UNITED ARAB EMIRATES
MINISTRY OF EDUCATION

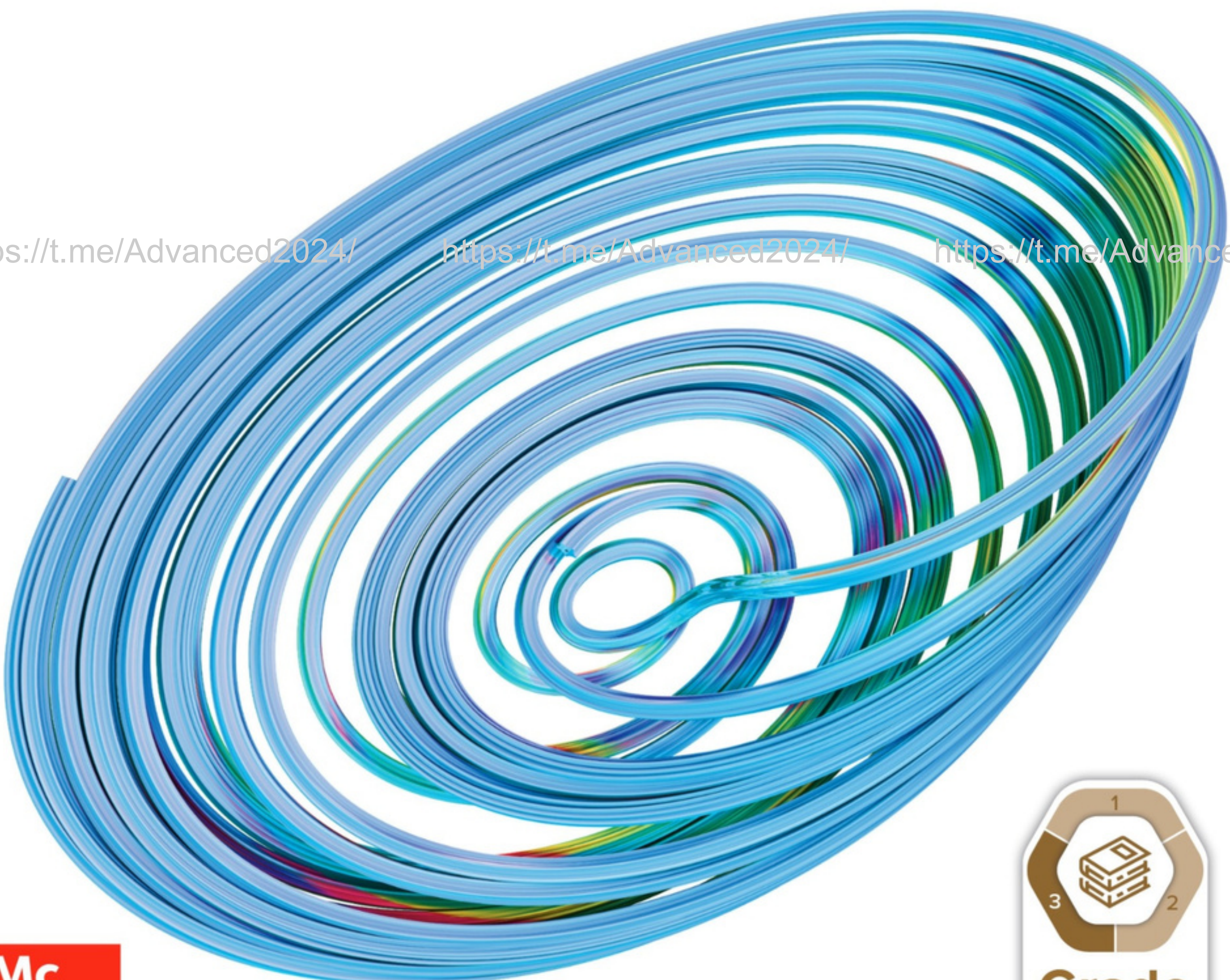
المدرسة الإماراتية

2022-2023

Mathematics

United Arab Emirates Edition

Solutions Manual



Mc Graw Hill

Grade 12 Advanced

Chapter 5

Applications of the Definite Integral

5.1 Area Between Curves

1. Area $= \int_1^3 \left[x^3 - \left(x^2 - 1\right)\right] dx$

$= \left(\frac{x^4}{4} - \frac{x^3}{3} + x\right)\Big|_1^3$

$= \left(\frac{81}{4} - \frac{27}{3} + 3\right) - \left(\frac{1}{4} - \frac{1}{3} + 1\right)$

$= \frac{160}{12} = \frac{40}{3}$

2. Area $= \int_0^2 [(x^2 + 2) - \cos x] dx$

$= \left(\frac{x^3}{3} + 2x - \sin x\right)\Big|_0^2 = \frac{20}{3} - \sin 2$

3. Area $= \int_{-2}^0 \left[e^x - (x - 1)\right] dx$

$= \left(e^x - \frac{x^2}{2} + x\right)\Big|_{-2}^0$

$= (1 - 0 + 0) - \left(e^{-2} - \frac{4}{2} + (-2)\right)$

$= 5 - e^{-2}$

4. Area $= \int_1^4 (x^2 - e^{-x}) dx$

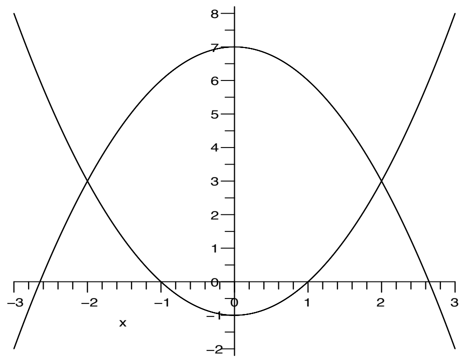
$= \left(\frac{x^3}{3} + e^{-x}\right)\Big|_1^4 = 21 + e^{-4} - e^{-1}$

5. Area $= \int_{-2}^2 \left[7 - x^2 - \left(x^2 - 1\right)\right] dx$

$= \left(8x - \frac{2x^3}{3}\right)\Big|_{-2}^2$

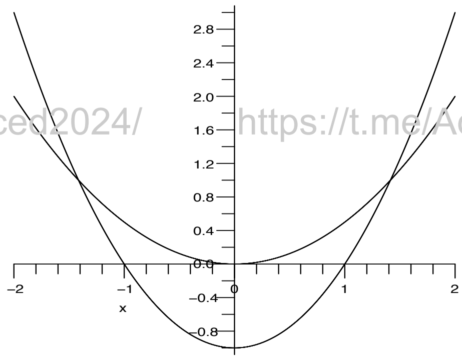
$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$

$= \frac{64}{3}$

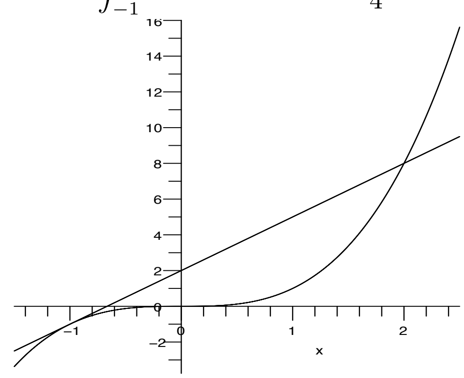


6. Area $= \int_{-\sqrt{2}}^{\sqrt{2}} \left[\frac{x^2}{2} - (x^2 - 1)\right] dx$

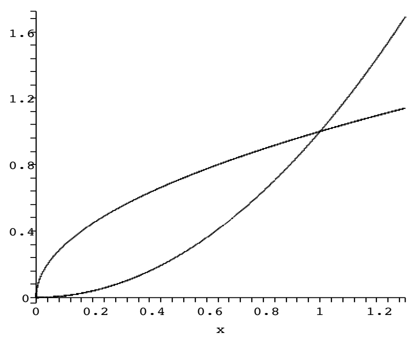
$= \frac{4\sqrt{2}}{3}$



7. Area $= \int_{-1}^2 (3x + 2 - x^3) dx = \frac{27}{4}$



8. Area $= \int_0^1 (\sqrt{x} - x^2) dx = \frac{1}{3}$

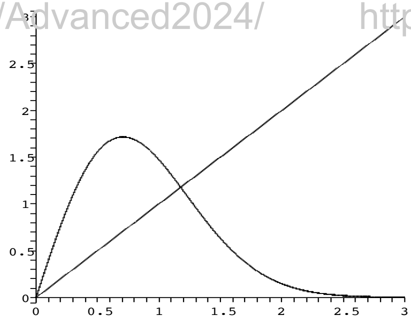


9. Area $= \int_0^{\sqrt{\ln 4}} \left(4xe^{-x^2} - x\right) dx$

$= -2e^{-x^2} - \dfrac{x^2}{2}\Big|_0^{\sqrt{\ln 4}}$

$= -2\left[\dfrac{1}{4} - 1\right] - \dfrac{\ln 4}{2}$

$= \dfrac{3 - \ln 4}{2}$



10. Area $= \int_{-1}^0 \left(\dfrac{2}{x^2+1} + x\right) dx$

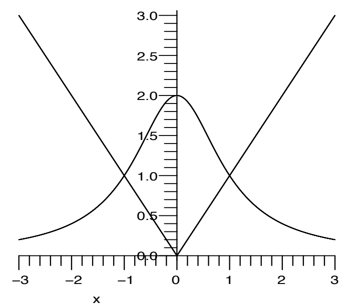
$+ \int_0^1 \left(\dfrac{2}{x^2+1} - x\right) dx$

$= (2\tan^{-1} x + \dfrac{x^2}{2})\Big|_{-1}^0$

$+(2\tan^{-1} x - \dfrac{x^2}{2})\Big|_0^1$

$= \left(\dfrac{\pi}{4} - \dfrac{1}{2}\right) + \left(\dfrac{\pi}{4} - \dfrac{1}{2}\right)$

$= \dfrac{\pi}{2} - 1$



11. Area $= \int_{-2}^0 \left[x - \dfrac{5x}{x^2+1}\right] dx$

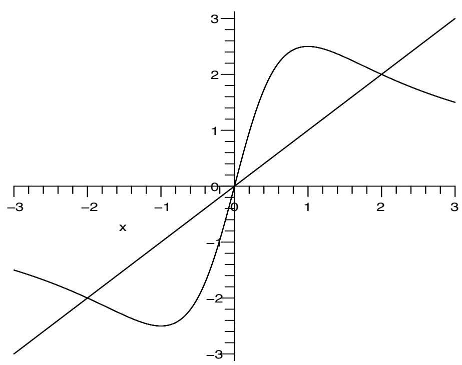
$+ \int_0^2 \left[\dfrac{5x}{x^2+1} - x\right] dx$

$= 2\int_0^2 \left[\dfrac{5x}{x^2+1} - x\right] dx$

$= 2\left(\dfrac{5}{2}\ln\left|x^2+1\right| - \dfrac{x^2}{2}\right)\Big|_0^2$

$= 5[\ln 5 - \ln 1] - [4 - 0]$

$= 5\ln 5 - 4$



12. Area $= \int_0^{\pi/4} (\cos x - \sin x) dx$

$+ \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx$

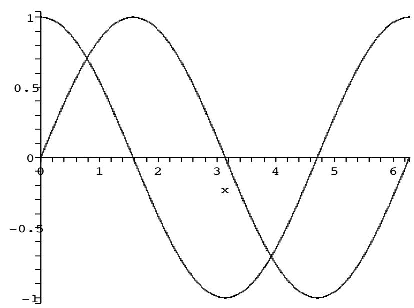
$+ \int_{5\pi/4}^{2\pi} (\cos x - \sin x) dx$

$= (\sin x + \cos x)|_0^{\pi/4}$

$+(-\cos x - \sin x)|_{\pi/4}^{5\pi/4}$

$+(\sin x + \cos x)|_{5\pi/4}^{2\pi}$

$= 4\sqrt{2}$

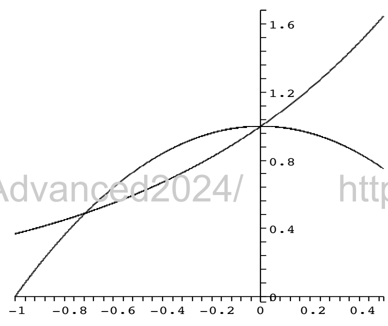


13. Area $= \int_{-.7145}^{0} (1 - x^2) - e^x dx$

$= \left(-e^x + x - \frac{x^3}{3}\right)\Big|_{-.7145}^{0}$

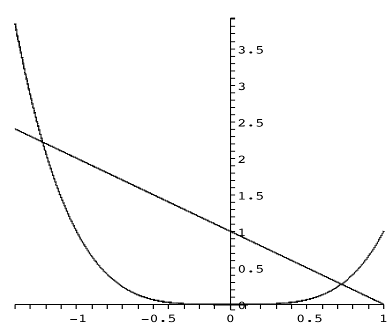
$= (-1 + 0 - 0) - (-1.08235)$

$= .08235$



14. Area $\approx \int_{-1.2207}^{0.72449} [(1 - x) - x^4] dx$

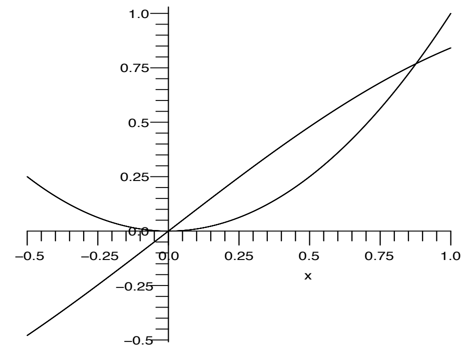
≈ 1.845787



15. Area $= \int_{0}^{.8767} \left(\sin x - x^2\right) dx$

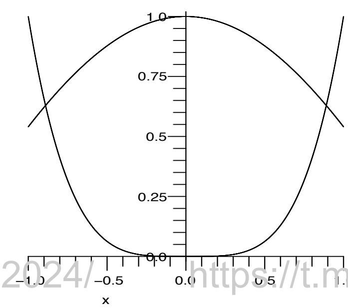
$= \left(-\cos x - \frac{x^3}{3}\right)\Big|_{0}^{.8767}$

$\approx .135697$



16. Area $\approx \int_{-.89055}^{0.89055} (\cos x - x^4) dx$

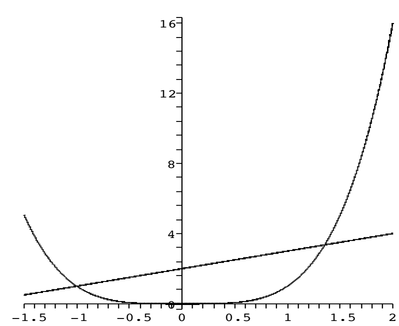
≈ 1.330782



17. Area $= \int_{-1}^{1.3532} \left(2 + x - x^4\right) dx$

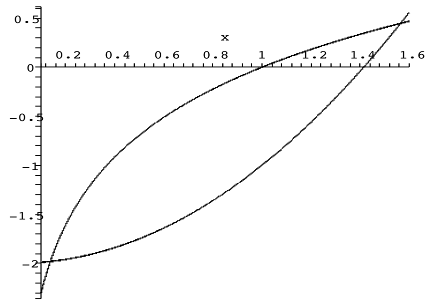
$= \left(2x + \frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_{-1}^{1.3532}$

$= 4.01449$



18. Area $\approx \int_{0.13793}^{1.5645} [\ln x - (x^2 - 2)] dx$

≈ 1.124448

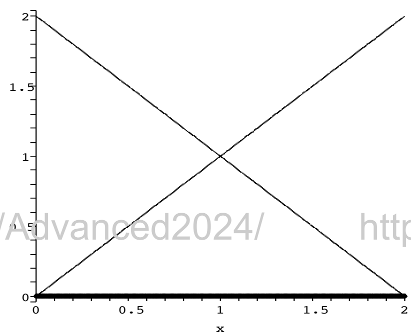


19. Area $= \int_0^1 [(2-y)-y]dy$

$= \int_0^1 [2-2y]dy$

$= \left(2y-y^2\right)\Big|_0^1$

$= 1-0 = 1$



Area of triangle $= \frac{1}{2}(base)(height)$

$= \frac{1}{2}\cdot(2)\cdot(1) = 1$

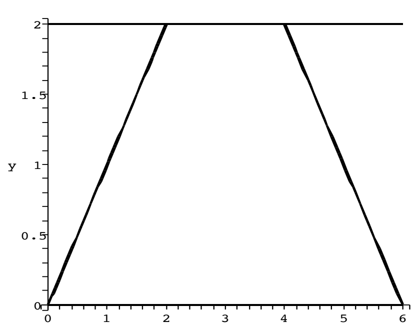
20. Area $= \int_0^2 [(6-y)-y]dy$

$= \int_0^2 (6-2y)dy$

$= \left(6y-y^2\right)\Big|_0^2$

$= (12-4)-(0-0)$

$= 8$



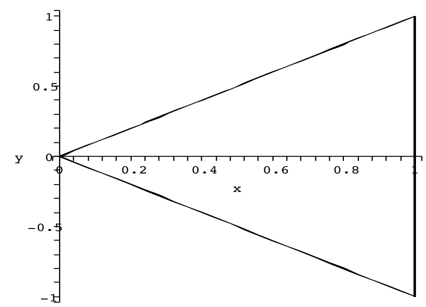
Area of Trapazium $= \frac{1}{2}(a+b)(h)$

$= \frac{1}{2}\cdot(8)\cdot(2) = 8$

21. Area $= \int_0^1 [x-(-x)]dx$

$= 2\int_0^1 xdx = x^2\Big|_0^1$

$= 1-0 = 1$



Area of triangle $= \frac{1}{2}(base)(height)$

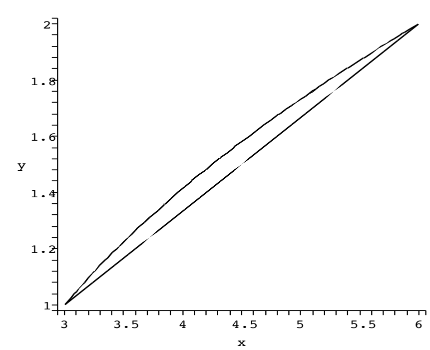
$= \frac{1}{2}\cdot(2)\cdot(1) = 1$

22. Area $= \int_1^2 \left[3y-\left(2+y^2\right)\right]dy$

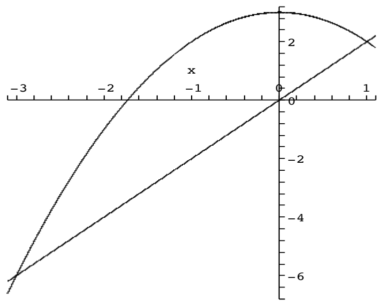
$= \left(\frac{3}{2}y^2-2y-\frac{y^3}{3}\right)\Big|_1^2$

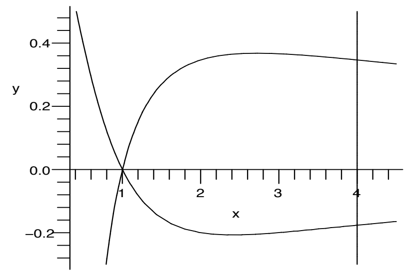
$= \left(6-4-\frac{8}{3}\right)-\left(\frac{3}{2}-2-\frac{1}{3}\right)$

$= \frac{1}{6}$

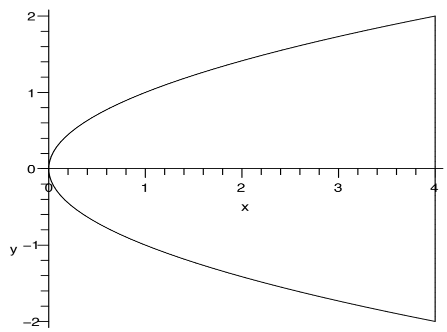


23. Area $= \int_{-3}^1 [(3-x^2)-2x]dx = \frac{32}{3}$

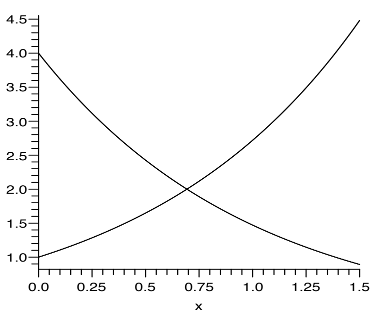




24. Area $= \int_{-2}^{2} (4 - y^2)dy = \frac{32}{3}$



25. Area $= \int_{0}^{\ln 2} (4e^{-x} - e^{x})dx \approx 1$



26. Area $= \int_1^4 \left(\frac{\ln x}{x} - \frac{1-x}{x^2+1} \right) dx$

$= \int_1^4 \frac{\ln x}{x} dx - \int_1^4 \frac{1}{x^2+1} dx$

$+ \frac{1}{2} \int_1^4 \frac{2x}{x^2+1}$

$= \left[\frac{\ln^2 x}{2} - \tan^{-1} x + \frac{1}{2} \ln |x^2 + 1| \right]_1^4$

$= \frac{\ln^4}{2} - \tan -14 + \frac{\ln 17}{2} + \frac{\pi}{4} - \frac{\ln 2}{2}$

27. $\int_0^{.4} f_c(x) \approx \frac{.4}{3(4)} \{f_c(0) + 4f_c(.1)$
$+ 2f_c(.2) + 4f_c(.3) + f_c(.4)\} = 291.67$

$\int_0^{.4} f_e(x) \approx \frac{.4}{3(4)} \{f_e(0) + 4f_e(.1)$
$+ 2f_e(.2) + 4f_e(.3) + f_e(.4)\} = 102.33$

$\frac{\int_0^{.4} f_c(x) - \int_0^{.4} f_e(x)}{\int_0^{.4} f_c(x)} \approx \frac{291.67 - 102.33}{291.67}$

$= .6491 \ldots.$

$1 - .6491 = .3508,$

so the proportion of energy retained is about 35.08%.

28. Energy $= \frac{\int_0^m [f_c(x) - f_e(x)]dx}{\int_0^m f_c(x)dx}$

$= \frac{\int_0^m f_c(x)dx}{\int_0^m f_c(x)dx} - \frac{\int_0^m f_e(x)dx}{\int_0^m f_c(x)dx}$

$= 1 - \frac{\int_0^m f_e(x)dx}{\int_0^m f_c(x)dx}$

$\int_0^{0.18} f_c(x)dx$

$\approx \frac{0.045}{3} [f_c(0) + 4f_c(0.045) + 2f_c(0.09)$
$+ 4f_c(0.135) + f_c(0.18)]$

$= \frac{0.045}{3} [0 + 4(200) + 2(500) + 4(1000)$
$+ 1800]$

$= 114$

$\int_0^{0.18} f_e(x)dx$

$\approx \frac{0.045}{3} [f_e(0) + 4f_e(0.045) + 2f_e(0.09)$
$+ 4f_e(0.135) + f_e(0.18)]$

$= \frac{0.045}{3} (0 + 4(125) + 2(350) + 4(700)$
$+ 1800)$

$= 87$

Putting these together gives the proportion of energy lost as

$$\text{Energy} \approx 1 - \frac{87}{114} \approx 0.2368.$$

29. $\int_0^3 f_s(x) \approx \frac{3}{3(4)}\{f_s(0) + 4f_s(.75) + 2f_s(1.5) + 4f_s(2.25) + f_s(3)\} = 860$

$\int_0^3 f_r(x) \approx \frac{3}{3(4)}\{f_r(0) + 4f_r(.75) + 2f_r(1.5) + 4f_r(2.25) + f_r(3)\} = 800$

$1 - \left(\frac{860 - 800}{860}\right) = .9302$

Energy returned by the tendon is 93.02%.

30. As in Exercise 28, the proportion of energy returned by the arch is given by

$$1 - \frac{\int_0^8 f_s(x)dx}{\int_0^8 f_r(x)dx}$$

$$\int_0^8 f_s(x)dx$$
$$\approx \frac{2}{3}[f_s(0) + 4f_s(2) + 2f_s(4) + 4f_s(6) + f_s(8)]$$
$$= \frac{2}{3}[0 + 4(300) + 2(1000) + 4(1800) + 3500]$$
$$\approx 8366.67$$

$$\int_0^8 f_r(x)dx$$
$$\approx \frac{2}{3}[f_r(0) + 4f_r(2) + 2f_r(4) + 4f_r(6) + f_r(8)]$$
$$= \frac{2}{3}[0 + 4(150) + 2(700) + 4(1300) + 3500]$$
$$\approx 7133.33$$

Putting these together gives the proportion of energy lost as

$$\text{Energy} \approx 1 - \frac{7133.33}{8366.67} \approx 0.1474.$$

31. $A = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{3-0}\int_0^3 x^2 dx$

$= \left(\frac{1}{3} \cdot \frac{x^3}{3}\right)\Big|_0^3 = \frac{27}{9} - 0 = 3$

Relative to the interval $[0, 3]$, the inequality $x^2 < 3$ holds only on the subinterval $[0, \sqrt{3})$. We find

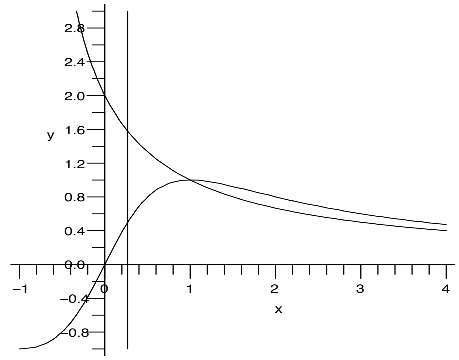
$\int_0^{\sqrt{3}} (3 - x^2)dx = \left(3x - \frac{x^3}{3}\right)\Big|_0^{\sqrt{3}}$

$= (3\sqrt{3} - \sqrt{3}) - (0 - 0)$
$= 2\sqrt{3}$, whereas

$\int_{\sqrt{3}}^3 (x^2 - 3)dx = \left(\frac{x^3}{3} - 3x\right)\Big|_{\sqrt{3}}^3$

$= (9 - 9) - (\sqrt{3} - 3\sqrt{3})$
$= 2\sqrt{3}$, the same.

32. Draw the graphs of the given functions, $y = \frac{2}{(x+1)}$ and $y = \frac{2x}{(x^2+1)}$ for $x > 0$.



It may be observed from the graph that these functions cut each other at a single point at $x = 1$. From the graph it is observed that the curve $y = \frac{2}{(x+1)}$ lies above the curve $y = \frac{2x}{(x^2+1)}$ for $0 \le x \le 1$, for $x > 1$, $y = \frac{2x}{(x^2+1)}$ lies above the curve $y = \frac{2}{(x+1)}$

Let us find the area bounded by these curves between $x = 0$ and $x = 1$. It is given by

$$\int_0^1 \left(\frac{2}{(x+1)} - \frac{2x}{(x^2+1)}\right)dx$$
$$= \left(\ln(x+1)^2 - \ln\left(x^2+1\right)\right)\Big|_0^1$$
$$= \ln 2 > \ln\left(\frac{3}{2}\right)$$
$$\Rightarrow 0 < t < 1$$

Therefore

$$\ln\left(\frac{3}{2}\right) = \int_0^t \left(\frac{2}{(x+1)} - \frac{2x}{(x^2+1)}\right)dx$$

i.e. $\ln\left(\frac{3}{2}\right) = \left(\ln(x+1)^2 - \ln\left(x^2+1\right)\right)\Big|_0^t$

or $\ln\left(\frac{3}{2}\right) = \ln\left(\frac{(t+1)^2}{(t^2+1)}\right)$

$\Rightarrow 3t^2 + 3 = 2(t^2 + t + 1)$

i.e. $t = 2 \pm \sqrt{3}$

But as $0 < t < 1$, we consider $t = 2 - \sqrt{3}$

33. Let $y_1 = ax^2 + bx + c$, $y_2 = mx + n$, and $u = y_1 - y_2$. If we assume that $a < 0$, then $y_1 > y_2$ on (A, B) and the area between the curves is given by the integral

$\int_A^B (y_1 - y_2)dx$
$= \int_A^B u dx = ux|_A^B - \int_A^B x du.$

By assumption, u is zero ($y_1 = y_2$) at both A and B, so the first part of the last expression is zero. We must now show that

$-\int_A^B x du = -\int_A^B x[2ax + (b - m)]dx$

is the same as

$|a|(B - A)^3/6$
$= |a|(B^3 - 3B^2A + 3BA^2 - A^3)/6.$

But again because $u = 0$ at both A and B, we know that

$aA^2 + bA + c = mA + n$ and
$aB^2 + bB + c = mB + n.$

By subtraction of the first from second, factoring out (and canceling) $B - A$, we learn $a(B + A) = m - b$, so that our target integral is also given by

$-2a \int_A^B x(x - \frac{A+B}{2})dx$
$= |a|\{2(B^3 - A^3)/3 - (A + B)(B^2 - A^2)/2\}$

and the student who cares enough can finish the details.

The case in which $a > 0 (y_2 > y_1)$ is not essentially different.

34. Perhaps the most straightforward way to handle this problem is by brute force. First, the area is given by

$$\text{Area} = \pm \int_A^B [(ax^3 + bx^2 + cx + d) - (kx^2 + mx + n)]dx$$
$$= \frac{a}{4}(A^4 - B^4) + \frac{(b-k)}{3}(B^3 - A^3) + \frac{(c-m)}{2}(B^2 - A^2) + (d - n)(B - A).$$

We can set up equations for the fact that the graphs meet at A and B. At A and B, we set the functions equal. At B, we set the derivatives equal.

$aA^3 + bA^2 + cA + d = kA^2 + mA + n$
$aB^3 + bB^2 + cB + d = kB^2 + mB + n$
$3aB^2 + 2bB + c = 2kB + m$

We now have a system of equations. We solve the last equation for m and plug the result in for m in the previous two equations. This transforms the three equations to

$aA^3 + (b - k)A^2 - 3aAB^2$
$- 2(b - k)AB + d - n = 0$
$- 2aB^3 - (b - k)B^2 + d - n = 0$
$m = 3aB^2 + 2(b - k)B + c.$

We solve the second equation for n and plug the result into the first equation which then gives

$aA^3 + (b - k)A^2 - 2(b - k)AB - 3aAB^2$
$+ 2aB^3 + (b - k)B^2 = 0$
$n = -2aB^3 - (b - k)B^2 + d$
$m = 3aB^2 + 2(b - k)B + c.$

Finally, solving the first equation for k gives $k = aA + 2aB + b$.

We now substitute m, then n and then finally k in to the equation for area. After simplifying this finally gives

$\text{Area} = \frac{\pm a(A - B)^4}{12}.$

35. Let the upper parabola be $y = y_1 = qx^2 + v + h$ and let the lower be $y = y_2 = px^2 + v$. They are to meet at $x = w/2$, so we must have

$qw^2/4 + h = pw^2/4$, hence
$h = (p - q)w^2/4$ or $(q - p)w^2 = -4h$.

Using symmetry, the area between the curves is given by the integral

$2\int_0^{w/2} (y_1 - y_2)dx$
$= 2\int_0^{w/2} [h + (q - p)x^2]dx$
$= 2[hw/2 + (q - p)w^3/24]$
$= w[h + (q - p)w^2/12]$
$= w[h - 4h/12] = (2/3)wh.$

36. Solve the equation $2 - x^2 = mx$ we get

$x = \frac{m \pm \sqrt{m^2 + 8}}{2}$

So the area between $y = 2 - x^2$ and $y = mx$ is

$$\int_{(m-\sqrt{m^2+8})/2}^{(m+\sqrt{m^2+8})/2} (2 - x^2 - mx)dx$$
$$= \left(2x - \frac{x^3}{3} - \frac{mx^2}{2}\right)\Bigg|_{(m-\sqrt{m^2+8})/2}^{(m+\sqrt{m^2+8})/2}$$
$$= \frac{1}{6}(m^2 + 8)^{3/2}$$

The minimum of $(m^2 + 8)^{3/2}/6$ happens when $m = 0$ and then

$\frac{1}{6}(m^2 + 8)^{3/2} = \frac{1}{6} \cdot 8^{3/2} = \frac{8\sqrt{2}}{3}$

37. Solve for x in $x - x^2 = L$ we get

$x = \frac{1 \pm \sqrt{1 - 4L}}{2}$

$A_1 = \int_0^{(1-\sqrt{1-4L})/2} [L - (x - x^2)]dx$

$$= \left(Lx - \frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_0^{(1-\sqrt{1-4L})/2}$$

$$A_2 = \int_{(1-\sqrt{1-4L})/2}^{(1+\sqrt{1-4L})/2} [(x-x^2) - L]dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3} - Lx\right)\Big|_{(1-\sqrt{1-4L})/2}^{(1+\sqrt{1-4L})/2}$$

By setting $A_1 = A_2$, we get the final answer $L = \frac{16}{3}$.

38. Solve for x in $x - x^2 = kx$ we get $x = 0, x = 1-k$

And the areas are

$$A_1 + A_2 = \int_0^1 (x - x^2)dx = \frac{1}{6}$$

$$A_2 = \int_0^{1-k} kx dx + \int_{1-k}^1 (x-x^2)dx$$

$$= \frac{kx^2}{2}\Big|_0^{1-k} + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{1-k}^1$$

$$= \frac{k(1-k)^2}{2} + \frac{1}{6} - \frac{(1-k)^2}{2} + \frac{(1-k)^3}{3}$$

$$= \frac{1}{6}\cdot[1-(1-k)^3]$$

We want $A_1 = A_2$, that is, we want $A_2 = 1/12$, that is,

$$1 - (1-k)^3 = \frac{1}{2}$$

$$(1-k)^3 = \frac{1}{2}$$

$$1 - k = \frac{1}{\sqrt[3]{2}}$$

$$k = 1 - \frac{1}{\sqrt[3]{2}}$$

39. (a) Consider $\int_0^2 (2x - x^2)\, dx$

The integrand consists of the two curves $y = 2x$ and $y = x^2$. Both these curves intersect, when $2x = x^2$ i.e. when$x = 0$ or $x = 2$. therefore The given integral represents the area between the curves $y = 2x$ and $y = x^2$ Which is A_2.

(b) Consider $\int_0^2 (4 - x^2)\, dx$

The integrand consists of two curves $y = 4$ and $y = x^2$. Both these curves intersect when $4 = x^2$ i.e. when $x = -2$ or $x = 2$. But we consider $x = 2$, as the area lies in the 1st Quadrant therefore the given integral represents the area between the curves $y = 4$ and $y = x^2$ which is $A_1 + A_2$.

(c) Consider $\int_0^4 (2 - \sqrt{y})\, dy$

Here the limits of integration correspond to the y-coordinates of the point of intersection of the two curves. This is because here the variable is y and not x. The integrand consists of two curves $x = 2$ and $x = \sqrt{y}$ (i.e. $y = x^2$ with $x > 0$). Both these curves intersect, when $2 = \sqrt{y}$ i.e. when $y = 4$. therefore The given integral represents the area between the curves $x = 2$ and $x = \sqrt{y}$ which is A_3

(d) Consider $\int_0^4 \left(\sqrt{y} - \frac{y}{2}\right) dy$

Here the limits of integration correspond to the y-coordinates of the point of intersection of the two curves. This is because here the variable is y and not x. The integrand consists of two curves $x = \sqrt{y}$ (i.e. $y = x^2$ with $x > 0$) and $x = \frac{y}{2}$. Both these curves intersect, when $\frac{y}{2} = \sqrt{y}$ i.e. when $y^2 - 4y = 0$ i.e. at $y = 0$ and $y = 4$. therefore the given integral represents the area between the curves $x = \sqrt{y}$ and $x = \frac{y}{2}$ which is A_2(same as part **(a)**).

40. (a) Consider the area $A_2 + A_3$. It may be observed from the part (a) of the Exercise 39 that, A_2 is the area bounded by the curves $y = 2x$, $y = x^2$ between the ordinates $x = 0$ and $x = 2$. It may also be observed from the part (c) of the Exercise 39 that, A_3 is the area bounded by the curves $x = 2$ and $y = x^2$ i.e. $x = \sqrt{y}$ therefore from the given figure $A_2 + A_3$ is the area bounded by the curves $y = 2x$ i.e. $x = \frac{y}{2}$ and $x = 2$. therefore

$$A_2 + A_3 = \int_0^4 \left(2 - \frac{y}{2}\right) dy.$$

Note that here we have y as the variable.

(b) Consider the area $A_1 + A_2$, refer part (b) of the Exercise 39 It is in fact the converse of that part.

(c) Consider the area A_1, from the given figure it may be observed that, A_1 is the area bounded by curves $y = 4$ and $y = 2x$. Between the ordinates $x = 0$ and $x = 2$. Therefore $A_1 = \int_0^2 (4 - 2x)\, dx$

(d) A_3 refer part (c) or the Exercise 39. Note that here we have y as the variable.

41. The area between two curves $y = \sin^2(x)$ and $y = 1$, for $0 \le x \le t$ is given by:

$f(t) = \int_0^t (1 - \sin^2 x)\,dx = \int_0^t (\cos^2 x)\,dx$

$= \frac{1}{2}\int_0^t (1 + \cos 2x)\,dx$

$= \frac{1}{2}[x]_0^t + \frac{1}{4}[\sin 2x]_0^t$

$\Rightarrow f(t) = \frac{1}{2}t + \frac{1}{4}\sin 2t$

For finding the critical points,
$f'(t) = 0,$ therefore
$\frac{1}{2} + \frac{1}{4}\cos 2t \cdot (2) = 0.$
$\Rightarrow 1 + \cos 2t = 0$
or $\cos 2t = -1$
$\Rightarrow 2t = n\pi$ for $n = 1, 3, 5, ...$
or $t = \frac{n\pi}{2}$ for $n = 1, 3, 5, ...$

Now, $f''(t) = -\sin 2t$ substituting the value of t in $f''(t)$, we get $f''(t) = 0$. Therefore, $t = \frac{n\pi}{2}$ for $n = 1, 3, 5, ...$ are the points of inflection.

42. Given $g(x)$ is a continuous function of x, for $x \ge 0$ and $|g(x)| \le 1$. $f(t)$ is the area between $y = g(x)$ and $y = 1$ for $0 \le x \le t$, therefore $f(t) = \int_0^t (1 - g(x))\,dx$. As $g(x)$ has the local maxima at $x = a$, $g'(a) = 0$ and $g''(a) < 0$.
Now from (1)
$f'(t) = (1 - g(t))$
$\Rightarrow f''(t) = -g'(t)$
$\Rightarrow f''(a) = -g'(a) = 0$
also $f'(a) = (1 - g(a)) \ge 0$.
Thus $f(t)$ has an point of inflection at $x = a$ and a need not be the critical point, it is only if $g(a) = 1$. If there is a local minima at $x = a$, then $g'(a) = 0$ and $g''(a) > 0$. This does not affect the answer.

43. $f(4) = 16.1e^{.07(4)} = 21.3$
$g(4) = 21.3e^{.04(4-4)} = 21.3$
21.3 represents the consumption rate (million barrels per year) at time $t = 4$ (1/1/74).

$\int_4^{10} \left(16.1e^{.07t} - 21.3e^{.04(t-4)}\right) dt$

$= \left(230e^{.07t} - 532.5e^{.04(t-4)}\right)\Big|_4^{10}$

$= 14.4$ million barrels saved

44. Area $= \int_0^{10} [76e^{0.03t} - (50 - 6e^{0.09t})]\,dt$
≈ 483.616
This area represents amount of wood used by firewood that was not replaced with new growth.

45. For $t \ge 0$,
$b(t) = 2e^{.04t} \ge 2e^{.02t} = d(t)$

$\int_0^{10} \left(2e^{.04t} - 2e^{.02t}\right) dt$

$= \left(50e^{.04t} - 100e^{.02t}\right)\Big|_0^{10}$

$= 2.45$ million people.
This number represents births minus deaths, hence population growth over the ten-year interval.

46. These curves intersect when

$T = \frac{\ln 3 - \ln 2}{.02} \approx 20.27325541$

The area between the curves for $0 \le t \le T$ is the decrease in population from $0 \le t \le T$ (because $b(t) < d(t)$ in this time period).

The area between the curves for $T \le t \le 30$ is the increase in population from $T \le t \le 30$ (because $b(t) > d(t)$ in this time period).

The change in population is given by the integral:

$\Delta P = \int_0^3 [b(t) - d(t))]\,dt$

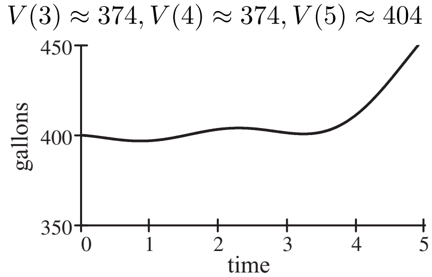
$= \int_0^3 2e^{0.04t} - 4e^{0.02t}\,dt$

≈ 7.3120 million people

47. Without formulae or tables, only rough or qualitative estimates are possible.

time	1	2	3	4	5
amount	397	403	401	412	455

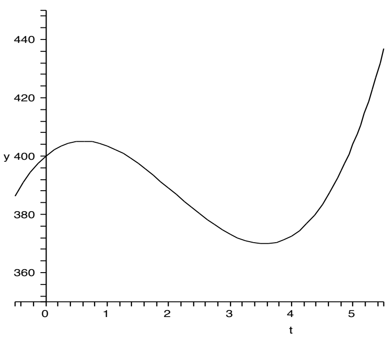
$V(3) \approx 374, V(4) \approx 374, V(5) \approx 404$



48. The change in amount of water is equal to the integral of the difference between the functions (the rate in minus the rate out). Approximating this integral:

$\int_0^1 (\text{Into} - \text{Out})\,dt \approx 0$

$\int_0^2 (\text{Into} - \text{Out})\,dt \approx -8$
$\int_0^3 (\text{Into} - \text{Out})\,dt \approx -26$
$\int_0^4 (\text{Into} - \text{Out})\,dt \approx -26$
$\int_0^5 (\text{Into} - \text{Out})\,dt \approx 4$
Therefore $V(1) = 400$, $V(2) \approx 392$,
$V(3) \approx 374$, $V(4) \approx 374$, $V(5) \approx 404$.



49. In this set-up, p is price and q is quantity. We find that $D(q) = S(q)$ only if $D(q) = S(q)$.

$$10 - \frac{q}{40} = 2 + \frac{q}{120} + \frac{q^2}{1200}$$

$12000 - 30q = 2400 + 10q + q^2$
$q^2 + 40q - 9600 = 0$
$(q - 80)(q + 120) = 0$

within the range of the picture only at $q = 80$. Thus $q^* = 80$ and $p^* = D(q^*) = S(q^*) = 8$.

Consumer surplus, as an area, is that part of the picture below the D curve, above $p = p^*$, and to the left of $Q = q^*$.

Numerically in this case the consumer surplus is

$$\int_0^{q^*} [D(q) - p^*]\,dq = \int_0^{80} \left(2 - \frac{q}{40}\right) dq$$
$$= 2q - \frac{q^2}{80}\Big|_0^{80} = 160 - 80 = 80.$$

The units are dollars (q counting items, p in dollars per item).

50. The intersection point is approximately $(q^*, p^*) = (76, 8)$. Therefore

$$PS = p^*q^* - \int_0^{q^*} S(q)\,dq$$
$$= (8)(76) - \int_0^{76} \left(2 + \frac{q}{120} + \frac{q^2}{1200}\right) dx$$
$$= \frac{86849}{225} \approx 386.00.$$

51. The curves, meeting as they do at 2 and 5, represent the derivatives C' and R'. The area (a) between the curves over the interval $[0, 2]$ is the loss resulting from the production of the first 2000 items. The area (b) between the curves over the interval $[2, 5]$ is the profit resulting from the production of the next 3000 items. The area (c), as the *sum* of the two previous (call it $(a) + (b)$), is *without meaning.* However, the *difference* $(b) - (a)$ would be the total profit on the first 5000 items, or, if negative, would represent the loss. The area (d) between the curves over the interval $[5, 6]$ represents the loss attributable to the (unprofitable) production of the next thousand items after the first 5000.

52. Profit increases when revenue is larger than cost. The point $x = 2$ represents a local minimum in profit. The point $x = 5$ represents a local maximum in profit.

5.2 Volume: Slicing, Disks and Washers

1. $V = \int_{-1}^{3} A(x)dx = \int_{-1}^{3} (x + 2)dx$
$= \left(\frac{x^2}{2} + 2x\right)\Big|_{-1}^{3} = \left(\frac{9}{2} + 6\right) - \left(\frac{1}{2} - 2\right)$
$= 12$

2. $V = \int_0^{10} 10e^{0.01x}dx = \left(1000e^{0.01x}\right)\Big|_0^{10}$
$= 1000(e^{0.1} - 1)$

3. $V = \pi \int_0^2 (4 - x)^2 dx = -\frac{\pi}{3}(4 - x)^3\Big|_0^2$
$= -\frac{\pi}{3}(8 - 64) = \frac{56\pi}{3}$

4. $V = \int_1^4 2(x + 1)^2 dx$
$= \int_1^4 (2x^2 + 4x + 2)dx = 78$

5. (a) $f(0) = 750, f(500) = 0$
$f(x) = -\frac{75}{50}x + 750$
$V = \int_0^{500} \left(-\frac{75}{50}x + 750\right)^2 dx$
$= \frac{50}{75} \cdot \left(\frac{750^3}{3} - 0\right) = 93,750,000 \text{ ft}^3$

(b) In this case, essentially the same integral is set up as in Part (a):

$$V = \int_0^{250} \left(\frac{750}{500}\right)^2 (500-y)^2 dy$$
$$= 82,031,250 \text{ cubic feet}$$

6. $f(0) = 300, f(160) = 0$

$$f(x) = -\frac{15}{8}x + 300$$
$$V = \int_0^1 60\left(-\frac{15}{8}x + 300\right)^2 dx$$
$$= \frac{8}{15} \cdot \left(\frac{300^3}{3} - 0\right) = 4,800,000 \text{ ft}^3$$

This volume is one-eighth of the volume in Example 2.1.

7. The key observation in this problem is that by simple proportions, had the steeple continued to a point it would have had height 36, hence 6 extra feet. One can copy the integration method, integrating only to 30, or one can subtract the volume of the missing "point" from the full pyramid. Either way the answer is $\frac{3^2 36}{3} - \left(\frac{1}{2}\right)^2 \cdot \frac{6}{3} = \frac{215}{2}$ ft^3.

8. This volume is easily computed using elementary geometry formulas. Using calculus and the triangular cross sections, the area of cross sections is 150, so the total volume is $V = \int_0^{60} 150 dx = 9000.$

9. $V = \int_0^{60} \pi x^2 dy = \pi \int_0^{60} 60[60-y] dy$

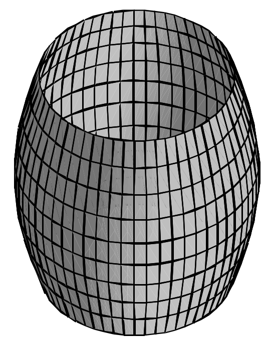
$$= 60\pi\left[60y - \frac{y^2}{2}\right]_0^{60} = 60\pi\left[60^2 - \frac{60^2}{2}\right]$$
$$= \frac{60^3\pi}{2} = 108000\pi \text{ ft}^3$$

10. The radius of the cross-section is given by $r = x$, therefore the volume is given by

$$V = \int_0^{120} \pi x^2 dy = \pi \int_0^{120} 120\,(120-y) dy$$
$$= 120\pi \cdot \left[120y - \frac{y^2}{2}\right]_0^{120}$$
$$= 120\pi\left[120^2 - \frac{120^2}{2}\right]$$
$$= \frac{120^3\pi}{2} = 864,000\pi ft^3.$$

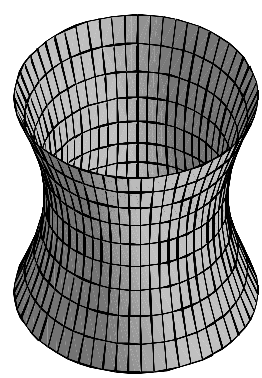
11. $V = \pi \int_0^{2\pi} \left(4 + \sin\frac{x}{2}\right)^2 dx$

$$= \pi \int_0^{2\pi} \left(16 + 8\sin\frac{x}{2} + \sin^2\frac{x}{2}\right) dx$$
$$= \pi\left(16x - 16\cos\frac{x}{2} + \frac{1}{2}x - \frac{1}{2}\sin x\right)\Big|_0^{2\pi}$$
$$= 33\pi^2 + 32\pi \text{ in}^3$$



12. $V = \int_0^{2\pi} \pi\left(4 - \sin\frac{x}{2}\right)^2 dx$

$$= \int_0^{2\pi} \pi\left(16 - 8\sin\frac{x}{2} + \sin^2\frac{x}{2}\right) dx$$
$$= 33\pi^2 - 32\pi \text{ in}^3$$



13. $V = \int_0^1 A(x) dx$

$$\approx \frac{1}{3(10)}[A(0) + 4A(.1) + 2A(.2)$$
$$+ 4A(.3) + 2A(.4) + 4A(.5)$$
$$+ 2A(.6) + 4A(.7) + 2A(.8)$$
$$+ 4A(.9) + A(1.0)]$$
$$= \frac{7.4}{30} \approx 0.2467\text{cm}^3$$

14. $V = \int_0^{1.2} A(x)dx$
$\approx \frac{0.2}{3}[f(0.0) + 4f(0.2) + 2f(0.4)$
$+ 4f(0.6) + 2f(0.8) + 4f(1.0)$
$+ f(1.2)]$
$= \frac{0.2}{3}[0 + 4(0.2) + 2(0.3) + 4(0.2)$
$+ 2(0.4) + 4(0.2) + 0]$
$\approx 0.253333.$

15. $V = \int_0^2 A(x)dx$
$\approx \frac{2}{3(4)}[A(0) + 4A(.5) + 2A(1)$
$+4A(1.5) + A(2)]$
$= 2.5 \text{ ft}^3$

16. $V = \int_0^{0.8} A(x)dx$
$\approx \frac{0.1}{3}[f(0.0) + 4f(0.1) + 2f(0.2)$
$+ 4f(0.3) + 2f(0.4) + 4f(0.5)$
$+ 2f(0.6) + 4f(0.7) + f(0.8)]$
$= \frac{0.1}{3}[2.0 + 4(1.8) + 2(1.7) + 4(1.6)$
$+ 2(1.8) + 4(2.0) + 2(2.1) + 4(2.2)$
$+ 2.4]$
≈ 1.533333

17. (a) $V = \pi \int_0^2 (2-x)^2 dx$
$= -\pi \left(\frac{(2-x)^3}{3}\right)\Big|_0^2$
$= \frac{8\pi}{3}$

(b) $V = \pi \int_0^2 \left[3^2 - \{3 - (2-x)\}^2\right] dx$
$= \pi \int_0^2 \left[9 - \{1+x\}^2\right] dx$
$= \pi \left[9x|_0^2 - \frac{(1+x)^3}{3}\Big|_0^2\right]$
$= \pi \left[18 - \frac{3^3 - 1^3}{3}\right] = \frac{28\pi}{3}$

18. (a) $V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[\left(4-x^2\right)^2 - \left(x^2\right)^2\right] dx$
$= \pi \left[16x - \frac{8x^3}{3}\right]\Big|_{-\sqrt{2}}^{\sqrt{2}}$
$= \pi \left(\frac{64\sqrt{2}}{3}\right)$

(b) $V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left(4-x^2\right)^2 - \left(x^2\right)^2 dx$
$= \pi \left(\frac{64\sqrt{2}}{3}\right)$

19. (a) $V = \pi \int_0^2 (y^2)^2 dy = \pi \int_0^2 y^4 dy$
$= \pi \left(\frac{y^5}{5}\right)\Big|_0^2 = \frac{32\pi}{5}$

(b) $V = \pi \int_0^2 (4)^2 dy$
$- \pi \int_0^2 (4 - y^2)^2 dy$
$= \pi \int_0^2 (-y^4 + 8y^2) dy$
$= \pi \left(-\frac{y^5}{5} + \frac{8y^3}{3}\right)\Big|_0^2$
$= \pi \left[\left(-\frac{32}{5} + \frac{64}{3}\right) - (0+0)\right]$
$= \frac{224\pi}{15}$

20. (a) $V = \pi \int_0^1 (\sqrt{y})^2 dy - \pi \int_0^1 \left(y^2\right)^2 dy$
$= \pi \left(\frac{y^2}{2} - \frac{y^5}{5}\right)\Big|_0^1$
$= \pi \left(\frac{1}{2} - \frac{1}{5}\right)$
$= \frac{3\pi}{10}$

(b) $V = \pi \int_0^1 \left(1 - y^2\right)^2 dy - \pi \int_0^1 (1 - \sqrt{y})^2 dy$
$= \pi \int_0^1 \left(y^4 - 2y^2 - y + 2\sqrt{y}\right) dy$
$= \pi \left(\frac{y^5}{5} - \frac{2y^3}{3} - \frac{y^2}{2} + \frac{4y^{\frac{3}{2}}}{3}\right)\Big|_0^1$
$= \pi \left(\frac{1}{5} - \frac{2}{3} - \frac{1}{2} + \frac{4}{3}\right) = \frac{11\pi}{30}$

21. (a) $V = 4\pi e^2 - \pi \int_1^{e^2} (\ln y)^2 dy$
$= 4\pi e^2$
$- \left[y(\ln y)^2 - 2y \ln y + 2y\right]\Big|_1^{e^2}$
$= 4\pi e^2 - (2e^2 - 2)$
$= 2\pi(e^2 + 1).$

(b) $V = \pi \int_0^2 (e^x + 2)^2\,dx - \pi \int_0^2 (2)^2 dx$

$= \pi \int_0^2 \left(e^{2x} + 4e^x\right) dx$

$= \pi \left(\frac{e^{2x}}{2} + 4e^x\right)\Big|_0^2$

$= \pi \left[\left(\frac{e^4}{2} + 4e^2\right) - \left(\frac{1}{2} + 4\right)\right]$

$= \pi \left(\frac{e^4}{2} + 4e^2 - \frac{9}{2}\right)$

22. (a) $V = \pi \int_{-\pi/4}^{\pi/4} [2^2 - (2 - \sec x)^2] dx$

$= \left(4\pi \int_{-\pi/4}^{\pi/4} \sec x dx\right)$

$= -\left(\pi \tan x|_{-\pi/4}^{\pi/4}\right)$

≈ 15.868

(b) $V = \pi \int_{-\pi/4}^{\pi/4} \sec^2 x dx$

$= \pi \tan x|_{-\pi/4}^{\pi/4} = 2\pi$

23. (a) $V = \pi \int_0^1 \left(\sqrt{\frac{x}{x^2+2}}\right)^2 dx$

$= \frac{\pi}{2} \ln|x^2 + 2|\big|_0^1$

$= \frac{\pi}{2} \ln \frac{3}{2} \approx 0.637$

(b) $V = \pi \int_0^1 \left[3^2 - \left(3 - \sqrt{\frac{x}{x^2+2}}\right)^2\right] dx$

$= \pi \int_0^1 \left(6\sqrt{\frac{x}{x^2+2}} - \frac{3x}{x^2+2}\right) dx$

$= 6\pi \int_0^1 \sqrt{\frac{x}{x^2+2}} dx$

$= -\frac{3\pi}{2} \ln|x^2 + 2|\Big|_0^1$

≈ 7.4721

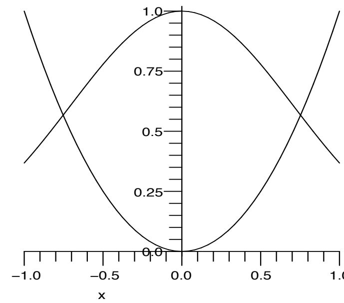
24. $e^{-x^2} = x^2$ when $x \approx \pm 0.753$

(a) $V = \pi \int_{0.753}^{0.753} [(e^{-x^2})^2 - (x^2)^2] dx$

≈ 3.113

(b) $V = \pi \int_{0.753}^{0.753} [(e^{-x^2} + 1)^2 - (x^2 + 1)^2] dx$

≈ 9.266



25. (a) $V = \int_0^4 \pi \left(\frac{4-y}{2}\right)^2 dy$

$= \frac{\pi}{4} \int_0^4 \left(16 - 8y + y^2\right) dy$

$= \frac{\pi}{4} \left[16y - 4y^2 + \frac{y^3}{3}\right]_0^4$

$= \frac{\pi}{4} \left[64 - 64 + \frac{64}{3}\right] = \frac{16\pi}{3}$

(b) $V = \int_0^2 \pi (4 - 2x)^2 dx$

$= \pi \int_0^2 \left(16 - 16x + 4x^2\right) dx$

$= \pi \left[16x - 16\frac{x^2}{2} + \frac{4x^3}{3}\right]_0^2$

$= \pi \left[32 - 32 + \frac{32}{3}\right] = \frac{32\pi}{3}$

(c) $V = \int_0^2 \pi (4)^2 dx - \int_0^2 \pi (2x)^2 dx$

$= \pi \int_0^2 \left(16 - 4x^2\right) dx$

$= \pi \left[16x - \frac{4x^3}{3}\right]_0^2$

$= \pi \left[32 - \frac{32}{3}\right] = \frac{64\pi}{3}$

(d) $V = \int_0^2 \pi (8 - 2x)^2 dx - \int_0^2 \pi (4)^2 dx$

$= \pi \int_0^2 \left(64 - 32x + 4x^2 - 16\right) dx$

$= \pi \left[48x - 32\frac{x^2}{2} + \frac{4x^3}{3}\right]_0^2$

$$= \pi \left[96 - 64 + \frac{32}{3}\right] = \frac{128\pi}{3}$$

(e) $V = \int_0^4 \pi (2)^2 dy - \int_0^4 \pi \left(\frac{y}{2}\right)^2 dy$

$$= \pi \int_0^4 \left(4 - \frac{y^2}{4}\right) dy$$

$$= \pi \left[4y - \frac{1}{4} \cdot \frac{y^3}{3}\right]_0^4$$

$$= \pi \left[16 - \frac{16}{3}\right] = \frac{32\pi}{3}$$

(f) $V = \int_0^4 \pi \left(\frac{8-y}{2}\right)^2 dy - \int_0^4 \pi (2)^2 dy$

$$= \pi \int_0^4 \left(\frac{64 - 16y + y^2}{4} - 4\right) dy$$

$$= \frac{\pi}{4} \left[64y - 16\frac{y^2}{2} + \frac{y^3}{3} - 16y\right]_0^4$$

$$= \pi \left[64 + \frac{64}{3}\right] = \frac{256\pi}{3}$$

26. (a) $V = \int_{-2}^{2} \pi \left(4 - x^2\right)^2 dx = \frac{512\pi}{15}$

(b) $V = \int_0^4 \pi (\sqrt{y})^2 dy = 8\pi$

(c) $V = \int_{-2}^{2} \pi \left[(6 - x^2)^2 - 2^2\right] dx$

$$= \frac{384\pi}{5}$$

(d) $V = \int_{-2}^{2} \pi \left[6^2 - (2 + x^2)^2\right] dx$

$$= \frac{1408\pi}{15}$$

(e) $V = \int_0^4 \pi \left[(2 + \sqrt{y})^2 - (2 - \sqrt{y})^2\right] dy$

$$= \int_0^4 8\pi\, y^{1/2} dy = \frac{16}{3}\pi y^{3/2}\Big|_0^4$$

$$= \frac{128}{3}\pi$$

(f) $V = \int_0^4 \pi \left[(4 + \sqrt{y})^2 - (4 - \sqrt{y})^2\right] dy$

$$= \int_0^4 16\pi\, y^{1/2} dy = \frac{32}{3}\pi y^{3/2}\Big|_0^4$$

$$= \frac{256}{3}\pi$$

27. (a) $V = \int_0^1 \pi (1)^2 dy - \int_0^1 \pi \left(\sqrt{y}\right)^2 dy$

$$= \pi \int_0^1 (1 - y) dy$$

$$= \pi \left(y - \frac{y^2}{2}\right)\Big|_0^1 = \frac{\pi}{2}$$

(b) $V = \int_0^1 \pi \left(x^2\right)^2 dx$

$$= \pi \frac{x^5}{5}\Big|_0^1 = \frac{\pi}{5}$$

(c) $V = \int_0^1 \pi \left(1 - \sqrt{y}\right)^2 dy$

$$= \pi \int_0^1 \left(1 - 2y^{1/2} + y\right) dy$$

$$= \pi \left(y - \frac{4}{3}y^{3/2} + \frac{y^2}{2}\right)\Big|_0^1 = \frac{\pi}{6}$$

(d) $V = \int_0^1 \pi (1)^2 dx - \int_0^1 \pi \left(1 - x^2\right)^2 dx$

$$= \pi \int_0^1 \left(2x^2 - x^4\right) dx$$

$$= \pi \left(\frac{2}{3}x^3 - \frac{x^5}{5}\right)\Big|_0^1 = \frac{7\pi}{15}$$

(e) $V = \int_0^1 \pi (2)^2 dy - \int \pi \left(1 + \sqrt{y}\right)^2 dy$

$$= \pi \int_0^1 \left(3 - 2y^{1/2} - y\right) dy$$

$$= \pi \left(3y - \frac{4}{3}y^{3/2} - \frac{y^2}{2}\right)\Big|_0^1 = \frac{7\pi}{6}$$

(f) $V = \int_0^1 \pi \left(x^2 + 1\right)^2 dx$

$$= - \int_0^1 \pi (1)^2 dx$$

$$= \pi \int_0^1 \left(x^4 + 2x^2\right) dx$$

$$= \pi \left(\frac{x^5}{5} + \frac{2}{3}x^3\right)\Big|_0^1 = \frac{13\pi}{15}$$

28. (a) $V = \int_0^1 \pi x^2 dx = \frac{\pi}{3}$

(b) $V = \int_{-1}^{0} \pi \left[1 - (1 + y)^2\right] dy$

$$+ \int_0^1 \pi \left[1 - (1 - y)^2\right] dy$$

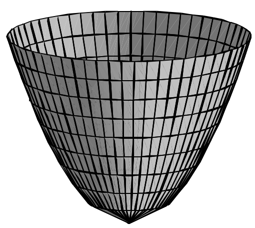
$$= \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3}$$

(c) $V = \int_0^1 \pi \left[(1 + x)^2 - (1 - x)^2\right] dx$

$$= 2\pi$$

(d) $V = \int_0^1 \pi\left[(1+x)^2 - (1-x)^2\right] dx$
$= 2\pi$

29. $V = \pi \int_0^h \left(\sqrt{\frac{y}{a}}\right)^2 dy$
$= \frac{\pi}{a}\int_0^h y dy = \frac{\pi h^2}{2a}$
The volume of a cylinder of height h and radius $\sqrt{h/a}$ is $h \cdot \pi(\sqrt{h/a})^2 = \frac{\pi h^2}{a}$



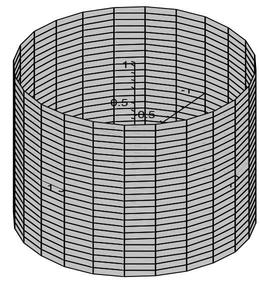
30. The confusing thing here is that the h of Exercise 29 is not the h of this problem. Realizing this,
$V = \frac{\pi(h/a)^2}{2a} = \frac{\pi h^2}{2a^3}$

31. We can choose either x or y to be our integration variable,
$V = \pi \int_{-1}^1 dx = \pi x|_{-1}^1 = 2\pi$

32. This is, of course, a solid ball. Notice that $y = \sqrt{1-x^2}$.
$V = \int_{-1}^1 \pi(\sqrt{1-x^2})^2 dx = \frac{4\pi}{3}$

33. The line connecting the two points $(0,1)$ and $(1,-1)$ has equation
$y = -2x + 1$ or $x = \frac{1-y}{2}$.
$V = \int_{-1}^1 \pi\left(\frac{1-y}{2}\right)^2 dy$
$= \pi\left(\frac{y}{4} - \frac{y^2}{4} + \frac{y^3}{12}\right)\Big|_{-1}^1 = \frac{2\pi}{3}$

34. The fact that the ratios is $3:2:1$ is easy to confirm since we know the volumes are 2π, $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$.



35. $V = \pi \int_{-r}^r \left(\sqrt{r^2-y^2}\right)^2 dy$
$= \pi \int_{-r}^r (r^2 - y^2) dy$
$= \pi \left(r^2 y - \frac{y^3}{3}\right)\Big|_{-r}^r = \frac{4}{3}\pi r^3$

36. $V = \int_0^h \pi\left(-\frac{r}{h}y + r\right)^2 dy = \frac{\pi r^2 h}{3}$

37. If we compute the two volumes using disks parallel to the base, we have identical cross sections, so the volumes are the same.

38. They have the same areas. This can be seen by using elementary geometrical formulas for area or by considering integrals. The area of the parallelograms is given by the integral of the heights of the line segments from 0 to 5. The heights of the line segments are equal.

39. (a) If each of these line segments is the base of square, then the cross-sectional area is evidently
$A(x) = 4(1-x^2)$.
The volume would be
$V_a = \int_{-1}^1 A(x)dx$
$= 2\int_0^1 A(x)dx = 8\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{16}{3}$.

(b) These segments I_x cannot be the literal "bases" of circles, because circles "sit" on a single point of tangency. They could however be diameters. Assuming so, the cross sectional area would be "$\pi/2$ times radius-squared" or $\pi(1-x^2)/2$. The resulting volume would be $\pi/8$ times the previous case, or $2\pi/3$.

40. (a) $V = \int_{-1}^0 [2(x+1)]^2 dx = \frac{4}{3}$

(b) Note that the area of an equilateral triangle with side length l is $\sqrt{3}l^2/4$. This means that for a slice we have
$A(x) = \sqrt{3}(x+1)^2/4$
and
$V = \int_{-1}^{0} \frac{\sqrt{3}(x+1)^2}{4}dx = \frac{\sqrt{3}}{12}$

41. Reasoning as in Exercise 39, the line segment I_x is $[x^2, 2-x^2], (1 \le x \le 1)$. The length of this segment is
$(2-x^2) - x^2 = 2(1-x^2)$,
hence in case (a)
$A(x) = 4(1-x^2)^2 = 4(1-2x^2+x^4)$.
The volume would again be
$V = 2\int_0^1 A(x)dx$
$= 8\left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1$
$= 8\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{64}{15}$.
With the same provisos as in Exercise 39, the answer to (b) would be $\pi/8$ times the (a)-case, or $8\pi/15$.
For (c), the volume would be $\sqrt{3}/4$ times the (a)-case, or $16\sqrt{3}/15$.

42. (a) In this case, $A(x) = (\ln x)^2$ and
$V = \int_1^2 (\ln x)^2 dx$
$= 2(\ln 2)^2 - 4\ln 2 + 2$.

(b) In this case, $A(x) = \frac{\pi}{2}\left(\frac{\ln x}{2}\right)^2$ and
$V = \int_1^2 \frac{\pi}{2}\left(\frac{\ln x}{2}\right)^2 dx$
$= \frac{(\ln 2)^2}{4} - \frac{\ln 2}{2} + \frac{1}{4}$.

