

$$\sqrt{16} = 4$$



$$y = mx + b$$



2025 - 2026
Grade 11 Advanced
Math End of Term 1 Exam Coverage

Module 5 - Exponential Functions

Module 6 - Logarithmic Functions

Module 7 - Rational Functions

Module 8 - Inferential Statistics

Module 9 - Trigonometric Functions

Name

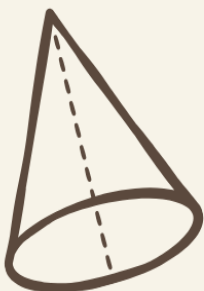
Grade

5510 - Fatma Al Zahraa Cycle 3 School

Teacher Dana Kamal Moh'd Abdel Jabbar



$$(a+b)^2$$



$$\frac{x}{1} + \frac{y}{2}$$



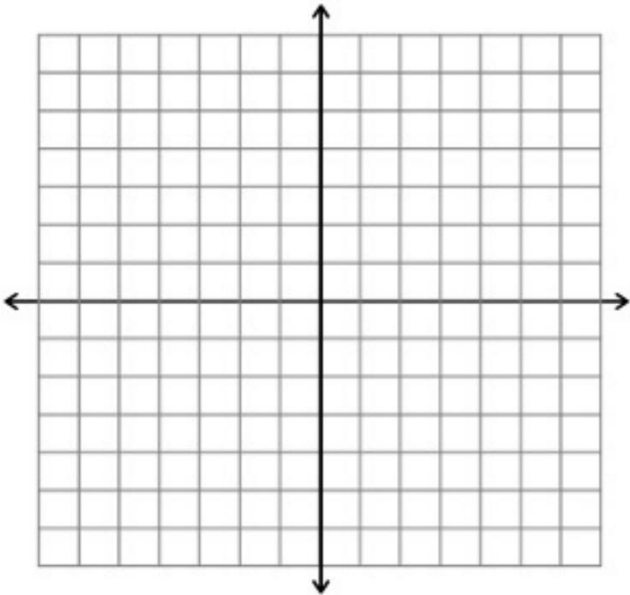
Example 1

Graph each function. Find the domain, range, y-intercept, asymptote, and end behavior.

1. $f(x) = 3^x$

Domain : Range :

Y-intercept : Asymptote :



End Behavior :

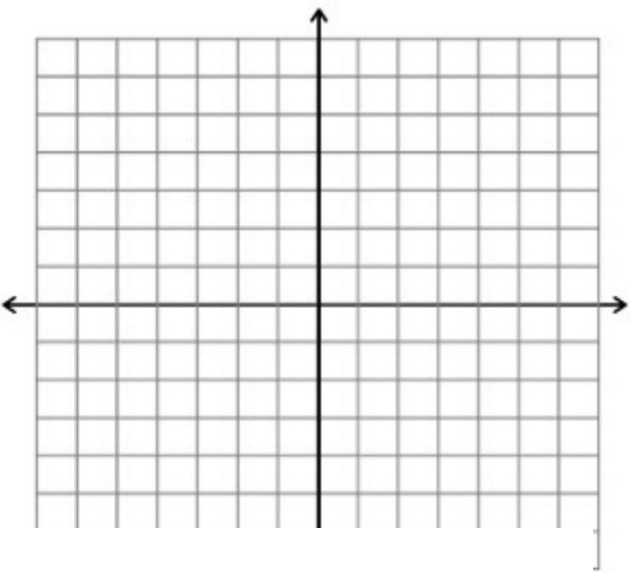
as $x \rightarrow -\infty$, $f(x) \rightarrow$

as $x \rightarrow +\infty$, $f(x) \rightarrow$

2. $f(x) = 5^x$

Domain : Range :

Y-intercept : Asymptote :



End Behavior :

as $x \rightarrow -\infty$, $f(x) \rightarrow$



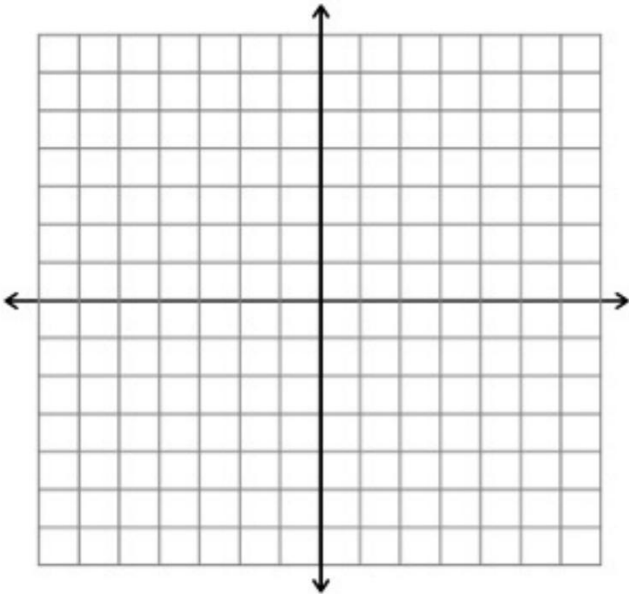
Example 1

Graph each function. Find the domain, range, y-intercept, asymptote, and end behavior.

3. $f(x) = 1.5^x$

Domain : Range :

Y-intercept : Asymptote :



End Behavior :

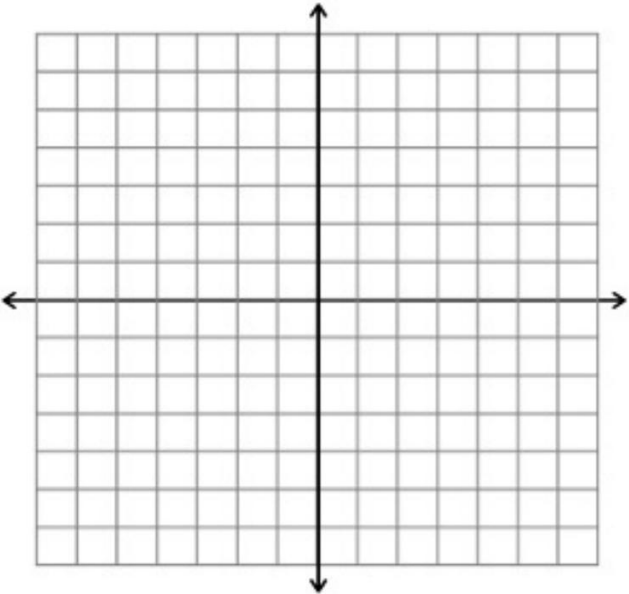
 $as\ x \rightarrow -\infty , f(x) \rightarrow$

 $as\ x \rightarrow +\infty , f(x) \rightarrow$

4. $f(x) = \left(\frac{5}{2}\right)^x$

Domain : Range :

Y-intercept : Asymptote :



End Behavior :

 $as\ x \rightarrow -\infty , f(x) \rightarrow$

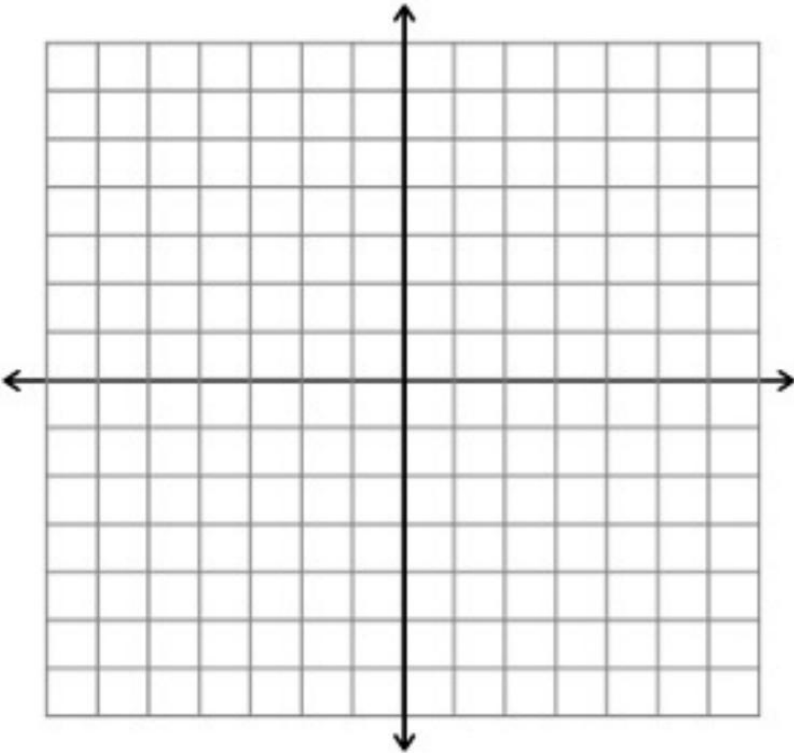
 $as\ x \rightarrow +\infty , f(x) \rightarrow$



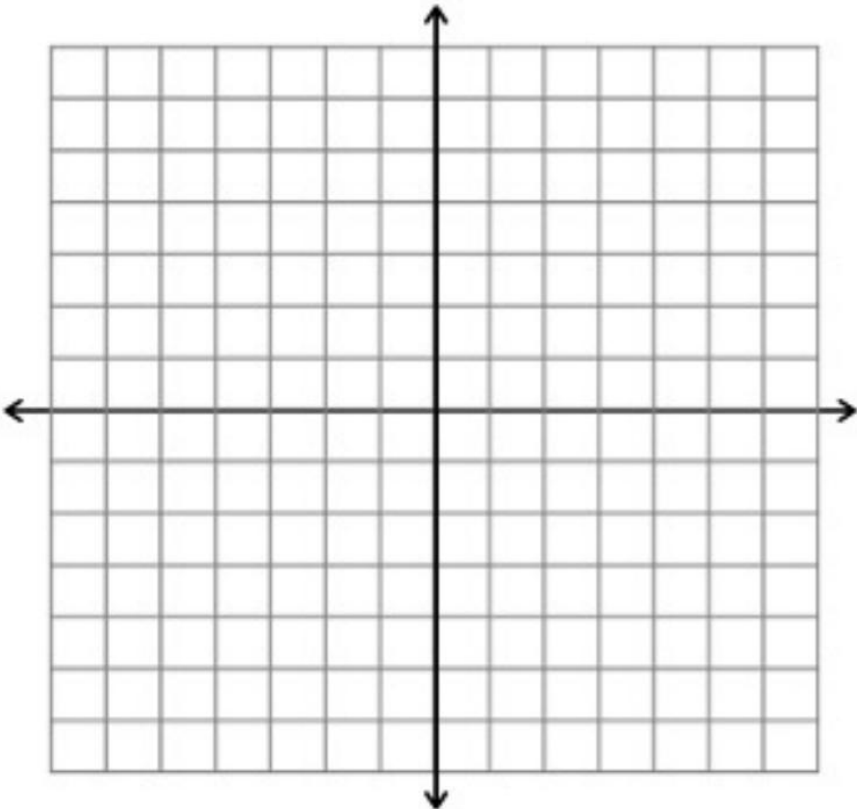
Example 2

Graph each function.

5. $f(x) = 2(3)^x$



6. $f(x) = -2(4)^x$

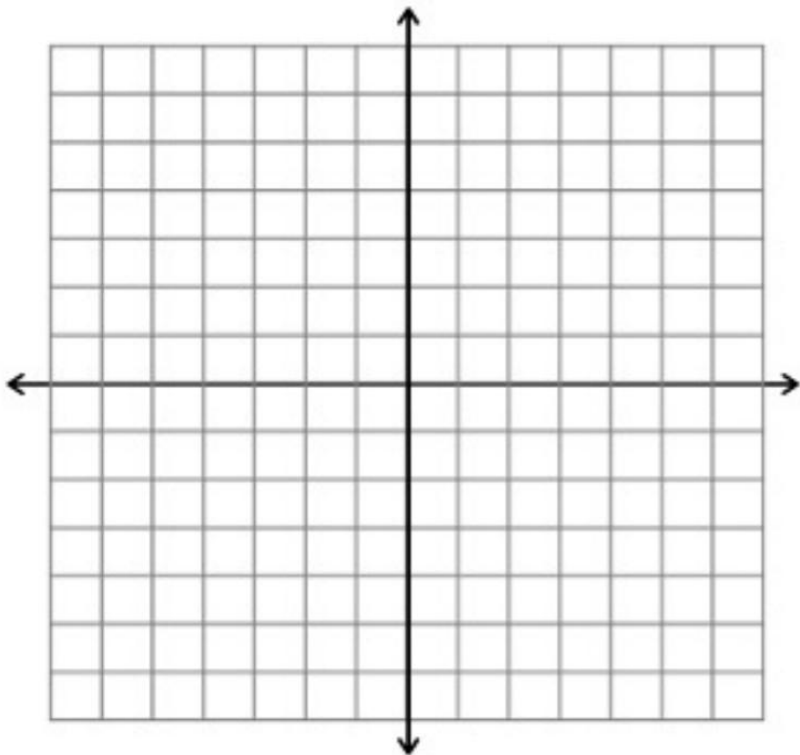




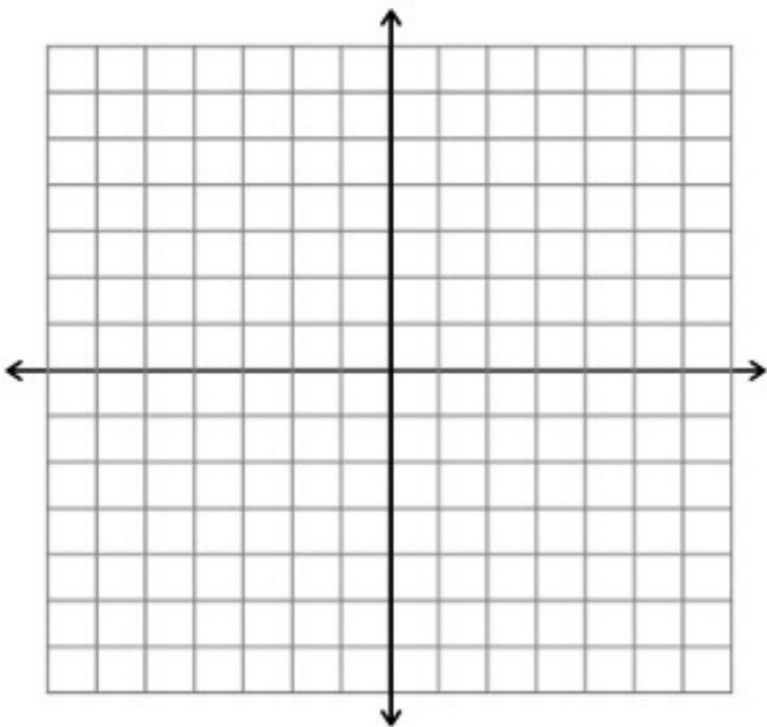
Example 2

Graph each function.

7. $f(x) = 4^x + 1 - 5$



8. $f(x) = 3^{2x} + 1$

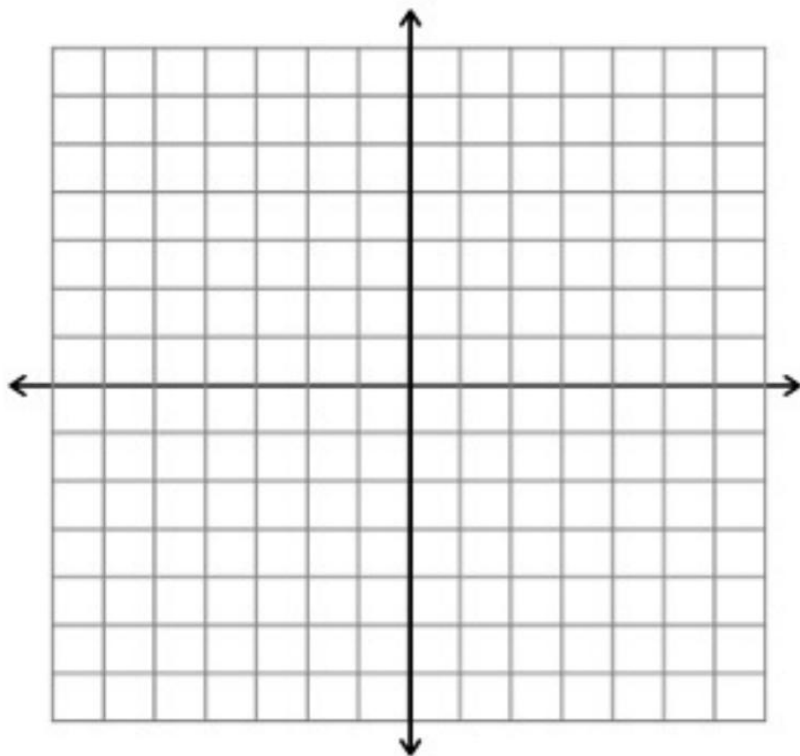




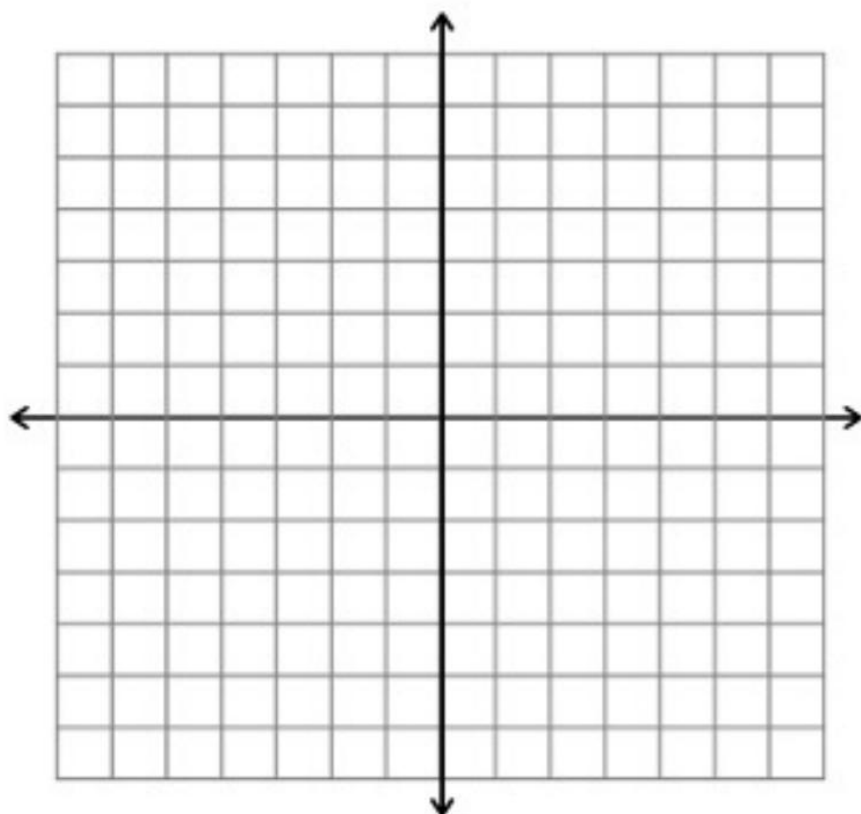
Example 2

Graph each function.

9. $f(x) = -0.4(3)^{x + 2} + 4$

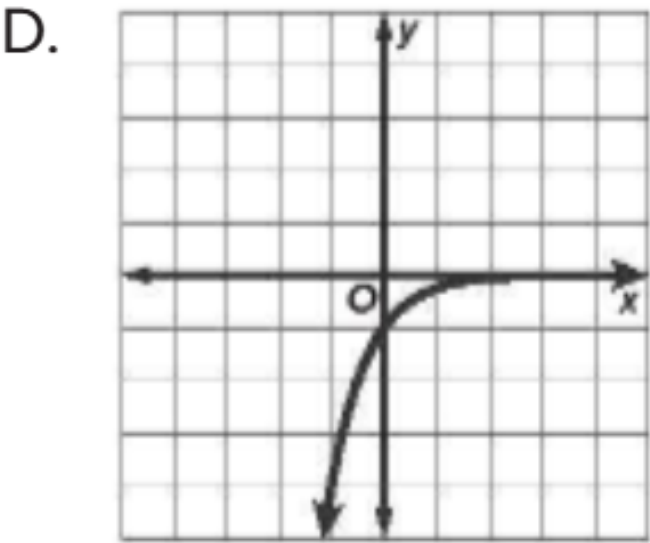
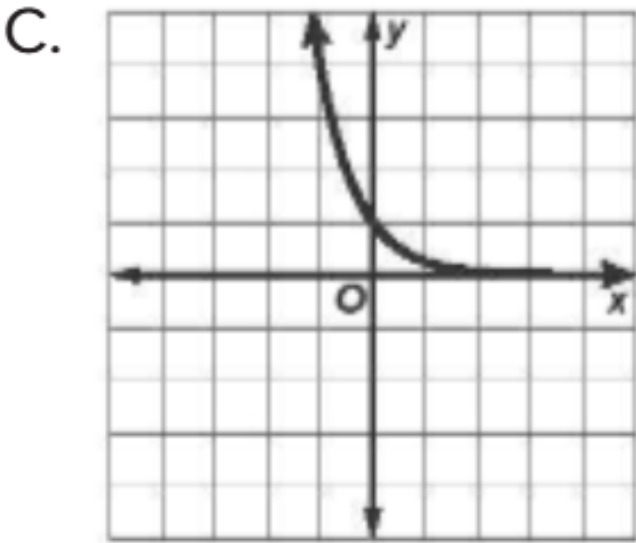
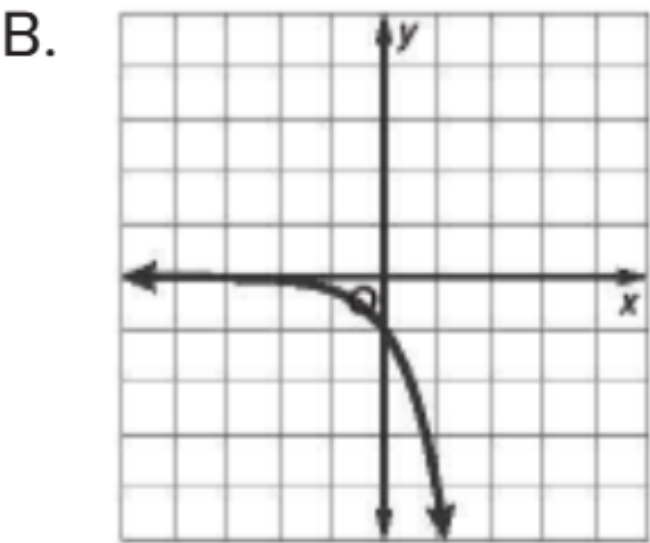
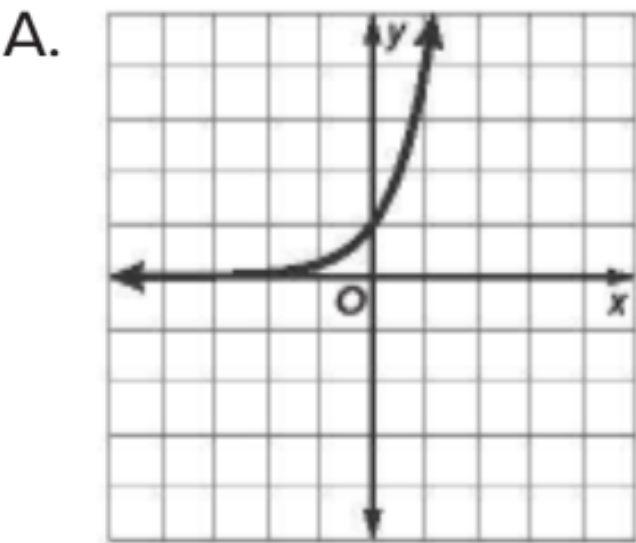


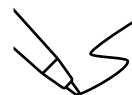
10. $f(x) = 1.5(2)^x + 6$





1. MULTIPLE CHOICE Select the graph of $f(x) = 4^x$. (Lesson 5-1)





Learn Solving Exponential Equations

In an **exponential equation**, the independent variable is an exponent.

