

Chapter (11)
First chapter in T2

Mathematics

Trigonometric Identities and Equations

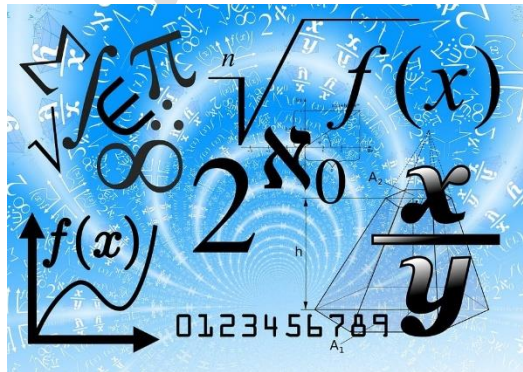
المتطابقات والمعادلات المثلثية
الوحدة الأولى من الفصل الثاني

11 Advanced

T2

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<https://t.me/+XPfMBpsQpVxjNzNk> <http://www.youtube.com/@imaths2022>

Chapter / Module Name	Section/Lesson Number	Section / Lesson Name
Trigonometric Identities and Equations	C11L1	Trigonometric Identities
Trigonometric Identities and Equations	C11L2	Verifying Trigonometric Identities
Trigonometric Identities and Equations	C11L3	Sum and Difference of Angles Identities
Trigonometric Identities and Equations	C11L4	Double Angle and Half Angle Identities
Trigonometric Identities and Equations	C11L5	Solving Trigonometric Equations

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Lesson11-1

Trigonometric Identities

Learning outcomes

- 1- Use trigonometric identities to find trigonometric values.
- 2- Use trigonometric identities to simplify expressions.

KeyConcept Basic Trigonometric Identities

Quotient Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta},$$

$$\cos \theta \neq 0$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta},$$

$$\sin \theta \neq 0$$

Reciprocal Identities

$$\sin \theta = \frac{1}{\csc \theta}, \csc \theta \neq 0$$

$$\cos \theta = \frac{1}{\sec \theta}, \sec \theta \neq 0$$

$$\tan \theta = \frac{1}{\cot \theta}, \cot \theta \neq 0$$

$$\csc \theta = \frac{1}{\sin \theta}, \sin \theta \neq 0$$

$$\sec \theta = \frac{1}{\cos \theta}, \cos \theta \neq 0$$

$$\cot \theta = \frac{1}{\tan \theta}, \tan \theta \neq 0$$

Pythagorean Identities

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

Cofunction Identities

$$\sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta$$

$$\cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta$$

$$\tan \left(\frac{\pi}{2} - \theta \right) = \cot \theta$$

Negative Angle Identities

$$\sin (-\theta) = -\sin \theta$$

$$\cos (-\theta) = \cos \theta$$

$$\tan (-\theta) = -\tan \theta$$

Example 1 Use Trigonometric Identities

a. Find the exact value of $\cos \theta$ if $\sin \theta = \frac{1}{4}$ and $90^\circ < \theta < 180^\circ$.

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b. Find the exact value of $\csc \theta$ if $\cot \theta = -\frac{3}{5}$ and $270^\circ < \theta < 360^\circ$.

Guided Practice

1A. Find $\sin \theta$ if $\cos \theta = \frac{1}{3}$ and $270^\circ < \theta < 360^\circ$.

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1B. Find $\sec \theta$ if $\sin \theta = -\frac{2}{7}$ and $180^\circ < \theta < 270^\circ$.

Example 2 Simplify an Expression

Simplify $\frac{\sin \theta \csc \theta}{\cot \theta}$.

Guided Practice

Simplify each expression.

2A. $\frac{\tan^2 \theta \csc^2 \theta - 1}{\sec^2 \theta}$

2B. $\frac{\sec \theta}{\sin \theta}(1 - \cos^2 \theta)$

Real-World Example 3 Simplify and Use an Expression

LIGHTING Refer to the beginning of the lesson.

- a. Solve the formula in terms of E .

$$\sec \theta = \frac{I}{ER^2}$$

b. Is the equation in part a equivalent to $R^2 = \frac{I \tan \theta \cos \theta}{E}$? Explain.

Guided Practice

3. Rewrite $\cot^2 \theta - \tan^2 \theta$ in terms of $\sin \theta$.

Check Your Understanding

Find the exact value of each expression if $0^\circ < \theta < 90^\circ$.

1. If $\cot \theta = 2$, find $\tan \theta$.

2. If $\sin \theta = \frac{4}{5}$, find $\cos \theta$.

3. If $\cos \theta = \frac{2}{3}$, find $\sin \theta$.

4. If $\cos \theta = \frac{2}{3}$, find $\csc \theta$.

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Simplify each expression.

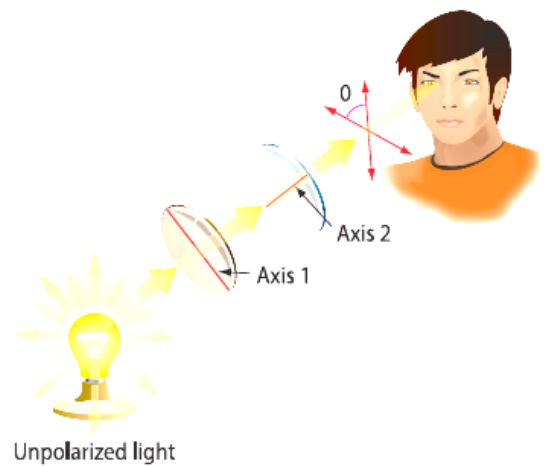
5. $\tan \theta \cos^2 \theta$

6. $\csc^2 \theta - \cot^2 \theta$

7. $\frac{\cos \theta \csc \theta}{\tan \theta}$

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8. **PERSEVERANCE** When unpolarized light passes through polarized sunglass lenses, the intensity of the light is cut in half. If the light then passes through another polarized lens with its axis at an angle of θ to the first, the intensity of the light is again diminished. The intensity of the emerging light can be found by using the formula $I = I_0 - \frac{I_0}{\csc^2 \theta'}$ where I_0 is the intensity of the light incoming to the second polarized lens, I is the intensity of the emerging light, and θ is the angle between the axes of polarization.