43. This time the line segment I_x is $[0, e^{-2x}]$, $(0 \le x \le \ln 5)$. If (a) this is the base of a square, the cross-sectional area is $A(x) = (e^{-2x})^2 = e^{-4x}$. The volume V_a would be the integral
$\int_0^{\ln 5} A(x)dx$
$= \int_0^{\ln 5} e^{-4x}dx = \frac{-e^{-4x}}{4}\Big|_0^{\ln 5}$
$= \frac{1-\left(\frac{1}{5}\right)^4}{4} = \frac{156}{625} = .2496$.
In the (b)-case, the segment I_x is the base of a *semi*circle, so the cross-sectional area would be
$\left(\frac{1}{2}\right)\pi\left(\frac{e^{-2x}}{2}\right)^2 = \left(\frac{\pi}{8}\right)e^{-4x}$.
The resulting volume V_b would be
$(\pi/8)V_a = \frac{39\pi}{1250} \approx .09802$.

44. (a) In this case, $A(x) = (x^2 - \sqrt{x})^2$ and
$V = \int_0^1 (x^2 - \sqrt{x})^2 dx = \frac{9}{70}$

(b) In this case,
$A(x) = \pi\left(\frac{x^2 - \sqrt{x}}{2}\right)^2$ and
$V = \int_0^1 \pi\left(\frac{x^2-\sqrt{x}}{2}\right)^2 dx = \frac{9}{280}$

45. We must estimate $\pi\int_0^3 [f(x)]^2 dx$.
The given table can be extended to give these respective values for
$f(x)2 : 4, 1.44, .81, .16, 1.0, 1.96, 2.56$.
Simpson's approximation to the integral would be
$\frac{3}{(3)(6)}\{4 + 4(1.44) + 2(.81)$
$+ 4(.16) + 2(1.0) + 4(1.96) + 2.56\}$

The sum in the braces is 24.42, and this must be multiplied by $\pi/6$ giving a final answer of 12.786.

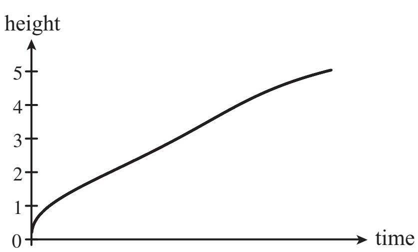
46. Use Simpson's rule.
$V = \int_0^2 \pi[f(x)]^2 dx$
$\approx \frac{\pi(0.25)}{3}[(4.0)^2 + 4(3.6)^2 + 2(3.4)^2$
$+ 4(3.2)^2 + 2(3.5)^2 + 4(3.8)^2 + 2(4.2)^2$
$+ 4(4.6)^2 + (5.0)^2]$
≈ 94.01216

47. In this problem, let $x = g(y)$ be the equation of the given curve describing the shape of the container. For each height y, let $V(y)$ be the volume of fluid in the container when the depth is y. Later we will estimate $V(y)$. For now, one knows that $V(y)$ is the integral of $\pi[g(y)]^2$, or by the fundamental theorem of calculus, that
$\frac{dV}{dy} = \pi[g(y)]^2$.
In actual practice, y and hence V are functions of t (time). Our primary interest is in y as a function of t, but we will obtain this information indirectly, first finding V as a function of y. It appears that $g(y)$ is about $2y$ for $0 < y < 1$,

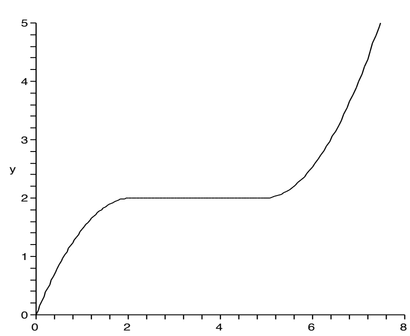
which leads to $[g(y)]^2 = 4y^2$, $V(y) = 4\pi y^3/3$ (on $0 < y < 1$), and $V(1) = 4\pi/3 = 4.2$. We'll keep the *formula* in mind for later, but for now will use the value at $y = 1$ and the crude trapezoidal estimate
$V(y+1) = V(y) + \pi[g^2(y) + g^2(y+1)]/2$
to compile the following table:

y	$g(y)$	$g^2(y)$	$V(y)$
1	2	4	4.2
2	2	9	24.6
3	3	9	52.9
4	3	9	81.2
5	4	16	120.4

The assumption of uniform flow rate amounts to $dV/dt =$ constant, and if we start the clock ($t = 0$) as we begin the flow, we get $V = kt$ for some k. The above table, supplemented by the formula when $y < 1$, can be read to give y (vertical) as a function of V (horizontal). But because $V = kt$, the graph looks exactly the same if the horizontal units are time. In the following picture, we have scaled it on the assumption of a flow rate of 120.4 cubic units per minute, a rate which requires one minute to fill the container. The previous formula $4\pi y^3/3 = V(= kt = (120.4)t)$ (on $0 < y < 1$), becomes $y = (3.06)t^{1/3}$ for very small t, and accounts for the (barely discernible) vertical tangent at $t = 0$.



48.



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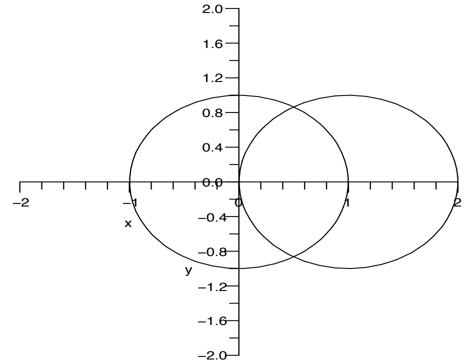


For the points of intersection, solve
$1 - (x-1)^2 = 1 - x^2$
thatis,$x^2 - 2x + 1 = x^2$
or $x = \frac{1}{2} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$
The desired volume V is the sum of the volume V_1 generated by revolving the arc of the circle $x^2 + y^2 = 1$ about the x-axis from $x = \frac{1}{2}$ to $x = 1$ and the volumeV_2 generated by revolving the arc of the circle $(x-1)^2 + y^2 = 1$ about the x-axis from $x = 0$ to $x = \frac{1}{2}$.

Therefore $V = V_1 + V_2$ where,

$$V_1 = \pi \int_{1/2}^{1} (1 - x^2)dx = \pi\left(x - \frac{x^3}{3}\right)\Bigg|_{1/2}^{1}$$

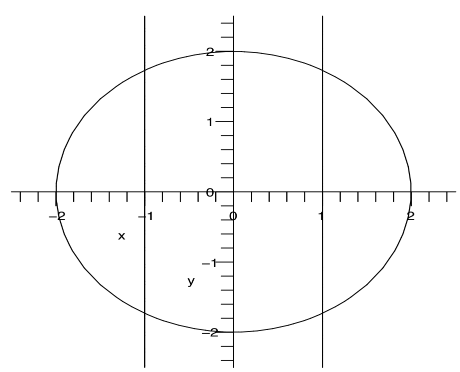
$$= \pi\left[\left(1 - \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{24}\right)\right] = \frac{5\pi}{24}$$

and $V_2 = \pi \int_{0}^{1/2} \left(1 - (x-1)^2\right)dx$

$$= \pi \int_{0}^{1/2} (2x - x^2)dx = \pi\left[x^2 - \frac{x^3}{3}\right]\Bigg|_{0}^{1/2}$$

$= \frac{5\pi}{24}$
$V = V_1 + V_2 \approx 1.308997$

50.



The required region is formed by intersection of revolving circle $x^2 + y^2 = 4$ about y-axis and revolving $x = 1, -4 \leq y \leq 4$about y-axis. Desired volume V is the volume obtained by revolving the shaded region R about the x-axis where R is bounded by $x = 0, x = 1$ and the arc of the circle $x^2 + y^2 = 4$

$x = 1 \Rightarrow y = \pm\sqrt{3}$
$R = R_1 + R_2 + R_3$
R1 is bounded by $x = 0, x^2 + y^2 = 4, y = \sqrt{3}$
R2 is bounded by $x = 0, y = \sqrt{3}, y = -\sqrt{3}$
R3 is bounded by $x = 0, x^2 + y^2 = 4, y = -\sqrt{3}$
Let V1 ,V2 ,V3 be the respective volumes obtained by revolving R1 , R2 , R3 about y-axis

$$V_1 = \int_{\sqrt{3}}^{2} \pi \left(4 - y^2\right) dy$$

$$= \pi \left[4y - \frac{y^3}{3}\right]\Bigg|_{\sqrt{3}}^{2} = \pi \left(\frac{16}{3} - \frac{8\sqrt{3}}{3}\right)$$

$$V_2 = \pi \int_{-\sqrt{3}}^{\sqrt{3}} 1 dy = 2\pi\sqrt{3}$$

$V_3 = V_1$
$V = V_1 + V_2 + V_3$
$= \frac{2\pi}{3}\left(16 - 5\sqrt{3}\right)$

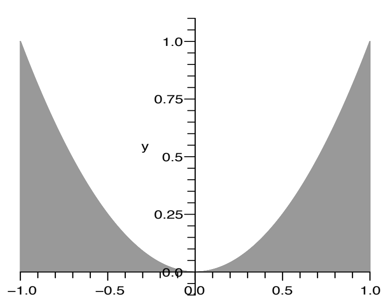
5.3 Volumes by Cylindrical Shells

1. Radius of a shell: $r = 2 - x$
Height of a shell: $h = x^2$

$$V = \int_{-1}^{1} 2\pi(2 - x)x^2 dx$$

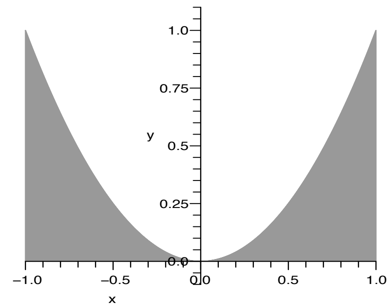
$$= 2\pi \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Bigg|_{-1}^{1}$$

$$= \frac{8\pi}{3}$$



2. Radius of a shell: $r = 2 + x$
Height of a shell: $h = x^2$

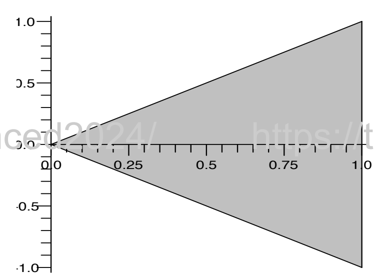
$$V = \int_{-1}^{1} 2\pi(2 + x)x^2 dx = \frac{8\pi}{3}$$



3. Radius of a shell: $r = x$
Height of a shell: $h = 2x$

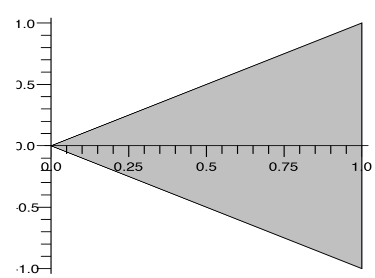
$$V = \int_{0}^{1} 2\pi x(2x) dx$$

$$= \frac{4\pi}{3}x^3\Bigg|_{0}^{1} = \frac{4\pi}{3}$$



4. Radius of a shell: $r = 2 - x$.
Height of a shell: $h = 2x$.

$$V = \int_{0}^{1} 2\pi(2 - x)(2x) dx = \frac{8\pi}{3}$$



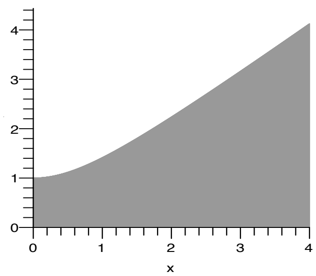
5. Radius of a shell: $r = x$.
eight of a shell: $h = f(x) = \sqrt{x^2 + 1}$.

$$V = \int_{0}^{4} 2\pi x \sqrt{x^2 + 1} dx$$

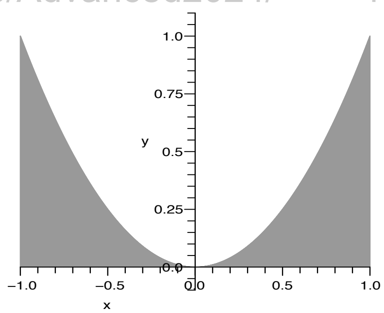
$$= \pi \int_0^4 2x\sqrt{x^2+1}\,dx$$

$$= \pi \left(\frac{2\left(x^2+1\right)^{\frac{3}{2}}}{3} \right) \Bigg|_0^4 = \frac{2\pi}{3}\left[(17)^{\frac{3}{2}} - 1\right]$$

$$\approx 144.7076$$



6. Radius of a shell: $r = 2 - x$.
Height of a shell: $h = f(x) = x^2$.
$V = \int_{-1}^{1} 2\pi\,(2-x)\,x^2 dx = \frac{8\pi}{3}$



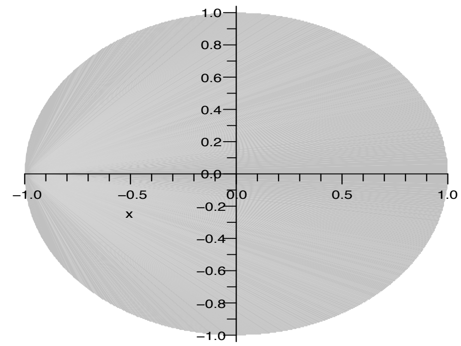
7. Radius of a shell: $r = 2 - y$.
Height of a shell: $h = f(y) = 2\sqrt{1-y^2}$.

$$V = \int_{-1}^{1} 2\pi\,(2-y)\,2\sqrt{1-y^2}dy$$

$$= 4\pi \int_{-1}^{1} (2-y)\sqrt{1-y^2}dy$$

$$= 8\pi \int_{-1}^{1} \sqrt{1-y^2}dy - 4\pi \int_{-1}^{1} y\sqrt{1-y^2}dy$$

$$= 16\pi\left(\frac{\pi}{4}\right) - 0 = 4\pi^2$$



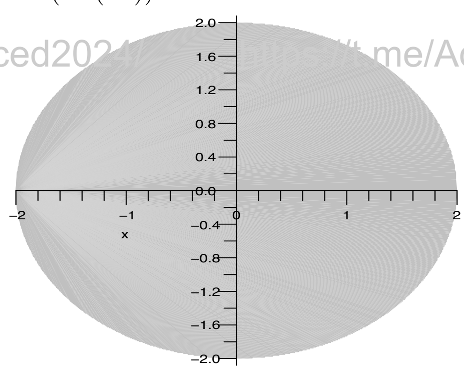
8. Radius of a shell: $r = 4 - y$.
Height of a shell: $h = f(y) = 2\sqrt{4-y^2}$.

$$V = \int_{-2}^{2} 2\pi\,(4-y)\,2\sqrt{4-y^2}dy$$

$$= 4\pi \int_{-2}^{2} (4-y)\sqrt{4-y^2}dy$$

$$= 2\left(8\pi \int_{-2}^{2} \sqrt{4-y^2}dy - 2\pi \int_{-2}^{2} y\sqrt{4-y^2}dy\right)$$

$$= 2\,(8\pi\,(2\pi)) - 0 = 32\pi^2$$



9. $V = \int_{-1}^{1} 2\pi(x+2)\left((2-x^2)-x^2\right)dx$

$$= 2\pi \int_{-1}^{1} \left(4 + 2x - 4x^2 - 2x^3\right)dx$$

$$= 2\pi\left(4x + x^2 - \frac{4x^3}{3} - \frac{x^4}{2}\right)\Bigg|_{-1}^{1}$$

$$= \frac{32\pi}{3}$$

10. $V = \int_{-1}^{1} 2\pi(2-x)\left((2-x^2)-x^2\right)dx$

$$= 2\pi \int_{-1}^{1} \left(4 - 2x - 4x^2 + 2x^3\right)dx$$

$$= 2\pi\left(4x - x^2 - \frac{4x^3}{3} + \frac{x^4}{2}\right)\Bigg|_{-1}^{1}$$

$$= \frac{32\pi}{3}$$

11. $V = \int_{-2}^{2} 2\pi(2+y)(4-y^2)dy$
$= 2\pi\left(8y + 2y^2 - \frac{2y^3}{3} - \frac{y^4}{4}\right)\Big|_{-2}^{2}$
$= \frac{128\pi}{3}$

12. $V = \int_{-2}^{2} 2\pi(2-y)(4-y^2)dy$
$= 2\pi\left(8y - 2y^2 - \frac{2y^3}{3} + \frac{y^4}{4}\right)\Big|_{-2}^{2}$
$= \frac{128\pi}{3}$

13. $V = \int_0^2 2\pi\,(3-x)\,(e^x - x - 1)\,dx$
$= 2\pi\int_0^2 \left((3-x)\,e^x - 2x + x^2 - 3\right)dx$
$= 2\pi\left[[(4-x)\,e^x - x^2 + \frac{x^3}{3} - 3x]\right]\Big|_0^2$
$= 2\pi\left[\left(2e^2 - 4 + \frac{8}{3} - 6\right) - (4-3)\right]$
≈ 21.6448

14. $V = \int_{-1}^{2} 2\pi(3-x)(x-(x^2-2))dx$
$= 2\pi\int_{-1}^{2}\left(6 + x - 4x^2 + x^3\right)dx$
$= 2\pi\left(6x + \frac{x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4}\right)\Big|_{-1}^{2}$
$= \frac{45\pi}{2}$

15. $V = \int_{-2}^{4} 2\pi(5-y)[9-(y-1)^2]dy$
$= \int_{-2}^{4}(y^3 - 7y^2 + 2y + 40)\ dy$
$= \left(\frac{y^4}{4} - \frac{7y^3}{3} + y^2 + 40y\right)\Big|_{-2}^{4}$
$= 288\pi$

16. $V = \int_{-2}^{4} 2\pi(3+y)[9-(y-1)^2]dy$
$= \int_{-2}^{4}(-y^3 - y^2 + 14y + 24)dy$
$= \left(-\frac{y^4}{4} - \frac{y^3}{3} + 7y^2 + 24y\right)\Big|_{-2}^{4}$
$= 288\pi$

17. (a) $V = \int_2^4 2\pi(y)\,(y-(4-y))\,dy$
$= 2\pi\int_2^4\left(2y^2 - 4y\right)dy$
$= 2\pi\left(\frac{2y^3}{3} - 2y^2\right)\Big|_2^4$
$= \frac{80\pi}{3}$

(b) $V = \int_0^2 2\pi(x)\,(4-(4-x))\,dx$
$= \int_2^4 2\pi(x)(4-x)dx$
$= 2\pi\left(\frac{x^3}{3}\right)\Big|_0^2 + 2\pi\left(2x^2 - \frac{x^3}{3}\right)\Big|_2^4$
$= 2\pi\left(\frac{8}{3} + \frac{16}{3}\right) = 16\pi$

(c) $V = \int_2^4 \pi\,(4-(4-y))^2\,dy$
$= \int_2^4 \pi(4-y)^2 dy$
$= \pi\int_2^4 y^2 dy$
$-\pi\int_2^4 (16 - 8y + y^2)dy$
$= \pi\int_2^4 (-16 + 8y)\,dy$
$= \pi\,(-16y + 4y^2)\big|_2^4 = 16\pi$

(d) $V = \int_2^4 2\pi(4-y)\,(y-(4-y))\,dy$
$= 2\pi\int_2^4\left(-2y^2 + 12y - 16\right)dy$
$= 2\pi\left(-\frac{2y^3}{3} + 6y^2 - 16y\right)\Big|_2^4$
$= \frac{16\pi}{3}$

18. (a) $V = \pi\int_{-2}^{0}\left[(x+4)^2 - (-x)^2\right]dx$
$= \pi\int_{-2}^{0}(8x+16)dx$
$= \pi\,(4x^2 + 16x)\big|_{-2}^{0}$
$= 32\pi$

(b) $V = 2\pi\int_{-2}^{0}(2+x)\cdot[(x+2)-(-x-2)]dx$
$= 2\pi\int_{-2}^{0}(2x^2 + 8x + 8)dx$
$= 2\pi\left(\frac{2x^3}{3} + 4x^2 + 8x\right)\Big|_{-2}^{0}$
$= \frac{32\pi}{3}$

(c) $V = 2\pi \int_{-2}^{0} (-x) \cdot [(x+2) - (-x-2)]dx$

$= 2\pi \int_{-2}^{0} (-2x^2 - 4x)dx$

$= 2\pi \left(-\frac{2x^3}{3} - 2x^2 \right)\Big|_{-2}^{0}$

$= \frac{16\pi}{3}$

(d) $V = \pi \int_{-2}^{0} (x+2)^2 dx$

$= \pi \int_{-2}^{0} (x^2 + 4x + 4)dx$

$= \pi \left(\frac{x^3}{3} + 2x^2 + 4x\right)\Big|_{-2}^{0}$

$= \frac{8\pi}{3}$

19. (a) Method of shells.

$V = \int_{-2}^{3} 2\pi(3-x)[x - (x^2 - 6)]dx$

$= \int_{-2}^{3} 2\pi(-x^3 - 4x^2 - 3x + 18)dx$

$= \frac{625\pi}{6}$

(b) Method of washers.

$V = \int_{-2}^{3} \pi[(x^2 - 6)^2 - x^2]dx$

$= \int_{-2}^{3} \pi(x^4 - 13x^2 + 36)dx$

$= \frac{250\pi}{3}$

(c) Method of shells.

$V = \int_{-2}^{3} 2\pi(3+x)[x - (x^2 - 6)]dx$

$= \int_{-2}^{3} 2\pi(x^3 - 2x^2 + 9x + 18)dx$

$= \frac{875\pi}{6}$

(d) Method of washers.

$V = \int_{-2}^{3} \pi[(6+x)^2 - (x^2)^2]dx$

$= \int_{-2}^{3} \pi(-x^4 + x^2 + 12x + 36)dx$

$= \frac{500\pi}{3}$

20. (a) $V = \pi \int_{-1}^{2} [(3+y)^2 - (y^2+1)^2]dy$

$= \pi \int_{-1}^{2} (-y^4 - y^2 + 6y + 8)dy$

$= \pi \left(-\frac{y^5}{5} - \frac{y^3}{3} + 3y^2 + 8y \right)\Big|_{-1}^{2}$

$= \frac{117\pi}{5}$

(b) $V = 2\pi \int_{-1}^{2} (y+1)[((2+y) - y^2]dy$

$= 2\pi \int_{-1}^{2} (-y^3 + 3y + 2)dy$

$= 2\pi \left(-\frac{y^4}{4} + \frac{3y^2}{2} + 2y \right)\Big|_{-1}^{2}$

$= \frac{27\pi}{2}$

(c) $V = \pi \int_{-1}^{2} [(4+y)^2 - (y^2+2)^2]dy$

$= \pi \int_{-1}^{2} (-y^4 - 3y^2 + 8y + 12)dy$

$= \pi \left(-\frac{y^5}{5} - y^3 + 4y^2 + 12y \right)\Big|_{-1}^{2}$

$= \frac{162\pi}{5}$

(d) $V = 2\pi \int_{-1}^{2} (y+2)[(2+y) - y^2]dy$

$= 2\pi \int_{-1}^{2} (-y^3 - y^2 + 4y + 4)\,dy$

$= 2\pi \left(-\frac{y^4}{4} - \frac{y^3}{3} + 2y^2 + 4y \right)\Big|_{-1}^{2}$

$= \frac{45\pi}{2}$

21. (a) $V = \int_{0}^{1} \pi(2-x)^2 dx$

$- \int_{0}^{1} \pi \left(x^2\right)^2 dx$

$= \pi \int_{0}^{1} (x^2 - 4x + 4)dx$

$- \pi \int_{0}^{1} x^4 dx$

$= \pi \int_{0}^{1} (-x^4 + x^2 - 4x + 4)dx$

$= \pi \left(\frac{x^5}{5} + \frac{x^3}{3} - 2x^2 + 4x \right)\Big|_{0}^{1}$

$= \frac{32\pi}{15}$

(b) $V = \int_{0}^{1} 2\pi x \left(2 - x - x^2\right) dx$

$= 2\pi \int_{0}^{1} \left(2x - x^2 - x^3\right) dx$

$= 2\pi \left(x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right)\Big|_{0}^{1}$

$$= \frac{5\pi}{6}$$

(c) $V = \int_0^1 2\pi(1-x)(2-x-x^2)dx$

$$= 2\pi \int_0^1 \left(x^3 - 3x + 2\right)dx$$

$$= 2\pi \left(\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right)\Big|_0^1$$

$$= \frac{3\pi}{2}$$

(d) $V = \int_0^1 \pi\left(2 - 2x^2\right)^2 dx$

$$= \int_0^1 \pi\left(2 - (2-x)\right)^2 dx$$

$$= \pi \int_0^1 (x^4 - 4x^2 + 4)dx - \pi \int_0^1 x^2 dx$$

$$= \pi \int_0^1 (x^4 - 5x^2 + 4)dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right)\Big|_0^1$$

$$= \frac{38\pi}{15}$$

22. (a) $V = \pi \int_0^1 [(2-x^2)^2 - x^2]dx$

$$= \pi \int_0^1 (x^4 - 5x^2 + 4)dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right)dx$$

$$= \frac{97\pi}{60}$$

(b) $V = 2\pi \int_0^1 x(2 - x^2 - x)dx$

$$= 2\pi \int_0^1 (-x^3 - x^2 + 2x)dx$$

$$= 2\pi \left(-\frac{x^4}{4} - \frac{x^3}{3} + x^2\right)dx$$

$$= \frac{3\pi}{5}$$

(c) $V = 2\pi \int_0^1 (x+1)(2 - x^2 - x)dx$

$$= 2\pi \int_0^1 (-x^3 - 2x^2 + x + 2)dx$$

$$= 2\pi \left(-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + 2x\right)dx$$

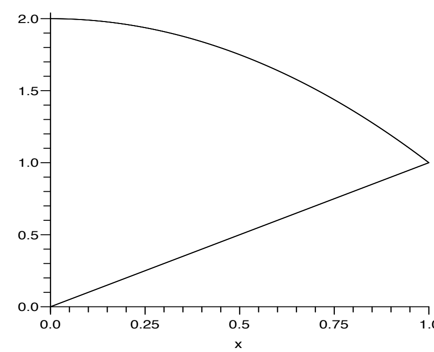
$$= \frac{21\pi}{10}$$

(d) $V = \pi \int_0^1 [(2 - x^2 + 1)^2$
$- (x+1)^2]dx$

$$= \pi \int_0^1 (x^4 - 7x^2 - 2x + 8)dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{7x^3}{3} - x^2 + 8x\right)dx$$

$$= \frac{187\pi}{60}$$



23. (a) $V = 2\pi \int_0^1 y(2 - y - y^2)dy$

$$= 2\pi \int_0^1 (-y^3 - y^2 + 2y)dy$$

$$= 2\pi \left(-\frac{y^4}{4} - y^3 3 + y^2\right)\Big|_0^1$$

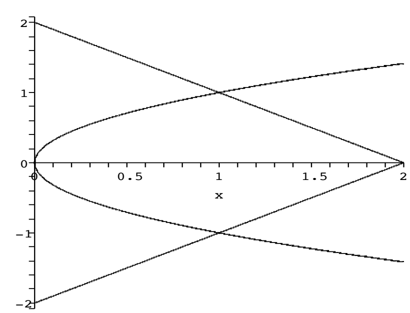
$$= \frac{5\pi}{6}$$

(b) $V = 2\pi \int_0^1 [(2-y)^2 - (y^2)^2]dy$

$$= 2\pi \int_0^1 (-y^4 + y^2 - 4y + 4)dy$$

$$= 2\pi \left(-\frac{y^5}{5} + \frac{y^3}{3} - 2y^2 + 4y\right)\Big|_0^1$$

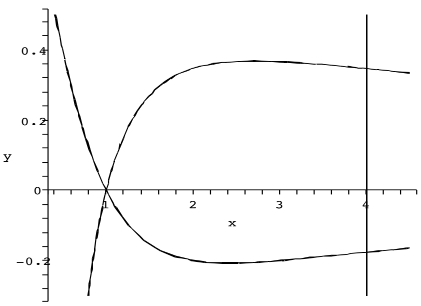
$$= \frac{64\pi}{15}$$



24. (a) $V \approx 2\pi \int_0^{0.79} y[(2-y) - \ln(y+1)]dy$
≈ 2.08

(b) $V \approx \pi \int_0^{0.79} [(2-y)^2 - \ln^2(y+1)]dy$

≈ 6.20

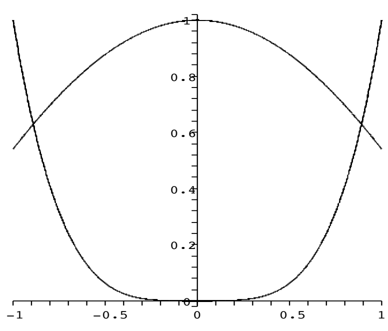


25. (a) $V \approx 2\pi \int_{-0.89}^{0.89} (2-x)\cdot(\cos x - x^4)\,dx$
≈ 16.72

(b) $V \approx \pi \int_{-0.89}^{0.89} [(2-x^4)^2 - (2-\cos x)^2]dx$
≈ 12.64

(c) $V \approx \pi \int_{-0.89}^{0.89} [(\cos x)^2 - (x^4)^2]dx$
≈ 4.09

(d) $V \approx 2\cdot 2\pi \int_{0}^{0.89} x(\cos x - x^4)dx$
≈ 2.99

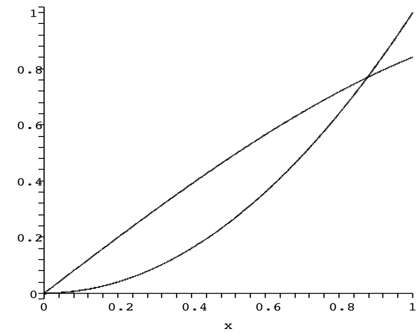


26. (a) $V \approx \pi \int_{0}^{0.85} [(1-x^2)^2 - (1-\sin x)^2]dx$
≈ 0.57

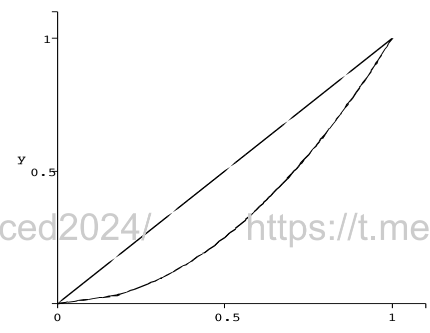
(b) $V \approx 2\pi \int_{0}^{0.85} (1-x)\cdot(\sin x - x^2)dx$
≈ 0.47

(c) $V \approx 2\pi \int_{0}^{0.85} x(\sin x - x^2)dx$
≈ 0.38

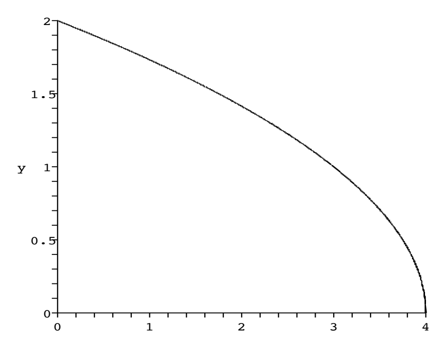
(d) $V \approx \pi \int_{0}^{0.85} [(\sin x)^2 - (x^2)^2]dx$
≈ 0.28



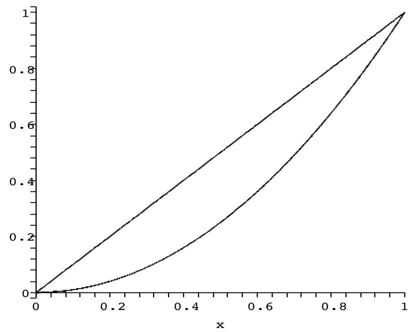
27. Axis of revolution: y-axis
Region bounded by: $x = \sqrt{y}, x = y$



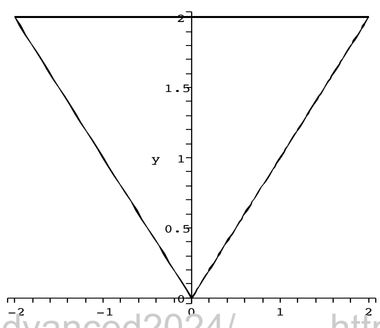
28. Axis of revolution: y-axis
Region bounded by:
$x = 4 - y^2, x = 0, y = 0$



29. Axis of revolution: y-axis
Region bounded by: $y = x, y = x^2$



30. Axis of revolution: $y = 4$
Region bounded by:
$y = x, y = -x, y = 2$



31. If the r-interval $[0, R]$ is partitioned by points r_i, the circular band

$$\{r_i^2 \le x^2 + y^2 \le r_{i+1}^2\}$$

has approximate area $c(r_i)\Delta r_i$ (length times thickness). The limit of the sum of these areas is $A = \lim \sum_{i=1}^{n} c(r_i)\Delta r_i = \int_0^R c(r)dr$ Because we know that $c(r) = 2\pi r$,
we can evaluate the integral, getting
$2\pi \frac{r^2}{2}\Big|_0^R = \pi R^2.$

32. If we think of the area of a circle of radius R as being built up as described in Problem 61, then
$A = \int_0^R 2\pi r dr$ Viewed as a function of R, the derivative is
$\frac{dA}{dR} = 2\pi R$ so this is, of course, not a coincidence.

33. The volume that we are looking for is twice the volume of a shell with radius x and height $\sqrt{1 - x^2}$.
In other words, The bead is mathematically the solid formed up from revolving the region bounded by $y = \sqrt{1 - x^2}, x = 1/2$
and the x-axis around the y-axis.
Therefore
$V = 2 \cdot \int_{1/2}^{1} 2\pi x\sqrt{1 - x^2}dx$
Let $u = 1 - x^2, du = -2xdx$,
and $V = 4\pi \int_{1/2}^{1} x\sqrt{1 - x^2}dx$
$= -\frac{1}{2}\, 4\pi \int_{3/4}^{0} u^{1/2}du$
$= 2\pi \cdot \frac{2}{3}\, u^{3/2}\Big|_0^{3/4}$
$= \frac{\sqrt{3}\pi}{2} cm^3.$

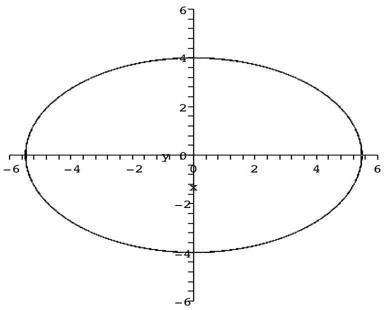
34. The size of the sphere is $4\pi/3$ cm^3, so we look for the value of c such that
$4\pi \int_c^1 x\sqrt{1 - x^2}dx = \frac{2}{3}\pi.$
$V = 4\pi \int_c^1 x\sqrt{1 - x^2}dx$
$= \frac{4}{3}\pi(1 - c^2)^{3/2} = \frac{2}{3}\pi$
Hence we want the size of the hole to be $c = \sqrt{1 - \sqrt{3}\frac{1}{4}} \approx 0.6$ cm.

35. $V = \int_0^1 x(1 - x^2)dx$
$= \int_0^1 (x - x^3)dx$
$= \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{4}$

$V_1 = \int_c^1 x(1 - x^2)dx$
$= \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_c^1 = \frac{1}{4} - \frac{c^2}{2} + \frac{c^4}{4}$
We want
$V - V_1 = \frac{1}{10}V$
Then
$\frac{c^2}{2} - \frac{c^4}{4} = \frac{1}{40}$
$c \approx 0.2265$

36. $V = 4\pi \int_0^4 y\sqrt{30\left(1 - \frac{y^2}{16}\right)}dy$
Let $u = 1 - y^2/16, du = -ydy/8$
$V = -32\sqrt{30}\pi \int_1^0 u^{1/2}du$

$= 32\sqrt{30}\pi \cdot \frac{2}{3} = \frac{64\sqrt{30}\pi}{3}$



5.4 Arc Length and Surface Area

1. For $n = 2$, the evaluations points are $0, 0.5, 1$
$s \approx s_1 + s_2$
$= \sqrt{(0 - 0.5)^2 + [f(0) - f(0.5)]^2}$
$+ \sqrt{(1 - 0.5)^2 + [f(1) - f(0.5)]^2}$
$= \sqrt{0.5^2 + 0.5^4} + \sqrt{0.5^2 + 0.75^2}$
≈ 1.460
For $n = 4$, the evaluations points:
$0, 0, 25, 0.5, 0.75, 1$
$s \approx \sum_{i=1}^{4} s_i \approx 1.474$

2. For $n = 2$, the evaluations points are $0, 0.5, 1$
$s \approx s_1 + s_2 \approx 1.566$
For $n = 4$, the evaluations points:
$0, 0, 25, 0.5, 0.75, 1$
$s \approx \sum_{i=1}^{4} s_i \approx 1.591$

3. For $n = 2$, the evaluations points are
$0, \pi/2, \pi$
$s \approx s_1 + s_2$
$= \sqrt{(\pi/2)^2 + [\cos(\pi/2) - \cos 0]^2}$
$+ \sqrt{(\pi/2)^2 + [\cos\pi - \cos(\pi/2)]^2}$
$= \sqrt{\pi^2 + 4} \approx 3.724$
For $n = 4$, the evaluations points:
$0, \pi/4, \pi/2, 3\pi/4, \pi$
$s \approx \sum_{i=1}^{4} s_i \approx 3.790$

4. For $n = 2$, the evaluation points are $1, 2, 3$
$s \approx s_1 + s_2$
$= \sqrt{1^2 + (\ln 2 - \ln 1)^2}$
$+ \sqrt{1^2 + (\ln 3 - \ln 2)^2}$
≈ 2.296
For $n = 4$, the evaluation points are
$1, 1.5, 2, 2.5, 3$
$s \approx \sum_{i=1}^{4} s_i \approx 4.161$

5. This is a straight line segment from $(0, 1)$ to $(2, 5)$. As such, its length is
$s = \sqrt{(5 - 1)^2 + (2 - 0)^2}$
$= \sqrt{20} = 2\sqrt{5}$

6. $s = \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1 - x^2}} dx$
$= \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} dx$
$= \left(\sin^{-1} x\right)\Big|_{-1}^{1} = \pi$

7. $y'(x) = 6x^{1/2}$, the arc length integrand is $\sqrt{1 + (y')^2} = \sqrt{1 + 36x}$.
Let $u = 1 + 36x$ then
$s = \int_{1}^{2} \sqrt{1 + 36x} dx$
$= \int_{37}^{73} \sqrt{u} \left(\frac{du}{36}\right)$
$= \frac{2}{3(36)} u^{3/2}\Big|_{37}^{73}$
$= \frac{1}{54}(73\sqrt{73} - 37\sqrt{37})$
≈ 7.3824

8. $s = \int_{0}^{1} \sqrt{1 + (e^{2x} - e^{-2x})^2} dx$
$= \int_{0}^{1} \sqrt{e^{4x} - 1 + e^{-4x}} dx$
≈ 3.056

9. $y'(x) = \frac{2x}{4} - \frac{1}{2x} = \frac{1}{2}\left(x - \frac{1}{x}\right)$
$1 + (y')^2 = 1 + \frac{1}{4}\left(x^2 - 2 + \frac{1}{x^2}\right)$
$= \frac{1}{4}\left(x^2 + 2 + \frac{1}{x^2}\right)$
$= \left[\frac{1}{2}\left(x + \frac{1}{x}\right)\right]^2$
$s = \frac{1}{2}\int_{1}^{2} \left(x + \frac{1}{x}\right) dx$

$$= \frac{1}{2}\left(\frac{x^2}{2} + \ln x\right)\Big|_1^2$$
$$= \frac{1}{2}\left(\frac{3}{2} + \ln 2\right)$$
$$\approx 1.0965$$

10. $y'(x) = \frac{1}{2}(x^2 + x^{-2})$
$$s = \int_1^3 \sqrt{1 + \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} dx$$
$$= \frac{1}{2}\int_1^3 \frac{\sqrt{x^8 + 6x^4 + 1}}{x^2} dx$$
$$\approx 5.152$$

11. $x'(y) = \frac{y^3}{2} - \frac{1}{2y^3} = \frac{1}{2}\left(y^3 - \frac{1}{y^3}\right)$
$$1 + (x')^2 = 1 + \frac{1}{4}\left(y^6 - 2 + \frac{1}{y^6}\right)$$
$$= \frac{1}{4}\left(y^6 + 2 + \frac{1}{y^6}\right)$$
$$= \left[\frac{1}{2}(y^3 + \frac{1}{y^3})\right]^2$$
$$s = \int_{-2}^{-1} \sqrt{1 + (x')^2}\, dy$$
$$= -\frac{-1}{-2}\int_{-2}^{-1}\left(y^3 + \frac{1}{y^3}\right) dy$$
$$= \frac{1}{2}\left(-\frac{y^4}{4}\Big|_{-2}^{-1} + \frac{1}{2y^2}\Big|_{-2}^{-1}\right)$$
$$= \frac{1}{2}\left(\frac{15}{4} + \frac{3}{8}\right) = \frac{33}{16}$$

12. Here $x(y) = e^{y/2} + e^{-y/2}$
$x'(y) = \frac{1}{2}\left(e^{y/2} - e^{-y/2}\right)$
Now
$$s = \int_{-1}^{1} \sqrt{1 + \left[\frac{1}{2}\left(e^{y/2} - e^{-y/2}\right)\right]^2} dy$$
$$= \frac{1}{2}\int_{-1}^{1}\left(e^{y/2} + e^{-y/2}\right) dy$$
$$= \int_0^1 \left(e^{y/2} + e^{-y/2}\right) dy$$
$$= 2\left(e^{y/2} - e^{-y/2}\right)\Big|_0^1 = 2\left(\frac{e-1}{\sqrt{e}}\right)$$

13. $y'(x) = \frac{x^{1/2}}{2} - \frac{x^{-1/2}}{2}$
$$= \frac{1}{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)$$
$$1 + (y')^2 = 1 + \frac{1}{4}\left(x - 2 + \frac{1}{x}\right)$$
$$= \frac{1}{4}\left(x + 2 + \frac{1}{x}\right)$$
$$= \left[\frac{1}{2}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\right]^2$$
$$s = \int_1^4 \sqrt{1 + (y')^2}$$
$$= \frac{1}{2}\int_1^4 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$
$$= \frac{x^{3/2}}{3}\Big|_1^4 + \sqrt{x}\Big|_1^4$$
$$= \frac{7}{3} + 1 = \frac{10}{3}$$

14. Here $f(x) = 2\ln\left(4 - x^2\right)$
$\Rightarrow f'(x) = \frac{-4x}{(4 - x^2)}$
$$1 + \left(f'(x)\right)^2 = 1 + \left(\frac{-4x}{(4 - x^2)}\right)^2 = \left(\frac{4 + x^2}{4 - x^2}\right)^2$$
Now, $s = \int_0^1 \left(\frac{4 + x^2}{4 - x^2}\right) dx = 2\ln(3) - 1$

15. $s = \int_{-1}^{1} \sqrt{1 + (3x^2)^2} dx$
$$= \int_{-1}^{1} \sqrt{1 + 9x^4} dx \approx 3.0957$$

16. $s = \int_{-2}^{2} \sqrt{1 + 9x^4} dx \approx 17.2607$

17. $s = \int_0^2 \sqrt{1 + (2 - 2x)^2} dx \approx 2.9578$

18. $s = \int_0^{\pi/4} \sqrt{1 + \sec^4 x} dx \approx 1.2780$

19. $s = \int_0^{\pi} \sqrt{1 + (-\sin x)^2} dx$
$$= \int_0^{\pi} \sqrt{1 + \sin^2 x} dx \approx 3.8201$$

20. $s = \int_1^3 \sqrt{1 + \frac{1}{x^2}} dx \approx 2.3020$

21. $s = \int_0^{\pi} \sqrt{1 + (x \sin x)^2} dx = 4.6984$

22. $s = \int_0^{\pi} \sqrt{1 + e^{-x}\sin^2 x} dx \approx 13.1152$

23. Here $f(x) = 10\left(e^{x/20} + e^{-x/20}\right)$

$\Rightarrow f'(x) = \frac{10}{20}\left(e^{x/20} - e^{-x/20}\right)$

$1 + \left(f'(x)\right)^2 = 1 + \left(\frac{1}{2}\left(e^{x/20} - e^{-x/20}\right)\right)^2$

$= \left(\frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)\right)^2$

Now,

$s = \int_{-20}^{20} \frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)dx$

$= \int_{0}^{20} \left(e^{x/20} + e^{-x/20}\right)dx$

$= 20\left(e^{x/20} - e^{-x/20}\right)\Big|_0^{20}$

$= 20\left(e - e^{-1}\right) \approx 47.0080$

24. $s = \int_{-30}^{30} \sqrt{1 + \left[\frac{1}{2}\left(e^{x/30} - e^{-x/30}\right)\right]^2}\,dx$

$= \int_{-30}^{30} \frac{1}{2}\left(e^{x/30} + e^{-x/30}\right)dx$

$= \left(15e^{x/30} - 15e^{-x/30}\right)\Big|_{-30}^{30}$

$= 30e - 30e^{-1} \approx 70.51207161ft.$

25. In Example 4.4, $y(x) = 5(e^{x/10} + e^{-x/10})$
$y(0) = 5(e^0 + e^0) = 10$
$y(-10) = y(10)$
$= 5(e^1 + e^{-1}) = 15.43$
sag $= 15.43 - 10 = 5.43$ ft

A lower estimate for the arc length given the sag would be

$2\sqrt{(10)^2 + (sag)^2}$

$= 2\sqrt{100 + 29.4849} \approx 22.76$

This looks good against the calculated arc length of 23.504.

26. If $x^{2/3} + y^{2/3} = 1$, then in the first quadrant, $y = (1 - x^{2/3})^{3/2}$ and taking only the first-quadrant case (which would produce one fourth of the total length s), we have $y = \frac{3}{2}(1 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)$

$= -x^{-1/3}(1 - x^{2/3})^{1/2}$
$(y')^2 = x^{-2/3}(1 - x^{2/3}) = x^{-2/3} - 1$

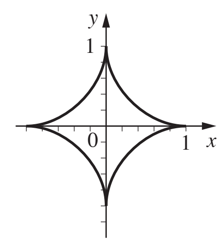
$s = 4\int_0^1 \sqrt{1 + y'^2}\,dx$

$= 4\int_0^1 \sqrt{x^{-2/3}}\,dx$

$= 4\int_0^1 x^{-1/3}\,dx$

$= 4\left(\frac{3}{2}\right)x^{2/3}\Big|_0^1 = 6$

There are some technicalities in fully justifying the preceding computation, since the integrand $(x^{-1/3})$ is unbounded at $x = 0$, but the conclusion is sound.



27. $y = 0$ when $x = 0$ and when $x = 60$, so the punt traveled 60 yards horizontally.

$y'(x) = 4 - \frac{2}{15}x = \frac{2}{15}(30 - x)$

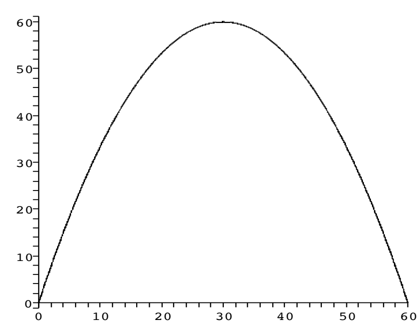
This is zero only when $x = 30$, at which point the punt was $(30)^2/15 = 60$ yards high.

$s = \int_0^{60} \sqrt{1 + \left(4 - \frac{2}{15}x\right)^2}\,dx$

≈ 139.4 yards

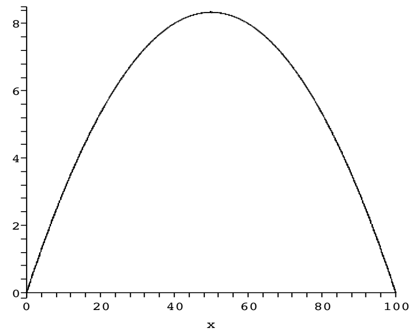
$v = \frac{s}{4 \text{ sec}} = \frac{139.4 \text{ yards}}{4 \text{ sec}} \cdot \frac{3 \text{ feet}}{1 \text{ yard}}$

$= 104.55$ ft/s



28. Since $y(100) = 0$, the ball traveled 100 yards. The maximum height of the ball is $y(50) = \frac{25}{3}$ yards. The arc length is $s = \int_0^{100} \sqrt{1 + \left[\frac{1}{300}(100 - 2x)\right]^2}\,dx$

≈ 101.82215 yards



29. $S = 2\pi \int_0^1 y\,ds$
$= 2\pi \int_0^1 x^2 \sqrt{1+(2x)^2}dx$
≈ 3.8097

30. $S = \int_0^\pi 2\pi \sin x \sqrt{1+\cos^2 x}dx$
≈ 14.42360

31. $S = 2\pi \int_0^2 y\,ds$
$= 2\pi \int_0^2 (2x-x^2)\sqrt{1+(2-2x)^2}dx$
≈ 10.9654

32. $S = \int_{-2}^0 2\pi(x^3-4x)\sqrt{1+(3x^2-4)^2}dx$
≈ 67.06557

33. $S = 2\pi \int_0^1 y\,ds$
$= 2\pi \int_0^1 e^x\sqrt{1+e^{2x}}dx \approx 22.9430$

34. $S = \int_1^2 2\pi \ln x \sqrt{1+\frac{1}{x^2}}dx$
≈ 2.86563

35. $S = 2\pi \int_0^{\pi/2} y\,ds$
$= 2\pi \int_0^{\pi/2} \cos x \sqrt{1+\sin^2 x}dx$
≈ 7.2117

36. $S = \int_1^2 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}dx \approx 8.28315$

37. $s_1 = \int_0^1 \sqrt{1+(6x^5)^2}dx$
$= \int_0^1 \sqrt{1+36x^{10}}dx \approx 1.672$

$s_2 = \int_0^1 \sqrt{1+(8x^7)^2}dx$
$= \int_0^1 \sqrt{1+64x^{14}}dx \approx 1.720$

$s_3 = \int_0^1 \sqrt{1+(10x^9)^2}dx$
$= \int_0^1 \sqrt{1+100x^{18}}dx \approx 1.75$

As $n \to \infty$, the length approaches 2, since one can see that the graph of $y = x^n$ on $[0,1]$ approaches a path consisting of the horizontal line segment from $(0,0)$ to $(1,0)$ followed by the vertical line segment from $(1,0)$ to $(1,1)$.

38. (a) For $0 \le x < 1$, we have $\lim_{n\to\infty} x^n = 0$ Therefore, the length of the limiting curve is 1 (the limiting curve is a horizontal line). Connecting the limiting curve to the endpoint at $(1,1)$ adds an additional length of 1 for a total length of 2.

(b) $y_1 = x^4, y_1' = 4x^3$
$y_2 = x^2, y_2' = 2x$
Since both are increasing for positive x, y_1 is "steeper" (y_2 is "flatter") if and only if $y_1' > y_2'$, i.e.,
$4x^3 > 2x,\ x^2 > \frac{1}{2},\ x > \sqrt{\frac{1}{2}}$

39. (a) $L_1 = \int_{-\pi/6}^{\pi/6} \sqrt{1+\cos^2 x}dx \approx 1.44829$

$L_2 = \sqrt{\left(\sin\frac{\pi}{6} - \sin\left(-\frac{\pi}{6}\right)\right)^2 + \left(\frac{2\pi}{6}\right)^2}$

≈ 1.44797 Hence

$\frac{L_2}{L_1} = \frac{1.44797}{1.44829} \approx .9998$

(b) $L_1 = \int_{-\pi/2}^{\pi/2} \sqrt{1+\cos^2 x}dx \approx 3.8202$
$L_2 = \sqrt{\left(2\sin\frac{\pi}{2}\right)^2 + (\pi)^2}$
$= \sqrt{\pi^2+4} = 3.7242$
Hence
$\frac{L_2}{L_1} \approx 0.9749$

40. (a) $L_1 = \int_3^5 \sqrt{1+(e^x)^2}dx \approx 128.3491$
$L_2 = \sqrt{2^2+(e^5-e^3)^2} \approx 128.3432$
Hence
$\frac{L_2}{L_1} \approx 0.9999$

(b) $L_1 = \int_{-5}^{-3} \sqrt{1+(e^x)^2}dx \approx 2.0006$

$L_2 = \sqrt{2^2+(e^{-5}-e^{-3})^2} \approx 2.0005$

Hence

$\frac{L_2}{L_1} \approx 0.9999$

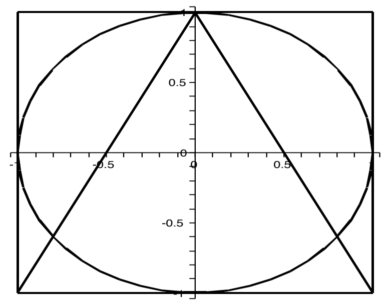
41. (a) Considering only the vertical segment $x = 1$, $(-1 < y < 1)$, the area after rotation, as an integral in y, would be

$2\pi \int_{y=-1}^{y=1} xds(y) = 2\pi \int_{-1}^{1} (1)\sqrt{1+0^2}dy$

$= 2\pi y|_{-1}^{1} = 4\pi$

(height times circumference)

The full solid of revolution is a cylinder with radius 1, and its top and bottom each have area $\pi(1)^2 = \pi$. Hence the total surface area is $4\pi + \pi + \pi = 6\pi$.



(b) $S = \int_{-1}^{1} 2\pi\sqrt{1-y^2}\sqrt{1+\left(\frac{y}{\sqrt{1-y^2}}\right)^2}dy$

$= \int_{-1}^{1} 2\pi\sqrt{1-y^2}\sqrt{\frac{1}{\sqrt{1-y^2}}}\,dy$

$= \int_{-1}^{1} 2\pi\,dy = 4\pi$

(c) The equation for the right segment of the triangle is $x = (1-y)/2$. Hence the resulting area is $2\pi \int_{y=-1}^{y=1} xds(y)$

$= 2\pi \int_{-1}^{1} \left(\frac{1-y}{2}\right)\sqrt{1+\left(-\frac{1}{2}\right)^2}dy$

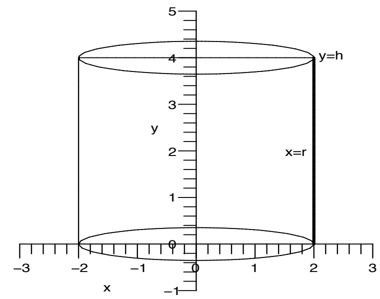
$= 2\pi \int_{-1}^{1} \left(\frac{1-y}{2}\right)\sqrt{\frac{5}{4}}dy$

$= \frac{\pi\sqrt{5}}{2}\left(y-\frac{y^2}{2}\right)\Big|_{-1}^{1} = \pi\sqrt{5}$

The full revolved figure is a cone with added base of radius 1 (and area π). Hence the total surface area $\pi\sqrt{5} + \pi(\sqrt{5}+1)\pi$.

(d) $6\pi : 4\pi : (\sqrt{5}+1)\pi = 3 : 2 : \tau$

42. (a) Surface area of a right circular cylinder of radius r and height h.



Consider a line $x = r$ and $0 \le y \le h$ rotating about the $y-axis$ to form a Right Circular Cylinder.

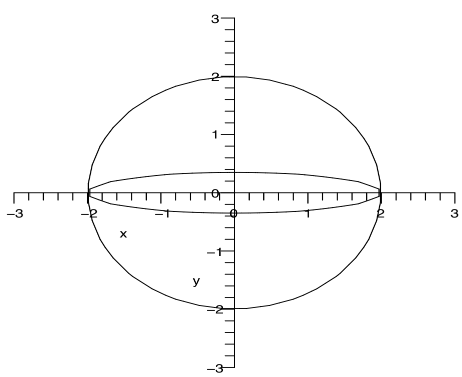
Here $f(y) = r$

Therefore, the surface area

$S = \int_0^h 2\pi f(y)\sqrt{1+(f'(y))^2}dy$

$= \int_0^h 2\pi r\sqrt{1+(0)^2}dy = 2\pi rh$

(b) Surface area of a sphere of radius r



Consider a semicircle of radius r with centre as the origin, its equation is $y = \sqrt{r^2-x^2}$

for $-r \leq x \leq r$ Rotating it about the $x-axis$ we get a sphere Here

$f(x) = \sqrt{r^2 - x^2}$

Therefore, the surface area

$$S = 2\pi \int_{-r}^{r} f(x)\sqrt{1 + (f'(x))^2}dx$$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}dx$$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{1 + \frac{x^2}{r^2 - x^2}}dx$$

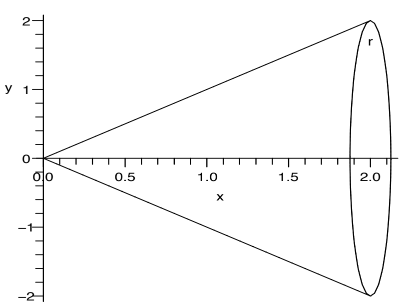
$$= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}}dx$$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}dx$$

$$= 2\pi \int_{-r}^{r} r\,dx$$

$$= 4\pi r^2$$

(c) Surface area of cone of radius r and height h



Consider a line $y = (\frac{r}{h})x$ Rotating it about the $x-axis$, we get a cone of radius r and height h Here

$f(x) = (\frac{r}{h})x$

Therefore, the surface area

$$S = 2\pi \int_{0}^{h} f(x)\sqrt{1 + (f'(x))^2}dx$$

$$= 2\pi \int_{0}^{h} \frac{rx}{h}\sqrt{1 + \left(\frac{r}{h}\right)^2}dx$$

$$= 2\pi \int_{0}^{h} \frac{rx}{h}\sqrt{\frac{r^2 + h^2}{h^2}}dx$$

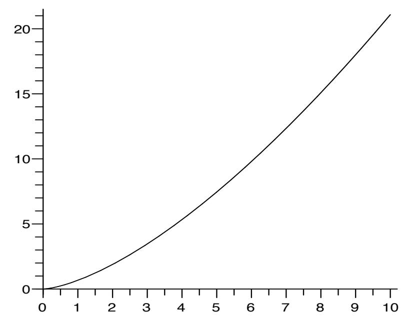
$$= 2\pi \int_{0}^{h} \frac{rx}{h^2}\sqrt{r^2 + h^2}dx$$

$$= \frac{2\pi r\sqrt{r^2 + h^2}}{h^2}\left(\frac{x^2}{2}\right)\Big|_0^h$$

$$= \pi r\sqrt{r^2 + h^2} = \pi r l$$

where $l = \sqrt{r^2 + h^2}$ is the slanted height of the cone.

43.



For the path along the positive $x-axis$, the equation of the path is $f(x) = 0$ Therefore $f'(x) = 0$ The distance covered along the $x-axis$ is

$$L_1 = \int_0^s \sqrt{1 + f'(x)}dx = \int_0^s dx \Rightarrow L_1 = s$$

Now, for the path along the curve

$y = \frac{2}{3}(x)^{3/2}$

The equation of the path is

$f(x) = \frac{2}{3}(x)^{3/2}$

Therefore

$f'(x) = \frac{2}{3} \cdot \frac{3}{2} \cdot x^{1/2} \Rightarrow f'(x) = x^{1/2}$

The distance covered along these curve is

$$L_2 = \int_0^s \sqrt{1 + f'(x)}dx = \int_0^s \sqrt{1+x}\,dx$$

$$L_2 = \frac{2}{3}(s+1)^{3/2} - \frac{2}{3}$$

(a) Consider $L_2 = 2L_1$

$$\frac{L_2}{L_1} = \frac{2(s+1)^{\frac{3}{2}} - 2}{3s} = 2$$

$\Rightarrow (s+1)^{\frac{3}{2}} = 3s + 1$ or

$(s+1)^3 = (3s+1)^2$

$\Rightarrow s^3 - 6s^2 - 3s = 0$

Thus $s = 0$ or $s = 6.464102$

or $s = -0.464102$

But $s > 0$,

therefore $s = 6.464102$

(b) Consider the motion of the person along the $x - axis$

Let $g(t)$ be the distance walked along the $x - axis$

Therefore $g(t) = t, 0 \leq t \leq x, \Rightarrow g^{'}(t) = 1$

Now, consider the motion of the person along the curve $y = \frac{2}{3}(x)^{3/2}$

$f(t) = \frac{2}{3}(t)^{3/2}$ is the distance walked along the curve $y = \frac{2}{3}(x)^{3/2}, 0 \leq t \leq x$

Therefore

$f(t) = \frac{2}{3}(t)^{3/2}, 0 \leq t \leq x \Rightarrow f^{'}(t) = \sqrt{t}$

The ratio of the speeds $= \frac{f^{'}(t)}{g^{'}(t)} = \frac{\sqrt{t}}{1} = 2$

$\Rightarrow t = 4$

44. (a) $\frac{d}{dx}\sqrt{2}\int_0^x \sqrt{1 - \frac{\sin^2 u}{3}}du$

$= \frac{1}{2}\sqrt{2} \cdot \sqrt{4 - 2\sin^2 x}$

$= \sqrt{1 + \cos^2 x}$

(b) $\frac{d}{dx}\left(\frac{1}{4}x\sqrt{1+16x^6} + \int \frac{3/4}{\sqrt{1+16x^6}}\,dx\right)$

$= \left(\frac{1}{4}\sqrt{1+16x^6} + \frac{12x^6}{\sqrt{1+16x^6}}\right) + \frac{3/4}{\sqrt{1+16x^6}}$

$= \frac{1/4(1+16x^6)}{\sqrt{1+16x^6}} + \frac{12x^6}{\sqrt{1+16x^6}} + \frac{3/4}{\sqrt{1+16x^6}}$

$= \frac{1+16x^6}{\sqrt{1+16x^6}} = \sqrt{1+16x^6}$

5.5 Projectile Motion

1. $y(0) = 80, y'(0) = 0$

2. $y(0) = 100, y'(0) = 0$

3. $y(0) = 60, y'(0) = 10$

4. $y(0) = 20, y'(0) = -4$

5. The initial conditions are
$y(0) = 30$ and $y'(0) = 0$
We want to find $y'(t)$ when $y(t) = 0$.

We start with the equation $y''(t) = -32$.
Integrating gives $y'(t) = -32t + c_1$.
From the initial velocity, we have
$0 = y'(0) = -32(0) + c_1$, and so $y'(t) = -32t$
Integrating again gives $y(t) = -16t^2 + c_2$.
From the initial position, we have
$30 = y(0) = -16(0) + c_2$ and so
$y(t) = -16t^2 + 30$.

Solving $y(t) = 0$ gives $t = \pm\sqrt{\frac{15}{8}}$ The positive solution is the solution we are interested in. This is the time when the diver hits the water. The diver's velocity is therefore
$y'\left(\sqrt{\frac{15}{8}}\right) = -32\sqrt{\frac{15}{8}} \approx -43.8$ ft/sec

6. The initial conditions are
$y(0) = 120$ and $y'(0) = 0$
We want to find $y'(t)$ when $y(t) = 0$. We start with the equation $y''(t) = -32$.
Integrating gives $y'(t) = -32t + c_1$.
From the initial velocity, we have
$0 = y'(0) = -32(0) + c_1$, and so $y'(t) = -32t$.

Integrating again gives $y(t) = -16t^2 + c_2$. From the initial position, we have
$120 = y(0) = -16(0) + c_2$ and so
$y(t) = -16t^2 + 120$.

Solving $y(t) = 0$ gives $t = \pm\sqrt{\frac{15}{2}}$. The positive solution is the solution we are interested in. This is the time when the diver hits the water. The diver's velocity is therefore
$y'\left(\sqrt{\frac{15}{2}}\right) = -32\sqrt{\frac{15}{2}}$ ft/sec

7. If an object is dropped (time zero, zero initial velocity) from an initial height of y_0, then the impact moment is $t_0 = \sqrt{y_0}/4$ and the impact velocity (ignoring possible negative sign) is $v_{\text{impact}} = 32t_0 = 8\sqrt{y_0}$
Therefore if the object is dropped from 30 ft, the impact velocity is
$8\sqrt{30} \approx 43.8178$ feet per second.
If dropped from 120 ft, impact velocity is
$8\sqrt{120} \approx 87.6356$ feet per second.
From 3000 ft, impact velocity is
$8\sqrt{3000} \approx 438.178$ feet per second.
From a height of $h\, y_0$, the impact velocity is
$8\sqrt{hy_0} = 8\sqrt{h}\sqrt{y_0} = \sqrt{h}\left(8\sqrt{y_0}\right)$,
which is to say that impact velocity increases by a factor of $\sqrt{h}$ when initial height increases by a factor of h.

8. Ignoring air friction we have initial conditions
$y(0) = 555.427$ and $y'(0) = 0$.

Integrating $y''(t) = -32$ gives
$y'(t) = -32t + c_1$. The initial condition gives

$0 = y'(0) = -32(0) + c_1$ and therefore
$y'(t) = -32t$.

Integrating again gives $y(t) = -16t + c_2$.

The initial condition gives
$555.427 = y(0) = -16(0) + c_2$ and therefore
$y(t) = -16t^2 + 555.427$.

We will assume that the baseball player catches the ball when it is 6 feet above the ground, so we solve
$6 = y(t) = -16t^2 + 555.427$. Solving gives $t \approx \pm 5.86$. We use the positive solution.
The velocity at this time is
$y'(5.86) = -16(5.86) = -93.75$ ft/sec
(If you assume the ball is caught at ground level, the ball will be going 94.27 ft/sec.)

9. As $y''(t) = -9.8, y'(t) = -9.8t + y'(0)$

Therefore, $y(t) = -4.9t^2 + y'(0)\,t + y(0)$

where $y(0)$ represents the height of the cliff and $y(4) = 0$.
Now, $y(4) = -4.9(16) + 4(0) + y(0)$

Thus, $y(0) = 78.4$ is the height of the cliff in meters.

10. Let $y(t)$ be the height of the boulder.
Therefore $y''(t) = -9.8; y(3) = 0$ and
$y'(0) = 0$
Thus, $y'(t) = -9.8t + y'(0)$ and
$y(t) = -4.9t^2 + y'(0)\,t + y(0)$
Thus,
$y(3) = -4.9(9) + y(0) \Rightarrow y(0) = 43.1$meters

11. Let $y(t)$ be the height at any time t.
Here $v'(t) = -9.8$
Therefore $v(t) = -9.8\,t + v(0) = -9.8t + 19.6$
or $y'(t) = -9.8\,t + 19.6$
$\Rightarrow y(t) = -4.9t^2 + 19.6\,t + y(0)$.

But $y(0) = 0$ therefore, $y(t) = -4.9t^2 + 19.6\,t$ which is the height at ay time t. Also the velocity at any instant t is
$v(t) = -9.8\,t + 19.6 = -9.8(t - 2)$

Now for the maximum height,
$v(t) = 0 \Rightarrow t = 2$.
Therefore, maximum height is
$y(2) = -4.9(2)^2 + 19.6(2) + y(0) = 19.6$

He remains in the air until $y(t) = 0$.
That is, till $-4.9t^2 + 19.6t = 0 \Rightarrow t = 0$ or $t = 4$
Therefore, the amount of time he spent in the air is 4sec.
The velocity with which he smacks back is
$v(4) = -9.8(4 - 2) = -19.6$ m/s

12. Let $y(t)$ be the height at any time t.
Here $v'(t) = -9.8$,
Therefore $v(t) = -9.8t + v(0)$
$\Rightarrow y'(t) = -9.8\,t + v(0)$
$\Rightarrow y(t) = -4.9t^2 + v(0)\,t + y(0)$.