Key Concept • Property of Equality for Exponential Equations

If $b > 0$ and $b \neq 1$, then $b^x = b^y$ if and only if $x = y$.

Example 1 Solve Exponential Equations Algebraically

Solve each equation.

a. $4^{5x+1} = 64^7$

$$4^{5x+1} = 64^7$$

Original equation

$$4^{5x+1} = (4^3)^7$$

Rewrite 64 as 4^3 .

$$4^{5x+1} = 4^{21}$$

Power of a Power

$$5x + 1 = 21$$

Property of Equality for Exponential Equations

$$5x = 20$$

Subtract 1 from each side.

$$x = 4$$

Divide each side by 5.

b. $\left(\frac{1}{2}\right)^{4x-16} = 16^{2x-5}$

$$\left(\frac{1}{2}\right)^{4x-16} = 16^{2x-5}$$

Original equation

$$(2^{-1})^{4x-16} = (2^4)^{2x-5}$$

$$\frac{1}{2} = 2^{-1}, 16 = 2^4$$

$$2^{-4x+16} = 2^{8x-20}$$

Power of a Power Property

$$-4x + 16 = 8x - 20$$

Property of Equality for Exponential Equations

$$-12x + 16 = -20$$

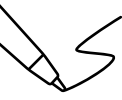
Subtract $8x$ from each side.

$$-12x = -36$$

Subtract 16 from each side.

$$x = 3$$

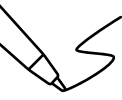
Divide each side by -12 .

**Example 1****Solve each equation.**

1. $25^{2x+3} = 25^{5x-9}$

2. $9^{8x-4} = 81^{3x+6}$

3. $4^{x-5} = 16^{2x-31}$

**Example 1****Solve each equation.**

4. $4^{3x-3} = 8^{4x-4}$

5. $9^{-x+5} = 27^{6x-10}$

6. $125^{3x-4} = 25^{4x+2}$

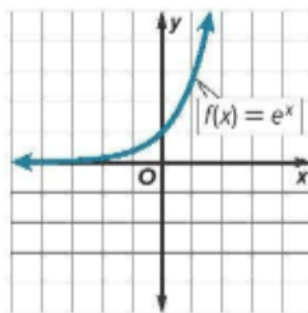


Learn Exponential Functions with Base e

The constant **e** has certain mathematical properties that make it a convenient base for exponential functions. **e** is the irrational number that $(1 + \frac{1}{n})^n$ approaches as n approaches ∞ . This value is approximately equal to 2.7182818...

Graphs of exponential functions with base **e** display the same general characteristics as other exponential functions.

- $f(x)$ approaches an asymptote of $y = 0$ as x approaches $-\infty$.
- $f(x)$ approaches ∞ as x approaches ∞ .
- The y -intercept is 1.



You can calculate continuously compounding interest by using the formula $A = Pe^{rt}$, where A is the amount in the account after t years, P is the principal amount invested, and r is the annual interest rate.

Example 2 Graph Functions with Base e

Consider $g(x) = -2e^{x-3} + 2$.

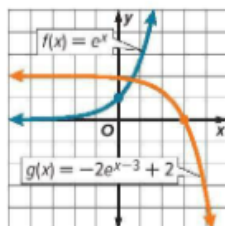
Part A Graph the function.

The function $g(x) = -2e^{x-3} + 2$ represents a transformation of the graph $f(x) = e^x$.

$a = -2$; The graph is reflected in the x -axis and stretched vertically.

$h = 3$; The graph is translated 3 units right.

$k = 2$; The graph is translated 2 units up.



Part B Determine the domain and range.

The domain is all real numbers. The range is all real numbers less than 2.

Part C Find the average rate of change.

Determine the average rate of change of $g(x)$ over each interval.

a. $[-4, -1]$

Based on the graph, the graph from -4 to -1 appears approximately horizontal. So, the average rate of change should be close to 0.

$$\begin{aligned} \frac{g(-1) - g(-4)}{-1 - (-4)} &\approx \frac{1.963 - 1.998}{3} && \text{Evaluate } g(-1) \text{ and } g(-4). \\ &\approx \frac{-0.035}{3} \text{ or } -0.012 && \text{Simplify.} \end{aligned}$$

b. $[0, 3]$

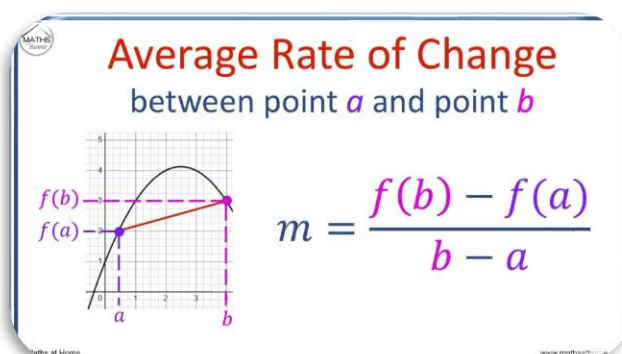
Based on the graph, the curve appears to pass through approximately $(0, 2)$ and $(3, 0)$. So, the average rate of change for the interval $[0, 3]$ should be about $\frac{0-2}{3-0}$ or $-\frac{2}{3}$.

$$\begin{aligned} \frac{g(3) - g(0)}{3 - 0} &\approx \frac{0 - 1.900}{3} && \text{Evaluate } g(3) \text{ and } g(0). \\ &\approx \frac{-1.900}{3} \text{ or } -0.633 && \text{Simplify.} \end{aligned}$$

c. $[4, 8]$

The graph is very steep for values greater than 4. So, the average rate of change from 4 to 8 should be a negative number with a large absolute value.

$$\begin{aligned} \frac{g(8) - g(4)}{8 - 4} &\approx \frac{-294.826 - (-3.437)}{4} && \text{Evaluate } g(8) \text{ and } g(4). \\ &\approx \frac{-291.389}{4} \text{ or } -72.847 && \text{Simplify.} \end{aligned}$$



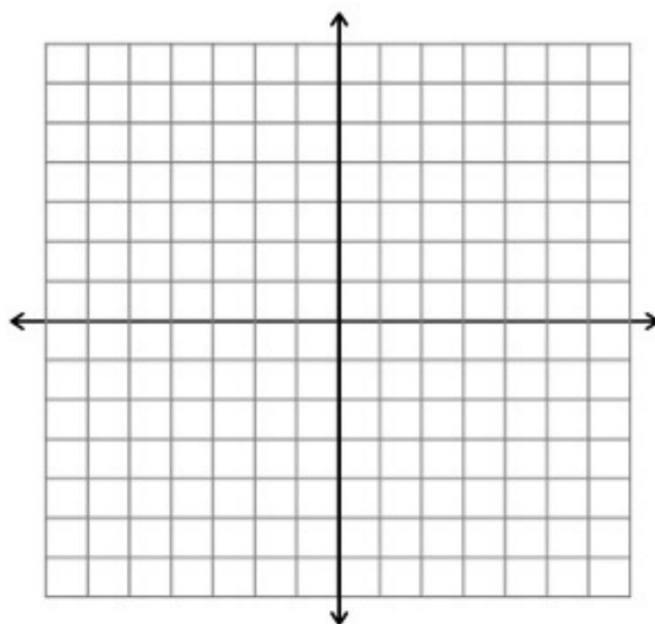


Check

Consider $g(x) = -e^{x+1} + 2$.

Part A

Graph the function.



Part B

Determine the domain and range of

$$g(x) = -e^{x+1} + 2.$$

Domain: _____?

Range: _____?

Part C

Find the average rate of change of $g(x) = -e^{x+1} + 2$ over each interval.

a. $[-6, -3] \approx$ _____?

b. $[-1, 1] \approx$ _____?

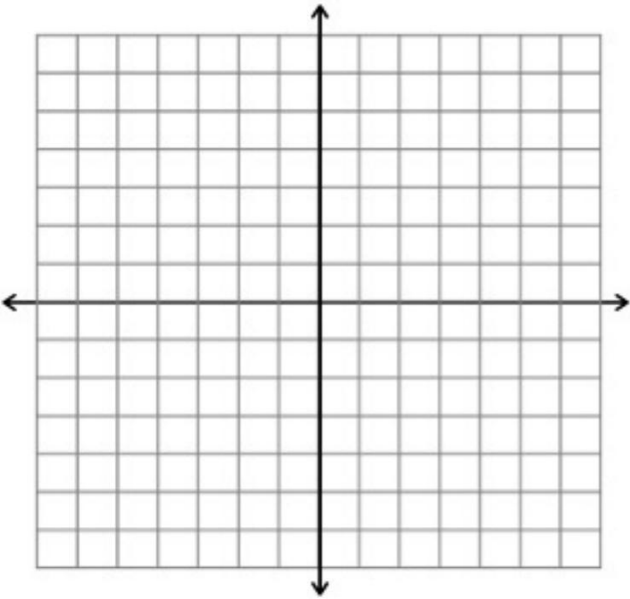
c. $[1, 4] \approx$ _____?



Example 2

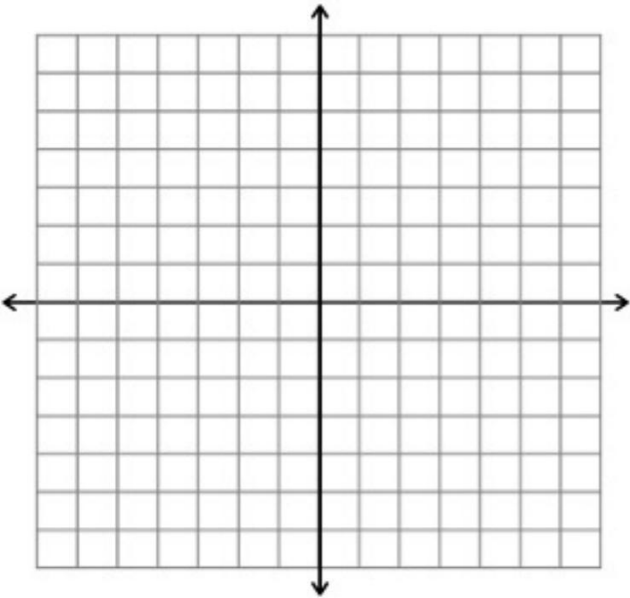
10. Consider the function $f(x) = 3e^{x-1} + 3$.

- a. Graph the function.
- b. Determine domain and range.
- c. Find the average rate of change over the interval $[-5, -2]$.



11. Consider the function $f(x) = 4e^{2x} - 1$.

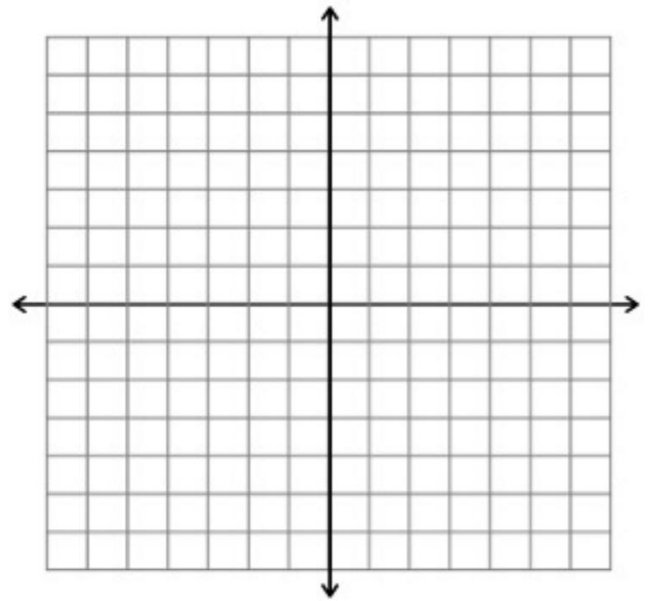
- a. Graph the function.
- b. Determine domain and range.
- c. Find the average rate of change over the interval $[-3, -1]$.





12. Consider the function $f(x) = -2e^{x+3} + 2$.

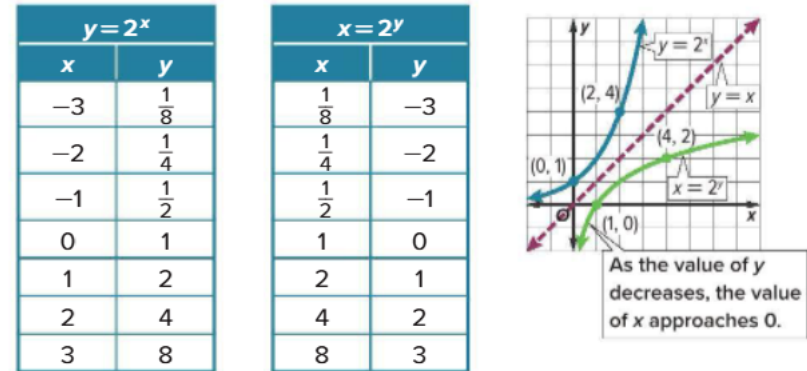
- Graph the function.
- Determine domain and range.
- Find the average rate of change over the interval $[-7, -4]$.





Learn Logarithmic Functions

Recall the exponential function $f(x) = 2^x$. You can graph its inverse by interchanging the x - and y -values and graphing the ordered pairs.



The inverse of $y = 2^x$ can be defined as $x = 2^y$. In general, the inverse of $y = b^x$ is $x = b^y$. In $x = b^y$, y is called the **logarithm**, base b , of x . This is usually written as $y = \log_b x$ and is read as *log base b of x* .

Key Concept • Logarithms with Base b	
Words	Let b and x be positive numbers, $b \neq 1$. The <i>logarithm</i> of x with base b is denoted $\log_b x$ and is defined as the exponent y that makes the equation $b^y = x$ true.
Symbols	Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that $\log_b x = y$ if and only if $b^y = x$.
Example	If $\log_2 8 = y$, then $2^y = 8$.

In the expression $\log_b x$, x is called the *argument*.

Example 1 Logarithmic to Exponential Form

Write each equation in exponential form.

- a. $\log_4 64 = 3$
 $\log_4 64 = 3 \rightarrow 64 = 4^3$
- b. $\log_3 \frac{1}{243} = -5$
 $\log_3 \frac{1}{243} = -5 \rightarrow \frac{1}{243} = 3^{-5}$
- c. $\log_9 3 = \frac{1}{2}$
 $\log_9 3 = \frac{1}{2} \rightarrow 3 = 9^{\frac{1}{2}}$

Check

Write $\log_{125} 5 = \frac{1}{3}$ in exponential form.

Example 2 Exponential to Logarithmic Form

Write each equation in logarithmic form.

- a. $7^6 = 117,649$
 $7^6 = 117,649 \rightarrow \log_7 117,649 = 6$
- b. $8^{-3} = \frac{1}{512}$
 $8^{-3} = \frac{1}{512} \rightarrow \log_8 \frac{1}{512} = -3$
- c. $8^{\frac{2}{3}} = 4$
 $8^{\frac{2}{3}} = 4 \rightarrow \log_8 4 = \frac{2}{3}$

Check

Write $2^9 = 512$ in logarithmic form.



Example 1

Write each equation in exponential form.

1. $\log_{15} 225 = 2$

2. $\log_3 \frac{1}{27} = -3$

3. $\log_5 \frac{1}{25} = 2$

4. $\log_3 243 = 5$

5. $\log_4 64 = 3$

6. $\log_4 32 = \frac{5}{2}$

Example 2

Write each equation in logarithmic form.

7. $2^7 = 128$

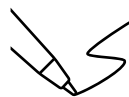
8. $3^{-4} = \frac{1}{81}$

9. $7^{-2} = \frac{1}{49}$

10. $\left(\frac{1}{7}\right)^3 = \frac{1}{343}$

11. $2^9 = 512$

12. $64^{\frac{2}{3}} = 16$



Learn Logarithmic Equations

A **logarithmic equation** contains one or more logarithms.

Key Concept • Property of Equality for Logarithmic Equations

Symbols

If b is a positive number other than 1, then
 $\log_b x = \log_b y$ if and only if $x = y$.

Example

If $\log_3 x = \log_3 7$, then $x = 7$. If $x = 7$, then $\log_3 x = \log_3 7$.

This property also holds true for inequalities.

Example 2 Solve a Logarithmic Equation by Using Properties of Equality

Solve $\log_5 (2x^2 - 6) = \log_5 4x$.

Step 1 Solve for x .

$$\log_5 (2x^2 - 6) = \log_5 4x \quad \text{Original equation}$$

$$2x^2 - 6 = 4x \quad \text{Property of Equality for Logarithmic Equations}$$

$$2x^2 - 4x - 6 = 0 \quad \text{Subtract } 4x \text{ from each side.}$$

$$x^2 - 2x - 3 = 0 \quad \text{Divide each side by 2.}$$

$$(x - 3)(x + 1) = 0 \quad \text{Factor.}$$

$$x - 3 = 0 \text{ or } x + 1 = 0 \quad \text{Zero Product Property}$$

$$x = 3 \quad x = -1 \quad \text{Solve each equation.}$$

Step 2 Check for extraneous solutions.

$$x = 3$$

$$x = -1$$

$$\log_5 (2 \cdot 3^2 - 6) \stackrel{?}{=} \log_5 (4 \cdot 3)$$

$$\log_5 [2 \cdot (-1)^2 - 6] \stackrel{?}{=} \log_5 [4 \cdot (-1)]$$

$$\log_5 (12) \stackrel{?}{=} \log_5 (12) \quad \checkmark$$

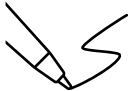
$$\log_5 (-4) \stackrel{?}{=} \log_5 (-4)$$

Because the logarithm of a negative number is undefined, -1 is an extraneous solution. So, $x = 3$.

Check

Solve $\log_{13} (-5x) = \log_{13} (-2x^2 + 3)$.

$$x = \underline{\hspace{2cm} ? \hspace{2cm}}$$



Learn Properties of Logarithms

Because logarithms are exponents, the properties of logarithms can be derived from the properties of exponents. For example, the Product Property of Logarithms can be derived from the Product of Powers Property of Exponents.

Key Concept • Product Property of Logarithms	
Words	The logarithm of a product is the sum of the logarithms of its factors.
Symbols	For all positive numbers b , m , and n , where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$.
Example	$\log_5 8(4) = \log_5 8 + \log_5 4$

You can use the Product Property of Logarithms to approximate logarithmic expressions.

Key Concept • Quotient Property of Logarithms	
Words	The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
Symbols	For all positive numbers b , m , and n , where $b \neq 1$ and $n \neq 0$, $\log_b \frac{m}{n} = \log_b m - \log_b n$.
Example	$\log_8 \frac{2}{3} = \log_8 2 - \log_8 3$

Key Concept • Power Property of Logarithms	
Words	The logarithm of a power is the product of the logarithm and the exponent.
Symbols	For any real number n , and positive numbers m and b , where $b \neq 1$, $\log_b m^n = n \log_b m$.
Example	$\log_2 3^7 = 7 \log_2 3$

Example 3 Product Property of Logarithms

Use $\log_3 5 \approx 1.465$ to approximate the value of $\log_3 405$.

$$\begin{aligned}\log_3 405 &= \log_3 (3^4 \cdot 5) \\ &= \log_3 3^4 + \log_3 5 \\ &= 4 + \log_3 5 \\ &\approx 4 + 1.465 \text{ or } 5.465\end{aligned}$$

$$405 = 3^4 \cdot 5$$

Product Property of Logarithms

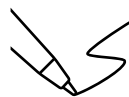
Inverse Property of Exponents and Logarithms

Replace $\log_3 5$ with 1.465.

Check

Use $\log_8 5 \approx 0.7740$ to approximate the value of $\log_8 320$.

$\log_8 320 \approx \underline{\hspace{1cm}}?$



Example 4 Quotient Property of Logarithms

Use $\log_3 5 \approx 1.465$ to approximate the value of $\log_3 \frac{9}{5}$.

$$\log_3 \frac{9}{5} = \log_3 \frac{3^2}{5}$$

$$\frac{9}{5} = \frac{3^2}{5}$$

$$= \log_3 3^2 - \log_3 5$$

Quotient Property of Logarithms

$$= 2 - \log_3 5$$

Inverse Property of Exponents and Logarithms

$$\approx 2 - 1.465 \text{ or } 0.535$$

Replace $\log_3 5$ with 1.465.

Check

Use $\log_6 5 \approx 0.8982$ to approximate the value of $\log_6 \frac{5}{1296}$.

$$\log_6 \frac{5}{1296} \approx \underline{\hspace{2cm}} ?$$

Example 5 Power Property of Logarithms

Use $\log_2 6 \approx 2.585$ to approximate the value of $\log_2 1296$.

$$\log_2 1296 = \log_2 6^4$$

$$1296 = 6^4$$

$$= 4 \log_2 6$$

Power Property of Logarithms

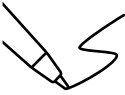
$$\approx 4(2.585) \text{ or } 10.34$$

Replace $\log_2 6$ with 2.585.

Check

Use $\log_2 3 \approx 1.5850$ to approximate the value of $\log_2 243$.

$$\log_2 243 \approx \underline{\hspace{2cm}} ?$$

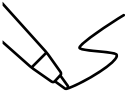


Example 2

Solve each equation.

7. $\log_4 (2x^2 - 4) = \log_4 2x$

8. $\log_5 (x^2 - 6) = \log_5 x$

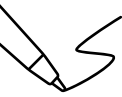


Example 2

Solve each equation.

9. $\log_3 (x^2 - 8) = \log_3 2x$

10. $\log_4 (2x^2 - 20) = \log_4 6x$

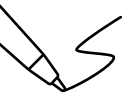


Example 2

Solve each equation.

11. $\log_2 (6x^2 + 1) = \log_2 5x$

12. $\log_6 (6x^2 - 3) = \log_6 7x$

**Examples 3 and 4**

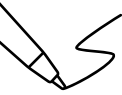
Use $\log_4 2 = 0.5$, $\log_4 3 \approx 0.7925$, and $\log_4 5 \approx 1.1610$ to approximate the value of each expression.

13. $\log_4 30$

14. $\log_4 20$

15. $\log_4 \frac{2}{3}$

16. $\log_4 \frac{4}{3}$

**Examples 3 and 4**

Use $\log_4 2 = 0.5$, $\log_4 3 \approx 0.7925$, and $\log_4 5 \approx 1.1610$ to approximate the value of each expression.

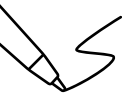
17. $\log_4 9$

18. $\log_4 8$

Example 5

Use $\log_2 3 \approx 1.5850$ and $\log_2 5 \approx 2.3219$ to approximate the value of each expression.

19. $\log_2 25$

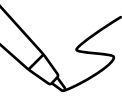
**Example 5**

Use $\log_2 3 \approx 1.5850$ and $\log_2 5 \approx 2.3219$ to approximate the value of each expression.

19. $\log_2 25$

20. $\log_2 27$

21. $\log_2 125$

**Example 5**

Use $\log_2 3 \approx 1.5850$ and $\log_2 5 \approx 2.3219$ to approximate the value of each expression.

22. $\log_2 625$

23. $\log_2 81$

24. $\log_2 243$



Learn Change of Base Formula

The Change of Base Formula allows you to write equivalent logarithmic expressions that have different bases.