- Simplify the formula in terms of $\cos \theta$.
- Use the simplified formula to determine the intensity of light that passes through a second polarizing lens with axis at 30° to the original.

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Practice and Problem Solving

Find the exact value of each expression if $0^\circ < \theta < 90^\circ$.

9. If $\cos \theta = \frac{3}{5}$, find $\csc \theta$.

10. If $\sin \theta = \frac{1}{2}$, find $\tan \theta$.

11. If $\sin \theta = \frac{3}{5}$, find $\cos \theta$.

12. If $\tan \theta = 2$, find $\sec \theta$.

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Find the exact value of each expression if $180^\circ < \theta < 270^\circ$.

13. If $\cos \theta = -\frac{3}{5}$, find $\csc \theta$.

14. If $\sec \theta = -3$, find $\tan \theta$.

15. If $\cot \theta = \frac{1}{4}$, find $\csc \theta$.

16. If $\sin \theta = -\frac{1}{2}$, find $\cos \theta$.

دعوة

Find the exact value of each expression if $270^\circ < \theta < 360^\circ$.

17. If $\cos \theta = \frac{5}{13}$, find $\sin \theta$.

18. If $\tan \theta = -1$, find $\sec \theta$.

19. If $\sec \theta = \frac{5}{3}$, find $\cos \theta$.

20. If $\csc \theta = -\frac{5}{3}$, find $\cos \theta$.

Simplify each expression.

21. $\sec \theta \tan^2 \theta + \sec \theta$

22. $\cos \left(\frac{\pi}{2} - \theta \right) \cot \theta$

23. $\cot \theta \sec \theta$

24. $\sin \theta (1 + \cot^2 \theta)$

25. $\sin \left(\frac{\pi}{2} - \theta \right) \sec \theta$

26. $\frac{\cos (-\theta)}{\sin (-\theta)}$

Simplify each expression.

28. $\frac{1 - \sin^2 \theta}{\sin^2 \theta}$

29. $\tan \theta \csc \theta$

30. $\frac{1}{\sin^2 \theta} - \frac{\cos^2 \theta}{\sin^2 \theta}$

31. $2(\csc^2 \theta - \cot^2 \theta)$

32. $(1 + \sin \theta)(1 - \sin \theta)$

33. $2 - 2 \sin^2 \theta$

$$38. \frac{\tan\left(\frac{\pi}{2} - \theta\right)\sec\theta}{1 - \csc^2\theta}$$

$$39. \frac{\cos\left(\frac{\pi}{2} - \theta\right) - 1}{1 + \sin(-\theta)}$$

$$31. 2(\csc^2\theta - \cot^2\theta)$$

$$32. (1 + \sin\theta)(1 - \sin\theta)$$

$$33. 2 - 2\sin^2\theta$$

Lesson 11-2

Verifying Trigonometric Identities

Learning outcomes

- 1- Verify trigonometric identities by transforming one side of an equation into the form of the other side.
- 2- Verify trigonometric identities by transforming each side of the equation into the same form.

KeyConcept Verifying Identities by Transforming One Side

Step 1 Simplify one side of an equation until the two sides of the equation are the same. It is often easier to work with the more complicated side of the equation.

Step 2 Transform that expression into the form of the simpler side.

Example 1 Transform One Side of an Equation

Verify that $\frac{\sin^2 \theta}{1 - \cos \theta} = 1 + \cos \theta$ is an identity.

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Guided Practice

1. Verify that $\cot^2 \theta - \cos^2 \theta = \cot^2 \theta \cos^2 \theta$ is an identity.

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Standardized Test Example 2 Simplify an Expression

$$\frac{\cos \theta \csc \theta}{\tan \theta} =$$

A $\cot \theta$

B $\csc \theta$

C $\cot^2 \theta$

D $\csc^2 \theta$

Guided Practice

2. $\tan^2 \theta (\cot^2 \theta - \cos^2 \theta) =$

F $\cot^2 \theta$

G $\tan^2 \theta$

H $\cos^2 \theta$

J $\sin^2 \theta$

KeyConcept Suggestions for Verifying Identities

- Substitute one or more basic trigonometric identities to simplify the expression.
- Factor or multiply as necessary. You may have to multiply both the numerator and denominator by the same trigonometric expression.
- Write each side of the identity in terms of sine and cosine only. Then simplify each side as much as possible.
- The properties of equality do not apply to identities as with equations. Do not perform operations to the quantities on each side of an unverified identity.

Example 3 Verify by Transforming Each Side

Verify that $1 - \tan^4 \theta = 2 \sec^2 \theta - \sec^4 \theta$ is an identity.

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GuidedPractice

3. Verify that $\csc^2 \theta - \cot^2 \theta = \cot \theta \tan \theta$ is an identity.

Check Your Understanding

PRECISION Verify that each equation is an identity.