But $y(0) = 0$.
Therefore, $y(t) = -4.9t^2 + v(0)\,t$ which is the height at any time t.
Now the maximum height is reached when
$y'(t) = 0$ that is when $t = \dfrac{v(0)}{9.8}$.
Therefore for the maximum height

$$y\left(\frac{v(0)}{9.8}\right) = -4.9\left(\frac{v(0)}{9.8}\right)^2 + v(0)\left(\frac{v(0)}{9.8}\right)$$

$$\Rightarrow 78.4 = -4.9\left(\frac{v(0)}{9.8}\right)^2 + v(0)\left(\frac{v(0)}{9.8}\right)$$

$$\Rightarrow \frac{(v(0))^2}{9.8}\left[-\frac{4.9}{9.8} + 1\right] = 78.4$$

$\Rightarrow v(0) = 39.2$ m/s

13. Reviewing the solution to Exercise 11, the difference is that $v(0)$ is unknown. However, we still see that
$y = -16t^2 + tv(0) = -t[16t - v(0)]$ (factoring, rather than completing the square). The second time that $y = 0$ can be seen to occur at time $t_2 = v(0)/16$, at which time
$v(t_2) = -32t_2 + v(0) = v(0)(-2 + 1) = -v(0)$

Now we see
$v(t) = -32t + v(0) = -32t + 16t_2$
$= -16(2t - t_2)$

The peak was therefore at time $t_2/2$, at which time the height was $-(t_2/2)[16t2/2 - v(0)]$
$= -(t2/2)[(v(0)/2) - v(0)]$
$= -(v(0)/32)[-v(0)/2] = v(0)^2/64$.

In summary, $y_{\max} = [v(0)/8]^2$ in this problem (and more generally, $y_{\max} = [v(0)/8]^2 + y(0)$).
If $y_{\max} = 20$ inches $= 5/3$ feet, then
$v(0)/8 = \sqrt{5/3}$, and
$v(0) = 8\sqrt{5/3} \approx 10.33$ feet per second.

This is considerably less than Michael Jordan's initial velocity of about 17 feet per second, but the difference in velocity is not as dramatic as in height (20 inches to 54 inches).

14. For a given initial velocity of v_0, the velocity and position are given by
$y' = -32t + v_0$
$y = -16t^2 + v_0 t$

The maximum occurs when $y' = 0$ or when

$t_0 = \dfrac{v_0}{32}$
and the maximum height is
$y(t_0) = -16\left(\dfrac{v_0}{32}\right)^2 + v_0\left(\dfrac{v_0}{32}\right) = \left(\dfrac{v_0}{8}\right)^2$
Therefore if the new initial velocity was $1.1v_0$ (an increase of 10%), the new maximum height would be
$\left(\dfrac{1.1v_0}{8}\right)^2 = 1.21\left(\dfrac{v_0}{8}\right)^2$
In other words, it would be an increase in height by 21%.

15. (a) If the initial conditions are
$y(0) = H$ and $y'(0) = 0$
Integrating $y''(t) = -32$ gives
$y'(t) = -32t + c_1$.
The initial condition gives
$y'(t) = -32t + v_0 = -32t$.
Integrating gives
$y(t) = -16t^2 + c_2$.
The initial condition gives
$y(t) = -16t^2 + H$.
The impact occurs when $y(t_0) = 0$ or when $t_0 = \sqrt{y_0}/4 = \sqrt{H}/4$. Therefore the impact velocity is
$y'(t_0) = -32t_0 = -8\sqrt{H}$

(b) If the initial conditions are
$y(0) = 0$ and $y'(0) = v_0$
Integrating $y''(t) = -32$ gives
$y'(t) = -32t + c_1$.
The initial condition gives
$y'(t) = -32t + v_0$.
Integrating gives
$y(t) = -16t^2 + v_0 t + c_2$.
The initial condition gives
$y(t) = -16t^2 + v_0 t$.
The maximum occurs when $y'(t) = 0$ or when $t = v_0/32$.
Therefore the maximum height is
$y\left(\dfrac{v_0}{32}\right) = -\dfrac{16v_0^2}{32^2} + \dfrac{v_0^2}{32} = \dfrac{v_0}{64}$.

16. (a) The time t_0 when the lead ball hits the ground satisfies
$179 = 12800 \ln\left(\cosh\left(\dfrac{t_0}{20}\right)\right)$
$\cosh\left(\dfrac{t_0}{20}\right) = e^{179/12800}$
$t_0 \approx 3.3526$
At time t_0, the height of the wood ball is
$179 - \dfrac{7225}{8}\ln\left(\cosh\left(\dfrac{16}{85}t_0\right)\right)$
$\approx 179 - 169.0337 = 9.9663$ ft

(b) The time t_1 that the wood ball need to hit the ground satisfies
$179 = \dfrac{7225}{8}\ln\left(\cosh\left(\dfrac{16}{85}t_1\right)\right)$
$\cosh\left(\dfrac{16}{85}t_1\right) = e^{1432/7225}$
$t_1 \approx 3.4562$
The wood ball need to be released about $t_1 = t_0 = 0.1036$ seconds earlier.

17. The starting point is
$y'' = -9.8$, $y'(0) = 98\sin(\pi/3) = 49\sqrt{3}$.
We get $y(t) = -4.9t^2 + ty'(0)$
$= -4.9t(t - [v(0)/4.9])$
$= -4.9t(t - 10\sqrt{3})$
The flight time is $10\sqrt{3}$. As to the horizontal range, we have $x'(t)$ constant and forever equal to $98\cos(\pi/3) = 49$. Therefore $x(t) = 49t$ and in this case, the horizontal range is $49(10\sqrt{3})$ (meters).

18. Here $y'(0) = 40\sin\left(\dfrac{\pi}{6}\right) = 20$
Therefore $y(t) = -4.9t^2 + 20t$
$= t(-4.9t + 20)$
$\Rightarrow$ the time of flight $= t = \dfrac{20}{4.9} = 4.082$

Now, for the horizontal range $x(t)$
$x'(t) = 40\cos\left(\dfrac{\pi}{6}\right) = 20\sqrt{3}$
Therefore
$x(t) = 20\sqrt{3}t$ and
$x(4.082) = 20(1.7321)(4.082) = 141.3919$
Repeating the same for the angle 60^0
$y'(0) = 40\sin\left(\dfrac{\pi}{3}\right) = 34.6410$
Therefore
$y(t) = -4.9t^2 + (34.6410)t$
$\Rightarrow y(t) = t(-4.9t + 34.6410)$
$\Rightarrow$ the time of flight $= t = \dfrac{34.6410}{4.9} = 7.0696$
Now, for the horizontal range $x(t)$
$x'(t) = 40\cos\left(\dfrac{\pi}{3}\right) = 20$
Therefore $x(t) = 20t$ and
$x(7.0696) = 20(7.0696) = 141.3919$

19. This problem modifies Example 5.5 by using a service angle of 6° (where the Example 5.5 used 7°) and no other changes. Here the serve hits the net.

Next we want to find the range for which the serve will be in.

If θ is the angle, then the initial conditions are
$x'(0) = 176\cos\theta$, $x(0) = 0$
$y'(0) = 176\sin\theta$, $y(0) = 10$

Integrating $x''(t) = 0$ and $y''(t) = -32$, then using the initial conditions gives
$x'(t) = 176\cos\theta$
$x(t) = 176(\cos\theta)t$
$y'(t) = -32t + 176\sin\theta$
$y(t) = -16t^2 + 176(\sin\theta)t + 10$

To make sure the serve is in, we see what happens at the net and then when the ball hits the ground. First, the ball passes the net when $x = 39$ or when $39 = 176(\cos\theta)t$. Solving gives $t = \dfrac{39}{176\cos\theta}$ Plugging this in for the function $y(t)$ gives

$$y\left(\frac{39}{176\cos\theta}\right) = -16\left(\frac{39}{176\cos\theta}\right)^2 + 176(\sin\theta)\left(\frac{39}{176\cos\theta}\right) + 10 = -\frac{1521}{1936}\sec^2\theta + 39\tan\theta + 10$$

We want to ensure that this value is greater than 3 so we determine the values of θ that give $y > 3$ (using a graphing calculator or CAS). This restriction means that we must have $-0.15752 < \theta < 1.5507$

Next, we want to determine when the ball hits the ground. This is when
$0 = y(t) = -16t^2 + 176(\sin\theta)t + 10$
We solve this equation using the quadratic formula to get

$$t = \frac{-176\sin\theta \pm \sqrt{176^2\sin^2\theta + 640}}{-32}$$

We are interested in the positive solution, so

$$t = \frac{176\sin\theta + \sqrt{176^2\sin^2\theta + 640}}{32}$$

Substituting this in to
$x(t) = 176(\cos\theta)t$ gives

$$x = 44\cos\theta\left(22\sin\theta + \sqrt{484\sin^2\theta + 10}\right)$$

We want to determine the values of θ that ensure that $x < 60$. Using a graphing calculator or a CAS gives $\theta < -0.13429$

Putting together our two conditions on θ now gives the possible range of angles for which the serve will be in:
$-0.15752 < \theta < -0.13429$

20. In these tennis problems, the issue is purely geometric. Time is irrelevant. One can obtain valuable information by eliminating time and writing y as a function of x. For example, with service angle of θ (in degrees below the horizontal), initial speed v_0, and initial height h, one has

$y(t) = -16t^2 - tv_0\sin\theta + h$,
$x(t) = tv_0\cos\theta$, and hence

$$y = f(x) = \frac{-16x^2}{v_0^2\cos^2\theta} - \frac{x\sin\theta}{\cos\theta} + h$$

Now one could put $x = 60$ (the serve would be in if $f(60) < 0$), or put $x = 39$ (the serve would clear the net if $f(39) > 3$. If one were to set $f(60) = 0$ and solve for v_0, one would obtain a critical speed (call it v_1) for the given (h, θ), *above which the serve would be out.* Solving $f(39) = 3$ one would obtain a second critical speed (call it v_2), *below which the serve would hit the net.* Below we tabulate v_1 and v_2 for $h = 10$ and selected values of θ.

In the 7° line, we see that it would be necessary to reduce the service speed to 149ft./sec. to get it in, and the net would not be a problem. The 7.6° line has these interesting features: the service at 176 ft./sec. is out, whereas the service at 170 ft./sec. is in.

h	θ	v_1	v_2
feet	degrees	ft./sec	ft./sec
10	7.0	149.0	105.7
10	7.6	171.5	117.4
10	8.0	193.6	127.8

21. Let $(x(t), y(t))$ be the trajectory. In this case
$y(0) = 6, x(0) = 0$
$y'(0) = 0, x'(0) = 130$
$x''(t) \equiv 0, x'(t) \equiv 130$
$x(t) = 130t$

This is 60 at time $t = 6/13$. Meanwhile,
$y''(t) = -32, y'(t) = -32t$
$y(t) = -16t^2 + 6$

$$y\left(\frac{6}{13}\right) = -16\left(\frac{6}{13}\right)^2 + 6 = \frac{438}{169}$$

$$y\left(\frac{6}{13}\right) \approx 2.59 \text{ ft}$$

22. If the initial speed is now 80 ft/s, the equations become
$x(t) = 80t$
$y(t) = -16t^2 + 6$

The ball crosses home plate when $x = 60$, or when $t = 3/4$. At the home plate, we then have,
$y(3/4) = -16(3/4)^2 + 6 = -3$

In other words, the ball is "under" the ground and the ball hits the ground before reaching home plate.

23. Let $(x(t), y(t))$ be the trajectory. In this case $5°$ is converted to $\pi/36$ radians.

$y(0) = 5, x(0) = 0$
$y'(0) = 120\sin\frac{\pi}{36} \approx 10.46$
$x'(0) = 120\cos\frac{\pi}{36} \approx 119.54$
$x''(0) \equiv 0$
$x'(t) \equiv 119.54$
$x(t) = 119.54t$
This is 120 when
$t = 120/119.54 = 1.00385\ldots$

Meanwhile,
$y''(t) = -32$
$y'(t) = -32t + 10.46$
$y(t) = -16t^2 + 10.46t + 5$
$y(1.00385) = -16(1.00385)^2$
$+ 10.46(1.00385) + 5$
$y(1.00385) \approx -.62$ ft

24. We are assuming that the height at 120 feet is the same as the release height 5. Let θ be the angle of release (above the horizontal).
We have
$y(t) = -16t^2 + 120t\sin\theta + 5$
$x(t) = 120t\cos\theta$

Thus $x(t)$ will be 120 when $t = 1/cos\theta$, at which time $y(t)$ will be 5 only if
$\frac{-16}{\cos^2\theta} + 120\frac{\sin\theta}{\cos\theta} = 0$
Hence if $120\sin\theta\cos\theta = 16$
$60\sin 2\theta = 16$
$2\theta = \sin^{-1}(16/60) = .2699\ldots,$
$\theta = .135$ (radians) or about $7.7°$

To find the aim, we need the length of the vertical leg of a right triangle with opposite angle $7.7°$, and adjacent leg 120 ft. Thus the player should aim
$120\tan(7.7°) \approx 120\tan(.135) \approx 16.2$ ft

above the first baseman's head.

25. (a) Assuming that the ramp height h is the same as the height of the cars, this problem seems to be asking for the initial speed v_0 required to achieve a horizontal flight distance of 125 feet from a launch angle of $30°$ above the horizontal. We may assume $x(0) = 0, y(0) = h$, and we find

$y'(0) = v_0\sin\frac{\pi}{6} = \frac{v_0}{2}$
$x'(0) = v_0\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}v_0$
$y''(t) \equiv -32,\ x''(t) \equiv 0$
$y'(t) = -32t + \frac{v_0}{2},\ x'(t) = \frac{\sqrt{3}}{2}v_0$
$y(t) = -16t^2 + \frac{v_0}{2}t + h,$
$x(t) = \frac{\sqrt{3}}{2}v_0 t.$

$x(t)$ will be 125 if $t = 250/\left(\sqrt{3}\nu_0\right)$ at which time we require that y be h. Therefore

$$-16\left(\frac{250}{\sqrt{3}v_0}\right)^2 + \frac{v_0}{2}\left(\frac{250}{\sqrt{3}v_0}\right) = 0$$
$$v_0 = \sqrt{\frac{8000}{\sqrt{3}}} \approx 68\text{ft/s}$$

(b) With an angle of $45° = \pi/4$, the equations become
$y'(0) = v_0\sin\frac{\pi}{4} = \frac{v_0}{\sqrt{2}}$
$x'(0) = v_0\cos\frac{\pi}{4} = \frac{v_0}{\sqrt{2}}$
$y''(t) = -32,\quad x''(t) = 0$
$y'(t) = -32t + \frac{v_0}{\sqrt{2}},\quad x'(t) = \frac{v_0}{\sqrt{2}}$
$y(t) = -16t^2 + \frac{v_0 t}{\sqrt{2}} + h,$
$x(t) = \frac{v_0 t}{\sqrt{2}}$

where h is the height of the ramp.

We now solve $x(t) = 125$ which gives
$t_0 = t = \frac{125\sqrt{2}}{v_0}$
At this distance, we want the car to be at a height h to clear the cars. This gives the equation $y(t_0) = h$, or

$$-16\left(\frac{125\sqrt{2}}{v_0}\right)^2 + \frac{125v_0\sqrt{2}}{v_0\sqrt{2}} + h = h$$

Solving for v_0 gives
$v_0 = 20\sqrt{10} \approx 63.24$ ft/s.

26. Let $(x(t), y(t))$ be the trajectory. In this case,
$y(0) = 256, x(0) = 0$
$y'(0) = 0, x'(0) = 100$
$y''(t) \equiv 32,\ x''(t) \equiv 0$
$y'(t) = -32t, y(t) = -16t^2 + 256$
$x'(t) = 100, x(t) = 100t$

y will be zero when $t = 4$, at which time x will be 400. This is the drift distance.

27. (a) In this case with
$\theta_0 = 0$ and $\omega = 1$
$x''(t) = -25\sin(4t)$
$x'(0) = x(0) = 0$
$x'(t) = \frac{25}{4}\cos 4t - \frac{25}{4}$

$x(t) = \frac{25}{16}\sin 4t - \frac{25}{4}t$

(b) With $\theta_0 = \frac{\pi}{2}$ and $\omega = 1$

$x''(t) = -25\sin\left(4t + \frac{\pi}{2}\right)$

$x'(0) = x(0) = 0$

$x'(t) = \frac{25}{4}\cos\left(4t + \frac{\pi}{2}\right)$

$x(t) = \frac{25}{16}\sin\left(4t + \frac{\pi}{2}\right) - \frac{25}{16}$

28. (a) With $\theta_0 = \frac{\pi}{4}$ and $\omega = 2$

$x''(t) = -25\sin\left(8t + \frac{\pi}{4}\right)$

$x'(0) = 0 = x(0)$

$x'(t) = \frac{25}{8}\cos\left(8t + \frac{\pi}{4}\right) - \frac{25\sqrt{2}}{16}$

$x(t) = \frac{25}{64}\sin\left(8t + \frac{\pi}{4}\right) - \frac{25\sqrt{2}}{16}t - \frac{25\sqrt{2}}{128}$

(b) With $\theta_0 = \frac{\pi}{4}$ and $\omega = 1$

$x''(t) = -25\sin(4t + \pi/4)$

$x'(0) = x(0) = 0$

$x'(t) = \frac{25}{4}\cos(4t + \pi/4) - \frac{25\sqrt{2}}{8}$

$x(t) = \frac{25}{16}\sin(4t + \pi/4)\frac{25t\sqrt{2}}{8} - \frac{25\sqrt{2}}{32}$

29. The initial conditions are
$s(0) = 0, s'(0) = 0.$

Integrating $s''(t) = -32$ gives
$s'(t) = -32t + c_1.$
The initial condition gives
$s'(t) = -32t.$
Integrating gives
$s(t) = -16t^2 + c_2.$
The initial condition gives
$s(t) = -16t^2.$
Realizing that -32 was given in feet per second^2,and we are using centimeters now, we use, 1 foot $= 30.48$ cms
and get
$s(t) = -487.68t^2$ cm

The yardstick is grabbed when $s(t_0) = -d$, that is when
$t_0 = \frac{\sqrt{d}}{487.68} \approx 0.045\sqrt{d}$

30. Using the result from Exercise 15,
$v_1 = 8\sqrt{H}.$

Now we need to compute how big v_2 is in order for the ball to rebound to cH.
The initial conditions are
$v(0) = v_2, s(0) = 0.$
Integrating $a(t) = -32$ gives
$v(t) = -16t + v(0) = -16t + v_2$
Integrating again we get
$s(t) = -8t^2 + v_2t + s(0) = -8t^2 + v_2t$
$s(t_0) = cH$ when $v(t_0) = 0$, that is when $t_0 = v_2/16$

$-8\left(\frac{v_2}{16}\right)^2 + v_2\left(\frac{v_2}{16}\right) = cH$

$\frac{v_2^2}{32} = cH$

$v_2 = \sqrt{32cH}$

Now the coefficient of restitution is
$\frac{v_2}{v_1} = \frac{\sqrt{32cH}}{8\sqrt{H}} = \sqrt{\frac{c}{2}}$

31. From Exercise 5, time of impact is
$t = \frac{\sqrt{30}}{4}$ seconds.
$2\frac{1}{2}$ somersaults corresponds to 5π radians of revolution.
Therefore the average angular velocity is
$\frac{5\pi}{\sqrt{30}/4} = \frac{20\pi}{\sqrt{30}} \approx 11.47$ rad/sec

32. The initial conditions are
$y(0) = 10$, $y'(0) = 160\sin 45°$
$x(0) = 0$, and $x'(0) = 160\cos 45°$

Integrating $x''(t) = 0$ and $y''(t) = -32$ and using the initial conditions gives
$x'(t) = 80\sqrt{2}$
$x(t) = (80\sqrt{2})t$
$y'(t) = -32t + 80\sqrt{2}$
$y(t) = -16t^2 + (80\sqrt{2})t + 10.$

We now want to solve for when $y(t) = 5$, which gives the equation
$-16t^2 + (80\sqrt{2})t + 10 = 5$
Solving gives
$t = \frac{-80\sqrt{2} \pm \sqrt{12800 + 640}}{-32} \approx -0.087, 7.16.$
We, of course, take the positive solution.
$x(7.16) = (80\sqrt{2})(7.16) \approx 810.1.$
So, place the net 810.1 feet away from the cannon.
$y'(7.16) = -32(7.16) + 80\sqrt{2} \approx 116.0$
Since we have $x' = 80\sqrt{2} \approx 113.1$, this means that the impact velocity is
$v = \sqrt{(x')^2 + (y')^2}$
$= \sqrt{(116.0)^2 + (113.1)^2} \approx 162.0$

which means the Flying Zucchini comes down squash.(We should have known this—the velocity at a height of 10 should have been equal to his initial velocity so his velocity at a height of of 5 should be slightly higher, which it is.)

33. Let $(x(t), y(t))$ be the trajectory of the center of the basketball. We are assuming that $y(0) = 6$, $x(0) = 0$, the angle of launch θ of the shot is $52°$ ($\theta = \frac{13\pi}{45}$ in radians) and the initial speed is 25 feet per second. Therefore

$y'(0) = 25 \sin \frac{13\pi}{45} \approx 19.70$

$x'(0) = 25 \cos \frac{13\pi}{45} \approx 15.39$

$y''(t) \equiv -32, x''(t) \equiv 0$
$y'(t) = -32t + 19.70, x'(t) \equiv 15.39$
$y(t) = -16t^2 + 19.70t + 6,$
$x(t) = 15.39t.$
x will be 15 when t is about $15/15.39 = .9746\ldots$, at which time y will be about
$-16(.9746\ldots)^2 + 19.70(.9746\ldots) + 6 \approx 10$

In other words, the center of the ball is at position $(15, 10)$ and the shot is good. More generally, with unknown θ, the number 19.70 is replaced by $25 \sin \theta$, while the number 15.39 is replaced by $25 \cos \theta$. y will be exactly 10 if

$$-16t^2 + 25t \sin \theta + 6 = 10$$
$$t = \frac{25 \sin \theta + \sqrt{625 \sin^2 \theta - 256}}{32}$$
$$x = 25t \cos \theta.$$

As a function of θ, this last expression is too complicated to use calculus (easily) to maximize and minimize it on the θ-interval $(48°, 57°)$, but quick spreadsheet calculations give these values:
(Observe that x is not a monotonic function of θ in this range. It takes its maximum when θ is between 52.4 and 52.5 degrees. The evidence is overwhelming that all the shots will be good.)

θ	t	x
degrees	seconds	feet
48.0	0.8757	14.6484
49.0	0.9021	14.7958
50.0	0.9274	14.9024
51.0	0.9516	14.9710
52.0	0.9748	15.0038
52.1	0.9771	15.0051
52.2	0.9793	15.0062
52.3	0.9816	15.0069
52.4	0.9838	15.0073
52.5	0.9861	15.0073
52.6	0.9883	15.0070
52.7	0.9905	15.0064
52.8	0.9928	15.0054
52.9	0.9950	15.0042
53.0	0.9972	15.0026
54.0	1.0187	14.9690
55.0	1.0394	14.9044
56.0	1.0594	14.8100
57.0	1.0787	14.6869

34. Let$(x(t), y(t))$ be the trajectory of the centre of the basketball.

Here $y(0) = 8, x(0) = 0, \theta = 30^0$ and $v = 27$.
Therefore y'(0)=27sin$\frac{\pi}{6} = 13.5$ and
$x'(0) = 27 \cos \frac{\pi}{6} = 23.3827$
$y''(t) \equiv -32 \Rightarrow y'(t) = -32t + 13.5,$
Or $y(t) = -16t^2 + 13.5t + 8$ also,
$x''(t) \equiv 0 \Rightarrow x'(t) \equiv 23.3827$

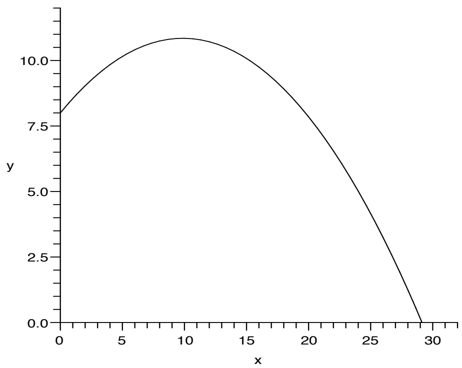
That is $x(t) = (23.3827)\, t$

(a) Consider $x(t) = 15$
$\Rightarrow t = \frac{15}{23.3827} \approx 0.6415,$
for which
$y(0.6415)$
$= -16(0.6415)^2 + 13.50\,(0.6415) + 8$
$= 10.0759$

Now, $y(t) = 10 \Rightarrow t \approx 0.6520$ for which

$x(0.6520) = (23.3827)\,(0.6520)$
≈ 15.2455

It is evident from the above calculations that the centre of the ball passes through $(15, 10.0759)$ and $(15.2455, 10)$. This means that the centre of the ball goes through the basket. The graph of the motion is as follows



(b) When $x(t) = 14.25 \Rightarrow t \approx 0.6094$ this gives $y(t) = 10.2849$.
That is $(14.25, 10.2845)$ lies on the curve. Therefore the minimum distance between the centre of the ball and the front rim is 0.2845. The minimum distance between the centre of the ball and the back rim at $(15.75, 10)$ is $0.5045'$.

(c) If the ball is of diameter, then its radius is . Since the minimum distance between the center of the ball and the front rim is less than the radius of the ball, the ball hits the front rim.

35. (a) $85^\circ = \frac{17}{36}\pi$ radiance.
$x'(0) = 100 \cdot \cos\left(\frac{17}{36}\pi\right) \approx 8.72$
$y'(0) = 100 \cdot \sin\left(\frac{17}{36}\pi\right) \approx 99.62$
$x''(0) = -20$
$y''(0) = 0$
$y(t) = 99.62t$
$x(t) = -10t^2 + 8.72t$
$y(t_0) = 90$ when $t_0 = 0.903$
$x(t_0) = x(0.903) \approx -0.29$
The ball just barely gets into the goal.

(b) Use the calculation from Exercise 35.(a),
$y(t_1) = 10$ when $t_1 = 0.100$
$x(t_1) = x(0.100) \approx 0.775$
The kick does not go around the wall.

36. Let $(x(t), y(t))$ be the trajectory of the ship. Some of our data is in feet, so we will take $g = -32$ in this problem. We have
$y''(t) = 32$
$y'(t) = -32t + y'(0)$
$y(t) = -16t^2 + y'(0)t + y(0)$
$x'(t) \equiv c$
$x(t) = ct + x(0)$
Solving for t, we have
$\frac{1}{c}(x - x(0)) = t.$

Substituting this expression for t in $y(t)$, we have
$y - y(0)$
$= -16\left[\frac{1}{c}(x - x(0))\right]^2 + y'(0)\left[\frac{1}{c}(x - x(0))\right]$
Hence the path is a parabola.
Turning to the question of the duration of weightlessness, we can assume $x(0) = 0$, and we know that $y'(t) = 0$ when $y - y(0) = 2500$. For this unknown time t_1 (the moment when y' is zero), we have $0 = -32t_1 + y'(0)$.
Therefore $t_1 = y'(0)/32$, and
$2500 = y(t_1) - y(0)$
$= -16\left[\frac{y'(0)}{32}\right]^2 + y'(0)\left[\frac{y'(0)}{32}\right]$
$= \frac{y'(0)^2}{64},$
hence $y'(0)2 = 64(2500)$
$y'(0) = 8(50) = 400$, and
$t_1 = 400/32 = 25/2$.
We now know that $y - y(0) = -16t^2 + 400t$ for all t.
The second time (t_2) that $y(t) = y(0)$ (after time zero) occurs when $t = 400/16 = 25$ seconds.
This is the duration of the weightless experience. Note that $t_2 = 2t_1$. The plane must pull out of the dive soon after this time.

37. Let $y(t)$ be the height of the first ball at time t, and let v_{0y} be the initial velocity. We can assume $y(0) = 0$. As usual, we have
$y'' = -32$, $y' = -32t + v_{0y}$,
$y = -16t2 + tv_{0y}$.
The second return to height zero is at time $t = 16/v_{0y}$. If this is to be $5/2$, then $v_{0y} = 40$. But the maximum occurs at time
$v_{0y}/32 = 5/4$
at which time the height $(y(5/4))$ is $-16(25/16) + 40(5/4) = 25$feet.
For eleven balls, the difference is that the second return to zero is to be at time $11/4$, hence $v_{0y} = 44$, and the maximum height is 30.25.

38. In this case, we start with initial conditions
$x'(0) = v_{0x}, x(0) = 0; y'(0) = v_{0y}, y(0) = 0$.
Integrating $x''(t) = 0$ and $y''(t) = -32$ and using the initial conditions gives
$x'(t) = v_{0x}$
$x(t) = v_{0x}t$
$y'(t) = -32t + v_{0y}$
$y(t) = -16t + v_{0y}t$
The ball is caught when $y(t) = 0$ so we solve

this equation to get $t = \frac{v_{0y}}{16}$. Plugging this into $x(t)$ gives the horizontal distance
$\omega = x\left(\frac{v_{0y}}{16}\right) = \frac{v_{0x}v_{0y}}{16}$.

39. The student must first study the solution to Exercise 38. Here we have the additional x-component of the motion, which as in so many problems is $x(t) = tv_{0x}$. With initial *speed* of v_0, and initial angle α from the *vertical,* we have $v_{0y} = v_0 \cos\alpha$ and
$v_{0x} = v_0 \sin\alpha$.The horizontal distance at elapsed time $v_{0y}/16$ (time of return to initial height) is by formula
$x(v_{0y}/16) = (v_{0y}/16)v_{0x}$ which defines ω.
As in Exercise 37, the maximum height occurs at time $v_{0y}/32$, and at this time the height h is
$-16(v_{0y}/32)^2 + v_{0y}(v_{0y}/32) = v_{0y}^2/64$
$= (v0y/64)(16\omega/v_{0x})$
$= (\omega/4)(\cos\alpha/\sin\alpha) = \omega/(4\tan\alpha)$.
Thus $\omega = 4h\tan\alpha$.

40. The linear approximation is $\tan^{-1} x = x$, i.e., $\tan x \approx x$ From Exercise 43, we have
$\omega = 4h\tan\alpha$

Applying the linearization gives
$\omega = 4h\tan\alpha \approx 4h\alpha$
or $\alpha \approx \frac{\omega}{4h}$
This shows that $\Delta\alpha \approx \frac{\Delta\omega}{4h}$

41. We must use the result
$\Delta\alpha \approx \frac{\Delta\omega}{4h}$ from Exercise 40.
With $h = 25$ from Exercise 51 (10 balls) and $\omega = 1$, we get
$\Delta\alpha$ about $1/100 = .01$ radians
or about $.6^\circ$

42. In this case, the height to juggle 11 balls is 30.25 feet. Therefore with $\Delta\omega = 1$, we get $\Delta\alpha \approx \frac{\Delta\omega}{4h} = \frac{1}{4(30.25)} \approx 0.0083$ rad or about 0.47°.

43. With trajectory (x, y), and assuming
$x(0) = 0$ and $y(0) = 0$, we have by now seen many times the conclusion $y = -gt^2 + tv\sin\theta$.

The return to ground level occurs at time $t = 2v\sin\theta/g$, at which time the horizontal range is $x = tv\cos\theta = v^2\sin(2\theta)/g$.

With $v = 60$ ft per second and $\theta = 25^\circ$, and on earth with $g = 32$, this is about 86 feet, a short chip shot. On the moon with $g = 5.2$, it is about 530.34 ft.

44. Let $((x(t), y(t))$ be the trajectory of the initial burst of water. If the angle of inclination of the hose is θ, we have the relations
$\tan\theta = m$
$\sin\theta = \frac{m}{\sqrt{1+m^2}}$
$\cos\theta = \frac{1}{\sqrt{1+m^2}}$
We assume $x(0)$ =0 and $y(0) = 0$ and then find
$y''(t) \equiv -32$
$y'(t) = -32t + v\sin\theta$
$y = y(t) = -16t^2 + tv\sin\theta$
$y = y(t) = -16t^2 + \frac{tvm}{\sqrt{1+m^2}}$
$x'(t) \equiv v\cos\theta$
$x = x(t) = tv\cos\theta = \frac{tv}{\sqrt{1+m^2}}$
Solving the last equation in the form
$t = \frac{x\sqrt{1+m^2}}{v}$
and inserting this in the y-formula, we find
$y = -16x^2\frac{(1+m^2)}{v^2} + mx$.

45. Let $(x(t), y(t))$ be the trajectory of the paint ball, and let $z(t)$ be the height of the target at time t. We do assume that
$y(0) = z(0)$ (target opposite shooter at timeof shot) and
$y'(0) = 0$ (aiming directly at the target, hence using an initially horizontal trajectory), and as a result $y - z$ has second derivative 0, and initial value 0.

However, this only tells us that
$y - z = [y'(0) - z'(0)]t = -z'(0)t$
and if the target is already in motion ($z'(0)$ not zero), the shot may miss at 20 feet or any distance.

If on the other hand, the target is stationary at the moment of the shot, then the shot hits at20 feet or any other distance.

46. In this problem, we have the falling object with initial conditions
$y_1'(0) = 0, y_1(0) = 100$.

The object that is launched from the ground has initial conditions
$y_2'(0) = 40, y_2(0) = 0$

We now integrate the equations
$y_1''(t) = -32$ and $y_2''(t) = -32$, using the initial conditions, to get

$y_1'(t) = -32t$
$y_1(t) = -16t^2 + 100$
$y_2'(t) = -32t + 40$
$y_2(t) = -16t^2 + 40t$

Now, we just solve $y_1(t) = y_2(t)$, or
$-16t^2 + 100 = -16t^2 + 40t$
Solving gives $t = 2.5$, so the objects collide after 2.5 seconds and this collision occurs at a height of $y_1(2.5) = 0$.

This may seem odd, but notice that the maximum height of the y_2 object is only 25 feet. What this means is that the y_2 object goes up and then down and then the y_1 object only catches the y_1 object when both objects actually hit the ground!

47. (a) The speed at the bottom is given by
$\frac{1}{2}mv^2 = mgH, v = \sqrt{2gH}$

(b) Use the result from (a)
$v = \sqrt{2gH} = \sqrt{2 \cdot 16g} = 4\sqrt{2g}$
$= 4\sqrt{2 \cdot 32} = 32\text{ft/s}$

(c) At half way down,
$\frac{1}{2}mv^2 + mh8 = mh16,$
$v = \sqrt{2 \cdot (16 - 8)g} = 4\sqrt{g}$
$= 4\sqrt{32} \approx 22.63\text{ft/s}$

(d) At half way down, the slope of the line tangent to $y = x^2$ is, $2 \cdot \sqrt{8} = 4\sqrt{2}$
Hence we know that
$\frac{v_y}{v_x} = 4\sqrt{2}$
At the same time,
$(v_y)^2 + (v_x)^2 = (4\sqrt{g})^2$
$v_x^2 = \frac{16g}{33}$

$v_x = 4\sqrt{\frac{g}{33}} \approx 3.939 \text{ ft/s}$

$v_y = 16\sqrt{\frac{2g}{33}} \approx 22.282 \text{ ft/s}$

48. First we compute the speed v of the bowling ball at the moment when it rolls right out of the window.

$30 = 16t_0^2, t_0 = \frac{\sqrt{30}}{4}$
$10 = t_0 v_0, v_0 = \frac{40}{\sqrt{30}}.$

From conservation of energy
$\frac{1}{2}mv^2 = mgh,$
$\frac{1}{2}m\left(\frac{40}{\sqrt{30}}\right)^2 = mgh$

$\frac{80}{3} = 32 \cdot h,$
$h = \frac{5}{6}$

The height of the ramp should be $\frac{5}{6}$.

5.6 Applications of Integration to Physics and Engineering

1. We first determine the value of the spring constant k. We convert to feet so that our units of work is in foot-pounds.
$5 = F(1/3) = \frac{k}{3}$ and so $k = 15$.
$W = \int_0^6 F(x)dx$
$= \int_0^{1/2} 15x dx = \frac{15}{8}$ foot-pounds.

2. We first determine the value of the spring constant k. We convert to feet so that our units of work is in foot-pounds.
$10 = F(1/6) = \frac{k}{6}$ and so $k = 60$.
$W = \int_0^3 F(x)dx$
$= \int_0^{1/4} 60x dx = \frac{15}{8}$ foot-pounds.

3. The force is constant (250 pounds) and the distance is 20/12 feet, so the work is
$W = Fd = (250)(20/12)$
$= 1250/3$foot-pounds.

4. The force is constant (300 pounds) and the distance is 6 feet, so the work is
$W = Fd = (300)(6) = 1800$ foot-pounds.

5. If x is between 0 and 30,000 feet, then the weight of the rocket at altitude x is

$10000 - \frac{1}{15}x.$

Therefore the work is
$\int_0^{30,000} \left(10{,}000 - \frac{x}{15}\right)dx$
$= \left(10{,}000x - \frac{x^2}{30}\right)\Big|_0^{30,000}$
$= 270{,}000{,}000$ ft-lb

6. If x is between 0 and 10,000 feet, then the weight of the rocket at altitude x is $8000 - \frac{x}{10}$. Therefore the work done is

$$W = \int_0^{10,000} \left(8000 - \frac{x}{10}\right) dx$$
$$= 60,800,000 \text{ft-lb}$$

7. The weight of the 40 feet long chain is 1000 pounds. Therefore the weight of the 30 feet long chain is 750 pounds. The force acting here is 750 pounds and the distance traced due to the applied force is 30 feet. Hence the work done is

$$W = Fd$$
$$= (750) \cdot (30)$$
$$= 22500 \text{ foot-pounds.}$$

8. Let x be the distance of the bucket from the initial position. Consequently x increases from 0 to 80. As the sand from the bucket leaks at rate of 2 lb/s, the weight of bucket at the distance x is $\left(100 - \frac{x}{2}\right)$. Therefore work done is

$$W = \int_0^{80} \left(100 - \frac{x}{2}\right) dx = \left(100x - \frac{x^2}{4}\right)_0^{80}$$
$$= 8000 - 1600$$
$$= 6400 \text{ ft-lb.}$$

9. (a) $W = \int_0^1 800x(1-x)dx$

$$= \left(400x^2 - \frac{800}{3}x^3\right)\Big|_0^1$$
$$= \frac{400}{3} \text{ mile-lb}$$
$$= 704{,}000 \text{ ft-lb}$$

(b) Horsepower is not equal to $800x(1-x)$ because this is the derivative with respect to distance and not with respect to time. Average horsepower is the ratio of total work done divided by time: $\frac{704,000 \text{ ft-lb}}{80 \text{ s}} = 16$ hp

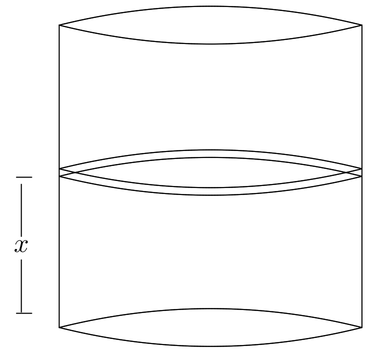
10. (a) $W = \int_0^{100} 62.4\pi(100x - x^2)(200 + x)dx$

$$= 62.4\pi \int_0^{100} \left(20{,}000x - 100x^2 - x^3\right) dx$$
$$= 8{,}168{,}140{,}899 \text{ ft-lb}$$

(b) This is the same as Exercise 10.(a) except the limits of integration change to reflect that the tank is only filled half way:

$$W = \int_0^{50} 62.4\pi(100x - x^2)(200 + x)dx$$
$$= 3,777,765,166 \text{ ft-lb}$$

11. (a)



Let x represent the distance measured (in ft) from the bottom of the tank, as shown in the above diagram. The entire tank corresponds to the interval
$0 \le x \le 9.843$ (1 mt = 3.281 ft).
Let us partition the tank into
$0 = x_0 < x_1 < x_2 < ... < x_n = 9.843.$

such that
$$x_i - x_{i-1} = \Delta x = \frac{9.843}{n}$$

for each $i = 1, 2, 3, , n.$

This partitions the tank into n layers, each corresponding to an interval $[x_{i-1}, x_i]$.

Let us consider a water layer corresponding to $[x_{i-1}, x_i]$, which is a cylinder of height Δx and radius 3.281 ft(1mt) . This layer must be pumped at a distance of $(9.843 - c_i)$ for $c_i \in [x_{i-1}, x_i]$

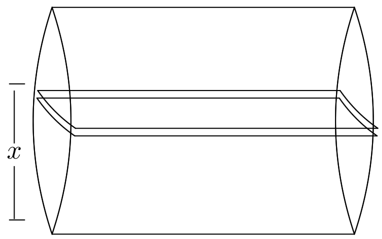
Thus the force exerted in doing so,is
$F_i \approx$ (Volume of the cylindrical slice)
$\times$ (Weight of the water per unit volume)
$\approx \pi(3.281)^2 (\Delta x) \times (62.4)$
$\approx 2110.31 (\Delta x)$

Thus the corresponding work done
$W_i = 2110.31 (9.843 - c_i) (\Delta x)$

Therefore the total work done

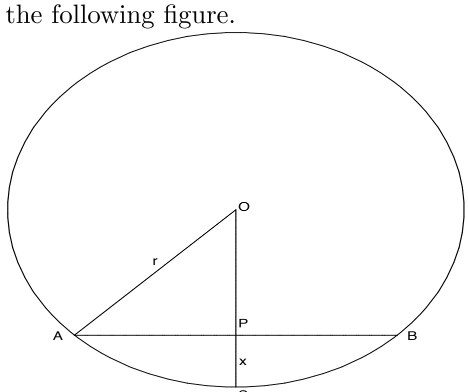
$$W = \lim_{n\to\infty} \sum_{i=1}^{n} (2110.31 (9.843 - c_i) (\Delta x))$$
$$= 2110.31 \int_0^{9.843} (9.843 - x)dx$$
$$= 2110.31 \left(9.843x - \frac{x^2}{2}\right)\Big|_0^{9.843}$$
$$= 102228.48 \text{ feet pounds}$$

(b)



Let x represent the distance measured (in ft) from the bottom of the tank, as shown in the above diagram. The entire tank corresponds to the interval $0 \le x \le 3.281$ (as 1mt = 3.281 ft). Let us partition the tank into
$0 = x_0 < x_1 < x_2 < ... < x_n = 3.281$. such that
$x_i - x_{i-1} = \Delta x = \dfrac{3.281}{n}$ for each i =1,2,3,,n. This partitions the tank into n layers, each corresponding to an interval $[x_{i-1}, x_i]$. Let us consider a water layer corresponding to $[x_{i-1}, x_i]$. Which is a cuboid of length 9.843, width $2\sqrt{6.562x - x^2}$ and height Δx.

The width is calculated with the help of the following figure.



In the above figure O is the centre of the circle of radius r. $OP = r - x$,
$AP = \sqrt{r^2 - (r-x)^2} = \sqrt{2rx - x^2}$;
$AB = 2\sqrt{2rx - x^2}$

The said layer must be pumped at a distance of $(2r - c_i)$ for $c_i \in [x_{i-1}, x_i]$. Thus the force exerted in doing so, is
$F_i \approx$ (Volume of the cuboid shaped slice)
$\times$ (Weight of the water per unit volume)
= (length × width × height) × (62.4)
$\approx \left(9.843 \times 2\sqrt{6.562x - x^2} \times \Delta x\right) \times$ (62.4)
$\approx 1228.41\sqrt{6.562x - x^2}\,(\Delta x)$
Thus the corresponding work done
$W_i = 1228.41\sqrt{6.562x - x^2}\,(6.562 - c_i)\,(\Delta x)$
Therefore the total work done

$$W = (1228.41) \times \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{6.562x - x^2}\,(6.562 - c_i)\,\Delta x\right)$$
$$= 1228.41 \int_0^{6.562} \sqrt{6.562x - x^2}\,(6.562 - x)dx$$
$$= 136304.64 \text{ feet pounds}$$

12. We set up our coordinates similar to Example 6.3, with x representing vertical distance from the vertex (the bottom of the tank). If slice the water in horizontal slices, these slices have radius $r = \dfrac{x}{2}$ and the volume of a cylindrical slice is $\pi r^2 \Delta x = \dfrac{\pi x^2}{4}\Delta x$. The weight density of water is 62.4, which gives the force exerted by this slice of water as $15.6\pi x^2 \Delta x$. This slice of water must travel up a distance of $10 - x$ and therefore the work required to pump this slice out of the tank is

$W_i \approx 15.6\pi x^2 \Delta x(10 - x)$

$\approx 15.6(10 - x)\pi x^2 \Delta x$

Now, we add up the work for all the slices and turn it into an integral.

$$W = \int_0^{10} 15.6(10 - x)\pi x^2 dx$$
$$= 15.6\pi \left(\frac{2500}{3}\right)$$
$\approx$ 40841 foot-pounds

13. $W = \displaystyle\int_0^{10} ax dx = \frac{100a}{2}$

$W_1 = \displaystyle\int_0^{c} ax dx = \frac{ac^2}{2}$

$W_1 = \dfrac{W}{2}$ gives $\dfrac{ac^2}{2} = \dfrac{1}{2}\dfrac{100a}{2}$

$c = \sqrt{50} \approx 7.1$ feet

The answer is greater than 5 feet because the deeper the laborer digs, the more distance it is required for him to lift the dirt out of the hole.

14. By calculation, the width at x feet depth is $5 - x/2$, therefore
$$W(x) = \int_0^x t\left(5 - \frac{t}{2}\right) dt = v52x^2 - \frac{1}{9}x^3$$

$W(6) = 66$
Solving $\frac{5}{2}x^2 - \frac{1}{9}x^3 = 33$ we get
$x \approx 4.0$ feet

15. We estimate the integral using Simpson's Rule:
$J = \int_0^{.0008} F(t)dt$
$\approx \frac{.0008}{3(8)}[0 + 4(1000) + 2(2100)$
$+ 4(4000) + 2(5000) + 4(5200)$
$+ 2(2500) + 4(1000) + 0]$
≈ 2.133
$2.13 = J = m\Delta v = .01\Delta v$
$\Delta v = 213$ ft/sec
The velocity after impact is therefore
$213 - 100 = 113$ ft/sec.

16. We compute the impulse using Simpson's rule:
$J \approx \frac{.6}{3(6)}[0 + 4(8000) + 2(16,000)$
$+ 4(24,000) + 2(15,000) + 4(9000)[5pt] \quad + 0]$
≈ 7533.3
$7533.3 = J = m\Delta v = 200\Delta v$
$\Delta v = 37.7$ ft/sec
Since the velocity after the crash is zero, this number is the estimated original velocity.

17. $F'(t)$ is zero at $t = 3$, and the maximum thrust is $F(3) = 30/e \approx 11.0364$

It is implicit in the drawing that the thrust is zero after time 6. Therefore the impulse is $\int_0^6 10te^{-t/3}dt = 90 - 270e^{-2} \approx 53.55$.

18. The impulse is
$J = \int_0^6 F(t)\, dt = 48$. The impulse of Exercise 17 was about 53.55 which means that the rocket of Exercise 17 would have greater velocity and therefore a higher altitude.

19. $m = \int_0^6 \left(\frac{x}{6} + 2\right) dx = 15$
$M = \int_0^6 x\left(\frac{x}{6} + 2\right) dx = 48$
Therefore,
$\bar{x} = \frac{M}{m} = \frac{48}{15} = \frac{16}{5} = 3.2$
So the center of mass is to the right of $x = 3$.

20. $m = \int_0^6 \left(3 - \frac{x}{6}\right) dx = 15$
$M = \int_0^6 x\left(3 - \frac{x}{6}\right) dx = 42$
So, therefore
$\bar{x} = \frac{M}{m} = \frac{42}{15} = \frac{14}{5} = 2.8$
So the center of mass is to the left of $x = 3$.

21. $m = \int_{-3}^{27} \left(\frac{1}{46} + \frac{x+3}{690}\right)^2 dx$
$= \frac{690}{3}\left(\frac{1}{46} + \frac{x+3}{690}\right)^3 \Big|_{-3}^{27}$
$\approx .0614$ slugs ≈ 31.5 oz

22. $m = \int_0^{32} \left(\frac{1}{46} + \frac{x+3}{690}\right)^2 dx$
≈ 0.08343 slugs ≈ 42.418 oz

23. $M = \int_{-3}^{27} x\left(\frac{1}{46} + \frac{x+3}{690}\right)^2 dx$
≈ 1.0208
$\bar{x} = \frac{M}{m} = \frac{1.0208}{.0614} \approx 16.6$ in.
This is 3 inches less than the bat of Example 6.5, a reflection of the translation three inches to the left on the number line.

24. $M = \int_0^{32} x\left(\frac{1}{46} + \frac{x+3}{690}\right)^2 dx$
≈ 1.72495

$\bar{x} = \frac{M}{m} = 20.6745$
Compared to the baseball bat of Example 6.5, this baseball bat is longer and therefore has more mass further out.

25. $m = \int_0^{30} .00468\left(\frac{3}{16} + \frac{x}{60}\right) dx$
$\approx .0614$ slugs
$M = \int_0^{30} .00468x\left(\frac{3}{16} + \frac{x}{60}\right) dx$
≈ 1.0969
weight $= m(32)(16) = 31.4$ oz

$\bar{x} = \frac{M}{m} = \frac{1.0969}{.0614} \approx 17.8$ in.

26. The center of mass of the wooden bat of Example 6.5 is at 19.6 inches. The center of mass of the aluminum bat of Exercise 25 is at 17.8 inches—moving the sweet spot to the inside.

27. Area of the base is $\frac{1}{2}(3 + 1) = 2$.

Area of the body is $1 \times 4 = 4$.

Area of the tip is $\frac{1}{2}(1 \times 1) = \frac{1}{2}$.

Base:
$m = \int_0^1 \rho(3-2x)dx = \frac{5}{12} \approx .4167.$
Body:
$m = \int_1^5 \rho dx = 12\rho$
$\bar{x} = \frac{M}{m} = 3$
Tip:
$m = \int_5^6 \rho(6-x)dx \approx 2.67\rho$
$\bar{x} = \frac{M}{m} = \frac{16}{3} \approx 5.33$

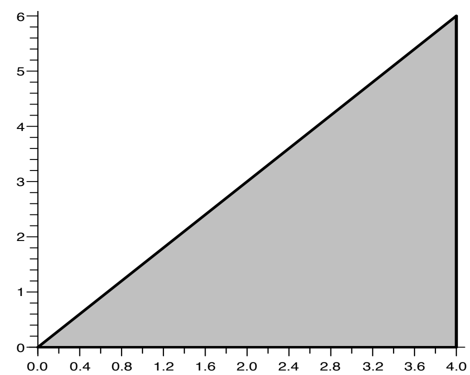
28. We use the coordinate system as in Exercise 29, with $x = 0$ corresponding to the left of the rocket.

From Exercise 27, the base has total mass $\frac{5}{6}\rho$ and center of mass at $x = \frac{5}{12}$.
From Exercise 27, the body has total mass 12ρ and center of mass at $x = 3$.
From Exercise 27, the tip has total mass $\frac{1}{2}\rho$ and center of mass at $x = \frac{16}{3}$.
The total mass of these three particles is $m = \frac{40}{3}\rho$ and the moment of these particles is

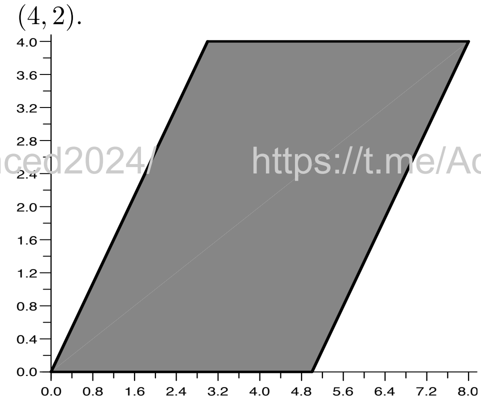
$$M = \left(\frac{5}{6}\rho\right)\left(\frac{5}{12}\right) + (12\rho)(3) + \left(\frac{1}{2}\rho\right)\left(\frac{16}{3}\right) = \frac{2809}{72}\rho$$

The center of mass of the system is
$$\overline{x} = \frac{M}{m} = \left(\frac{2809}{72}\rho\right)\left(\frac{3}{40\rho}\right) = \frac{2809}{960} \approx 2.926$$

29. The x-coordinate of the centroid is the same as the center of mass from $x = 0$ to $x = 4$ with density $\rho(x) = \frac{3}{2}x$, hence
$$\bar{x} = \frac{M}{m} = \frac{\int_0^4 3/2 \cdot x^2 dx}{\int_0^4 3/2 \cdot x dx} = \frac{8}{3}$$
The y-coordinate of the centroid is the same as the center of mass from $y = 0$ to $y = 6$ with density $\rho(y) = 6 - \frac{2}{3}y$, hence
$$\bar{y} = \frac{M}{m} = \frac{\int_0^6 2/3 \cdot \left(6y - \frac{2}{3}y^2\right)\, dy}{\int_0^6 2/3 \cdot \left(6 - \frac{2}{3}y\right) dy} = 2$$
So the center of the given triangle is the point $(8/3, 2)$.



30. Again we need to find both the x-coordinate and y-coordinate of the centroid. But in this case, since everything is symmetric, in fact we can easily see that the centroid is going to be $(4, 2)$.



31. This time the x-coordinate of the centroid is obviously $x = 0$, so the question remains to find the y-coordinate.

This is the same as finding the center of mass from $y = 0$ to $y = 4$ with density $\rho(y) = \sqrt{4-y}$, hence

$$\bar{y} = \frac{M}{m} = \frac{\int_0^4 y\sqrt{4-y}\, dy}{\int_0^4 \sqrt{4-y} dy} = \frac{-\int_4^0 (4u^{1/2} - u^{3/2})\, du}{-\int_4^0 u^{1/2}\, du} = \frac{(8/3 \cdot u^{3/2} - 2/5 \cdot u^{5/2})\big|_0^4}{2/3 \cdot u^{3/2}\big|_0^4} = \frac{8}{5}$$

So the centroid is the pint $(0, 8/5)$.

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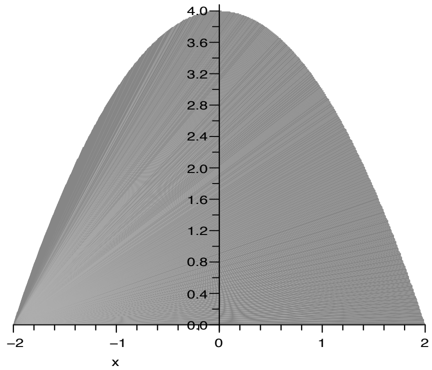
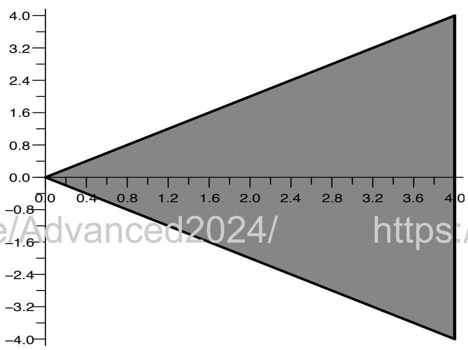


32. This time the y-coordinate is obviously $y = 0$. The x-coordinate can found using the density $\rho(x) = 2x$, from $x = 0$ to $x = 4$, and

$$\bar{x} = \frac{M}{m} = \frac{\int_0^4 2x^2 dx}{\int_0^4 2x dx} = \frac{8}{3}$$

So the centroid is $(8/3, 0)$.



33. With x the depth, the horizontal width is a linear function of x, given by $x + 40$. Hence,

$$F = \int_0^{60} 62.4x(x+40)dx$$

$$= 62.4\left(\frac{x^3}{3} + 20x^2\right)\Big|_0^{60} = 8{,}985{,}600 \text{ lb}$$

34. In this case, we just change the limits of integration.

$$F = \int_{10}^{60} 62.4x(x+40)\,dx = 8,840,000 \text{ lb}$$

35. Let x be the vertical deviation above the center of the window, the horizontal width of the window is given by $2\sqrt{25 - x^2}$, depth of water $40 + x$, and hydrostatic force

$$62.4\int_{-5}^{5}(x+40)2\sqrt{25-x^2}dx$$

$$= 62.4\int_{-5}^{5} 2x\sqrt{25-x^2}dx$$

$$+ 62.4(40)\int_{-5}^{5} 2\sqrt{25-x^2}dx$$

$\approx 196,035$ pounds.

36. Let x be the distance from the surface of the water. For a given value of x, the width of the window is constant, 40. The force exerted on the window by a slice of water, of depth x is $F_i \approx (62.4)(40)x\Delta x$.

We sum these forces up over the height of the window and turn it into an integral:

$$F = \int_0^{10}(62.5)(10)xdx = 31,250 \text{ lb.}$$

37. Assuming that the center of the circular window descends to 1000 feet, then by the previous principle, after converting the three inch radius to 1/4 feet, we get $F = 12{,}252$ pounds. An alternate calculation in which x is the deviation downward from the top edge of the window, would be

$$F = \int_0^{0.5} 62.4(999.75 + x)$$

$$\cdot\, 2\sqrt{(0.25)^2 - (0.25 - x)^2}dx$$

$$= \int_0^{0.5} 124.8(999.75 + x)\sqrt{0.5x - x^2}dx$$

$\approx 12,252$ lb

38. Due to the fact that the size of the watch is so small, we can assume that the force will be approximately the same regardless of orientation of the watch.
The hydrostatic force is given by $F = \rho d A$ where, ρ is the density of the water (62.4), d is the depth (60), and A is the area, $A = \pi(1/12)^2$.
Putting these together gives
$F \approx (62.4)(60)(\pi/144) \approx 81.68$ lb.

39. (100 tons)(20 miles/hr)

$$= \frac{(100 \cdot 2000 \text{ lbs})(20 \cdot 5280 \text{ ft})}{3600\text{sec}}$$

$\approx 5{,}866{,}667$ ft-lb/s

$$= \frac{5{,}866{,}667}{550} \text{ hp}$$

$\approx 10,667$ hp

40. This is a matter of slicing and approximating. Divide the subinterval $[a, b]$ into n equal subintervals. Then, we take the limit as $n \to \infty$, which turns the Riemann sum into an integral.

$$J \approx \sum_{i=1}^{n} F(t_i)\Delta t.$$

$$J = \lim_{n\to\infty} \sum_{i=1}^{n} F(t_i)\Delta t = \int_a^b F(t)\,dt$$

41. The bat in Exercise 23 models the bat of Example 6.5 choked up 3 in.

From Example 6.5:
$f(x) = \left(\frac{1}{46} + \frac{x}{690}\right)^2$;
$\int_{-3}^{27} f(x)\cdot x^2 dx \approx 27.22.$
From Exercise 23:
$f(x) = \left(\frac{1}{46} + \frac{x+3}{690}\right)^2$;
$\int_{-3}^{27} f(x)\cdot x^2 dx \approx 20.54.$
Reduction in moment:
$\frac{27.22 - 20.54}{27.22} \approx 24.5\%$

42. $m = \int_0^{28} \left(\frac{1}{46} + \frac{x}{690}\right)^2 dx$
$+ \int_{28}^{30} \left(\frac{1}{92} + \frac{x}{690}\right)^2 dx$
≈ 0.05918 slugs.
$M = \int_0^{28} x\left(\frac{1}{46} + \frac{x}{690}\right)^2 dx$
$+ \int_{28}^{30} x\left(\frac{1}{92} + \frac{x}{690}\right)^2 dx$
≈ 1.1398 slugs
$\bar{x} = \frac{M}{m} \approx 19.258$
The center of mass moves in.

43. $\int_{-a}^{a} 2\rho x^2 b\sqrt{1 - \frac{x^2}{a^2}}dx = \frac{1}{4}\rho\pi a^3 b$

44. If the racket was solid wood, then the second moment would be
$M_0 = \int_{-a}^{a} 2\rho b x^2\sqrt{1 - \frac{x^2}{a^2}}\,dx = \rho\frac{\pi}{4}a^3 b$
But, the racket is not solid wood. We have to subtract the contribution to the second moment from the empty space. This amount is equal to the second moment of a smaller wood racket:
$M_1 = \int_{-(a-w)}^{a-w} 2\rho(b-w)x^2$
$\cdot\sqrt{1 - \frac{x^2}{(a-w)^2}}\,dx$
$= \rho\frac{\pi}{4}(a-w)^3(b-w)$
Therefore the second moment is
$M = M_0 - M_1$
$= \rho\frac{\pi}{4}\left[a^3 b - (a-w)^3(b-w)\right]$

45. Using the formula in Exercise 42, we find that the moments are 1323.8 for the wooden racket, 1792.9 for the mid-sized racket, and 2361.0 for the oversized racket. The ratios are
$\frac{\text{mid}}{\text{wood}} \approx 1.35,\ \frac{\text{over}}{\text{wood}} \approx 1.78$

46. $\frac{dM}{da} = \rho\frac{\pi}{4}\left[3a^2 b - 3(a-w)^2(b-w)\right]$
Since $a > a-w$ and $b > b-w$
$\frac{dM}{da} > 0.$
Therefore as a increases, M increases.
$\frac{dM}{dw} = \rho\frac{\pi}{4}\left[3(a-w)^2(b-w) + (a-w)^3\right]$
It is easy to see that $\frac{dM}{dw} > 0$. Therefore as w increases M increases making the racket more stable.

5.7 Probability

1. $f(x) = 4x^3 \geq 0$ for $0 \leq x \leq 1$ and
$\int_0^1 4x^3 dx = x^4\Big|_0^1 = 1 - 0 = 1$

2. $f(x) = \frac{3}{8}x^2 \geq 0$ on the interval $[0,2]$ and
$\int_0^2 \frac{3}{8}x^2 dx = 1.$

3. $f(x) = x + 2x^3 \geq 0$ for $0 \leq x \leq 1$ and
$\int_0^1 (x + 2x^3)dx = \frac{x^2}{2} + \frac{x^4}{2}\Big|_0^1 = 1$

4. $f(x) = \cos x \geq 0$ over $[0, \pi/2]$ and
$\int_0^{\pi/2} \cos x dx = 1.$

5. $f(x) = \frac{1}{2}\sin x \geq 0$ over $[0,\pi]$ and
$\int_0^{\pi} \frac{1}{2}\sin x dx = \frac{1}{2} - \cos x\Big|_0^{\pi} = 1.$

6. $f(x) = e^{-x/2} \geq 0$ over $[0, \ln 4]$ and
$\int_0^{\ln 4} e^{-x/2}dx = -2e^{-x/2}\Big|_0^{\ln 4} = 1.$

7. We solve for c:
$1 = \int_0^1 cx^3 dx = \frac{c}{4}$ which gives $c = 4$.

8. We solve for c:
$1 = \int_0^1 cx + x^2 dx = \frac{c}{2} + \frac{1}{3}$
which gives $c = \frac{4}{3}$.

9. We solve for c:
$1 = \int_0^1 ce^{-4x} dx = -\frac{c}{4}(e^{-4} - 1)$
which gives $c = \frac{4}{1 - e^{-4}}$.

10. We solve for c:
$1 = \int_0^2 2ce^{-cx} dx = 2 - 2e^{-2c}$
which gives $c = \frac{1}{2}\ln 2$.

11. We solve for c:
$1 = \int_0^1 \frac{c}{1 + x^2} = c\tan^{-1}x\big|_0^1$
$= c\left(\frac{\pi}{4} - 0\right) = c\frac{\pi}{4}$
which gives $c = \frac{4}{\pi} \approx 1.2732$

12. We solve for c:
$1 = \int_0^1 \frac{c}{\sqrt{1 - x^2}} = c\sin^{-1}x\big|_0^1$
$= c\left(\frac{\pi}{2} - 0\right) = c\frac{\pi}{2}$
$\Rightarrow\ c = \frac{2}{\pi} \approx 0.6366$

13. $P(70 \le x \le 72)$
$= \int_{70}^{72} \frac{.4}{\sqrt{2\pi}} e^{-.08(x-68)^2} dx \approx 0.157$

14. $P(76 \le X \le 80)$
$= \int_{76}^{80} \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2} dx \approx 0.00068634$

15. $P(84 \le x \le 120)$
$= \int_{84}^{120} \frac{.4}{\sqrt{2\pi}} e^{-.08(x-68)^2} dx \approx 7.76 \times 10^{-11}$

16. $P(14 \le X \le 60)$
$= \int_{14}^{60} \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2} dx \approx 0.00068714$

17. $P\left(0 \le x \le \frac{1}{4}\right) = \int_0^{1/4} 6e^{-6x} dx$
$= -e^{-6x}\big|_0^{1/4} = (-e^{-3/2} + 1) \approx .77687$

18. $P(0 \le X \le 0.5) = \int_0^{0.5} 6e^{-6x} dx \approx 0.95021$

19. $P(1 \le x \le 2) = \int_1^2 6e^{-6x} dx$
$= -e^{-6x}\big|_1^2 = (-e^{-12} + e^{-6}) \approx .00247$

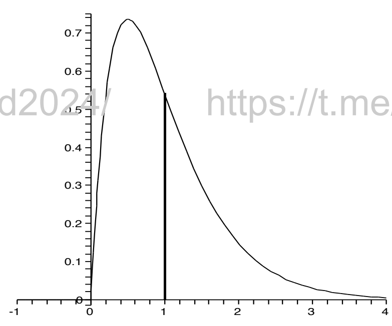
20. $P(3 \le X \le 10) = \int_3^{10} 6e^{-6x} dx$
$\approx 1.52300 \times 10^{-8}$

21. $P(0 \le x \le 1) = \int_0^1 4xe^{-2x} dx$
$= 1 - 3e^{-2} \approx .594$

22. $P(1 \le X \le 2) = \int_1^2 4xe^{-2x} dx \approx 0.31443$

23. Mean: $\int_0^{10} x(4xe^{-2x}) dx \approx 0.9999995$

24. The maximum is at $x = \frac{1}{2}$ and the mean is at $x \approx 0.31443$.



25. (a) Mean: $\mu = \int_a^b xf(x)dx = \int_0^1 3x^3 dx$
$= \frac{3}{4} = 0.75$

(b) Median, we must solve for m:
$\frac{1}{2} = \int_a^m f(x)dx = \int_0^m 3x^2 dx = m^3$
which gives $m = \frac{1}{\sqrt[3]{2}} \approx 0.7937$.

26. (a) Mean: $\mu = \int_a^b xf(x)dx = \int_0^1 4x^4 dx$
$= \frac{4}{5} = 0.8$

(b) Median, we must solve for m:
$\frac{1}{2} = \int_a^m f(x)dx = \int_0^m 4x^3 dx = m^4$
which gives $m = \frac{1}{\sqrt[4]{2}} \approx 0.8409$.