Key Concept • Change of Base Formula	
Symbols	For all positive numbers a , b , and n , where $a \neq 1$ and $b \neq 1$, $\log_a n = \frac{\log_b n}{\log_b a}$ $\log_b n$ ← log base b of the original number $\log_b a$ ← log base b of old base
Example	$\log_8 17 = \frac{\log_{10} 17}{\log_{10} 8}$



Example 5 Change of Base Formula

Evaluate $\log_2 11$ by writing it in terms of common logarithms. Round to the nearest ten-thousandth.

$$\log_2 11 = \frac{\log_{10} 11}{\log_{10} 2}$$
$$\approx 3.4594$$

Change of Base Formula

Use a calculator.

Check

Evaluate $\log_8 30$. Round to the nearest ten-thousandth.

$\log_8 30 \approx$ _____ ?

Example 5

Express each logarithm in terms of common logarithms. Then approximate its value to the nearest ten-thousandth.

21. $\log_4 22$

22. $\log_{12} 200$

23. $\log_2 50$
24. $\log_8 15$

25. $\log_3 2$

26. $\log_5 0.4$



Example 3 Simplify Logarithmic Expressions

Write $\frac{1}{3} \ln 8 - \ln 3 + \ln 9$ as a single logarithm.

$$\frac{1}{3} \ln 8 - \ln 3 + \ln 9$$

Original expression

$$= \ln 8^{\frac{1}{3}} - \ln 3 + \ln 9$$

Power Property of Logarithms

$$= \ln 2 - \ln 3 + \ln 9$$

$$8^{\frac{1}{3}} = \sqrt[3]{8} \text{ or } 2$$

$$= \ln \frac{2}{3} + \ln 9$$

Quotient Property of Logarithms

$$= \ln 6$$

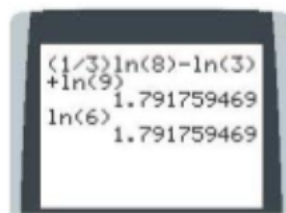
Product Property of Logarithms

Check

Use a calculator to verify the solution.

Because both entries result in the same value,

$$\frac{1}{3} \ln 8 - \ln 3 + \ln 9 = \ln 6.$$



Check

Write $\ln 7 - \frac{1}{3} \ln 27 + \ln 6$ as a single logarithm.

Example 3

Write each expression as a single logarithm.

19. $3 \ln 3 - \ln 9$

**Example 3**

Write each expression as a single logarithm.

20. $4 \ln 16 - \ln 256$

21. $2 \ln x + 2 \ln 4$

22. $3 \ln 4 + 3 \ln 3$

**Example 3**

Write each expression as a single logarithm.

23. $\ln 125 - 2 \ln 5$

24. $3 \ln 10 + 2 \ln 100$

25. $4 \ln \frac{1}{3} - 6 \ln \frac{1}{9}$



Example 3

Write each expression as a single logarithm.

26. $7 \ln \frac{1}{2} + 5 \ln 2$

27. $8 \ln x - 4 \ln 5$



Learn Using Logarithms to Solve Exponential Growth Problems

Exponential functions with base e are frequently used in an alternate form of the Exponential Growth Formula to represent situations involving continuous growth.

Key Concept • Continuous Exponential Growth

Exponential growth can be modeled by the function

$$f(x) = ae^{kt},$$

where a is the initial value, t is time in years, and k is a constant representing the rate of continuous growth.

The Continuous Exponential Growth equation $y = ae^{kt}$ is often used to represent population growth and is the same as the continuously compounded interest formula.

Continuously Compounded Interest	Continuous Exponential Growth
$A = Pe^{rt}$	$y = ae^{kt}$
P = initial amount	a = initial population
A = amount at time t	y = population at time t
r = interest rate	k = rate of continuous growth

Example 1 Continuous Exponential Growth

POPULATION In 2016, the population of Florida was 20.61 million people. In 2000, it was 15.98 million.

Part A Write an exponential growth equation.

Because the population is growing as time passes, use the year as the independent variable and the population as the dependent variable. Find the value of k , Florida's relative growth rate. Then write an equation that represents the population of Florida t years after 2000.

Since t is the number of years after 2000, 15.98 represents the initial population, and 20.61 represents the population 16 years after 2000, or at $t = 16$.

$y = ae^{kt}$	Formula for continuous exponential growth
$20.61 = 15.98e^{k(16)}$	$y = 20.61$, $a = 15.98$, and $t = 2016 - 2000$ or 16
$\frac{20.61}{15.98} = e^{16k}$	Divide each side by 15.98.
$\ln \frac{20.61}{15.98} = \ln e^{16k}$	Property of Equality for Logarithmic Equations
$\ln \frac{20.61}{15.98} = 16k$	$\ln e^x = x$
$\frac{\ln \frac{20.61}{15.98}}{16} = k$	Divide each side by 16.
$0.0159 \approx k$	Use a calculator.

Florida's relative rate of growth is about, 0.0159, or about 1.59%.

$y = ae^{kt}$	Formula for continuous exponential growth
$y = 15.98e^{0.0159t}$	$a = 15.98$ and $k = 0.0159$

So, the population of Florida t years after 2000 can be modeled by the equation $y = 15.98e^{0.0159t}$.

Part B Predict when the population will reach 25 million people.

Use your equation from **Part A** to predict the population.

$y = 15.98e^{0.0159t}$	Original equation
$25 = 15.98e^{0.0159t}$	$y = 25$
$1.564 = e^{0.0159t}$	Divide each side by 15.98.
$\ln 1.564 = \ln e^{0.0159t}$	Property of Equality for Logarithmic Equations
$\ln 1.564 = 0.0159t$	$\ln e^x = x$
$\frac{\ln 1.564}{0.0159} = t$	Divide each side by 0.0159.
$28.129 \approx t$	Use a calculator.

Florida's population will reach 25 million people approximately 28.129 years after 2000, or in 2028.

Part C Compare the populations of Florida and California.

California's population in 2000 was 33.9 million, and its population growth can be modeled by $y = 33.9e^{0.0092t}$. Determine when Florida's population will surpass California's.

$15.98e^{0.0159t} > 33.9e^{0.0092t}$	Formula for exponential growth
$\ln 15.98e^{0.0159t} > \ln 33.9e^{0.0092t}$	Prop. of Ineq. for Logarithms
$\ln 15.98 + \ln e^{0.0159t} > \ln 33.9 + \ln e^{0.0092t}$	Product Property of Logarithms
$\ln 15.98 + 0.0159t > \ln 33.9 + 0.0092t$	$\ln e^x = x$
$\ln 15.98 + 0.0067t > \ln 33.9$	Subtract 0.0092t from each side.
$0.0067t > \ln 33.9 - \ln 15.98$	Subtract $\ln 15.98$ from each side.
$t > \frac{\ln 33.9 - \ln 15.98}{0.0067}$	Divide each side by 0.0067.
$t > 112.25$	Use a calculator.

Assuming that this trend continues, Florida's population is on track to surpass California's in 2112.



Check

SCIENCE An experiment starts with 20 bacteria A cells. After 45 minutes, there are 710 bacteria A cells.

Part A

Write the equation that models the number of bacteria A cells y after t minutes. Round the value of k to the nearest thousandth.

Part B

After how many minutes will there be 1000 bacteria A cells? Round to the nearest tenth if necessary.

_____ minutes

Part C

Bacteria B grows exponentially according to the model $y = 52e^{0.064t}$. After how many minutes will there be more bacteria A cells than bacteria B cells? Round to the nearest tenth if necessary.

_____ minutes

**Example 1**

1. **POPULATION** In 2000, the world population was estimated to be 6.124 billion people. In 2005, it was 6.515 billion.
 - a. Write an exponential growth equation to represent the population y in billions t years after 2000.
 - b. Use the equation to predict the year in which the world population reached 7.5 billion people.



2. CONSUMER AWARENESS Jason wants to buy a new HD television but he thinks that if he waits, the quality of HD televisions will improve. The television he wants to buy costs \$2500 now, and based on pricing trends, Jason thinks that the price will increase by 4% each year.

a. Write an exponential growth equation to represent the price y of a new HD television t years from now.

b. Use the equation to predict when a new HD television will cost \$3000.

c. Jason decides to wait to buy a new television and saves his money. He puts \$2200 in a savings account with 4.7% annual interest compounded continuously. Determine when the amount in his savings will exceed the cost of a new television.



Example 1 Simplify a Rational Expression

Simplify $\frac{x^2 - 2x - 24}{2x^3 + 6x^2 - 8x}$, and state when the original expression is undefined.

Part A Simplify.

$$\begin{aligned}
 \frac{x^2 - 2x - 24}{2x^3 + 6x^2 - 8x} &= \frac{x^2 - 2x - 24}{2x(x^2 + 3x - 4)} && \text{Distributive Property} \\
 &= \frac{(x + 4)(x - 6)}{2x(x + 4)(x - 1)} && \text{Factor the numerator and denominator.} \\
 &= \frac{(x - 6)}{2x(x - 1)} \cdot \frac{\overset{1}{\cancel{(x + 4)}}}{\underset{1}{\cancel{(x + 4)}}} && \text{Eliminate common factors.} \\
 &= \frac{x - 6}{2x(x - 1)} && \text{Simplify.}
 \end{aligned}$$

Part B State when the expression is undefined.

The expression is undefined when the denominator is equal to 0.

Because the question asks when the original expression is undefined, consider the denominator before the common factors are eliminated. Thus, the expression is undefined when $2x(x + 4)(x - 1) = 0$. By the Zero Product Property, the expression is undefined when $2x = 0$ or $x = 0$, when $x + 4 = 0$ or $x = -4$, and when $x - 1 = 0$ or $x = 1$.

Check

Simplify $\frac{x^2 + 2x + 1}{4x^2 + 3x - 1}$, and state when the original expression is undefined.

Part A Select the simplified expression.

- A. 1
- B. $\frac{x + 1}{4x - 1}$
- C. $\frac{(x + 1)^2}{(4x + 1)(x - 1)}$
- D. $\frac{1}{4x - 1}$

Part B State when the expression is undefined.

$x = \underline{\quad ? \quad}, \underline{\quad ? \quad}$



Example 2 Simplify by Using -1

Simplify $\frac{(6x^2 - 5xy)(x + 2y)}{(x + y)(5y - 6x)}$.

$$\frac{(6x^2 - 5xy)(x + 2y)}{(x + y)(5y - 6x)} = \frac{x(6x - 5y)(x + 2y)}{(x + y)(5y - 6x)}$$

Factor.

$$= \frac{x(-1)(5y - 6x)(x + 2y)}{(x + y)(5y - 6x)}$$

$$6x - 5y = -1(5y - 6x)$$

$$= \frac{(-x)(x + 2y)}{x + y} \text{ or } -\frac{x(x + 2y)}{x + y}$$

Simplify.

Check

Select the simplified form of $\frac{(7y - 3x)(5x - 1)}{(5x^3 + x^2)(3x - 7y)}$.

A. $-\frac{5x - 1}{x^2(5x + 1)}$

B. $\frac{(7y - 3x)(5x - 1)}{x^2(5x + 1)(3x - 7y)}$

C. $\frac{1}{x^2}$

D. $\frac{5x - 1}{5x^3 + x^2}$

**Example 1**

Simplify each expression, and state when the original expression is undefined.

1.
$$\frac{x(x-3)(x+6)}{x^2+x-12}$$

2.
$$\frac{y^2(y^2+3y+2)}{2y(y-4)(y+2)}$$

3.
$$\frac{(x^2-9)(x^2-z^2)}{4(x+z)(x-3)}$$

**Example 1**

Simplify each expression, and state when the original expression is undefined.

4.
$$\frac{(x^2 - 16x + 64)(x + 2)}{(x^2 - 64)(x^2 - 6x - 16)}$$

5.
$$\frac{x^2(x + 2)(x - 4)}{6x(x^2 + x - 20)}$$

6.
$$\frac{3y(y - 8)(y^2 + 2y - 24)}{15y^2(y^2 - 12y + 32)}$$

**Example 2**

Simplify each expression.

7.
$$\frac{x^2 - 5x - 14}{28 + 3x - x^2}$$

8.
$$\frac{9x^2 - x^3}{x^2 - 3x - 54}$$

**Example 2**

Simplify each expression.

9.
$$\frac{(x - 4)(x^2 + 2x - 48)}{(36 - x^2)(x^2 + 4x - 32)}$$

10.
$$\frac{16 - c^2}{c^2 + c - 20}$$



Learn Adding and Subtracting Rational Expressions

Just as with rational numbers in fractional form, to add or subtract two rational expressions that have unlike denominators, you must first find the least common denominator (LCD). The LCD is the least common multiple (LCM) of the two denominators.

To find the LCM of two or more numbers or polynomials, factor them. The LCM contains each factor the greatest number of times it appears as a factor.

Key Concept • Adding Rational Expressions

Words: To add rational expressions, find the least common denominator. Rewrite each expression with the LCD. Then add.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}.$$

Key Concept • Subtracting Rational Expressions

Words: To subtract rational expressions, find the least common denominator. Rewrite each expression with the LCD. Then subtract.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$,

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad-bc}{bd}.$$

Example 1 Add and Subtract Rational Expressions with Monomial Denominators

Simplify each expression.

a. $\frac{7a}{4b} + \frac{4c^2}{10}$

$$\begin{aligned}\frac{7a}{4b} + \frac{4c^2}{10} &= \frac{7a}{4b} \cdot \frac{5}{5} + \frac{4c^2}{10} \cdot \frac{2b}{2b} \\ &= \frac{35a}{20b} + \frac{8bc^2}{20b} \\ &= \frac{35a + 8bc^2}{20b}\end{aligned}$$

The LCD is $20b$.

Multiply.

Add.

b. $\frac{2x}{9y} - \frac{7y}{6z}$

$$\begin{aligned}\frac{2x}{9y} - \frac{7y}{6z} &= \frac{2x}{9y} \cdot \frac{2z}{2z} - \frac{7y}{6z} \cdot \frac{3y}{3y} \\ &= \frac{4xz}{18yz} - \frac{21y^2}{18yz} \\ &= \frac{4xz - 21y^2}{18yz}\end{aligned}$$

The LCD is $18yz$.

Multiply.

Subtract.



Example 2 Add and Subtract Rational Expressions with Polynomial Denominators

Simplify $\frac{2x+1}{x^2+2x-15} - \frac{7}{5x-15}$.

$$\frac{2x+1}{x^2+2x-15} - \frac{7}{5x-15}$$

$$= \frac{2x+1}{(x-3)(x+5)} - \frac{7}{5(x-3)}$$

$$= \frac{5(2x+1)}{5(x-3)(x+5)} - \frac{7(x+5)}{5(x-3)(x+5)}$$

$$= \frac{10x+5-7x-35}{5(x-3)(x+5)}$$

$$= \frac{3x-30}{5(x-3)(x+5)}$$

Factor the denominators.

The LCD is $5(x-3)(x+5)$.

Distributive Property

Subtract.

Check

Simplify $\frac{3x}{4x^2+4} - \frac{2x^2}{x^4-1}$.



Examples 1 and 2

Simplify each expression.

1. $\frac{3}{x} + \frac{5}{y}$

2. $\frac{3}{8p^2r} + \frac{5}{4p^2r}$

3. $\frac{2c - 7}{3} + 4$



Examples 1 and 2

Simplify each expression.

4. $\frac{2}{m^2p} + \frac{5}{p}$

5. $\frac{12}{5y^2} - \frac{2}{5yz}$

6. $\frac{7}{4gh} + \frac{3}{4h^2}$



Examples 1 and 2

Simplify each expression.

7. $\frac{3}{w-3} - \frac{2}{w^2-9}$

8. $\frac{3t}{2-x} + \frac{5}{x-2}$

9. $\frac{k}{k-n} - \frac{k}{n-k}$



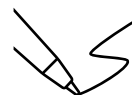
Examples 1 and 2

Simplify each expression.

10. $\frac{4z}{z-4} + \frac{z+4}{z+1}$

11. $\frac{n}{n-3} + \frac{2n+2}{n^2-2n-3}$

12. $\frac{3}{y^2+y-12} - \frac{2}{y^2+6y+8}$



Key Concept • Transformations of Reciprocal Functions

$$g(x) = \frac{a}{x-h} + k$$

 h – horizontal translation

If $h > 0$, the graph of $f(x)$ is translated h units right.
 If $h < 0$, the graph of $f(x)$ is translated $|h|$ units left.
 The *vertical asymptote* is at $x = h$.

 k – vertical translation

If $k > 0$, the graph of $f(x)$ is translated k units up.
 If $k < 0$, the graph of $f(x)$ is translated $|k|$ units down.
 The *horizontal asymptote* is at $f(x) = k$.

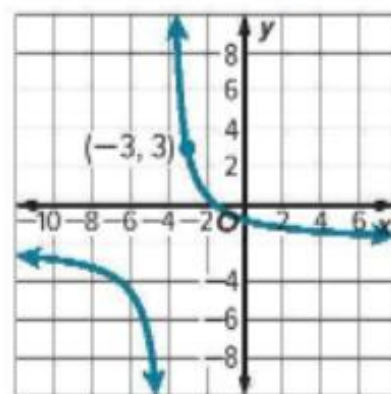
 a – orientation and shape

If $|a| > 1$, the graph is stretched vertically.
 If $0 < |a| < 1$, the graph is compressed vertically.

Example 5 Write a Reciprocal Function from a Graph

Identify the values of a , h , and k . Then write a function for the graph $g(x) = \frac{a}{x-h} + k$.

The asymptotes are $x = -4$ and $y = -2$. So, $h = -4$ and $k = -2$. Use a point on the graph to solve for a .



$$g(x) = \frac{a}{x-h} + k$$

$$3 = \frac{a}{(-3) - (-4)} + (-2)$$

$$3 = a - 2$$

$$5 = a$$

General form of reciprocal function

$$h = -4, k = -2, \text{ and } (x, g(x)) = (-3, 3)$$

Simplify.

Add 2 to each side.

Use the values of a , h , and k to write the function.

$$g(x) = \frac{5}{x+4} - 2$$

$$a = 5, h = -4, \text{ and } k = -2$$

Check

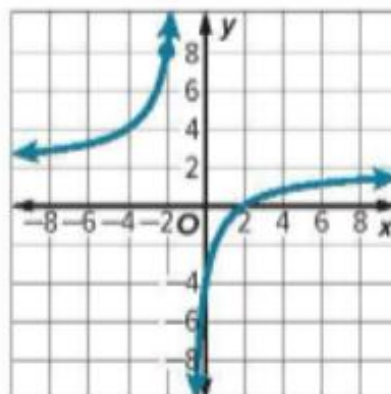
Identify the values of a , h , and k for the graph.

$$a = \underline{\quad?}, h = \underline{\quad?}, k = \underline{\quad?}$$

Write a function for the graph

$$g(x) = \frac{a}{x-h} + k.$$

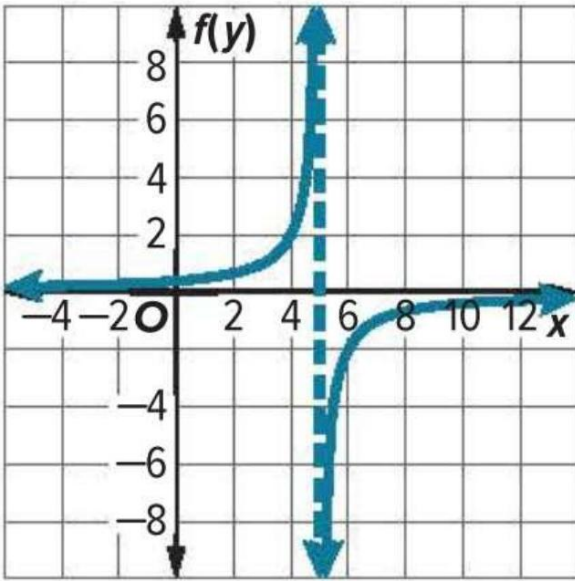
$$g(x) = \underline{\quad?}$$



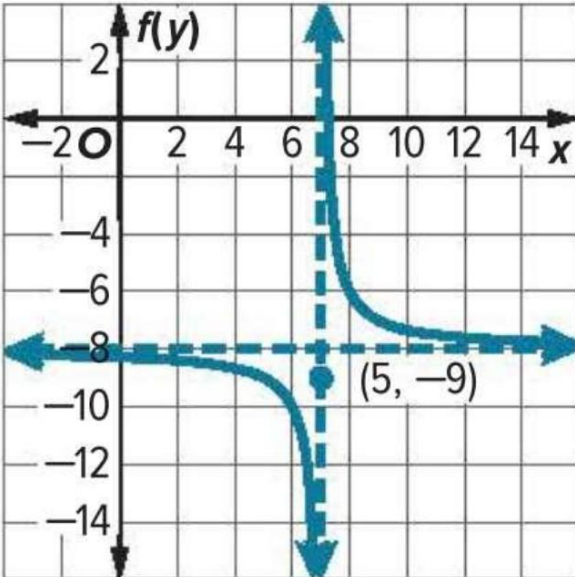
Example 5

Identify the values of a , h , and k . Then write a function for the graph $g(x) = \frac{a}{x-h} + k$.

17.



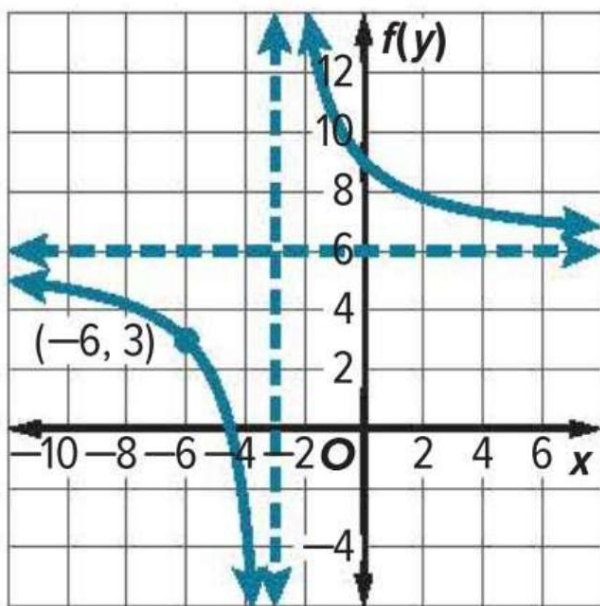
18.



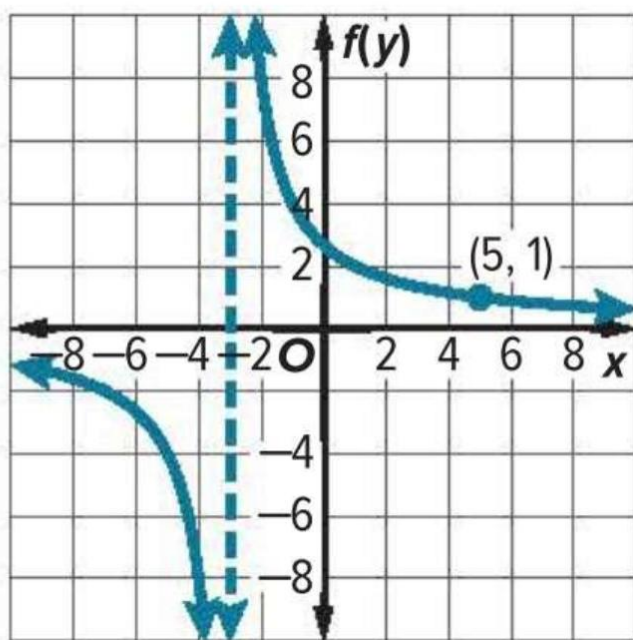
Example 5

Identify the values of a , h , and k . Then write a function for the graph $g(x) = \frac{a}{x-h} + k$.