1. $\cot \theta + \tan \theta = \frac{\sec^2 \theta}{\tan \theta}$

2. $\cos^2 \theta = (1 + \sin \theta)(1 - \sin \theta)$

3. $\sin \theta = \frac{\sec \theta}{\tan \theta + \cot \theta}$

4. $\tan^2 \theta = \frac{1 - \cos^2 \theta}{\cos^2 \theta}$

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5. $\tan^2 \theta \csc^2 \theta = 1 + \tan^2 \theta$

6. $\tan^2 \theta = (\sec \theta + 1)(\sec \theta - 1)$

7. **MULTIPLE CHOICE** Which expression can be used to form an identity with $\frac{\tan^2 \theta + 1}{\tan^2 \theta}$?

A $\sin^2 \theta$

B $\cos^2 \theta$

C $\tan^2 \theta$

D $\csc^2 \theta$

Verify that each equation is an identity.

8. $\cos^2 \theta + \tan^2 \theta \cos^2 \theta = 1$

9. $\cot \theta (\cot \theta + \tan \theta) = \csc^2 \theta$

10. $1 + \sec^2 \theta \sin^2 \theta = \sec^2 \theta$

11. $\sin \theta \sec \theta \cot \theta = 1$

12. $\frac{1 - \cos \theta}{1 + \cos \theta} = (\csc \theta - \cot \theta)^2$

13. $\frac{1 - 2 \cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta - \cot \theta$

$$14. \tan \theta = \frac{\sec \theta}{\csc \theta}$$

$$15. \cos \theta = \sin \theta \cot \theta$$

$$16. (\sin \theta - 1)(\tan \theta + \sec \theta) = -\cos \theta$$

$$17. \cos \theta \cos (-\theta) - \sin \theta \sin (-\theta) = 1$$

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Verify that each equation is an identity.

$$19. \sec \theta - \tan \theta = \frac{1 - \sin \theta}{\cos \theta}$$

$$21. \sec \theta \csc \theta = \tan \theta + \cot \theta$$

$$23. (\sin \theta + \cos \theta)^2 = \frac{2 + \sec \theta \csc \theta}{\sec \theta \csc \theta}$$

$$25. \csc \theta - 1 = \frac{\cot^2 \theta}{\csc \theta + 1}$$

$$27. \sin \theta \cos \theta \tan \theta + \cos^2 \theta = 1$$

$$29. \csc^2 \theta = \cot^2 \theta + \sin \theta \csc \theta$$

$$31. \sin^2 \theta + \cos^2 \theta = \sec^2 \theta - \tan^2 \theta$$

$$20. \frac{1 + \tan \theta}{\sin \theta + \cos \theta} = \sec \theta$$

$$22. \sin \theta + \cos \theta = \frac{2 \sin^2 \theta - 1}{\sin \theta - \cos \theta}$$

$$24. \frac{\cos \theta}{1 - \sin \theta} = \frac{1 + \sin \theta}{\cos \theta}$$

$$26. \cos \theta \cot \theta = \csc \theta - \sin \theta$$

$$28. (\csc \theta - \cot \theta)^2 = \frac{1 - \cos \theta}{1 + \cos \theta}$$

$$30. \frac{\sec \theta - \csc \theta}{\csc \theta \sec \theta} = \sin \theta - \cos \theta$$

$$32. \sec \theta - \cos \theta = \tan \theta \sin \theta$$

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When simplified, would the expression be equal to 1 or -1 ?

35. $\cot(-\theta) \tan(-\theta)$

36. $\sin \theta \csc(-\theta)$

37. $\sin^2(-\theta) + \cos^2(-\theta)$

38. $\sec(-\theta) \cos(-\theta)$

39. $\sec^2(-\theta) - \tan^2(-\theta)$

40. $\cot(-\theta) \cot\left(\frac{\pi}{2} - \theta\right)$

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Simplify the expression to either a constant or a basic trigonometric function.

41. $\frac{\tan\left(\frac{\pi}{2} - \theta\right)\csc \theta}{\csc^2 \theta}$

42. $\frac{1 + \tan \theta}{1 + \cot \theta}$

43. $(\sec^2 \theta + \csc^2 \theta) - (\tan^2 \theta + \cot^2 \theta)$

44. $\frac{\sec^2 \theta - \tan^2 \theta}{\cos^2 x + \sin^2 x}$

45. $\tan \theta \cos \theta$

46. $\cot \theta \tan \theta$

47. $\sec \theta \sin\left(\frac{\pi}{2} - \theta\right)$

48. $\frac{1 + \tan^2 \theta}{\csc^2 \theta}$

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Lesson 11-3

Sum and Difference of Angles Identities

Learning outcomes

- 1- Find values of sine and cosine by using sum and difference identities.
- 2- Verify trigonometric identities by using sum and difference identities.

Key Concept Sum and Difference Identities

Sum Identities

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

Difference Identities

- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Example 1 Find Trigonometric Values

Find the exact value of each expression.

a. $\sin 105^\circ$

b. $\cos(-120^\circ)$

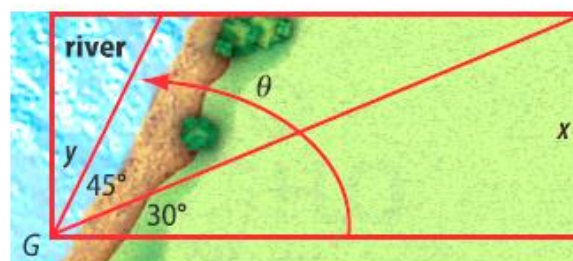
Guided Practice

1A. $\sin 15^\circ$

1B. $\cos (-15^\circ)$

Real-World Example 2 Sum and Difference of Angles Identities

A geologist measures the angle between one side of a rectangular lot and the line from her position to the opposite corner of the lot as 30° . She then measures the angle between that line and the line to the point on the property where a river crosses as 45° . She stands 100 meters from the opposite corner of the property. How far is she from the point at which the river crosses the property line?