27. (a) Mean: $\mu = \int_a^b xf(x)dx$
$= \int_0^1 x\left(\frac{4/\pi}{1+x^2}\right)dx \approx 0.4413$

(b) Median, we must solve for m:
$\frac{1}{2} = \int_a^m f(x)dx$
$= \int_0^m \left(\frac{4/\pi}{1+x^2}\right)dx$
$= \frac{4}{\pi}\left(\tan^{-1}x\right)\Big|_0^m$
$= \frac{4}{\pi}\tan^{-1}m$
$\Rightarrow m = \tan\frac{\pi}{8} \approx 0.4142$

28. (a) Mean: $\mu = \int_a^b xf(x)dx$
$= \int_0^1 x\left(\frac{2/\pi}{\sqrt{1-x^2}}\right)dx$
≈ 0.6366

(b) Median, we must solve for m:
$\frac{1}{2} = \int_a^m f(x)dx$
$= \int_0^m \left(\frac{2/\pi}{\sqrt{1-x^2}}\right)dx$
$= \frac{2}{\pi}\sin^{-1}x\Big|_0^m$
$= \frac{2}{\pi}\left(\sin^{-1}m - 0\right)$
$= \frac{2}{\pi}\sin^{-1}m$
$\Rightarrow m = \sin\frac{\pi}{4} \approx 0.7071$

29. (a) Mean: $\mu = \int_a^b xf(x)dx$
$= \int_0^\pi \frac{1}{2}x\sin x dx$
$= \frac{1}{2}(\sin x - x\cos x)\Big|_0^\pi = \frac{\pi}{2}$

(b) Median, we must solve for m:
$\frac{1}{2} = \int_a^m f(x)dx$
$= \int_0^m \sin x dx = \frac{1}{2}(1-\cos m)$
which gives
$m = \cos^{-1}(0) = \frac{\pi}{2} \approx 1.57.$

30. (a) Mean: $\mu = \int_a^b xf(x)dx$
$= \int_0^{\pi/2} x\cos x dx$
$= \frac{\pi}{2} - 1 \approx 0.57080$

(b) Median, we must solve for m:
$\frac{1}{2} = \int_a^m f(x)dx$
$= \int_0^m \cos x dx = \sin m$
which gives $m = \frac{\pi}{6} \approx 0.5236.$

31. Density $f(x) = ce^{-4x}, [0,b], b > 0$
$1 = \int_0^b ce^{-4x}dx$
$= -\frac{c}{4}e^{-4x}\Big|_0^b = -\frac{c}{4}\left(e^{-4b}-1\right)$
$c = \frac{4}{1-e^{-4b}}$
As $b \to \infty$, $c \to 4$

32. From Exercise 31, $c = \frac{4}{1-e^{-4b}}$
$\mu = \int_0^b cxe^{-4x}dx$
$= \frac{c}{16}\left[1 - e^{-4b}(1+4b)\right]$
$= \frac{1-e^{-4b}(1+4b)}{4(1-e^{-4b})}$
Now, taking the limit,
$\lim_{b\to\infty} \mu = \frac{1}{4}$

33. Density $f(x) = ce^{-6x}, [0,b], b > 0$
$1 = \int_0^b ce^{-6x}dx$
$= \frac{-c}{6}e^{-6x}\Big|_0^b = -\frac{c}{6}\left(e^{-6b}-1\right)$
$c = \frac{6}{1-e^{-6b}}$
As $b \to \infty$, $c \to 6$
$\mu = \int_0^b xce^{-6x}dx$
$= \frac{ce^{-6c}}{36}(-6x-1)\Big|_0^b$
$= \frac{ce^{-6b}}{36}(-6b-1) + \frac{c}{36}$
As $b \to \infty$, $\mu \to \frac{1}{6}$

34. $c = \frac{A}{1-e^{-ab}}$
$\mu = \frac{1-e^{-ab}(1+ab)}{a(1-e^{-ab})}$
$\lim_{b\to\infty} \mu = \frac{1}{a}$

35. To find the probability of these events, we add the probabilities.

(a) $P(X \geq 5) = 0.0514 + 0.0115 + 0.0016 + 0.0001 = 0.0646$

(b) $P(X \leq 4) = 0.0458 + 0.1796 + 0.2953 + 0.2674 + 0.1473$
$= 0.9354$

(c) $P(X \geq 6) = 0.0115 + 0.0016 + 0.0001$
$= 0.0132$

(d) $P(X = 3 \text{ or } X = 4)$
$= 0.2674 + 0.1473$
$= 0.4147$

36. (a) $P(X = 2 \text{ or } X = 3) = 0.441 + 0.343$
$= 0.784$

(b) $P(X \geq 1) = 0.189 + 0.441 + 0.343 = 0.973$

37. (a) Suppose the statement is not true. Then there must be a game before which the player's winning percentage is smaller than 75% and after which the player's winning percentage is greater than 75%. Then there are integers a and b (note that $a \geq m, b \geq n$ and $a - b = m - n$), such that
$\frac{a}{b} < \frac{3}{4}$ and $\frac{a+1}{b+1} > \frac{3}{4}$. Then
$4a < 3b,$ and $4a + 4 > 3b + 3$
$3b + 4 > 4a + 4 > 3b + 3.$

But there is no integer between the two numbers $3b + 4$ and $3b + 3$, and thus such situation will never happen. Thus there must be a game after which the player's winning percentage is exactly 75%.

(b) Using the same argument as in the previous problem, we can conclude that:

If after a certain game, a game player's winning percentage is strictly less than $100\frac{k}{k+1}$, and then the player wins several games in a row so that the winning percentage exceeds $100\frac{k}{k+1}$, then at some point in this process the player's winning percentage is exactly $100\frac{k}{k+1}$.

38. First the first quartile, we solve
$0.25 = \int_0^c \ln 2 e^{-(\ln 2)x/2} dx$
$= 2\left(1 - e^{-(\ln 2)c/2}\right)$
Solving gives
$c = -2\ln(7/8)/\ln 2 \approx 0.3853$ days.

For the third quartile, we solve
$0.75 = \int_0^c \ln 2 e^{-(\ln 2)x/2} dx$
$= 2\left(1 - e^{-(\ln 2)c/2}\right)$
Solving gives
$c = -2\ln(5/8)/\ln 2 \approx 1.3561$ days.

39. $f(x) = \frac{.4}{\sqrt{2\pi}} e^{-.08(x-68)^2}$

$f'(x) = \frac{-.064}{\sqrt{2\pi}}(x - 68)e^{-.08(x-68)^2}$

$f''(x) = \frac{-.064}{\sqrt{2\pi}} e^{-.08(x-68)^2}$

$\cdot \left(1 - .16(x - 68)^2\right)$

The second derivative is zero when
$x - 68 = \pm 1/\sqrt{0.16} = \pm 1/0.4 = \pm 5/2$
Thus the standard deviation is $\frac{5}{2}$.

40. For this, we have $\mu = 68$ and $\sigma = \frac{5}{2}$.
$P(\mu - \sigma \leq X \leq \mu + \sigma)$
$= P(65.5 < X < 70.5) \approx 0.6827$
$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$
$= P(63 < X < 73) \approx 0.9545$
$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma)$
$= P(60.5 < X < 75.5) \approx 0.9973$

41. $f'(p) = mp^{m-1}(1 - p)^{n-m}$
$\quad - (n - m)p^m(1 - p)^{n-m-1}$

$f'(p) = 0$ when $p = \frac{m}{n}$ and

$f'(p) \begin{cases} < 0 & \text{if } p < m/n \\ > 0 & \text{if } p > m/n \end{cases}$

Hence $f(p)$ is maximized when $p = \frac{m}{n}$.
In common senses, in order for an event to happen m times in n tries, the probability of the event itself should be about m/n.

42. In the picture, although it might appear that $y > 1/2$, the conditions are that $0 \leq y \leq 1/2$, and the labeling in the drawing implies that the lower line is the closer. This is indeed always an allowable assumption (by turning the picture upside down if necessary). In the right triangle whose hypotenuse is the lower half-needle, the vertical side is of length $(\sin\theta)/2$. Therefore the needle hits the lower line if $y - (\sin\theta)/2 \leq 0$, or if $y \leq (\sin\theta)/2$. As to the actual probability ratio, the denominator is just $\pi/2$, while the numerator is

$$-\frac{\cos\theta}{2}\bigg|_0^\pi = \frac{-\cos\pi + \cos 0}{2} = \frac{2}{2} = 1.$$

The total probability of hitting a line is thus $2/\pi \approx 63.66\%$.

43. To find the maximum, we take the derivative and set it equal to zero:
$f'(x) = -2ax(bx-1)(bx+1)e^{-b^2x^2} = 0$. This gives critical numbers $x = 0, \pm\dfrac{1}{b}$.
Since this will be a pdf for the interval $[0, 4m]$, we only have to check that there is a maximum at $\dfrac{1}{b}$. An easy check shows that
$f'(x) > 0$ on the interval $\left[0, \dfrac{1}{b}\right]$ and
$f'(x) < 0$ for $x > \dfrac{1}{b}$. Therefore there is a maximum at $x = m = \dfrac{1}{b}$ (the most common speed).

To find a in terms of m, we want the total probability equal to 1. Since $m = \dfrac{1}{b}$, we also make the substitution $b = \dfrac{1}{m}$.
$$1 = \int_0^{4m} ax^2 e^{-x^2/m^2}\,dx$$
Solving for a gives
$$a = \left(\int_0^{4m} x^2 e^{-x^2/m^2}\,dx\right)^{-1}$$
Note: this integral is not expressible in terms of elementary functions, so we will leave it like this. Using a CAS, one can find that
$a \approx 2.2568m^{-3}$

44. $f(t) = t^{-3/2}e^{0.38t-100/t}$
$\int_0^{40} k \cdot f(t)dt = 1$ for $k = 0.000318$.
$$\int_{20}^{30} 0.000318 \cdot f(t)dt \approx 0.0134$$

45. The probability of a $2k$-goal game ending in a $k-k$ tie is
$$(2k) = \frac{(2k)\cdots(k+1)}{(k)\cdots(1)}p^k(1-p)^k$$
$f(2k) < f(2k-2)$ for general k.
$$\frac{f(2k)}{f(2k-2)} = 2\frac{2k-1}{k}p\,(1-p)$$
Here $\dfrac{2k-1}{k} = 2 - \dfrac{1}{k} < 2$.
On the other hand,
$$\left(p - \frac{1}{2}\right)^2 \geq 0, p^2 - p + \frac{1}{4} \geq 0$$
$$p - p^2 \leq \frac{1}{4}, p(1-p) \leq \frac{1}{4}$$
Now we get $\dfrac{f(2k)}{f(2k-2)} = 2\dfrac{2k-1}{k}p\,(1-p)$
$< 2 \cdot 2 \cdot \dfrac{1}{4} = 1$. So $f(2k) < f(2k-2)$. In other words, the probability of a tie is decreasing as the number of goals increases.

46. The probability HTT appears first is the mean of that probability over the four possibilities for the first two coin tosses.

Let P(HT) be the probability HTT appears first following HT.

Suppose the first two throws are HH. Then the third throw can be either H or T. If it's H, then we are back in the same position: the preceding two throws are HH. But if it's T, then player B has won. So the probability of player A winning in this case is 0. Putting the two possibilities for the third throw together, as a mean, the probability that player A wins following HH is:
$$\text{P (HH)} = \frac{1}{2} \times \text{P (HH)} + \frac{1}{2} \times 0 = \frac{1}{2}\text{P (HH)}.$$
Now suppose the first two throws are HT. If the third throw is H, then neither player has won, and the probability HTT will ultimately win is (by definition) P(TH). (The last two throws were TH.) On the other hand, if the third throw is T, then player A has won! So this time the weighted mean for the probability that player A wins, following HT is:
$$\text{P (HH)} = \frac{1}{2} \times \text{P (TH)} + \frac{1}{2} \times 1 = \frac{1}{2}\text{P (TH)} + \frac{1}{2}$$
Similarly, we get
$$\text{P (TH)} = \frac{1}{2} \times \text{P (HH)} + \frac{1}{2} \times \text{P(HT)} \text{ and}$$
$$\text{P (TT)} = \frac{1}{2} \times \text{P (TH)} + \frac{1}{2} \times \text{P(TT)}.$$
Therefore, we have
P(HH) = 0
P(HT) = P(HT)/4 + 1/2 P(HT) = 2/3
P(TH) = P(HT)/2 = 1/3
P(TT) = P(TH) P(TT) = 1/3
The mean of these four results gives us the probability of HTT appearing before HHT is 1/3. Hence, the probability of HHT appearing before HTT is 2/3. Therefore, player B is twice as likely to win.

47. (a) The functions $f(x)$ and $g(x)$ are the pdfs, such that $f(x) = a + bx + cx^2$;
$f(x^2) = g(x)$.
Therefore by definition,
$f(x); g(x) \geq 0$ and
$$\int_0^1 f(x)dx = \int_0^1 g(x)dx = 1$$
Consider $f(x) = a + bx + cx^2$ and
$g(x) = f(x^2) = a + bx^2 + cx^4$.

Thus, $1 = \int_0^1 f(x)dx$
$= \int_0^1 \left(a + bx + cx^2\right) dx$
$= \left(ax + b\frac{x^2}{2} + c\frac{x^3}{3}\right)\Big|_0^1$
$\Rightarrow a + \frac{b}{2} + \frac{c}{3} = 1.....(1)$
and $1 = \int_0^1 g(x)dx$
$= \int_0^1 \left(a + bx^2 + cx^4\right) dx$
$= \left(ax + b\frac{x^3}{3} + c\frac{x^5}{5}\right)\Big|_0^1$
$\Rightarrow a + \frac{b}{3} + \frac{c}{5} = 1.....(2)$
Solving (1) and (2), we get,
$b = -\frac{4c}{5}; a = 1 + \frac{c}{15};$
Thus $f(x) = 1 + \frac{c}{15} - \frac{4c}{5}x + cx^2$
or $f(x) = \frac{\left(15cx^2 - 12cx + c + 15\right)}{15}$

(b) Mean of pdf g:
$\mu = \int_a^b xg(x)dx$
$= \int_0^1 x\frac{\left(15cx^4 - 12cx^2 + c + 15\right)}{15}dx$
$= \frac{1}{15}\int_0^1 \left(15cx^5 - 12cx^3 + (c+15)x\right) dx$
$= \frac{1}{15}\left(\frac{15cx^6}{6} - \frac{12cx^4}{4} + \frac{(c+15)x^2}{2}\right)\Big|_0^1$
$= 0.5$

Ch. 5 Review Exercises

1. Area $= \int_0^\pi \left(x^2 + 2 - \sin x\right) dx$
$= \left(\frac{x^3}{3} + 2x + \cos x\right)\Big|_0^\pi$
$= \frac{\pi^3}{3} + 2\pi - 2$

2. Area $= \int_0^1 (e^x - e^{-x})\, dx$
$= (e^x + e^{-x})\Big|_0^1 = e + e^{-1} - 2$

3. Area $= \int_0^1 x^3 - \left(2x^2 - x\right)\, dx$
$= \left(\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{x^2}{2}\right)\Big|_0^1 = \frac{1}{12}$

4. First solve $x^2 - 3 = -x^2 + 5$ to find that the intersections points are $x = -2, 2$.

Area $= \int_{-2}^2 [(-x^2 + 5) - (x^2 - 3)]\, dx$
$= \left(-\frac{2}{3}x^3 + 8x\right)\Big|_{-2}^2 = \frac{64}{3}.$

5. Solving $e^{-x} = 2 - x^2$ we get
$x \approx -0.537, 1.316$

Area $\approx \int_{-.537}^{1.316} \left(2 - x^2 - e^x\right) dx$
$= \left(2x - \frac{x^3}{3} + e^{-x}\right)\Big|_{-.537}^{1.316} \approx 1.452$

6. First solve $y^2 = 1 - y$ to find that the intersections points are $y = \frac{-1 \pm \sqrt{5}}{2}$.

Area $= \int_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} [(1 - y) - y^2]\, dy$
$= \left(y - \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}}$
$= \frac{5\sqrt{5}}{6}.$

7. Area $= \int_0^1 x^2\, dx + \int_1^2 (2 - x)\, dx$
$= \frac{x^3}{3}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2$
$= \frac{1}{3} + (4 - 2) - \left(2 - \frac{1}{2}\right) = \frac{5}{6}$

8. Area $= \int_0^2 x^2\, dx = \frac{8}{3}$

9. If P is the population at time t, the equation is

$P'(t) =$ birth rate $-$ death rate
$= (10 + 2t) - (4 + t) = 6 + t$

Thus $P = 6t + t^2/2 + P(0)$, so at time $t = 6$,

$$P(6) = 36 + 18 + 10{,}000 = 10{,}054.$$

Alternatively,

$A = \int_0^6 [(10 + 2t) - (4 + t)]dt$

$$= \int_0^6 (6+t)dt = \left(6t + \frac{t^2}{2}\right)\Big|_0^6 = 54$$

population $= 10{,}000 + 54 = 10{,}054$

10. For this we use Simpson's rule on the function $(f-g)(x)$.

$$\int_0^2 [f(x) - g(x)]\, dx$$

$\approx \frac{0.2}{3}[(3.2-1.2)+4(3.5-1.5)+2(3.8-1.6) + 4(3.7-2.2)+2(3.2-2.0)+4(3.4-2.4) +2(3.0-2.2)+4(2.8-2.1)+2(2.3-2.3) +4(2.9-2.8)+(3.4-2.4)]$

$\approx 2.1733.$

11. $V = \int_0^2 \pi(3+x)^2\, dx$

$= \pi \int_0^2 (9 + 6x + x^2)\, dx$

$= \pi \left(9x + 3x^2 + \frac{x^3}{3}\right)\Big|_0^2$

$= \frac{98\pi}{3}$

12. If we consider slices perpendicular to the x-axis, then the area of a slice is equal to $(10 + 2x)(4+x)$ (length times depth). We integrate the areas from $x = 0$ to $x = 2$:

Area $= \int_0^2 (10+2x)(4+x)\, dx$

$= \frac{364}{3} \approx 121.33$ cubic feet.

13. Use trapezoidal estimate:

$$V = 0.4\left(\frac{0.4}{2} + 1.4 + 1.8 + 2.0 + 2.1 + 1.8 + 1.1 + \frac{0.4}{2}\right)$$

≈ 4.2

14. (a) $V = \int_0^1 \pi x^4\, dx = \frac{\pi}{5}$

(b) $V = \int_0^1 \pi(1-y)\, dy = \frac{\pi}{2}$

(c) $V = \int_0^1 \pi[(2-\sqrt{y})^2 - 1]\, dy = \frac{5\pi}{6}$

(d) $V = \int_0^1 \pi[(2+x^2)^2 - 2]\, dx = \frac{53}{15}$

15. (a) $V = \int_{-2}^2 \pi(4)^2\, dx - \int_{-2}^2 \pi(x^2)^2\, dx$

$= \pi \int_{-2}^2 (16 - x^4)\, dx$

$= \pi \left(16x - \frac{x^5}{5}\right)\Big|_{-2}^2$

$= \frac{256\pi}{5}$

(b) $V = \int_0^4 \pi(\sqrt{y})^2 dy = \pi \int_0^4 y dy$

$= \frac{\pi y^2}{2}\Big|_0^4 = 8\pi$

(c) $V = \int_0^4 \pi(2+\sqrt{y})^2 dy - \int_0^4 \pi(2-\sqrt{y})^2 dy$

$= \pi \int_0^4 (4 + 4y^{1/2} + y)dy - \pi \int_0^4 (4 - 4y^{1/2} + y)dy$

$= \pi \int_0^4 (8y^{1/2})dy$

$= 8\pi \cdot \frac{2}{3} y^{3/2}\Big|_0^4 = \frac{128\pi}{3}$

(d) $V = \int_{-2}^2 \pi(6)^2\, dx - \int_{-2}^2 \pi(x^2+2)^2\, dx$

$= \pi \int_{-2}^2 (-x^4 - 4x^2 + 32)\, dx$

$= \pi \left(-\frac{x^5}{5} - \frac{4x^3}{3} + 32x\right)\Big|_{-2}^2$

$= \frac{1408\pi}{15}$

16. (a) $V = \int_0^2 \pi(4x^2 - x^2)\, dx = 8\pi$

(b) $V = \int_0^2 \pi\left(y^2 - \frac{y^2}{4}\right) dy + \int_2^4 \pi\left(4 - \frac{y^2}{4}\right) dy$

$= 2\pi + \frac{10\pi}{3} = \frac{16\pi}{3}$

(c) V

$= \int_0^2 \pi\left[(1+y)^2 - \left(1+\frac{y}{2}\right)^2\right] dy + \int_2^4 \pi\left[9 - \left(1+\frac{y}{2}\right)^2\right] dy$

$= 4\pi + \frac{16\pi}{3} = \frac{28\pi}{3}$

(d) $V = \int_0^2 \pi[(4-x)^2 - (4-2x)^2]\,dx$
$= 8\pi$

17. (a) $V = \int_0^1 2\pi y((2-y)-y)dy$
$= 2\pi \int_0^1 (2y - 2y^2)dy$
$= 2\pi \left(y^2 - \frac{2y^3}{3} \right)\Big|_0^1$
$= \frac{2\pi}{3}$

(b) $V = \int_0^1 \pi(2-y)^2 dy$
$- \int_0^1 \pi(y)^2 dy$
$= \pi \int_0^1 (4-4y)dy$
$= \pi\ (4y - 2y^2)\big|_0^1 = 2\pi$

(c) $V = \int_0^1 \pi((2-y)+1)^2 dy$
$- \int_0^1 \pi(y+1)^2 dy$
$= \pi \int_0^1 (9 - 6y + y^2)dy$
$- \pi \int_0^1 (y^2 + 2y + 1)dy$
$= \pi \int_0^1 (8 - 8y)dy$
$= \pi\ (8y - 4y^2)\big|_0^1 = 4\pi$

(d) $V = \int_0^1 2\pi(4-y)((2-y)-y)dy$
$= 2\pi \int_0^1 (8 - 10y + 2y^2)dy$
$= 2\pi \left(8y - 5y^2 + \frac{2y^3}{3} \right)\Big|_0^1$
$= \frac{22\pi}{3}$

18. (a) Method of shells.
$V = \int_0^2 2\pi y[(4-y^2) - (y^2-4)]\,dy$
$= 16\pi$

(b) $V = \int_{-2}^2 \pi(4-y^2)^2\,dy = \frac{512\pi}{15}$

(c) $V = \int_{-2}^2 \pi[(8-y^2)^2 - y^4]\,dy$
$= \frac{512\pi}{3}$

(d) Method of shells.
$V = \int_{-2}^2 2\pi(4-y)[(4-y^2)$
$- (y^2-4)]\,dy$
$= \frac{208\pi}{3}$

19. $s = \int_{-1}^1 \sqrt{1 + (4x^3)^2}\,dx \approx 3.2$

20. $s = \int_{-1}^0 \sqrt{1 + (2x+1)^2}\,dx \approx 1.14779$

21. $s \int_{-2}^2 \sqrt{1 + \left(\frac{e^{x/2}}{2}\right)^2}\,dx \approx 4.767$

22. $s = \int_0^\pi \sqrt{1 + 4\cos^2 2x}\,dx \approx 5.27037$

23. $S = \int_0^1 2\pi(1-x^2)\sqrt{1+4x^2}\,dx$
≈ 5.483

24. $S = \int_0^1 2\pi x^3 \sqrt{1+9x^4}\,dx \approx 3.56312$

25. $h''(t) = -32$
$h(0) = 64, h'(0) = 0$
$h'(t) = -32t$
$h(t) = -16t^2 + 64$

This is zero when $t = 2$, at which time $h'(2) = -32(2) = -64$. The speed at impact is reported as 64 feet per second.

26. In this case we have the equations
$h''(t) = -32$
$h(0) = 64 \quad h'(0) = 4$
$h'(t) = -32t + 4$
$h(t) = -16t^2 + 4t + 64$

This is zero when
$t = t_0 = \frac{1 + \sqrt{257}}{8}$

Therefore the velocity at impact is
$h'(t_0) = \frac{-32(1+\sqrt{257})}{8} + 4$
$= -4\sqrt{257} \approx -64.125$ ft/s

27. $y''(t) = -32, x''(t) = 0,$
$y(0) = 0, x(0) = 0$
$y'(0) = 48 \sin\left(\frac{\pi}{9}\right)$
$x'(0) = 48 \cos\left(\frac{\pi}{9}\right)$
$y'(0) \approx 16.42, x'(0) \approx 45.11$

$y'(t) = -32t + 16.42$
$y(t) = -16t^2 + 16.42t$

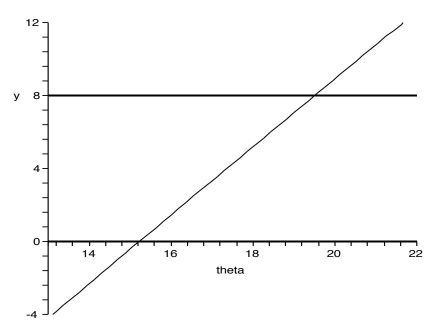
This is zero at $t = 1.026$. Meanwhile,

$x'(t) \equiv 45.11$
$x(t) = 45.11t$
$x(1.026) = 45.11(1.026) \approx 46.3$ ft This is the horizontal range.

28. In this case we have the equations
$y''(t) = -32, x''(t) = 0$
$y(0) = 6, x(0) = 0$
$y'(0) = 48 \sin \frac{\pi}{9}, \quad x'(0) = 48 \cos \frac{\pi}{9}$
$y'(t) = -32t + 48 \sin \frac{\pi}{9}$
$x'(t) = 48 \cos \frac{\pi}{9}$
$y(t) = -16t^2 + 48t \sin \frac{\pi}{9} + 6$
$x(t) = 48t \cos \frac{\pi}{9}$
We now solve $y(t) = 0$ or
$-16t^2 + 48t \sin \frac{\pi}{9} + 6 = 0$
which gives $t \approx 1.3119$, this is the time of flight.
The horizontal range is
$x(1.3119) \approx 59.17$ feet.

29. $y(0) = 6, x(0) = 0$
$y'(0) = 80 \sin \left(\frac{2\pi}{45}\right) \approx 11.13,$
$x'(0) = 80 \cos \left(\frac{2\pi}{45}\right) \approx 79.22$
$y''(t) = -32, x''(t) = 0$
$y'(t) = -32t + 11.13$
$y(t) = -16t^2 + 11.13t + 6$
$x'(t) = 79.22$
$x(t) = 79.22t$
This is 120 (40 yards) when t is about 1.51. At this time, the vertical height (if still in flight) would be
$y(1.51) = -16(1.51)^2 + 11.13(1.51) + 6$
$= -13.6753,$
Since this is negative, we conclude the ball is not still in flight, has hit the ground, and was not catchable.

30. If we repeat Exercise 29, but we'll leave the angle as θ (we will plug in $\theta = 24° = \frac{2\pi}{15}$ later too).
Our equations become
$y(0) = 6, \quad x(0) = 0$
$y'(0) = 80 \sin \theta, \quad x'(0) = 80 \cos \theta$
$y''(t) = -32, \quad x''(t) = 0$
Integrating and using the initial conditions gives
$y'(t) = -32t + 80 \sin \theta$
$x'(t) = 80 \cos \theta$
$y(t) = -16t^2 + 80t \sin \theta + 6$
$x(t) = 80t \cos \theta$
We solve for the time when the ball is 40 yards down the field:
$120 = x(t) = 80t \cos \theta$
Solving gives
$t_0 = t = \frac{3}{2} \sec \theta$
The height at this time is
$$y(t_0) = -16 \left(\frac{3}{2} \sec \theta\right)^2 + 80 \left(\frac{3}{2} \sec \theta\right) \sin \theta + 6$$
$$= -36 \sec^2 \theta + 120 \tan \theta + 6$$
Let us say that the ball is catchable if it is between 0 and 8 feet high when the ball reaches the 40 yard point (the player can dive or jump to catch a low or high ball). To determine when this occurs, we graph the function and see that for the ball to be catchable it must be thrown with angle in the range:
$15.23° < \theta < 19.51°$



31. $h''(t) = -32$
$h'(0) = v_0$
$h(0) = 0$
$h'(t) = -32t + v_0$
This is zero at $t = v_0/32$.
$h\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0^2}{32^2}\right) + \frac{v_0^2}{32} = \frac{v_0^2}{64}$
If this is to be 128, then clearly v_0 must be
$$\sqrt{(64)(128)} = 64\sqrt{2} \text{ ft/sec.}$$
Impact speed from ground to ground is the same as launch speed, which can be verified by first finding the time t of return to ground:
$-16t^2 + v_0 t = 0$

$t = v_0/16$
and then compiling

$$h'(v_0/16) = -32(v_0/16) + v_0 = -v_0$$

32. We want to determine how far in the x-direction the drop travels. We have initial conditions
$x'(0) = 100$, $x(0) = 0$
$y'(0) = 0$, $y(0) = 120$
$x'(t) = 100$, $x(t) = 100t$,
$y'(t) = -32t$, $y(t) = -16t^2 + 120$

We first solve $0 = y = -16t^2 + 120$ to get $t = \sqrt{\frac{15}{2}}$. This is when the supplies hit the ground. We plug this into the equation $x(t)$ to determine how far the supplies traveled.

$$x\left(\sqrt{\frac{15}{2}}\right) = 100\sqrt{\frac{15}{2}} \approx 273.86$$

So, the supplies should be dropped 273.86 feet before the target.

33. $F = kx,\ 60 = k \cdot 1,\ k = 60$

$W = \int_0^{2/3} 60x\,dx = 30x^2\Big|_0^{2/3}$

$= \frac{30 \cdot 4}{9} = \frac{40}{3}$ ft-lb

34. Remember to convert miles to feet.

$W = \int_0^8 (800 + 2x)\,dx$

$= 6464$ mile-pounds
$= 3.413 \times 10^7$ foot-pounds.

35. $m = \int_0^4 \left(x^2 - 2x + 8\right) dx$

$= \left(\frac{x^3}{3} - x^2 + 8x\right)\Big|_0^4 = \frac{112}{3}$

$M = \int_0^4 x\left(x^2 - 2x + 8\right)\,dx$

$= \int_0^4 \left(x^3 - 2x^2 + 8x\right) dx$

$= \left(\frac{x^4}{4} - \frac{2x^3}{3} + 4x^2\right)\Big|_0^4 = \frac{256}{3}$

$\bar{x} = \frac{M}{m} = \frac{\frac{256}{3}}{\frac{112}{3}} = \frac{256}{112} = \frac{16}{7}$

Center of mass is greater than 2 because the object has greater density on the right side of the interval $[0, 4]$.

36. $m = \int_0^2 (x^2 - 2x + 8)\,dx = \frac{44}{3}$.

$M = \int_0^2 x(x^2 - 2x + 8)\,dx = \frac{44}{3}$.

$\overline{x} = \frac{M}{m} = 1$

The center of mass is at one because the density function is symmetrical about the point $x = 1$. (The graph of $y = x^2 - 2x + 8$ is a parabola with vertex at $x = 1$.)

37. $F = \int_0^{80} 62.4x(140 - x)\,dx$

$= 62.4 \int_0^{80} (140x - x^2)\,dx$

$= 62.4\left(70x^2 - \frac{x^3}{3}\right)\Big|_0^{80}$

$= 62.4(80)^2(130/3)$

$\approx 17{,}305{,}600$ lb

38. $F = \int_5^{10} 62.4(20)x\,dx = 46800$ lb

39. $J \approx \frac{0008}{3(8)}\{0 + 4(800) + 2(1600)$

$+ 4(2400) + 2(3000) + 4(3600)$
$+2(2200) + 4(1200) + 0\}$
$= 1.52$

$J = m\Delta v$

$1.52 = .01\Delta v$
$\Delta v = 152$ ft/s
$152 - 120 = 32$ ft/s

40. $J = \int_0^2 3000t(2 - t)\,dt = 4000$

Since $J = m\Delta v$, we have $\Delta v = \frac{4000}{100} = 40$ and the speed before the collision must have been 40 feet per second (about 23.7 miles per hour).

41. $f(x) = x + 2x^3$ on $[0, 1]$
$f(x) \geq 0$ for $0 \leq x \leq 1$ and

$$\int_0^1 \left(x + 2x^3\right) dx = \left(\frac{x^2}{2} + \frac{x^4}{2}\right)\Big|_0^1 = 1$$

42. The function is positive on the interval, and

$$\int_0^{\ln 2} \frac{8}{3}e^{-2x}\,dx = 1.$$

43.
$$1 = \int_1^2 \frac{c}{x^2}\,dx = \left.\frac{-c}{x}\right|_1^2 = \frac{-c}{2} + c = \frac{c}{2}$$
Therefore $c = 2$

44. We want to solve for c:
$$1 = \int_0^4 ce^{-2x}\,dx = \frac{c}{2}(1 - e^{-8})$$
Solving gives
$$c = \frac{2}{1 - e^{-8}}.$$

45. (a) $P(x < .5) = \int_0^{.5} 4e^{-4x}\,dx$
$= -e^{-4x}\big|_0^{.5} = 1 - e^{-2} \approx .864$

(b) $P(.5 \le x \le 1) = \int_{.5}^1 4e^{-4x}\,dx$
$= -e^{-4x}\big|_{.5}^1 = -e^{-4} + e^{-2} \approx .117$

46. (a) $P\left(X < \frac{1}{12}\right) = \int_0^{1/12} 9xe^{-3x}\,dx$
$= 1 - \frac{5}{4}e^{-1/4} \approx 0.026499$

(b) $P\left(\frac{1}{2} < X < 1\right) = \int_{1/2}^1 9xe^{-3x}\,dx$
$= \frac{5}{2}e^{-3/2} - 4e^{-3} \approx 0.35868$

47. (a) $\mu = \int_0^1 x\left(x + 2x^3\right)\,dx$
$= \left.\frac{x^3}{3} + \frac{2x^5}{5}\right|_0^1 = \frac{11}{15} \approx 0.7333$

(b) $\frac{1}{2} = \int_0^c \left(x + 2x^3\right)dx$
$= \left.\frac{x^2}{2} + \frac{x^4}{2}\right|_0^c = \frac{c^2}{2} + \frac{c^4}{2}$
Therefore $c^2 + c^4 = 1$,
$$c = \sqrt{\frac{-1 + \sqrt{5}}{2}} \approx 0.786$$

48. (a) $\mu = \int_0^{\ln 2} \frac{8}{3}xe^{-2x}\,dx$
$= \frac{1}{2} - \frac{1}{3}\ln 2 \approx 0.26895$

(b) For the median, we have to solve the equation
$$0.5 = \int_0^m \frac{8}{3}e^{-2x}\,dx = \frac{4}{3}(1 - e^{2m})$$
Solving gives
$$m = \frac{1}{2}\ln(8/5) \approx 0.23500$$

CHAPTER 6

Chapter 6

Integration Techniques

6.1 Review of Formulas and Techniques

1. $\int e^{ax}dx = \frac{1}{a}e^{ax} + c,\ \text{for } a \neq 0.$

2. $\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c,\ \text{for } a \neq 0.$

3. $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}}\left(\frac{1}{a}\right)dx$

Let $u = \frac{x}{a}, du = \frac{1}{a}dx.$

$= \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + c$

$= \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0.$

4. $\int \frac{b}{|x|\sqrt{x^2 - a^2}}dx$

$= \int \frac{b}{|x|\sqrt{\left(\frac{x}{a}\right)^2 - 1}}\left(\frac{1}{a}\right)dx$

Let $u = \frac{x}{a}, du = \frac{1}{a}dx$ and $|au| = |x|$.

$= \int \frac{b}{|au|\sqrt{u^2-1}}\,du$

$= \frac{b}{|a|}\int \frac{1}{|u|\sqrt{u^2-1}}\,du$

$= \frac{b}{|a|}\sec^{-1}(u) + c$

$= \frac{b}{|a|}\sec^{-1}\left(\frac{x}{a}\right) + c,\ \ a > 0.$

5. $\int \sin(6t)dt = -\frac{1}{6}\cos(6t) + c$

6. $\int \sec 2t \tan 2t\, dt = \frac{1}{2}\sec 2t + c$

7. $\int (x^2+4)^2 dx = \int (x^4 + 8x^2 + 16)dx$

$= \frac{x^5}{5} + \frac{8}{3}x^3 + 16x + c$

8. $\int x(x^2+4)^2 dx = \int (x^5 + 8x^3 + 16x)dx$

$= \frac{x^6}{6} + 2x^4 + 8x^2 + c$

9. $\frac{3}{16+x^2}dx = \frac{3}{4}\tan^{-1}\frac{x}{4} + c$

10. $\frac{2}{4+4x^2}dx = \frac{1}{2}\tan^{-1}x + c$

11. $\int \frac{1}{\sqrt{3-2x-x^2}}dx$

$= \int \frac{1}{\sqrt{4-(x+1)^2}}dx = \arcsin\left(\frac{x+1}{2}\right) + c$

12. $\int \frac{x+1}{\sqrt{3-2x-x^2}}dx$

$= -\frac{1}{2}\int \frac{-2(x+1)}{\sqrt{4-(x+1)^2}}dx$

$= -\frac{1}{2}\cdot 2[4-(x+1)^2]^{1/2} + C$

$= -\sqrt{4-(x+1)^2} + c$

13. $\int \frac{4}{5+2x+x^2}dx$

$= 4\int \frac{1}{4+(x+1)^2}dx = 2\tan^{-1}\left(\frac{x+1}{2}\right) + c$

14. $\int \frac{4x+4}{5+2x+x^2}dx$

$= 2\int \frac{2(x+1)}{4+(x+1)^2}dx = 2\ln|4+(x+1)^2| + c$

15. $\int \frac{4t}{5+2t+t^2}dt$

$= \int \frac{4t+4}{5+2t+t^2}dt - \int \frac{4}{5+2t+t^2}dt$

$= 2\ln\left|4+(t+1)^2\right| - 2\tan^{-1}\left(\frac{t+1}{2}\right) + c$

16. $\int \frac{t+1}{t^2+2t+4}dt = \int \frac{2(t+1)}{(t+1)^2+3}dt$

$= \frac{1}{2}\ln\left|(t+1)^2+3\right| + c$

17. $\int e^{3-2x}dx = -\frac{1}{2}e^{3-2x} + c$

18. $\int 3e^{-6x}dx = -\frac{3}{6}e^{-6x} + c$

19. Let $u = 1 + x^{2/3}, du = \frac{2}{3}x^{-1/3}dx$
$\int \frac{4}{x^{1/3}(1+x^{2/3})}dx = 4\left(\frac{3}{2}\right)\int u^{-1}du$
$= 6\ln|u| + C = 6\ln|1+x^{2/3}| + c$

20. Let $u = 1 + x^{3/4}, du = \frac{3}{4}x^{-1/4}dx$
$\int \frac{2}{x^{1/4}+x}dx = \int \frac{2}{x^{1/4}(1+x^{3/4})}dx$
$= 2\left(\frac{4}{3}\right)\int u^{-1}du = \frac{8}{3}\ln|u| + C$
$= \frac{8}{3}\ln|1+x^{3/4}| + c$

21. Let $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx$
$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin u du$
$= -2\cos u + C = -2\cos\sqrt{x} + c$

22. Let $u = \frac{1}{x}, du = -\frac{1}{x^2}dx$
$\int \frac{\cos(1/x)}{x^2}dx = -\int \cos u du$
$= -\sin u + C = -\sin\frac{1}{x} + c$

23. Let $u = \sin x, du = \cos x dx$
$\int_0^{\pi} \cos x e^{\sin x} dx = \int_0^0 e^u du = 0$

24. Let $u = \tan x, du = \sec^2 x dx$
$\int_0^{\pi/2} \sec^2 x e^{\tan x} dx = \int_0^1 e^u du$
$= e^u \Big|_0^1 = e - 1$

25. $\int_{-\pi/4}^0 \sec x \tan x dx$
$= \sec x \Big|_{-\pi/4}^0 = 1 - \sqrt{2}$

26. $\int_{\pi/4}^{\pi/2} \csc^2 x dx = -\cot x \Big|_{\pi/4}^{\pi/2} = 1$

27. Let $u = x^3, du = 3x^2dx$
$\frac{x^2}{1+x^6}dx = \frac{1}{3}\int \frac{1}{1+u^2}du$
$= \frac{1}{3}\tan^{-1} u + C = \frac{1}{3}\tan^{-1} x^3 + c$

28. $\int \frac{x^5}{1+x^6}dx = \frac{1}{6}\ln(1+x^6) + c$

29. $\frac{1}{\sqrt{4-x^2}}dx = \sin^{-1}\frac{x}{2} + c$

30. Let $u = e^x, du = e^x dx$
$\frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{1}{\sqrt{1-u^2}}du$
$= \sin^{-1} u + C = \sin^{-1} e^x + c$

31. Let $u = x^2, du = 2xdx$
$\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$
$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} x^2 + c$

32. Let $u = 1 - x^4, du = -4x^3dx$
$\int \frac{2x^3}{\sqrt{1-x^4}}dx = -\frac{1}{2}\int u^{-1/2}du$
$= -u^{1/2} + C = -(1-x^4)^{1/2} + c$

33. $\int \frac{1+x}{1+x^2}dx$
$= \int \frac{1}{1+x^2}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx$
$= \tan^{-1} x + \frac{1}{2}\ln|1+x^2| + c$

34. $\int \frac{1}{\sqrt{x}+x}dx$
$= \int x^{-1/2}\cdot\frac{1}{1+x^{1/2}}dx$
$= 2\ln|1+x^{1/2}| + c$

35. $\int \frac{\ln x^2}{x}dx = 2\int \ln x\left(\frac{1}{x}\right)dx$
Let $u = \ln x, du = \frac{1}{x}dx.$
$= 2\int u\,du = u^2 + c = (\ln x)^2 + c$

36. $\int_1^3 e^{2\ln x}dx = \int_1^3 x^2dx = \frac{x^3}{3}\Big|_1^3 = \frac{26}{3}$

37. $\int_3^4 x\sqrt{x-3}dx$
$= \int_3^4 (x-3+3)\sqrt{x-3}dx$
$= \int_3^4 (x-3)^{3/2}dx + 3\int_3^4 (x-3)^{1/2}dx$
$= \frac{2}{5}(x-3)^{5/2}\Big|_3^4 + 3\cdot\frac{2}{3}(x-3)^{3/2}\Big|_3^4 = \frac{12}{5}$

38. $\int_0^1 x(x-3)^2dx$
$= \int_0^1 (x^3 - 6x^2 + 9x)dx$
$= \left(\frac{x^4}{4} - 2x^3 + \frac{9}{2}x^2\right)\Big|_0^1 = \frac{11}{4}$

39. $\int_1^4 \frac{x^2+1}{\sqrt{x}} dx$
$= \int_1^4 x^{3/2} dx + \int_1^4 x^{-1/2} dx$
$= \frac{2}{5} x^{5/2} \Big|_1^4 + 2x^{1/2} \Big|_1^4 = \frac{72}{5}$

40. $\int_{-2}^0 x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} \Big|_{-2}^0 = \frac{e^{-4}-1}{2}$

41. $\int \frac{5}{3+x^2} dx = \frac{5}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + c$
$\int \frac{5}{3+x^3} dx$: N/A

42. $\int \sin(3x) dx = \frac{1}{3} \int \sin(3x) 3dx$
Let $u = 3x, du = 3dx$.
$= \frac{1}{3} \int (\sin u) du = -\frac{1}{3} \cos u + c$
$= -\frac{1}{3} \cos(3x) + c.$

$\int \sin^3 x dx = \int (\sin^2 x) \sin x dx$
$= \int (1 - \cos^2 x) \sin x dx$

Let $u = \cos x, du = -\sin x dx$.
$= \int (1-u^2)(-du) = \int u^2 du - \int du$
$= \frac{u^3}{3} - u = \frac{\cos^3 x}{3} - \cos x.$

43. $\int \ln x dx$: N/A
Substituting $u = \ln x$,
$\int \frac{\ln x}{2x} dx = \frac{1}{4} \ln^2 x + c$

44. Substituting $u = x^4$
$\int \frac{x^3}{1+x^8} dx = \frac{1}{4} \arctan x^4 + c$
$\int \frac{x^4}{1+x^8} dx$: N/A

45. $\int e^{-x^2} dx$: N/A
Substituting $u = -x^2$
$\int x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + c$

46. $\int \sec x dx$: N/A
$\int \sec^2 x dx = \tan x + c$

47. $\int_0^2 f(x) dx$
$= \int_0^1 \frac{x}{x^2+1} dx + \int_1^2 \frac{x^2}{x^2+1} dx$
$= \frac{1}{2} \ln|x^2+1| \Big|_0^1 + \int_1^2 \left(1 - \frac{1}{x^2+1}\right) dx$
$= \frac{1}{2} \ln 2 + (x - \arctan x) \Big|_1^2$
$= \frac{\ln 2}{2} + 1 + \frac{\pi}{4} - \arctan 2$

48. $\int \frac{4x+1}{2x^2+4x+10} dx$
$= \int \frac{4x+4}{2x^2+4x+10} dx - \int \frac{3}{2x^2+4x+10} dx$
$= \ln|2x^2+4x+10| - \frac{3}{2} \int \frac{1}{(x+1)^2+4} dx$
$= \ln|2x^2+4x+10| - \frac{3}{4} \tan^{-1}\left(\frac{x+1}{2}\right) + c$

49. $\int \frac{1}{(1+x^2)} dx = \tan^{-1}(x) + c.$
$\int \frac{x}{(1+x^2)} dx = \frac{1}{2} \int \frac{2x}{(1+x^2)} dx$
$= \frac{1}{2} \ln(1+x^2) + c.$
$\int \frac{x^2}{(1+x^2)} dx = \int \frac{x^2+1-1}{(1+x^2)} dx$
$= \int \frac{(x^2+1)}{(x^2+1)} dx - \int \frac{1}{(1+x^2)} dx$
$= \int dx - \int \frac{1}{(1+x^2)} dx$
$= x - \tan^{-1}(x) + c.$
$\int \frac{x^3}{(1+x^2)} dx = \frac{1}{2} \int \frac{x^2}{(1+x^2)} 2x dx$
Let $u = x^2, du = 2x dx$.
$= \frac{1}{2} \int \frac{u}{1+u} du = \frac{1}{2} \int \frac{u+1-1}{1+u} du$
$= \frac{1}{2} \left\{ \int \frac{u+1}{1+u} du - \int \frac{1}{1+u} du \right\}$
$= \frac{1}{2} \left\{ \int du - \int \frac{1}{1+u} du \right\}$
$= \frac{1}{2} (u - \ln(1+u)) + c$
$= \frac{1}{2} x^2 - \frac{1}{2} \ln(1+x^2) + c.$
Hence we can generalize this as follows,
$\int \left(\frac{x^n}{1+x^2}\right) dx$
$= \frac{1}{n-1} x^{n-1} - \int \left(\frac{x^{n-2}}{1+x^2}\right) dx$

50. $\int \frac{x}{1+x^4} dx = \frac{1}{2} \int \frac{1}{1+x^4} 2x dx$

Let $u = x^2, du = 2xdx$.

$= \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u) + c$

$= \frac{1}{2}\tan^{-1}\left(x^2\right) + c.$

$\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int \frac{1}{1+x^4}4x^3dx$

Let $u = 1+x^4, du = 4x^3$.

$= \frac{1}{4}\int \frac{1}{u}du = \frac{1}{4}\ln(u) + c$

$= \frac{1}{4}\ln\left(1+x^4\right) + c.$

$\int \frac{x^5}{1+x^4}\,dx$

$= \frac{1}{2}\int \frac{x^4}{1+x^4}2xdx$

Let $u = x^2, du = 2xdx$.

$= \frac{1}{2}\int \frac{u^2}{1+u^2}\,du = \frac{1}{2}\int \frac{u^2+1-1}{1+u^2}\,du$

$= \frac{1}{2}\left\{\int \frac{u^2+1}{1+u^2}du - \int \frac{1}{1+u^2}\,du\right\}$

$= \frac{1}{2}\left\{\int du - \int \frac{1}{1+u^2}\,du\right\}$

$= \frac{1}{2}\left\{u - \tan^{-1}(u)\right\} + c$

$= \frac{1}{2}\left\{x^2 - \tan^{-1}\left(x^2\right)\right\} + c.$

Hence we can generalize this as follows,

$\int \frac{x^{4n+1}}{1+x^4}dx = \frac{1}{2}\left\{\frac{x^{2n-2}}{n-1}\right\} - \int \frac{x^{4(n-1)+1}}{1+x^4}dx$

and

$\int \frac{x^{4n+3}}{1+x^4}dx = \frac{1}{4}\left\{\frac{x^{2n}}{n}\right\} - \int \frac{x^{4(n-1)+3}}{1+x^4}dx$

6.2 Integration by Parts

1. Let $u = x,\ dv = \cos x dx$

$du = dx, v = \sin x$.

$\int x\cos x dx = x\sin x - \int \sin x dx$

$= x\sin x + \cos x + c$

2. Let $u = x, dv = \sin 4x dx$

$du = dx, v = -\frac{1}{4}\cos 4x$

$\int x\sin 4x\,dx$

$= -\frac{1}{4}x\cos 4x - \int -\frac{1}{4}\cos 4x\,dx$

$= -\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + c.$

3. Let $u = x, dv = e^{2x}dx$

$du = dx,\ v = \frac{1}{2}e^{2x}$.

$\int xe^{2x}dx = \frac{1}{2}\,xe^{2x} - \int \frac{1}{2}\,e^{2x}dx$

$= \frac{1}{2}\,xe^{2x} - \frac{1}{4}\,e^{2x} + c.$

4. Let $u = \ln x, dv = x\,dx$

$du = \frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$.

$\int x\ln x\,dx = \frac{1}{2}\,x^2\ln x - \int \frac{1}{2}\,x\,dx$

$= \frac{1}{2}\,x^2\ln x - \frac{1}{4}\,x^2 + c.$

5. Let $u = \ln x,\ dv = x^2dx$

$du = \frac{1}{x}dx,\ v = \frac{1}{3}\,x^3$.

$\int x^2\ln x dx = \frac{1}{3}x^3\ln x - \int \frac{1}{3}\,x^3\cdot\frac{1}{x}dx$

$= \frac{1}{3}\,x^3\ln x - \frac{1}{3}\int x^2dx$

$= \frac{1}{3}\,x^3\ln x - \frac{1}{9}\,x^3 + c.$

6. Let $u = \ln x, du = \frac{1}{x}\,dx$.

$\int \frac{\ln x}{x}dx = \int udu = \frac{u^2}{2} + c = \frac{1}{2}(\ln x)^2 + c.$

7. Let $u = x^2,\ dv = e^{-3x}dx$

$du = 2xdx,\ v = -\frac{1}{3}e^{-3x}$

$I = \int x^2e^{-3x}dx$

$= -\frac{1}{3}x^2e^{-3x} - \int\left(-\frac{1}{3}e^{-3x}\right)\cdot 2xdx$

$= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3x}dx$

Let $u = x,\ dv = e^{-3x}dx$

$du = dx,\ v = -\frac{1}{3}e^{-3x}$

$I = -\frac{1}{3}x^2e^{-3x}$

$\quad + \frac{2}{3}\left[-\frac{1}{3}\,xe^{-3x} - \int\left(-\frac{1}{3}\,e^{-3x}\right)dx\right]$

$= -\frac{1}{3}\,x^2e^{-3x} - \frac{2}{9}\,xe^{-3x} + \frac{2}{9}\int e^{-3x}dx$

$= -\frac{1}{3}\,x^2e^{-3x} - \frac{2}{9}\,xe^{-3x} - \frac{2}{27}\,e^{-3x} + c$

8. Let $u = x^3, du = 3x^2\,dx$.

$\int x^2e^{x^3}\,dx = \frac{1}{3}\int e^u\,dx = \frac{1}{3}e^u + c$

$= \frac{1}{3}e^{x^3} + c.$

9. Let $I = \int e^x\sin 4x dx$

$u = e^x,\ dv = \sin 4x dx$

$du = e^x dx,\ v = -\frac{1}{4}\cos 4x$

$I = -\frac{1}{4}e^x \cos 4x - \int \left(-\frac{1}{4}\cos 4x\right) e^x dx$
$= -\frac{1}{4}e^x \cos 4x + \frac{1}{4}\int e^x \cos 4x dx$

Use integration by parts again, this time let $u = e^x,\ dv = \cos 4x dx$

$du = e^x dx,\ v = \frac{1}{4}\sin 4x$

$I = -\frac{1}{4}e^x \cos 4x + \frac{1}{4}\left(\frac{1}{4}e^x \sin 4x - \int \frac{1}{4}(\sin 4x)e^x dx\right)$

$I = -\frac{1}{4}e^x \cos 4x + \frac{1}{16}e^x \sin 4x - \frac{1}{16}I$

So,

$\frac{17}{16}I = -\frac{1}{4}e^x \cos 4x + \frac{1}{16}e^x \sin 4x + c_1$

$I = -\frac{4}{17}e^x \cos 4x + \frac{1}{17}e^x \sin 4x + c$

10. Let, $u = e^{2x}, dv = \cos x\, dx$ so that, $du = 2e^{2x}\ dx$ and $v = \sin x$.

$\int e^{2x}\cos x\, dx$
$= e^{2x}\sin x - 2\int e^{2x}\sin x\ dx$

Let, $u = e^{2x}$, $dv = \sin x\, dx$ so that, $du = 2e^{2x}\ dx$ and $v = -\cos x$.

$\int e^{2x}\sin x\, dx$
$= -e^{2x}\cos x + 2\int e^{2x}\cos x\ dx$

$\int e^{2x}\cos x\, dx$
$= e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x\, dx$

Now we notice that the integral on both of these is the same, so we bring them to one side of the equation.

$5\int e^{2x}\cos x\, dx$
$= e^{2x}\sin x + 2e^{2x}\cos x + c_1$

$\int e^{2x}\cos x\, dx$
$= \frac{1}{5}e^{2x}\sin x + \frac{2}{5}e^{2x}\cos x + c$

11. Let $I = \int \cos x \cos 2x dx$

and $u = \cos x, dv = \cos 2x dx$

$du = \sin x dx,\ v = \frac{1}{2}\sin 2x$

$I = \frac{1}{2}\cos x \sin 2x - \int \frac{1}{2}\sin 2x(-\sin x)dx$
$= \frac{1}{2}\cos x \sin 2x + \frac{1}{2}\int \sin x \sin 2x dx$

Let,$u = \sin x, dv = \sin 2x dx$

$du = \cos x dx\ v = -\frac{1}{2}\cos 2x$

$I = \frac{1}{2}\cos x \sin 2x + \frac{1}{2}\left[-\frac{1}{2}\cos 2x \sin x - \int \left(-\frac{1}{2}\cos 2x\right)\cos x dx\right]$

$= \frac{1}{2}\cos x \sin 2x \quad - \frac{1}{4}\cos 2x \sin x + \frac{1}{4}I dx$

So,

$\frac{3}{4}I = \frac{1}{2}\cos x \sin 2x - \frac{1}{4}\cos 2x \sin x + c_1$

$I = \frac{2}{3}\cos x \sin 2x - \frac{1}{3}\cos 2x \sin x + c$

12. Here we use the trigonometric identity: $\sin 2x = 2\sin x \cos x$.

We then make the substitution $u = \sin x, du = \cos x\ dx$.

$\int \sin x \sin 2x\ dx = \int 2\sin^2 x \cos x\ dx$

$= \int 2u^2\ du = \frac{2}{3}u^3 + c = \frac{2}{3}\sin^3 x + c$

This integral can also be done by parts, twice. If this is done, an equivalent answer is obtained:

$\frac{1}{3}\cos x \sin 2x - \frac{2}{3}\cos 2x \sin x + c$

13. Let $u = x,\ dv = \sec^2 x dx$

$du = dx,\ v = \tan x$

$\int x\sec^2 x dx = x\tan x - \int \tan x dx$

$= x\tan x - \int \frac{\sin x}{\cos x}dx$

Let $u = \cos x$, $du = -\sin x dx$

$\int x\sec^2 x dx = x\tan x + \int \frac{1}{u}\, du$
$= x\tan x + \ln|u| + c$
$= x\tan x + \ln|\cos x| + c$

14. Let $u = (\ln x)^2,\ dv = dx$

$du = 2\frac{\ln x}{x}dx,\ v = x$

$I = \int (\ln x)^2 dx$
$= x(\ln x)^2 - \int x \cdot 2\frac{\ln x}{x}dx$
$= x(\ln x)^2 - 2\int \ln x dx$

Integration by parts again,

$u = \ln x,\ dv = dx du = \frac{1}{x}dx,\ v = x$

$I = x(\ln x)^2 - 2\left[x\ln x - \int x\cdot\frac{1}{x}dx\right]$

$= x(\ln x)^2 - 2x\ln x + 2\int dx$

$= x(\ln x)^2 - 2x\ln x + 2x + c$

15. Let $u = x^2$, $dv = xe^{x^2}dx$ so that, $du = 2x\ dx$ and $v = \frac{1}{2}e^{x^2}$ (v is obtained using substitution).

$\int x^3 e^{x^2}\ dx = \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2}\ dx$

$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$

16. Let $u = x^2, dv = \left(\frac{x}{(4+x^2)^{3/2}}\right)dx$

$du = 2xdx, v = -\frac{1}{\sqrt{4+x^2}}$

$\int \frac{x^3}{(4+x^2)^{3/2}}dx = \int x^2\left(\frac{x}{(4+x^2)^{3/2}}\right)dx$

$= -\frac{x^2}{\sqrt{4+x^2}} + \int \frac{1}{\sqrt{4+x^2}}2xdx$

$= -\frac{x^2}{\sqrt{(4+x^2)}} + 2\sqrt{(4+x^2)} + c.$

17. Let $u = \ln(\sin x),\ dv = \cos xdx$

$du = \frac{1}{\sin x}\cdot\cos xdx,\ v = \sin x$

$I = \int \cos x\ln(\sin x)dx$

$= \sin x\ln(\sin x)$

$\quad - \int \sin x\cdot\frac{1}{\sin x}\cdot\cos xdx$

$= \sin x\ln(\sin x) - \int \cos xdx$

$= \sin x\ln(\sin x) - \sin x + c$

18. This is a substitution $u = x^2$.

$\int x\sin x^2 dx = \frac{1}{2}\int \sin udu$

$= -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos x^2 + c.$

19. Let $u = x,\ dv = \sin 2xdx$

$du = dx,\ v = -\frac{1}{2}\cos 2x$

$\int_0^1 x\sin 2xdx$

$= -\frac{1}{2}x\cos 2x\Big|_0^1 - \int_0^1\left(-\frac{1}{2}\cos 2x\right)dx$

$= -\frac{1}{2}(1\cos 2 - 0\cos 0) + \frac{1}{2}\int_0^1 \cos 2xdx$

$= -\frac{1}{2}\cos 2 + \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^1$

$= -\frac{1}{2}\cos 2 + \frac{1}{4}(\sin 2 - \sin 0)$

$= -\frac{1}{2}\cos 2 + \frac{1}{4}\sin 2$

20. Let $u = 2x, dv = \cos x\ dx$

$du = 2\ dxandv = \sin x.$

$\int_0^\pi 2x\cos xdx = 2x\sin x|_0^\pi - 2\int_0^\pi \sin xdx$

$= (2x\sin x + 2\cos x)|_0^\pi = -4.$

21. $\int_0^1 x^2\cos\pi xdx$

Let $u = x^2, dv = \cos\pi xdx$,

$du = 2xdx, v = \frac{\sin\pi x}{\pi}.$

$\int_0^1 x^2\cos\pi xdx = x^2\frac{\sin\pi x}{\pi}\Big|_0^1 - \int_0^1 \frac{\sin\pi x}{\pi}2xdx$

$= (0-0) - \frac{2}{\pi}\int_0^1 x\sin(\pi x)\,dx$

$= -\frac{2}{\pi}\int_0^1 x\sin(\pi x)\,dx$

Let $u = x, dv = \sin(\pi x)dx$,

$du = dx, v = -\frac{\cos(\pi x)}{\pi}.$

$-\frac{2}{\pi}\int_0^1 x\sin(\pi x)dx$

$= -\frac{2}{\pi}\left\{-\frac{x\cos(\pi x)}{\pi}\Big|_0^1 - \int_0^1 -\frac{\cos(\pi x)}{\pi}dx\right\}$

$= -\frac{2}{\pi}\left\{(-\frac{\cos\pi}{\pi} - 0) + \frac{1}{\pi}\left[\frac{\sin(\pi x)}{\pi}\right]_0^1\right\}$

$= -\frac{2}{\pi}\left\{\frac{1}{\pi} + \frac{1}{\pi}(0-0)\right\} = -\frac{2}{\pi^2}$

22. $\int_0^1 x^2 e^{3x}dx$

Let $u = x^2, dv = e^{3x}dx$,

$du = 2xdx, v = \frac{e^{3x}}{3}.$

$\int_0^1 x^2 e^{3x}dx = \frac{x^2 e^{3x}}{3}\Big|_0^1 - \int_0^1 \frac{e^{3x}}{3}2xdx$

$= \frac{1}{3}(e^3 - 0) - \frac{2}{3}\int_0^1 xe^{3x}dx.$

Let $u = x, dv = e^{3x}dx$,

$dv = dx, v = \frac{e^{3x}}{3}.$

$\frac{e^3}{3} - \frac{2}{3}\int_0^1 xe^{3x}dx$

$= \frac{e^3}{3} - \frac{2}{3}\left\{x\ \frac{e^{3x}}{3}\Big|_0^1 - \int_0^1 \frac{e^{3x}}{3}dx\right\}$

$= \frac{e^3}{3} - \frac{2}{3}\left\{\left(\frac{e^3}{3}\right) - \int_0^1 \frac{e^{3x}}{3}dx\right\}$

$= \frac{e^3}{3} - \frac{2}{3}\left\{\left(\frac{e^3}{3}\right) - \left[\frac{e^{3x}}{9}\right]_0^1\right\}$

$= \frac{e^3}{3} - \frac{2}{3}\left\{\left(\frac{e^3}{3}\right) - \frac{1}{9}(e^3 - 1)\right\}$

$= \frac{e^3}{3} - \frac{2e^3}{9} + \frac{2}{27}\left(e^3 - 1\right)$
$= \frac{e^3}{3} - \frac{2e^3}{9} + \frac{2e^3}{27} - \frac{2}{27} = \frac{5e^3}{27} - \frac{2}{27}$

23. $\int_1^{10} \ln 2x dx$
Let $u = \ln 2x, dv = dx$
$du = \frac{1}{x}dx, v = x.$
$\int_1^{10} \ln(2x)dx = x\ln(2x)|_1^{10} - \int_1^{10} x\frac{1}{x}dx$
$= (10\ln(20) - \ln 2) - \int_1^{10} dx$
$= (10\ln(20) - \ln 2) - [x]_1^{10}$
$= (10\ln(20) - \ln 2) - (10 - 1)$
$= (10\ln(20) - \ln 2) - 9.$

24. Let, $u = \ln x, dv = x\ dx$
$du = \frac{1}{x}\ dx, v = \frac{x^2}{2}.$
$\int_1^2 x\ln x dx = \left.\frac{1}{2}x^2\ln x\right|_1^2 - \int_1^2 \frac{1}{2}x dx$
$= \left.\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\right|_1^2 = 2\ln 2 - \frac{3}{4}.$

25. $\int x^2 e^{ax} dx$
Let $u = x^2, dv = e^{ax}dx,$
$du = 2xdx, v = \frac{e^{ax}}{a}.$
$\int x^2 e^{ax}dx = x^2\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}2xdx$
$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\int xe^{ax}dx.$
Let $u = x, dv = e^{ax}dx,$
$dv = dx, v = \frac{e^{ax}}{a}.$
$\frac{x^2e^{ax}}{a} - \frac{2}{a}\int xe^{ax}dx$
$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\left\{x\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}dx\right\}$
$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\left\{\frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}\right\} + c$
$= \frac{x^2e^{ax}}{a} - \frac{2xe^{ax}}{a^2} + \frac{2e^{ax}}{a^3} + c, a \neq 0.$

26. $\int x\sin(ax)\,dx$
Let $u = x, dv = \sin ax dx,$
$du = dx, v = -\frac{\cos ax}{a}.$
$\int x\sin(ax)\,dx$
$= x\frac{-\cos(ax)}{a} - \int -\frac{\cos(ax)}{a}dx$
$= -\frac{x\cos(ax)}{a} + \frac{\sin(ax)}{a^2} + c,\ a \neq 0.$

27. $\int (x^n)(\ln x)\,dx = \int (\ln x)\ (x^n)\,dx$
Let $u = \ln x, dv = x^n dx,$
$du = \frac{1}{x}dx, v = \frac{x^{n+1}}{(n+1)}.$
$\int (\ln x)(x^n)\,dx$
$= (\ln x)\frac{x^{n+1}}{(n+1)} - \int \frac{x^{n+1}}{(n+1)}\frac{dx}{x}$
$= \frac{x^{n+1}(\ln x)}{(n+1)} - \int \frac{x^n}{(n+1)}dx$
$= \frac{x^{n+1}(\ln x)}{(n+1)} - \frac{x^{n+1}}{(n+1)^2} + c, n \neq -1.$

28. $\int (\sin ax)\ (\cos bx)\ dx$
Let $u = \sin ax, dv = (\cos bx)\ dx$
$du = a(\cos ax)\,dx, v = \frac{\sin bx}{b}.$
$\int \sin ax\ \cos bx\, dx$
$= (\sin ax)\ \frac{\sin bx}{b} - \int a\left(\frac{\sin bx}{b}\right)(\cos ax)\,dx$
$= \frac{(\sin ax)(\sin bx)}{b} - \frac{a}{b}\int (\cos ax)(\sin bx)\ dx$
Let $u = \cos ax, dv = \sin bx dx,$
$du = -a(\sin ax)\,dx, v = -\frac{\cos bx}{b}.$
$\frac{\sin ax\sin bx}{b} - \frac{a}{b}\int \cos ax\sin bx\,dx$
$= \frac{\sin ax\sin bx}{b} - \frac{a}{b}\left\{\cos ax\frac{-\cos bx}{b} - \int \frac{-\cos bx}{b}(-\sin ax)\ adx\right\}$
$= \frac{\sin ax\sin bx}{b} - \frac{a}{b}\left\{\frac{-\cos ax\cos bx}{b} - \frac{a}{b}\int \cos bx\sin ax\,dx\right\}$
$= \frac{\sin ax\sin bx}{b} + \frac{a\cos ax\cos bx}{b^2} + \left(\frac{a}{b}\right)^2\int \sin ax\ \cos bx\,dx$
$\int \sin ax\ \cos bx\,dx$
$= \frac{\sin ax\sin bx}{b} + \frac{a\cos ax\cos bx}{b^2} + \left(\frac{a}{b}\right)^2\int \sin ax\ \cos bx\,dx$
$\int \sin ax\ \cos bx\,dx - \left(\frac{a}{b}\right)^2\int \sin ax\ \cos bx\,dx$
$= \frac{\sin ax\sin bx}{b} + \frac{a\cos ax\cos bx}{b^2}$

$\left(1-\frac{a^2}{b^2}\right)\int \sin ax\ \cos bx\,dx$
$=\frac{\sin ax\sin bx}{b}+\frac{a\cos ax\cos bx}{b^2}$
$\int \sin ax\ \cos bx\,dx$
$=\left(\frac{b^2}{b^2-a^2}\right)\left(\frac{\sin ax\sin bx}{b}+\frac{a\cos ax\cos bx}{b^2}\right)$
$\int \sin ax\ \cos bx\,dx$
$=\left(\frac{1}{b^2-a^2}\right)(b\sin ax\sin bx+a\cos ax\cos bx)$,
$a\neq 0\ b\neq 0$.