19.



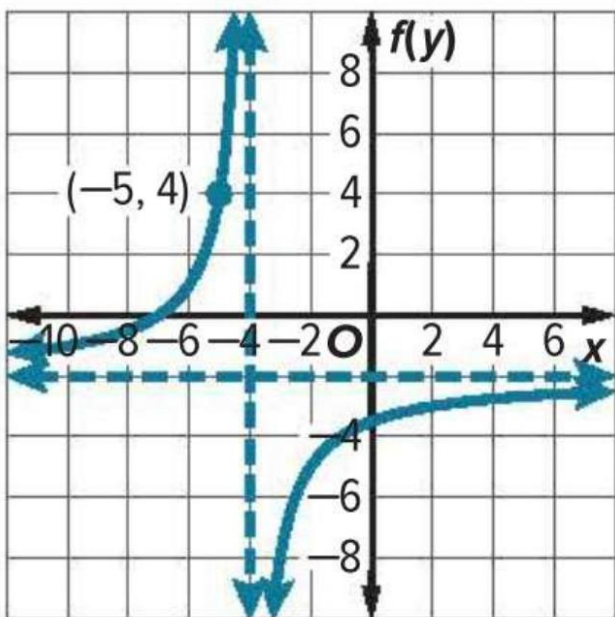
20.



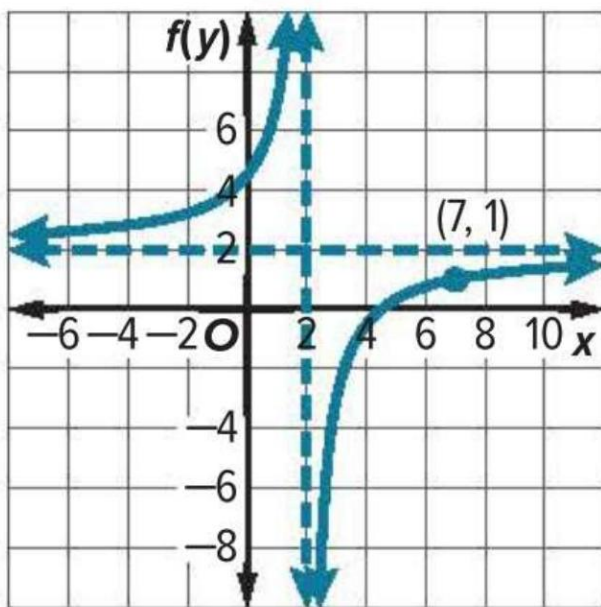
Example 5

Identify the values of a , h , and k . Then write a function for the graph $g(x) = \frac{a}{x-h} + k$.

21.



22.





Learn Graphing Rational Functions with Vertical and Horizontal Asymptotes

A **rational function** has an equation of the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$.

Key Concept • Vertical and Horizontal Asymptotes

If $f(x) = \frac{a(x)}{b(x)}$, $a(x)$ and $b(x)$ are polynomial functions with no common

factors other than 1, and $b(x) \neq 0$, then:

- $f(x)$ has a vertical asymptote whenever $b(x) = 0$.
- $f(x)$ has at most one horizontal asymptote.
 - If the degree of $a(x)$ is greater than the degree of $b(x)$, there is no horizontal asymptote.
 - If the degree of $a(x)$ is less than the degree of $b(x)$, the horizontal asymptote is the line $y = 0$.
 - If the degree of $a(x)$ equals the degree of $b(x)$, the horizontal asymptote is the line $y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$.

Learn Graphing Rational Functions with Oblique Asymptotes

An **oblique asymptote**, or slant asymptote, is neither horizontal nor vertical.

Key Concept • Oblique Asymptotes

If $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions with no common factors other than 1 and $b(x) \neq 0$, then $f(x)$ has an oblique asymptote if the degree of $a(x)$ minus the degree of $b(x)$ equals 1.

The equation of the asymptote is $f(x) = \frac{a(x)}{b(x)}$ with no remainder.

In some cases, graphs of rational functions may have **point discontinuity**, which looks like a hole in the graph. This is because the function is undefined at that point. If the original function is undefined for $x = a$ but the related rational expression of the function in simplest form is defined for $x = a$, then there is a point discontinuity or hole in the graph at $x = a$.

Key Concept • Point Discontinuity

If $f(x) = \frac{a(x)}{b(x)}$, $b(x) \neq 0$, and $x - c$ is a factor of both $a(x)$ and $b(x)$, then there is a point discontinuity at $x = c$.



Examples: 11 - 22 Pages : 344

Example 4 Graph with Oblique AsymptotesConsider $f(x) = \frac{x^2 + 2x + 1}{x + 4}$.**Part A** Find the zeros.

$$x^2 + 2x + 1 = 0$$

Set $a(x) = 0$.

$$(x + 1)^2 = 0$$

Factor.

$$x + 1 = 0$$

Take the square root of each side.

$$x = -1$$

Subtract 1 from each side.

There is a zero at $x = -1$.**Part B** Find the asymptotes.

Find the vertical asymptote.

$$x + 4 = 0$$

Set $b(x) = 0$.

$$x = -4$$

Subtract 4 from each side.

There is a vertical asymptote at $x = -4$.**Part B** Find the asymptotes.

Find the vertical asymptote.

$$x + 4 = 0$$

Set $b(x) = 0$.

$$x = -4$$

Subtract 4 from each side.

There is a vertical asymptote at $x = -4$.

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

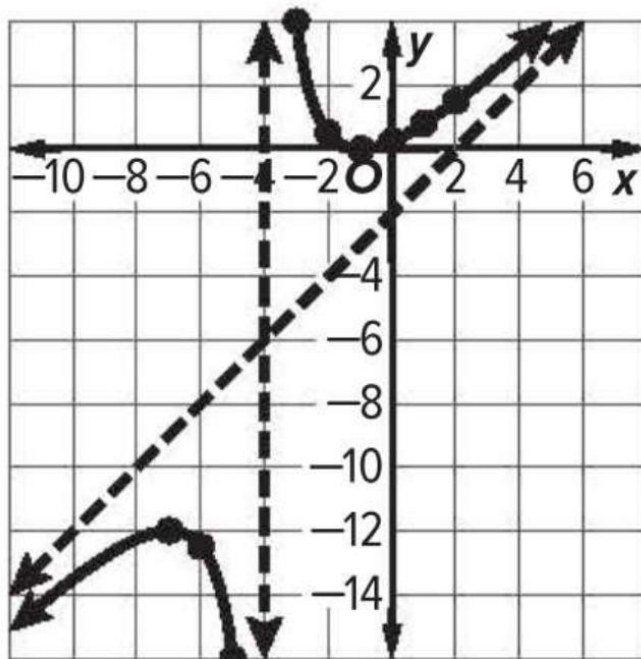
The difference between the degree of the numerator and the degree of the denominator is 1, so there is an oblique asymptote. To find the oblique asymptote, divide the numerator by the denominator.

$$\begin{array}{r}
 x - 2 \\
 x + 4 \overline{) x^2 + 2x + 1} \\
 \underline{(-)x^2 + 4x} \\
 -2x + 1 \\
 \underline{(-)-2x - 8} \\
 9
 \end{array}$$

The equation of the asymptote is the quotient excluding the remainder. So, there is an oblique asymptote at $f(x) = x - 2$.**Part C** Graph $f(x) = \frac{x^2 + 2x + 1}{x + 4}$.

Graph the asymptotes. Then, make a table of values and graph the ordered pairs.

x	$f(x)$
-7	-12
-6	-12.5
-5	-16
-3	4
-2	0.5
-1	0
0	0.25
1	0.8
2	1.5





Check

Find the asymptotes of $g(x) = \frac{2x^2 + 3x + 2}{2x + 1}$.

Then graph the function.

There is a vertical asymptote at $x = \underline{\hspace{1cm}} ? \underline{\hspace{1cm}}$.

There is an oblique asymptote at $f(x) = x + \underline{\hspace{1cm}} ? \underline{\hspace{1cm}}$.

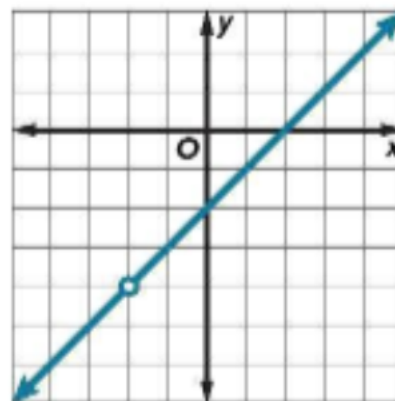
Example 5 Graph with Point Discontinuity

Graph $f(x) = \frac{x^2 - 4}{x + 2}$. Find the point discontinuity.

Notice that $\frac{x^2 - 4}{x + 2} = \frac{(x + 2)(x - 2)}{x + 2}$ or $x - 2$.

However, because the denominator of the original function cannot be 0, there is a discontinuity at $x + 2 = 0$ or $x = -2$.

Therefore, the graph of $f(x) = \frac{x^2 - 4}{x + 2}$ is the graph of $f(x) = x - 2$ with a hole or point discontinuity at $x = -2$.



Check

Consider $f(x) = \frac{(x + 3)^2}{x + 3}$. Find the point discontinuity. Then graph the function.

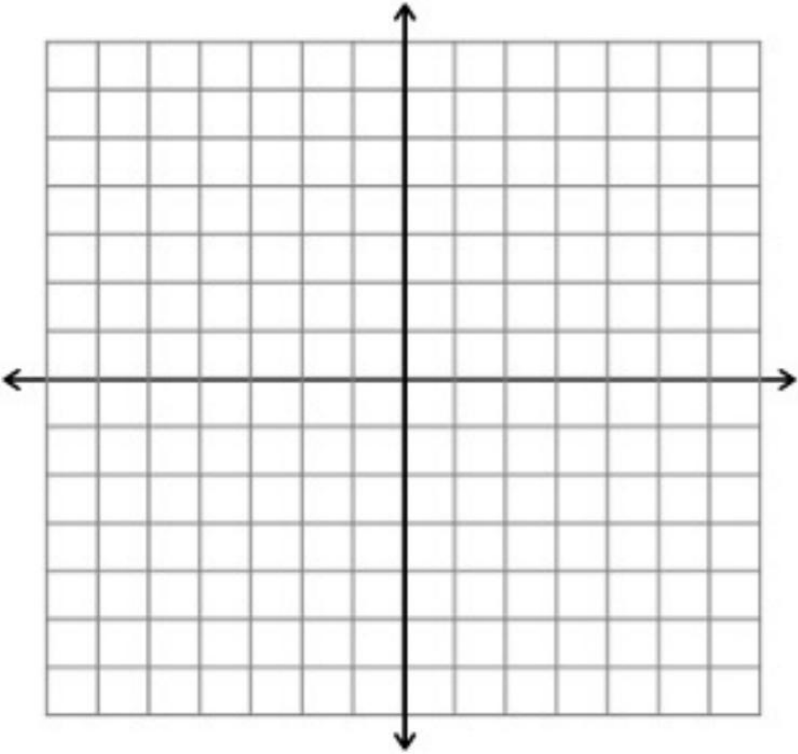
$f(x) = \frac{(x + 3)^2}{x + 3}$ has a point discontinuity at $x = \underline{\hspace{1cm}} ? \underline{\hspace{1cm}}$.



Example 4

Find the zeros and asymptotes of each function. Then graph each function.

11. $f(x) = \frac{(x - 4)^2}{x + 2}$

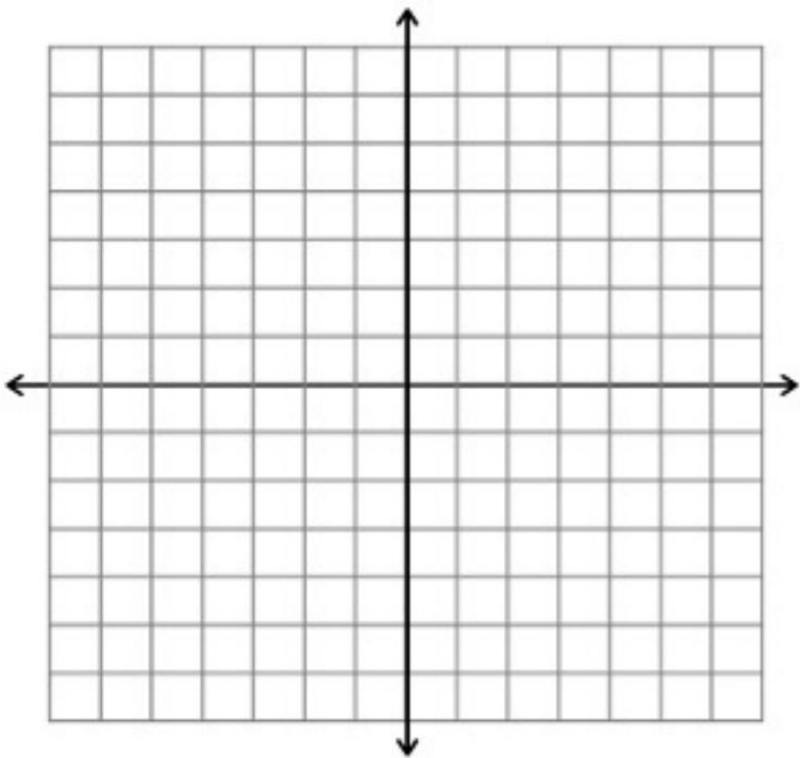




Example 4

Find the zeros and asymptotes of each function. Then graph each function.

12. $f(x) = \frac{(x + 3)^2}{x - 5}$

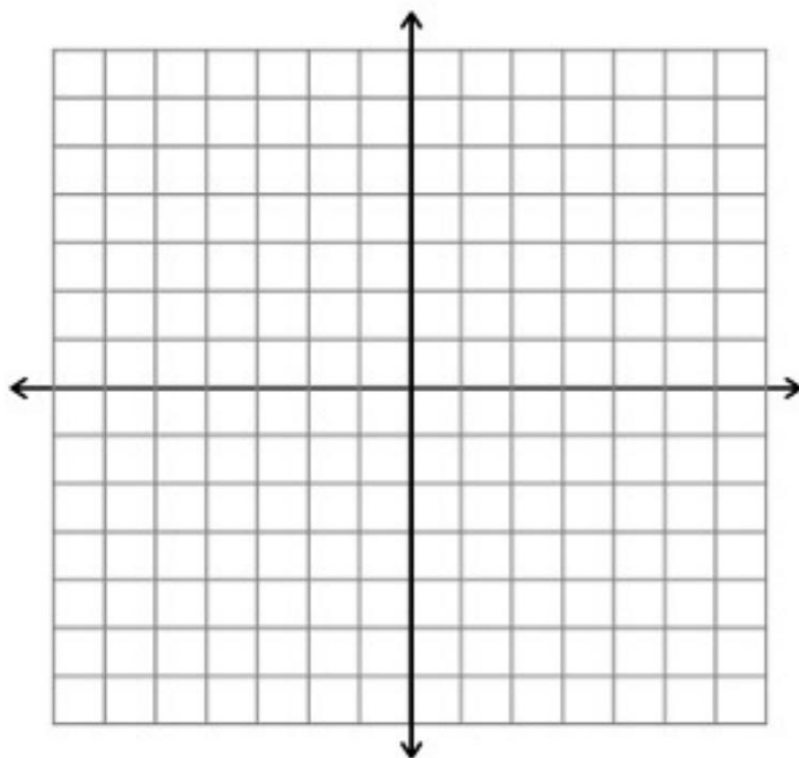




Examples: 11 - 22 Pages : 344

Example 4**Find the zeros and asymptotes of each function. Then graph each function.**

13. $f(x) = \frac{6x^2 + 4x + 2}{x + 2}$

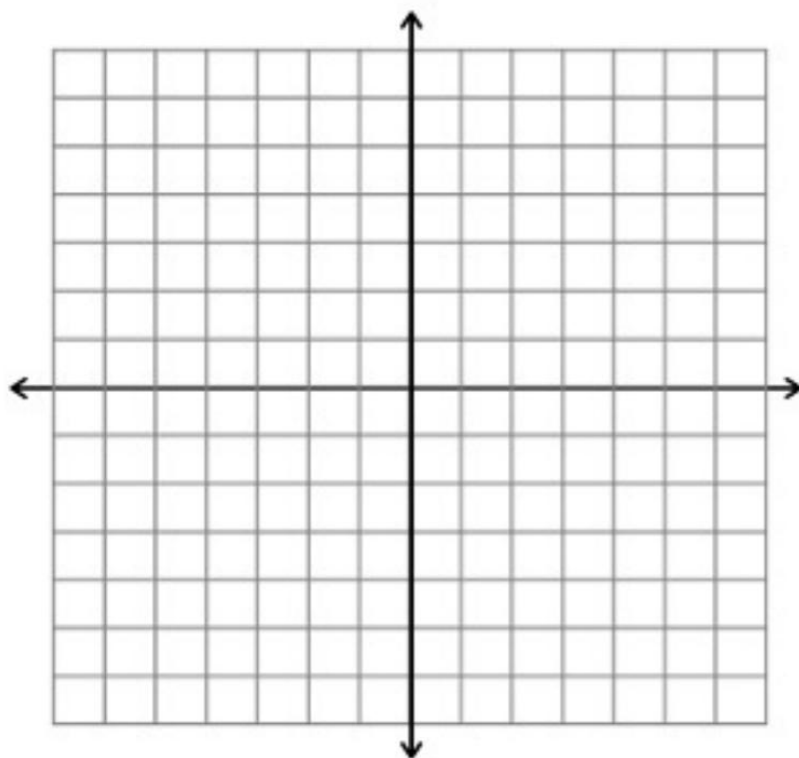




Examples: 11 - 22 Pages : 344

Example 4**Find the zeros and asymptotes of each function. Then graph each function.**

14. $f(x) = \frac{2x^2 + 7x}{x - 2}$

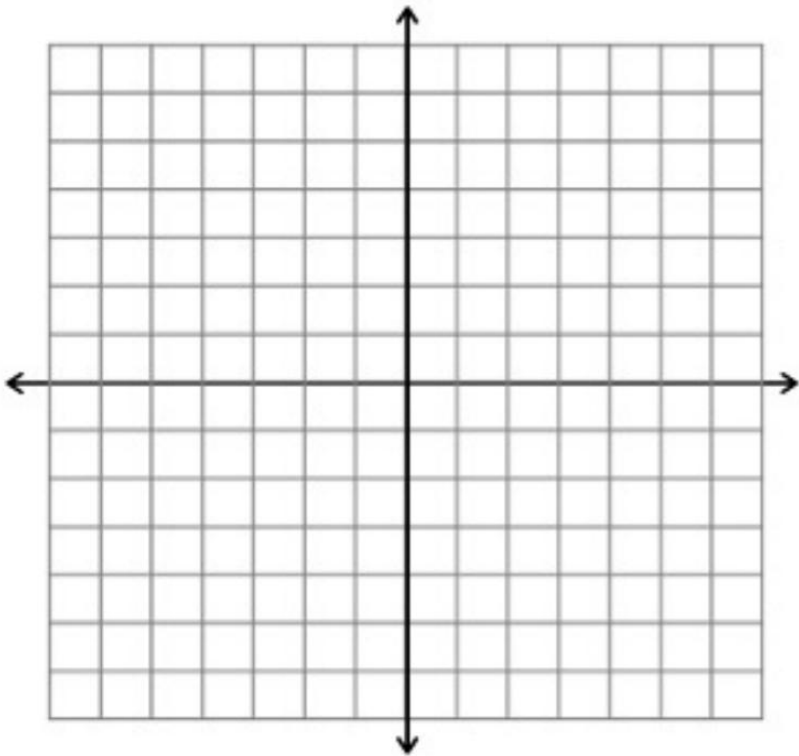




Example 4

Find the zeros and asymptotes of each function. Then graph each function.

15. $f(x) = \frac{3x^2 + 8}{2x - 1}$

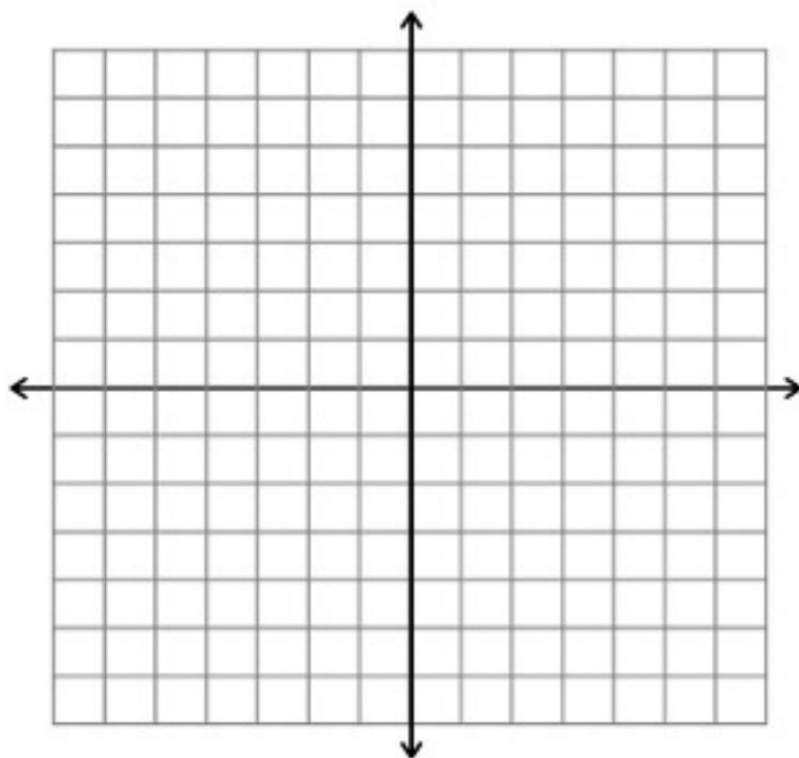




Examples: 11 - 22 Pages : 344

Example 4**Find the zeros and asymptotes of each function. Then graph each function.**

16. $f(x) = \frac{2x^2 + 5}{3x + 4}$

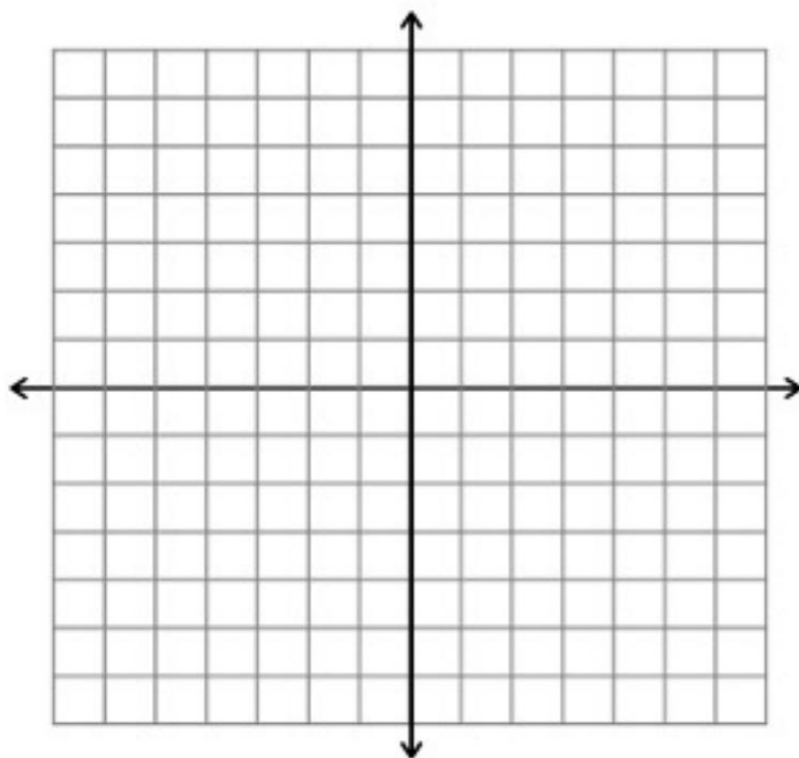




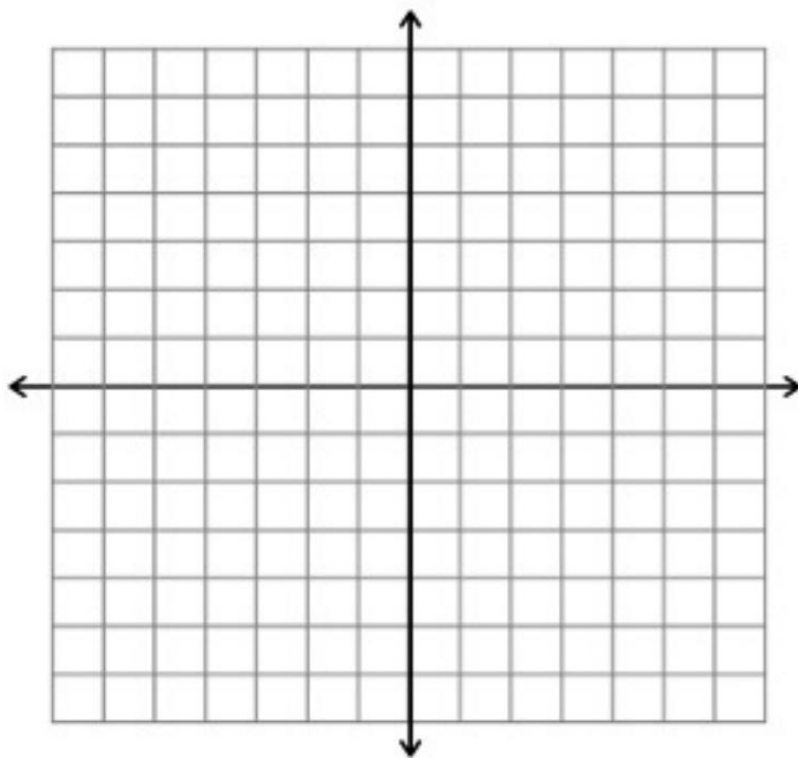
Example 5

Graph each function. Find the point discontinuity.