Guided Practice

2. The harmonic motion of an object can be described by $x = 4 \cos \left(2\pi t - \frac{\pi}{4} \right)$, where x is distance from the equilibrium point in centimeters and t is time in minutes. Find the exact distance from the equilibrium point at 45 seconds.

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Example 3 Verify Trigonometric Identities

Verify that each equation is an identity.

a. $\cos (90^\circ - \theta) = \sin \theta$

b. $\sin \left(\theta + \frac{\pi}{2} \right) = \cos \theta$

Guided Practice

3A. $\sin (90^\circ - \theta) = \cos \theta$

3B. $\cos (90^\circ + \theta) = -\sin \theta$

Check Your Understanding

Find the exact value of each expression.

1. $\cos 165^\circ$

2. $\cos 105^\circ$

3. $\cos 75^\circ$

4. $\sin (-30^\circ)$

5. $\sin 135^\circ$

6. $\sin (-210^\circ)$

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Verify that each equation is an identity.

8. $\sin (90^\circ + \theta) = \cos \theta$

9. $\cos \left(\frac{3\pi}{2} - \theta \right) = -\sin \theta$

10. $\tan \left(\theta + \frac{\pi}{2} \right) = -\cot \theta$

11. $\sin (\theta + \pi) = -\sin \theta$

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Practice and Problem Solving

Find the exact value of each expression.

12. $\sin 165^\circ$

13. $\cos 135^\circ$

14. $\cos \frac{7\pi}{12}$

15. $\sin \frac{\pi}{12}$

16. $\tan 195^\circ$

17. $\cos \left(-\frac{\pi}{12}\right)$

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18. **ELECTRONICS** In a certain circuit carrying alternating current, the formula $c = 2 \sin (120t)$ can be used to find the current c in amperes after t seconds.
- Rewrite the formula using the sum of two angles.
 - Use the sum of angles formula to find the exact current at $t = 1$ second.

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Verify that each equation is an identity.

19. $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$

20. $\cos(60^\circ + \theta) = \sin(30^\circ - \theta)$

21. $\cos(180^\circ + \theta) = -\cos \theta$

22. $\tan(\theta + 45^\circ) = \frac{1 + \tan \theta}{1 - \tan \theta}$

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Find the exact value of each expression.

24. $\tan 165^\circ$

25. $\sec 1275^\circ$

26. $\sin 735^\circ$

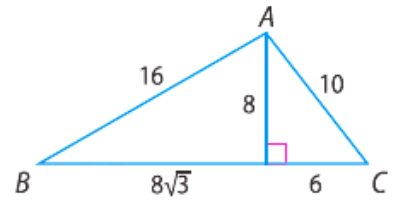
27. $\tan \frac{23\pi}{12}$

28. $\csc \frac{5\pi}{12}$

29. $\cot \frac{113\pi}{12}$

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31. **QUILTING** As part of a quilt that is being made, the quilter places two right triangular swatches together to make a new triangular piece. One swatch has sides 6 centimeters, 8 centimeters, and 10 centimeters long. The other swatch has sides 8 centimeters, $8\sqrt{3}$ centimeters, and 16 centimeters long. The pieces are placed with the sides of eight centimeters against each other, as shown in the figure, to form triangle ABC .



- What is the exact value of the sine of angle BAC ?
- What is the exact value of the cosine of angle BAC ?
- What is the measure of angle BAC ?
- Is the new triangle formed from the two triangles also a right triangle?

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33.  **MULTIPLE REPRESENTATIONS** In this problem, you will disprove the hypothesis that $\sin(A + B) = \sin A + \sin B$.

a. **Tabular** Copy and complete the table.

b. **Graphical** Assume that B is always 15° less than A . Use a graphing calculator to graph $y = \sin(x + x - 15)$ and $y = \sin x + \sin(x - 15)$ on the same screen.

c. **Analytical** Determine whether $\cos(A + B) = \cos A + \cos B$ is an identity. Explain your reasoning.

A	B	$\sin A$	$\sin B$	$\sin(A + B)$	$\sin A + \sin B$
30°	90°				
45°	60°				
60°	45°				
90°	30°				

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Verify that each equation is an identity.

$$34. \sin (A + B) = \frac{\tan A + \tan B}{\sec A \sec B}$$

$$36. \sec (A - B) = \frac{\sec A \sec B}{1 + \tan A \tan B}$$

$$35. \cos (A + B) = \frac{1 - \tan A \tan B}{\sec A \sec B}$$

$$37. \sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B$$

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Mid-Chapter Quiz

Lessons 11-1 through 11-3

Simplify each expression. (Lesson 12-1)

1. $\cot \theta \sec \theta$

2. $\frac{1 - \cos^2 \theta}{\sin^2 \theta}$

3. $\frac{1}{\cos \theta} - \frac{\sin^2 \theta}{\cos \theta}$

4. $\cos \left(\frac{\pi}{2} - \theta \right) \csc \theta$

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5. **HISTORY** In 1971, the flag of the United Arab Emirates was adopted. For this flag, $\tan \theta = \frac{31.5}{51}$. Find $\sin \theta$.