29. Let$u=\cos^{n-1}x,\ dv=\cos x dx$
$du=(n-1)(\cos^{n-2}x)(-\sin x)dx, v=\sin x$
$\int \cos^n xdx$
$=\sin x\cos^{n-1}x$
$-\int(\sin x)(n-1)(\cos^{n-2}x)(-\sin x)dx$
$=\sin x\cos^{n-1}x$
$+\int(n-1)(\cos^{n-2}x)(\sin^2 x)dx$
$=\sin x\cos^{n-1}x$
$+\int(n-1)(\cos^{n-2}x)(1-\cos^2 x)dx$
$=\sin x\cos^{n-1}x$
$+\int(n-1)(\cos^{n-2}x-\cos^n x)dx$
Thus, $\int\cos^n xdx$
$=\sin x\cos^{n-1}x+\int(n-1)\cos^{n-2}xdx$
$-(n-1)\int\cos^n xdx.$
$n\int\cos^n xdx=\sin x\cos^{n-1}x$
$+(n-1)\int\cos^{n-2}xdx$
$\int\cos^n xdx$
$=\frac{1}{n}\sin x\cos^{n-1}x+\frac{n-1}{n}\int\cos^{n-2}xdx$

30. Let $u=\sin^{n-1}x, dv=\sin x\ dx$
$du=(n-1)\sin^{n-2}x\cos x, v=-\cos x.$
$\int\sin^n xdx$
$=-\sin^{n-1}x\cos x$
$+(n-1)\int\cos^2 x\sin^{n-2}xdx$
$=-\sin^{n-1}x\cos x$
$+(n-1)\int(1-\sin^2 x)\sin^{n-2}xdx$
$=-\sin^{n-1}x\cos x$
$-(n-1)\int\sin^{n-2}xdx$
$+(n-1)\int\sin^n xdx$
$n\int\sin^n xdx$
$=-\sin^{n-1}x\cos x$
$-(n-1)\int\sin^{n-2}xdx$
$\int\sin^n xdx=-\frac{1}{n}\sin^{n-1}x\cos x$
$-\frac{n-1}{n}\int\sin^{n-2}xdx$

31. $\int x^3e^xdx=e^x(x^3-3x^2+6x-6)+c$

32. $\int\cos^5 xdx$
$=\frac{1}{5}\cos^4\sin x+\frac{4}{5}\int\cos^3 xdx$
$=\frac{1}{5}\cos^4\sin x$
$+\frac{4}{5}\left(\frac{1}{3}\cos^2 x\sin x+\frac{2}{3}\int\cos xdx\right)$
$=\frac{1}{5}\cos^4\sin x+\frac{4}{15}\cos^2 x\sin x$
$+\frac{8}{15}\sin x+c$

33. $\int\cos^3 xdx$
$=\frac{1}{3}\cos^2 x\sin x+\frac{2}{3}\int\cos xdx$
$=\frac{1}{3}\cos^2 x\sin x+\frac{2}{3}\sin x+c$

34. $\int\sin^4 xdx$
$=-\frac{1}{4}\sin^3 x\cos x+\frac{3}{4}\int\sin^2 xdx$
$=-\frac{1}{4}\sin^3 x\cos x+\frac{3}{4}\left(\frac{1}{2}x-\frac{1}{4}\sin 2x\right)$

35. $\int_0^1 x^4e^xdx$
$=e^x(x^4-4x^3+12x^2-24x+24)\Big|_0^1$
$=9e-24$

36. Using the work done in Exercise 34,
$\int_0^{\pi/2}\sin^4 xdx$
$=\left(-\frac{1}{4}\sin^3 x\cos x+\frac{3}{8}x-\frac{3}{16}\sin 2x\right)\Big|_0^{\pi/2}$
$=\frac{3\pi}{16}$

37. $\int_0^{\pi/2}\sin^5 xdx$

$= -\frac{1}{5}\sin^4 x\cos x\Big|_0^{\pi/2} + \frac{4}{5}\int_0^{\pi/2}\sin^3 x dx$

$= -\frac{1}{5}\sin^4 x\cos x\Big|_0^{\pi/2}$

$+\frac{4}{5}\left(-\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x\right)\Big|_0^{\pi/2}$

(Using Exercise 30)

$= -\frac{1}{5}\left(\sin^4\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} - \sin^4 0\cos 0\right)$

$+\frac{4}{5}\left(-\frac{1}{3}\sin^2\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} - \frac{2}{3}\cos\frac{\pi}{2}\right)$

$= \frac{8}{15}$

38. Here we will again use the work we did in Exercise 34.

$\int \sin^6 x dx$

$= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\int\sin^4 x dx$

$= -\frac{1}{6}\sin^5 x\cos x$

$+\frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{8}x - \frac{3}{16}\sin 2x\right) + c$

$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x$

$+\frac{15}{48}x - \frac{15}{96}\sin 2x + c$

We now just have to plug in the endpoints:

$\int_0^{\pi/2}\sin^6 x dx$

$= \left(-\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x\right.$

$\left.+\frac{15}{48}x - \frac{15}{96}\sin 2x\right)\Big|_0^{\pi/2}$

$= \frac{15\pi}{96}$

39. m even :

$\int_0^{\pi/2}\sin^m x dx$

$= \frac{(m-1)(m-3)\ldots 1}{m(m-2)\ldots 2}\cdot\frac{\pi}{2}$

m odd:

$\int_0^{\pi/2}\sin^m x dx$

$= \frac{(m-1)(m-3)\ldots 2}{m(m-2)\ldots 3}$

40. m even:

$\int_0^{\pi/2}\cos^m x dx$

$= \frac{\pi(n-1)(n-3)(n-5)\cdots 1}{2n(n-2)(n-4)\cdots 2}$

m odd:

$\int_0^{\pi/2}\cos^m x dx$

$= \frac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3}.$

41. Let $u = \cos^{-1} x,\ dv = dx$

$du = -\frac{1}{\sqrt{1-x^2}}dx, v = x$

$I = \int\cos^{-1} x dx$

$= x\cos^{-1} x - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$

$= x\cos^{-1} x + \int\frac{x}{\sqrt{1-x^2}}dx$

Substituting $u = 1-x^2, du = -2xdx$

$I = x\cos^{-1} x + \int\frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right)$

$= x\cos^1 x - \frac{1}{2}\int u^{-1/2}du$

$= x\cos^{-1} x - \frac{1}{2}\cdot 2u^{1/2} + c$

$= x\cos^{-1} x - \sqrt{1-x^2} + c$

42. Let $u = \tan^{-1} x, dv = dx$

$du = \frac{1}{1+x^2}dx, v = x$

$I = \int\tan^{-1} x dx = x\tan^{-1} x - \int\frac{x}{1+x^2}dx$

Substituting $u = 1+x^2$,

$I = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c.$

43. Substituting $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx$

$I = \int\sin\sqrt{x}dx = 2\int u\sin u du$

$= 2(-u\cos u + \sin u) + c$

$= 2(-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}) + c$

44. Substituting $w = \sqrt{x}$

$dw = \frac{1}{2\sqrt{x}}dx = \frac{1}{2w}dx$

$I = \int e^{\sqrt{x}}dx = \int 2we^w dx$

Next, using integration by parts

$u = 2w,\ dv = e^w dw$

$du = 2dw,\ v = e^w$

$I = 2we^w - 2\int e^w dw$

$= 2we^w - 2e^w + c = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + c$

45. Let $u = \sin(\ln x),\ dv = dx$

$du = \cos(\ln x)\frac{dx}{x},\ v = x$

$I = \int\sin(\ln x)dx$

$= x\sin(\ln x) - \int\cos(\ln x)dx$

Integration by parts again,
$u = \cos(\ln x), dv = dx$
$du = -\sin(\ln x)\dfrac{dx}{x}, v = x$
$\int \cos(\ln x)dx$
$= x\cos(\ln x) + \int \sin(\ln x)dx$
$I = x\sin(\ln x) - x\cos(\ln x) - I$
$2I = x\sin(\ln x) - x\cos(\ln x) + c_1$
$I = \dfrac{1}{2}x\sin(\ln x) - \dfrac{1}{2}x\cos(\ln x) + c$

46. Let $u = 4 + x^2, du = 2xdx$
$I = \int x\ln(4+x^2)dx$
$= \dfrac{1}{2}\int \ln u du \quad = \dfrac{1}{2}(u\ln u - u) + C$
$= \dfrac{1}{2}[(4+x^2)\ln(4+x^2) - 4 - x^2] + c$

47. Let $u = e^{2x}, du = 2e^{2x}dx$
$I = \int e^{6x}\sin(e^{2x})dx = \dfrac{1}{2}\int u^2\sin u du$
Let $v = u^2,\ dw = \sin u du$
$dv = 2udu,\ w = -\cos u$
$I = \dfrac{1}{2}\left(-u^2\cos u + 2\int u\cos u du\right)$
$= -\dfrac{1}{2}u^2\cos u + \int u\cos u du$
$= -\dfrac{1}{2}u^2\cos u + (u\sin u + \cos u) + c$
$= -\dfrac{1}{2}e^{4x}\cos(e^{2x}) + e^{2x}\sin(e^{2x})$
$+ \cos(e^{2x}) + c$

48. Let $u = \sqrt[3]{x} = x^{1/3}, du = \dfrac{1}{3}x^{-2/3}dx,$
$3u^2du = dx$
$I = \int \cos x^{1/3}dx = 3\int u^2\cos u du$
Let $v = u^2, dw = \cos u du$
$dv = 2udu, w = \sin u$
$I = 3\left(u^2\sin u - 2\int u\sin u du\right)$
$= 3u^2\sin u - 6\int u\sin u du$
$= 3u^2\sin u - 6\left(-u\cos u + \int \cos u du\right)$
$= 3u^3\sin u + 6u\cos u - 6\sin u + c$
$= 3x\sin\sqrt[3]{x} + 6\sqrt[3]{x}\cos\sqrt[3]{x} - 6\sin\sqrt[3]{x} + c$

49. Let $u = \sqrt[3]{x} = x^{1/3}, du = \dfrac{1}{3}x^{-2/3}dx,$
$3u^2du = dx$
$I = \int e^{\sqrt[3]{x}}dx = 3\int u^2e^u du$
$= 3\left(u^2e^u - 2\int ue^u du\right)$
$= 3u^2e^u - 6\left(ue^u - \int e^u du\right)$
$= 3u^2e^u - 6ue^u + 6e^u + c$
Hence $\int_0^8 e^{\sqrt[3]{x}}dx = \int_0^2 3u^2e^u du$
$= \left(3u^2e^u - 6ue^u + 6e^u\right)\Big|_0^2 = 6e^2 - 6$

50. Let $u = \tan^{-1}x, dv = xdx$
$du = \dfrac{dx}{1+x^2},\ v = \dfrac{x^2}{2}$
$I = \int x\tan^{-1}x dx$
$= \tan^{-1}x\dfrac{x^2}{2} - \dfrac{1}{2}\int \dfrac{x^2}{1+x^2}dx$
$= \tan^{-1}x\dfrac{x^2}{2}$
$- \dfrac{1}{2}\left[\int 1dx - \int \dfrac{1}{1+x^2}dx\right]$
$= \tan^{-1}x\dfrac{x^2}{2} - \dfrac{1}{2}\left(x - \tan^{-1}x\right) + C$
$= \tan^{-1}x\dfrac{x^2}{2} - \dfrac{x}{2} + \dfrac{1}{2}\tan^{-1}x + c$
Hence $\int_0^1 x\tan^{-1}x dx$

$= \left(\tan^{-1}x\dfrac{x^2}{2} - \dfrac{x}{2} + \dfrac{1}{2}\tan^{-1}x\right)\Big|_0^1 = \dfrac{\pi}{4} - \dfrac{1}{2}$

51. n times. Each integration reduces the power of x by 1.

52. 1 time. The first integration by parts gets rid of the $\ln x$ and turns the integrand into a simple integral. See, for example, Problem 4.

53. (a) As the given problem, $\int x\sin x^2dx$ can be simplified by substituting $x^2 = u$, we can solve the example using substitution method.

(b) As the given integral, $\int x^2\sin x\,dx$ can not be simplified by substitution method and can be solved using method of integration by parts.

(c) As the integral,$\int x\ln x\,dx$ can not be simplified by substitution and can be solved using the method of integration by parts.

(d) As the given problem, $\int \dfrac{\ln x}{x}dx$ can be simplified by substituting , $\ln x = u$ we can solve the example by substitution method.

54. (a) As this integral, $\int x^3e^{4x}dx$ can not be simplified by substitution method and can be solved by using the method of integration by parts.

(b) As the given problem, $\int x^3 e^{x^4} dx$ can be simplified by substituting $x^4 = u$, we can solve the example using the substitution method.

(c) As the given problem, $\displaystyle\int x^{-2} e^{\frac{4}{x}} dx$ can be simplified by substituting $\dfrac{1}{x} = u$, we can solve the example using the substitution method.

(d) As this integral, $\int x^2 e^{-4x} dx$ can not be simplified by substitution and can be solved by using the method of integration by parts.

55. First column: each row is the derivative of the previous row; Second column: each row is the antiderivative of the previous row.

56.

	$\sin x$	
x^4	$-\cos x$	$+$
$4x^3$	$-\sin x$	$-$
$12x^2$	$\cos x$	$+$
$24x$	$\sin x$	$-$
24	$-\cos x$	$+$

$$\int x^4 \sin x dx$$
$$= -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + c$$

57.

	$\cos x$	
x^4	$\sin x$	$+$
$4x^3$	$-\cos x$	$-$
$12x^2$	$-\sin x$	$+$
$24x$	$\cos x$	$-$
24	$\sin x$	$+$

$$\int x^4 \cos x dx$$
$$= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x + c$$

58.

	e^x	
x^4	e^x	$+$
$4x^3$	e^x	$-$
$12x^2$	e^x	$+$
$24x$	e^x	$-$
24	e^x	$+$

$$\int x^4 e^x dx$$
$$= (x^4 - 4x^3 + 12x^2 - 24x + 24)e^x + c$$

59.

	e^{2x}	
x^4	$e^{2x}/2$	$+$
$4x^3$	$e^{2x}/4$	$-$
$12x^2$	$e^{2x}/8$	$+$
$24x$	$e^{2x}/16$	$-$
24	$e^{2x}/32$	$+$

$$\int x^4 e^{2x} dx$$
$$= \left(\frac{x^4}{2} - x^3 + \frac{3x^2}{2} - \frac{3x}{2} + \frac{3}{4}\right) e^{2x} + c$$

60.

	$\cos 2x$	
x^5	$\sin 2x/2$	$+$
$5x^4$	$-\cos 2x/4$	$-$
$20x^3$	$-\sin 2x/8$	$+$
$60x^2$	$\cos 2x/16$	$-$
$120x$	$\sin 2x/32$	$+$
120	$-\cos 2x/64$	$-$

$$\int x^5 \cos 2x dx$$
$$= \frac{1}{2}x^5 \sin 2x \quad + \frac{5}{4}x^4 \cos 2x$$
$$- \frac{20}{8}x^3 \sin 2x - \frac{60}{16}x^2 \cos 2x$$
$$+ \frac{120}{32}x \sin 2x + \frac{120}{64} \cos 2x + c$$

61.

	e^{-3x}	
x^3	$-e^{-3x}/3$	$+$
$3x^2$	$e^{-3x}/9$	$-$
$6x$	$-e^{-3x}/27$	$+$
6	$e^{-3x}/81$	$-$

$$\int x^3 e^{-3x} dx$$
$$= \left(-\frac{x^3}{3} - \frac{x^2}{3} - \frac{2x}{9} - \frac{2}{27}\right) e^{-3x} + c$$

62.

	x^2	
$\ln x$	$x^3/3$	$+$
x^{-1}	$x^4/12$	$+$
$-x^{-2}$	$x^5/60$	$+$

The table will never terminate.

63. (a) Use the identity
$\cos A \cos B$
$= \dfrac{1}{2}[\cos(A - B) + \cos(A + B)]$
This identity gives
$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx) dx$$

$= \int_{-\pi}^{\pi} \frac{1}{2}[\cos((m-n)x)$
$+ \cos((m+n)x)]dx$
$= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n}\right.$
$\left.+\frac{\sin((m+n)x)}{m+n}\right]\Big|_{-\pi}^{\pi}$
$= 0$
It is important that $m \neq n$ because otherwise $\cos((m-n)x) = \cos 0 = 1$

(b) Use the identity
$\sin A \sin B$
$= \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
This identity gives
$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx$
$= \int_{-\pi}^{\pi} \frac{1}{2}[\cos((m-n)x)$
$- \cos((m+n)x)]dx$
$= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n}\right.$
$\left.-\frac{\sin((m+n)x)}{m+n}\right]\Big|_{-\pi}^{\pi}$
$= 0$
It is important that $m \neq n$ because otherwise $\cos((m-n)x) = \cos 0 = 1$

64. (a) Use the identity
$\cos A \sin B$
$= \frac{1}{2}[\sin(B+A) - \sin(B-A)]$
This identity gives
$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx$
$= \int_{-\pi}^{\pi} \frac{1}{2}[\sin((n+m)x)$
$- \sin((n-m)x)]\,dx$
$= \frac{1}{2}\left[-\frac{\cos((n+m)x)}{n+m}\right.$
$\left.+\frac{\cos((n-m)x)}{n-m}\right]\Big|_{-\pi}^{\pi}$
$= 0$

(b) We have seen that
$\int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\cos(2x) + c$
Hence by letting $u = nx$:
$\int_{-\pi}^{\pi} \cos^2(nx)dx$
$= \frac{1}{n}\int_{-n\pi}^{n\pi} \cos^2 u du$
$= \frac{1}{n}\left(\frac{1}{2}u + \frac{1}{4}\cos(2u)\right)\Big|_{-n\pi}^{n\pi} = \pi$
And then $\int_{-\pi}^{\pi} \sin^2(nx)dx$
$= \int_{-\pi}^{\pi} (1-\cos^2(nx))dx$
$= \int_{-\pi}^{\pi} dx - \int_{-\pi}^{\pi} \cos^2(nx)dx$
$= 2\pi - \pi = \pi$

65. The only mistake is the misunderstanding of antiderivatives. In this problem, $\int e^x e^{-x} dx$ is understood as a group of antiderivatives of $e^x e^{-x}$, not a fixed function. So the subtraction by $\int e^x e^{-x} dx$ on both sides of
$\int e^x e^{-x} dx = -1 + \int e^x e^{-x} dx$
does not make sense.

66. $V = \pi\int_0^{\pi} (x\sqrt{\sin x})^2 dx = \pi\int_0^{\pi} x^2 \sin x dx$
Using integration by parts twice we get
$\int x^2 \sin x dx$
$= -x^2\cos x + 2\int x\cos x dx$
$= -x^2\cos x + 2(x\sin x - \int \sin x dx)$
$= -x^2\cos x + 2x\sin x + 2\cos x + c$
Hence,
$V = (-x^2\cos x + 2x\sin x + 2\cos x)\Big|_0^{\pi}$
$= \pi^2 - 4 \approx 5.87$

67. Let $u = \ln x,\ dv = e^x dx$
$du = \frac{dx}{x},\ v = e^x$
$\int e^x \ln x dx = e^x \ln x - \int \frac{e^x}{x} dx$
$\int e^x \ln x dx + \int \frac{e^x}{x} dx = e^x \ln x + C$
Hence,
$\int e^x\left(\ln x + \frac{1}{x}\right) dx = e^x \ln x + c$

68. We can guess the formula:
$\int e^x(f(x) + f'(x))dx = e^x f(x) + c$
and prove it by taking the derivative:
$\frac{d}{dx}(e^x f(x)) = e^x f(x) + e^x f'(x)$

$= e^x (f(x) + f'(x))$

69. Consider, $\int_0^1 f''(x)g(x)\,dx$

Choose $u = g(x)$ and $dv = f''(x)dx$,
so that $du = g'(x)\,dx$ and , $v = f'(x)$.

Hence, we have

$\int_0^1 g(x)f''(x)dx$

$= g(x)\, f'(x)|_0^1 - \int_0^1 f'(x)g'(x)\,dx$

$= (g(1)\, f'(1) - g(0)\, f'(0))$

$\quad - \int_0^1 g'(x)f'(x)\,dx$

From the given data.

$= (0-0) - \int_0^1 g'(x)f'(x)\,dx.$

Choose, $u = g'(x)$ and $dv = f'(x)dx$,
so that,$du = g''(x)\,dx$ and $v = f(x)$.

Hence, we have

$-\int_0^1 g'(x)f'(x)\,dx$

$= -\left\{ g'(x)\, f(x)|_0^1 - \int_0^1 f(x)\, g''(x)\, dx \right\}$

$= -\{(g'(1)\, f(1) - g'(0)\, f(0))$

$\quad - \int_0^1 f(x)\, g''(x)\, dx\}$

From the given data.

$= -\left\{ (0-0) - \int_0^1 f(x)\, g''(x)\, dx \right\}$

$= \int_0^1 f(x)\, g''(x)\, dx.$

70. Consider,

$\int_a^b f''(x)(b-x)\,dx = \int_a^b (b-x)\, f''(x)\,dx$

Choose $u = (b-x)$ and $dv = f''(x)\,dx$,
so that $du = -dx$ and $v = f'(x)$.

Hence, we have:

$\int_a^b (b-x)f''(x)\,dx$

$= (b-x)\, f'(x)|_a^b + \int_a^b f'(x)\,dx$

$= (0 - [(b-a)\, f'(a)]) + \int_a^b f'(x)\,dx$

$= -[(b-a)\, f'(a)] + f(x)|_a^b$

$= -[(b-a)\, f'(a)] + f(b) - f(a)$

$\int_a^b f''(x)(b-x)\,dx$

$= -[(b-a)\, f'(a)] + f(b) - f(a)$

$f(b) = f(a) + (b-a)\, f'(a)$

$\quad + \int_a^b f''(x)(b-x)\,dx$

Consider $\int_0^b x \sin(b-x)\,dx$

$= \int_0^b (b-x)\sin x dx = \int_0^b (\sin x)(b-x)\,dx$

Now, consider

$f(x) = x - \sin x \Rightarrow f'(x) = 1 - \cos x$

and $f''(x) = \sin x$.

Therefore, using

$f(b) = f(a) + f'(a)(b-a)$

$\quad + \int_a^b f''(x)(b-x)\,dx,$

we get

$b - \sin b = 0 - \sin 0 + f'(0)(b-0)$

$\quad + \int_0^b (\sin x)(b-x)\,dx$

$\Rightarrow |\sin b - b| = \left| \int_0^b x \sin(b-x)\,dx \right|.$

Further,

$|\sin b - b| = \left| \int_0^b x \sin(b-x)\,dx \right| \le \left| \int_0^b x dx \right|,$

as $\sin(b-x) \le 1$.

Thus, $|\sin b - b| \le \dfrac{b^2}{2}$.

Therefore the error in the approximation $\sin x \approx x$ is at most $\dfrac{1}{2}x^2$.

6.3 Trigonometric Techniques of Integration

1. Let $u = \sin x$, $du = \cos x dx$

$\int \cos x \sin^4 x dx = \int u^4 du$

$= \frac{1}{5}u^5 + c = \frac{1}{5}\sin^5 x + c$

2. Let $u = \sin x, du = \cos x dx$

$\int \cos^3 x \sin^4 x dx = \int (1-u^2)u^4 du$

$= \frac{u^5}{5} - \frac{u^7}{7} + c$

$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$

3. Let $u = \sin 2x, du = 2\cos 2x dx$.

$\int_0^{\pi/4} \cos 2x \sin^3 2x dx$

$= \frac{1}{2}\int_0^1 u^3 du = \frac{1}{2}\left[\frac{u^4}{4}\right]_0^1 = \frac{1}{8}$

4. Let $u = \cos 3x, du = -3\sin x dx$.
$\int_{\pi/4}^{\pi/3} (\cos^3 3x)(\sin^3 3x)\,dx$
$= -\frac{1}{3}\int_{-1/\sqrt{2}}^{-1} u^3(1-u^2)\,du$
$= -\frac{1}{3}\left[\frac{u^4}{4} - \frac{u^6}{6}\right]_{\frac{-1}{\sqrt{2}}}^{-1}$
$= -\frac{1}{3}\left(\frac{3}{16} - \frac{7}{48}\right) = -\frac{1}{72}$

5. Let $u = \cos x, du = -\sin x dx$
$\int_0^{\pi/2} \cos^2 x \sin x dx = \int_1^0 u^2(-du)$
$= \left(-\frac{1}{3}u^3\right)\Big|_1^0 = \frac{1}{3}$

6. Let $u = \cos x, du = -\sin x dx$
$\int_{-\pi/2}^{0} \cos^3 x \sin x dx = -\int_0^1 u^3 du = -1$

7. $\int \cos^2(x+1)\,dx$
$= \frac{1}{2}\int (1 + \cos 2(x+1))dx$
$= \frac{1}{2}x + \frac{1}{4}(\sin 2(x+1)) + c.$

8. Let $u = x - 3, du = dx$
$\int \sin^4(x-3)dx = \int \sin^4 u du$
$= \int (\sin^2 u)^2 du$
$= \int \frac{(1-\cos 2u)}{2} \times \frac{(1-\cos 2u)}{2} du$
$= \int \frac{1}{4}(1 - 2\cos 2u + \cos^2 2u)$
$= \frac{1}{4}\int \left[1 - 2\cos 2u + \frac{1}{2}(1 + \cos 4u)\right] du$
$= \frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\cos 4u + c$
$= \frac{3}{8}(x-3) - \frac{1}{4}\sin 2(x-3)$
$+ \frac{1}{32}\cos 4(x-3) + c.$

9. Let $u = \sec x,\ du = \sec x \tan x dx$
$\int \tan x \sec^3 x dx$
$= \int \tan x \sec x \sec^2 x dx$
$= \int u^2 du = \frac{1}{3}u^3 + c = \frac{1}{3}\sec^3 x + c$

10. Let $u = \cot x, du = -\csc^2 x dx$
$\int \cot x \csc^4 x dx$
$= -\int \cot x(1 + \cot^2 x)\cdot \csc^2 x dx$
$= -\int (u + u^3)du = -\frac{u^2}{2} - \frac{u^4}{4} + C$
$= -\frac{\cot^2 x}{2} - \frac{\cot^4}{4} + c$

11. Let $u = x^2 + 1$, so that $du = 2xdx$.
$\int x\tan^3(x^2+1)(\sec(x^2+1))\,dx$
$= \frac{1}{2}\int \tan^3 u(\sec u)\,du$
$= \frac{1}{2}\int \left[(\sec^2 u - 1)\tan u(\sec u)\right]du$
Let $\sec u = t, dt = \tan u \sec u du$
$= \frac{1}{2}\int (t^2 - 1)dt = \frac{1}{2}\left[\frac{t^3}{3} - t\right] + c$
$= \frac{1}{2}\left[\frac{\sec^3 u}{3} - \sec u\right] + c$
$= \frac{1}{6}\sec^3(x^2+1) - \frac{1}{2}\sec(x^2+1) + c.$

12. Let $u = 2x + 1$, so that $du = 2dx$.
$\int \tan(2x+1).\sec^3(2x+1)\,dx$
$= \frac{1}{2}\int \tan u.\sec u.\sec^2 u du$
$= \frac{1}{2}\int \sec^2 u \tan u \sec u du$
Let $t = \sec u$, so that $dt = \tan u \sec u du$.
$= \frac{1}{2}\int t^2 dt = \frac{1}{2}\left[\frac{t^3}{3}\right] + c$
$= \frac{1}{2}\left[\frac{\sec^3 u}{3}\right] + c = \frac{1}{6}\sec^3(2x+1) + c.$

13. Let $u = \cot x, du = (-\csc^2 x)\,dx$
$\int \cot^2 x \csc^4 x dx = \int \cot^2 x(1 + \cot^2 x)\csc^2 x dx$
$= -\int u^2(1+u^2)du$
$= -\frac{u^3}{3} - \frac{u^5}{5} + c$
$= -\frac{(\cot x)^3}{3} - \frac{(\cot x)^5}{5} + c.$

14. Let $u = \cot x, du = (-\csc^2 x)\,dx$.
$\int \cot^2 x \csc^2 x dx = -\int u^2 du$
$= -\frac{u^3}{3} + c = \frac{\cot^3 x}{3} + c.$

15. Let $u = \tan x, du = \sec^2 x dx$
$\int_0^{\pi/4} \tan^4 x \sec^4 x dx$
$= \int_0^{\pi/4} \tan^4 x \sec^2 x \sec^2 x dx$

$$= \int_0^{\pi/4} \tan^4 x(1+\tan^2 x)\sec^2 x dx$$
$$= \int_0^1 u^4(1+u^2)\,du$$
$$= \int_0^1 (u^4+u^6)du \ = \frac{u^5}{5}+\frac{u^7}{7}\Big|_0^1 = \frac{12}{35}$$

16. Let $u = \tan x, du = \sec^2 x dx.$
$$\int_{\pi/4}^{\pi/4} \tan^4 x \sec^2 x dx$$
$$= \int_{-1}^1 u^4 du = \frac{u^5}{5}\Big|_{-1}^1 = \frac{2}{5}$$

17. $\int \cos^2 x \sin^2 x dx$
$$= \int \frac{1}{2}(1+\cos 2x)\cdot\frac{1}{2}(1-\cos 2x)dx$$
$$= \frac{1}{4}\int (1-\cos^2 2x)dx$$
$$= \frac{1}{4}\int \left[1-\frac{1}{2}(1+\cos 4x)\right]dx$$
$$= \frac{1}{4}\left(\frac{1}{2}x-\frac{1}{8}\sin 4x\right)+c$$
$$= \frac{1}{8}x-\frac{1}{32}\sin 4x+c$$

18. $\int (\cos^2 x+\sin^2 x)dx = \int 1dx = x+c$

19. Let $u = cosx, du = -\sin x dx$
$$\int_{-\pi/3}^0 \sqrt{\cos x}\sin^3 x dx$$
$$= \int_{-\pi/3}^0 \sqrt{\cos x}(1-\cos^2 x)\sin x dx$$
$$= \int_{1/2}^1 \sqrt{u}(1-u^2)(-du)$$
$$= \int_{1/2}^1 (u^{5/2}-u^{1/2})du$$
$$= \left[\frac{2}{7}u^{7/2}-\frac{2}{3}u^{3/2}\right]\Big|_{1/2}^1 = \frac{25}{168}\sqrt{2}-\frac{8}{21}$$

20. Let $u = \cot x, \quad du = -\csc^2 x dx$
$$\int_{\pi/4}^{\pi/2} \cot^2 x\csc^4 x dx$$
$$= \int_{\pi/4}^{\pi/2} \cot^2 x\csc^2 x\csc^2 x dx$$
$$= \int_{\pi/4}^{\pi/2} \cot^2 x(1+\cot^2 x)\csc^2 x dx$$
$$= -\int_1^0 u^2(1+u^2)du$$
$$= -\left[\frac{u^3}{3}+\frac{u^5}{5}\right]\Big|_1^0 = \frac{1}{3}+\frac{1}{5} = \frac{8}{15}$$

21. Let $x = 3\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$dx = 3\cos\theta\, d\theta$
$$\int \frac{1}{x^2\sqrt{9-x^2}}dx = \int \frac{3\cos\theta}{9\sin^2\theta\cdot 3\cos\theta}d\theta$$
$$= \frac{1}{9}\int \csc^2\theta d\theta = -\frac{1}{9}\cot\theta + C$$
By drawing a diagram, we see that if
$x = \sin\theta$, then $\cot\theta = \frac{\sqrt{9-x^2}}{x}$.
Thus the integral $= -\frac{\sqrt{9-x^2}}{9x}+c$

22. Let $x = 4\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$,
$dx = 4\cos\theta d\theta$
$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{\cos\theta}{16\sin^2\theta\cos\theta}d\theta$$
$$= \frac{1}{16}\int \csc^2\theta d\theta = -\frac{1}{16}\cot\theta + c$$
$$= -\frac{\sqrt{16-x^2}}{16x}+c$$

23. Let $x = \ 4\sin\theta$, so that $dx = 4\cos\theta d\theta$.
$$\int \frac{x^2}{\sqrt{16-x^2}}dx = \int \frac{\left(16\sin^2\theta\right)4\cos\theta}{\sqrt{16-(4\sin\theta)^2}}d\theta$$
$$= 64\int \frac{\left(\sin^2\theta\right)\cos\theta}{\sqrt{16-16\sin^2\theta}}d\theta$$
$$= 64\int \frac{\left(\sin^2\theta\right)\cos\theta}{4\sqrt{\left(1-\sin^2\theta\right)}}d\theta$$
$$= 16\int \frac{\sin^2\theta\cos\theta}{\cos\theta}d\theta = 16\int \sin^2\theta d\theta$$
$$= 16\int \left(\frac{1-\cos 2\theta}{2}\right)d\theta$$
$$= 8\left[\int d\theta - \int (\cos 2\theta)\,d\theta\right]$$
$$= 8\left[\theta - \frac{\sin 2\theta}{2}\right]+c$$
$$= 8\sin^{-1}\left(\frac{x}{4}\right) - 4\sin\left[2\sin^{-1}\left(\frac{x}{4}\right)\right]+c.$$
$$= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2}+c$$

24. Let $x = 3\sin\theta,$ so that $dx = 3\cos\theta d\theta$.
$$\int \frac{x^3}{\sqrt{9-x^2}}dx$$
$$= \int \frac{27\left(\sin^3\theta\right)}{\sqrt{9-(3\sin\theta)^2}}(3\cos\theta)\,d\theta$$
$$= 81\int \frac{\sin^3\theta}{\sqrt{9-9\sin^2\theta}}(\cos\theta)\,d\theta$$
$$= 81\int \left(\frac{\sin^3\theta}{3\cos\theta}\right)\cos\theta d\theta = 27\int \sin^3\theta d\theta$$
$$= 27\int \left(\frac{3\sin\theta - \sin 3\theta}{4}\right)d\theta$$

$= \frac{27}{4}\left[3\int \sin\theta d\theta - \int \sin 3\theta d\theta\right]$
$= \frac{27}{4}\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right] + c$
$= \frac{27}{4}\left\{-3\cos\left[\sin^{-1}\left(\frac{x}{3}\right)\right]\right.$
$\left. + \frac{\cos\left[\sin^{-1}\left(\frac{x}{3}\right)\right]}{3}\right\} + c.$

25. This is the area of a quarter of a circle of radius 2,
$\int_0^2 \sqrt{4-x^2}dx = \pi$

26. Let $u = 4 - x^2$, $du = -2xdx$
$\int_0^1 \frac{x}{\sqrt{4-x^2}}dx = -\int_4^3 \frac{du}{2\sqrt{u}}$
$= -u^{1/2}\Big|_4^3 = 2 - \sqrt{3}$

27. Let $x = 3\sec\theta, dx = 3\sec\theta\tan\theta d\theta$.
$I = \int \frac{x^2}{\sqrt{x^2-9}}dx$
$= \int \frac{27\sec^2\theta\sec\theta\tan\theta}{\sqrt{9\sec^2\theta - 9}}d\theta$
$= \int 9\sec^3\theta d\theta$
Use integration by parts.
Let $u = \sec\theta$ and $dv = \sec^2\theta d\theta$. This gives
$\int \sec^3\theta d\theta$
$= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta$
$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta$
$= \sec\theta\tan\theta + \int \sec\theta d\theta - \int \sec^3\theta d\theta$
$2\int \sec^3\theta d\theta$
$= \sec\theta\tan\theta + \int \sec\theta d\theta$
$\int \sec^3\theta d\theta$
$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta$
This leaves us to compute $\int \sec\theta d\theta$.
For this notice if $u = \sec\theta + \tan\theta$ then $du = \sec\theta\tan\theta + \sec^2\theta$.
$\int \sec\theta d\theta$
$= \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\, d\theta$
$= \int \frac{1}{u}du = \ln|u| + c$
$= \ln|\sec\theta + \tan\theta| + c$

Putting all these together and using $\sec\theta = \frac{x}{3}$, $\tan\theta = \frac{\sqrt{x^2-9}}{3}$:
$\int \frac{x^2}{\sqrt{x^2-9}}dx = \int 9\sec^3\theta\, d\theta$
$= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\int \sec\theta\, d\theta$
$= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta + \tan\theta| + c$
$= \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{x^2-9}}{3}\right)$
$+ \frac{9}{2}\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + c$
$= \frac{x\sqrt{x^2-9}}{2} + \frac{9}{2}\ln\left|\frac{x + \sqrt{x^2-9}}{3}\right| + c$

28. Let $u = x^2 - 1, du = 2xdx$
$\int x^3\sqrt{x^2-1}dx$
$= \frac{1}{2}\int x^2\sqrt{x^2-1}(2x)dx$
$= \frac{1}{2}\int (u+1)\sqrt{u}du$
$= \frac{1}{2}\int u^{3/2} + u^{1/2}\, du$
$= \frac{1}{2}\left(\frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right) + c$
$= \frac{1}{5}(x^2-1)^{5/2} + \frac{1}{3}(x^2-1)^{3/2} + c$

29. Let $x = 2\sec\theta, dx = 2\sec\theta\tan\theta d\theta$
$\int \frac{2}{\sqrt{x^2-4}}dx = \int \frac{4\sec\theta\tan\theta}{2\tan\theta}d\theta$
$= 2\int \sec\theta d\theta$
$= 2\ln|2\sec\theta + 2\tan\theta| + c$
$= 2\ln\left|x + \sqrt{x^2-4}\right| + c$

30. Let $x = 2\sec\theta, dx = 2\sec\theta\tan\theta d\theta$
$\int \frac{x}{\sqrt{x^2-4}}dx = \int \frac{4\sec^2\theta\tan\theta}{2\tan\theta}d\theta$
$= 2\int \sec^2\theta d\theta = 2\tan\theta + C = \sqrt{x^2-4} + c$

31. $\int \frac{\sqrt{4x^2-9}}{x}dx = \int \frac{\sqrt{4x^2-9}}{4x^2}4xdx$
Let $u = \sqrt{4x^2-9}$,
$du = \frac{1}{2\sqrt{4x^2-9}}8xdx = \frac{1}{2u}8xdx$
or $udu = 4xdx$.
Hence, we have
$\int \frac{\sqrt{4x^2-9}}{x}dx$

$$= \int \frac{u}{u^2+9} u du = \int \frac{u^2}{u^2+9} du$$
$$= \int \frac{u^2+9-9}{u^2+9} du = \int du - \int \frac{9}{u^2+9} du$$
$$= u - 9\tan^{-1}\left(\frac{u}{3}\right) + c$$
$$= \sqrt{4x^2-9} - 9\tan^{-1}\left(\frac{\sqrt{4x^2-9}}{3}\right) + c.$$

32. Let $x = 2\sec\theta, dx = 2\tan\theta\sec\theta d\theta$.

$$\int \frac{\sqrt{x^2-4}}{x^2} dx$$
$$= \int \frac{\sqrt{4\sec^2\theta - 4}}{4\sec^2\theta} (2\tan\theta\sec\theta)\, d\theta$$
$$= \int \frac{2\tan\theta}{4\sec^2\theta} (2\tan\theta\sec\theta)\, d\theta$$
$$= \int \frac{\tan^2\theta}{\sec\theta} d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta} d\theta$$
$$= \int \sec\theta d\theta - \int \frac{1}{\sec\theta} d\theta$$
$$= \int \sec\theta d\theta - \int \cos\theta d\theta$$
$$= \ln|\sec\theta + \tan\theta| - \sin\theta + c$$
$$= \ln\left|\sec\left[\sec^{-1}\left(\frac{x}{2}\right)\right] + \tan\left[\sec^{-1}\left(\frac{x}{2}\right)\right]\right|$$
$$- \sin\left[\sec^{-1}\left(\frac{x}{2}\right)\right] + c$$
$$= \ln\left|\left(\frac{x}{2}\right) + \tan\left[\sec^{-1}\left(\frac{x}{2}\right)\right]\right|$$
$$- \sin\left[\sec^{-1}\left(\frac{x}{2}\right)\right] + c.$$

33. Let $x = 3\tan\theta, dx = 3\sec^2\theta d\theta$

$$\int \frac{x^2}{\sqrt{9+x^2}} dx$$
$$= \int \frac{27\tan^2\theta\sec^2\theta}{\sqrt{9+9\tan^2\theta}} d\theta$$
$$= \int 9\tan^2\theta\sec\theta d\theta$$
$$= 9\int (\sec^2\theta - 1)\sec\theta d\theta$$
$$= 9\int \sec^3\theta d\theta - 9\int \sec\theta d\theta$$
$$= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| + c$$
$$= \frac{9}{2}\left(\frac{\sqrt{9+x^2}}{3}\right)\left(\frac{x}{3}\right)$$
$$- \frac{9}{2}\ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + c$$
$$= \frac{x\sqrt{9+x^2}}{2} - \frac{9}{2}\ln\left|\frac{x+\sqrt{9+x^2}}{3}\right| + c$$

34. Let $x = 2\sqrt{2}\tan\theta, dx = 2\sqrt{2}\sec^2\theta d\theta$

$$\int x^3\sqrt{8+x^2} dx$$
$$= \int (16\sqrt{2}\tan^3\theta)(2\sqrt{2}\sec\theta) d\theta$$
$$= 64\int \tan^3\theta\sec\theta\, d\theta$$
$$= 64\int (\sec^2\theta - 1)(\sec\theta\tan\theta\, d\theta)$$
$$= 64\int (u^2-1)du = \frac{64}{3}u^3 - 64u + c$$
$$= \frac{64}{3}\sec^3\theta - 64\sec\theta + c$$
$$= \frac{64}{3}\left(\frac{\sqrt{8+x^2}}{2\sqrt{2}}\right)^3 - 64\left(\frac{\sqrt{8+x^2}}{2\sqrt{2}}\right) + c$$
$$= \frac{2\sqrt{2}}{3}(8+x^2)^{3/2} - 16\sqrt{2}(8+x^2)^{1/2} + c$$

35. Let $x = 4\tan\theta, dx = 4\sec^2\theta d\theta$

$$\int \sqrt{16+x^2} dx$$
$$= \int \sqrt{16+16\tan^2\theta} \cdot 4\sec^2\theta d\theta$$
$$= 16\int \sec^3\theta d\theta$$
$$= 16\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta d\theta\right)$$
$$= 8\sec\theta\tan\theta + 8\int \sec\theta d\theta$$
$$= 8\sec\theta\tan\theta + 8\ln|\sec\theta + \tan\theta| + c$$
$$= \frac{1}{2}x\sqrt{16+x^2}$$
$$+ 8\ln\left|\frac{1}{4}\sqrt{16+x^2} + \frac{x}{4}\right| + c$$

36. Let $x = 2\tan\theta, dx = 2\sec^2\theta d\theta$

$$\int \frac{1}{\sqrt{4+x^2}} dx = \int \frac{2\sec^2\theta}{2\sec\theta} d\theta$$
$$= \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c$$
$$= \ln\left|\frac{x+\sqrt{4+x^2}}{2}\right| + c$$

37. Let $u = x^2 + 8, du = 2xdx$

$$\int_0^1 x\sqrt{x^2+8} dx = \frac{1}{2}\int_8^9 u^{1/2} du$$
$$= \frac{1}{3}u^{3/2}\Big|_8^9 = \frac{27-16\sqrt{2}}{3}$$

38. Let $x = 3\tan\theta, dx = 3\sec^2\theta\, d\theta$

$$I = \int x^2\sqrt{x^2+9} dx$$
$$= \int 27\tan^2\theta\sec^2\theta\sqrt{9\tan^2\theta + 9} dx$$
$$= 81\int \tan^2\theta\sec^3\theta dx$$
$$= 81\int (\sec^2\theta - 1)\sec^3\theta dx$$

$= 81\int(\sec^5\theta - \sec^3\theta)dx$

To compute $\int\sec^5\theta\, d\theta$, we use integration by parts with $u = \sec^3\theta$ and $dv = \sec^2\theta d\theta$.

$\int\sec^5\theta\, d\theta$

$= \sec^3\theta\tan\theta - \int 3\sec^3\theta\tan^2\theta d\theta$

$= \sec^3\theta\tan\theta \quad - 3\int\sec^3\theta(\sec^2\theta - 1)d\theta$

$= \sec^3\theta\tan\theta - 3\int(\sec^5\theta - \sec^3\theta)d\theta$

$4\int\sec^5\theta d\theta$

$= \sec^3\theta\tan\theta + 3\int\sec^3\theta d\theta \int\sec^5\theta d\theta$

$= \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{4}\int\sec^3\theta d\theta$

To compute $\int\sec^3\theta d\theta$ and $\int\sec\theta\, d\theta$, see Exercise 27.

Putting all this together gives:

$I = 81\int(\sec^5\theta - \sec^3\theta)dx$

$= \frac{81}{4}\sec^3\theta\tan\theta + \frac{243}{4}\int\sec^3\theta d\theta$

$-81\int\sec^3\theta d\theta$

$= \frac{81}{4}\sec^3\theta\tan\theta - \frac{81}{4}\int\sec^3\theta d\theta$

$= \frac{81}{4}\sec^3\theta\tan\theta - \frac{81}{8}\sec\theta\tan\theta$

$\quad - \frac{81}{8}\ln|\sec\theta + \tan\theta| + c$

We don't worry about the result being in terms of x since this is a definite integral. Our limits of integration are $x = 0$ and $x = 2$. In terms of θ this means the limits of integration correspond to $\theta = 0$ and $\tan\theta = \frac{2}{3}$.

$\int_0^2 x^2\sqrt{x^2+9}dx$

$= \left(\frac{81}{4}\sec^3\theta\tan\theta - \frac{81}{8}\sec\theta\tan\theta\right.$

$\quad \left.-\frac{81}{8}\ln|\sec\theta + \tan\theta|\right)\Big|_{x=0}^{x=2}$

$= \left(\frac{81}{4}\left(\frac{\sqrt{13}}{3}\right)^3\left(\frac{2}{3}\right) - \frac{81}{8}\left(\frac{\sqrt{13}}{3}\right)\left(\frac{2}{3}\right)\right.$

$\quad \left.-\frac{81}{8}\ln\left|\frac{\sqrt{13}}{3} + \frac{2}{3}\right|\right)$

$\quad -\left(\frac{81}{4}(1)(0) - \frac{81}{8}(1)(0) - \frac{81}{8}\ln|1+0|\right)$

$= \frac{17\sqrt{13}}{4} - \frac{81}{8}\ln\left|\frac{2+\sqrt{13}}{3}\right|$

39. Let $x = \tan\theta, dx = \sec^2\theta d\theta$.

$\int\frac{x^3}{\sqrt{1+x^2}}dx = \int\left(\frac{\tan^3\theta}{\sec\theta}\right)\sec^2\theta d\theta$

$= \int\left(\tan^2\theta\right)(\tan\theta\sec\theta)\, d\theta$

Let $t = \sec\theta,\ dt = \tan\theta\sec\theta d\theta$.

$= \int\left(\sec^2\theta - 1\right)\tan\theta\sec\theta d\theta$

$= \int\left(t^2 - 1\right)dt = \left[\frac{t^3}{3} - t\right] + c$

$= \left[\frac{\sec^3\theta}{3} - \sec\theta\right] + c$

$= \left[\frac{\sec^3\left(\tan^{-1}x\right)}{3} - \sec\left(\tan^{-1}x\right)\right] + c.$

40. Let $x = 2\tan\theta, d\theta = \left(2\sec^2\theta\right)d\theta$.

$\int\frac{x+1}{\sqrt{4+x^2}}dx$

$= \int\left(\frac{2\tan\theta + 1}{\sqrt{4+4\tan^2\theta}}\right)2\sec^2\theta d\theta$

$= \int\left(\frac{2\tan\theta + 1}{2\sec\theta}\right)\left(2\sec^2\theta\right)d\theta$

$= \int(2\tan\theta + 1)(\sec\theta)\, d\theta$

$= 2\int\sec\theta\tan\theta d\theta + \int\sec\theta d\theta$

$= 2\sec\theta + \ln|\sec\theta + \tan\theta| + c$

$= 2\sec\left[\tan^{-1}\left(\frac{x}{2}\right)\right] + \ln\left|\sec\left[\tan^{-1}\left(\frac{x}{2}\right)\right]\right.$

$\quad \left.+\tan\left[\tan^{-1}\left(\frac{x}{2}\right)\right]\right| + c$

$= 2\sec\left[\tan^{-1}\left(\frac{x}{2}\right)\right]$

$\quad + \ln\left|\sec\left[\tan^{-1}\left(\frac{x}{2}\right)\right] + \left(\frac{x}{2}\right)\right| + c.$

41. $\int\frac{x}{\sqrt{x^2+4x}}dx$

$= \frac{1}{2}\int\frac{2x+4-4}{\sqrt{x^2+4x}}dx$

$= \frac{1}{2}\int\frac{2x+4}{\sqrt{x^2+4x}}dx - \frac{1}{2}\int\frac{4}{\sqrt{x^2+4x}}dx$

Let $u = x^2 + 4x, du = (2x+4)\, dx$.

$= \frac{1}{2}\int\frac{du}{\sqrt{u}} - \frac{1}{2}\int\frac{4}{\sqrt{x^2+4x-4+4}}dx$

$= u^{1/2} - \frac{1}{2}\int\frac{4}{\sqrt{(x+2)^2 - 4}}dx$

$= \sqrt{(x^2+4x)}$

$\quad - 2\log\left[\left(x^2+4x\right) + \sqrt{(x+2)^2 - 4}\right] + c.$

42. $\int\frac{2}{\sqrt{x^2-6x}}dx = \int\frac{2}{\sqrt{x^2-6x+9-9}}dx$

$= \int \frac{2}{\sqrt{(x-3)^2-9}}dx$
Let $u = x-3,\ du = dx.$
$= \int \frac{2}{\sqrt{u^2-9}}du$
Let $u = 3\sec\theta,\ du = 3\sec\theta\tan\theta d\theta.$
$= \int \frac{2}{\sqrt{(3\sec\theta)^2-9}} 3\sec\theta\tan\theta d\theta$
$= 2\int \frac{1}{\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta d\theta$
$= 2\int \frac{1}{\tan\theta}\sec\theta\tan\theta d\theta$
$= 2\int \sec\theta d\theta \ = 2\ln|\sec\theta+\tan\theta| + c$
$= 2\ln\left|\sec\left(\sec^{-1}\left(\frac{u}{3}\right)\right) + \tan\left(\sec^{-1}\left(\frac{u}{3}\right)\right)\right| + c$
$= 2\ln\left|\left(\frac{u}{3}\right) + \tan\left(\sec^{-1}\left(\frac{u}{3}\right)\right)\right| + c$
$= 2\ln\left|\left(\frac{x-3}{3}\right) + \tan\left(\sec^{-1}\left(\frac{x-3}{3}\right)\right)\right| + c.$

43. $\int \frac{x}{\sqrt{10+2x+x^2}}dx$

$= \int \frac{x}{\sqrt{9+1+2x+x^2}}dx$
$= \int \frac{x}{\sqrt{(x+1)^2+9}}dx$
$= \int \frac{x+1-1}{\sqrt{(x+1)^2+9}}dx$
$= \int \frac{x+1}{\sqrt{(x+1)^2+9}}dx - \int \frac{1}{\sqrt{(x+1)^2+9}}dx$
Let $u = x+1, du = dx.$
$= \int \frac{u}{\sqrt{u^2+9}}du - \int \frac{1}{\sqrt{u^2+9}}du$
$= \frac{1}{2}\int \frac{2u}{\sqrt{u^2+9}}du - \int \frac{1}{\sqrt{u^2+3^2}}du$
Let $t = u^2+9,\ dt = 2udu.$
$= \frac{1}{2}\int \frac{dt}{\sqrt{t}}dt - \log\left[u+\sqrt{u^2+3^2}\right] + c$
$= \sqrt{t} - \log\left[u+\sqrt{u^2+3^2}\right] + c$
$= \sqrt{u^2+9} - \log\left[u+\sqrt{u^2+9}\right] + c$
$= \sqrt{(x+1)^2+9} - \log\left[(x+1)+\sqrt{(x+1)^2+9}\right] + c.$

44. $\int \frac{2}{\sqrt{4x-x^2}}dx = \int \frac{2}{\sqrt{4-4+4x-x^2}}dx$
$= \int \frac{2}{\sqrt{4-(x-2)^2}}dx$
Let $u = x-2,\ du = dx.$
$= \int \frac{2}{\sqrt{4-u^2}}du$
Let $u = 2\sin\theta,\ du = 2\cos d\theta.$
$= \int \frac{2}{\sqrt{4-(2\sin\theta)^2}} 2\cos\theta d\theta$
$= 2\int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta$
$= 2\int \frac{1}{\cos\theta}\cos\theta d\theta = 2\int d\theta = 2\theta + c$
$= 2\sin^{-1}\left(\frac{u}{2}\right) + c = 2\sin^{-1}\left(\frac{x-2}{2}\right) + c.$

45. Using $u = \tan x$, gives
$\int \tan x\sec^4 xdx$
$= \int \tan x(1+\tan^2 x)\sec^2 xdx$
$= \int u(1+u^2)du = \int (u+u^3)du$
$= \frac{1}{2}u^2 + \frac{1}{4}u^4 + c$
$= \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + c$
Using $u = \sec x$, gives
$\int \tan x\sec^4 xdx$
$= \int \tan x\sec x\sec^3 xdx$
$= \int u^3 du = \frac{1}{4}u^4 + c = \frac{1}{4}\sec^4 x + c$

46. Using $u = \tan x$ gives
$\int \tan^3 x\sec^4 xdx = \int u^3(u^2+1)du$
$= \frac{u^6}{6} + \frac{u^4}{4} + c_1$
$= \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + c_2$
Using $u = \sec x$ gives
$\int \tan^3 x\sec^4 xdx = \int (u^2-1)u^3\ du$
$= \frac{u^6}{6} - \frac{u^4}{4} = \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4}$
$= \frac{(\tan^2 x+1)^3}{6} - \frac{(\tan^2 x+1)^2}{4}$
$= \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} - \frac{1}{12} + c_1$
$= \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + c_2$

47. (a) This is using integration by parts followed by substitution

$u = \sec^{n-2} x,\ dv = \sec^2 x dx$
$du = (n-2)\sec^{n-2} x \tan x dx,\ v = \tan x$

$$I = \int \sec^n x dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2}(\sec^2 x - 1)dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int (\sec^n x - \sec^{n-2} x)dx$$
$$= \sec^{n-2} x \tan x - (n-2)I + (n-2)\int \sec^{n-2} xdx\ (n-1)I$$
$$= \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} xdx$$
$$I = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} xdx$$

(b) $\int \sec^3 xdx$
$$= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec xdx$$
$$= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + c$$

(c) $\int \sec^4 xdx$
$$= \frac{1}{3}\sec^3 x \tan x + \frac{2}{3}\int \sec^2 xdx$$
$$= \frac{1}{3}\sec^3 x \tan x + \frac{2}{3}\tan x + c$$

(d) $\int \sec^5 xdx$
$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 xdx$$
$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + c$$

48. Make the substitution $x = a\sin\theta$.
$$\frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}dx$$
$$= \frac{4b}{a}\int_0^{\pi/2} a\cos\theta\sqrt{a^2 - a^2\sin^2\theta}d\theta$$
$$= 4b\int_0^{\pi/2} a\cos^2\theta d\theta$$
$$= 4ab\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right)\Big|_0^{\pi/2} = ab\pi$$

49. $\int \csc xdx = \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x}dx$
$$= \int \frac{(\csc x)\cot x + \csc^2 x}{\csc x + \cot x}dx$$
Let $u = \csc x + \cot x$,
$du = -(\csc x)\cot x - \csc^2 x$.
$$= -\int \frac{1}{u}du = -\ln|u| + c$$
$$= -\ln|\csc x + \cot x| + c.$$
$$= \ln|\csc x - \cot x| + c.$$
$$\int \csc^3 xdx = \int \csc x.\csc^2 xdx$$
$u = \csc x, dv = \csc^2 xdx$
$du = -\csc x.\cot x, v = -\cot x$
$$\int \csc^3 xdx = -\csc x.\cot x - \int (-\cot x)(-\csc x.\cot x)dx$$
$$= -\csc x\cot x - \int \left(\csc x.\cot^2 x\right)dx$$
$$= -\csc x\cot x - \int \csc x.\left(\csc^2 x - 1\right)dx$$
$$= -\csc x\cot x - \int \left(\csc^3 x\right)dx + \int \csc xdx$$
$$2\int \csc^3 xdx = -\csc x\cot x + \int \csc xdx$$
$$= -\csc x\cot x + \ln|\csc x - \cot x| + c$$
$$\int \csc^3 xdx = \frac{1}{2}\left(-\csc x\cot x + \ln|\csc x - \cot x|\right) + c$$

50. $\int \frac{1}{\cos x - 1}dx$
$$= \int \frac{\cos x + 1}{(\cos x - 1)(\cos x + 1)}dx$$
$$= -\int \frac{\cos x + 1}{\sin^2 x}dx$$
$$= -\int \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right)dx$$
$$= -\int \csc x(\cot x + \csc x)dx$$
$$= \int -\csc x\cot x - \csc^2 xdx$$
$$= \int (-\csc x\cot x)dx + \int \left(-\csc^2 x\right)dx$$
$= \csc x + \cot x + c$ and,
$$\int \frac{1}{\cos x + 1}dx$$
$$= \int \frac{\cos x - 1}{(\cos x - 1)(\cos x + 1)}dx$$
$$= -\int \frac{\cos x - 1}{\sin^2 x}dx$$
$$= -\int \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right)dx$$
$$= -\int \csc x(\cot x - \csc x)dx$$
$$= \int -\csc x\cot x + \csc^2 xdx$$
$$= \int (-\csc x\cot x)dx - \int \left(-\csc^2 x\right)dx$$
$$= \csc x - \cot x + c$$

51. Using a CAS we get

(Ex 3.2) $\int \cos^4 x \sin^3 x dx$

$= -\frac{1}{7}\sin x^2 \cos x^5 - \frac{2}{35} \cos x^5 + c$

(Ex 3.3) $\int \sqrt{\sin x} \cos^5 x dx$

$= \frac{2}{11}\sin x^{11/2} - \frac{4}{7}\sin x^{7/2}$
$+ \frac{2}{3}\sin x^{3/2} + c$

(Ex 3.5) $\int \cos^4 x dx$

$= \frac{1}{4}\cos x^3 \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + c$

(Ex 3.6) $\int \tan^3 x \sec^3 x dx$

$= 1/5 \, \frac{\sin x^4}{\cos x^5} + 1/15 \frac{\sin x^4}{\cos x^3}$
$- 1/15 \, \frac{\sin x^4}{\cos x} - 1/15 \, \sin x^2 \cos x$
$- 2/15 \, \cos x + c$

Obviously my CAS used different techniques. The answers given by the book are simpler.

