a, b]$ such that $f(c) = f_{ave}$



Q1) Find the Average Value of a Function $f(x) = 3x^2 + 1$ on the interval $[0, 2]$ and then Find the value of c that satisfy the conclusion of the Integral Mean Value Theorem

Q2) A factory produces 100 devices per day, so if the profit of the factory is given by the formula $p(x) = 200x + 5000$ dirhams

Find the average value of profit and then find the value at which the average value is achieved

If the average value of the function $f(x)$ over the interval $[0,2]$ is 3 and the average value of the function $f(x)$ over the interval

$[2,6]$ is 1 Find

$$\int_0^6 f(x) - 5) dx$$

The Fundamental Theorem of Calculus

- Learn the Fundamental Theorem of Calculus (Part I) and use it to compute various definite integrals.
- Learn the Fundamental Theorem of Calculus (Part II) and use it to compute derivatives of functions defined as definite integrals.
- Write the equation of a tangent line at a given point to a function defined as definite integral.

*(The Fundamental Theorem of Calculus.)***(The Fundamental Theorem of Calculus. Part I)**

If f is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Compute

1) $\int_1^2 (3x^2 + 2x - 5)dx$

2) $\int_0^1 \left(\frac{2x}{x^2+1} + e^{2x} + \sqrt{x} \right) dx$

3) $\int_0^1 \left(\frac{5}{x^2+1} + \sin \pi x \right) dx$

4) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{12}{\pi} - \sec^2 x + \cos x \right) dx$

(The Fundamental Theorem of Calculus. Part II)

If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$, on $[a, b]$

$$\text{If } y = \int_{h(x)}^{g(x)} f(t)dt \quad \text{then} \quad \frac{dy}{dx} = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$$

$$\text{find } \frac{dy}{dx}$$

$$1) \quad y = \int_{x^2}^{\sin x} (5x - 2)dx$$

$$2) \quad y = \int_3^{\ln x} (5 \tan x - e^{3x})dx$$

$$3) \quad y = \int_{\tan x}^7 (3x^2)dx$$

Q2) $f(x) = \int_x^{x^2} (2t - 4) dx$ find

1) $f(2) = \dots$

2) $f'(2) = \dots$

Q3) Find the equation of the tangent line to the function $f(x) = \int_0^{x^2} 4x dx$ at $x=1$

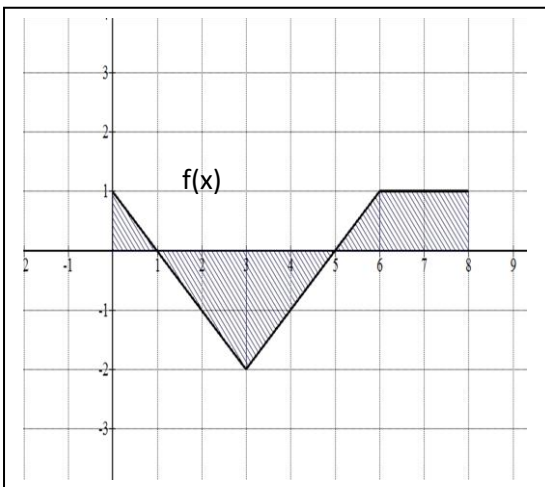
Q4

$H(x) = \int_0^x f(x) dx$ find

1) $H(8) = \dots\dots\dots$

2) $H'(3) = \dots\dots\dots$

3) $H''(5) = \dots\dots\dots$



Integration by Substitution

Compute integrals using substitution.

Evaluate

1) $\int 2x(x^2 + 1)^{10} dx$

2) $\int 3x^2 \sqrt{x^3 + 5} dx$

3) $\int \frac{(\sqrt{x}+3)^5}{\sqrt{x}} dx$

4) $\int \frac{1+\ln x}{x} dx$

$$5) \int 2x \sin x^2 dx$$

$$\int e^x \sec e^x \tan e^x dx \quad 6)$$

$$7) \int \frac{\tan^{-1} x}{1+x^2} dx$$

$$8) \int \cos x \sin^5 x dx$$

$$9) \int 2x (5 + x)^5 dx$$

$$10) \int 2x^3 (x^2 + 1)^5 dx$$

$$11) \int_0^1 \frac{3x^2}{1+x^6} dx$$

$$12) \int_0^1 2x(x^2 + 1)^4 dx$$

13) $f(0)=2, f(2)=7, f(1)=4$ find $\int_0^1 f'(x) x^2 + 1) dx \dots$

14) $f(0)=1, f(3)=2$ find $\int_0^3 8f'(x)(f(x))^3 dx$