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Find the value of each expression. (Lesson 12-1)

6. $\sin \theta$, if $\cos \theta = \frac{3}{5}$; $0^\circ < \theta < 90^\circ$

7. $\csc \theta$, if $\cot \theta = \frac{1}{2}$; $270^\circ < \theta < 360^\circ$

8. $\tan \theta$, if $\sec \theta = \frac{4}{3}$; $0^\circ < \theta < 90^\circ$

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9. **MULTIPLE CHOICE** Which of the following is equivalent

to $\frac{\cos \theta}{1 - \sin^2 \theta}$? (Lesson 12-1)

A $\cos \theta$

B $\csc \theta$

C $\tan \theta$

D $\sec \theta$

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10. **AMUSEMENT PARKS** Suppose a child on a merry-go-round is seated on an outside horse. The diameter of the merry-go-round is 16 meters. The angle of inclination is represented by the equation $\tan \theta = \frac{v^2}{gR}$, where R is the radius of the circular path, v is the speed in meters per second, and g is 9.8 meters per second squared. (Lesson 12-1)

- a. If the sine of the angle of inclination of the child is $\frac{1}{5}$, what is the angle of inclination made by the child?
- b. What is the velocity of the merry-go-round?
- c. If the speed of the merry-go-round is 3.6 meters per second, what is the value of the angle of inclination of a rider?

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Verify that each of the following is an identity. (Lesson 12-2)

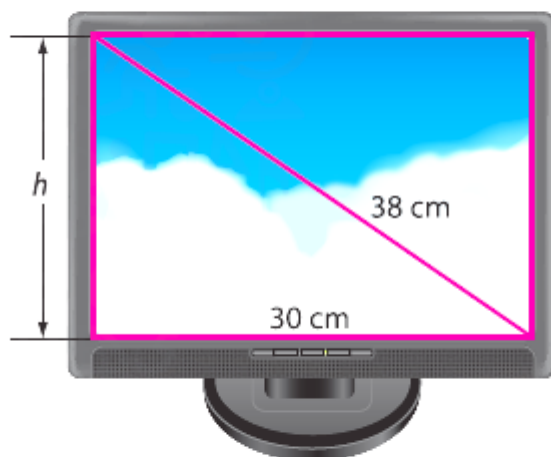
$$11. \cot^2 \theta + 1 = \frac{\cot \theta}{\cos \theta \cdot \sin \theta} \quad 12. \frac{\cos \theta \csc \theta}{\cot \theta} = 1$$

$$13. \frac{\sin \theta \tan \theta}{1 - \cos \theta} = (1 + \cos \theta) \sec \theta$$

$$14. \tan \theta (1 - \sin \theta) = \frac{\cos \theta \sin \theta}{1 + \sin \theta}$$

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15. **COMPUTER** The front of a computer monitor is usually measured along the diagonal of the screen as shown below. (Lesson 12-2)



- a. Find h .
- b. Using the diagram shown, show that $\cot \theta = \frac{\cos \theta}{\sin \theta}$.

Verify that each of the following is an identity. (Lesson 12-2)

$$16. \tan^2 \theta + 1 = \frac{\tan \theta}{\cos \theta \cdot \sin \theta}$$

$$17. \frac{\sin \theta \cdot \sec \theta}{\sec \theta - 1} = (\sec \theta + 1) \cot \theta$$

$$18. \sin^2 \theta \cdot \tan^2 \theta = \tan^2 \theta - \sin^2 \theta$$

$$19. \cot \theta (1 - \cos \theta) = \frac{\cos \theta \cdot \sin \theta}{1 + \cos \theta}$$

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Find the exact value of each expression. (Lesson 12-3)

20. $\cos 105^\circ$

21. $\sin (-135^\circ)$

22. $\tan 15^\circ$

23. $\cot 75^\circ$

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24. **MULTIPLE CHOICE** What is the exact value of $\cos \frac{5\pi}{12}$?
(Lesson 12-3)

F $\sqrt{2}$

H $\frac{\sqrt{6} - \sqrt{2}}{4}$

G $\frac{\sqrt{6} + \sqrt{2}}{2}$

J $\frac{\sqrt{6} + \sqrt{2}}{4}$

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25. Verify that $\cos 30^\circ \cos \theta + \sin 30^\circ \sin \theta = \sin 60^\circ$
 $\cos \theta + \cos 60^\circ \sin \theta$ is an identity. (Lesson 12-3)

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Lesson 11-4

Double-Angle and Half-Angle Identities

Learning outcomes

- 1- Find values of sine and cosine by using double-angle identities.
- 2- Find values of sine and cosine by using half-angle identities.

KeyConcept Double-Angle Identities

The following identities hold true for all values of θ .

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta & \cos 2\theta &= \cos^2 \theta - \sin^2 \theta & \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\ \cos 2\theta &= 2 \cos^2 \theta - 1 \\ \cos 2\theta &= 1 - 2 \sin^2 \theta\end{aligned}$$

Example 1 Double-Angle Identities

Find the exact value of $\sin 2\theta$ if $\sin \theta = \frac{2}{3}$ and θ is between 0° and 90° .

GuidedPractice

1. Find the exact value of $\sin 2\theta$ if $\cos \theta = -\frac{1}{3}$ and $90^\circ < \theta < 180^\circ$.

Example 2 Double-Angle Identities

Find the exact value of each expression if $\sin \theta = \frac{2}{3}$ and θ is between 0° and 90° .

a. $\cos 2\theta$

b. $\tan 2\theta$

Guided Practice

Find the exact value of each expression if $\cos \theta = -\frac{1}{3}$ and $90^\circ < \theta < 180^\circ$.

2A. $\cos 2\theta$

2B. $\tan 2\theta$

KeyConcept Half-Angle Identities

The following identities hold true for all values of θ .

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad \tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}, \cos \theta \neq -1$$

Example 3 Half-Angle Identities

a. Find the exact value of $\cos \frac{\theta}{2}$ if $\sin \theta = -\frac{4}{5}$ and θ is in the third quadrant.

b. Find the exact value of $\cos 67.5^\circ$

$$\cos 67.5^\circ = \cos \frac{135^\circ}{2}$$

GuidedPractice

3. Find the exact value of $\sin \frac{\theta}{2}$ if $\sin \theta = \frac{2}{3}$ and θ is in the second quadrant.

Guided Practice

Find each value.