52. (a) $-\frac{1}{7}\sin^2 x \cos^5 x - \frac{2}{35}\cos^5 x$
$= -\frac{1}{7}(1-\cos^2 x)\cos^5 x - \frac{2}{35}\cos^5 x$
$= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x$
The conclusion is $c = 0$

(b) $-\frac{2}{15}\tan x - \frac{1}{15}\sec^2 x \tan x$
$+ \frac{1}{5}\sec^4 x \tan x$
$= -\frac{2}{15}\tan x - \frac{1}{15}(1+\tan^2 x)\tan x$
$+ \frac{1}{5}(1+\tan^2 x)^2 \tan x$
$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x$
The conclusion is $c = 0$

53. The average power

$$= \frac{1}{\frac{2\pi}{\omega}} \int_0^{2\pi/\omega} RI^2 \cos^2(\omega t)\, dt$$
$$= \frac{\omega RI^2}{2\pi} \int_0^{2\pi/\omega} \frac{1}{2}\left[1 + \cos(2\omega t)\right]\, dt$$
$$= \frac{\omega RI^2}{4\pi} \left[t + \frac{1}{2\omega}\sin(2\omega t) \right]\Bigg|_0^{2\pi/\omega}$$
$$= \frac{\omega RI^2}{4\pi} \left[\frac{2\pi}{\omega} + \frac{1}{2\omega}\sin\left(\frac{4\omega\pi}{\omega}\right) - 0 \right] = \frac{1}{2}RI^2$$

6.4 Integration of Rational Functions Using Partial Fractions

1. $\frac{x-5}{x^2-1} = \frac{x-5}{(x+1)(x-1)}$
$= \frac{A}{x+1} + \frac{B}{x-1}$

$x - 5 = A(x-1) + B(x+1)$
$x = -1 : -6 = -2A; A = 3$
$x = 1 : -4 = 2B; B = -2$

$\frac{x-5}{x^2-1} = \frac{3}{x+1} - \frac{2}{x-1}$
$\int \frac{x-5}{x^2-1} dx = \int \left(\frac{3}{x+1} - \frac{2}{x-1} \right) dx$
$= 3\ln|x+1| - 2\ln|x-1| + c$

2. $\frac{5x-2}{x^2-4} = \frac{5x-2}{(x+2)(x-2)}$
$= \frac{A}{x+2} + \frac{B}{x-2}$

$5x - 2 = A(x-2) + B(x+2)$
$x = -2 : -12 = -4A; A = 3$
$x = 2 : 8 = 4B; B = 2$

$\frac{5x-2}{x^2-4} = \frac{3}{x+2} + \frac{2}{x-2}$
$\int \frac{5x-2}{x^2-4} dx = \int \left(\frac{3}{x+2} + \frac{2}{x-2} \right) dx$
$= 3\ln|x+2| + 2\ln|x-2| + c$

3. $\frac{6x}{x^2-x-2} = \frac{6x}{(x-2)(x+1)}$
$= \frac{A}{x-2} + \frac{B}{x+1}$

$6x = A(x+1) + B(x-2)$
$x = 2 : 12 = 3A; A = 4$
$x = -1 : -6 = -3B; B = 2$
$\frac{6x}{x^2-x-2} = \frac{4}{x-2} + \frac{2}{x+1}$
$\int \frac{6x}{x^2-x-2} dx$
$= \int \left(\frac{4}{x-2} + \frac{2}{x+1} \right) dx$
$= 4\ln|x-2| + 2\ln|x+1| + c$

4. $\frac{3x}{x^2-3x-4} = \frac{3x}{(x+1)(x-4)}$
$= \frac{A}{x+1} + \frac{B}{x-4}$

$3x = A(x-4) + B(x+1)$
$x = -1 : -3 = -5A; A = \frac{3}{5}$
$x = 3 : 12 = 5B; B = \frac{12}{5}$

$$\frac{3x}{x^2-3x-4} = \frac{3/5}{x+1} + \frac{12/5}{x-4}$$

$$\int \frac{3x}{x^2-3x-4}dx = \int \left(\frac{3/5}{x+1} + \frac{12/5}{x-4}\right) dx = \frac{3}{5}\ln|x+1| + \frac{12}{5}\ln|x-4| + c$$

5. $\frac{-x+5}{x^3-x^2-2x} = \frac{-x+5}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$

$-x+5 = A(x-2)(x+1) + Bx(x+1) + cx(x-2)$
$x = 0 : 5 = -2A : A = -\frac{5}{2}$
$x = 2 : 3 = 6B : B = \frac{1}{2}$
$x = -1 : 6 = 3C : C = 2$

$$\frac{-x+5}{x^3-x^2-2x} = -\frac{5/2}{x} + \frac{1/2}{x-2} + \frac{2}{x+1}$$

$$\int \frac{-x+5}{x^3-x^2-2x}dx = \int \left(-\frac{5/2}{x} + \frac{1/2}{x-2} + \frac{2}{x+1}\right) dx = -\frac{5}{2}\ln|x| + \frac{1}{2}\ln|x-2| + 2\ln|x+1| + c$$

6. $\frac{3x+8}{x^3+5x^2+6x} = \frac{3x+8}{x(x+2)(x+3))} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+3}$

$3x+8 = A(x+2)(x+3) + Bx(x+3) + cx(x+2)$
$x = 0 : 8 = 6A; A = \frac{4}{3}$
$x = -2 : 2 = -2B; B = -1$
$x = -3 : -1 = 3C; C = -\frac{1}{3}$

$$\frac{3x+8}{x^3+5x^2+6x} = \frac{4/3}{x} - \frac{1}{x+2} - \frac{1/3}{x+3}$$

$$\int \frac{3x+8}{x^3+5x^2+6x}dx = \int \left(\frac{4/3}{x} - \frac{1}{x+2} - \frac{1/3}{x+3}\right) dx = \frac{4}{3}\ln|x| - \ln|x+2| - \frac{1}{3}\ln|x+3| + c$$

7. $\frac{5x-23}{6x^2-11x-7} = \frac{5x-23}{(2x+1)(3x-7)} = \frac{A}{2x+1} + \frac{B}{3x-7}$

$5x-23 = A(3x-7) + B(2x+1)$
$x = -\frac{1}{2} : -\frac{51}{2} = -\frac{17}{2}A; A = 3$
$x = \frac{7}{3} : -\frac{34}{3} = \frac{17}{3}B; B = -2$

$$\frac{5x-23}{6x^2-11x-7} = \frac{3}{2x+1} - \frac{2}{3x-7}$$

$$\int \frac{5x-23}{6x^2-11x-7}dx = \int \left(\frac{3}{2x+1} - \frac{2}{3x-7}\right) dx = \frac{3}{2}\ln|2x+1| - \frac{2}{3}\ln|3x-7| + c$$

8. $\frac{3x+5}{5x^2-4x-1} = \frac{3x+5}{(5x+1)(x-1)} = \frac{A}{5x+1} + \frac{B}{x-1}$

$3x+5 = A(x-1) + B(5x+1)$
$x = -\frac{1}{5} : \frac{22}{5} = -\frac{6}{5}A; A = -\frac{11}{3}$
$x = 1 : 8 = 6B; B = \frac{4}{3}$

$$\frac{3x+5}{5x^2-4x-1} = -\frac{11/3}{5x+1} + \frac{4/3}{x-1}$$

$$\int \frac{3x+5}{5x^2-4x-1}dx = \int \left(-\frac{11/3}{5x+1} + \frac{4/3}{x-1}\right) dx = -\frac{11}{15}\ln|5x+1| + \frac{4}{3}\ln|x-1| + c$$

9. $\frac{x-1}{x^3+4x^2+4x} = \frac{x-1}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$x-1 = A(x+2)^2 + Bx(x+2) + Cx$
$x = 0 : -1 = 4A; A = -\frac{1}{4}$
$x = -2 : -3 = -2C; C = \frac{3}{2}$
$x = 1 : 0 = 9A + 3B + C; B = \frac{1}{4}$

$$\frac{x-1}{x^3+4x^2+4x} = -\frac{1/4}{x} + \frac{1/4}{x+2} + \frac{3/2}{(x+2)^2}$$

$$\int \frac{x-1}{x^3+4x^2+4x}dx$$
$$= \int \left(-\frac{1/4}{x} + \frac{1/4}{x+2} + \frac{3/2}{(x+2)^2}\right) dx$$
$$= -\frac{1}{4}\ln|x| + \frac{1}{4}\ln|x+2| - \frac{3}{2(x+2)} + c$$

10. $\dfrac{4x-5}{x^3-3x^2} = \dfrac{4x-5}{x^2(x-3)}$
$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$
$$4x-5 = Ax(x-3) + B(x-3) + Cx^2$$
$$= (A+C)x^2 + (-3A+B)x + (-3B)$$
$$B = \frac{5}{3}; A = -\frac{7}{9}; C = \frac{7}{9}$$
$$\frac{4x-5}{x^3-3x^2} = -\frac{7/9}{x} + \frac{5/3}{x^2} + \frac{7/9}{x-3}$$
$$\int \frac{4x-5}{x^3-3x^2}dx$$
$$= \int \left(-\frac{7/9}{x} + \frac{5/3}{x^2} + \frac{7/9}{x-3}\right) dx$$
$$= -\frac{7/9}{\ln}|x| - \frac{5}{3}\frac{1}{x} + \frac{7}{9}\ln|x-3| + c$$

11. $\dfrac{x+2}{x^3+x} = \dfrac{x+2}{x(x^2+1)}$

$$= \frac{A}{x} + \frac{Bx+C}{x^2+1}$$
$$x+2 = A(x^2+1) + (Bx+C)x$$
$$= Ax^2 + A + Bx^2 + Cx$$
$$= (A+B)x^2 + Cx + A$$
$$A = 2; C = 1; B = -2$$
$$\frac{x+2}{x^3+x} = \frac{2}{x} + \frac{-2x+1}{x^2+1}$$
$$\int \frac{x+2}{x^3+x}dx = \int \left(\frac{2}{x} + \frac{-2x+1}{x^2+1}\right) dx$$
$$= \int \left(\frac{2}{x} - \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right) dx$$
$$= 2\ln|x| - \ln(x^2+1) + \tan^{-1}x + c$$

12. $\dfrac{1}{x^3+4x} = \dfrac{1}{x(x^2+4)}$
$$= \frac{A}{x} + \frac{Bx+C}{x^2+4}$$
$$1 = A(x^2+1) + (Bx+C)x$$
$$1 = (A+B)x^2 + Cx + A$$
$$A = 1; B = -1; C = 0$$
$$\frac{1}{x^3+4x} = \frac{1}{x} + \frac{-x}{x^2+4}$$
$$\int \frac{1}{x^3+4x}dx$$
$$= \int \left(\frac{1}{x} + \frac{-x}{x^2+4}\right) dx$$
$$= \ln|x| - \frac{1}{2}\ln(x^2+4) + c$$

13. $\dfrac{4x^2-7x-17}{6x^2-11x-10}$
$$= \frac{2}{3} + \frac{1}{3}\frac{x-31}{(2x-5)(3x+2)}$$
$$= \frac{2}{3} + \frac{1}{3}\left[\frac{A}{2x-5} + \frac{B}{3x+2}\right]$$
$$x-31 = A(3x+2) + B(2x-5)$$
$$x = \frac{5}{2} : -\frac{57}{2} = \frac{19}{2}A, A = -3;$$
$$x = -\frac{2}{3} : -\frac{95}{3} = -\frac{19}{3}B, B = 5;$$
$$\frac{4x^2-7x-17}{6x^2-11x-10}$$
$$= \frac{2}{3} + \frac{1}{3}\left[\frac{-3}{2x-5} + \frac{5}{3x+2}\right]$$
$$\int \frac{4x^2-7x-17}{6x^2-11x-10}dx$$
$$= \int \left(\frac{2}{3} - \frac{1}{2x-5} + \frac{5/3}{3x+2}\right) dx$$
$$= \frac{2}{3}x - \frac{1}{2}\ln|2x-5| + \frac{5}{9}\ln|3x+2| + c$$

14. $\dfrac{x^3+x}{x^2-1} = x + \dfrac{2x}{(x+1)(x-1)}$
$$= x + \frac{A}{x+1} + \frac{B}{x-1}$$
$$2x = A(x-1) + B(x+1)$$
$$A = B = 1$$
$$\frac{x^3+x}{x^2-1} = x + \frac{1}{x+1} + \frac{1}{x-1}$$
$$\int \frac{x^3+x}{x^2-1}dx$$
$$= \int \left(x + \frac{1}{x+1} + \frac{1}{x-1}\right) dx$$
$$= \frac{x^2}{2} + \ln|x+1| + \ln|x-1| + c$$

15. $\dfrac{2x+3}{x^2+2x+1} = \dfrac{2x+3}{(x+1)^2}$
$$= \frac{A}{x+1} + \frac{B}{(x+1)^2}$$
$$2x+3 = A(x+1) + B$$
$$x = -1 : B = 1; A = 2$$

$$\frac{2x+3}{x^2+2x+1} = \frac{2}{x+1} + \frac{1}{(x+1)^2}$$
$$\int \frac{2x+3}{x^2+2x+1}dx$$
$$= \int \left(\frac{2}{x+1} + \frac{1}{(x+1)^2}\right) dx$$
$$= 2\ln|x+1| - \frac{1}{x+1} + c$$

16. $\frac{2x}{x^2-6x+9} = \frac{2x}{(x-3)^2}$
$$= \frac{A}{x-3} + \frac{B}{(x-3)^2}$$
$2x = A(x-3) + B$
$A = 2; B = 6$

$$\frac{2x}{x^2-6x+9} = \frac{2}{x-3} + \frac{6}{(x-3)^2}$$
$$\int \frac{2x}{x^2-6x+9}dx$$
$$= \int \left(\frac{2}{x-3} + \frac{6}{(x-3)^2}\right) dx$$
$$= 2\ln|x-3| - \frac{6}{x-3} + c$$

17. $\frac{x^3-4}{x^3+2x^2+2x} = 1 + \frac{-2x^2-2x-4}{x(x^2+2x+2)}$

$$= 1 + \frac{A}{x} + \frac{Bx+c}{x^2+2x+2}$$
$-2x^2-2x-4 = A(x^2+2x+2) + (Bx+c)x$
$= (A+B)x^2 + (2A+c)x + 2A$
$A = -2; B = 0; C = 2$

$$\frac{x^3-4}{x^3+2x^2+2x}$$
$$= 1 + \frac{-2}{x} + \frac{2}{x^2+2x+2}$$
$$\int \frac{x^3-4}{x^3+2x^2+2x}dx$$
$$= \int \left(1 + \frac{-2}{x} + \frac{2}{(x+1)^2+1}\right) dx$$
$$= x - 2\ln|x| + 2\tan^{-1}(x+1) + c$$

18. $\frac{4}{x^3-2x^2+4x} = \frac{4}{x(x^2-2x+4)}$
$$= \frac{A}{x} + \frac{Bx+C}{x^2-2x+4}$$
$4 = A(x^2-2x+4) + (Bx+C)x$
$= (A+B)x^2 + (-2A+C)x + 4A$
$A = 1; B = -1; C = 2$

$$\frac{4}{x^3-2x^2+4x} = \frac{1}{x} + \frac{-x+2}{x^2-2x+4}$$
$$\int \frac{4}{x^3-2x^2+4x}dx$$
$$= \int \left(\frac{1}{x} + \frac{-x+2}{x^2-2x+4}\right) dx$$
$$= \int \left(\frac{1}{x} - \frac{1}{2}\frac{2x-2}{x^2-2x+4} + \frac{1}{(x-1)^2+3}\right) dx$$
$$= \ln|x| - \frac{1}{2}\ln(x^2-2x+4)$$
$$+ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

19. $\frac{3x^3+1}{x^3-x^2+x-1}$
$$= 3 + \frac{3x^2-3x+4}{x^3-x^2+x-1}$$
$$= 3 + \frac{3x^2-3x+4}{(x^2+1)(x-1)}$$
$$= 3 + \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$
$3x^2-3x+4 = (Ax+B)(x-1) + C(x^2+1)$
$= Ax^2 - Ax + Bx - B + Cx^2 + C$
$x = 1 : 4 = 2C; C = 2$
$A + c = 3 : A = 1$
$-A + B = -3 : B = -2$

$$\frac{3x^3+1}{x^3-x^2+x-1} = 3 + \frac{x-2}{x^2+1} + \frac{2}{x-1}$$
$$\int \frac{3x^3+1}{x^3-x^2+x-1}dx$$
$$= \int \left(3 + \frac{x-2}{x^2+1} + \frac{2}{x-1}\right) dx$$
$$= \int \left(3 + \frac{x}{x^2+1} - \frac{2}{x^2+1} + \frac{2}{x-1}\right) dx$$
$$= 3x + \frac{1}{2}\ln(x^2+1) - 2\tan^{-1}x$$
$$+ 2\ln|x-1| + c$$

20. $\frac{2x^4+9x^2+x-4}{x^3+4x} = 2x + \frac{x^2+x-4}{x(x^2+4)}$
$$= 2x + \frac{A}{x} + \frac{Bx+C}{x^2+4}$$
$x^2+x-4 = A(x^2+4) + (Bx+C)x$
$= (A+B)x^2 + Cx + 4A$
$A = -1; B = 2; C = 1$

$$\frac{2x^4+9x^2+x-4}{x^3+4x} = 2x - \frac{1}{x} + \frac{2x+1}{x^2+4}$$
$$= 2x - \frac{1}{x} + \frac{2x}{x^2+4} + \frac{1}{x^2+4}$$
$$\int \frac{2x^4+9x^2+x-4}{x^3+4x}dx$$
$$= \int \left(2x - \frac{1}{x} + \frac{2x}{x^2+4} + \frac{1}{x^2+4}\right) dx$$
$$= x^2 - \ln|x| + \ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

21. $\dfrac{x^3+x+2}{x^2+2x-8} = x-2+\dfrac{11}{x+4}+\dfrac{2}{x-2}$

$$\int \frac{x^3+x+2}{x^2+2x-8}dx$$
$$= \int \left(x-2+\frac{11}{x+4}+\frac{2}{x-2}\right)dx$$
$$= \frac{x^2}{2}-2x+11\ln|x+4|$$
$$+2\ln|x-2|+c$$

22. $\dfrac{x^2+1}{x^2-5x-6} = -\dfrac{2/7}{x+1}+\dfrac{37/7}{x-6}$

$$\int \frac{x^2+1}{x^2-5x-6}dx$$
$$= \int \left(-\frac{2/7}{x+1}+\frac{37/7}{x-6}\right)dx$$
$$= -\frac{2}{7}\ln|x+1|+\frac{37}{7}\ln|x-6|+c$$

23. $\dfrac{x+4}{x^3+3x^2+2x} = \dfrac{2}{x}+\dfrac{1}{x+2}-\dfrac{3}{x+1}$

$$\int \frac{x+4}{x^3+3x^2+2x}dx$$
$$= \int \left(\frac{2}{x}+\frac{1}{x+2}-\frac{3}{x+1}\right)dx$$
$$= 2\ln|x|+\ln|x+2|-3\ln|x+1|+c$$

24. $\dfrac{1}{x^3-1} = \dfrac{1/3}{(x-1)}-\dfrac{(x+2)/3}{(x^2+x+1)}$

$$\int \frac{1}{x^3-1}dx$$
$$= \frac{1}{3}\int \frac{1}{(x-1)}-\frac{x+2}{(x^2+x+1)}dx$$
$$= \frac{1}{3}\int \frac{1}{(x-1)}-\frac{1}{2}\frac{2x+4}{(x^2+x+1)}dx$$
$$= \frac{1}{3}\int \frac{1}{(x-1)}-\frac{1}{2}\frac{2x+1}{(x^2+x+1)}$$
$$-\frac{1}{2}\frac{3}{(x^2+x+1)}dx$$
$$= \frac{1}{3}\int \frac{1}{(x-1)}-\frac{1}{2}\frac{2x+1}{(x^2+x+1)}$$
$$-\frac{1}{2}\frac{3}{(x+1/2)^2+3/4}dx$$
$$= \frac{1}{3}\Bigg[\ln|x-1|-\frac{1}{2}\ln\left|x^2+x+1\right|$$
$$-\sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\Bigg]+c$$

25. Let $u=x^4-x, du=\left(4x^3-1\right)dx$.

$$\int \frac{\left(4x^3-1\right)}{x^4-x}dx = \int \frac{du}{u}$$
$$= \ln|u|+c = \ln\left|x^4-x\right|+c.$$

26. Let $u=x^2, du=(2x)\,dx$.

$$\int \frac{x}{x^4+1}dx = \frac{1}{2}\int \frac{2x}{x^4+1}dx$$
$$= \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan(u)+c$$
$$= \frac{1}{2}\tan\left(x^2\right)+c.$$

27. $\dfrac{4x-2}{16x^4-1} = \dfrac{-4x+1}{4x^2+1}+\dfrac{1}{2x+1}$

$$\int \frac{4x-2}{16x^4-1}dx$$
$$= \int \left(\frac{-4x+1}{4x^2+1}+\frac{1}{2x+1}\right)dx$$
$$= \int \left(-\frac{1}{2}\frac{8x}{4x^2+1}+\frac{1}{4x^2+1}+\frac{1}{2x+1}\right)dx$$
$$= -\frac{1}{2}\ln|4x^2+1|+\frac{1}{2}\tan^{-1}(2x)$$
$$+\frac{1}{2}\ln|2x+1|+c$$

28. $\dfrac{3x+7}{x^4-16} = \dfrac{13/32}{x-2}-\dfrac{1/32}{x+2}-\dfrac{3x/8+7/8}{x^2+4}$

$$\int \frac{3x+7}{x^4-16}dx$$
$$= \int \left(\frac{13/32}{x-2}-\frac{1/32}{x+2}-\frac{3x/8+7/8}{x^2+4}\right)dx$$
$$= \int \Bigg(\frac{13/32}{x-2}-\frac{1/32}{x+2}\quad-\frac{3}{16}\frac{2x}{x^2+4}$$
$$-\frac{7}{8}\frac{1}{x^2+4}\Bigg)dx$$
$$= \frac{13}{32}\ln|x-2|-\frac{1}{32}\ln|x+2|$$
$$-\frac{3}{16}\ln(x^2+4)-\frac{7}{16}\tan^{-1}\frac{x}{2}+c$$

29. $\dfrac{x^3+x}{3x^2+2x+1}$

$$= \frac{x}{3}-\frac{2}{9}+\frac{1}{9}\frac{10x+2}{3x^2+2x+1}$$
$$\int \frac{x^3+x}{3x^2+2x+1}dx$$
$$= \int \left(\frac{x}{3}-\frac{2}{9}+\frac{1}{9}\frac{10x+2}{3x^2+2x+1}\right)dx$$
$$= \int \Bigg(\frac{x}{3}-\frac{2}{9}+\frac{1}{9}\frac{5}{3}\frac{6x+2}{3x^2+2x+1}$$
$$-\frac{1}{9}\frac{4}{3}\frac{1}{3(x+1/3)^2+2/3}\Bigg)dx$$
$$= \frac{x^2}{6}-\frac{2}{9}x+\frac{5}{27}\ln(3x^2+2x+1)$$
$$-\frac{2\sqrt{2}}{27}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+c$$

30. $\dfrac{x^3-2x}{2x^2-3x+2}$
$= \dfrac{x}{2}+\dfrac{4}{3}+\dfrac{1}{4}\dfrac{21x-6}{2x^2-3x+2}$

$\displaystyle\int \frac{x^3-2x}{2x^2-3x+2}dx$
$= \displaystyle\int\left(\frac{x}{2}+\frac{4}{3}+\frac{1}{4}\frac{21x-6}{2x^2-3x+2}\right)dx$
$= \displaystyle\int\left(\frac{x}{2}+\frac{4}{3}+\frac{21}{16}\frac{4x-3}{2x^2-3x+2}\right.$
$\left.+\dfrac{39}{32}\dfrac{1}{(x-3/4)^2+7/16}\right)dx$
$= \dfrac{x^2}{4}+\dfrac{4}{3}x+\dfrac{21}{16}\ln(2x^2-3x+2)$
$-\dfrac{39\sqrt{7}}{56}\tan^{-1}\left(\dfrac{4x-3}{\sqrt{7}}\right)+c$

31. $\dfrac{4x^2+3}{x^3+x^2+x}=\dfrac{3}{x}+\dfrac{x-3}{x^2+x+1}$

$\displaystyle\int \frac{4x^2+3}{x^3+x^2+x}dx$
$= \displaystyle\int\left(\frac{3}{x}+\frac{x-3}{x^2+x+1}\right)dx$
$= \displaystyle\int\left(\frac{3}{x}+\frac{x+1/2}{x^2+x+1}-\frac{7/2}{x^2+x+1}\right)dx$
$= 3\ln|x|+\dfrac{1}{2}\ln|x^2+x+1|$
$-\dfrac{7}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+c$

32. $\dfrac{4x+4}{x^4+x^3+2x^2}=\dfrac{1}{x}+\dfrac{2}{x^2}+\dfrac{-x-3}{x^2+x+2}$

$\displaystyle\int \frac{4x+4}{x^4+x^3+2x^2}dx$
$= \displaystyle\int\left(\frac{1}{x}+\frac{2}{x^2}+\frac{-x-3}{x^2+x+2}\right)dx$
$= \ln|x|-\dfrac{2}{x}-\dfrac{1}{2}\ln(x^2+x+2)$
$-\dfrac{5}{\sqrt{7}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{7}}\right)+c$

33. Let $u=x^2, dv=(\sin x)\,dx$
So that $du=(2x)\,dx$ and $v=-\cos x$.
$\displaystyle\int x^2\sin x dx$
$= x^2(-\cos x)-\displaystyle\int(-\cos x)(2x)\,dx$
$= -x^2\cos x+2\displaystyle\int x(\cos x)\,dx$
Let $u=x, dv=\cos x dx$,
so that $du=dx$ and $v=\sin x$.
$\displaystyle\int x^2\sin x dx$
$= -x^2\cos x+2\displaystyle\int x\cos x dx$
$= -x^2\cos x+2\{x\sin x+\cos x\}+c.$

34. Let $u=x, dv=e^{2x}dx$.
so that $du=dx$ and $v=\dfrac{e^{2x}}{2}$.
$\displaystyle\int xe^{2x}dx = x\frac{e^{2x}}{2}-\int\frac{e^{2x}}{2}dx$
$= x\dfrac{e^{2x}}{2}-\dfrac{e^{2x}}{4}+c$

35. Let $u=(\sin^2x-4)$,
so that $du=2\sin x\cos x\,dx$.
$\displaystyle\int\frac{\sin x\cos x}{\sin^2x-4}dx = \frac{1}{2}\int\frac{du}{u}$
$= \dfrac{1}{2}\ln|u|+c=\dfrac{1}{2}\ln\left|\sin^2x-4\right|+c$

36. Let $t=e^x,\ dt=e^xdx$ and $e^{3x}=t^3$
$\displaystyle\int\frac{2e^x}{e^{3x}+e^x}dx=\int\frac{2}{t^3+t}dt.$
$= \displaystyle\int\frac{2}{t}-\frac{2t}{t^2+1}dt = 2\ln|t|-\ln\left|t^2+1\right|+c$
$= 2\ln|e^x|-\ln\left|e^{2x}+1\right|+c$

37. $\dfrac{4x^2+2}{(x^2+1)^2}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{(x^2+1)^2}$
$4x^2+2=(Ax+B)(x^2+1)+(Cx+D)$
$= Ax^3+Bx^2+(A+C)x+(B+D)$
$A=0; B=4; C=0; D=-2$

$\dfrac{4x^2+2}{(x^2+1)^2}=\dfrac{4}{x^2+1}+\dfrac{-2}{(x^2+1)^2}$

38. $\dfrac{x^3+2}{(x^2+1)^2}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{(x^2+1)^2}$
$x^3+2=(Ax+B)(x^2+1)+cx+D$
$= Ax^3+Bx^2+(A+c)x+(B+D)$
$A=1; B=0; C=-1; D=2$

$\dfrac{x^3+2}{(x^2+1)^2}=\dfrac{x}{x^2+1}+\dfrac{-x+2}{(x^2+1)^2}$

39. $\dfrac{4x^2+3}{(x^2+x+1)^2}$
$= \dfrac{Ax+B}{x^2+x+1}+\dfrac{Cx+D}{(x^2+x+1)^2}$
$4x^2+3=(Ax+B)(x^2+x+1)+cx+D$
$= Ax^3+Ax^2+Ax+Bx^2+Bx+B+cx+D$
$A=0$
$A+B=4 : B=4$
$A+B+c=0 : C=-4$
$B+D=3 : D=-1$

$$\frac{4x^2+3}{(x^2+x+1)^2}$$
$$= \frac{4}{x^2+x+1} - \frac{4x+1}{(x^2+x+1)^2}$$

40. $\frac{x^4+x^3}{(x^2+4)^2} = 1 + \frac{x^3-8x^2-8}{(x^2+4)^2}$
$= 1 + \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$

$x^3 - 8x^2 - 8 = (Ax+B)(x^2+4) + cx + D$
$= Ax^3 + Bx^2 + (4A+c)x + (4B+D)$
$A = 1; B = -8; C = -4; D = 24$

$$\frac{x^4+x^3}{(x^2+4)^2} = 1 + \frac{x-8}{x^2+4} + \frac{-4x+24}{(x^2+4)^2}$$

41. Let $u = x^3+1$, $du = 3x^2dx$
$\int \frac{3}{x^4+x}dx = \int \frac{3x^2}{x^3(x^3+1)}dx$
$= \int \frac{1}{(u-1)u}\,du$
$= \int \left(\frac{1}{u-1} - \frac{1}{u}\right) du$
$= \ln|u-1| - \ln|u| + c$
$= \ln\left|\frac{u-1}{u}\right| + c$

$= \ln\left|\frac{x^3}{x^3+1}\right| + c$

On the other hand, we can let
$u = \frac{1}{x}$, $du = -\frac{1}{x^2}dx$
$\int \frac{3}{x^4+x}dx = -\int \frac{3u^2}{1+u^3}\,du$
$= -\ln|1+u^3| + c = -\ln|1+1/x^3| + c$

To see that the two answers are equivalent, note that
$\ln\left|\frac{x^3}{x^3+1}\right| = -\ln\left|\frac{x^3+1}{x^3}\right| = -\ln|1+1/x^3|$

42. Let $u = x^2+1, du = 2xdx$

$\int \frac{2}{x^3+x}dx = \int \frac{2x}{x^2(x^2+1)}dx$
$= \int \frac{du}{u(u-1)} = \ln\left|\frac{u-1}{u}\right| + c$
$= \ln\left|\frac{x^2}{x^2+1}\right| + c$

Let $u = \frac{1}{x}, du = -\frac{1}{x^2}dx$
$\int \frac{2}{x^3+x}dx = -\int \frac{2u}{1+u^2}du$
$= -\ln|1+u^2| + c = -\ln\left|1+\frac{1}{x^2}\right| + c$

To see that the two answers are equivalent, note that
$\ln\left|\frac{x^2}{x^2+1}\right| = -\ln\left|\frac{x^2+1}{x^2}\right| = -\ln\left|1+\frac{1}{x^2}\right|$

43. (a) Partial fractions
(b) Substitution method
(c) Substitution and Partial fractions.
(d) Substitution

44. (a) Partial fractions
(b) Substitution and Partial fractions.
(c) Partial fractions
(d) Partial fractions

45. $\int \sec^3 x dx = \int \frac{\cos x}{(1-\sin^2 x)^2}dx$
Let $u = \sin x$, so that $du = \cos x dx$.
$\int \frac{\cos x\,dx}{(1-\sin^2 x)^2} = \int \frac{du}{(1-u^2)^2}$
$= \int \frac{1}{(1-u)^2(1+u)^2}du$
By partial fractions,
$$\frac{1}{(1-u)^2(1+u)^2} = \frac{1}{4}\left(\frac{1}{(1-u)} + \frac{1}{(1-u)^2} + \frac{1}{(1+u)} + \frac{1}{(1+u)^2}\right)$$
Hence, $\int \sec^3 x dx$
$= \frac{1}{4}\left[-\ln|1-u| + \frac{1}{(1-u)} + \ln|1+u| - \frac{1}{(1+u)}\right] + c$
$= \frac{1}{4}\left[-\ln|1-\sin x| + \frac{1}{(1-\sin x)} + \ln|1+\sin x| - \frac{1}{(1+\sin x)}\right] + c$

6.5 Integration Table and Computer Algebra Systems

1. $\int \frac{x}{(2+4x)^2}dx$
$= \frac{2}{16(2+4x)} + \frac{1}{16}\ln|2+4x| + c$
$= \frac{1}{8(2+4x)} + \frac{1}{16}\ln|2+4x| + c$

2. $\int \frac{x^2}{(2+4x)^2}\,dx$
$= \frac{1}{64}\left(2+4x-\frac{4}{2+4x}-4\ln|2+4x|\right)+c$

3. Substitute $u=1+e^x$
$\int e^{2x}\sqrt{1+e^x}dx = \int (u-1)\sqrt{u}\,du$
$= \int (u^{3/2}-u^{1/2})\,du$
$= \frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}+c$
$= \frac{2}{5}(1+e^x)^{5/2}-\frac{2}{3}(1+e^x)^{3/2}+c$

4. Substitute $u=e^x$
$\int e^{3x}\sqrt{1+e^{2x}}\,dx = \int u^2\sqrt{1+u^2}\,du$
$= \frac{1}{8}u(1+2u^2)\sqrt{1+u^2}$
$\quad -\frac{1}{8}\ln|u+\sqrt{1+u^2}|+c$
$= \frac{1}{8}e^x(1+2e^{2x})\sqrt{1+e^{2x}}$
$\quad -\frac{1}{8}\ln|e^x+\sqrt{1+e^{2x}}|+c$

5. Substitute $u=2x$

$\int \frac{x^2}{\sqrt{1+4x^2}}dx$
$= \frac{1}{8}\int \frac{u^2}{\sqrt{1+u^2}}\,du$
$= \frac{1}{8}\left[\frac{u}{2}-\sqrt{1+u^2}\right.$
$\quad \left.-\frac{1}{2}\ln(u+\sqrt{1+u^2})\right]+c$
$= \frac{1}{8}x\sqrt{1+4x^2}$
$\quad -\frac{1}{16}\ln(2x+\sqrt{1+4x^2})+c$

6. Substitute $u=\sin x$
$\int \frac{\cos x}{\sin^2 x(3+2\sin x)}\,dx$
$= \int \frac{1}{u^2(3+2u)}\,du$
$= \frac{2}{9}\ln\left|\frac{3+2u}{u}\right|-\frac{1}{2u}+c$
$= \frac{2}{9}\left|\frac{3+2\sin x}{\sin x}\right|-\frac{1}{3\sin x}+c$

7. Substitute $u=t^3$
$\int t^8\sqrt{4-t^6}dt$
$= \frac{1}{3}\int u^2\left(\sqrt{4-u^2}\right)du$
$= \frac{1}{3}\left[\frac{u}{8}\left(2u^2-4\right)\sqrt{4-u^2}+\frac{16}{8}\sin^{-1}\frac{u}{2}\right]+c$
$= \frac{1}{24}t^3\left(2t^6-4\right)\sqrt{4-t^6}+\frac{2}{3}\sin^{-1}\frac{t^3}{2}+c$
$\int_0^1 t^8\sqrt{4-t^6}dt = \frac{\pi}{9}-\frac{\sqrt{3}}{12}$

8. Substitute $u=e^t$
$\int \sqrt{16-e^{2t}}dt = \int \frac{\sqrt{16-u^2}}{u}du$
$= \sqrt{16-u^2}-4\ln\left|\frac{4+\sqrt{16-u^2}}{u}\right|+c$
$= \sqrt{16-e^{2t}}-4\ln\left|\frac{4+\sqrt{16-e^{2t}}}{e^t}\right|+c$
$\int_0^{\ln 4}\sqrt{16-e^{2t}}dt = -\sqrt{15}+4\ln\left(\sqrt{15}+4\right)$

9. Substitute $u=e^x$
$\int \frac{e^x}{\sqrt{e^{2x}+4}}dx = \int \frac{1}{\sqrt{u^2+4}}\,du$
$= \ln(u+\sqrt{4+u^2})+c$
$= \ln(e^x+\sqrt{4+e^{2x}})+c$
$\int_0^{\ln 2}\frac{e^x}{\sqrt{e^{2x}+4}}dx = \ln\left(\frac{2\sqrt{2}+2}{1+\sqrt{5}}\right)$

10. Substitute $u=x^2$
$\int_{\sqrt{3}}^{2}\frac{x\sqrt{x^4-9}}{x^2}\,dx = \frac{1}{2}\int_3^4\frac{\sqrt{u^2-9}}{u}\,du$
$= \frac{1}{2}\left(\sqrt{u^2-9}-3\sec^{-1}\frac{|u|}{3}\right)\Big|_3^4$
$= \frac{\sqrt{7}}{2}-\frac{3}{2}\sec^{-1}\left(\frac{4}{3}\right)$

11. Substitute $u=x-3$
$\int \frac{\sqrt{6x-x^2}}{(x-3)^2}dx$
$= \int \frac{\sqrt{(u+3)(6-(u+3))}}{u^2}\,du$
$= \int \frac{\sqrt{9-u^2}}{u^2}\,du$
$= -\frac{1}{u}\sqrt{9-u^2}-\sin^{-1}\frac{u}{3}+c$
$= -\frac{1}{x-3}\sqrt{9-(x-3)^2}$
$\quad -\sin^{-1}\left(\frac{x-3}{3}\right)+c$

12. Substitute $u=\tan x$
$\int \frac{\sec^2 x}{\tan x\sqrt{8\tan x-\tan^2 x}}\,dx$
$= \int \frac{1}{u\sqrt{8u-u^2}}\,du$
$= -\frac{\sqrt{8u-u^2}}{4u}+c$

$$= -\frac{\sqrt{8\tan x - \tan^2 x}}{4\tan x} + c$$

13. $\int \tan^6 u du$

$$= \frac{1}{5}\tan^5 u - \int \tan^4 u du$$

$$= \frac{1}{5}\tan^5 u - \left[\frac{1}{3}\tan^3 u - \int \tan^2 u du\right]$$

$$= \frac{1}{5}\tan^5 u - \frac{1}{3}\tan^3 u + \tan u - u + c.$$

14. $\int \csc^4 u du$

$$= -\frac{1}{3}\csc^2 u \cot u + \frac{2}{3}\int \csc^2 u du$$

$$= -\frac{1}{3}\csc^2 u \cot u - \frac{2}{3}\cot u + c.$$

15. Substitute $u = \sin x$

$$\int \frac{\cos x}{\sin x\sqrt{4+\sin x}}dx = \int \frac{1}{u\sqrt{4+u}}\,du$$

$$= \frac{1}{\sqrt{4}}\ln\left|\frac{\sqrt{4+u}-2}{\sqrt{4+u}+2}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{4+\sin x}-2}{\sqrt{4+\sin x}+2}\right| + c$$

16. Substitute $u = x^2$

$$\int \frac{x^5}{\sqrt{4+x^2}}\,dx = \frac{1}{2}\int \frac{u^2}{\sqrt{4+u^2}}\,du$$

$$= \left(\frac{1}{2}\right)\frac{2}{15}(3u^2 - 16u + 128)\sqrt{4+u} + c$$

$$= \frac{1}{15}(3x^4 - 16x^2 + 128)\sqrt{4+x^2} + c$$

17. Substitute $u = x^2$

$$\int x^3\cos x^2 dx = \frac{1}{2}\int u\cos u\,du$$

$$= \frac{1}{2}(\cos u + u\sin u) + c$$

$$= \frac{1}{2}\cos x^2 + \frac{1}{2}x^2\sin x^2 + c$$

18. Substitute $u = x^2$

$$\int x\sin(3x^2)\cos(4x^2)\,dx$$

$$= \frac{1}{2}\int \sin(3u)\cos(4u)\,du$$

$$= \frac{1}{2}\left(\frac{\cos u}{2} - \frac{\cos 7u}{14}\right) + c$$

$$= \frac{\cos x^2}{4} - \frac{\cos 7x^2}{28} + c$$

19. Substitute $u = \cos x$

$$\int \frac{\sin 2x}{\sqrt{1+\cos x}}dx = \int \frac{2\sin x\cos x}{\sqrt{1+\cos x}}dx$$

$$= -2\int \frac{u}{\sqrt{1+u}}du$$

$$= -2\left[\frac{2}{3}(u-2)\sqrt{1+u}\right] + c$$

$$= -\frac{4}{3}(\cos x - 2)\sqrt{1+\cos x} + c$$

20. Substitute $u = x^2$

$$\int \frac{x\sqrt{1+4x^2}}{x^4}\,dx = \frac{1}{2}\int \frac{\sqrt{1+4u}}{u^2}\,du$$

$$= -\frac{\sqrt{1+4u}}{2u} + \ln\left[\frac{\sqrt{1+4u}-1}{\sqrt{1+4u}+1}\right] + c$$

$$= -\frac{\sqrt{1+4x^2}}{2x^2} + \ln\left[\frac{\sqrt{1+4x^2}-1}{\sqrt{1+4x^2}+1}\right] + c$$

21. Substitute $u = \sin t$

$$\int \frac{\sin^2 t\cos t}{\sqrt{\sin^2 t + 4}}dt$$

$$= \int \frac{u^2}{\sqrt{u^2+4}}\,du$$

$$= \frac{u}{2}\sqrt{4+u^2} - \frac{4}{2}\ln(u + \sqrt{4+u^2}) + c$$

$$= \frac{1}{2}\sin t\sqrt{4+\sin^2 t}$$

$$- 2\ln\left(\sin t + \sqrt{4+\sin^2 t}\right) + c$$

22. Substitute $u = \sqrt{t}$

$$\int \frac{\ln\sqrt{t}}{\sqrt{t}}\,dt = 2\int \ln u\,du$$

$$= 2u\ln u - 2u + c = 2\sqrt{t}\ln\sqrt{t} - 2\sqrt{t} + c$$

23. Substitute $u = -\frac{2}{x^2}$

$$\int \frac{e^{-2/x^2}}{x^3}dx = \frac{1}{4}\int e^u\,du$$

$$= \frac{1}{4}e^u + c = \frac{1}{4}e^{-2/x^2} + c$$

24. Substitute $u = 2x^2$

$$\int x^3 e^{2x^2}\,dx = \frac{1}{8}\int ue^u\,du$$

$$= \frac{1}{8}(u-1)e^u + c = \frac{1}{8}(2x^2-1)e^{2x^2} + c$$

25. $\int \frac{x}{\sqrt{4x-x^2}}dx$

$$= -\sqrt{4x-x^2} + 2\cos^{-1}\left(\frac{2-x}{2}\right) + c$$

26. $\int e^{5x}\cos 3x\,dx$

$$= \frac{1}{34}(5\cos 3x + 3\sin 3x)e^{5x} + c$$

27. Substitute $u = e^x$

$$\int e^x\tan^{-1}(e^x)dx = \int \tan^{-1}u\,du$$

$= u\tan^{-1}u - \frac{1}{2}\ln(1+u^2) + c$
$= e^x\tan^{-1}e^x - \frac{1}{2}\ln(1+e^{2x}) + c$

28. Substitute $u = 4x$
$\int (\ln 4x)^3\,dx = \frac{1}{4}\int (\ln u)^3\,dx$
$= \frac{1}{4}\left(u(\ln u)^3 - 3\int (\ln u)^2\,dx\right)$
$= \frac{1}{4}u(\ln u)^3$
$- \frac{3}{4}\left(u(\ln u)^2 - 2u\ln u + 2u\right) + c$
$= x(\ln 4x)^3 - 3x(\ln u)^2 + 6x\ln 4x - 6x + c$

29. Answer depends on CAS used.

30. Answer depends on CAS used.

31. Any answer is wrong because the integrand is undefined for all $x \neq 1$.

32. Answer depends on CAS used.

33. Answer depends on CAS used.

34. Answer depends on CAS used.

35. Answer depends on CAS used.

36. Maple gives the result: $\dfrac{b\pi}{\sqrt{\dfrac{1}{a^2}}}$

37. If the CAS is unable to compute an antiderivative, $\int f(x)\,dx$ is generally printed showing this inability.

6.6 Improper Integrals

1. (a) improper, function not defined at $x = 0$
(b) not improper, function continuous on entire interval
(c) not improper, function continuous on on entire interval

2. (a) improper, interval is infinite
(b) improper, function not defined at $x = 0$
(c) improper, interval is infinite

3. (a) $\int_0^1 x^{-1/3}dx = \lim_{R\to 0^+}\int_R^1 x^{-1/3}dx$
$= \lim_{R\to 0^+} \frac{3}{2}x^{2/3}\Big|_R^1$
$= \lim_{R\to 0^+} \frac{3}{2}\left(1 - R^{2/3}\right) = \frac{3}{2}$

(b) $\int_0^1 x^{-4/3}dx = \lim_{R\to 0^+}\int_R^1 x^{-4/3}dx$
$= \lim_{R\to 0^+} (-3x^{-1/3})\Big|_R^1$
$= \lim_{R\to 0^+} (-3)(1 - R^{-1/3}) = \infty$
So the original integral diverges.

4. (a) $\int_1^\infty x^{-4/5}dx = \lim_{R\to\infty}\int_1^R x^{-4/5}dx$
$= \lim_{R\to\infty} 5x^{1/5}\Big|_1^R$
$= \lim_{R\to\infty} 5R^{1/5} - 5 = \infty$
So the original integral diverges.

(b) $\int_1^\infty x^{-6/5}dx = \lim_{R\to\infty}\int_1^R dx$
$= \lim_{R\to\infty} -5x^{-1/5}\Big|_1^R$
$= \lim_{R\to\infty} -5R^{-1/5} + 5 = 5$

5. (a) $\int_0^1 \frac{1}{\sqrt{1-x}}dx = \lim_{R\to 1^-}\int_0^R \frac{1}{\sqrt{1-x}}dx$
$= \lim_{R\to 1^-} - 2\sqrt{1-x}\Big|_0^R$
$= \lim_{R\to 1^-} -2(\sqrt{1-R} - 1) = 2$

(b) $\int_1^5 \frac{2}{\sqrt{5-x}}dx = \lim_{R\to 5^-}\int_1^R \frac{2}{\sqrt{5-x}}dx$
$= \lim_{R\to 5^-} - 4\sqrt{5-x}\Big|_1^R$
$= \lim_{R\to 5^-} -4(\sqrt{5-R} - 2) = -8$

6. (a) $\int_0^1 \frac{2}{\sqrt{1-x^2}}dx = \lim_{R\to 1^-}\int_0^R \frac{2}{\sqrt{1-x^2}}dx$
$= \lim_{R\to 1^-} 2\sin^{-1}x\Big|_0^R$
$= \lim_{R\to 1^-} 2(\sin^{-1}R - \sin^{-1}0)$
$= 2\left(\frac{\pi}{2} - 0\right) = \pi$

(b) $\int_0^{1/2} \frac{2}{x\sqrt{1-x^2}}dx$
$= \lim_{R\to 0^+}\int_R^{1/2} \frac{2}{x\sqrt{1-x^2}}dx$
$= \lim_{R\to 0^+} -2\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)\Bigg|_R^{1/2} = \infty$
Therefore the original integral diverges.

7. (a) $\int_0^\infty xe^x dx = \lim_{R\to\infty}\int_0^R xe^x dx$
$= \lim_{R\to\infty} (xe^x - e^x)\Big|_0^R$

$= \lim_{R\to\infty} e^R(R-1)+1 = \infty$

So the original integral diverges.

(b) Substitute $u=-2x$

$I = \int_1^{\infty} x^2e^{-2x}dx = -\frac{1}{8}\int_{-2}^{-\infty} u^2e^u du$

$= \frac{1}{8}\int_{-\infty}^{-2} u^2e^u du$

$= \frac{1}{8}\lim_{R\to-\infty}\int_R^{-2} u^2e^u du$

$= \frac{1}{8}\lim_{R\to-\infty}(u^2e^u - 2ue^u + 2e^u)\Big|_R^{-2}$

$= \frac{10}{8}e^{-2} + \frac{1}{8}\lim_{R\to-\infty} e^R(-R^2+2R-2)$

But, $\lim_{R\to-\infty} e^R(-R^2+2R-2)$

$= \lim_{R\to\infty} e^{-R}(-R^2-2R-2)$

$= \lim_{R\to\infty} \frac{-R^2-2R-2}{e^R}$

$= \lim_{R\to\infty} \frac{-2R-2}{e^R} = \lim_{R\to\infty}\frac{-2}{e^R} = 0$

Hence, $I = \frac{5}{4}e^{-2}$

8. (a) Substitute $u=3x$

$I = \lim_{-\infty\to 1}\int_{-\infty}^{1} x^2e^{3x}dx$

$= \frac{1}{27}\int_{-\infty}^{3} u^2e^u du$

$= \frac{1}{27}\lim_{R\to-\infty}(u^2e^u - 2ue^u + 2e^u)\Big|_R^{3}$

$= \frac{5}{27}e^3 - \frac{1}{27}\lim_{R\to-\infty} e^R(R^2-2R+2)$

But, $\lim_{R\to\infty} e^R(R^2-2R+2)$

$= \lim_{R\to\infty} e^{-R}(R^2+2R+2)$

$= \lim_{R\to\infty}\frac{R^2+2R+2}{e^R} = 0$

Hence, $I = \frac{5}{27}e^3$

(b) Substitute $u=-4x$

$I = \int_{-\infty}^{0} xe^{-4x}dx$

$= \frac{1}{16}\int_{-\infty}^{0} ue^u du$

$= \frac{1}{16}\lim_{R\to-\infty}\int_R^0 ue^u du$

$= \frac{1}{16}\lim_{R\to-\infty}(ue^u - e^u)\Big|_R^0$

$= -\frac{1}{16} + \frac{1}{16}\lim_{R\to-\infty} e^R(R-1)$

But, $\lim_{R\to-\infty} e^R(R-1)$

$= \lim_{R\to\infty} e^{-R}(-R-1) = 0$

Hence, $I = -\frac{1}{16}$

9. (a) $\int_{-\infty}^{-1}\frac{1}{x^2}dx = \lim_{R\to-\infty}\int_R^{-1}\frac{1}{x^2}dx$

$= \lim_{R\to-\infty} -\frac{1}{x}\Big|_R^{-1}$

$= 1 + \lim_{R\to-\infty}\frac{1}{R} = 1$

$\int_{-1}^{0}\frac{1}{x^2}dx = \lim_{R\to0^+}\int_{-1}^{R}\frac{1}{x^2}dx$

$= \lim_{R\to0^+} -\frac{1}{x}\Big|_{-1}^{R}$

$= -1 - \lim_{R\to0^+}\frac{1}{R} = \infty$

So the original integral diverges.

(b) $\int_{-\infty}^{-1}\frac{1}{\sqrt[3]{x}}dx = \lim_{R\to-\infty}\int_R^{-1}\frac{1}{\sqrt[3]{x}}dx$

$= \lim_{R\to-\infty}\frac{3}{2}x^{2/3}\Big|_R^{-1}$

$= \frac{3}{2} + \frac{3}{2}\lim_{R\to-\infty} R^{2/3} = \infty$

So the original integral diverges.

10. (a) $\int_0^{\infty}\cos x dx = \lim_{R\to\infty}\int_0^R \cos x dx$

$= \lim_{R\to\infty}\sin x\Big|_0^R$

$= \lim_{R\to\infty}(\sin R - \sin 0)$

So the original integral diverges.

(b) $\int_0^{\infty}\cos x e^{-\sin x}dx$

$= \lim_{R\to\infty}\int_0^R \cos x e^{-\sin x}dx$

$= \lim_{R\to\infty} -e^{-\sin x}\Big|_0^R$

$= \lim_{R\to\infty} -e^{-\sin R} + 1$

So the original integral diverges.

11. (a) $\int_0^1 \ln x dx$

$= \lim_{R\to0^+}\int_R^1 \ln x dx$

$= \lim_{R\to0^+}(x\ln x - x)\Big|_R^1$

$= \lim_{R\to0^+}(-1 - R\ln R + R)$

$= -1 - \lim_{R\to0^+}\frac{\ln R}{1/R}$

$= -1 - \lim_{R\to0^+}\frac{1/R}{-1/R^2}$

$= -1 + \lim_{R\to0^+} R = -1$

(b) $\int_0^{\pi/2} \sec^2 x dx$
$= \lim_{R\to\pi/2^-} \int_0^R \sec^2 x dx$
$= \lim_{R\to\pi/2^-} \tan x \Big|_0^R$
$= \lim_{R\to\pi/2^-} \tan R - \tan 0 = \infty$
Therefore the original integral diverges.

12. (a) $\int_0^{\pi/2} \cot x dx$
$= \lim_{R\to 0^+} \int_R^{\pi/2} \frac{\cos x}{\sin x} dx$
$= \lim_{R\to 0^+} \ln|\sin x| \Big|_R^{\pi/2}$
$= \ln|\sin(\pi/2)| - \lim_{R\to 0^+} \ln|\sin R| = \infty$
So the original integral diverges.

(b) $\int_0^{\pi/2} \tan x dx$
$= \lim_{R\to\pi/2} \int_0^R \frac{\sin x}{\cos x} dx$
$= \lim_{R\to\pi/2} -\ln|\cos x| \Big|_0^R$
$= \lim_{R\to\pi/2} (-\ln|\cos R| + \ln 1) = \infty$
So the original integral diverges.

13. (a) $\int_0^3 \frac{2}{x^2-1} dx$
$= \int_0^3 \left(-\frac{1}{x+1} + \frac{1}{x-1}\right) dx$
$= \lim_{R\to 1^-} \int_0^R \left(-\frac{1}{x+1} + \frac{1}{x-1}\right) dx$
$+ \lim_{R\to 1^+} \int_R^3 \left(-\frac{1}{x+1} + \frac{1}{x-1}\right) dx$
Both of these integrals behave like
$\lim_{R\to 0^+} \int_R^1 \frac{1}{x} dx$
$= \lim_{R\to 0^+} (\ln 1 - \ln R)$
$= \lim_{R\to 0^+} \ln\left(\frac{1}{R}\right) = \infty$
So the original integral diverges.

(b) $\int_1^4 \frac{2x}{x^2-1} dx$
$= \lim_{R\to 1^+} \int_R^4 \frac{2x}{x^2-1} dx$
$= \lim_{R\to 1^+} \ln(x^2-1)\big|_R^4$
$= \lim_{R\to 1^+} \ln 15 - \ln(R^2-1) = \infty$
$\int_{-4}^1 \frac{2x}{x^2-1} dx$
$= \lim_{R\to 1^-} \int_{-4}^R \frac{2x}{x^2-1} dx$
$= \lim_{R\to 1^-} \ln(x^2-1)\big|_{-4}^R$
$= \lim_{R\to 1^-} \ln(R^2-1) - \ln 15 = \infty$
So the original integral diverges.

14. (a) $\int_0^\pi x\sec^2 x dx$
$= \int_0^{\pi/2} x\sec^2 x dx + \int_{\pi/2}^\pi x\sec^2 x dx$
$= \lim_{R\to\pi/2^-} (x\tan x + \ln|\cos x|)\big|_0^R$
$+ \lim_{R\to\pi/2^+} (x\tan x + \ln|\cos x|)\big|_R^\pi$
$= \infty$
So the original integral diverges.

(b) $\int_0^2 \frac{2}{x^3-1} dx$
$= \int_0^1 \frac{2}{x^3-1} dx + \int_1^2 \frac{2}{x^3-1} dx$
$= \lim_{R\to 1^-} \int_0^R \frac{2}{x^3-1} dx$
$+ \lim_{R\to 1^+} \int_R^2 \frac{2}{x^3-1} dx$
$= \lim_{R\to 1^-} 2\left(-\frac{\ln\left(x^2+x+1\right)}{6}\right.$
$\left.-\frac{\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}} + \frac{\ln(x-1)}{3}\right)\Bigg|_0^R$
$+ \lim_{R\to 1^+} 2\left(-\frac{\ln\left(x^2+x+1\right)}{6}\right.$
$\left.-\frac{\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}} + \frac{\ln(x-1)}{3}\right)\Bigg|_0^R$
$= \infty$

15. (a) $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$
$= \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^\infty ds \frac{1}{1+x^2} dx$
$= \lim_{R\to-\infty} \int_R^0 \frac{1}{1+x^2} dx$
$+ \lim_{R\to\infty} \int_0^R \frac{1}{1+x^2} dx$
$= \lim_{R\to\infty} \tan^{-1} x\big|_R^0 + \lim_{R\to\infty} \tan^{-1} x\big|_0^R$
$= \lim_{R\to-\infty} (\tan^{-1} 0 - \tan^{-1} R)$
$+ \lim_{R\to\infty} (\tan^{-1} R - \tan^{-1} 0)$
$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi$

(b) $\int_1^2 \frac{1}{x^2-1}dx$
$= \lim_{R\to 1^+} \int_R^1 \frac{1}{x^2-1}dx$
$= \lim_{R\to 1^+} \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)\Big|_R^2$
$= \lim_{R\to 1^+} \frac{1}{2}\ln\left(\frac{1}{3}\right) - \frac{1}{2}\ln\left(\frac{R-1}{R+1}\right)$
$= \infty$
Therefore the original integral diverges.

16. (a) $\int_0^2 \frac{x}{x^2-1}dx$
$= \int_0^1 \frac{x}{x^2-1}dx + \int_1^2 \frac{x}{x^2-1}dx$
$= \lim_{R\to 1^-} \int_0^R \frac{x}{x^2-1}dx$
$+ \lim_{R\to 1^+} \int_R^2 \frac{x}{x^2-1}dx$
$= \lim_{R\to 1^-} \frac{1}{2}\ln|x^2-1|\Big|_0^R$
$+ \lim_{R\to 1^+} \frac{1}{2}\ln|x^2-1|\Big|_R^2$
$= \lim_{R\to 1^-} \left(\frac{1}{2}\ln|R^2-1| - \frac{1}{2}\ln|-1|\right)$
$= -\infty$
So the original integral diverges.

(b) $\int_0^2 \frac{1}{(x-2)^2}dx$
$= \lim_{R\to 2^-} \int_0^R \frac{1}{(x-2)^2}dx$
$= \lim_{R\to 2^-} \frac{1}{2-x}\Big|_0^R$
$= \lim_{R\to 2^-} \frac{1}{2-R} - \frac{1}{2} = \infty$
So the original integral diverges.

17. (a) Substitute $u = \sqrt{x}$
$\int \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx = \int 2e^{-u}du$
Hence $\int_0^\infty \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx$
$= \lim_{R\to 0^+} \int_R^1 \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx$
$+ \lim_{R\to\infty} \int_1^R \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx$
$= \lim_{R\to 0^+} \left(\frac{-2}{e^{\sqrt{x}}}\right)\Big|_R^1 + \lim_{R\to\infty} \left(\frac{-2}{e^{\sqrt{x}}}\right)\Big|_1^R$
$= \lim_{R\to 0^+} \left(\frac{-2}{e} + \frac{2}{e^R}\right)$
$+ \lim_{R\to\infty} \left(\frac{-2}{e} + \frac{2}{e^R}\right)$
$= 1 + 1 = 2$

(b) $\int_0^{\pi/2} \tan x dx$
$= \lim_{R\to\pi/2^-} \int_0^R \tan x dx$
$= \lim_{R\to\pi/2^-} -\ln\cos x\Big|_0^R$
$= \lim_{R\to\pi/2^-} (-\ln\cos R) = \infty$
Therefore the original integral diverges.

18. (a) Substitute $u = e^x$
$I = \int_0^\infty \frac{e^x}{e^{2x}+1}dx$
$= \int_1^\infty \frac{1}{u^2+1}dx$
$= \lim_{R\to\infty} \int_1^R \frac{1}{u^2+1}dx$
$= \lim_{R\to\infty} \tan^{-1}u\Big|_1^R$
$= \lim_{R\to\infty} \left(\tan^{-1}R - \tan^{-1}1\right)$
$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

(b) Substitute $u = \tan^{-1}x$
$I = \int_0^\infty \frac{x}{\sqrt{x^2+1}}dx$
$= \int_0^{\pi/2} \tan u \left(\sqrt{\tan^2 u + 1}\right) du$
$= \lim_{R\to\pi/2} \int_0^R \tan u (\sec u)\, du$
$= \lim_{R\to\pi/2} \sec u\Big|_0^R$
$= \lim_{R\to\pi/2^-} \sec R - \sec 0 = \infty$

Therefore the original integral diverges.

19. (a) $I_p = \int_0^1 x^{-p}dx = \lim_{R\to 0^+} \int_R^1 x^{-p}dx$
$= \lim_{R\to 0^+} \left(\frac{x^{-p+1}}{-p+1}\right)\Big|_R^1 = \lim_{R\to 0^+} \frac{1-R^{-p+1}}{-p+1}$
We need $p < 1$ for the above limit to converge. If this is the case,
$I_p = \frac{1}{-p+1}$.

(b) $I_p = \int_1^\infty x^{-p}dx = \lim_{R\to\infty} \int_1^R x^{-p}dx$
$= \lim_{R\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^R = \lim_{R\to\infty} \frac{R^{-p+1}-1}{-p+1}$
We need $p > 1$ for the above limit to

converge.

(c) There are three cases.
Case 1: $p > -1$
$$\int_0^\infty x^p dx = \lim_{R\to\infty} \int_0^R x^p dx$$
$$= \lim_{R\to\infty} \left. \frac{x^{p+1}}{p+1} \right|_0^R = \lim_{R\to\infty} \frac{R^{p+1}}{p+1} = \infty$$
So $\int_{-\infty}^{\infty} x^p dx$ diverges.
Case 2: $p = -1$
We have already seen that $\int_{-\infty}^{\infty} \frac{1}{x} dx$ diverges.
Case 3: $p < -1$
$$\int_0^1 x^p dx = \lim_{R\to 0^+} \int_R^1 x^p dx$$
$$= \lim_{R\to 0^+} \left. \frac{x^{p+1}}{p+1} \right|_R^1$$
$$= \lim_{R\to 0^+} \frac{1 - R^{p+1}}{p+1} = \infty$$
So $\int_{-\infty}^{\infty} x^p dx$ diverges.

20. (a) Case1: If $r \geq 0$
Substitute $u = rx$
$$I = \int_0^\infty x e^{rx} dx$$
$$= \frac{1}{r^2} \lim_{R\to\infty} \int_0^R u e^u du$$
$$= \frac{1}{r^2} \lim_{R\to\infty} \left(u e^u - e^u\right)\Big|_0^R$$
$$= \frac{1}{r^2} \lim_{R\to\infty} e^R (R-1) - \frac{1}{r^2} = \infty$$
So $\int_0^\infty x e^{rx} dx$ diverges for $r \geq 0$.

Case2: For $r < 0$,
Substitute $u = -rx$
$$I = \int_0^\infty x e^{rx} dx$$
$$= \frac{1}{r^2} \lim_{R\to\infty} \int_0^{-R} u e^u du$$
$$= -\frac{1}{r^2} \lim_{R\to\infty} \int_{-R}^0 u e^u du$$
$$= -\frac{1}{r^2} \lim_{R\to\infty} \left(u e^u - e^u\right)\Big|_{-R}^0$$
$$= -\frac{1}{r^2} \lim_{R\to\infty} e^{-R} (-R-1) - \frac{1}{r^2}$$
$$= 0 - \frac{1}{r^2} = -\frac{1}{r^2}$$
Therefore, for all $r < 0$ the integral $\int_0^\infty x e^{rx} dx$ converges.

(b) Case1: If $r > 0$
Substitute $u = rx$
$$I = \int_{-\infty}^0 x e^{rx} dx$$
$$= \frac{1}{r^2} \lim_{R\to -\infty} \int_R^0 u e^u du$$
$$= \frac{1}{r^2} \lim_{R\to -\infty} \left(u e^u - e^u\right)\Big|_R^0$$
$$= \frac{1}{r^2} - \frac{1}{r^2} \lim_{R\to -\infty} e^R (R-1)$$
$$= \frac{1}{r^2} - 0 = \frac{1}{r^2}$$
So $\int_{-\infty}^0 x e^{rx} dx$ converges for $r > 0$.

Case2: For $r \leq 0$,
the integral $\int_{-\infty}^0 x e^{rx} dx$ diverges.
Therefore, for all $r < 0$
the integral $\int_0^\infty x e^{rx} dx$ converges.

21. $0 < \frac{x}{1+x^3} < \frac{x}{x^3} = \frac{1}{x^2}$
$$\int_1^\infty \frac{1}{x^2} dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^2} dx$$
$$= \lim_{R\to\infty} \left(-\frac{1}{x}\right)\Big|_1^R$$
$$= \lim_{R\to\infty} \left(-\frac{1}{R} + 1\right) = 1$$
So $\int_1^\infty \frac{x}{1+x^3} dx$ converges.

22. $\frac{x^2-2}{x^4+3} \leq \frac{3x^2}{x^4} = 3x^{-2}$
$$\int_1^\infty 3x^{-2} dx = \lim_{R\to\infty} \int_1^R 3x^{-2} dx$$
$$= \lim_{R\to\infty} \left. \frac{-3}{x} \right|_1^R = \lim_{R\to\infty} \frac{-3}{R} + 3 = 3$$
So $\int_1^\infty \frac{x^2-2}{x^4+3} dx$ converges.

23. $\frac{x}{x^{3/2}-1} > \frac{x}{x^{3/2}} = \frac{1}{x^{1/2}} > 0$
$$\int_2^\infty x^{-1/2} dx = \lim_{R\to\infty} \int_2^R x^{-1/2} dx$$
$$= \lim_{R\to\infty} 2\sqrt{x}\Big|_2^R$$
$$= \lim_{R\to\infty} (2\sqrt{R} - 2\sqrt{2}) = \infty$$
So $\int_2^\infty \frac{x}{x^{3/2}-1} dx$ diverges.

24. $\dfrac{2+\sec^2 x}{x} \geq \dfrac{1}{x}$

$\displaystyle\int_1^\infty \frac{1}{x}dx = \lim_{R\to\infty}\int_1^R \frac{1}{x}dx$

$= \displaystyle\lim_{R\to\infty} \ln|x|\Big|_1^R = \lim_{R\to\infty} \ln|R| = \infty$

So $\displaystyle\int_1^\infty \frac{2+\sec^2 x}{x}dx$ diverges.

25. $0 < \dfrac{3}{x+e^x} < \dfrac{3}{e^x}$

$\displaystyle\int_0^\infty \frac{3}{e^x}dx = \lim_{R\to\infty}\int_0^R \frac{3}{e^x}dx$

$= \displaystyle\lim_{R\to\infty} \left(-\frac{3}{e^x}\right)\Big|_0^R$

$= \displaystyle\lim_{R\to\infty} \left(-\frac{3}{e^R}+3\right) = 3$

So $\displaystyle\int_0^\infty \frac{3}{x+e^x}dx$ converges.

26. $e^{-x^3} < e^{-x}$

$\displaystyle\int_1^\infty e^{-x}dx = \lim_{R\to\infty}\int_1^R e^{-x}dx$

$= \displaystyle\lim_{R\to\infty} -e^{-x}\Big|_1^R = \lim_{R\to\infty} -e^{-R}+e^{-1}$

$= e^{-1}$.

So $\displaystyle\int_1^\infty e^{-x^3}dx$ converges.

27. $\dfrac{\sin^2 x}{1+e^x} \leq \dfrac{1}{1+e^x} < \dfrac{1}{e^x}$

$\displaystyle\int_0^\infty \frac{1}{e^x}dx = \lim_{R\to\infty}\int_0^R \frac{1}{e^x}dx$

$= \displaystyle\lim_{R\to\infty} (-e^{-x})\Big|_0^R$

$= \displaystyle\lim_{R\to\infty} (-e^{-R}+1) = 1$

So $\displaystyle\int_0^\infty \frac{\sin^2 x}{1+e^x}dx$ converges.

28. $\dfrac{\ln x}{e^x+1} < \dfrac{x}{e^x}$

$\displaystyle\int_2^\infty \frac{x}{e^x}dx = \lim_{R\to\infty}\int_2^R xe^{-x}dx$

$= \displaystyle\lim_{R\to\infty} (-xe^{-x}-e^{-x})\Big|_2^R$

$= \displaystyle\lim_{R\to\infty} e^{-R}(-R-1)+3e^{-2}$

and $\displaystyle\lim_{R\to\infty} e^{-R}(-R-1)$

$= \displaystyle\lim_{R\to\infty} \frac{-R-1}{e^R} = \lim_{R\to\infty}\frac{-1}{e^R} = 0$

So $\displaystyle\int_2^\infty \frac{\ln x}{e^x+1}dx$ converges.

29. $\dfrac{x^2e^x}{\ln x} > e^x$

$\displaystyle\int_2^\infty e^x dx = \lim_{R\to\infty}\int_2^R e^x dx$

$= \displaystyle\lim_{R\to\infty} e^x\Big|_2^R$

$= \displaystyle\lim_{R\to\infty} (e^R - e^2) = \infty$

So $\displaystyle\int_2^\infty \frac{x^2e^x}{\ln x}dx$ diverges.

30. $e^{x^2+x+1} > e^x$

$\displaystyle\int_1^\infty e^x dx = \lim_{R\to\infty}\int_1^R e^x dx$

$= \displaystyle\lim_{R\to\infty} e^x\Big|_1^R = \lim_{R\to\infty}(e^R - e) = \infty$

So $\displaystyle\int_1^\infty e^{x^2+x+1}dx$ diverges.

31. Let $u = \ln 4x,\ dv = xdx$

$du = \dfrac{dx}{x}, v = \dfrac{x^2}{2}$

$\displaystyle\int x\ln 4x dx = \frac{1}{2}x^2\ln 4x - \frac{1}{2}\int xdx$

$= \dfrac{1}{2}x^2\ln 4x - \dfrac{x^2}{4}+c$

$I = \displaystyle\int_0^1 x\ln 4x dx = \lim_{R\to\ 0^+}\int_R^1 x\ln 4x dx$

$= \displaystyle\lim_{R\to\ 0^+}\left(\frac{1}{2}x^2\ln 4x - \frac{x^2}{4}\right)\Big|_R^1$

$= \displaystyle -\frac{1}{4} - \lim_{R\to\ 0^+}\left(\frac{1}{2}R^2\ln 4R - \frac{R^2}{4}\right)$

$= \displaystyle -\frac{1}{4} - \frac{1}{2}\lim_{R\to\ 0^+} R^2\ln 4R$

$\displaystyle\lim_{R\to\ 0^+} R^2\ln 4R = \lim_{R\to\ 0^+}\frac{\ln 4R}{R^{-2}}$

$= \displaystyle\lim_{R\to\ 0^+}\frac{R^{-1}}{-2R^{-3}} = \lim_{R\to\ 0^+}\frac{R^2}{-2} = 0$

Hence $I = -\dfrac{1}{4}$

32. Let $u = x, dv = e^{-2x}dx$

$du = dx,\ v = -\dfrac{1}{2}e^{-2x}$

$\displaystyle\int xe^{-2x}dx = -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}dx$

$\displaystyle = -\frac{x}{2}e^{-2x} - e^{-2x}$

$\displaystyle\int_0^\infty xe^{-2x}dx = \lim_{R\to\infty}\int_0^R xe^{-2x}dx$

$= \displaystyle\lim_{R\to\infty}\left(-\frac{x}{2}e^{-2x} - e^{-2x}\right)\Big|_0^R$

$= \displaystyle\lim_{R\to\infty} e^{-2R}(-R/2-1)+1$

$= \displaystyle\lim_{R\to\infty} \frac{-R/2-1}{e^{2R}}+1$

$= \lim_{R\to\infty} \frac{-2}{2e^{2R}} + 1 = 1$

33. The volume is finite:
$V = \pi \int_1^\infty \frac{1}{x^2} dx = \pi \lim_{R\to\infty} \int_1^R \frac{1}{x^2} dx$
$= \pi \lim_{R\to\infty} -\frac{1}{x}\Big|_1^R$
$= \pi \lim_{R\to\infty} \left(-\frac{1}{R} + 1\right) = \pi$
The surface area is infinite:
$S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}} dx$
$\frac{1}{x}\sqrt{1 + \frac{1}{x^4}} > \frac{1}{x}$
and $\int_1^\infty \frac{1}{x} dx = \lim_{R\to\infty} \int_1^R \frac{1}{x} dx$
$= \lim_{R\to\infty} \ln|x|\Big|_1^R = \lim_{R\to\infty} \ln R = \infty$

34. The integral $\int_{-\infty}^\infty x^3 dx$ diverges:
$\int_0^\infty x^3 dx = \lim_{R\to\infty} \int_0^R x^3 dx$
$= \lim_{R\to\infty} \frac{x^4}{4}\Big|_0^R = \lim_{R\to\infty} \frac{R^4}{4} = \infty$
The limit $\lim_{R\to\infty} \int_{-R}^R x^3 dx = 0$:
$\lim_{R\to\infty} \int_{-R}^R x^3 dx = \lim_{R\to\infty} \frac{x^4}{4}\Big|_{-R}^R$
$= \lim_{R\to\infty} \left(\frac{R^4}{4} - \frac{R^4}{4}\right) = \lim_{R\to\infty} 0 = 0$

35. True, this statement can be proved using the integration by parts:
$\int f(x) dx = x f(x) - \int g(x) dx,$
where $g(x)$ is some function related to $f(x)$.

36. False, consider $f(x) = \frac{1}{x}$

37. False, consider $f(x) = \ln x$

38. True, this statement is best understood graphically.

39. (a) Substitute $u = \sqrt{k}x$
$\int_{-\infty}^\infty e^{-kx^2} dx = \frac{1}{\sqrt{k}} \int_{-\infty}^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{\sqrt{k}}$

(b) We use integration by parts
($u = e^{-x^2}, v = x$):
$\int e^{-x^2} dx = xe^{-x^2} + 2\int x^2 e^{-x^2} dx$
Since the graph of the function xe^{-x^2} is anti-symmetric across the y-axis,
$\lim_{R\to\infty} \left(xe^{-x^2}\Big|_{-R}^0 + xe^{-x^2}\Big|_0^R \right) = 0$
Then we have
$\int_{-\infty}^\infty e^{-x^2} dx = 2\int_{-\infty}^\infty x^2 e^{-x^2} dx$
And the conclusion is
$\int_{-\infty}^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

40. (a) Since $k > 0$, we have
$\int_0^\infty \frac{\sin kx}{x} dx = \int_0^\infty \frac{\sin kx}{kx} (k) dx$
Let $u = kx, du = kdx$.
$= \int_0^\infty \frac{\sin u}{u} du = \frac{\pi}{2}.$

(b) Since $k < 0$, assume $k = -m$, where $m > 0$.
$\int_0^\infty \frac{\sin kx}{x} dx = \int_0^\infty \frac{\sin(-m)x}{x} dx$
$= -\int_0^\infty \frac{\sin mx}{x} dx$
$= -\int_0^\infty \frac{\sin mx}{mx} (m) dx$
Let $u = mx$, so that $du = mdx$
$= -\int_0^\infty \frac{\sin u}{u} du = -\frac{\pi}{2}.$

(c) Since $k > 0$, we have
$\int_0^\infty \frac{\sin^2 kx}{x^2} dx = \int_0^\infty \frac{\sin^2 kx}{(kx)^2} (k^2) dx$
Let $u = kx,\ du = kdx$.
$= \int_0^\infty \frac{\sin u}{u} k du = k\frac{\pi}{2}.$

(d) Since $k < 0$, assume $k = -m$, where $m > 0$.
$\int_0^\infty \frac{\sin^2 kx}{x^2} dx = \int_0^\infty \frac{\sin^2[(-m)x]}{x^2} dx$
$= \int_0^\infty \frac{\sin^2 mx}{x^2} dx$
$= \int_0^\infty \frac{\sin^2 mx}{(mx)^2} (m^2) dx$
Let $u = mx, du = mdx$.
$= \int_0^\infty m\left(\frac{\sin^2 u}{u^2}\right) du = m\frac{\pi}{2} = -\frac{k\pi}{2}.$

41. Since $\frac{x}{x^5+1} \approx \frac{1}{x^4}$,
$\int_1^\infty \frac{x}{x^5+1} dx \approx \int_1^\infty \frac{1}{x^4} dx.$
we have $\int_1^\infty \frac{1}{x^4} dx$ converges to -$\frac{1}{3}$.