17. $f(x) = \frac{x^2 - 2x - 8}{x - 4}$



18. $f(x) = \frac{x^2 + 4x - 12}{x - 2}$

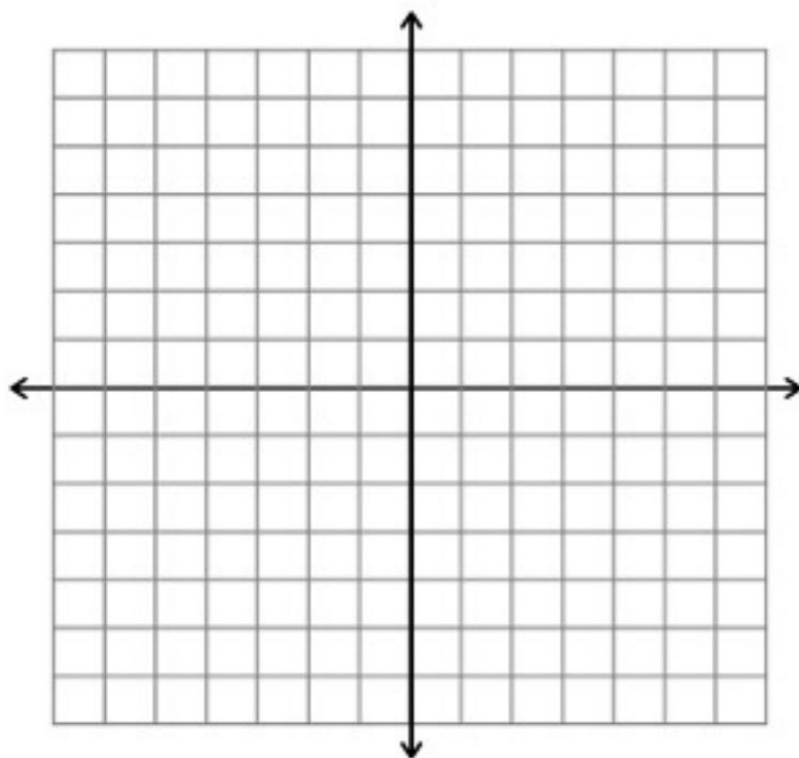




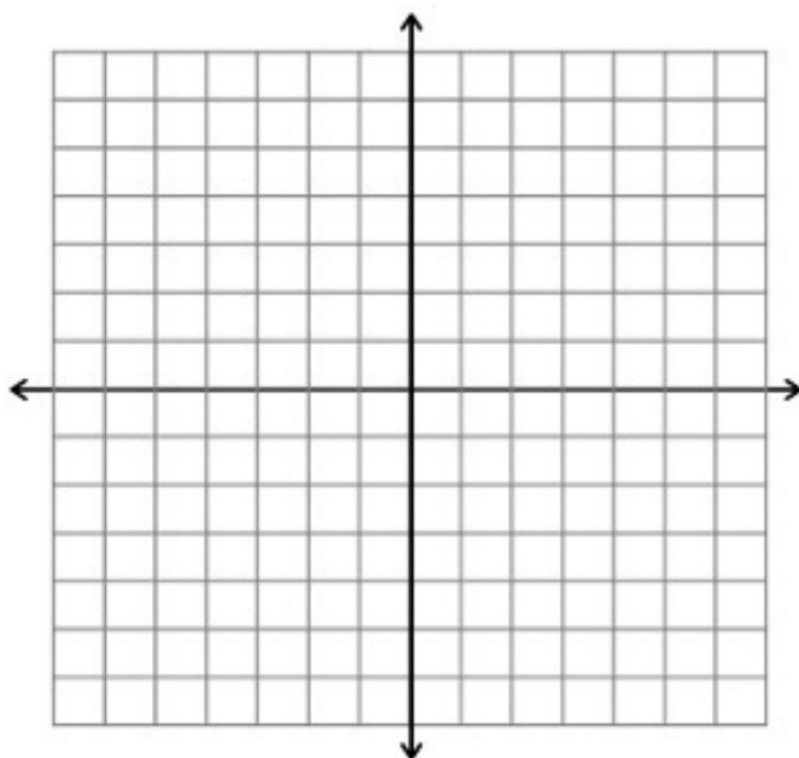
Example 5

Graph each function. Find the point discontinuity.

19. $f(x) = \frac{x^2 - 25}{x + 5}$



20. $f(x) = \frac{x^2 - 64}{x - 8}$

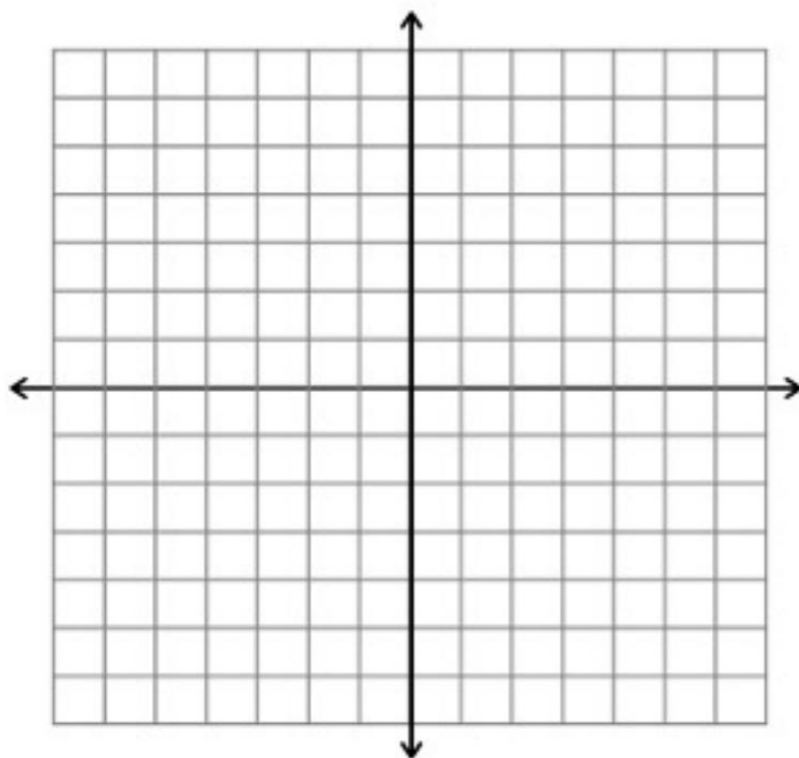




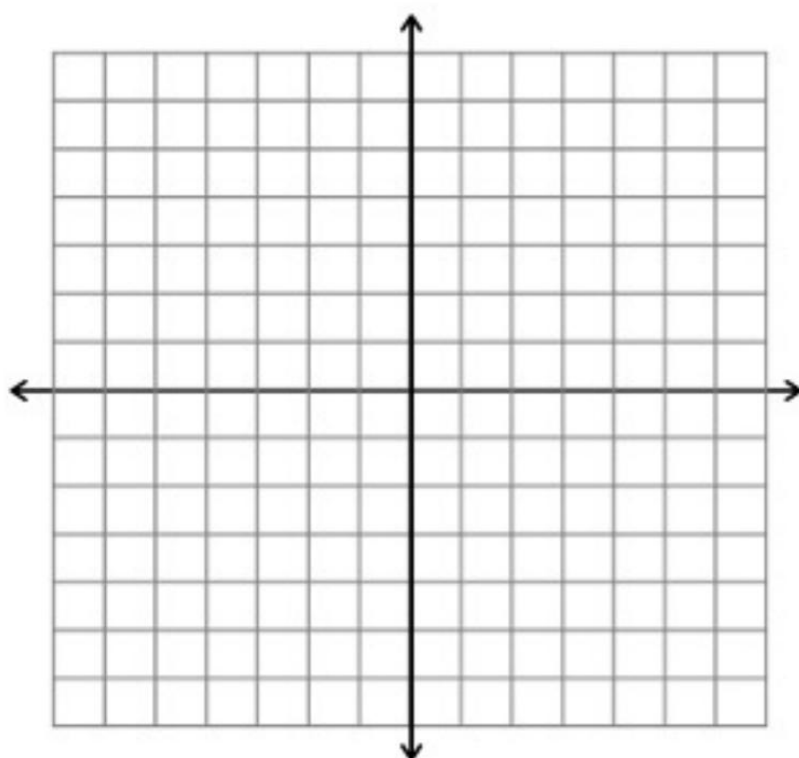
Example 5

Graph each function. Find the point discontinuity.

21. $f(x) = \frac{(x - 4)(x^2 - 4)}{x^2 - 6x + 8}$



22. $f(x) = \frac{(x + 5)(x^2 + 2x - 3)}{x^2 + 8x + 15}$





Learn Direct Variation and Joint Variation

Two quantities x and y are related by a **direct variation** if y is equal to a constant k times x . The constant k is called the **constant of variation**.

Key Concept • Direct Variation

y varies directly as x if there is some nonzero constant k such that $y = kx$.

The graph of a direct variation function, such as $f(x) = 4x$, is a line through the origin. A direct variation is a special case of an equation in slope-intercept form, $y = mx + b$, where $m = k$ and $b = 0$. The slope of a direct variation equation is its constant of variation.

If you know that y varies directly as x and one set of values, you can use a proportion to find another set of corresponding values.

Joint variation occurs when one quantity varies directly as the product of two or more other quantities.

Key Concept • Joint Variation

y varies jointly as x and z if there is some nonzero constant k such that $y = kxz$.

Example 2 Joint Variation

Suppose y varies jointly as x and z . Find y when $x = 4$ and $z = -3$, if $y = -15$ when $x = -6$ and $z = 1$.

Use a proportion that relates the values.

$$\frac{y_1}{x_1 z_1} = \frac{y_2}{x_2 z_2} \quad \text{Joint variation}$$

$$\frac{y_1}{4(-3)} = \frac{-15}{-6(1)} \quad x_1 = 4, z_1 = -3, y_2 = -15, x_2 = -6, \text{ and } z_2 = 1$$

$$\frac{y_1}{-12} = \frac{-15}{-6} \quad \text{Simplify.}$$

$$-12\left(\frac{y_1}{-12}\right) = -12\left(\frac{-15}{-6}\right) \quad \text{Divide each side by } -6.$$

$$y_1 = -30 \quad \text{Simplify.}$$

Check

Suppose y varies jointly as x and z . Find y when $x = -4$ and $z = -5$, if $y = -135$ when $x = 5$ and $z = -9$. $y_2 = 60$

Examples: 7 – 12 Pages : 351

**Example 2**

If a varies jointly as b and c , find a when $b = 4$ and $c = -3$.

7. $a = -96$ when $b = 3$ and $c = -8$

8. $a = -60$ when $b = -5$ and $c = 4$

9. $a = -108$ when $b = 2$ and $c = 9$

**Example 2**

If a varies jointly as b and c , find a when $b = 4$ and $c = -3$.

10. $a = 24$ when $b = 8$ and $c = 12$

11. If y varies jointly as x and z , and $y = 18$ when $x = 2$ and $z = 3$, find y when x is 5 and z is 6.

12. If y varies jointly as x and z , and $y = -16$ when $x = 4$ and $z = 2$, find y when x is -1 and z is 7.

**Example 1****Solve each equation. Check your solutions.**

1.
$$\frac{2x + 3}{x + 1} = \frac{3}{2}$$

2.
$$\frac{-12}{y} = y - 7$$

3.
$$\frac{9}{x - 7} - \frac{7}{x - 6} = \frac{13}{x^2 - 13x + 42}$$

4.
$$\frac{13}{y + 3} - \frac{12}{y + 4} = \frac{18}{y^2 + 7y + 12}$$

Examples: 1 – 12 Pages : 361

**Example 1****Solve each equation. Check your solutions.**

5. $\frac{14}{x-2} - \frac{18}{x+1} = \frac{22}{x^2 - x - 2}$

6. $\frac{2}{a+2} + \frac{10}{a+5} = \frac{36}{a^2 + 7a + 10}$



Example 2

Solve each equation. Check your solutions.

7.
$$\frac{x}{2x-1} + \frac{3}{x+4} = \frac{21}{2x^2 + 7x - 4}$$

8.
$$\frac{2}{y-5} + \frac{y-1}{2y+1} = \frac{2}{2y^2 - 9y - 5}$$

**Example 2**

Solve each equation. Check your solutions.

9.
$$\frac{x - 8}{2x + 2} + \frac{x}{2x + 2} = \frac{2x - 3}{x + 1}$$

10.
$$\frac{12p + 19}{p^2 + 7p + 12} - \frac{3}{p + 3} = \frac{5}{p + 4}$$

**Example 2**

Solve each equation. Check your solutions.

11.
$$\frac{2f}{f^2 - 4} + \frac{1}{f - 2} = \frac{2}{f + 2}$$

12.
$$\frac{8}{t^2 - 9} + \frac{4}{t + 3} = 1$$



Learn Randomness and Bias

Statistics is an area of mathematics that deals with collecting, analyzing, and interpreting data.

A characteristic of a population is called a **parameter**. Because it is not always practical to collect data from a large population, it may be collected from a representative sample. A characteristic of a sample is called a **statistic**.

A **population** consists of all the members of a group of interest. A **bias** is an error that results in a misrepresentation of a population.

Example 2 Identify Sample Bias

TRAFFIC The traffic commission took a random sample of travelers using Route 15 and determined the average daily commute on this road during rush hour. After viewing the results, the commission proposed expanding the road from 2 lanes to 4 lanes. Identify bias and potential interests in the sample.

Route 15 Congestion

Distance: 10.6 miles
Average Commute:
26.3 minutes
Standard Deviation:
4.4 minutes
Sample Size: 10

Step 1 Identify the purpose of the sample.

The purpose is to evaluate the length of the commute in order to determine whether road expansion is necessary.

Step 2 Identify the bias in the sample. While the sample is random, a sample size of 10 is too small to accurately reflect all of the cars that use the road.

Step 3 Identify interests. The bias in the sample size may lead to a sample average that does not reflect the true average commute of the population. This bias could serve the interests of contractors who hope to win the business of expanding the road.

Example 3 Identify Biased Questions

FOOD A restaurant owner wants to determine which kinds of meals she should add to her menu: vegetarian options or more traditional meat-based dishes. She releases a survey to her customers asking the following question: *Do you prefer a plain salad or a delicious steak?* Identify any bias in the question.

Step 1 Identify the purpose of the question. The purpose is to determine which dishes are most popular with customers.

Step 2 Identify the bias in the sample. The question provides descriptions of each kind of food, and the description for the steak favors it over the salad.



Examples 2 and 3

Identify each sample or question as *biased* or *unbiased*. Explain your reasoning.

4. A random sample of eight people is asked to select their favorite food for a survey about Americans' food preferences.
5. Every tenth student at band camp is asked to name his or her favorite band for a survey about the campers.
6. Every fifth person entering a museum is asked to name his or her favorite type of book to read for a survey about reading interests of people in the city.
7. Do you think that the workout facility needs a new treadmill and racquetball court?
8. Which is your favorite type of music, pop, or country?
9. Are you a member of any after-school clubs?
10. Don't you agree that employees should pack their lunch?



Learn Theoretical and Experimental Probability

Recall that the probability of an event is the ratio of the number of outcomes in which a specified event occurs to the total number of possible outcomes.

The **theoretical probability** is what is expected to happen, and the **experimental probability** is calculated using data from an actual experiment.

The Law of Large Numbers states that as a sample size increases, the variation in a data set decreases. So, as the sample size increases, experimental probabilities will more closely resemble their associated probabilities.

Example 1 Find Probabilities

A student tossed a fair eight-sided die 200 times and recorded the results. Find the theoretical and experimental probabilities of rolling an 8.

Theoretical Probability

The theoretical probability is what is expected to happen. Because the die is fair, each side has a $\frac{1}{8}$ chance of being the result. Thus, the theoretical probability is $\frac{1}{8}$, or 12.5%.

Number on Die	Frequency
1	28
2	19
3	24
4	22
5	21
6	18
7	26
8	42

Experimental Probability

The experimental probability is based on the data collected from the experiment. Because the die was thrown 200 times and 42 of those throws landed on 8, the experimental probability is $\frac{42}{200}$, or 21%.

Check

A student tossed a fair die 150 times and recorded the results. Find the theoretical and experimental probabilities of rolling a 5. Write your answer as a percentage rounded to the nearest tenth, if necessary.

Number	Frequency
1	51
2	16
3	23
4	19
5	24
6	17

The theoretical probability of rolling a 5 is $\frac{?}{6}$ %.

The experimental probability of rolling a 5 is $\frac{?}{150}$ %.

Example 1

1. A student spun a spinner with 4 equal sections 100 times and recorded the results.
- a. Find the theoretical probability of spinning blue. Write your answer as a percentage rounded to the nearest tenth, if necessary.

Spinner Section	Frequency
Red	35
Blue	38
Green	13
Yellow	14

- b. Find the experimental probability of spinning blue. Write your answer as a percentage rounded to the nearest tenth, if necessary.

2. A student flipped a coin 125 times and recorded the results.
- a. Find the theoretical probability of the coin landing on heads. Write your answer as a percentage rounded to the nearest tenth, if necessary.

Coin Result	Frequency
Heads	73
Tails	52

- b. Find the experimental probability of the coin landing on heads. Write your answer as a percentage rounded to the nearest tenth, if necessary.

3. A fair 6-sided die is rolled 150 times.
- a. Find the theoretical probability of rolling a 3. Write your answer as a percentage rounded to the nearest tenth, if necessary.

Number on Die	Frequency
1	32
2	18
3	27
4	16
5	33
6	24

- b. Find the experimental probability of rolling a 3. Write your answer as a percentage rounded to the nearest tenth, if necessary.

4. **MULTIPLE CHOICE** Leonard randomly selected a card from a standard deck of playing cards, recorded the suit, and returned the card. He followed this set of steps 120 times. The results are shown.

Suit	Frequency
Heart	28
Diamond	37
Spade	34
Club	21

Which statement about the results is true?

(Lesson 8-2)

- A. The theoretical probability of selecting a heart is less than the experimental probability of selecting a heart.
- B. The theoretical probability of selecting a diamond is less than the experimental probability of selecting a diamond.
- C. The experimental probability of selecting a club is greater than the theoretical probability of selecting a club.
- D. The experimental probability of selecting a heart is equal to the experimental probability of selecting a diamond.



Learn Describing Distributions

Measures of center and measures of spread or variation are often called **descriptive statistics**. To analyze data sets, we often look at their **distribution**, which is a graph or table that shows the theoretical frequency of each possible data value.

Symmetric Distribution	Negatively Skewed Distribution	Positively Skewed Distribution
The mean and median are approximately equal, and the data are approximately symmetric about the center.	Typically, the median is greater than the mean and less data is on the left side of the graph.	Typically, the mean is greater than the median and less data is on the right side of the graph.
 6 8 10 12 14 16 18 20 22 mean = 13.1 median = 13.2	 6 8 10 12 14 16 18 20 22 mean = 14.8 median = 16.7	 6 8 10 12 14 16 18 20 22 mean = 12.7 median = 11.6

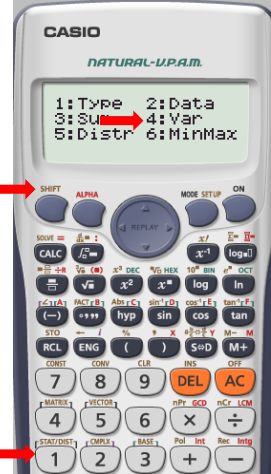
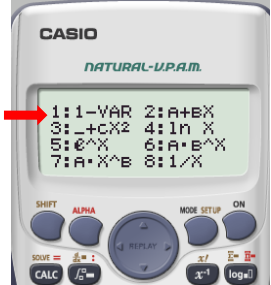
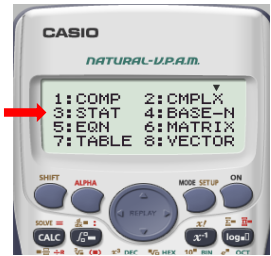
A bimodal distribution has two distinct peaks. When a distribution is symmetric, the mean accurately reflects the center of the data. However, when a distribution is skewed, the mean is not as reliable. **Outliers**, which are extremely high or low values when compared to the rest of the values in a data set, have a strong effect on the mean of a data set. When a distribution is skewed or has outliers, the mean lies away from the majority of the data toward the tail and may not accurately describe of the data.

In a symmetric distribution, the data are distributed symmetrically about the center. The center is measured with the mean, and the spread is measured with **variance** and **standard deviation**, which describe how the data deviate from the mean.

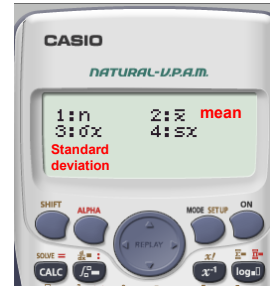
(No need for graphing calculator)

Example: 1 Page : 388 | Examples : 1 , 2 Page : 391 | Example : 7 Page 413

Using Calculator :



Shift + 1 , then 4



Example 1 Find a Standard Deviation

TRACK A coach recorded the times of each track member for a 400-meter race. Find and interpret the standard deviation of the data.