4A. $\sin 135^\circ$

4B. $\cos \frac{7\pi}{8}$

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Example 5 Verify Identities

Verify that $\frac{\cos 2\theta}{1 + \sin 2\theta} = \frac{\cot \theta - 1}{\cot \theta + 1}$ is an identity.

Guided Practice

5. Verify that $4 \cos^2 x - \sin^2 2x = 4 \cos^4 x$.

Check Your Understanding

PRECISION Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$.

1. $\sin \theta = \frac{1}{4}; 0^\circ < \theta < 90^\circ$

2. $\sin \theta = \frac{4}{5}; 90^\circ < \theta < 180^\circ$

3. $\cos \theta = -\frac{5}{13}; \frac{\pi}{2} < \theta < \pi$

4. $\cos \theta = \frac{3}{5}; 270^\circ < \theta < 360^\circ$

5. $\tan \theta = -\frac{8}{15}; 90^\circ < \theta < 180^\circ$

6. $\tan \theta = \frac{5}{12}; \pi < \theta < \frac{3\pi}{2}$

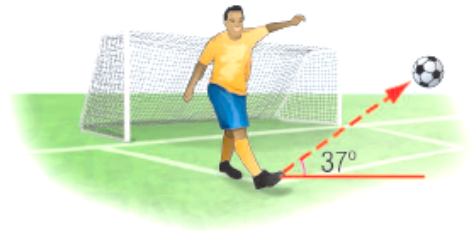
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Find the exact value of each expression.

7. $\sin \frac{\pi}{8}$

8. $\cos 15^\circ$

9. **SOCCER** A soccer player kicks a ball at an angle of 37° with the ground with an initial velocity of 16 meters per second. The distance d that the ball will go in the air if it is not blocked is given by $d = \frac{2v^2 \sin \theta \cos \theta}{g}$. In this formula, g is the acceleration due to gravity and is equal to 10 meters per second squared, and v is the initial velocity.
- Simplify this formula by using a double-angle identity.
 - Using the simplified formula, how far will this ball go?



Verify that each equation is an identity.

10. $\tan \theta = \frac{1 - \cos 2\theta}{\sin 2\theta}$

11. $(\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta$

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Practice and Problem Solving

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$.

12. $\sin \theta = \frac{2}{3}$; $90^\circ < \theta < 180^\circ$

13. $\sin \theta = -\frac{15}{17}$; $\pi < \theta < \frac{3\pi}{2}$

14. $\cos \theta = \frac{3}{5}$; $\frac{3\pi}{2} < \theta < 2\pi$

15. $\cos \theta = \frac{1}{5}$; $270^\circ < \theta < 360^\circ$

16. $\tan \theta = \frac{4}{3}$; $180^\circ < \theta < 270^\circ$

17. $\tan \theta = -2$; $\frac{\pi}{2} < \theta < \pi$

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Find the exact value of each expression.

18. $\sin 75^\circ$

19. $\sin \frac{3\pi}{8}$

20. $\cos \frac{7\pi}{12}$

21. $\tan 165^\circ$

22. $\tan \frac{5\pi}{12}$

23. $\tan 22.5^\circ$

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Verify that each equation is an identity.

$$26. \tan 2\theta = \frac{2}{\cot \theta - \tan \theta}$$

$$28. \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{\sin \theta}{2}$$

$$27. 1 + \frac{1}{2} \sin 2\theta = \frac{\sec \theta + \sin \theta}{\sec \theta}$$

$$29. \tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$$

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Verify that each equation is an identity.

$$26. \tan 2\theta = \frac{2}{\cot \theta - \tan \theta}$$

$$28. \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{\sin \theta}{2}$$

$$27. 1 + \frac{1}{2} \sin 2\theta = \frac{\sec \theta + \sin \theta}{\sec \theta}$$

$$29. \tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$$

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Find the exact values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

31. $\cos \theta = \frac{4}{5}; 0^\circ < \theta < 90^\circ$

32. $\sin \theta = \frac{1}{3}; 0 < \theta < \frac{\pi}{2}$

33. $\tan \theta = -3; 90^\circ < \theta < 180^\circ$

34. $\sec \theta = -\frac{4}{3}; 90^\circ < \theta < 180^\circ$

35. $\csc \theta = -\frac{5}{2}; \frac{3\pi}{2} < \theta < 2\pi$

36. $\cot \theta = \frac{3}{2}; 180^\circ < \theta < 270^\circ$

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11-5

Solving Trigonometric Equations

Learning outcomes

- 1 Solve trigonometric equations.
- 2 Find extraneous solutions from trigonometric equations.

Example 1 Solve Equations for a Given Interval

Solve $\sin \theta \cos \theta - \frac{1}{2} \cos \theta = 0$ if $0 \leq \theta \leq 180^\circ$.

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Guided Practice

1. Find all solutions of $\sin 2\theta = \cos \theta$ if $0 \leq \theta \leq 2\pi$.

Example 2 Infinitely Many Solutions

Solve $\cos \theta + 1 = 0$ for all values of θ if θ is measured in radians.

Guided Practice

2A. Solve $\cos 2\theta + \cos \theta + 1 = 0$ for all values of θ if θ is measured in degrees.

2B. Solve $2 \sin \theta = -1$ for all values of θ if θ is measured in radians.