Hence $\int_1^\infty \frac{x}{x^5+1}dx$ also converges.
Let $f(x)=\frac{x}{x^5+1}$ and $g(x)=\frac{1}{x^4}$.
So that, we have, $0<f(x)<g(x)$.
By Comparison test,
$\int_1^\infty \frac{x}{x^5+1}dx<\int_1^\infty \frac{1}{x^4}dx=-\frac{1}{3}$.

42. (a) Let $f(x)=\frac{1}{\sqrt{x^2}}$ and $g(x)=\frac{x}{\sqrt{x^3-1}}$.
So that, we have, $0<f(x)<g(x)$.
By Comparison test,
$\int_2^\infty \frac{1}{\sqrt{x^2}}dx \quad < \quad \int_2^\infty \frac{x}{\sqrt{x^3-1}}dx$, and
$f(x)=\frac{1}{\sqrt{x^2}}$ diverges.
Hence $g(x)=\frac{x}{\sqrt{x^3-1}}$ also diverges.

(b) Let $f(x)=\frac{x}{\sqrt{x^5-1}}$and $g(x)=\frac{1}{x^{5/4}}$.
So that, we have, $0<f(x)<g(x)$.
By Comparison test,
$\int_2^\infty \frac{x}{\sqrt{x^5-1}}dx \quad < \quad \int_2^\infty \frac{1}{x^{5/4}}dx$, and
$g(x)=\frac{1}{x^{5/4}}$ converges.

Hence $f(x)=\frac{x}{\sqrt{x^5-1}}$ also converges.

(c) Let $f(x)=\frac{x}{\sqrt{x^5+x-1}}$ and
$g(x)=\frac{x}{\sqrt{x^5-1}}$.
So that, we have, $0<f(x)<g(x)$.
By Comparison test,
$\int_2^\infty \frac{x}{\sqrt{x^5+x-1}}dx<\int_2^\infty \frac{x}{\sqrt{x^5-1}}dx$,
and $g(x)=\frac{x}{\sqrt{x^5-1}}$ converges.
Hence $f(x) = \frac{x}{\sqrt{x^5+x-1}}$ also converges.

43. Substitute $u=\frac{\pi}{2}-x$

$$\int_0^{\pi/2}\ln(\sin x)dx=-\int_{\pi/2}^0 \ln(\sin(\pi/2-u))du$$
$$=\int_0^{\pi/2}\ln(\cos u)du=\int_0^{\pi/2}\ln(\cos x)dx$$

Moreover,

$$2\int_0^{\pi/2}\ln(\sin x)dx$$
$$=\int_0^{\pi/2}\ln(\cos x)dx+\int_0^{\pi/2}\ln(\sin x)dx$$
$$=\int_0^{\pi/2}\left[\ln(\cos x)+\ln(\sin x)\right]dx$$
$$=\int_0^{\pi/2}\ln(\sin x\cos x)dx$$
$$=\int_0^{\pi/2}[\ln(\sin(2x))-\ln 2]dx$$
$$=\int_0^{\pi/2}\ln(\sin(2x))dx-\frac{\pi}{2}\ln 2$$
$$=\frac{1}{2}\int_0^{\pi}\ln(\sin x)dx-\frac{\pi}{2}\ln 2$$

Hence,

$$2\int_0^{\pi/2}\ln(\sin x)dx$$
$$=\frac{1}{2}\int_0^{\pi}\ln(\sin x)dx-\frac{\pi}{2}\ln 2$$

On the other hand, we notice that the graph of $\sin x$ is symmetric over the interval $[0,\pi]$ across the line $x=\pi/2$, hence

$$\int_0^{\pi}\ln(\sin x)dx=2\int_0^{\pi/2}\ln(\sin x)dx$$

and then

$$\frac{1}{2}\int_0^{\pi}\ln(\sin x)dx=\int_0^{\pi/2}\ln(\sin x)dx$$

So we get $\int_0^{\pi/2}\ln(\sin x)dx=-\frac{\pi}{2}\ln 2$

44. $\int_0^1 (\ln x)^n dx=\lim_{t\to 0^+}\int_t^1 (\ln x)^n dx$

$$=\lim_{t\to 0^+}\left[x(\ln x)^n|_t^1-n\int_t^1 (\ln x)^{n-1}dx\right]$$

Using#112 from the table.

$$=\lim_{t\to 0^+}\left[\left(0-t(\ln t)^2\right)\right]$$
$$-\lim_{t\to 0^+}\left[n\int_t^1 (\ln x)^{n-1}dx\right]$$
$$=0-\lim_{t\to 0^+}\left[n\int_t^1 (\ln x)^{n-1}dx\right].$$

Continuing in the same manner,

$$\int_0^1 (\ln x)^n dx$$
$$=(-1)^{n-1}n!\left[\lim_{t\to 0^+}\int_t^1 (\ln x)\,dx\right]$$
$$=(-1)^{n-1}n!\left[\lim_{t\to 0^+}(x\ln x-x)\right]_t^1$$
$$=(-1)^{n-1}n!\left[\lim_{t\to 0^+}((0-t\ln t)-(1-t))\right]$$
$$=(-1)^n n!.$$

45. Improper because $\tan(\pi/2)$ is not defined. The two integrals

$$\int_0^{\pi/2}\frac{1}{1+\tan x}dx=\int_0^{\pi/2}f(x)dx$$

because the two integrand only differ at one value of x, and that except for this value, everything is proper.

$$g(x) = \begin{cases} \dfrac{\tan x}{1+\tan x} \text{ if } 0 \le x < \dfrac{\pi}{2} \\ 0 \text{ if } x = \dfrac{\pi}{2} \end{cases}$$

Substitute $u = x - \dfrac{\pi}{2}$ followed by $w = -u$

$$\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \int_{-\pi/2}^{0} \frac{1}{1-\cot u}\,du$$
$$= -\int_{\pi/2}^{0} \frac{1}{1+\cot w}\,dw$$
$$= \int_0^{\pi/2} \frac{1}{1+\dfrac{\cos w}{\sin w}}\,dw$$
$$= \int_0^{\pi/2} \frac{\sin w}{\sin w+\cos w}\,dw$$
$$= \int_0^{\pi/2} \frac{\tan w}{\tan w+1}\,dw$$
$$= \int_0^{\pi/2} \frac{\tan x}{\tan x+1}\,dx$$

Moreover,

$$\int_0^{\pi/2} \frac{\tan x}{\tan x+1}\,dx + \int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$$
$$= \int_0^{\pi/2} \left(\frac{\tan x}{1+\tan x} + \frac{1}{1+\tan x}\right)dx$$
$$= \int_0^{\pi/2} 1\,dx = x\Big|_0^{\pi/2} = \frac{\pi}{2}$$

Hence $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$

46. As in exercise.45, We have

$$\int_0^{\pi/2} \frac{1}{1+\tan^k x}dx = \int_0^{\pi/2} \frac{\tan^k x}{1+\tan^k x}dx$$

hence,

$$2\int_0^{\pi/2} \frac{1}{1+\tan^k x}dx$$
$$= \int_0^{\pi/2} \frac{1}{1+\tan^k x}dx + \int_0^{\pi/2} \frac{\tan^k x}{1+\tan^k x}dx$$
$$= \int_0^{\pi/2} 1dx = \frac{\pi}{2}$$

therefore,

$$\int_0^{\pi/2} \frac{1}{1+\tan^k x}dx = \frac{\pi}{4}$$

47. Use integration by parts twice, first time let $u = -\dfrac{1}{2}x^3, dv = ds - 2xe^{-x^2}dx$

second time

let $u = -\dfrac{1}{2}x, dv = -2xe^{-x^2}dx$

$$\int x^4 e^{-x^2}dx$$
$$= -\frac{1}{2}x^3e^{-x^2} + \int \frac{3}{2}x^2e^{-x^2}dx$$
$$= -\frac{1}{2}x^3e^{-x^2} + \frac{3}{2}\left(-\frac{1}{2}xe^{-x^2} + \frac{1}{2}\int e^{-x^2}dx\right)$$
$$= -\frac{1}{2}x^3e^{-x^2} - \frac{3}{4}xe^{-x^2} + \frac{3}{4}\int e^{-x^2}dx$$

Putting integration limits to all the above, and realizing that when taking limits to $\pm\infty$, all multiples of e^{-x^2} as shown in above will go to 0 (we have seen this a lot of times before). Then we get

$$\int_{-\infty}^{\infty} x^4e^{-x^2}dx = \frac{3}{4}\int_{-\infty}^{\infty} e^{-x^2}dx = \frac{3}{4}\sqrt{\pi}$$

This means when $n = 2$, the statement

$$\int_{-\infty}^{\infty} x^{2n}e^{-x^2}dx = \frac{(2n-1)\cdots 3\cdot 1}{2^n}\sqrt{\pi}$$

is true. (We can also check that the case for $n = 1$ is correct.) For general n, supposing that the statement is true for all $m < n$, then integration by parts gives

$$\int x^{2n}e^{-x^2}dx$$
$$= -\frac{1}{2}x^{2n-1}e^{-x^2} + \frac{2n-1}{2}\int x^{2n-2}e^{-x^2}dx$$

and hence

$$\int_{-\infty}^{\infty} x^{2n}e^{-x^2}dx$$
$$= \frac{2n-1}{2}\int_{-\infty}^{\infty} x^{2n-2}e^{-x^2}dx$$
$$= \frac{2n-1}{2}\cdot\frac{(2n-3)\cdots 3\cdot 1}{2^{n-1}}\sqrt{\pi}$$
$$= \frac{(2n-1)\cdots 3\cdot 1}{2^n}\sqrt{\pi}$$

48. Substitute $u = \sqrt{a}x$

$$\int e^{-ax^2}dx = \frac{1}{\sqrt{a}}\int e^{-u^2}du$$

and then

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\frac{\pi}{a}}$$

Ignoring issues of convergence, the derivatives can be taken from the integrand, we get the following:

1st derivative

$$\frac{d}{da}\int_{-\infty}^{\infty} e^{-ax^2}dx = \frac{d}{da}\sqrt{\frac{\pi}{a}}$$
$$-\int_{-\infty}^{\infty} x^2e^{-ax^2}dx = -\frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

2nd derivative

$$\frac{d^2}{da^2}\int_{-\infty}^{\infty} e^{-ax^2}dx = \frac{d^2}{da^2}\sqrt{\frac{\pi}{a}}$$
$$\int_{-\infty}^{\infty} x^4e^{-ax^2}dx = \frac{3}{4}\sqrt{\frac{\pi}{a^5}}$$

… nth derivative

$$\frac{d^n}{da^n}\int_{-\infty}^{\infty} e^{-ax^2}dx = \frac{d^n}{da^n}\sqrt{\frac{\pi}{a}}$$

$(-1)^n \int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx$
$= (-1)^n \frac{(2n-1)\cdots 3\cdot 1}{2^n}\sqrt{\frac{\pi}{a^{2n+1}}}$
Setting $a = 1$, we get the result of Exercise 47.

49. (a) $\int_0^{\infty} ke^{-2x}dx = \lim_{R\to\infty}\int_0^R ke^{-2x}dx$
$= -\frac{k}{2}\lim_{R\to\infty} e^{-2x}\Big|_0^R$
$= -\frac{k}{2}\lim_{R\to\infty}(e^{-2R}-1) = \frac{k}{2} = 1$
So $k = 2$

(b) $\int_0^{\infty} ke^{-4x}dx = \lim_{R\to\infty}\int_0^R ke^{-4x}dx$
$= -\frac{k}{4}\lim_{R\to\infty} e^{-4x}\Big|_0^R$
$= -\frac{k}{4}\lim_{R\to\infty}(e^{-4R}-1) = \frac{k}{4} = 1$
So $k = 4$

(c) If $r > 0$:
$\int_0^{\infty} ke^{-rx}dx = \lim_{R\to\infty}\int_0^R ke^{-rx}dx$
$= -\frac{k}{r}\lim_{R\to\infty} e^{-rx}\Big|_0^R$
$= -\frac{k}{r2}\lim_{R\to\infty}(e^{-rR}-1) = \frac{k}{r} = 1$
So $k = r$

50. (a) Substitute $u = 2x$
$\int_0^{\infty} kxe^{-2x}dx = \lim_{R\to\infty}\int_0^R kxe^{-2x}dx$
$= \frac{k}{4}\lim_{R\to\infty}\int_0^R ue^{-u}dx$
$= \frac{k}{4}\lim_{R\to\infty}(ue^{-u}+e^{-u})\Big|_0^R$
$= \frac{k}{4}\lim_{R\to\infty}\left(\frac{R+1}{e^R}-1\right) = -\frac{k}{4} = 1$
So $k = -4$

(b) Substitute $u = 4x$
$\int_0^{\infty} kxe^{-4x}dx = \lim_{R\to\infty}\int_0^R kxe^{-4x}dx$
$= \frac{k}{16}\lim_{R\to\infty}\int_0^R ue^{-u}dx$
$= \frac{k}{16}\lim_{R\to\infty}(ue^{-u}+e^{-u})\Big|_0^R$
$= \frac{k}{16}\lim_{R\to\infty}\left(\frac{R+1}{e^R}-1\right) = -\frac{k}{16} = 1$
So $k = -16$

(c) If $r > 0$:
Substitute $u = rx$
$\int_0^{\infty} kxe^{-rx}dx = \lim_{R\to\infty}\int_0^R kxe^{-rx}dx$
$= \frac{k}{r^2}\lim_{R\to\infty}\int_0^R ue^{-u}dx$
$= \frac{k}{r^2}\lim_{R\to\infty}(ue^{-u}+e^{-u})\Big|_0^R$
$= \frac{k}{r^2}\lim_{R\to\infty}\left(\frac{R+1}{e^R}-1\right) = -\frac{k}{r^2} = 1$
So $k = -r^2$
If $r \le 0$:
The integral $\int_0^{\infty} kxe^{-rx}dx$ diverges for any value of k, so there is no value of k to make the function $f(x) = k$ a pdf.

51. From Exercise 49 (c) we know that this r has to be positive.
Substitute $u = rx$
$\mu = \int_0^{\infty} xf(x)dx = \int_0^{\infty} rxe^{-rx}dx$
$= \lim_{R\to\infty}\int_0^R rxe^{-rx}dx$
$= \frac{1}{r}\lim_{R\to\infty}\int_0^R ue^{-u}du$
$= \frac{1}{r}\lim_{R\to\infty} e^{-u}(-u-1)\Big|_0^R$
$= \lim_{R\to\infty}\frac{-R-1}{e^R} + \frac{1}{r} = 0 + \frac{1}{r} = \frac{1}{r}$

52. From Exercise 50 (c) we know that this r has to be positive.
Substitute $u = rx$
$\mu = \int_0^{\infty} xf(x)dx = \int_0^{\infty} r^2x^2e^{-rx}dx$
$= \lim_{R\to\infty}\int_0^R r^2x^2e^{-rx}dx$
$= \frac{1}{r}\lim_{R\to\infty}\int_0^R u^2e^{-u}du$
$= \frac{1}{r}\lim_{R\to\infty} e^{-u}(-u^2-2u-2)\Big|_0^R$
$= \lim_{R\to\infty}\frac{-R^2-2R-2}{e^R} + \frac{2}{r} = 0 + \frac{2}{r} = \frac{2}{r}$

53. $\int_0^{35}\frac{1}{40}e^{-x/40}dx = -e^{-x/40}\Big|_0^{35} = 1 - e^{-35/40}$

$P(x > 35) = 1 - \text{above } = e^{-35/40}$

$\int_0^{40}\frac{1}{40}e^{-x/40}dx = -e^{-x/40}\Big|_0^{40} = 1 - e^{-40/40}$

$P(x > 40) = 1 - \text{ above } = e^{-40/40}$

$\int_0^{45}\frac{1}{40}e^{-x/40}dx = -e^{-x/40}\Big|_0^{45} = 1 - e^{-45/40}$

$P(x > 45) = 1 - \text{ above } = e^{-45/40}$

Hence,

$P(x > 40|x > 35) = \dfrac{P(x > 40)}{P(x > 35)}$
$= \dfrac{e^{-40/40}}{e^{-35/40}} = e^{-5/40} \approx 0.8825$
$P(x > 45|x > 40) = \dfrac{P(x > 45)}{P(x > 40)}$
$= \dfrac{e^{-45/40}}{e^{-40/40}} = e^{-5/40} \approx 0.8825$

54. (a) Following Exercise 53, we get
$P(x > m+n|x > n) = \dfrac{P(x > m+n)}{P(x > n)}$
$= \dfrac{1 - \int_0^{m+n} \frac{1}{40} e^{-x/40} dx}{1 - \int_0^{m} \frac{1}{40} e^{-x/40} dx}$
$= \dfrac{e^{-(m+n)/40}}{e^{-n/40}} = e^{-m/40}$

(b) $\int_0^A ce^{-cx} dx = -e^{-cx}\Big|_0^A = 1 - e^{-cA}$
$P(x > m+n|x > n) = \dfrac{P(x > m+n)}{P(x > n)}$
$= \dfrac{1 - \int_0^{m+n} ce^{-cx} dx}{1 - \int_0^{m} ce^{-cx} dx dx} = \dfrac{e^{-c(m+n)}}{e^{-cn}}$
$= e^{-cm}$

55. (a) For $x \ge 0$,
$F_1(x) = \int_{-\infty}^{x} f_1(t) dt = \int_0^x f_1(t) dt$
$= \int_0^x 2e^{-2t} dt = -e^{-2t}\Big|_0^x = 1 - e^{-2x}$
$\Omega_1(r) = \dfrac{\int_r^\infty [1 - F_1(x)] dx}{\int_{-\infty}^r F_1(x) dx}$
$= \dfrac{\int_r^\infty e^{-2x} dx}{\int_0^r (1 - e^{-2x}) dx}$
$= \dfrac{\frac{1}{2} e^{-2r}}{r + \frac{1}{2} e^{-2r} - \frac{1}{2}} = \dfrac{e^{-2r}}{2r + e^{-2r} - 1}$

(b) For $0 \le x \le 1$, $F_2(x) = \int_{-\infty}^{x} f_2(t) dt$
$= \int_0^x f_2(t) dt = \int_0^x 1\, dt = t\Big|_0^x = x$
$\Omega_2(r) = \dfrac{\int_r^\infty [1 - F_2(x)] dx}{\int_{-\infty}^r F_2(x) dx}$
$= \dfrac{\int_r^1 (1 - x) dx}{\int_0^r x dx} = \dfrac{\frac{1}{2} - r + \frac{r^2}{2}}{\frac{r^2}{2}}$
$= \dfrac{1 - 2r + r^2}{r^2}$

(c) $\mu_1 = \int_{-\infty}^{\infty} x f_1(x) dx$
$= \int_0^\infty 2xe^{-2x} dx$
$= \lim_{R\to\infty} \int_0^R 2xe^{-2x} dx$
$= \lim_{R\to\infty} e^{-2x}(-x - 1/2)\Big|_0^R$
$= \lim_{R\to\infty} e^{-2R}(R + 1/2) + \dfrac{1}{2} = \dfrac{1}{2}$
$\mu_2 = \int_{-\infty}^{\infty} x f_2(x) dx$
$= \int_0^1 x dx = \dfrac{x^2}{2}\Big|_0^1$
$= \dfrac{1}{2}$
$\mu_1 = \mu_2$ and when $r = 1/2$
$\Omega_1(1/2) = \dfrac{e^{-2\cdot 1/2}}{2 \cdot 1/2 + e^{-2\cdot 1/2} - 1} = 1$
$\Omega_2(1/2) = \dfrac{1 - 2 \cdot 1/2 + (1/2)^2}{(1/2)^2} = 1$

(d) The graph of $f_2(x)$ is more stable than that of $f_1(x)$.
$f_1(x) > f_2(x)$ for $0 < x < 0.34$
and $f_1(x) < f_2(x)$ for $x > 1$.

(e) $\Omega_1(r) = 1 - \dfrac{2r - 1}{e^{-2r} + 2r - 1}$
$\Omega_2 r = 1 - \dfrac{2r - 1}{r^2}$
and
$r^2 - (e^{-2r} + 2r - 1) = e^{-2r} + (r - 1)^2 > 0$
This means
when $r < 1/2$, $\Omega_1(r) < \Omega_2(r)$
when $r > 1/2$, $\Omega_1(r) > \Omega_2(r)$
In terms of this example, we see that the riskier investment is only disadvantageous when r small, and will be better when r large.

56. Following Exercise 54(b),
$R(t) = P(x \ge t) = P(x > t)$
$= 1 - \int_0^t f(x) dx = 1 - \int_0^t ce^{-cx} dx$
$= 1 - (1 - e^{-ct}) = e^{-ct}$

57. Graph of $p_1(x)$:

$$\int_0^1 p_1(x)\, dx = \int_0^1 1 dx = 1,$$

Graph of $p_2(x)$:

Similarly,
$$\int_0^1 p_2(x)\, dx = \int_0^{1/2} 4x dx + \int_{1/2}^1 (4 - 4x)\, dx$$
$$= 2x^2\Big|_0^{1/2} + \left(4x - 2x^2\right)_{1/2}^1$$
$$= \left(\frac{1}{2} - 0\right) + \left((4 - 2) - \left(2 - \frac{1}{2}\right)\right) = 1.$$

The Boltzmann integral
$I(p_1) = \int_0^1 p_1(x) \ln p_1(x)\, dx$
$= \int_0^1 1 \ln 1 dx = 0.$
Also, $I(p_2) = \int_0^1 p_2(x) \ln p_2(x)\, dx$
$= \int_0^{1/2} 4x \ln(4x)\, dx$
$+ \int_{1/2}^1 (4-4x) \ln(4-4x)\, dx$
Let $u = 4x, du = 4dx$
and $t = 4-4x, dt = -4dx$
$= \frac{1}{4}\int_0^2 u \ln u du - \frac{1}{4}\int_2^0 t \ln t dt$
$= \frac{1}{2}\left(\frac{1}{2}u^2 \ln u - \frac{1}{4}u^2\right)_0^2$
$= \frac{1}{2}(2\ln 2 - 1) = 0.193147.$

For the pdf $p_2(x)$, the probability at $x = \frac{1}{2}$ is maximum which is equal to $\frac{1}{2}$.The probability decreases as x tends to 0 or 1.

$$p_3(x) = \begin{cases} 2x & 0 \le x < \frac{1}{4} \\ 10x-2 & \frac{1}{4} \le x < \frac{1}{2} \\ 8-10x & \frac{1}{2} \le x < \frac{3}{4} \\ 2-2x & \frac{1}{4} < x \le 1 \end{cases}$$

Graph of $p_3(x)$:
$\int_0^1 p_3(x)\, dx = \int_0^{1/4} 2x dx + \int_{1/4}^{1/2} (10x-2) dx$
$+ \int_{1/2}^{3/4} (8-10x) dx + \int_{3/4}^1 (2-2x)\, dx$
$= 1.$

Also, The Boltzmann integral
$I(p_3) = \int_0^1 p_3(x) \ln p_3(x)\, dx$
$= \int_0^{1/4} 2x \ln(2x)\, dx$
$+ \int_{1/4}^{1/2} (10x-1) \ln(10x-1) dx$
$+ \int_{1/2}^{3/4} (8-10x) \ln(8-10x) dx$
$+ \int_{3/4}^1 (2-2x) \ln(2-2x)\, dx$
$= 0.42.$

Ch. 6 Review Exercises

1. Substitute $u = \sqrt{x}$
$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = 2\int e^u\, du = 2e^u + c = 2e^{\sqrt{x}} + c$

2. Substitute $u = \frac{1}{x}$
$\int \frac{\sin(1/x)}{x^2}\, dx = -\int \sin u\, du$
$= \cos u + c = \cos(1/x) + c$

3. Use the table of integrals,
$\int \frac{x^2}{\sqrt{1-x^2}}\, dx = -\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x + c$

4. Use the table of integrals,
$\int \frac{2}{\sqrt{9-x^2}}\, dx = 2\sin^{-1}\frac{x}{3} + c$

5. Use integration by parts, twice:
$\int x^2 e^{-3x}\, dx$
$= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int x e^{-3x}\, dx$
$= -\frac{1}{3}x^2 e^{-3x}$
$+ \frac{2}{3}\left(-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\, dx\right)$
$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + c$

6. Substitute $u = x^3$
$\int x^2 e^{-x^3}\, dx = \frac{1}{3}\int e^{-u}\, du = \frac{1}{3}e^{-x^3} + c$

7. Substitute $u = x^2$
$\int \frac{x}{1+x^4}\, dx = \frac{1}{2}\int \frac{du}{1+u^2}$
$= \frac{1}{2}\tan^{-1} u + c = \frac{1}{2}\tan^{-1} x^2 + c$

8. $\frac{x^3}{1+x^4}\, dx = \frac{1}{4}\ln(1+x^4) + c$

9. $\frac{x^3}{4+x^4}\, dx = \frac{1}{4}\ln(4+x^4) + c$

10. Substitute $u = x^2$
$\int \frac{x}{4+x^4}\, dx = \frac{1}{2}\int \frac{du}{4+u^2}$
$= \frac{1}{4}\tan^{-1}\frac{u}{2} + c = \frac{1}{4}\tan^{-1}\frac{x^2}{2} + c$

11. $\int e^{2\ln x}\, dx = \int x^2\, dx = \frac{x^3}{3} + c$

12. $\int \cos 4x\, dx = \frac{1}{4}\sin 4x + c$

13. Integration by parts,

$$\int_0^1 x\sin 3x\,dx$$
$$= -\frac{1}{3}x\cos 3x\Big|_0^1 + \frac{1}{3}\int_0^1 \cos 3x\,dx$$
$$= -\frac{1}{3}\cos 3 + \frac{1}{9}\sin 3x\Big|_0^1$$
$$= -\frac{1}{3}\cos 3 + \frac{1}{9}\sin 3$$

14. Substitute $u = x^2$

$$\int_0^1 x\sin 4x^2\,dx = \int_0^1 \frac{1}{2}\sin 4u\,du$$
$$= -\frac{1}{8}\cos 4u\Big|_0^1 = \frac{1}{8}(1-\cos 4)$$

15. Use the table of integrals

$$\int_0^{\pi/2} \sin^4 x\,dx$$
$$= -\frac{1}{4}\sin^3 x\cos x\Big|_0^{\pi/2}$$
$$+ \frac{3}{4}\left(\frac{x}{2} - \frac{1}{2}\sin x\cos x\right)\Big|_0^{\pi/2}$$
$$= \frac{3\pi}{16}$$

16. Use the table of integrals

$$\int_0^{\pi/2} \cos^3 x\,dx$$
$$= \left(\frac{2}{3}\sin x + \frac{1}{3}\sin x\cos^2 x\right)\Big|_0^{\pi/2} = \frac{2}{3}$$

17. Use integration by parts,

$$\int_{-1}^1 x\sin \pi x\,dx$$
$$= -\frac{1}{\pi}x\cos \pi x\Big|_{-1}^1 + \frac{1}{\pi}\int_{-1}^1 \cos \pi x\,dx$$
$$= \frac{2}{\pi} + \frac{1}{\pi^2}\sin \pi x\Big|_{-1}^1 = \frac{2}{\pi}$$

18. Use integration by parts, twice

$$\int_0^1 x^2\cos \pi x\,dx$$
$$= \frac{1}{\pi}x^2\sin \pi x\Big|_0^1 - \frac{2}{\pi}\int_0^1 x\sin \pi x\,dx$$
$$= -\frac{2}{\pi}\left(-\frac{1}{\pi}x\cos \pi x\Big|_0^1 + \frac{1}{\pi}\int_0^1 \cos \pi x\,dx\right)$$
$$= -\frac{2}{\pi}\left(\frac{1}{\pi} + \frac{1}{\pi^2}\sin \pi x\Big|_0^1\right) = -\frac{2}{\pi^2}$$

19. Use integration by parts

$$\int_1^2 x^3\ln x\,dx = \frac{x^4}{4}\ln x\Big|_1^2 - \frac{1}{4}\int_1^2 x^3\,dx$$
$$= 4\ln 2 - \frac{x^4}{16}\Big|_1^2 = 4\ln 2 - \frac{15}{16}$$

20. $\displaystyle\int_0^{\pi/4} \sin x\cos x\,dx = \int_0^{\pi/4} \frac{1}{2}\sin 2x\,dx$

$$= -\frac{1}{4}\cos 2x\Big|_0^{\pi/4} = \frac{1}{4}$$

21. Substitute $u = \sin x$

$$\int \cos x\sin^2 x\,dx = \int u^2\,du$$
$$= \frac{u^3}{3} + c = \frac{\sin^3 x}{3} + c$$

22. Substitute $u = \sin x$

$$\int \cos x\sin^3 x\,dx = \int u^3\,du$$
$$= \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c$$

23. Substitute $u = \sin x$

$$\int \cos^3 x\sin^3 x\,dx = \int (1-u^2)u^3\,du$$
$$= \frac{u^4}{4} - \frac{u^6}{6} + c = \frac{3\sin^4 x - 2\sin^6 x}{12} + c$$

24. Substitute $u = \cos x$

$$\int \cos^4 x\sin^3 x\,dx = -\int u^4(1-u^2)\,du$$
$$= -\frac{u^5}{5} + \frac{u^7}{7} + c = \frac{-7u^5 + 5u^7}{35} + c$$

25. Substitute $u = \tan x$

$$\int \tan^2 x\sec^4 x\,dx = \int u^2(1+u^2)\,du$$
$$= \frac{u^3}{3} + \frac{u^5}{5} + c = \frac{5\tan^3 x + 3\tan^5 x}{15} + c$$

26. Substitute $u = \tan x$

$$\int \tan^3 x\sec^2 x\,dx = \int u^3\,du$$
$$= \frac{u^4}{4} + c = \frac{\tan^4 x}{4} + c$$

27. Substitute $u = \sin x$

$$\int \sqrt{\sin x}\cos^3 x\,dx$$
$$= \int u^{1/2}(1-u^2)\,du = \frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2} + c$$
$$= \frac{2}{3}\sin^{3/2} x - \frac{2}{7}\sin^{7/2} x + c$$

28. Substitute $u = \sec x$

$$\int \tan^3 x\sec^3 x\,dx = \int (u^2-1)u^2\,du$$
$$= \frac{u^5}{5} - \frac{u^3}{3} + c = \frac{3\sec^5 x - 5\sec^3 x}{15} + c$$

29. Complete the square,

$$\int \frac{2}{8+4x+x^2}\,dx$$
$$= \int \frac{2}{(x+2)^2+2^2}\,dx = \tan^{-1}\left(\frac{x+2}{2}\right) + c$$

30. Complete the square,
$$\int \frac{3}{\sqrt{-2x-x^2}}\,dx$$
$$= \int \frac{3}{\sqrt{1-(x-1)^2}}\,dx = 3\sin^{-1}(x-1)+c$$

31. Use the table of integrals,
$$\int \frac{2}{x^2\sqrt{4-x^2}}\,dx = -\frac{\sqrt{4-x^2}}{2x}+c$$

32. Substitute $u = 9 - x^2$
$$\int \frac{x}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int \frac{du}{u^{1/2}}$$
$$= -\frac{1}{3}u^{3/2}+c = -\frac{1}{3}(9-x^2)^{3/2}+c$$

33. Substitute $u = 9 - x^2$
$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int \frac{(9-u)}{u^{1/2}}\,du$$
$$= -\frac{9}{2}\int u^{-1/2}\,du + \frac{1}{2}\int u^{1/2}\,du$$
$$= -9u^{1/2}+\frac{1}{3}u^{3/2}+c$$
$$= -9(9-x^2)^{1/2}+\frac{1}{3}(9-x^2)^{3/2}+c$$

34. Substitute $u = x^2 - 9$

$$\int \frac{x^3}{\sqrt{x^2-9}}\,dx = \frac{1}{2}\int (u+9)u^{-1/2}\,du$$
$$= \frac{1}{3}u^{3/2}+9u^{1/2}+c$$
$$= \frac{1}{3}(9-x^2)^{3/2}+9(9-x^2)^{1/2}+c$$

35. Substitute $u = x^2 + 9$
$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int (u-9)u^{-1/2}\,du$$
$$= \frac{1}{3}u^{3/2}-9u^{1/2}+c$$
$$= \frac{1}{3}(x^2+9)^{3/2}-9(x^2+9)^{1/2}+c$$

36. Substitute $u = x + 9$
$$\int \frac{4}{\sqrt{x+9}}\,dx = 4\int \frac{du}{\sqrt{u}}$$
$$= 8u^{1/2}+c = 8\sqrt{x+9}+c$$

37. Use the method of PFD
$$\int \frac{x+4}{x^2+3x+2}\,dx$$
$$= \int \left(\frac{3}{x+1}+\frac{-2}{x+2}\right)dx$$
$$= 3\ln|x+1| - 2\ln|x+2|+c$$

38. Use the method of PFD
$$\int \frac{5x+6}{x^2+x-12}\,dx$$
$$= \int \left(\frac{3}{x-3}+\frac{3}{x+4}\right)dx$$
$$= 3\ln|x-3| + 3\ln|x+4|+c$$

39. Use the method of PFD
$$\int \frac{4x^2+6x-12}{x^3-4x}\,dx$$
$$= \int \left(\frac{3}{x}+\frac{-1}{x+2}+\frac{2}{x-2}\right)dx$$
$$= 3\ln|x| - \ln|x+2| + 2\ln|x-2|+c$$

40. Use the method of PFD
$$\int \frac{5x^2+2}{x^3+x}\,dx = \int \left(\frac{2}{x}+\frac{3x}{x^2+1}\right)dx$$
$$= 2\ln|x| + \frac{3}{2}\ln(x^2+1)+c$$

41. Use the table of integrals,
$$\int e^x\cos 2x\,dx$$
$$= \frac{(\cos 2x + 2\sin 2x)e^x}{5}+c$$

42. Substitute $u = x^2$ followed by integration by parts
$$\int x^3\sin x^2\,dx = \frac{1}{2}\int u\sin u\,du$$
$$= -\frac{1}{2}u\cos u + \frac{1}{2}\int \cos u\,du$$
$$= -\frac{1}{2}u\cos u + \frac{1}{2}\sin u + c$$
$$= -\frac{1}{2}x^2\cos x^2 + \frac{1}{2}\sin x^2 + c$$

43. Substitute $u = x^2 + 1$
$$\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int u^{1/2}\,du$$
$$= \frac{1}{3}u^{3/2}+c = \frac{1}{3}(x^2+1)^{3/2}+c$$

44. Use the table of integrals
$$\int \sqrt{1-x^2}\,dx$$
$$= \frac{1}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x+c$$

45. $\dfrac{4}{x^2-3x-4} = \dfrac{A}{x+1}+\dfrac{B}{x-4}$
$$4 = A(x-4)+B(x+1)$$
$$= (A+B)x + (-4A+B)$$
$$A = -\frac{4}{5};\ B = \frac{4}{5}$$
$$\frac{4}{x^2-3x-4} = \frac{-4/5}{x+1}+\frac{4/5}{x-4}$$

46. $\dfrac{2x}{x^2+x-6} = \dfrac{A}{x-2}+\dfrac{B}{x+3}$
$$2x = A(x+3)+B(x-2)$$
$$= (A+B)x + (3A-2B)$$

$A = \frac{4}{5}; B = \frac{6}{5}$
$\frac{2x}{x^2+x-6} = \frac{4/5}{x-2} + \frac{6/5}{x+3}$

47. $\frac{-6}{x^3+x^2-2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$
$-6 = A(x-1)(x+2) + Bx(x+2) + cx(x-1)$
$A = -3; B = -2; C = -1$
$\frac{-6}{x^3+x^2-2x} = \frac{-3}{x} + \frac{-2}{x-1} + \frac{-1}{x+2}$

48. $\frac{x^2-2x-2}{x^3+x} = \frac{A}{x} + \frac{Bx+c}{x^2+1}$
$x^2-2x-2 = A(x^2+1) + (Bx+c)x$
$= (A+B)x^2 + cx + A$
$A = -2; B = 3; C = -2$
$\frac{x^2-2x-2}{x^3+x} = \frac{-2}{x} + \frac{3x-2}{x^2+1}$

49. $\frac{x-2}{x^2+4x+4} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$
$x-2 = A(x+2) + B$
$A = 1; B = -4$
$\frac{x-2}{x^2+4x+4} = \frac{1}{x+2} + \frac{-4}{(x+2)^2}$

50. $\frac{x^2-2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$
$x^2-2 = (Ax+B)(x^2+1) + cx + D$
$A = 0; B = 1; C = 0; D = -3$

$\frac{x^2-2}{(x^2+1)^2} = \frac{1}{x^2+1} + \frac{-3}{(x^2+1)^2}$

51. Substitute $u = e^{2x}$
$\int e^{3x}\sqrt{4+e^{2x}}\,dx$
$= \int e^{2x}\sqrt{4e^{2x}+e^{4x}}dx = \frac{1}{2}\int \sqrt{4u+u^2}\,du$
$= \frac{1}{2}\int \sqrt{(u+2)^2-4}\,du$
$= \frac{1}{4}(u+2)\sqrt{(u+2)^2-4}$
$- \ln|(u+2) + \sqrt{(u+2)^2-4}| + c$
$= \frac{(e^{2x}+2)\sqrt{4e^{2x}+e^{4x}}}{4}$
$- \ln\left|(e^{2x}+2) + \sqrt{4e^{2x}+e^{4x}}\right| + c$

52. Substitute $u = x^2$
$\int x\sqrt{x^4-4}\,dx = \frac{1}{2}\int \sqrt{u^2-4}\,du$
$= \frac{u\sqrt{u^2-4}}{4} - \ln|u + \sqrt{u^2-4}| + c$
$= \frac{x^2\sqrt{x^4-4}}{4} - \ln|x^2 + \sqrt{x^4-4}| + c$

53. $\int \sec^4 x\,dx$
$= \frac{1}{3}\sec^2 x\tan x + \frac{2}{3}\int \sec^2 x\,dx$
$= \frac{1}{3}\sec^2 x\tan x + \frac{2}{3}\tan x + c$

54. $\int \tan^5 x\,dx$
$= \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx$
$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \int \tan x\,dx$
$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + c$

55. Substitute $u = 3 - x$
$\int \frac{4}{x(3-x)^2}\,dx = -4\int \frac{1}{(3-u)u^2}\,du$
$= \frac{4}{9}\ln\left|\frac{3-u}{u}\right| + \frac{4}{3u} + c$
$= \frac{4}{9}\ln\left|\frac{x}{3-x}\right| + \frac{4}{3(3-x)} + c$

56. Substitute $u = \sin x$
$\int \frac{\cos x}{\sin^2 x(3+4\sin x)}\,dx = \int \frac{du}{u^2(3+4u)}$

$= \frac{4}{9}\ln\left|\frac{3+4u}{u}\right| - \frac{1}{3u} + c$
$= \frac{4}{9}\ln\left|\frac{3+4\sin x}{\sin x}\right| - \frac{1}{3\sin x} + c$

57. $\int \frac{\sqrt{9+4x^2}}{x^2}\,dx = \int \frac{2\sqrt{\frac{9}{4}+x^2}}{x^2}\,dx$
$= 2\left(\frac{-\sqrt{\frac{9}{4}+x^2}}{x} + \ln\left|x + \sqrt{9/4+x^2}\right|\right) + c$
$= -\frac{\sqrt{9+4x^2}}{x} + 2\ln\left|x + \sqrt{\frac{9}{4}+x^2}\right| + c$

58. $\int \frac{x^2}{\sqrt{4-9x^2}}\,dx = \frac{1}{3}\int \frac{x^2}{\sqrt{4/9-x^2}}\,dx$
$= -\frac{1}{6}x\sqrt{4/9-x^2} + \frac{2}{27}\sin^{-1}\frac{3x}{2} + c$

59. $\int \frac{\sqrt{4-x^2}}{x}\,dx$
$= \sqrt{4-x^2} - 2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + c$

60. $\int \frac{x^2}{(x^6-4)^{3/2}}\,dx = \frac{1}{3}\int \frac{1}{(u^2-4)^{3/2}}\,dx$

$$= \frac{1}{3}\int \frac{2\sec\theta\tan\theta}{(4\sec^2\theta - 4)^{3/2}}\,dx$$
$$= \frac{1}{3}\int \frac{\sec\theta\tan\theta}{\tan^3\theta}\,dx$$
$$= \frac{1}{3}\int \csc\theta\cot\theta\,dx = -\frac{x^3}{3\sqrt{x^6-4}} + c$$

61. Substitute $u = x^2 - 1$
$$\int_0^1 \frac{x}{x^2-1}\,dx = \int_{-1}^0 \frac{du}{2u}$$
$$= \lim_{R\to 0^-}\int_{-1}^R \frac{du}{2u} = \lim_{R\to 0^-} \ln|u|\,\Big|_{-1}^0$$
This limit does not exist, so the integral diverges.

62. Substitute $u = x - 4$
$$\int_4^{10} \frac{2\,dx}{\sqrt{x-4}} = \int_0^6 \frac{2\,du}{\sqrt{u}}$$
$$= \lim_{R\to 0^+}\int_R^6 2u^{-1/2}\,du = \lim_{R\to 0^+} 4u^{1/2}\,\Big|_R^6$$
$$= \lim_{R\to 0^+}(4\sqrt{6} - 4\sqrt{R}) = 4\sqrt{6}$$

63. $\displaystyle\int_1^\infty \frac{3}{x^2}\,dx = \lim_{R\to\infty}\int_1^R \frac{3}{x^2}\,dx$
$$= \lim_{R\to\infty} -\frac{3}{x}\Big|_1^R = \lim_{R\to\infty} -\frac{3}{R} + 3 = 3$$

64. Use integration by parts,
$$\int_1^\infty xe^{-3x}\,dx = \lim_{R\to\infty}\int_1^R xe^{-3x}\,dx$$
$$= \lim_{R\to\infty}\left[e^{-3x}\left(-\frac{x}{3} - \frac{1}{9}\right)\right]\Big|_1^R$$
$$= \lim_{R\to\infty} e^{-3R}\left(-\frac{R}{3} - \frac{1}{9}\right) + \frac{4e^{-3}}{9}$$
$$= \frac{4e^{-3}}{9}$$

65. $\displaystyle\int_0^\infty \frac{4}{4+x^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{4}{4+x^2}\,dx$
$$= \lim_{R\to\infty} 2\tan^{-1}\frac{x}{2}\Big|_0^R = \lim_{R\to\infty} 2\tan^{-1}R = \pi$$

66. $\displaystyle\int_0^\infty xe^{-x^2}\,dx = \lim_{R\to\infty} -\frac{e^{-x^2}}{2}\Big|_0^R = \frac{1}{2}$
$$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{R\to\infty} -\frac{e^{-x^2}}{2}\Big|_{-R}^0 = -\frac{1}{2}$$
So $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx = \frac{1}{2} - \frac{1}{2} = 0$

67. $\displaystyle\int_0^2 \frac{3}{x^2}\,dx = \lim_{R\to 0^+}\int_R^2 \frac{3}{x^2}\,dx$
$$= \lim_{R\to 0^+} -\frac{3}{x}\Big|_R^2 = \infty$$
So the original integral diverges.

68. $\displaystyle\int_1^2 \frac{x\,dx}{1-x^2} = \lim_{R\to 1^+}\int_R^2 \frac{x\,dx}{1-x^2}$
$$= \lim_{R\to 1^+} -\frac{1}{2}\ln|1-x^2|\Big|_R^2 = \infty$$
So the original integral diverges.

69. If $c(t) = R$, then the total amount of dye is
$$\int_0^T c(t)\,dt = \int_0^T R\,dt = RT$$
If $c(t) = 3te^{2Tt}$, then we can use integration by parts to get
$$\int_0^T 3te^{2Tt}\,dt$$
$$= \frac{3t}{2T}e^{2Tt}\Big|_0^T - \int_0^T \frac{3}{2T}e^{2Tt}\,dt$$
$$= \frac{3}{2}e^{2T^2} - \frac{3}{4T^2}e^{2Tt}\Big|_0^T$$
$$= \frac{3}{2}e^{2T^2} - \frac{3}{4T^2}e^{2T^2} + \frac{3}{4T^2}$$
Since $R = c(T) = 3Te^{2T^2}$

The cardiac output is
$$\frac{RT}{\int_0^T c(t)\,dt} = \frac{3T^2e^{2T^2}}{\frac{3}{2}e^{2T^2} - \frac{3}{4T^2}e^{2T^2} + \frac{3}{4T^2}}$$
$$= \frac{RT^3}{3T^2e^{2T^2}/2 - 3e^{2T^2}/4 + 3/4}$$

70. With $u = \ln(x+1)$ and $v = x$
$$\int \ln x + 1\,dx$$
$$= x\ln(x+1) - \int \frac{x}{x+1}\,dx$$
$$= x\ln(x+1) - \int\left(1 - \frac{1}{x+1}\right)dx$$
$$= x\ln(x+1) - x + \ln(x+1) + c$$
With $u = \ln(x+1)$ and $v = x+1$
$$\int \ln x + 1\,dx$$
$$= (x+1)\ln(x+1) - \int \frac{x+1}{x+1}\,dx$$
$$= (x+1)\ln(x+1) - x + c$$
The two answers are the same.

71. $\displaystyle f_{n,\text{ave}} = \frac{1}{e^n}\int_0^{e^n} \ln x\,dx$
$$= \frac{1}{e^n}\lim_{R\to 0}\int_R^{e^n} \ln x\,dx$$
$$= \frac{1}{e^n}\lim_{R\to 0}(x\ln x - x)\Big|_R^{e^n}$$
$$= \frac{1}{e^n}\lim_{R\to 0}(ne^n - e^n - R\ln R + R) = n - 1$$

72. First we notice that

$$\lim_{\Delta t\to 0}\frac{P(t<x<t+\Delta t)}{\Delta t}$$
$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t} f(x)\,dx = f(t)$$

And then the failure rate function

$$\lim_{\Delta t\to 0}\frac{P(x<t+\Delta t|x>t)}{\Delta t}$$
$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\frac{P(t<x<t+\Delta t)}{P(x>t)}$$
$$=\lim_{\Delta t\to 0}\frac{P(t<x<t+\Delta t)}{\Delta t}\cdot\frac{1}{R(t)}=\frac{f(t)}{R(t)}$$

73. $R(t)=P(x>t)=1-\int_0^t ce^{-cx}\,dx$

$$=1-(-e^{-cx})\Big|_0^t = 1-(1-e^{-ct})=e^{-ct}$$

Hence

$$\frac{f(t)}{R(t)}=\frac{ce^{-ct}}{e^{-ct}}=c$$

74. (a) $P(x>s)=1-\int_0^s xe^{-x}\,dx$

$$=1-(-xe^{-x}-e^{-x})\Big|_0^s$$
$$=1-(1-se^{-s}-e^{-s})$$
$$=(s+1)e^{-s}$$

$P(x>s+t|x>s)$

$$=\frac{P(x>s+t)}{P(x>s)}$$
$$=\frac{(s+t+1)e^{-s-t}}{(s+1)e^{-s}}=e^{-t}+\frac{t}{1+s}e^{-t}$$

(b) Take the derivative w.r.t s:

$$\frac{d}{ds}\left(e^{-t}+\frac{t}{1+s}e^{-t}\right) = -e^{-t}\frac{t}{(1+s)^2}$$

When $t>0$, since $e^{-t}>0$ and $(1+s)^2>0$, the above derivative is negative, so the function $P(x>s+t|x>s)$ is decreasing w.r.t. s.

75. We use a CAS to see that

$$\int_{90}^{100}\frac{1}{\sqrt{450\pi}}e^{-(x-100)^2/450}\,dx\approx 24.75\%$$

We can use substitution to get

$$\frac{1}{\sqrt{450\pi}}\int_a^\infty e^{-(x-100)^2/450}\,dx$$
$$=\frac{1}{\sqrt{\pi}}\int_{\frac{a-90}{\sqrt{450}}}^\infty e^{-u^2}\,du$$

Since $\int_{-\infty}^\infty e^{-x^2}\,dx=\sqrt{\pi}$,

$$\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$$

So we want to find the value of a so that

$$\int_0^{\frac{a-90}{\sqrt{450}}} e^{-u^2}\,du = 0.49\sqrt{\pi}$$

Using a CAS we find

$$\frac{a-90}{\sqrt{450}}\approx 1.645, a\approx 125$$

Some body being called a genius need to have a IQ score of at least 125.

76. $I(1)=\int_0^\infty\frac{1}{(1+x^2)}\,dx=\tan^{-1}x\big|_0^\infty=\frac{\pi}{2}$

Now, $I(n+1)$

$$=\left\{\left[\frac{1}{(1+x^2)^n}\int\frac{1}{(1+x^2)}\,dx\right]_0^\infty - \int_0^\infty\left(\frac{d\left(\left(1+x^2\right)^{-n}\right)}{dx}\int\frac{1}{(1+x^2)}\,dx\right)dx\right\}$$
$$=\left[\frac{\tan^{-1}x}{(1+x^2)^n}\right]_0^\infty+2n\int_0^\infty\frac{x\tan^{-1}x}{(1+x^2)^n}\,dx$$
$$\Rightarrow\ I(n+1)\ =\ 2n\int_0^\infty\frac{x\tan^{-1}x}{(1+x^2)^n}\,dx\quad ...(1)$$

Also, $I(n+1)=\int_0^\infty\frac{1+x^2-x^2}{(1+x^2)^{n+1}}dx$

$$=I(n)-\int_0^\infty\frac{x^2}{(1+x^2)^{n+1}}dx$$
$$=I(n)-\left\{\left[\frac{1}{(1+x^2)^n}\int\frac{x^2}{(1+x^2)}\,dx\right]_0^\infty-\int_0^\infty\left(\frac{d\left(\left(1+x^2\right)^{-n}\right)}{dx}\int\frac{x^2}{(1+x^2)}\,dx\right)dx\right\}$$
$$=I(n)-\left\{\left[\frac{x-\tan^{-1}x}{(1+x^2)^{n+1}}\right]_0^\infty+2n\int_0^\infty\frac{x\left(x-\tan^{-1}x\right)}{(1+x^2)^{n+1}}dx\right\}$$
$$ds=I(n)-2n\int_0^\infty\frac{\left(x^2-x\tan^{-1}x\right)}{(1+x^2)^{n+1}}dx$$
$$=I(n)-2n\int_0^\infty\frac{x^2}{(1+x^2)^{n+1}}\,dx+2n\int_0^\infty\frac{x\tan^{-1}x}{(1+x^2)^{n+1}}\,dx$$

Therefore,

$$I(n+1)=I(n)-2n\int_0^\infty\frac{x^2}{(1+x^2)^{n+1}}\,dx+I(n+1)$$

(using (1))

$I(n+1)$

$$=I(n)-2n\left(I(n)-I(n+1)\right)+I(n+1)$$

Hence proved.

As,$I\left(n+1\right)=\dfrac{2n-1}{2n}I\left(n\right)$
$I\left(n\right)=\dfrac{2n-3}{2n-2}I\left(n-1\right)$
$I\left(n-1\right)=\dfrac{2n-5}{2n-4}I\left(n-2\right)$
$I\left(n-2\right)=\dfrac{2n-7}{2n-6}I\left(n-3\right)$
and so on therefore,
$I\left(2\right)=\dfrac{3}{4}I(1),$
$I\left(1\right)=\dfrac{3}{4}\cdot\dfrac{\pi}{2},$

Thus, $I\left(n\right) = \dfrac{2n-3}{2n-2}\cdot\dfrac{2n-5}{2n-4}\cdot\dfrac{2n-7}{2n-6}\cdots$
$I\left(1\right)I\left(n\right)=\dfrac{1}{2}\cdot\dfrac{3}{4}\cdots\dfrac{2n-3}{2n-2}\cdot\dfrac{\pi}{2}$

Chapter 7

First-Order Differential Equations

7.1 Modeling with Differential Equations

1. Exponential growth with $k = 4$ so we can use Equation (1.4) to arrive at the general solution of $y = Ae^{4t}$. The initial condition gives $2 = A$ so the solution is $y = 2e^{4t}$.

2. Exponential growth with $k = 3$ so we can use Equation (1.4) to arrive at the general solution of $y = Ae^{3t}$. The initial condition gives $-2 = A$ so the solution is $y = -2e^{3t}$.

3. Exponential growth with $k = -3$ so we can use Equation (1.4) to arrive at the general solution of $y = Ae^{-3t}$. The initial condition gives $5 = A$ so the solution is $y = 5e^{-3t}$.

4. Exponential growth with $k = -2$ so we can use Equation (1.4) to arrive at the general solution of $y = Ae^{-2t}$. The initial condition gives $-6 = A$ so the solution is $y = -6e^{-2t}$.

5. Exponential growth with $k = 2$ so we can use Equation (1.4) to arrive at the general solution of $y = Ae^{2t}$. The initial condition gives $2 = Ae^2, A = \dfrac{2}{e^2}$ so the solution is $y = \dfrac{2}{e^2}e^{2t}$.

6. Exponential growth with $k = -1$ so we can use Equation (1.4) to arrive at the general solution of $y = Ae^{-t}$. The initial condition gives $2 = Ae^{-1}$, or $A = 2e$ and so the solution is $y = 2e^{-t+1}$.

7. Exponential growth with $k = 1$. We can use equation 1.9, to arrive at the general solution $y(t) = Ae^t + 50$. The initial condition gives $A = 20$ so the solution is $y(t) = 20e^t + 50$.

8. Exponential growth with $k = 0.1$. We can use equation 1.9, to arrive at the general solution $y(t) = Ae^{0.1t} + 100$. The initial condition gives $A = 20$ so the solution is $y(t) = 20e^{0.1t} + 100$.

9. (a) The doubling time of the bacterial culture is 1hour. Hence, in 3 hours the population of bacteria will be 3200.

(b) The equation for population must be
$y(t) = 400e^{kt}$
We know that in 1 hour, the population is 800, so
$800 = y(1) = 400e^k$.
Solving for k gives $k = \ln 2$.
$y(t) = 400e^{t\ln 2}$

(c) After 3.5 hours, the population is
$y(3.5) = 400e^{3.5\ln 2}$
$= 400 \times 2^{3.5} \approx 4525$cells.

10. (a) The bacterial culture is increased by 4 times in two hours. Hence in 6 hours the population of bacteria will be 6400.

(b) The equation for population must be
$y(t) = 100e^{kt}$
We know that in 2 hours, the population is 400, so
$400 = y(2) = 100e^{2k}$.
Solving for k gives $k = \ln 2$.
$y(t) = 100e^{t\ln 2}$

(c) After 7 hours, the population is
$y(7) = 100e^{7\ln 2} = 100 \times 128$
$= 12800$ cells.

11. (a) The initial population of 100 bacteria will increase to 200 in four hours. Hence the population of bacteria will reach 400 in 8 hours.

(b) The equation for population must be
$y(t) = 100e^{kt}$
We know that in 4 hours, the population doubles, so
$200 = y(4) = 100e^{k4}$
Solving for k give $k = (\ln 2)/4$ and
$y(t) = 100e^{t(\ln 2)/4}$

(c) To determine when the population reaches 6,000, we solve
$y(t) = 6,000$ or $6000 = 100e^{t(\ln 2)/4}$
Solving gives
$t = \dfrac{4\ln 60}{\ln 2} \approx 23.628$ hours.

12. (a) The initial population of 200 bacteria will increase to 600 in five hours. Hence the population of bacteria will reach 5400 in 15 hours.

(b) The equation for population must be
$y(t) = 200e^{kt}$
We know that in 3 hours, the population triples, so
$600 = y(3) = 200e^{3k}$
Solving for k gives
$k = (\ln 3)/3$ and $y(t) = 200e^{t(\ln 3)/3}$

(c) To determine when the population reaches 20,000, we solve
$y(t) = 20,000$ or $20000 = 200e^{t(\ln 3)/3}$
Solving gives
$t = \dfrac{3\ln 100}{\ln 3} \approx 12.575$ hours.

13. With t measured in minutes, and
$y = Ae^(kt) = 10^8 e^{kt}$

on the time interval $(0, T)$ (during which no treatment is given), the condition on T is that 10% of the population at time T (surviving after the treatment) will be the same as the initial population.

In other words, $10^8 = (.1)10^8 e^{kT}$.
This gives
$e^{kT} = 10$ and $T = \ln(10)/k$.

To get k we use the given doubling time

$t_d = 20$. Since we always have $t_d = \ln(2)/k$, this leads to $k = \ln(2)/20$ and
$T = \dfrac{\ln(10)}{\ln(2)/20} = \dfrac{20\ln(10)}{\ln(2)} \approx 66.44$ minutes.

14. We will assume that the number of acres to sustain the growing population grows at a constant exponential rate. This means that the number of acres requires is given by
$N(t) = Ae^{rt}$

where $N(t)$ is given in billions of acres (this is not necessary, but it simplifies the constants). We will assume that $t = 0$ corresponds to the year 1950.

In this case we know that $N(0) = 1$ and $N(30) = 2$. This gives us $A = 1$ and we can solve for r:
$2 = e^{30r}$

which gives us $r = \dfrac{\ln 2}{30} \approx 0.0231$.

We now want to find when $N(t) = 3.2$

so we solve the equation $3.2 = e^{rt}$

Solving gives $t = \dfrac{\ln 3.2}{r} \approx 50.34$ which means that this occurs in the year 2000.

15. Given $y(t) = Ae^{rt}$, the doubling time t_d obeys
$2A = Ae^{rt_d}, 2 = e^{rt_d}$
$rt_d = \ln 2, t_d = \dfrac{\ln 2}{r}$ as desired.

16. The equation for amount of the substance is
$y(t) = Ae^{rt}$

To find the halving time, we solve
$\dfrac{A}{2} = Ae^{rt}$ for t.

Solving gives $t = \dfrac{\ln(1/2)}{r} = -\dfrac{\ln 2}{r}$

Notice that since $r < 0$, this value of t is positive. In fact, this formula is essentially the same formula for doubling time (the difference being that the value for r is either positive or negative depending on if we are in the growth or decay situation).

17. Using the formula in exercise 16, we find the decay constant is
$r = -\dfrac{\ln 2}{28}$
Thus the formula for the amount of substance is
$y(t) = Ae^{rt}$

(a) After 84 years,
$y(84) = Ae^{84r} \approx 0.125A$. Hence, this is about 12.5% of original amount of Strontium-90.

(b) After 100 years,
$y(100) = Ae^{100r} \approx 0.084A$.
Thus,this is about 8.4% of original amount of Strontium-90.

18. Using the formula in Exercise 16, we find the decay constant is
$r = -\dfrac{\ln 2}{0.7 \times 10^9}$
Thus the formula for the amount of substance is
$y(t) = 50e^{rt}$

(a) After 100 years,
$y(100) = 50e^{100r} \approx 49.9999995$ grams. Hence, approximately 49.99995% of original ^{235}U will remain after 100 years.

(b) After 1000 years,
$y(1000) = 50e^{1000r} \approx 49.9999995$ grams. Hence, approximately 49.99995% of original ^{235}U will remain after 1000 years.

19. Using the formula in Exercise 16, we have $3 = -(\ln 2)/r$ and therefore $r = -(\ln 2)/3$. Thus the formula for amount of substance is $y(t) = Ae^{-t(\ln 2)/3}$

The initial condition gives $A = 0.4$ and so $y(t) = 0.4e^{-t(\ln 2)/3}$

(a) For $y(t) = 0.1$,
We get, $0.1 = 0.4e^{-t(\ln 2)/3}$.
Solving for t gives
$t = \dfrac{3\ln(4)}{\ln(2)} = 6$ hours.
Thus the amount will drop below 0.1 mg after 6 hours.

(b) For $y(t) = 0.01$,
We get, $0.01 = 0.4e^{-t(\ln 2)/3}$.
Solve for t gives
$t = \dfrac{3\ln(40)}{\ln(2)} = 15.97$ hours.
Thus the amount will drop below 0.01 mg after 15.97 hours.

20. Using the formula in Exercise 16, we have $2.8 = -(\ln 2)/r$ and therefore $r = -(\ln 2)/2.8$. Thus the formula for amount of substance is $y(t) = Ae^{-t(\ln 2)/2.8}$

The initial condition gives $A = 0.4$ and so $y(t) = 0.4e^{-t(\ln 2)/2.8}$

(a) For $y(t) = 0.1$,
We get, $0.1 = 0.4e^{-t(\ln 2)/2.8}$.
Solving for t gives
$t = \dfrac{2.8\ln(4)}{\ln(2)} = 5.6$ hours.
Thus the amount will drop below 0.1 mg after 5.6 hours.

(b) For $y(t) = 0.01$,
We get, $0.01 = 0.4e^{-t(\ln 2)/2.8}$.
Solve for t gives
$t = \dfrac{2.8\ln(40)}{\ln(2)} = 14.9$ hours.
Thus the amount will drop below 0.01 mg after 14.9 hours.

21. The half-life is 5730 years, so $r = -\dfrac{\ln 2}{5730}$

Solving for t in
$y(t) = 0.20A = Ae^{-rt}$ gives
$t = \dfrac{5730\ln(5)}{\ln(2)} \approx 13{,}305$ years.

22. The half-life is 5730 years, so $r = -\dfrac{\ln 2}{5730}$

The proportion of the carbon-14 left is therefore equal to $e^{r10^6} \approx 2.912 \times 10^{-52}$.

23. Newton's Law of Cooling gives
$y(t) = Ae^{kt} + T_a$ with $T_a = 70$.
We have $y(0) = 200$ so
$200 = A + 70$ and $A = 130$
We have $y(1) = 180$ so
$180 = y(1) = 130e^k + 70$ and $k = \ln\left(\dfrac{110}{130}\right)$.
The temperature will be 120 when
$120 = y(t) = 130e^{\ln(110/130)t} + 70$ and
$t = \dfrac{\ln(5/13)}{\ln(11/13)} \approx 5.720$ minutes.

24. Newton's Law of Cooling gives
$y(t) = Ae^{kt} + T_a$ with $T_a = 70$.
We have $y(0) = 200$ so
$200 = y(0) = A + 70$ and $A = 130$.
After one minute we have $y(1) = 160$ and
$160 = y(1) = 130e^k + 70$
Solving for k gives $k = \ln\dfrac{9}{13}$.
The bowl in Exercise 23 reaches it temperature in about 5.720 minutes. At this time, the temperature of this bowl will be:
$y(5.720) = 130e^{k(5.720)} + 70 \approx 85.87$ degrees.

25. (a) Using Newton's Law of Cooling
$y = Ae^{kt} + T_a$ with $T_a = 70, y(0) = 50$,
we get $50 = Ae^0 + 70, A = -20$
so that $y(t) = -20e^{kt} + 70$.
If, after two minutes, the temperature is 56 degrees, $56 = -20e^{k2} + 70$
$e^{2k} = \dfrac{14}{20} = 0.7$
$2k = \ln 0.7, k = \dfrac{1}{2}\ln 0.7$
Therefore, $y(t) = -20e^{(\ln 0.7)t/2} + 70$.

(b) From (a.), the equation for the temperature of the drink is
$y(t) = -20e^{(\ln 0.7)t/2} + 70$
After 10 minutes, the temperature is
$y(10) \approx 66.64$ degrees
The drink warms to 66° when
$66 = y(t) = -20e^{(\ln 0.7)t/2} + 70$
Solving for t gives $t \approx 9.025$ minutes

26. (a) The problem is that the rate of cooling is not constant

(b) The coffee will cool quicker when it is hotter. Therefore the serving temperature was greater than 180 degrees.

(c) With t the time elapsed since serving, with the ambient temperature 68 degrees and if the temperature is 160 degrees when $t = 20$, then
$y(t) = Ae^{kt} + T_a, 160 = Ae^{k\cdot 20} + 68$

$Ae^{20k} = 92$ After 22 minutes the temperature is 158 degrees,
$158 = Ae^{22k} + 68, Ae^{22k} = 90$
$e^{2k} = \frac{Ae^{22k}}{Ae^{20k}} = \frac{90}{92}, k = \frac{1}{2}\ln\frac{90}{92}$
Therefore, $y(t) = Ae^{\frac{1}{2}(\ln 90ff92)t} + 68$
Using the first set of numbers,
$Ae^{20\cdot\frac{1}{2}\ln\frac{90}{92}} = 92$
$A = \frac{92}{e^{10\ln\frac{90}{92}}} \approx 114.615$
$y(t) = 114.615e^{\frac{1}{2}(\ln\frac{90}{92})t} + 68$
The serving temperature is
$y(0) = 114.615e^0 + 68 = 182.615$ minutes.

27. Using Newton's Law of Cooling with ambient temperature 70 degrees, initial temperature 60 degrees, and with time t (in minutes) elapsed since 10:07, we have
$y(t) = Ae^{kt} + 70, 60 = Ae^{0k} + 70$
$= A + 70, A = -10$
and $y(t) = -10e^{kt} + 70$ (for the martini).
Two minutes later, its temperature is 61 degrees. Hence,
$61 = -10e^{k2} + 70, e^{2k} = \frac{9}{10}$
$2k = \ln\frac{9}{10}, k = \frac{1}{2}\ln\frac{9}{10} = \frac{1}{2}\ln .9$
Therefore, $y(t) = -10e^{(\frac{1}{2}\ln .9)t} + 70$
The temperature is 40 degrees at elapsed time t only if
$40 = -10e^{(\frac{1}{2}\ln .9)t} + 70$
$t = \frac{2\ln 3}{\ln .9} \approx -20.854$ or about 21 minutes *before* 10:07 p.m. The time was 9:46p.m.

28. Here the unknown is the initial temperature, $T = y(0)$. The equation for temperature of the coffee is $y(t) = Ae^{kt} + 70$
Using the initial temperature gives the equation $T = A + 70$, so $A = T - 70$ and the equation for the temperature is now given by $y(t) = (T - 70)e^{kt} + 70$
The value for k will not change (k does not depend on initial conditions) and therefore
$k = \frac{1}{2}\ln(95/110)$
We want the temperature at 5 minutes to be 120, so this gives the equation
$120 = y(5) = (T - 70)e^{5k} + 70$
Solving for T gives
$T = \frac{50}{e^{5k}} + 70 \approx 142.13$ degrees.