400m Race Times (seconds)	
57.1	55.9
59.3	54.9
54.6	50.3
55.2	53.5

Step 1 Find the mean, μ .

$$\mu = \frac{57.1 + 59.3 + 54.6 + 55.2 + 55.9 + 54.9 + 50.3 + 53.5}{8}$$
$$= 55.1$$

The mean running time for the team is 55.1 seconds.

Step 2 Find the squares of the differences, $(\mu - x_n)^2$.

$(55.1 - 57.1)^2 = 4.00$	$(55.1 - 55.9)^2 = 0.64$
$(55.1 - 59.3)^2 = 17.64$	$(55.1 - 54.9)^2 = 0.04$
$(55.1 - 54.6)^2 = 0.25$	$(55.1 - 50.3)^2 = 23.04$
$(55.1 - 55.2)^2 = 0.01$	$(55.1 - 53.5)^2 = 2.56$

Step 3 Find the sum.

Find the sum of the values from **Step 2**.

$$4.00 + 17.64 + 0.25 + 0.01 + 0.64 + 0.04 + 23.04 + 2.56$$
$$= 48.18$$

Step 4 Divide by the number of values.

Divide the sum from **Step 3** by the number of running times.

$$\frac{48.18}{8} = 6.0225$$
 This is the variance.

Step 5 Take the square root of the variance.

$$\sqrt{6.0225} \approx 2.45$$
 This is the standard deviation.

The standard deviation is about 2.45. This is small compared with the run times, which means that the majority of the team members' times are close to the mean of 55.1 seconds, and almost all of the times will likely fall within 2 standard deviations of the mean, or between 50.2 and 60.0 seconds.

Check

How do you know that your solution is reasonable? (*Hint:* The majority of the data should be within one standard deviation of the mean, and almost all of the data should be within two.)

Over half of the times are within one standard deviation of the mean, which is the interval [52.65, 57.55] All of the data are within two standard deviations of the mean, or the interval [50.2, 60].

Check

BREAKFAST The manager of a cafeteria tracked the number of waffles that were sold during breakfast every day. Find and interpret the standard deviation of the data. Round your answers to the nearest tenth.

Number Sold	
36	42
48	40
44	56
57	53

Part A The standard deviation is approximately ? waffles.

Part B The majority of the sales were between ? and ? waffles.

(No need for graphing calculator)



Example 3 Compare Distributions

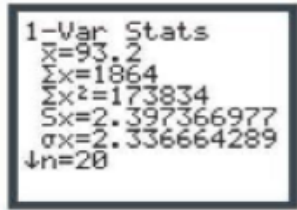
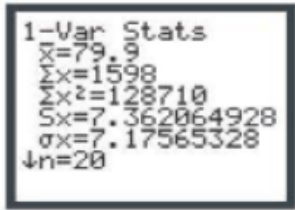
COURSES The exam scores for two Algebra 2 classes are shown.
Use the mean and standard deviation to compare the distributions.

Mr. Jackson's Algebra 2 Class					Ms. Hettrick's Algebra 2 Class				
95	96	95	94	90	73	86	83	79	78
98	92	91	94	92	66	82	79	83	77
92	92	94	93	93	70	85	84	76	91
96	95	88	94	90	68	73	89	92	84

Part A Find the mean and standard deviation of the distributions.

Measures of center and spread are used to compare symmetric distributions of data.

Using a TI-84 calculator, enter the data for Mr. Jackson's class in **L1** and the data for Ms. Hettrick's class in **L2**. To display the statistics, press **stat**, access the **CALC** menu, and select **1-Var Stats**. For Mr. Jackson's class, enter **L1** and press **enter** again. For Ms. Hettrick's class, enter **L2** and press **enter** again instead.

Mr. Jackson's Class	Ms. Hettrick's Class
	

Part B Compare the distributions.

Mr. Jackson's class has a mean score of 93.2 and standard deviation of 2.3. Ms. Hettrick's class has a mean score of 79.9 and a standard deviation of 7.2. This means that Mr. Jackson's class scored higher on average, and that the scores in Ms. Hettrick's class are more spread out from the mean.

Check

TRAINING Twenty trainees were provided with a 100-point pretest before their training began. Once their training was complete, they were each given a 100-point posttest that was very similar to the pretest. Use the mean and standard deviation to compare the distributions.

Pretest Scores					Posttest Scores				
80	60	85	75	90	73	88	84	88	85
70	65	70	75	50	83	93	94	94	93
55	65	80	95	65	93	97	99	96	81
90	60	85	70	75	89	96	93	93	93

The average score was higher for the _____. The scores were more varied in the _____.



Example: 1 Page : 388 | Examples : 1 , 2 Page : 391 | Example : 7 Page 413

Example 1

1. **BARBER** A barber wants to purchase new professional shears from a Web site. The prices of all of the shears are shown in the table. Use the standard deviation formula to find and interpret the standard deviation of the data. Round your answers to the nearest cent.

Cost of Shears (\$)			
50	165	55	79
84	68	38	42

2. **READING** Ms. Sanchez keeps track of the total number of books each student in the book club reads during the school year. Use the standard deviation formula to find and interpret the standard deviation of the data. Round your answers to the nearest tenth.

Books Read		
9	6	12
8	9	14
10	13	8

7. **MULTI-SELECT** Zain is analyzing two sets of data. (Lesson 8-3)

Set A				
35	28	35	33	32
32	31	30	36	49
36	33	29	34	37

Set B				
22	26	26	24	27
38	35	31	22	30
24	33	30	25	38

- Select all the true statements about each data set.
- A. Data set A has a higher average value.
 - B. Data set B has a higher average value.
 - C. Data set A is more varied.
 - D. Data set B is more varied.

Example 6 Use z-Values to Locate Position

Find z if $X = 24$, $\mu = 19$, and $\sigma = 3.8$.

$$z = \frac{X - \mu}{\sigma}$$
 Formula for z-value

$$z = \frac{24 - 19}{3.8}$$
 $X = 24, \mu = 19$, and $\sigma = 3.8$

$$z = \frac{5}{3.8}$$
 Subtract.

$$z \approx 1.316$$
 Divide.

Check

Find z if $X = 106.3$, $\mu = 88.8$, and $\sigma = 9.6$. Round your answer to the nearest thousandth. $z = \underline{\hspace{1cm}}?$

Example 7 Find Area Under the Standard Normal Curve by Using a Table

Use a Standard Normal Distribution Table to find the area under the normal curve within the interval $z < -1.18$.

Step 1 Identify the probability for the left endpoint, $z = -1.18$.

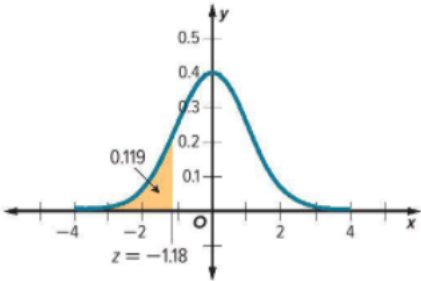
Using a Standard Normal Distribution Table, move down the left column until you reach the correct tenths place, -1.1 . Then move across the row until you reach the hundredths place, 0.08 .

z	0.00	...	0.08	0.09
-3.4	0.0003	...	0.0003	0.0002
⋮	...	...	...	...
-1.1	0.1357	...	0.1190	0.1170

The probability associated with $z = -1.18$ is 0.1190.

Step 2 Identify the area under the curve within the interval.

As shown in the graph, 0.119 represents the area under the curve to the left of $z = -1.18$.



Check

Use a table to find the area under the normal curve within $z < 0.19$. Round your answer to the nearest ten thousandth.

The area is $\underline{\hspace{1cm}}?$

Example 8 Find Area Under the Standard Normal Curve by Using a Calculator

INTERNET TRAFFIC The number of daily hits to a local news Web site is normally distributed with $\mu = 98,452$ hits and $\sigma = 10,325$ hits. Find the probability that the Web site will get at least 100,000 hits on a given day, $P(X > 100,000)$.



Step 1 Find the corresponding z-value for $X = 100,000$.

$$z = \frac{X - \mu}{\sigma}$$
 Formula for z-value

$$z = \frac{100,000 - 98,452}{10,325}$$
 $X = 100,000, \mu = 98,452$, and $\sigma = 10,325$

$$z \approx 0.150$$
 Simplify.

Step 2 Find the probability.

The area under the curve when $z > 4$ is negligible. So $z = 4$ can be used as an upper bound for finding area. You can use a graphing calculator to find the area between $z = 0.150$ and $z = 4$.

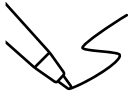
Press **2nd** **[DISTR]** and select **normalcdf**. Enter the interval and press **enter** to display the area.

The area is 0.44. Therefore, the probability of the Web site getting at least 100,000 hits in one day is 44.0%.

```
normalcdf(0.150,
4)
.4403506018
```

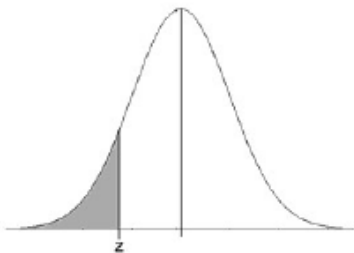
Check

CUSTOMER SERVICE The length of time that a customer spends waiting to be connected to a customer service representative is normally distributed with a mean of 21.3 seconds and a standard deviation of 3.8 seconds. Find the probability that a wait time will be greater than 30 seconds. $\underline{\hspace{1cm}}?$ %.



Standard Normal Cumulative Probability Table

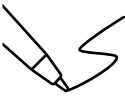
Cumulative probabilities for NEGATIVE z-values are shown in the following table:



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

A normal distribution curve is shown. The area under the curve to the left of a point labeled z on the horizontal axis is shaded gray. The vertical axis passes through the peak of the curve.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	



Example 6

Find the z -value for each standard normal distribution.

8. $\sigma = 9.8$, $X = 55.4$, and $\mu = 68.34$

9. $\sigma = 11.6$, $X = 42.80$, and $\mu = 68.2$

10. $\sigma = 11.9$, $X = 119.2$, and $\mu = 112.4$

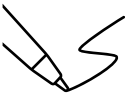
Example 7

Use a table to find the area under the normal curve for each interval.

11. $z > 0.58$

12. $z < -1.56$

13. $-2.29 < z < 2.76$



Example 8

14. TESTING The scores on a test administered to prospective employees are normally distributed with a mean of 100 and a standard deviation of 12.3.

- a. What percent of the scores are between 70 and 80?
- b. What percent of the scores are over 115?
- c. If 75 people take the test, how many would you expect to score lower than 75?



Learn Degrees and Radians

Angles can also be measured in units called **radians**, which are based on arc length. A radian is the measure of an angle θ in standard position with a terminal side that intercepts an arc with the same length as the radius of the circle. The circumference of a circle is $2\pi r$ and an angle that measures 1 radian has an arc length of r . So, one complete revolution around a circle equals 2π radians.

Key Concept • Convert Between Degrees and Radians

Degrees to Radians	Radians to Degrees
To convert a degree measure to radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180^\circ}$.	To convert a radian measure to degrees, multiply the number of radians by $\frac{180^\circ}{\pi \text{ radians}}$.

There are special angles with equivalent degree and radian measures that may be helpful to memorize. Other special angles are multiples of these angles.

$$30^\circ = \frac{\pi}{6} \text{ radians}$$

$$45^\circ = \frac{\pi}{4} \text{ radians}$$

$$60^\circ = \frac{\pi}{3} \text{ radians}$$

$$90^\circ = \frac{\pi}{2} \text{ radians}$$

Example 4 Convert Degrees to Radians

Rewrite -100° in radians.

$$-100^\circ = -100^\circ \cdot \frac{\pi \text{ radians}}{180^\circ}$$

Multiply by conversion factor.

$$= \frac{-100\pi}{180} \text{ or } -\frac{5\pi}{9} \text{ radians}$$

Simplify.

Check

Rewrite -220° in radians. Write the solution as a fraction in simplest form.

_____ radians

Example 5 Convert Radians to Degrees

Rewrite $\frac{11\pi}{4}$ in degrees.

$$\frac{11\pi}{4} = \frac{11\pi}{4} \text{ radians} \cdot \frac{180^\circ}{\pi \text{ radians}}$$

Multiply by conversion factor.

$$= \frac{11\pi}{4} \text{ radians} \cdot \frac{180^\circ}{\pi \text{ radians}}$$

Cancel units.

$$= \frac{1980^\circ}{4} \text{ or } 495^\circ$$

Simplify.



A **central angle of a circle** has its vertex at the center of the circle and sides that are radii. Central angles intercept an arc of the circle. The endpoints of the intercepted arc lie on the rays of the central angle. If you know the measure of a central angle and the radius of the circle, you can find the length of the arc that is intercepted by the angle.

Key Concept • Arc Length

Words: For a circle with radius r and central angle θ (in radians), the arc length s equals the product of r and θ .

Symbols: $s = r\theta$

 Example 6 Find Arc Length

TRAFFIC A traffic circle, or roundabout, is a circular roadway at the intersection of two or more streets that allows cars to travel through more continuously than a traffic light or stop sign. The diameter of a traffic circle is 160 feet. How far does a car travel in the roundabout if it goes three-fourths of the way around?

Step 1 Find the central angle in radians.

$$\theta = \frac{3}{4} \cdot 2\pi \text{ or } \frac{3\pi}{2}$$

The angle is $\frac{3}{4}$ of a rotation.

Step 2 Find the arc length.

$$\begin{aligned} s &= r\theta \\ &= 80 \cdot \frac{3\pi}{2} \\ &= 120\pi \\ &\approx 376.99 \end{aligned}$$

Formula for arc length

$$r = 0.5d = 0.5(160) \text{ or } 80 \text{ and } \theta = \frac{3\pi}{2}$$

Multiply.

Use a calculator.

A car that travels three-fourths of the way around the traffic circle will travel about 377 feet.

Check

SURVEYING If a surveyor’s wheel with a radius of 15 inches completes $\frac{13}{20}$ of a rotation, what is the total distance traveled in feet? Round to the nearest hundredth if necessary.

 ? ft



Mixed Exercises

REGULARITY Rewrite each degree measure in radians and each radian measure in degrees.

54. 18°

55. 6°

56. -72°

57. -820°

58. 4π

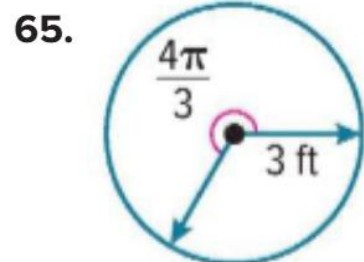
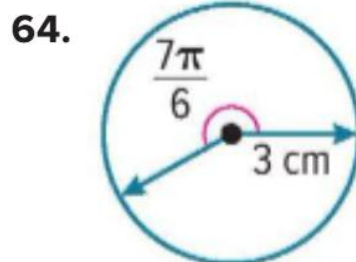
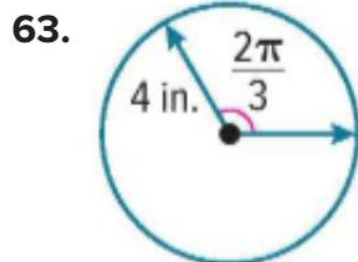
59. $\frac{5\pi}{2}$

60. $-\frac{9\pi}{2}$

61. $-\frac{7\pi}{12}$

62. -270°

Find the length of each arc. Round to the nearest tenth.





Learn Trigonometric Functions with Reference Angles

For a nonquadrantal angle θ in standard position, its **reference angle** is the acute angle θ' formed by the terminal side and the x-axis. The rules for finding the measures of reference angles vary depending on the quadrant in which the terminal side is located.

If the terminal side is in:

- Quadrant I, then $\theta' = \theta$.
- Quadrant II, then $\theta' = 180^\circ - \theta$ or $\theta' = \pi - \theta$.
- Quadrant III, then $\theta' = \theta - 180^\circ$ or $\theta' = \theta - \pi$.
- Quadrant IV, then $\theta' = 360^\circ - \theta$ or $\theta' = 2\pi - \theta$.

You can use reference angles to evaluate trigonometric functions for any angle θ . The sign of the function is determined by the quadrant in which the terminal side of θ lies.

Reciprocal Identities

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

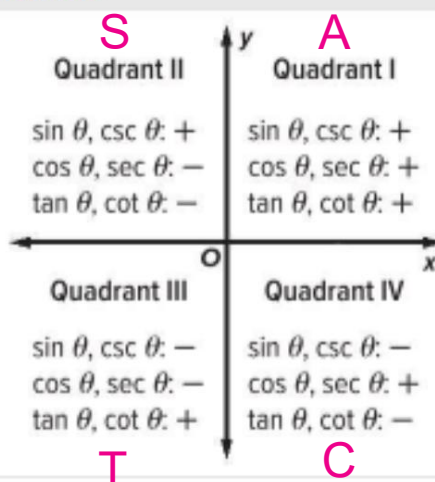
$$\cot \theta = \frac{1}{\tan \theta}$$

Key Concept • Evaluating Trigonometric Functions

Step 1 Find the measure of the reference angle θ' .

Step 2 Evaluate the trigonometric function for θ' .

Step 3 Determine the sign of the trigonometric function values. Use the quadrant in which the terminal side of θ lies.



All Students Take Calculus

Example 6 Use a Reference Angle to Find a Trigonometric Value

Find the exact value of $\tan \frac{7\pi}{4}$.

Part A Find the measure of the reference angle.

The terminal side of $\frac{7\pi}{4}$ lies in Quadrant IV.

$$\begin{aligned}\theta' &= 2\pi - \theta \\ &= 2\pi - \frac{7\pi}{4} \text{ or } \frac{\pi}{4}\end{aligned}$$

Find the reference angle θ' .

$$\theta = \frac{7\pi}{4}$$

Part B Use the reference angle to evaluate the function.

$$\begin{aligned}\tan \frac{7\pi}{4} &= -\tan \frac{\pi}{4} \\ &= -\tan 45^\circ \\ &= -1\end{aligned}$$

The tangent function is negative in Quadrant IV.

$$\frac{\pi}{4} \text{ radians} = 45^\circ$$

$$\tan 45^\circ = 1$$

Check

Find the exact value of $\sec 225^\circ$.

**Example 6****PRECISION** Find the exact value of each trigonometric function.

40. $\tan 330^\circ$

41. $\cos \left(-\frac{11\pi}{4} \right)$

42. $\cot 30^\circ$

43. $\csc \frac{\pi}{4}$

44. $\sin (-150^\circ)$

45. $\tan \left(-\frac{\pi}{4} \right)$

Reciprocal Identities

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$



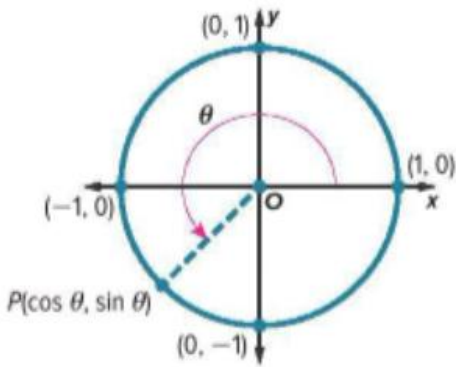
Key Concept • Functions on a Unit Circle

Words: If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$.

Symbols: $P(x, y) = P(\cos \theta, \sin \theta)$

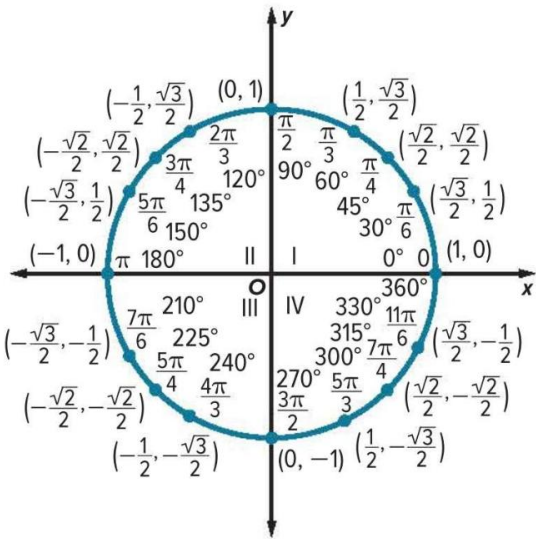
Example:

If $\theta = \frac{5\pi}{4}$,
 $P(x, y) = P\left(\cos \frac{5\pi}{4}, \sin \frac{5\pi}{4}\right)$.



Study Tip

Sine and Cosine To help you remember that for (x, y) on the unit circle, $x = \cos \theta$ and $y = \sin \theta$, notice that alphabetically x comes before y and *cosine* comes before *sine*.



Example 1 Find Sine and Cosine Given a Point on the Unit Circle

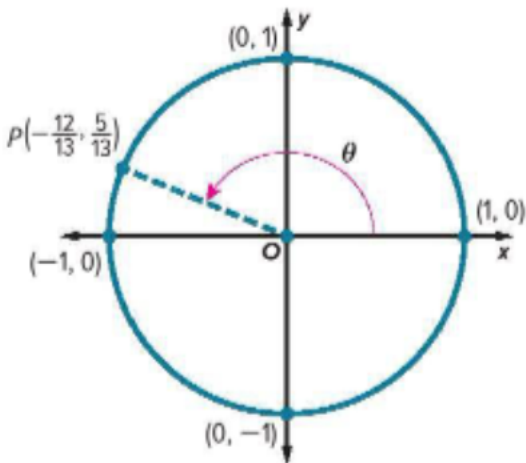
The terminal side of θ in standard position intersects the unit circle at $P\left(-\frac{12}{13}, \frac{5}{13}\right)$.

Find $\cos \theta$ and $\sin \theta$.

$P\left(-\frac{12}{13}, \frac{5}{13}\right) = P(\cos \theta, \sin \theta)$

$\cos \theta = -\frac{12}{13}$

$\sin \theta = \frac{5}{13}$



Check

The terminal side of θ in standard position intersects the unit circle at $P\left(-\frac{4}{5}, -\frac{3}{5}\right)$. Find $\cos \theta$ and $\sin \theta$. Write the solutions as decimals.

$\cos \theta = \underline{\hspace{1cm}}$ $\sin \theta = \underline{\hspace{1cm}}$



Example 1

The terminal side of angle θ in standard position intersects the unit circle at each point P . Find $\cos \theta$ and $\sin \theta$.

1. $P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

2. $P(0, -1)$

3. $P\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$

4. $P\left(-\frac{4}{5}, -\frac{3}{5}\right)$

5. $P\left(\frac{1}{6}, -\frac{\sqrt{35}}{6}\right)$

6. $P\left(\frac{\sqrt{7}}{4}, \frac{3}{4}\right)$

(No need for graphing calculator)

Examples : 5 & 6 Page : 449 & 450 | Examples : 17 – 20 Page : 452



Learn Graphing Sine and Cosine Functions

Like other functions you have studied, trigonometric functions can be graphed on the coordinate plane. Sine and cosine functions are oscillating functions. The **oscillation** of a function refers to how much the graph of the function varies between its extreme values as it approaches positive or negative infinity. The **midline** is the line about which a graph oscillates, so it is halfway between the maximum and minimum. The **amplitude** of the graph of a sine or cosine function equals half the difference between the maximum and minimum values of the function.

Sine and cosine functions, like other functions, can be transformed. You can use the graphs of the parent functions to graph $y = a \sin bx$ and $y = a \cos bx$. For functions of the form $y = a \sin bx$ and $y = a \cos bx$, $|a|$ is the amplitude and $\frac{360^\circ}{|b|}$ is the period.