Extraneous Solutions

Example 4 Determine Whether a Solution Exists

Solve each equation.

a. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$ if $0 \leq \theta \leq 2\pi$

b. $\sin \theta = 1 + \cos \theta$ if $0^\circ \leq \theta < 360^\circ$

Guided Practice

4A. $\sin^2 \theta + 2 \cos^2 \theta = 4$

4B. $\cos^2 \theta + 3 = 4 - \sin^2 \theta$

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Example 5 Solve Trigonometric Equations by Using Identities

Solve $2 \sec^2 \theta - \tan^4 \theta = -1$ for all values of θ if θ is measured in degrees.

Guided Practice

Solve each equation.

5A. $\sin \theta \cot \theta - \cos^2 \theta = 0$

5B. $\frac{\cos \theta}{\cot \theta} + 2 \sin^2 \theta = 0$

Check Your Understanding

REGULARITY Solve each equation if $0^\circ \leq \theta \leq 360^\circ$.

1. $2 \sin \theta + 1 = 0$

2. $\cos^2 \theta + 2 \cos \theta + 1 = 0$

3. $\cos 2\theta + \cos \theta = 0$

4. $2 \cos \theta = 1$

5. $\cos \theta = -\frac{\sqrt{3}}{2}$

6. $\sin 2\theta = -\frac{\sqrt{3}}{2}$

7. $\cos 2\theta = 8 - 15 \sin \theta$

8. $\sin \theta + \cos \theta = 1$

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Solve each equation for all values of θ if θ is measured in radians.

9. $4 \sin^2 \theta - 1 = 0$

10. $2 \cos^2 \theta = 1$

11. $\cos 2\theta \sin \theta = 1$

12. $\sin \frac{\theta}{2} + \cos \frac{\theta}{2} = \sqrt{2}$

13. $\cos 2\theta + 4 \cos \theta = -3$

14. $\sin \frac{\theta}{2} + \cos \theta = 1$

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Solve each equation for all values of θ if θ is measured in degrees.

15. $\cos 2\theta - \sin^2 \theta + 2 = 0$

16. $\sin^2 \theta - \sin \theta = 0$

17. $2 \sin^2 \theta - 1 = 0$

18. $\cos \theta - 2 \cos \theta \sin \theta = 0$

19. $\cos 2\theta \sin \theta = 1$

20. $\sin \theta \tan \theta - \tan \theta = 0$

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21. **LIGHT** The number of hours of daylight d in Hartford, Connecticut, may be approximated by the equation $d = 3 \sin \frac{2\pi}{365}t + 12$, where t is the number of days after March 21.
- On what days will Hartford have exactly $10\frac{1}{2}$ hours of daylight?
 - Using the results in part a, tell what days of the year have at least $10\frac{1}{2}$ hours of daylight. Explain how you know.

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Solve each equation.

22. $\sin^2 2\theta + \cos^2 \theta = 0$

23. $\tan^2 \theta + 2 \tan \theta + 1 = 0$

24. $\cos^2 \theta + 3 \cos \theta = -2$

25. $\sin 2\theta - \cos \theta = 0$

26. $\tan \theta = 1$

27. $\cos 8\theta = 1$

28. $\sin \theta + 1 = \cos 2\theta$

29. $2 \cos^2 \theta = \cos \theta$

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Practice and Problem Solving

Solve each equation for the given interval.

30. $\cos^2 \theta = \frac{1}{4}$; $0^\circ \leq \theta \leq 360^\circ$

32. $\sin 2\theta - \cos \theta = 0$; $0 \leq \theta \leq 2\pi$

34. $2 \sin \theta + \sqrt{3} = 0$; $180^\circ < \theta < 360^\circ$

31. $2 \sin^2 \theta = 1$; $90^\circ < \theta < 270^\circ$

33. $3 \sin^2 \theta = \cos^2 \theta$; $0 \leq \theta \leq \frac{\pi}{2}$

35. $4 \sin^2 \theta - 1 = 0$; $180^\circ < \theta < 360^\circ$

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Solve each equation for all values of θ if θ is measured in radians.

36. $\cos 2\theta + 3 \cos \theta = 1$

37. $2 \sin^2 \theta = \cos \theta + 1$

38. $\cos^2 \theta - \frac{3}{2} = \frac{5}{2} \cos \theta$

39. $3 \cos \theta - \cos \theta = 2$

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Solve each equation for all values of θ if θ is measured in degrees.

40. $\sin \theta - \cos \theta = 0$

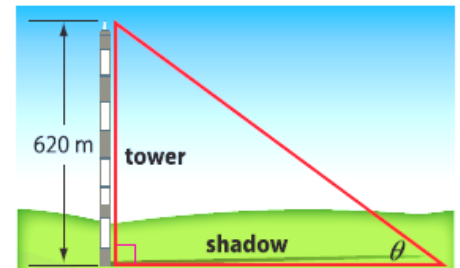
41. $\tan \theta - \sin \theta = 0$

42. $\sin^2 \theta = 2 \sin \theta + 3$

43. $4 \sin^2 \theta = 4 \sin \theta - 1$

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44. **ELECTRONICS** One of the tallest structures in the world is a television transmitting tower located near Fargo, North Dakota, with a height of 620 meters. What is the measure of θ if the length of the shadow is 1.6 kilometers?



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Solve each equation.

45. $2 \sin^2 \theta = 3 \sin \theta + 2$

46. $2 \cos^2 \theta + 3 \sin \theta = 3$

47. $\sin^2 \theta + \cos 2\theta = \cos \theta$

48. $2 \cos^2 \theta = -\cos \theta$

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Solve each equation if θ is measured in radians.