You can also use x-intercepts to help graph the functions. The x-intercepts for one cycle of the sine and cosine functions are:

$y = a \sin bx$	$y = a \cos bx$
$(0, 0), \left(\frac{1}{2} \cdot \frac{360^\circ}{b}, 0\right), \left(\frac{360^\circ}{b}, 0\right)$	$\left(\frac{1}{4} \cdot \frac{360^\circ}{b}, 0\right), \left(\frac{3}{4} \cdot \frac{360^\circ}{b}, 0\right)$

Key Concept • Sine and Cosine Functions

Parent	$y = \sin x$	$y = \cos x$
Graph		
Domain	all real numbers	all real numbers
Range	$\{y \mid -1 \leq y \leq 1\}$	$\{y \mid -1 \leq y \leq 1\}$
Amplitude	1	1
Midline	$y = 0$	$y = 0$
Period	360°	360°
Oscillation	between -1 and 1	between -1 and 1

Learn Modeling with Sine and Cosine Functions

The sine and cosine functions are sometimes referred to as sinusoidal functions. A **sinusoidal function** is a function that can be produced by translating, reflecting, or dilating the sine function. The **frequency** of a sinusoidal function is the number of cycles in a given unit of time. This frequency is the reciprocal of the period of the function. So, if the period is $\frac{1}{100}$ of a second, then the frequency is 100 cycles per second.

Key Concept • Modeling with $y = a \sin bx$ and $y = a \cos bx$

Step 1 Use the amplitude to find a .

Step 2 Use the period to find b . The period is the reciprocal of the frequency. It can be written as $\frac{360^\circ}{|b|}$ or $\frac{2\pi}{|b|}$.

Step 3 Write the function.



(No need for graphing calculator)

Examples : 5 & 6 Page : 449 & 450 | Examples : 17 – 20 Page : 452

Example 5 Characteristics of the Sine and Cosine Functions

SPRINGS An object on a spring oscillates according to the function $y = 40 \cos \pi t$, where y is the distance in centimeters above its equilibrium position at time t in seconds.

Part A Find the period and frequency, and describe them in the context of the situation.

The period is $\frac{2\pi}{|b|}$. Since $b = \pi$, the period is $\frac{2\pi}{\pi}$ or 2.

The frequency is $\frac{1}{\text{period}}$ or $\frac{1}{2}$.

Therefore, the object completes $\frac{1}{2}$ of a cycle per second, and it will reach the maximum distance above the equilibrium point every 2 seconds.

Part B Identify the domain and range in the context of the situation.

The domain of $y = 40 \cos \pi t$ is all real numbers. Because time cannot be negative, the relevant domain in the context of the situation is $[0, \infty)$.

The range is $[-40, 40]$. This references the farthest away that the object can get from its point of equilibrium. With the equilibrium at the center, the object can be as much as 40 centimeters from the center in either direction. This is verified by the amplitude.

Check

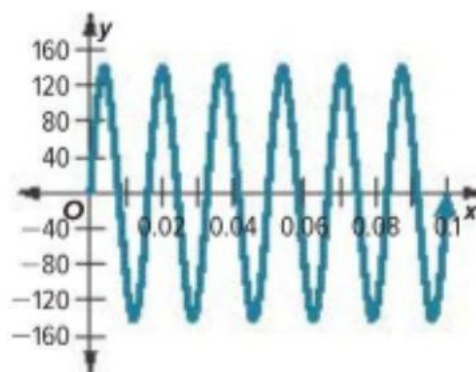
ELECTRONICS The electric power delivered for household use in the United States is 110 volts. The function $y = 100\sqrt{2} \sin 120\pi t$ represents the effective current, where t is the time in seconds.

Part A Find the period and frequency, and describe them in the context of the situation.

period = ? frequency = ?

The current completes ? cycle(s) per second, and will reach the maximum every ? second(s).

Part B Graph the function.





Example 6 Model Periodic Situations

ELECTRICITY The voltage supplied by an electrical outlet can be represented by a periodic function. The voltage oscillates between -120 and 120 volts, with a frequency of 50 cycles per second. Write and graph a function for the voltage v as a function of time t .

Step 1 The voltage oscillates from -120 to 120 .

The maximum is 120 . The minimum is -120 .

The amplitude is 120 . $a = 120$

Step 2 The frequency is 50 cycles per second.

The period is $\frac{1}{\text{frequency}}$ or $\frac{1}{50}$.

$$\text{period} = \frac{2\pi}{b}$$

$$\frac{1}{50} = \frac{2\pi}{b}$$

$$b = 100\pi$$

Period in terms of b

Substitute.

Solve.

Step 3 Use t instead of x to write the function as a function of time.

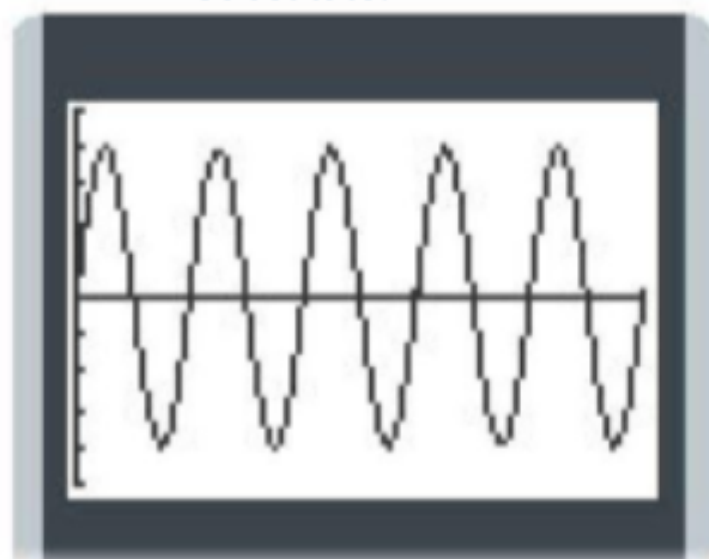
$$y = a \sin bt$$

$$y = 120 \sin 100\pi t$$

General equation

Substitute.

Step 4 Use a graphing calculator to graph the function.



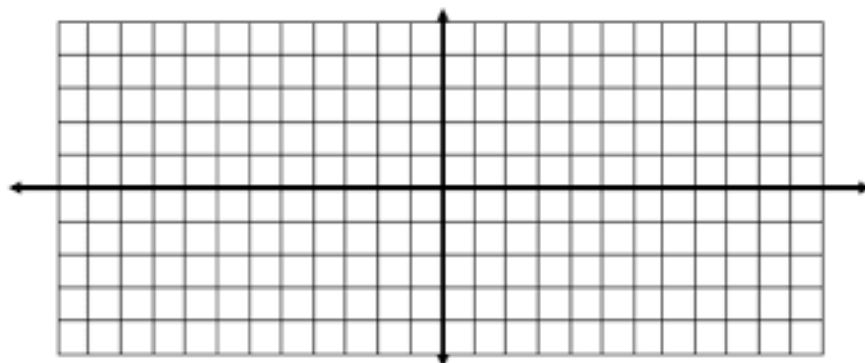
$[0, 0.1]$ scl: 0.01 by $[-150, 150]$ scl: 30



Example 5

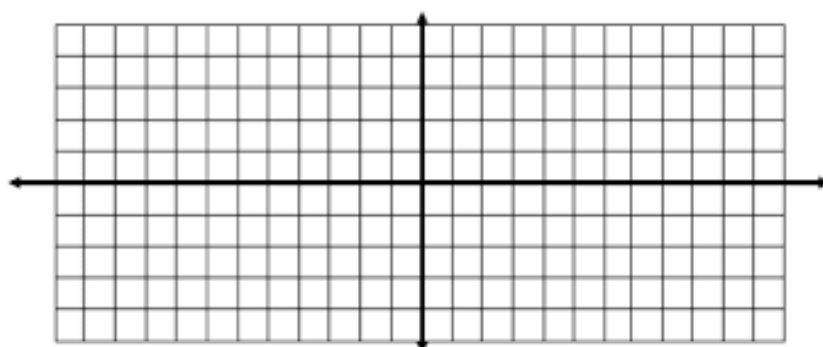
17. USE A MODEL An object on a spring oscillates using the function $y = 25 \cos \pi t$, where y is the distance in inches from its equilibrium position in t seconds.

- Find and describe the period and frequency.
- Identify the domain and range in the context of the situation.



18. SWINGS Suppose a tire swing is rotated $\frac{\pi}{5}$ radians and released. The function $y = \frac{\pi}{5} \cos 2t$ represents the displacement of the swing at time t for a frequency of radians per second.

- Find the period and frequency, and describe them in the context of the situation.
- Identify the domain and range in the context of the situation.



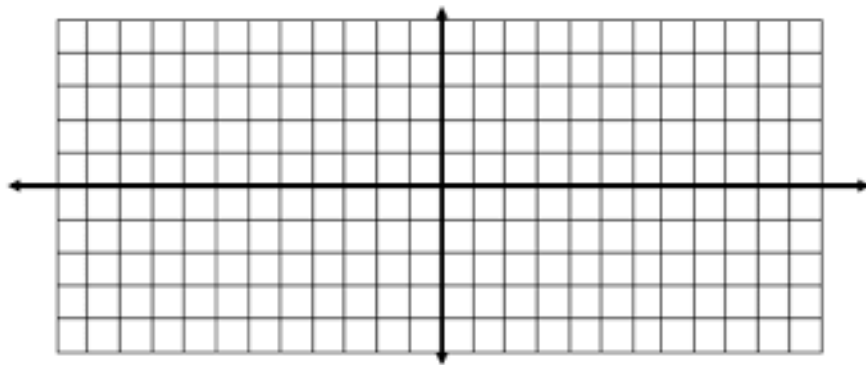
(No need for graphing calculator)

Examples : 5 & 6 Page : 449 & 450 | Examples : 17 – 20 Page : 452

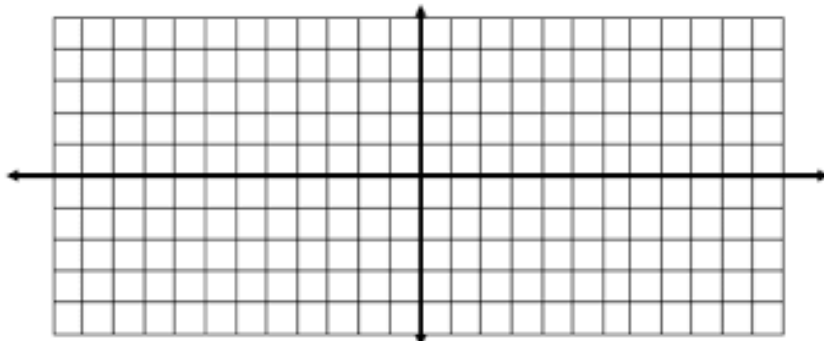


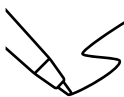
Example 6

- 19. REASONING** A boat that is tied to a dock moves vertically up and down with the waves. Delray watches the boat for 30 seconds and notes that the boat moves up and down a total of 6 times. The difference between the boat's highest point and lowest point is 3 feet. Write and graph a trigonometric function that models the boat's vertical position x seconds after she began watching. Assume that when Delray began watching the boat, it was at its highest point and that its average vertical position was 0 feet.



- 20. FERRIS WHEELS** A Ferris wheel at a state fair has a diameter of 65 feet and makes 4 complete revolutions each minute. Santiago boards a car of the Ferris wheel at the car's lowest point, and he rides for 2 minutes. Write and graph a trigonometric function that models his height above or below the axle of the Ferris wheel θ minutes after the ride starts.





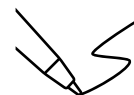
Learn Graphing Tangent Functions

Because $\tan x = \frac{\sin x}{\cos x}$, the tangent function is undefined when $\cos x = 0$, which occurs at $x = (90 + 180n)^\circ$, where n is an integer. As a result, the graph of the tangent function has asymptotes where $x = (90 + 180n)^\circ$.

Key Concept • Graphs of Tangent Functions		
Parent Function	$y = \tan x$	Graph
Domain	$\{x \mid x \neq (90 + 180n)^\circ, n \text{ is an integer}\}$	
Range	all real numbers	
Amplitude	undefined	
Period	180°	
Number of x-Intercepts in One Cycle	1	
Midline	$y = 0$	

For the graph of $y = a \tan bx$, the period is $\frac{180^\circ}{|b|}$, there is no amplitude, and the asymptotes are at $x = \frac{(90 + 180n)^\circ}{|b|}$, where n is an integer.

The value of a determines the vertical dilation, where $|a| > 1$ results in a vertical stretch and $0 < |a| < 1$ results in a vertical compression. The function is reflected in the x -axis when $a < 0$. The value of b determines the horizontal dilation, where $|b| > 1$ results in a horizontal compression and $0 < |b| < 1$ results in a horizontal stretch.



Example 1 Graph a Tangent Function with a Dilation

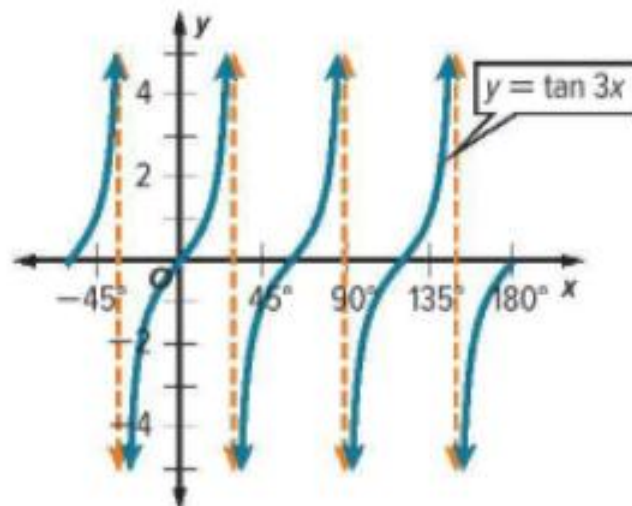
Find the period, asymptotes, x-intercepts, midline, and transformations of $y = \tan 3x$. Then graph the function.

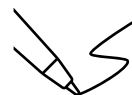
Step 1 Analyze the function.

- For a tangent function in the form $y = a \tan bx$, $\frac{180^\circ}{|b|}$ represents the period. Because $b = 3$, the period is $\frac{180^\circ}{|3|}$ or 60° .
- The asymptotes occur at $x = \frac{(90 + 180n)^\circ}{|3|}$ or $(60n + 30)^\circ$, where n is an odd integer.
- x-intercepts occur at integer multiples of the period. The period of this function is 60° , so the x-intercepts are $0^\circ, 60^\circ, 120^\circ, 180^\circ, \dots$
- The midline of $y = \tan 3x$ is $y = 0$.
- Because $a = 1$, the function is not vertically dilated in relation to the parent function. Because $b = 3$, the function is compressed horizontally in relation to the parent function.

Step 2 Graph the function.

Use the period, asymptotes, and x-intercepts to graph the function. Notice that $y = \tan 3x$ is compressed horizontally in relation to the parent function.





Check

Consider $y = \tan 0.25x$.

Part A Identify the period, asymptotes, x-intercepts, and transformations of $y = \tan 0.25x$ for $-2\pi \leq x \leq 6\pi$.
period: ?

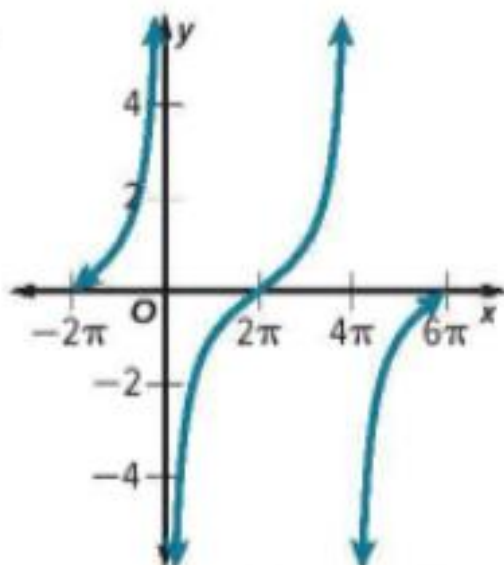
asymptotes: ?

x-intercepts: ?

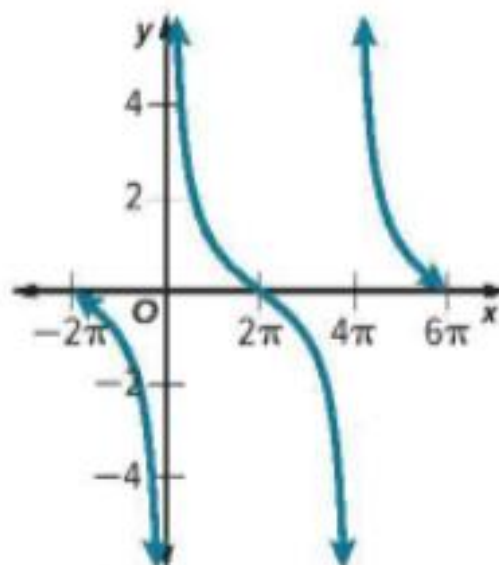
The function is ? in relation to the parent function.

Part B Select the graph of $y = \tan 0.25x$.

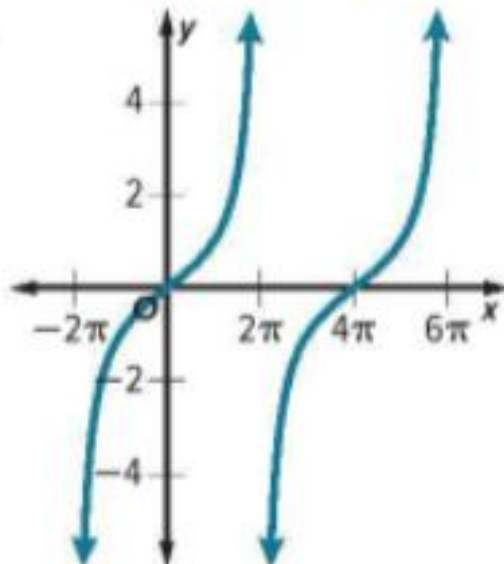
A.



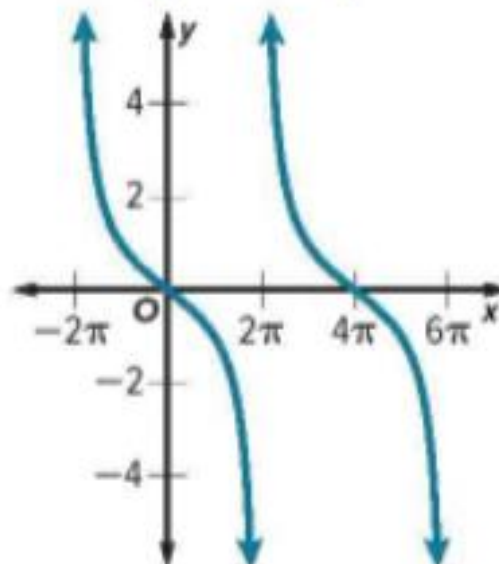
B.

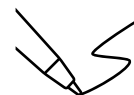


C.



D.





Example 2 Graph a Tangent Function with a Dilation and a Reflection

Find the period, asymptotes, x-intercepts, midline, and transformations of $y = -\frac{1}{3} \tan 2x$. Then graph the function.

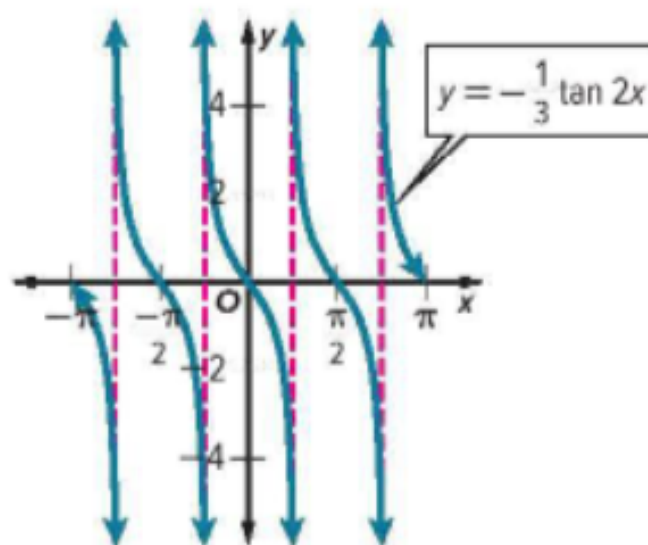
Step 1 Analyze the function.

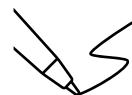
- The period is $\frac{\pi}{|2|}$ or $\frac{\pi}{2}$.
- The asymptotes occur at $\frac{\frac{\pi}{2} + \pi n}{|b|}$, where n is an odd integer. The asymptotes are at $x = \frac{\frac{\pi}{2} + \pi n}{|2|}$ or $\frac{\pi}{4} + \frac{\pi n}{2}$, where n is an odd integer.
- x-intercepts occur at integer multiples of the period. The period of this function is $\frac{\pi}{2}$, so the x-intercepts are $-\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots$
- The midline of $y = -\frac{1}{3} \tan 2x$ is $y = 0$.
- Because $a = -\frac{1}{3}$, the function is compressed vertically and reflected in the x-axis in relation to the parent function. Because $b = 2$, the function is compressed horizontally in relation to the parent function.

Step 2 Graph the function.

Use the period, asymptotes, and x-intercepts to graph the function.

Notice that $y = -\frac{1}{3} \tan 2x$ is compressed vertically, compressed horizontally, and reflected in the x-axis in relation to the parent function.





Check

Consider $y = -2 \tan \frac{3}{2}x$.

Part A

Identify the period, asymptotes, x-intercepts, and transformations

of $y = -2 \tan \frac{3}{2}x$ for
 $-60^\circ \leq x \leq 180^\circ$.

period: ?

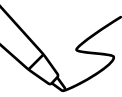
asymptotes: ?

x-intercepts: ?

The function is ? vertically,
? horizontally and
 reflected in the ? in relation
 to the parent function.

Part B

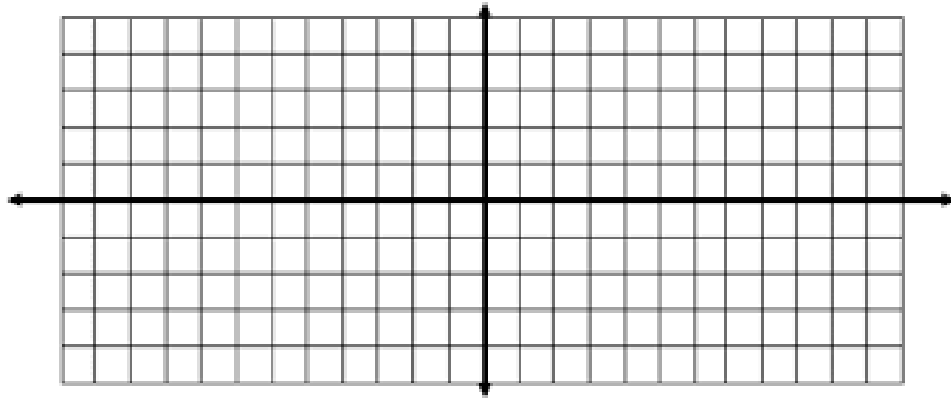
Graph $y = -2 \tan \frac{3}{2}x$.



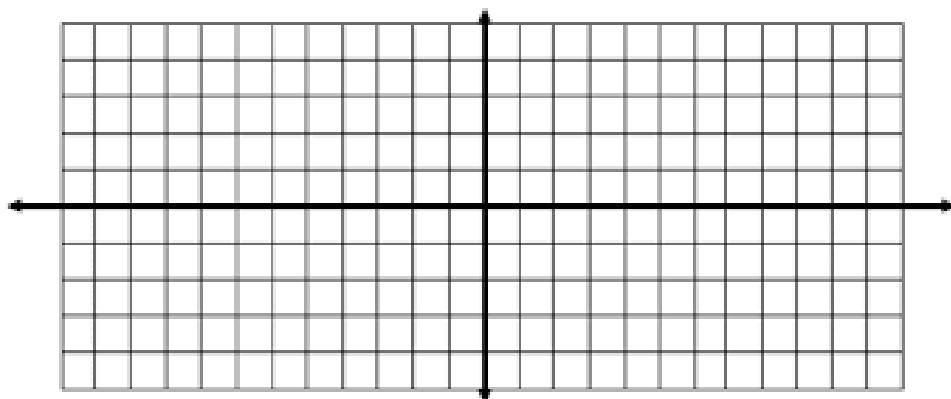
Examples 1 and 2

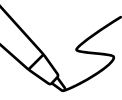
Find the period, asymptotes, x-intercepts, midline, and transformations of each tangent function. Then graph the function.

1. $y = \tan 5x$



2. $y = \tan 4x$

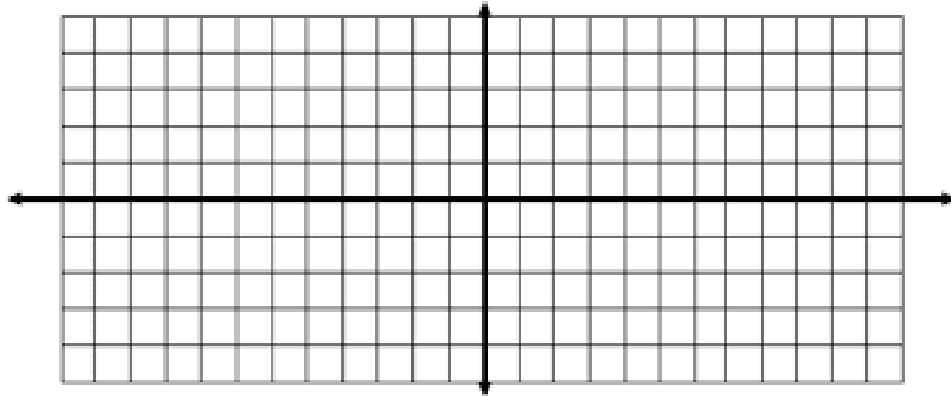




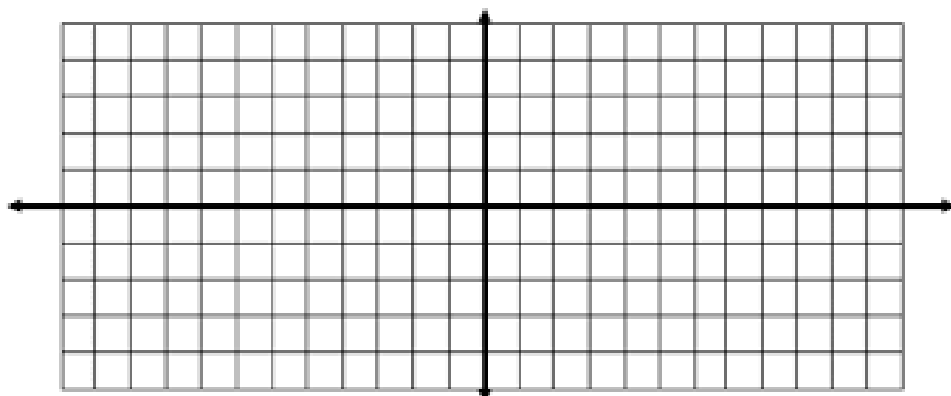
Examples 1 and 2

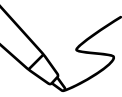
Find the period, asymptotes, x-intercepts, midline, and transformations of each tangent function. Then graph the function.

3. $y = \tan 2x$



4. $y = \frac{1}{2} \tan x$

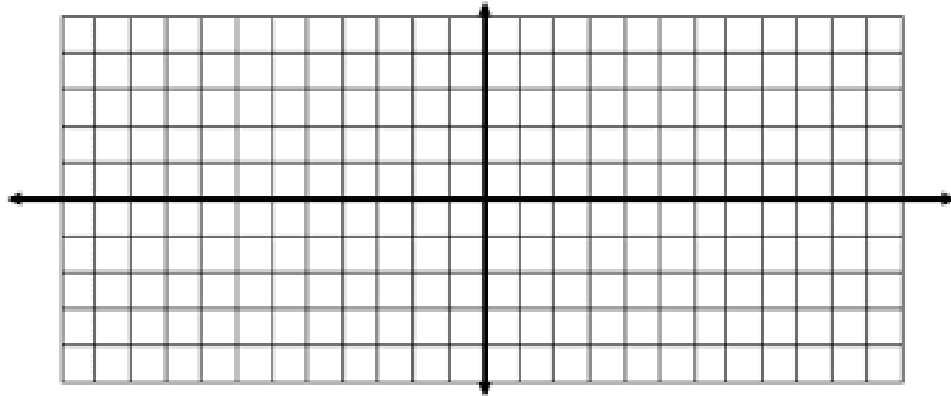




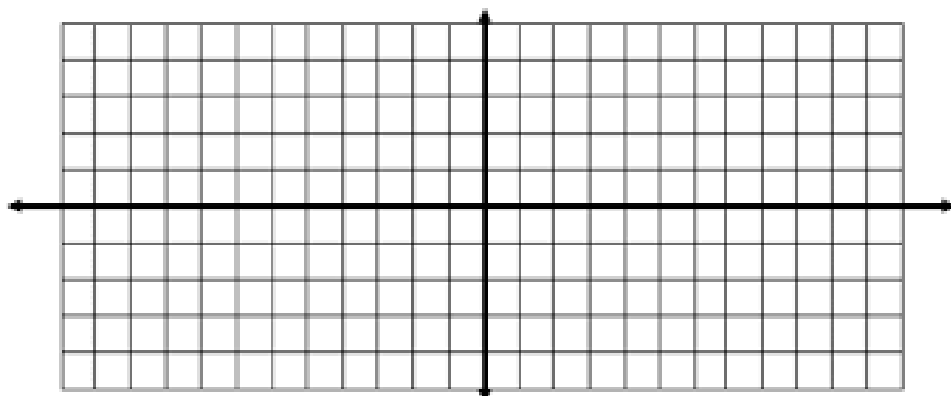
Examples 1 and 2

Find the period, asymptotes, x-intercepts, midline, and transformations of each tangent function. Then graph the function.

5. $y = 2 \tan \frac{1}{2}x$



6. $y = -\frac{1}{2} \tan 2x$



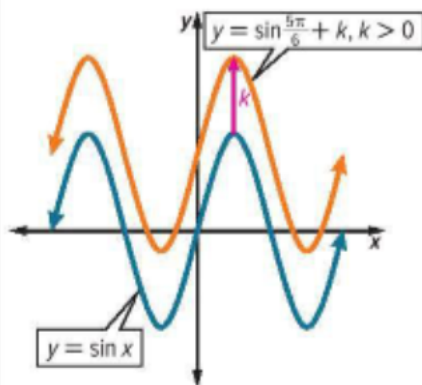


Learn Vertical Translations of Trigonometric Functions

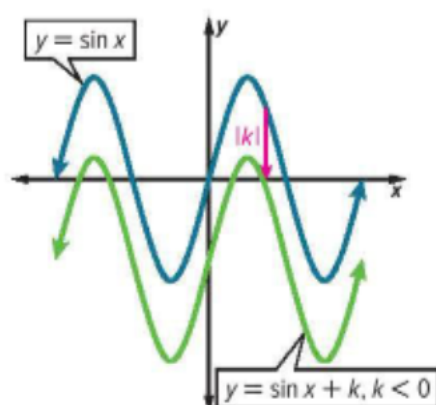
A vertical translation of the graph of a trigonometric function is called a **vertical shift**. By adding or subtracting a positive constant k , the graph is translated up or down in relation to the parent function.

Key Concept • Vertical Shift

The vertical shift of the functions $y = a \sin b\theta + k$, $y = a \cos b\theta + k$, and $y = a \tan b\theta + k$ is k , where $b > 0$.



If $k > 0$, the parent function is translated up k units.



If $k < 0$, the parent function is translated down $|k|$ units.

When a trigonometric function is shifted vertically k units, the midline is $y = k$, which is the line about which the graph of a function oscillates. You can use the midline to help sketch graphs of functions with vertical shifts.

Example 3 Graph a Vertical Shift

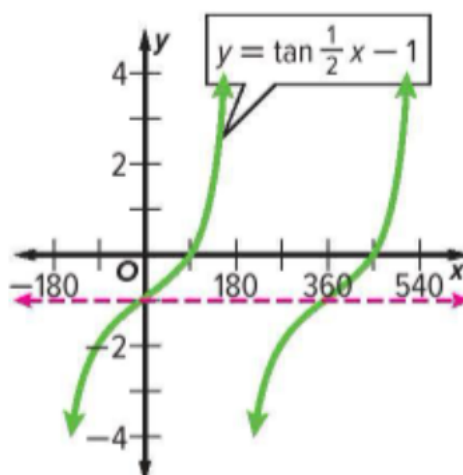
State the amplitude, period, phase shift, vertical shift, and midline equation of $y = \tan \frac{1}{2}x - 1$. Then graph the function and state the domain and range.

Step 1 Analyze the function.

For $y = \tan \frac{1}{2}x - 1$, $a = 1$, $b = \frac{1}{2}$, and $k = -1$. So the period is $\frac{180^\circ}{\frac{1}{2}}$ or 360° , the vertical shift is down 1 unit, and the midline is $y = -1$.

Step 2 Graph the function.

For $y = \tan \frac{1}{2}x$, the vertical asymptotes are at $x = (360n + 180)^\circ$, where n is an integer. Translate the function down 1 unit. The graph of $y = \tan \frac{1}{2}x - 1$ is stretched horizontally and shifted down 1 unit in relation to the parent function. The domain is $\{x \mid x \neq 180n^\circ, \text{ where } n \text{ is an integer}\}$ and the range is all real numbers.

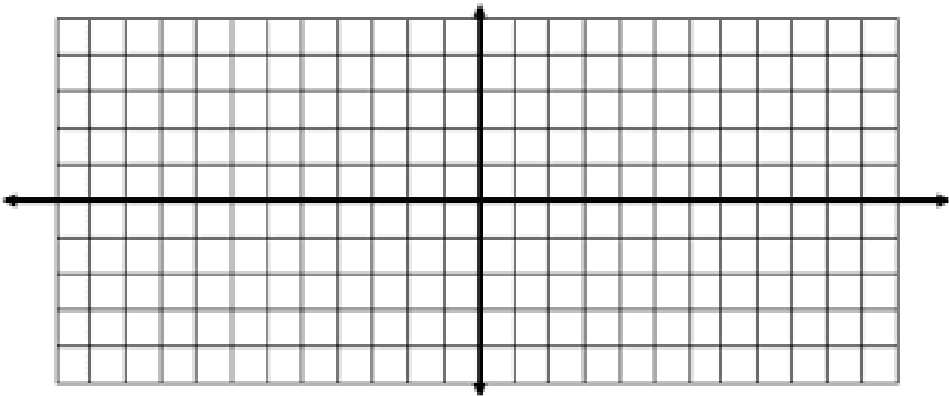




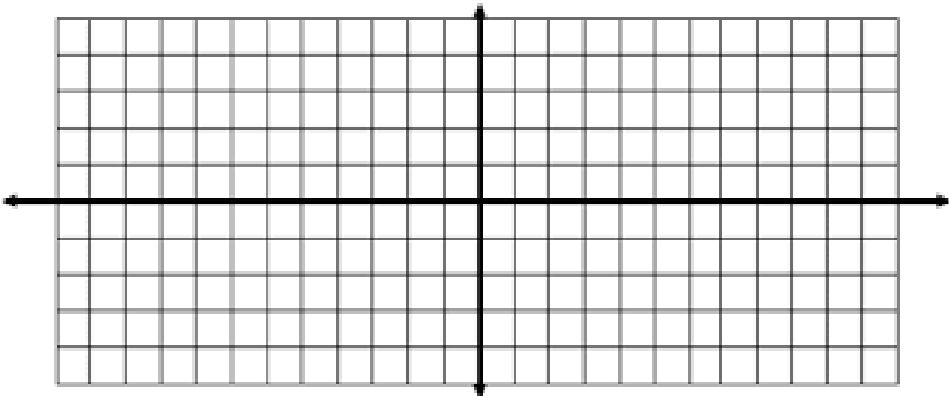
Example 3

State the amplitude, period, phase shift, vertical shift, and midline equation of each function. Then graph the function and state the domain and range.

13. $y = \cos \theta + 3$



14. $y = \tan \theta - 1$

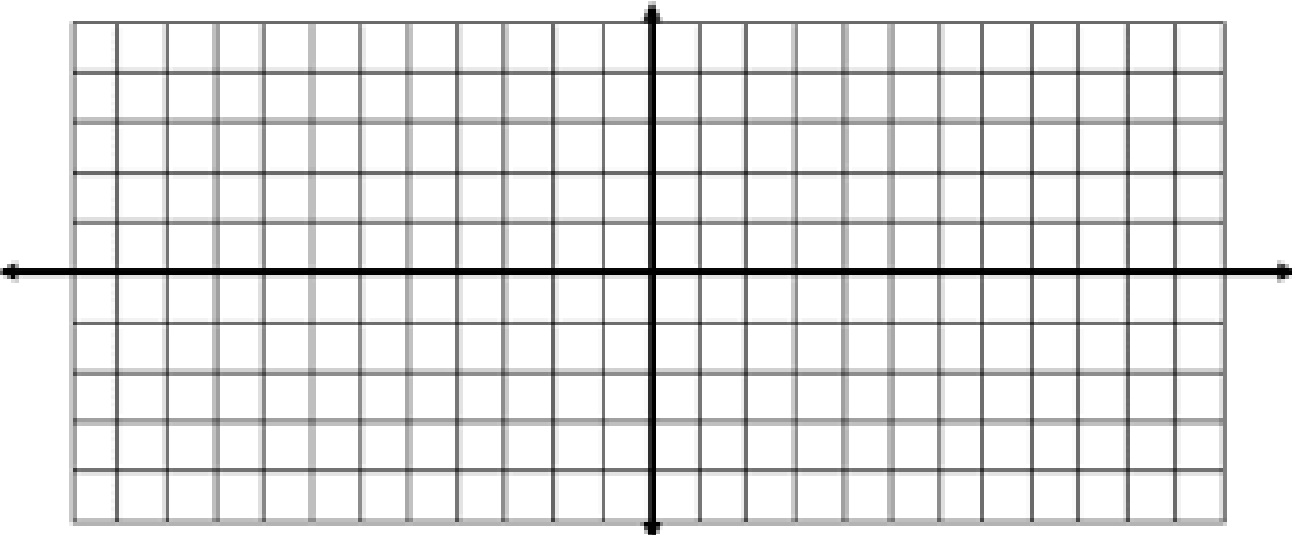




Example 3

State the amplitude, period, phase shift, vertical shift, and midline equation of each function. Then graph the function and state the domain and range.

15. $y = \tan \theta + \frac{1}{2}$

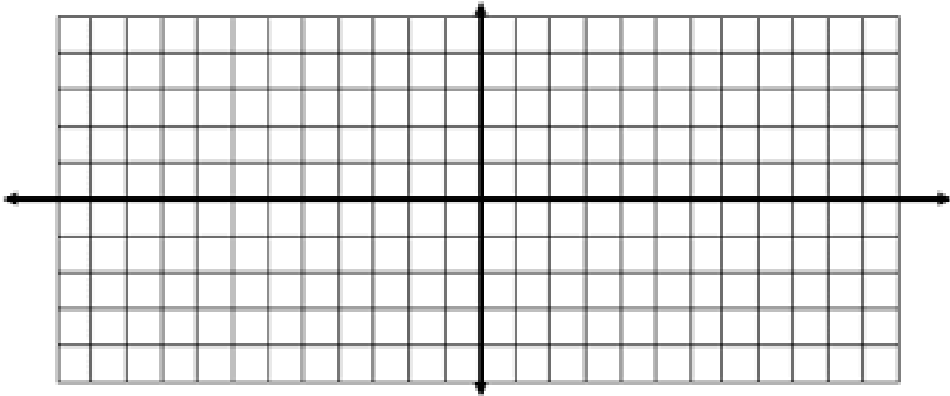




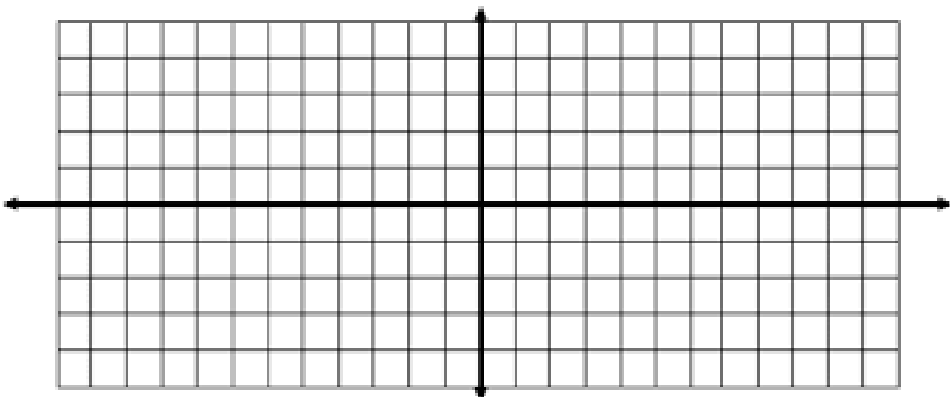
Example 3

State the amplitude, period, phase shift, vertical shift, and midline equation of each function. Then graph the function and state the domain and range.

16. $y = 2 \cos \theta - 5$



17. $y = 2 \sin \theta - 4$

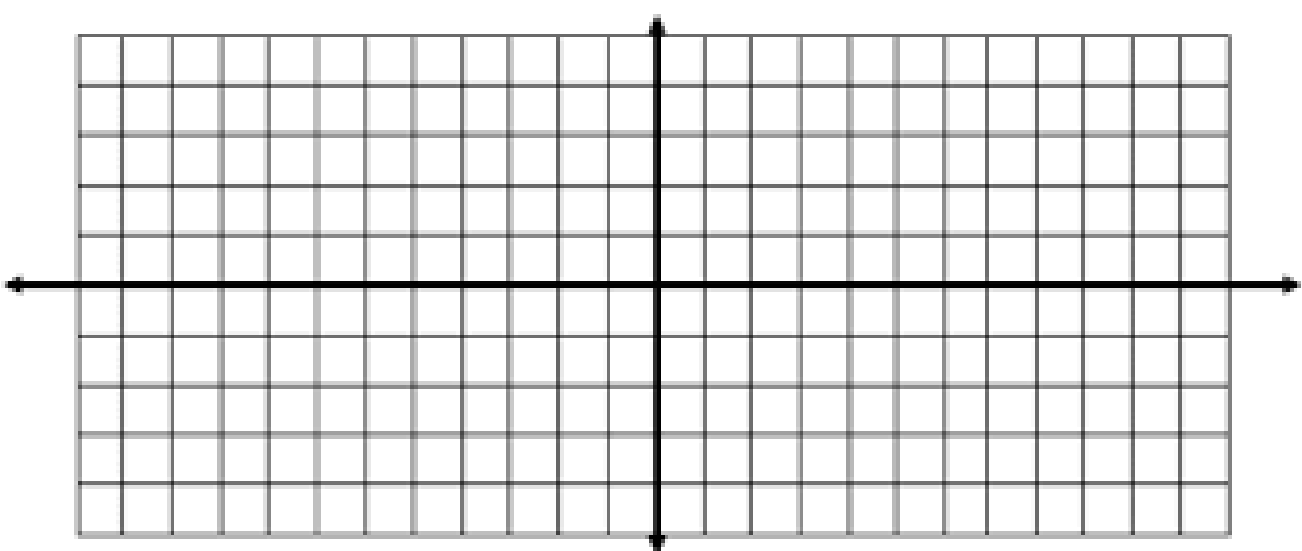




Example 3

State the amplitude, period, phase shift, vertical shift, and midline equation of each function. Then graph the function and state the domain and range.

18. $y = \frac{1}{3} \sin \theta + 7$





Learn Inverse Trigonometric Functions

If you know the value of a trigonometric function for an angle, you can use the inverse function to find the angle measure. Because the inverse of a function transposes the x - and y -values, the inverse of $y = \sin x$ is $x = \sin y$.

The relation $x = \sin y$ is not a function because there are many values of y for each value of x . Therefore, the domain must be restricted in order for the inverse to be a function. Domains for the inverses of the cosine and tangent functions must also be restricted. The values in the restricted domain are called **principal values**. Trigonometric functions with domains restricted to the principal values are denoted with a capital letter. **Inverse trigonometric functions** are inverses of the trigonometric functions with restricted domains.

Key Concept • Inverse Trigonometric Functions			
Inverse Function	Symbols	Domain	Range
Arcsine	$y = \operatorname{Arcsin} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ or
	$y = \sin^{-1} x$		$-90^\circ \leq y \leq 90^\circ$
Arccosine	$y = \operatorname{Arccos} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$ or
	$y = \cos^{-1} x$		$0^\circ \leq y \leq 180^\circ$
Arctangent	$y = \operatorname{Arctan} x$	all real numbers	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ or
	$y = \tan^{-1} x$		$-90^\circ \leq y \leq 90^\circ$

Example 1 Evaluate Inverse Trigonometric Functions

Find $\tan^{-1} \sqrt{3}$. Write angle measures in degrees and radians.

Find the angle θ for $-90^\circ \leq \theta \leq 90^\circ$ with a tangent value of $\sqrt{3}$.

Because $\tan \theta = \frac{\sin \theta}{\cos \theta}$, use the unit circle to find a point where the ratio of sine to cosine is $\sqrt{3}$ to 1. Every special angle on the unit circle can be written as a value over 2. A point on the unit circle at which

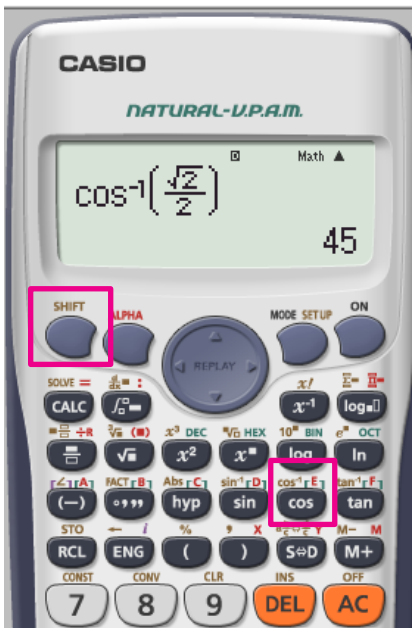
$\sin \theta = \frac{\sqrt{3}}{2}$ and $\cos \theta = \frac{1}{2}$ has a tangent value of $\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$ or $\sqrt{3}$.

When $\theta = 60^\circ$, $\tan \theta = \sqrt{3}$. So, $\tan^{-1} \sqrt{3} = 60^\circ$ or $\frac{\pi}{3}$.

Check

Find $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$. Write the angle measure in degrees.

$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = ?^\circ$



**Example 1**

Find each value. Write angle measures in degrees and radians.

1. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

2. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

3. $\arccos\left(-\frac{1}{2}\right)$

4. $\arctan \sqrt{3}$

5. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

6. $\tan^{-1}(-1)$

7. $\sin^{-1}\frac{\sqrt{2}}{2}$

8. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

9. $\arcsin 1$