50. $(\cos \theta)(\sin 2\theta) - 2 \sin \theta + 2 = 0$

51. $2 \sin^2 \theta + (\sqrt{2} - 1) \sin \theta = \frac{\sqrt{2}}{2}$

Solve each equation if θ is measured in degrees.

52. $\sin 2\theta + \frac{\sqrt{3}}{2} = \sqrt{3} \sin \theta + \cos \theta$

53. $1 - \sin^2 \theta - \cos \theta = \frac{3}{4}$

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Solve each equation.

54. $2 \sin \theta = \sin 2\theta$

55. $\cos \theta \tan \theta - 2 \cos^2 \theta = -1$

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Practice Test

1. **MULTIPLE CHOICE** Which expression is equivalent to $\sin \theta + \cos \theta \cot \theta$?

- A $\cot \theta$ C $\sec \theta$
B $\tan \theta$ D $\csc \theta$

2. Verify that $\cos (30^\circ - \theta) = \sin (60^\circ + \theta)$ is an identity.

3. Verify that $\cos (\theta - \pi) = -\cos \theta$.

4. **MULTIPLE CHOICE** What is the exact value of $\sin \theta$, if

$\cos \theta = -\frac{3}{5}$ and $90^\circ < \theta < 180^\circ$?

- F $\frac{5}{3}$
G $\frac{\sqrt{34}}{8}$
H $-\frac{4}{5}$
J $\frac{4}{5}$

Find the value of each expression.

5. $\cot \theta$, if $\sec \theta = \frac{4}{3}$; $270^\circ < \theta < 360^\circ$
6. $\tan \theta$, if $\cos \theta = -\frac{1}{2}$; $90^\circ < \theta < 180^\circ$
7. $\sec \theta$, if $\csc \theta = -2$; $180^\circ < \theta < 270^\circ$
8. $\cot \theta$, if $\csc \theta = -\frac{5}{3}$; $270^\circ < \theta < 360^\circ$
9. $\sec \theta$, if $\sin \theta = \frac{1}{2}$; $0^\circ \leq \theta < 90^\circ$

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Verify that each of the following is an identity.

10. $\sin \theta (\cot \theta + \tan \theta) = \sec \theta$

11. $\frac{\cos^2 \theta}{1 - \sin \theta} = \frac{\cos \theta}{\sec \theta - \tan \theta}$

12. $(\tan \theta + \cot \theta)^2 = \csc^2 \theta \sec^2 \theta$

13. $\frac{1 + \sec \theta}{\sec \theta} = \frac{\sin^2 \theta}{1 - \cos \theta}$

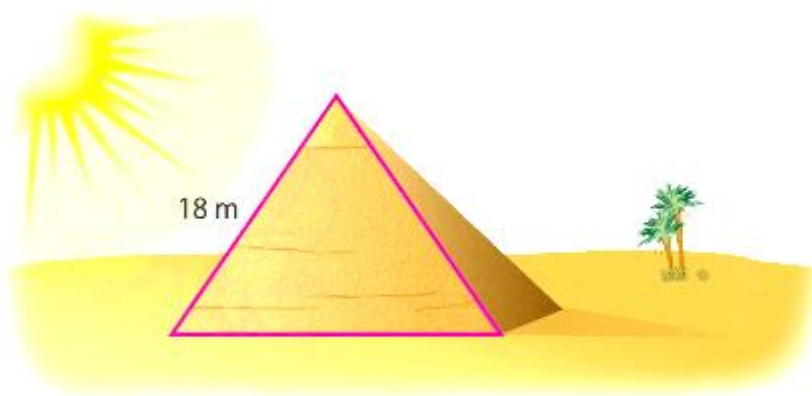
14. $\frac{\sin \theta}{1 - \cos \theta} = \csc \theta + \cot \theta$

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15. **MULTIPLE CHOICE** What is the exact value of $\tan \frac{\pi}{8}$?

- A $\frac{\sqrt{2-\sqrt{3}}}{2}$
- B $\sqrt{2} - 1$
- C $1 - \sqrt{2}$
- D $-\frac{\sqrt{2-\sqrt{3}}}{2}$

16. **HISTORY** Some researchers believe that the builders of ancient pyramids, such as the Great Pyramid of Khufu, may have tried to build the faces as equilateral triangles. Later they had to change to other types of triangles. Suppose a pyramid is built such that a face is an equilateral triangle of side length 18 meters.



- a. Find the height of the equilateral triangle.
- b. Use the formula $\sin 2\theta = 2 \sin \theta \cos \theta$ and the measures of the equilateral triangle and its height to show that $\sin 2(30^\circ) = \sin 60^\circ$. Find the exact values.

Find the exact value of each expression.

17. $\cos(-225^\circ)$

18. $\sin 480^\circ$

19. $\cos 75^\circ$

20. $\sin 165^\circ$

21. **ROCKETS** A model rocket is launched with an initial velocity of 20 meters per second. The range of a projectile is given by the formula $R = \frac{v^2}{g} \sin 2\theta$, where R is the range, v is the initial velocity, g is acceleration due to gravity or 9.8 meters per second squared, and θ is the launch angle. What angle is needed in order for the rocket to reach a range of 25 meters?

Solve each equation for all values of θ if θ is measured in radians.

22. $2 \cos^2 \theta - 3 \cos \theta - 2 = 0$

23. $2 \sin 3\theta - 1 = 0$

Solve each equation for $0^\circ \leq \theta \leq 360^\circ$ if θ is measured in degrees.

24. $\cos 2\theta + \cos \theta = 2$

25. $\sin \theta \cos \theta - \frac{1}{2} \sin \theta = 0$

تم بحمد الله

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<https://t.me/+XPfMBpsQpVxjNzNk> <http://www.youtube.com/@imaths2022>