29. Annual: $A = 1000(1 + 0.08)^1 \approx \$1080.00$

Monthly: $A = 1000\left(1 + \frac{0.08}{12}\right)^{12} \approx \$1083.00$

Daily: $A = 1000\left(1 + \frac{0.08}{365}\right)^{365} \approx \$1083.28$

Continuous: $A = 1000e^{(0.8)1} \approx \$1083.29$

30. Annual:$A = 1000(1 + 0.08)^5 \approx \$1469.33$

Monthly:$A = 1000\left(1 + \frac{0.08}{12}\right)^{60} \approx \$1489.85$

Daily: $A = 1000\left(1 + \frac{0.08}{365}\right)^{5\cdot 365} \approx \$1491.76$

Continuous: $A = 1000e^{(0.8)5} \approx \$1491.83$

31. (a) Person A:
$A = 10{,}000e^{.12\cdot 20} = \$110{,}231.76$
Person B:
$B = 20,000e^{.12\cdot 10} = \$66{,}402.34$

(b) At 4% interest:
Person A:
$A = 10,000e^{(0.04)20} \approx \$22,255.41$
Person B:
$A = 20,000e^{(0.04)10} \approx \$29,836.49$

(c) To find the rate so that A and B are even, we solve, $10,000e^{10r} = 20,000$
Solving gives $r = \ln 2/2 \approx 6.93\%$

32. (a) Let t be the number of years after 1985. Then, assuming continuous compounding at rate r,
$9800 = 34e^{r\cdot 10}, e^{10r} = \frac{9800}{34}$
$r = \frac{1}{10}\ln\left(\frac{9800}{34}\right) \approx .566378$
Therefore,
$A = 34e^{\frac{1}{10}\ln(\frac{9800}{34})t} = 34\left(\frac{9800}{34}\right)^{t/10}$

(b) In 2005, $t = 20$ and
$A = 34\left(\frac{9800}{34}\right)^2 = \$2{,}824{,}705.88$

(c) The equation for the value of the cards is $y(t) = Pe^{rt}$.
We take $t = 0$ to correspond to the year 1985 which means that $P = 22$.
To determine k we use
$32 = y(10) = 22e^{10r}$
Solving for r gives , $r = \frac{1}{10}\ln(32/22)$
The value in 2005 is then given by
$y(20) = 22e^{20r} \approx \$46.55$

33. With a constant depreciation rate of 10%, the value of the \$40,000 item after ten years would

be,
$40{,}000(e^{-(0.1)10}) = 40{,}000e^{-1} \approx \$14{,}715.18$
and after twenty years
$40{,}000(e^{-(0.1)20}) = 40{,}000e^{-2} \approx \$5{,}413.41$
By the straight line method, assuming a value of zero after 20 years, the value would be \$20,000 after ten years.

34. The value of the asset is given by $v(t) = Pe^{rt}$ where $P = 400,00$ and $r = -0.4$.
After 5 years, the value is $v(5) \approx 54,134$.
After 10 years, the value is $v(10) \approx 7326$.
For the \$ 40,000 asset with linear depreciation, we have $v(t) = 40000 - 4000t$
In this case, after 5 years, the value is $v(5) \approx \$20,000$.
After 10 years, the value is $v(10) = \$0$.

35. The problem with comparing tax rates for the income bracket $[16K, 20K]$ over a thirteen year time interval, is that due to inflation, the persons in this income bracket in 1988 have less purchasing power than those in the same bracket in 1975, and a lower tax rate may or may not compensate. To quantify and illustrate, assume a 5.5% annual inflation rate. This would translate into a loss of purchasing power amounting to
$41/(1.055)^{13} = 1/(2.006) \approx 1/2$,
which is essentially to say that in terms of comparable purchasing power, the income bracket $[16K, 20K]$ in 1988 corresponds to an income bracket of $[8K, 10K]$ in 1975. One should then go back and look at the tax rate for the latter bracket in 1975. Only if that tax rate exceeds the 1988 rate (15%) for the bracket $[16K, 20K]$ should one consider that taxes have genuinely gone down.

36. Adjusting for inflation, \$16,000 in 1975 was worth $16,000(1.055)^{13} \approx \$32,092$
In 1975, the tax rate on \$16,000 was 28%. In 1988, the tax rate on \$32,092 was also 28%. This means that the tax rates were roughly equal.

37. $T_1 = 30,000 \cdot 0.15 + (40{,}000 - 30{,}000) \cdot 0.28$
$= \$7300$
$T_2 = 30{,}000 \cdot 0.15 + (42{,}000 - 30{,}000) \cdot 0.28$
$= \$7860$
$T_1 + .05T_1 = \$7665$

The tax T_2 on the new salary is greater than the adjusted tax $(1.05T_1)$ on the old salary.

38. What happened is that the amount taxed at 15% remains \$30,000. If this figure is also adjusted for inflation then the amount of tax owed remains the same. In other words, if the first $\$30,000(1.05) = \$31500$ is taxed at 15% and the rest is taxed at 28%.

39. Fitting a line to the first two data points on the plot of time vs. the natural log of the population ($y = \ln(P(x))$) produces the linear function $y = 1.468x + 0.182$,
which is equivalent to fitting the original date with the exponential function
$P(x) = e^{1.468x+0.182}$ or
$P(x) = 1.200e^{1.468x}$

40. (a) As in Exercise 39, we let x denote time and $y = \ln P$. We pick the second and fourth data point to fit a line to (any two data points are fine to use and will give slightly different answers). In this case, the points are $(1, \ln 15)$ $(3, \ln 33)$
The equation of the line connecting these two points is
$\ln P = y = 0.394x + 3.102$
Exponentiating this equation gives
$P = e^y = e^{0.394x+3.102} = 22.242\, e^{0.394x}$

(b) As in Exercise 39, we let x denote time and $y = \ln P$. We pick the second and fourth data point to fit a line to (any two data points are fine to use and will give slightly different answers). In this case, the points are $(1, \ln 16)$ $(3, \ln 11)$
The equation of the line connecting these two points is
$\ln P = y = -0.18735x + 2.9599$
Exponentiating this equation gives
$P = e^y = e^{-0.18735x+2.9599}$
$= 19.297\, e^{-0.18735x}$

41. As in Exercise 39, we let x denote time (with $x = 0$ corresponding to the year 1960) and let $y = \ln P$. Looking at the graph of the modified data, we decide to use the first and last data points. In this case, the points are
$(0, \ln 7.5)$ $(30, \ln 1.6)$
The equation of the line connecting these two points is $\ln P = y = -0.0515x + 2.0149$
Exponentiating this equation gives
$P = e^y = e^{-0.0515x+2.0149} = 7.5\, e^{-0.0515x}$

42. As in Exercise 39, we let x denote time (with $x = 0$ corresponding to the year 1960) and let $y = \ln P$. Looking at the graph of the modified data, we decide to use the first and last data points. In this case, the points are
$(0, \ln 69.9)$ $(30, \ln 75.2)$

The equation of the line connecting these two points is $\ln P = y = 0.013790x + 4.2471$
Exponentiating this equation gives
$P = e^y = e^{0.013790x+4.2471} = 69.9\,e^{0.013790x}$

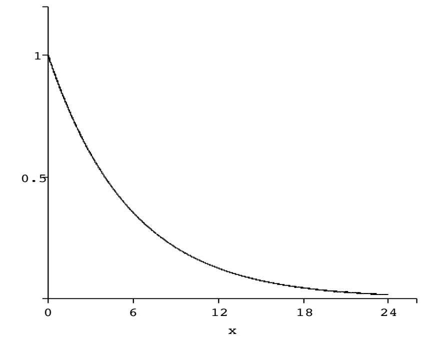
43. Consider the equation,
$y = x - \frac{1}{2} + ce^{-2x}$
differentiating both sides by x
$y' = 1 - 2ce^{-2x}$
substituting for ce^{-2x}
$= 1 - 2\left[y - x + \frac{1}{2}\right] \quad = 2x - 2y$
$y' + 2y = 2x.$

44. Consider the equation,
$y = \sqrt{(3x^2 + c)}$
differentiating both sides with respect to x
$y' = \frac{1}{2\sqrt{(3x^2+c)}} \times 6x = \frac{3x}{\sqrt{(3x^2+c)}} = \frac{3x}{y}.$

45. With known conclusion
$y = Ae^{-rt}$, $A = 150$, $t = 24$, and $r = \ln(2)/t_h$ we find that with $t_h = 31$ we get
$y = 150(1/2)^{(24/31)} = 87.7$, and with $t_h = 46$ we get $y = 150(1/2)^{(24/46)} = 104.5$.
The difference is about 17 days, at 19% not a dramatically large percentage of the smaller base of $88(105/88 = 1.19)$. If one had expected the two numbers to be proportional to the half lives, one would have expected the difference to come in at 48% $(46/31 = 1.48)$ and would definitely consider the 19% to be far less than anticipated.

46. We use the formula of Exercise 16.
If the half-life is 2 days then $r = -\frac{\ln 2}{2}$ and in two weeks the proportion remaining would be $e^{-14(\ln 2)/2} \approx 0.007813$ (so about 0.78%). If the half-life is 3 days then
$r = -\frac{\ln 2}{3}$ and in two weeks the proportion remaining would be
$e^{-14(\ln 2)/3} \approx 0.03917 = 3.9.\%$

47. In this case, with $t_h = 4$, $A = 1$, $y = Ae^{-rt}$, and $r = \ln(2)/4$, one finds $y = (1/2)^{(t/4)}$.
The curve is a typical exponential, declining from a value of 1 at $t = 0$ to
$1/2^6 = 1/64 = .016$ at $t = 24$.



48. If the half-life is 1 hour, the decay rate is
$r = -\frac{\ln 2}{1} = -\ln 2$
We assume that the drug is taken every 6 hours. When the drug is initially taken, the initial amount of 1 gm:
$y_1 = 1e^{-t\ln 2} = 2^{-t}$
After 6 hours, the amount left is
$y_1(6) = 2^{-6} = 0.015625$
When the 2nd dose is taken, the initial amount will be 1 gm plus the amount left from the 1st dose:
$y_2 = (1.015625)e^{-t\ln 2} = (1.015625)2^{-t}$
After 6 hours, the amount left is
$y_2(6) = (1.015625)2^{-6} \approx 0.0158691$
When the 3rd dose is taken, the initial amount will be 1 gm plus the amount left from the 2nd dose:
$y_3 = (1.0158691)e^{-t\ln 2} = (1.0158691)2^{-t}$
After 6 hours, the amount left is
$y_3(6) = (1.0158691)2^{-6} \approx 0.015873$
When the 4th dose is taken, the initial amount will be 1 gm plus the amount left from the 3rd dose:
$y_4 = (1.015873)e^{-t\ln 2} = (1.015873)2^{-t}$
After 6 hours, the amount left is
$y_4(6) = (1.015873)2^{-6} \approx 0.015873$

49. With r the rate of continuous compounding, the value of an initial amount X after t years is Xe^{rt}. If the goal is P, then the relation is $P = Xe^{rt}$ or $X = Pe^{-rt}$. With $r = .08$, $t = 10$, $P = 10{,}000$, we find,
$X = 10{,}000e^{-.8} = \$4493.29$.

50. The present value is
$PV = \$40,000e^{2\sqrt{t}}e^{-0.06t}$
$= \$40,000e^{2\sqrt{t}-0.06t}$
The best time to sell is when this is at a maximum (because this is when it is worth the most). To maximize PV, we can maximize $2\sqrt{t} - 0.06t$. This maximum occurs when $t \approx 278$.

51. $\int_0^T e^{-rt}dt = \frac{1-e^{-rT}}{r}$
$\int_0^T te^{-rt}dt = \frac{-Te^{-rT}}{r} + \frac{1-e^{-rT}}{r^2}$,
With $r = .05$ and $T = 3$, we find
for (A): $60{,}000(20)\ (1-e^{-.15}) = \$167{,}150$

for (B): we get the above *plus*
$(3000)-60e^{-.15}+400(1-e^{-.15})$
$= 12{,}223$ for a total of \$179,373

for (C), the exponentials cancel, and the answer is simply
$\int_0^3 60000dt = \$180{,}000.$

52. (a) $\int_0^3 60,000e^{0.05(3-t)}\,dt \approx \$194,201.09$

(b) $\int_0^3 (60,000+3,000t)e^{0.05(3-t)}\,dt$
$\approx \$208,402.18$

(c) $\int_0^3 60,000e^{0.05t}e^{0.05(3-t)}\,dt$
$\approx \$209,130.16$

53. (a) The comparison is to be made between three years of accumulation of \$1,000,000 versus the accumulation of four annual payments of \$280,000 at times $0,1,2,3$, then the respective figures are $1{,}000{,}000(1.08)^3 = 1{,}259{,}712$ versus $280{,}000(1.083+1.082+1.08+1)$
$= 1{,}261{,}711$.
One should take the annuity.

(b) If we got the \$1 million lump sum, then the amount received after 3 years at the rate of interest of 6% is,
$1000000\times(1.06)^3 \approx 1000000\times 1.191016$
$= \$1191016.$
If the amount is received in installments of \$280000 at the starting of every year, the amount received is,
$280000(1+1.06+1.06^2+1.06^3)$
$= 280000\times 4.374616 \approx 1224892.$
One should take the installments

(c) If we got the \$1 million lump sum, then the amount received after 3 years at the rate of interest of 10% is,
$1000000\times(1.1)^3 \approx 1000000\times 1.331$
$= \$1331000.$
If the amount is received in installments of \$280000 at the starting of every year, the amount received is,
$280000(1+1.1+1.1^2+1.1^3)$
$= 280000\times 4.641 \approx \$1299480.$
One should take the annuity

54. The actual doubling time for money invested at 8% is obtained by solving
$2 = e^{0.08t}$ which gives 8.66 years.
In general, the doubling time is
$\frac{\ln 2}{r} \approx \frac{0.69314}{r}$
(hence the "Rule of 69"). 72 is used because most interest is not compounded continuously. For example, if 10% interest is compounded once a year, it takes 7.27 years to double.

7.2 Separable Differential Equations

1. (a) Separable.
$\frac{y'}{\cos y} = 3x+1$
(b) Not separable.

2. (a) Separable.
$\frac{y'}{\cos y - 1} = 2x$
(b) Not separable.

3. (a) Separable.
$y' = y(x^2+\cos x)$
$\frac{y'}{y} = x^2+\cos x$
(b) Not separable.

4. (a) Not separable.
(b) Separable.
$y'-1 = x^3-2x$

5. $\frac{1}{y}y' = x^2+1$
$\int \frac{1}{y}dy = \int (x^2+1)dx$
$\ln|y| = \frac{x^3}{3}+x+c$
$y = e^{x^3/3+x+c} = Ae^{x^3/3+x}$

6. $\frac{1}{y-1}y' = 2x$
$\int \frac{1}{y-1}dy = \int 2xdx$
$\ln|y-1| = x^2+c$
$y-1 = e^{x^2+c}$
$y = 1+Ae^{x^2}$

7. $\frac{1}{y^2}y' = 2x^2$
$\int \frac{1}{y^2}dy = \int 2x^2dx$
$-\frac{1}{y} = \frac{2x^3}{3} + c$
$y = -\frac{1}{2x^3/3 + c}$

8. $\frac{1}{y^2+1}y' = 2$
$\int \frac{1}{y^2+1}dy = \int 2dx$
$\arctan y = 2x + c$
$y = \tan(2x + c)$

9. $yy' = \frac{6x^2}{1+x^3}$
$\int ydy = \int \frac{6x^2}{1+x^3}dx$
$\frac{1}{2}y^2 = 2\ln|1+x^3| + c$
$y = \pm\sqrt{4\ln|1+x^3| + c}$

10. $(y+1)y' = 3x$
$\int (y+1)dy = \int 3xdx$
$\frac{y^2}{2} + y = \frac{3}{2}x^2 + c$

11. $y' = \frac{2x}{y}e^{y-x}, \quad y' = \frac{2x}{y} \times \frac{e^y}{e^x}$
$y'ye^{-y} = 2xe^{-x}$
$\int y'ye^{-y}dx = \int 2xe^{-x}dx$
$\int ye^{-y}(y'dx) = \int 2xe^{-x}dx$
or $\int ye^{-y}dy = \int 2xe^{-x}dx$
$\int ye^{-y}dy = -ye^{-y} - e^{-y} + c$
and $\int xe^{-x}dx = -xe^{-x} - e^{-x} + c$
$-ye^{-y} - e^{-y} = 2(-xe^{-x} - e^{-x}) + c$
$-ye^{-y} - e^{-y} = -2xe^{-x} - 2e^{-x} + c.$

12. $\frac{y'}{\sqrt{1-y^2}} = \frac{1}{x\ln x}$
$\int \frac{1}{\sqrt{1-y^2}}dy = \int \frac{1}{x\ln x}dx$
$\arcsin y = \ln(\ln x) + c$
$y = \sin[\ln(\ln x) + c]$

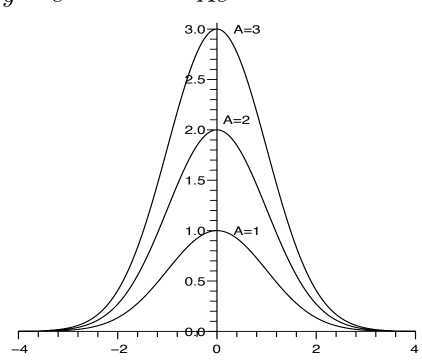
13. $y' = \frac{\cos x}{\sin y}$
$(\sin y)\, y' = \cos x$
$\int (\sin y)\, y'(x)dx = \int (\cos x)dx$
or $\int (\sin y)\, dy = \int (\cos x)dx$
$\cos y = -\sin x + c.$

14. $\sec^2 yy' = x$
$\int \sec^2 ydy = \int xdx$
$\tan y = \frac{x^2}{2} + c$
$y = \tan^{-1}\left(\frac{x^2}{2} + c\right)$

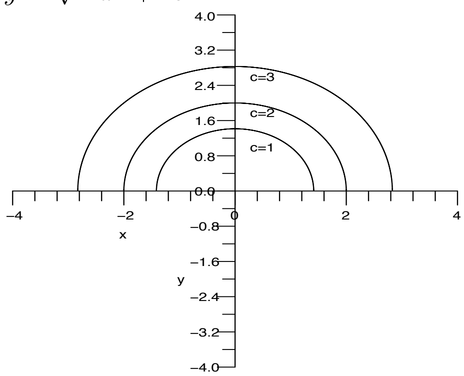
15. $\frac{1}{y}y' = \frac{x}{1+x^2}$
$\int \frac{1}{y}dy = \int \frac{x}{1+x^2}dx$
$\ln|y| = \frac{1}{2}\ln|1+x^2| + c$
$y = e^{\frac{1}{2}\ln|1+x^2|+c} = k\sqrt{1+x^2}$

16. $yy' = \frac{2}{x+1}$
$\int ydy = \int \frac{2}{x+1}dx$
$\frac{y^2}{2} = 2\ln|x+1| + c$

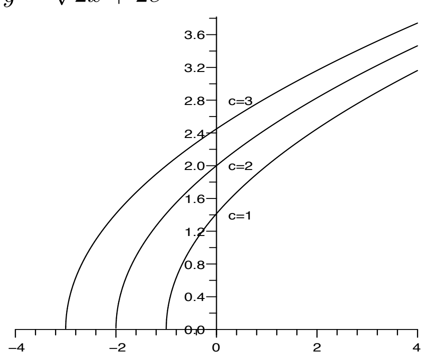
17. $y' = -xy, \ \frac{y'}{y} = -x$
$\int \frac{y'}{y}dx = \int -xdx$
$\int \frac{1}{y}(y'dx) = -\int xdx$
$\ln|y| = -\frac{x^2}{2} + c$
$y = e^{\left(-\frac{x^2}{2}+c\right)} = Ae^{-\frac{x^2}{2}}$



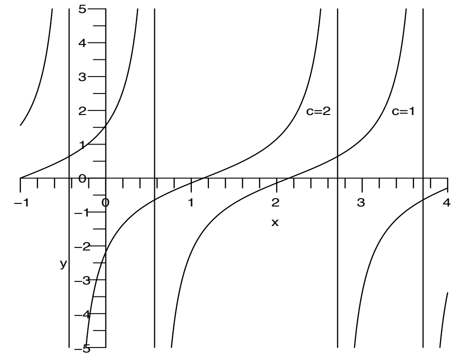
18. $y' = -\dfrac{x}{y},\ y'y = -x$
$\displaystyle\int y'y\,dx = \int -x\,dx$
$\displaystyle\int y\,(y'dx) = -\int x\,dx$
$\dfrac{y^2}{2} = -\dfrac{x^2}{2} + c$
$y^2 = -x^2 + 2c$
$y = \sqrt{-x^2 + 2c}$



19. $y' = \dfrac{1}{y},\ y'y = 1$
$\int y'y\,dx = \int dx$
$\int y\,(y'dx) = \int dx$
$\dfrac{y^2}{2} = x + c$
$y^2 = 2x + 2c$
$y = \sqrt{2x + 2c}$



20. $y' = 1 + y^2$
$\dfrac{y'}{1+y^2} = 1$
$\displaystyle\int \frac{y'}{1+y^2}dx = \int dx$
$\displaystyle\int \frac{1}{1+y^2}\,(y'dx) = \int dx$
$\displaystyle\int \frac{1}{1+y^2}dy = \int dx$
$\arctan y = x + c$
$y = \tan(x + c).$



21. $\dfrac{y'}{y} = 3(x+1)^2$
$\ln y = (x+1)^3 + c$
$y = ke^{(x+1)^3}$
Using the initial condition,
$1 = ke, k = \dfrac{1}{e}$
$y = \dfrac{1}{e}\,e^{(x+1)^3}$

22. $y^2y' = x - 1$
$\displaystyle\int y^2dy = \int (x-1)dx$
$\dfrac{y^3}{3} = \dfrac{x^2}{2} - x + c$
Using the initial condition,
$\dfrac{2^3}{3} = \dfrac{0^2}{2} - 0 + c, c = \dfrac{8}{3}$
$\dfrac{y^3}{3} = \dfrac{x^2}{2} - x + \dfrac{8}{3}$

23. $yy' = 4x^2$
$\dfrac{y^2}{2} = \dfrac{4x^3}{3} + c$
Using the initial condition,
$\dfrac{2^2}{2} = c = 2$
$\dfrac{y^2}{2} = \dfrac{4x^3}{3} + 2$

24. $yy' = x - 1$
$\displaystyle\int ydy = \int x - 1dx$
$\dfrac{y^2}{2} = \dfrac{x^2}{2} - x + c$
Using the initial condition,
$\dfrac{(-2)^2}{2} = \dfrac{0^2}{2} - 0 + c, c = \dfrac{8}{3}$
$\dfrac{y^2}{2} = \dfrac{x^2}{2} - x + \dfrac{8}{3}$

25. $\dfrac{y'}{4y} = \dfrac{1}{x+3}$
$\dfrac{\ln|y|}{4} = \ln|x+3| + c$

$\ln|y| = 4\ln|x+3| + c$
$|y| = k(|x+3|)^4$
Using the initial condition,
$|1| = k(1)^4, k = 1$
$|y| = (|x+3|)^4$

26. $(4y+1)y' = 3x$
$\int(4y+1)dy = \int 3x dx$
$2y^2 + y = \frac{3x^2}{2} + c$
Using the initial condition,
$2(4)^2 + 4 = \frac{3(1)^2}{2} + c, c = \frac{69}{2}$
$2y^2 + y = \frac{3x^2}{2} + \frac{69}{2}$

27. $\cos y\, y' = 4x$
$\sin y = 2x^2 + c.$
Using the initial condition,
$0 = \sin(0) = \sin y(0) = 0 + c = c$
$\sin y = 2x^2$
$y = \arcsin(2x^2)$
$(-1/\sqrt{2} < x < 1/\sqrt{2})$

28. $(\cot y)y' = \frac{1}{x}$
$\int \cot y dy = \int \frac{1}{x} dx$
$\ln|\sin y| = \ln|x| + c$
$\sin y = Ax$
Using the initial condition,
$\sin\frac{\pi}{2} = A, A = 1$
$\sin y = x$

29. For this problem we have $M = 2$ and $k = 3$. Using these and the initial condition, we solve for A.
$1 = \frac{2Ae^{3(2)(0)}}{1 + Ae^{3(2)(0)}} = \frac{2A}{1+A},$
$A = 1$
$y = \frac{2e^{6t}}{1+e^{6t}}$

30. For this problem we have $M = 3$ and $k = 1$. Using these and the initial condition, we solve for A.
$2 = \frac{3Ae^{3(0)}}{1 + Ae^{3(0)}} = \frac{3A}{1+A},$
$A = 2$
$y = \frac{6e^{3t}}{1+3e^{3t}}$

31. For this problem we have $M = 5$ and $k = 2$. Using these and the initial condition, we solve for A.
$4 = \frac{5Ae^{10(0)}}{1 + Ae^{10(0)}} = \frac{5A}{1+A},$
$A = 4$
$y = \frac{20e^{10t}}{1+4e^{10t}}$

32. For this problem we have $M = 2$ and $k = 1$. Using these and the initial condition, we solve for A.
$1 = \frac{2Ae^{2(0)}}{1 + Ae^{2(0)}} = \frac{2A}{1+A},$
$A = 1$
$y = \frac{2e^{2t}}{1+e^{2t}}$

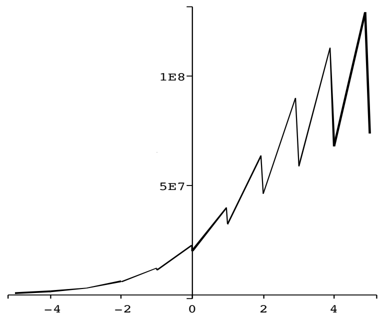
33. For this problem we have $M = 1$ and $k = 1$. Using these and the initial condition, we solve for A.
$\frac{3}{4} = \frac{Ae^{(0)}}{1 + Ae^{(0)}} = \frac{A}{1+A},$
$A = 3$
$y = \frac{3e^{t}}{1+3e^{t}}$

34. For this problem we have $M = 3$ and $k = 1$. Using these and the initial condition, we solve for A.

$0 = \frac{3Ae^{3(0)}}{1 + Ae^{3(0)}} = \frac{3A}{1+A},$
$A = 0$
$y = 0$

35. (a) Substituting $r = Mk$ in
$y' = ry\left(1 - \frac{y}{M}\right)$ we get
$y' = Mk\left(1 - \frac{y}{M}\right) = ky(M-y)$
$\frac{1}{y(M-y)}y' = k$
Adapting the solution
$y = \frac{MAe^{Mkt}}{1+Ae^{Mkt}}$ in (2.7) with $r = Mk$,
we find $y = \frac{MAe^{rt}}{1+Ae^{rt}}$
In this case with $r = .71$, $M = 8\times10^7$ and $y(0) = 2\times10^7$, we find
$2\times10^7 = y(0) = \frac{8\times10^7 A}{1+A}.$
Therefore $\frac{A}{1+A} = \frac{2}{8} = \frac{1}{4}, A = 1/3,$
and after routine simplification we find
$y(t) = \frac{(8\times10^7)e^{.71t}}{3+e^{.71t}}$

(b)



(c) The biomass of halibut is given by
$y = \dfrac{(8 \times 10^7)e^{0.71t}}{3 + e^{0.71t}}$
The carrying capacity is 8×10^7 so we solve: $0.9\left(8 \times 10^7\right) = \dfrac{(8 \times 10^7)e^{0.71t}}{3 + e^{0.71t}}$
Solving gives $t \approx 4.642$ years

36. (a) $\left|\dfrac{y}{M-y}\right| = Ae^{Mkt}$ with $A > 0$. Under the circumstances $y > M$, the ratio is negative, and the resolution is
$\dfrac{y}{M-y} = -Ae^{Mkt}$.
This further resolves as
$y = -MAe^{Mkt} + yAe^{Mkt}$,
which eventually becomes
$y = \dfrac{MAe^{Mkt}}{Ae^{Mkt} - 1} = \dfrac{MAe^{rt}}{Ae^{rt} - 1}$.

(b) From Part (a), $y = \dfrac{MAe^{rt}}{Ae^{rt} - 1}$
Our initial condition is $y(0) = 3 \times 10^8$ which gives
$3 \times 10^8 = y(0) = \dfrac{(8 \times 10^7)A}{A - 1}$
$\dfrac{15}{4}(A - 1) = A, A = \dfrac{15}{11}$
After routine simplification this gives the equation $y = \dfrac{(12 \times 10^8)e^{0.71t}}{15e^{0.71t} - 11}$
We now want to solve
$y = 1.1M = (1.1)(8 \times 10^7)$ or
$(1.1)(8 \times 10^7) = \dfrac{(12 \times 10^8)e^{0.71t}}{15e^{0.71t} - 11}$
Solving gives $t \approx 2.94$ years

37. (a) Let A be the accumulated value at time t and d be the amount of the deposits made yearly, then A satisfies
$A' = 0.06A + d$
This differential equation separates to
$\dfrac{A'}{0.06A + d} = 1$ and integrates to
$\dfrac{\ln(0.06A + d)}{0.06} = t + c$ or
$0.06A + d = ke^{0.06t}$
At time $t = 0$, A is the unknown initial investment P,
hence $k = .06P + 2000$, and so
$.06A + 2000 = (.06P + 2000)e^{.06t}$.
If we want $A = 1{,}000{,}000$ at $t = 20$, we must have
$62000 = (.06P + 2000)e^{1.2}$
$P = \dfrac{62000e^{-1.2} - 2000}{.06} \approx \$277{,}901$

(b) As in Part (a), if A is the accumulated value at time t and d is the amount of the deposits made yearly, then A satisfies
$A' = 0.06A + d$
This differential equation separates to
$\dfrac{A'}{0.06A + d} = 1$ and integrates to
$\dfrac{\ln(0.06A + d)}{0.06} = t + c$ or
$0.06A + d = ke^{0.06t}$
We know that $A(0) = 10{,}000$ which gives
$0.06(10{,}000) + d = k$
and therefore $k = d + 600$ and
$0.06A + d = (d + 600)e^{0.06t}$
We want to find d when $t = 20$ and
$A = 1{,}000{,}000$:
$0.06(1{,}000{,}000) + d = (d + 600)e^{0.06(20)}$
$60{,}000 + d = (d + 600)e^{1.2}$
Solving for d gives
$d = \dfrac{60{,}000 - 600e^{1.2}}{e^{1.2} - 1} \approx \$25{,}002.16$

38. We start with $A'(t) = 0.08, A(t) - 12P$
$A(0) = 150{,}000$
where P is the payment made each month.
Solving this differential equation:
$\dfrac{A'}{0.08A - 12P} = 1$
$\dfrac{\ln(0.08A - 12P)}{0.08} = t + c$
$0.08A - 12P = ke^{0.08t}$
Using the initial condition gives
$k = 12000 - 12P$ We set $A(30) = 0$

(a) Solve for P:
$-12P = (12000 - 12P)e^{2.4}$
$P = \dfrac{12000e^{2.4}}{12(e^{2.4} - 1)} \approx \$1099.77$
Total amount paid:
$(30)(12)(1099.77) = \$395{,}917$
Total interest:
$395{,}917 - 150{,}000 = \$245{,}917$

(b) Reworking Exercise 38.(a):
$A'(t) = 0.075A(t) - 12P$

$A(0) = 150,000$
where P is the payment made each month. Solving this differential equation:
$\dfrac{A'}{0.075A - 12P} = 1$
$\dfrac{\ln(0.075A - 12P)}{0.075} = t + c$
$0.075A - 12P = ke^{0.075t} \quad (k = e^{0.075c})$
Using the initial condition gives
$k = 11250 - 12P$
We set $A(30) = 0$ and solve for P:
$-12P = (11250 - 12P)e^{2.25}$
$P = \dfrac{11250e^{2.25}}{12(e^{2.25} - 1)}$
$P \approx \$1047.95$
Total amount paid:
$(30)(12)(1047.95) = \$377,262$
Total interest:
$377,262 - 150,000 = \$227,262$
The half-percent decrease in interest decreases the total interest by $18655.

(c) Reworking Exercise 38.(a):
$A'(t) = 0.08A(t) - 12P$
$A(0) = 150,000$
where P is the payment made each month. Solving this differential equation:
$\dfrac{A'}{0.08A - 12P} = 1$
$0.08A - 12P = ke^{0.08t}$
$k = 12000 - 12P$
We set $A(15) = 0$ and solve for P:
$-12P = (12000 - 12P)e^{1.2}$
$P = \dfrac{12000e^{1.2}}{12(e^{1.2} - 1)} \approx \$1430.01$
The monthly payments are increased by about $330.
Total amount paid:
$(15)(12)(1430.01) = \$257,582$
The total amount is decreased by about $138,335.
Total interest:
$257,582 - 150,000 = \$107,582$

(d) Reworking Exercise 38.(a):
$A'(t) = 0.08A(t) - 12P$
$A(0) = 125,000$
where P is the payment made each month. Solving this differential equation:
$\dfrac{A'}{0.08A - 12P} = 1$
$0.08A - 12P = ke^{0.08t}$
$k = 10000 - 12P$ We set $A(30) = 0$ and solve for P:
$-12P = (10000 - 12P)e^{2.4}$
$P = \dfrac{10000e^{2.4}}{12(e^{2.4} - 1)} \approx \$916.47$
Total amount paid:
$(30)(12)(916.47) = \$329,930$
Total interest:
$329,930 - 125,000 = \$204,930$
By adding an additional down payment of $25000, the total interest is decreased by about $41000.

39. (a) Starting with $A' = .08A + 10{,}000$ with the initial condition $A(0) = 0$.
Solving gives $.08A + 10{,}000 = 10{,}000e^{.08t}$.
At time $t = 10$ we have
$A = \dfrac{10{,}000(e^{.8} - 1)}{.08} = \$153{,}193$
This would be the amount in his fund at age 40, and it would accumulate in the next 25 years to
$153{,}193e^{(.08)25} = \$1{,}131{,}949$.

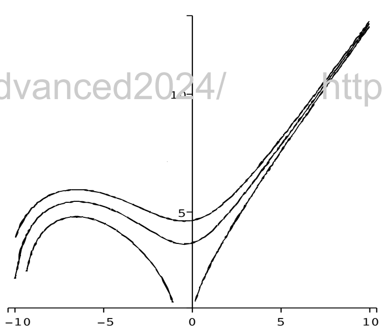
(b) We set up and solve the initial value problem:
$\dfrac{dA}{dt} = 0.08A + 20000, \quad A(0) = 0$
$\dfrac{1}{0.08} \ln|0.08A + 20000| = t + c$
$12.5 \ln|0.08(0) + 20000| = 0 + c$
$c = 12.5 \ln 20000$
At age 65, $t = 25$ and we have the equation
$12.5 \ln|0.08A + 20000|$
$= 20 + 12.5 \ln 20000$
Solving for A gives
$A = \dfrac{20000(e^{1.6} - 1)}{.08} \approx \$998,258$

(c) Following the conditions of Part (a), replacing however the 8% by an unknown force r, we come after ten years of payment and twenty-five additional years of accumulation to
$10{,}000\dfrac{(e^{10r} - 1)}{r}e^{25r}$.
For contrast, if the payment rate $10,000$ is replaced by $20,000$, and the payment interval of ten years is replaced by twenty-five years, we come to an accumulation after the twenty-five years of
$20{,}000\dfrac{(e^{25r} - 1)}{r}$.
This number is to be compared to the previous. Equating the two expressions leads to
$2(e^{25r} - 1) = e^{35r} - e^{25r}$ or
$3e^{25r} - 2 = e^{35r}$.

The equation can only be solved with the help of some form of technology, but the answer of r about .105(10.5%) can at least be checked.

40. $\dfrac{dA}{dt} = 0.1A - d, \quad A(0) = 1,000,000$
$10\ln|0.1A - d| = t + c$
$A = e^{t/10+c/10} + 10d$
$A = Be^{t/10} + 10d$
$1,000,000 = B + 10d$
$B = 1,000,000 - 10d$
$A = (1,000,000 - 10d)e^{t/10} + 10d$
We now want to determine d so that $A(30) = 0$
$0 = (1,000,000 - 10d)e^3 + 10d$
$d = \dfrac{1,000,000e^3}{10(e^3-1)} \approx \$105,240$

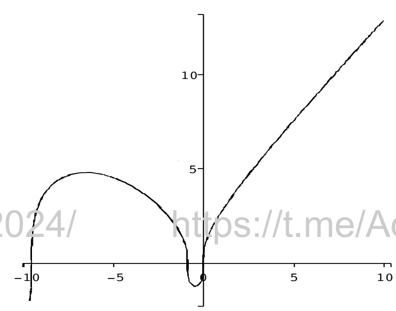
41. (a) Starting from
$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x + 3c}$
with $y(0) = 0$, we have $c = 0$. Therefore,
$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x}$



(b) The solution given in Part (a) is
$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x}$,
Notice that
$y' = \dfrac{3x^2+21x+9}{3\left(x^3 + \frac{21}{2}x^2 + 9x\right)^{2/3}}$
and this solution has a vertical tangent line at $x = 0$.

(c) Given $y' = \dfrac{x^2+7x+3}{y^2}$, that $y'(x)$ does not exist for a given x if $y(x) = 0$.
We see that $y(x) = 0$ if
$-3c = x^3 + \left(\frac{21}{2}\right)x^2 + 9x$
The cubic polynomial on the right, call it $h(x)$, has its derivative given by
$h'(x) = 3x^2 + 21x + 9 = 3(x2 + 7x + 3)$, and the roots of $h'(x)$ are
$x_1 = \dfrac{-7-\sqrt{37}}{2} \approx -6.5414$
$x_2 = \dfrac{-7+\sqrt{37}}{2} \approx -.4586$
The effect is that $h(x)$ has a relative maximum at x_1 and a relative minimum at x_2, and so the equation $-3c = h(x)$ has three solutions when $-3c$ lies between the relative minimum and the relative maximum, i.e., if $h(x_2) < -3c < h(x_1)$, or when
$\dfrac{-h(x_1)}{3} < c < \dfrac{-h(x_2)}{3}$
Therefore,
$c_1 = -\left(\dfrac{217+37\sqrt{37}}{12}\right) \approx -36.84$
$c_2 = \dfrac{-217+37\sqrt{37}}{12} \approx .67185.$

42. (a)



(b) When $c = c_2, h(x) = -3c_2$
In effect, $h(x) + 3c_2$
$= (x-x_2)^2\left(x + \dfrac{3c_2}{(x_2)^2}\right).$
Now, in the solution $y(x)$ to the differential equation, we have
$3y^2(x)y'(x) = \dfrac{d}{dx}(y^3) = \dfrac{d}{dx}(h(x)+3c_2)$
$= h'(x) = (x-x_1)(x-x_2)$,
, while$y^2(x) = [y^3(x)]^{2/3} = [h(x)+3c_2]^{2/3}$
$= (x-x_2)^{4/3}(x-x_3)^{2/3}$.
Now we can see that
$y'(x) = \dfrac{h'(x)}{3y^2(x)}$
$= \dfrac{(x-x_1)}{3(x-x_2)^{1/3}(x-x_3)^{2/3}}$
and this will become unbounded if x approaches either x_2 or x_3. These are the two points of vertical tangency.

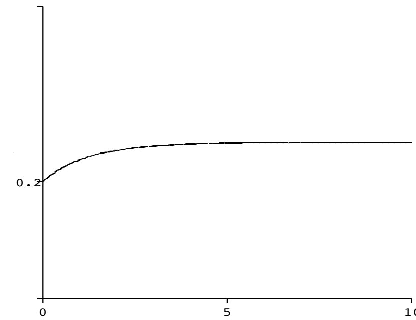
(c) Looking at Exercise s 41.(a) and 41.(b), the denominator of y' is
$3\left(x^3 + \frac{21}{2}x^2 + 9x\right)^{2/3}$

Setting this to 0 gives the equation
$x^3 + \frac{21}{2}x^2 + 9x = 0$

Solving gives $x = 0$ and
$x = \frac{-21 \pm 3\sqrt{33}}{4} \approx -9.5584, -0.94158$

43. When the given numbers are substituted for the given symbols, the differential equation becomes
$x' = (.4 - x)(.6 - x) - .625x^2$
$= \frac{3}{8}x^2 - x + \frac{6}{25} = \frac{3}{8}\left(x - \frac{12}{5}\right)\left(x - \frac{4}{15}\right).$
When separated it takes the form
$\frac{x'}{(x-b)(x-a)} = r$
in which $b = 12/5$, $a = 4/15 < b$, and $r = 3/8$.
By partial fractions we find
$\frac{1}{(x-b)(x-a)} = \frac{1}{(b-a)}\left\{\frac{1}{(x-b)}\frac{1}{(x-a)}\right\}$
and after integration we find
$\frac{1}{(b-a)}\ln\left|\frac{x-b}{x-a}\right| = rt + c_1$
or in this case with
$b - a = (36/15) - (4/15) = 32/15$,
$\ln\left|\frac{x - 12/5}{x - 4/15}\right| = \frac{32}{15}\left(\frac{3}{8}t + c_1\right)$
$= \frac{4}{5}t + c_2 \left(c_2 = \frac{32}{15}c_1\right).$
Using the given initial condition
$x = .2 = 1/5$ when $t = 0$, we find
$c_2 = \ln|(11/5)/(1/15)| = \ln(33)$,
$\ln\left|\frac{x - 12/5}{33\,(x - 4/15)}\right| = \frac{4}{5}t$ and
$\frac{x - 12/5}{33\,(x - 4/15)} = \pm e^{4t/5} = e^{4t/5}$
(the choice of sign is + since the left side is 1 when $x = 1/5$).

Concluding the algebra we find
$\frac{5x - 12}{11(15x - 4)} = e^{4t/5}$,
$5x - 12 = 11(15x - 4)e^{4/5}$,
$x = \frac{12 - 44e^{4t/5}}{5 - 11(15)e^{4t/5}} = \frac{4}{5}\left(\frac{3 - 11e^{4t/5}}{1 - 33e^{4t/5}}\right)$, and
it is apparent that $x \to \frac{4}{15}$ as $t \to \infty$.



44. The text should read (b) $x(0) = 0.6$.
In both cases, the general solution to the differential equation is as in Exercise 43:
$x = \frac{4\left(ke^{4t/5} - 3\right)}{5\left(3ke^{4t/5} - 1\right)}$
Notice that regardless of initial condition,
$\lim_{t\to\infty} x(t) = \frac{4}{15}$

(a) Using the initial condition $x(0) = 0.3$ gives $k = -21$ and the solution is
$x = \frac{4\left(-21e^{4t/5} - 3\right)}{5\left(-63e^{4t/5} - 1\right)}$

(b) Using the initial condition $x(0) = 0.6$ gives $k = -1.8$ and the solution is
$x = \frac{4\left(-1.8e^{4t/5} - 3\right)}{5\left(-5.4e^{4t/5} - 1\right)}$
This situation is impossible because the initial $x(0) = c = 0.6$. We are given that $b + c = 0.6$ and $c = d$ which means that $c < 0.6$, contradicting the initial condition.

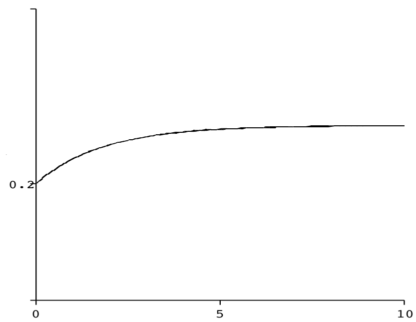
45. After beginning,
$x' = .6(.5 - x)(.6 - x) - .4x(0 + x)$
$= .6(.3 - 1.1x + x^2) - .4x^2$
$= .2x^2 - .66x + .18$
$= \frac{1}{5}\left(x^2 - \frac{33}{10}x + \frac{9}{10}\right)$
$= \frac{1}{5}(x - 3)\left(x - \frac{3}{10}\right).$
The parameters b, a, r are respectively $3, 3/10, 1/5$.
We jump ahead to
$\ln\left|\frac{x-3}{x - 3/10}\right| = \frac{27}{10}\left(\frac{t}{5} + c_1\right) = \frac{27}{50}t + c_2.$
In this case with $x = .2 = 1/5$ when $t = 0$, we find
$c_2 = \ln\left|\frac{(1/5) - 3}{(1/5) - (3/10)}\right| = \ln 28$,
$\frac{x-3}{28\,(x - 3/10)} = \pm e^{27/50} = e^{27t/50}$,

and the conclusion is

$5(x-3) = 14(10x-3)e^{27t/50}$,

$x = \dfrac{15-42e^{27t/50}}{5-140e^{27t/50}} = \dfrac{42-15e^{-27t/50}}{140-5e^{-27t/50}}$

$= \dfrac{3}{5}\left(\dfrac{14-5e^{-27t/50}}{28-e^{-27t/50}}\right)$.



46. $x'(t) = (0.6-x)(0.4-x) - 0.4x(0.1+x)$

$x(0) = 0.2$

$\dfrac{x'(t)}{x^2-1.4x+0.2} = 1$

$\displaystyle\int \frac{25}{15x^2-26x+6}dx = t+c$

$\dfrac{25\sqrt{79}}{158}\ln\left|\dfrac{15x-\sqrt{79}-13}{15x+\sqrt{79}-13}\right| = t+c$

$\dfrac{15x-\sqrt{79}-13}{15x+\sqrt{79}-13} = ce^{rt}$

where $r = \dfrac{158}{25\sqrt{79}}$.

With $t=0$ and $x=0.2$, you can solve to get $C \approx 17$.

$x = \dfrac{\sqrt{79}+13+(\sqrt{79}-13)Ce^{rt}}{15(1-Ce^{rt})}$

47. (a) We find

$y' = .025y(8-y) - .2 = -.025(y^2-8y+8)$

$= -\dfrac{1}{40}(y-b)(y-a)$, in which

$b = 4+\sqrt{8}, a = 4-\sqrt{8}$.

This leads to $\ln\left|\dfrac{y-b}{y-a}\right| = -\dfrac{1}{40}\left(2\sqrt{8}\right)t + c_2$

and with $y(0) = 8$ we have

$\ln\left|\dfrac{8-b}{8-a}\right| = c_2$,

$\ln\left|\dfrac{(y-b)(8-a)}{(y-a)(8-b)}\right| = \dfrac{-t\sqrt{8}}{20}$,

$\dfrac{y-b}{y-a} = \dfrac{8-b}{8-a}e^{-\dfrac{t\sqrt{8}}{20}}$.

We can see that as $t \to \infty$ the right side goes to zero, hence also the left side, and hence

$y \to b = 4+\sqrt{8} = 6.828427$

This represents an eventual fish population of about 682,800.

(b) We set up the differential equation. In this case we have to complete the square:

$y' = 0.025y(8-y) - 0.6$

$= -0.025(y^2-8y+24)$

$= -0.25[(y-4)^2+8]$

$\dfrac{y'}{(y-4)^2+8} = -0.025$

$\displaystyle\int \frac{1}{(y-4)^2+8}\,dy = -0.025t+c$

To integrate, we will use the substitution $u = \dfrac{y-4}{\sqrt{8}}$ which gives us

$\displaystyle -0.025t+c = \int \frac{1}{(y-4)^2+8}\,dy$

$\displaystyle = \int \frac{\sqrt{8}}{8(u^2+1)}du = \frac{1}{2\sqrt{2}}\tan^{-1}u$

$= \dfrac{1}{2\sqrt{2}}\tan^{-1}\left(\dfrac{y-4}{\sqrt{8}}\right)$

Manipulating gives

$y = 4+2\sqrt{2}\tan\left(\dfrac{-t\sqrt{2}}{20}+k\right)$

The initial condition, $y(0) = 8$ gives us $k = \tan^{-1}\sqrt{2} \approx 0.9553$ and therefore

$y = 4+2\sqrt{2}\tan\left(-\dfrac{t\sqrt{2}}{20}+0.9553\right)$

And, if you graph y, it is easy to see that $y(27.02) = 0$ and therefore the fish all die off in about 27 years.

48. The equilibrium solutions are the algebraic solutions to the quadratic equation $.025P(8-P) - R = 0$, or $P^2-8P-40R = 0$. In the process of studying Exercise 47.b ($R = .2$) we found it convenient to factor the left side (P was y at the time) and the roots were $b = 4+\sqrt{8}$ and $a = 4-\sqrt{8}$. In Exercise 47.b, the corresponding equation ($R = .6$) would be

$0 = P^2-8P+40R = P^2-8P+24$.

But this equation has no real roots, hence no equilibrium populations.

49. $P' = .05P(8-P) - .6$

$= -\dfrac{1}{20}(P^2-8P+12)$

$= -\dfrac{1}{20}(P-6)(P-2)$

Following well-established procedures, we come to

$\ln\left|\dfrac{P-6}{P-2}\right| = -\dfrac{1}{5}t + c_2,$

$\dfrac{P-6}{P-2} = Ae^{-\frac{t}{5}}, (A = \pm e^{c_2} \text{ or zero})$

We learn from this relation that the ratio $(P-6)/(P-2)$ never changes sign, always negative if the initial condition has $P(0)$ in the interval $(2,6)$. Clearly in this case the exponential approaches zero as $t \to \infty$ and P approaches 6. This last conclusion is true even if $P(0) > 6$.

If on the other hand $0 \le P(0) < 2$, the ratio is forever positive, and we find eventually $\dfrac{P-6}{P-2} = \dfrac{P(0)-6}{P(0)-2}e^{-t/5}$.

Here the right side is a positive decreasing function of t and so must be the left side. The effect is that P itself is decreasing (not obvious) and reaches the value zero when

$e^{-t/5} = 3\dfrac{P(0)-2}{P(0)-6}$ or when

$t = 5\ln\dfrac{6-P(0)}{3[2-P(0)]} = 5\ln\dfrac{6-P(0)}{6-3P(0)}$

In the ratio inside the (second) ln, the numerator is clearly more than the denominator, which is itself positive. This is some moment of positive time, after which the population is zero and no further activity occurs.

50. Comparing Exercise's 47.(b) and 49, we see that the equations are the same except for the natural growth rates (0.2 in Exercise 47.b, 0.4 in Exercise 49). The fish in Exercise 64 die out whereas the fish population in Exercise 49 approaches a limiting population.

51. The differential equation is

$r'(t) = k[r(t) - S]$. This separates as

$\dfrac{r'}{r-S} = k$, and solves as

$\ln(r-S) = kt + c$.

In this case $S = 1000$, $r(0) = 14{,}000$, and $r(4) = 8{,}000$.

Putting $t = 0$, we see that the constant c is $\ln 13{,}000$, we learn

$\ln\dfrac{r-1000}{13{,}000} = kt,$

and putting $t = 4$,

$\ln\dfrac{7}{13} = \ln\dfrac{7{,}000}{13{,}000} = 4k.$

Assembling the available information, we find $\ln\dfrac{r-1000}{13{,}000} = kt = \dfrac{t}{4}(4k) = \dfrac{t}{4}\ln\dfrac{7}{13}$, and

$r = 1{,}000 + 13{,}000\left(\dfrac{7}{13}\right)^{t/4},$

or equivalently $r = 1 + 13e^{-.15476t}$ thousands.

52. The amount of grain is

$A(t) = -1000t + 6000$

The differential equation for $S(t)$ is

$S'(t) = 0.02A(t) = -20t + 120$

$S(0) = 0,$

We solve this to get

$S(t) = -10t^2 + 120t$

$S(6) = 360$

53. From the differential equation, with $z = y'/y$, we find $z = k(M-y)$. This is a line in the (y,z)-plane. The z-intercept is M and the slope is $-k$.

We estimate the derivative, y', at each point by using the adjacent point and computing the slope:

t	2	3	4	5
y	1197	1291	1380	1462
y'		94	89	82
$z = y'/y$		0.073	0.064	0.056

We now plot the (y,z) data and find a slope and intercept. By looking at the graph or by picking two points you can see that slope is about -9.4×10^{-5} and the z-intercept is about 2037. This gives us $M \approx 2037$

54. If $y' = ky(M-y)$, then by the product rule

$y'' = k[y'(M-y) - yy'] = ky'[M-2y].$

This will be zero when $y = M/2$. In what follows, we make exception of the two equilibrium solutions $y \equiv 0$ and $y \equiv M$. With any other solution, $y \ne 0$, $y \ne M$, and $y' \ne 0$. Thus whatever time t_0 (if any) at which y becomes $M/2$ is sure to be an inflection time. Moreover, there can be *no circumstances of inflection other than* $y = M/2$. Such a time $t_0 > 0$ is bound to occur if and only if $0 < y(0) < M/2$, in which case the time t_0 is unique.

55. The given differential equation is

$\dfrac{dv}{dt} = 9.8 - 0.002v^2$

$\dfrac{dv}{dt} = -\dfrac{2}{1000}\left(v^2 - 4900\right)$

$k = -\dfrac{2}{1000}$

$\dfrac{dv}{dt} = k\,(v+70)\,(v-70)$

As the value of k is a negative number, the parameters b and a $(b > a)$ are $b = 70$ and $a = -70$.

Thus, the solution is
$\ln\left|\dfrac{v-70}{v+70}\right| = 140kt + c$
Given that $v(0) = 0,$ we find $c = 0$ and
$\dfrac{70-v}{70+v} = e^{140kt}$

Because $k < 0$, the right hand side goes to zero as t goes to infinity. Therefore, $v \to 70$.This is the terminal velocity.

56. The tangent line to $y = f(x)$ at $x = a$ passes through the point $(a, f(a))$ with slope $f'(a)$, and hence the equation is
$y - f(a) = f'(a)(x - a)$

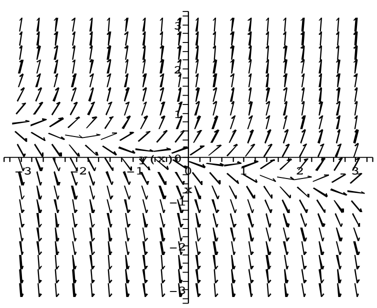
We find the $x-$ and $y-$ intersections of this line
$y = f(a) - af'(a),$
$x = \dfrac{-f(a) + af'(a)}{f'(a)}$
and hence the area is
$A(a) = [f(a) - af'(a)] \cdot \dfrac{-f(a) + af'(a)}{f'(a)}$
$= -\dfrac{1}{2}\left[a^2 f'(a) - 2af(a) + \dfrac{f(a)^2}{f'(a)}\right]$ and
$\dfrac{dA}{da} = -\dfrac{1}{2}\left[2af'(a) - 2f(a)\right]$

Setting $\dfrac{dA}{da} = 0$ we get
$2af'(a) = 2f(a), f'(a) = f(a)$

This means that a curve such that A is the same for any choice of $a > 0$ satisfies
$y'(x) = y(x)$ for all x.
Hence
$\dfrac{dy}{dx} = y, \dfrac{dy}{y} = dx, \ln|y| = x + c$
$y = ke^x$

7.3 Direction Fields and Euler's Method

1.



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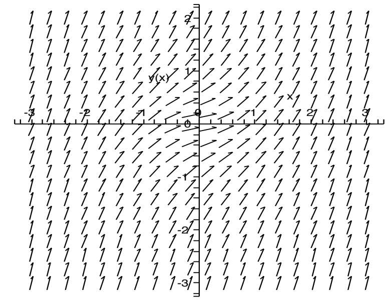

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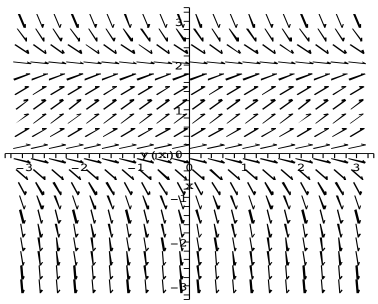

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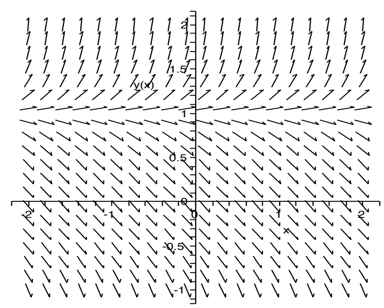

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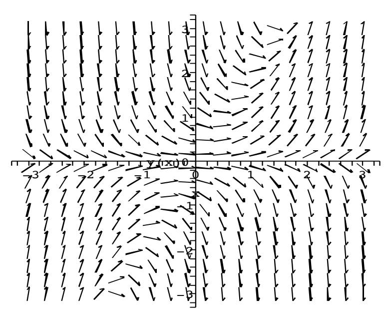

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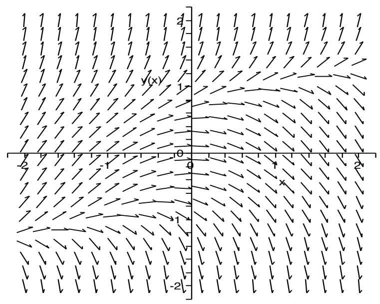


7. Field C

8. Field B

9. Field D

10. Field F

11. Field A

12. Field E

13. For $h = 0.1$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.0	1	0
1	0.1	1	0.2
2	0.2	1.02	.408
3	0.3	1.0608	.63648
10	1.0	2.334633363	4.669266726
20	2.0	29.49864321	117.9945728

For $h = 0.05$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.00	1	0
1	0.05	1	.10
2	0.10	1.0050	.201000
3	0.15	1.01505000	.3045150000
20	1.00	2.510662314	5.021324628
40	2.00	39.09299942	156.3719977

14. For $h = 0.1$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.0	2	0
1	0.1	2	0.05
2	0.2	2.005	0.0997506
3	0.3	2.01498	0.148885
10	1.0	2.21504	0.45146
20	2.0	2.80022	0.714229

For $h = 0.05$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.00	2	0
1	0.05	2	0.025
2	0.10	2.00125	0.0499688
3	0.15	2.00375	0.0748597
20	1.00	2.22563	0.449312
40	2.00	2.81443	0.710622

15. First for $h = 0.1$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.0	1	3
1	0.1	1.3	3.51
2	0.2	1.651	3.878199
3	0.3	2.0388199	3.998493015
10	1.0	3.847783601	.58569576
20	2.0	3.999018724	0.00392415

For for $h = 0.05$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.00	1	3
1	0.05	1.15	3.2775
2	0.10	1.313875	3.529232484
3	0.15	1.490336624	3.740243243
20	1.00	3.818763110	.69210075
40	2.00	3.997787406	0.00884548

16. For $h = 0.1$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.0	2	0
1	0.1	2	0.025
2	0.2	2.0025	0.0498752
3	0.3	2.00749	0.0744416
10	1.0	2.10745	0.225158
20	2.0	2.39672	0.348172

For $h = 0.05$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.00	2	0
1	0.05	2	0.0125
2	0.10	2.00062	0.0249844
3	0.15	2.00187	0.0374298
20	1.00	2.11272	0.224036
40	2.00	2.40349	0.346214

17. For $h = 0.1$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.0	3	-3
1	0.1	2.7	-2.604837418
2	0.2	2.439516258	-2.258247011
3	0.3	2.213691557	-1.954509778
10	1.0	1.300430235	$-.6683096762$
20	2.0	.9587323942	-0.0940676774

For $h = 0.05$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.0	3	-3
1	0.05	2.85	-2.801229424
2	0.10	2.709938529	-2.614775947
3	0.15	2.579199732	-2.439907708
20	1.00	1.334942742	$-.7028221832$
40	2.00	.9795316061	$-.1148668893$

18. For $h = 0.1$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.0	1	0.841471
1	0.1	1.08415	0.873905
2	0.2	1.17154	0.881349
3	0.3	1.25967	0.86199
10	1.0	1.67065	-0.00498132
20	2.0	0.418744	-3.59339

For $h = 0.05$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.00	1	0.841471
1	0.05	1.04207	0.860952
2	0.10	1.08512	0.87436
3	0.15	1.12884	0.881416
20	1.00	1.6513	-0.003238
40	2.00	0.291667	-3.71245

19. For $h = 0.1$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.0	1	1.0
1	0.1	1.10	1.095445115
2	0.2	1.209544512	1.187242398
3	0.3	1.328268752	1.276036344
10	1.0	2.395982932	1.842819289
20	2.0	4.568765342	2.562960269

For $h = 0.05$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.00	1	1
1	0.05	1.05	1.048808848
2	0.10	1.102440442	1.096558454
3	0.15	1.157268365	1.143358371
20	1.00	2.420997836	1.849593965
40	2.00	4.620277218	2.572989937

20. For $h = 0.1$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.0	4	4
1	0.1	4.4	4.40114
2	0.2	4.84011	4.84424
3	0.3	5.32454	5.33298
10	1.0	10.3981	10.4461
20	2.0	27.0677	27.1414

For $h = 0.05$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.00	4	4
1	0.05	4.2	4.2003
2	0.10	4.41001	4.41115
3	0.15	4.63057	4.633
20	1.00	10.6384	10.6853
40	2.00	28.326	28.3965

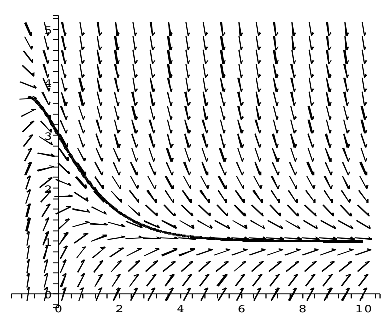
21. (a) The exact solution to Exercise 13 is
$y(x) = e^{x^2}$
$y(1) \approx 2.718281828$
$y(2) \approx 54.59815003$

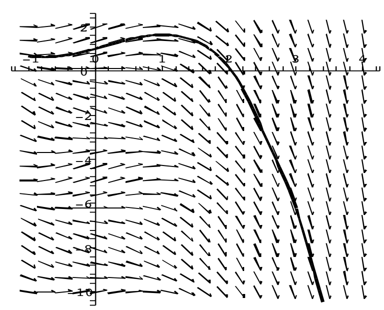
(b) The exact solution to Exercise 14 is
$y(x) = \sqrt{x^2 + 4}$
$y(1) \approx 2.236067977$
$y(2) \approx 2.828427124$

22. (a) The exact solution to Exercise 15 is
$y(x) = \dfrac{4}{1 + 3e^{-4x}}$
$y(1) \approx 3.791659974$
$y(2) \approx 3.995978495$

(b) The exact solution to Exercise 16 is
$y(x) = \dfrac{1}{2}(12x^2 + 64)^{1/3}$
$y(1) \approx 2.117911792$
$y(2) \approx 2.410142264$

23.





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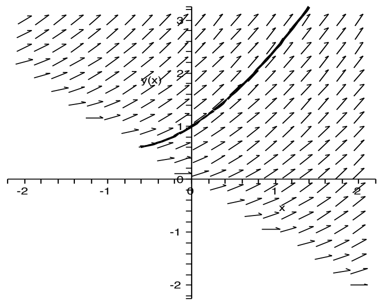
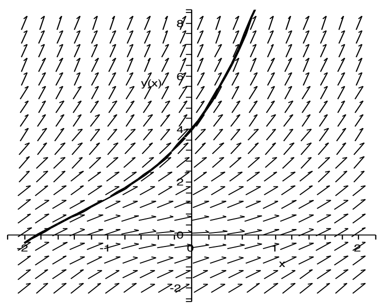
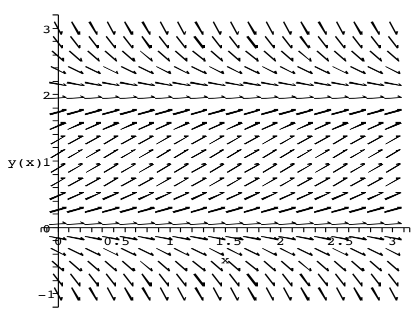
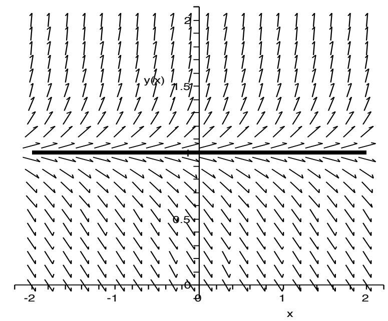




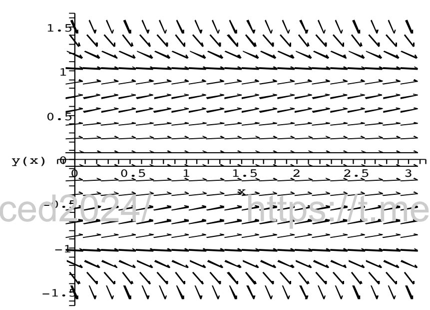
25. Equilibrium solutions come from $y' = 0$, which only occur when $y = 0$ or $y = 2$. From the direction field, $y = 0$ is seen to be an unstable equilibrium and $y = 2$ is seen to be a stable equilibrium.



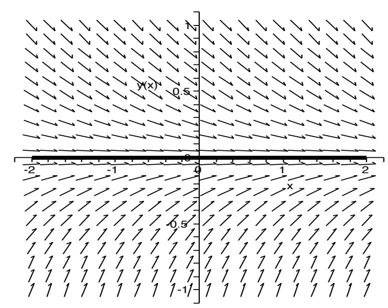
26. Equilibrium solutions come from $y' = 0$, which only occur when $y = 1$. From the direction field, $y = 1$ is seen to be an unstable equilibrium.



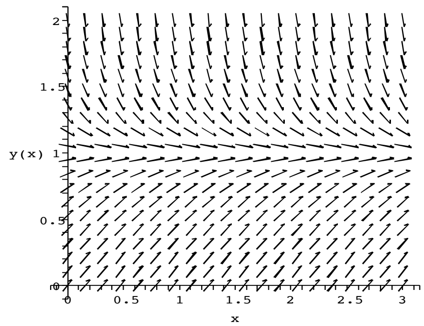
27. Equilibrium solutions come from $y' = 0$, which only occur when $y = 0$ or $y = \pm 1$. From the direction field, $y = 0$ and $y = -1$ are seen to be an unstable equilibrium and $y = 1$ is seen to be a stable equilibrium.



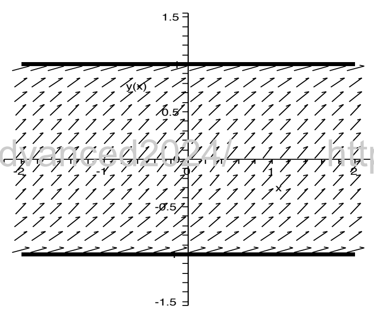
28. Equilibrium solutions come from $y' = 0$, which only occur when $e^{-y} = 1$ or when $y = 0$. From the direction field, $y = 0$ is seen to be a stable equilibrium.



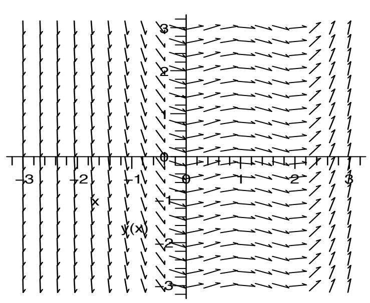
29. Equilibrium solutions come from $y' = 0$, which only occur when $y = 1$. From the direction field, $y = 1$ is seen to be a stable equilibrium.



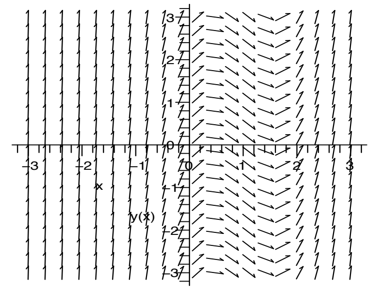
30. Equilibrium solutions come from $y' = 0$, which only occur when $y^2 = 1$ or when $y = -1$ and $y = 1$. From the direction field, $y = -1$ is seen to be an unstable equilibrium and $y = 1$ is seen to be a stable equilibrium.



31.



32.



33. Using Euler's method:

For $h = 0.1$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.00	3.0000	8.0000
1	0.10	3.8000	13.4400
2	0.20	5.1440	25.4607
3	0.30	7.6901	58.1372
4	0.40	13.5038	181.3525
5	0.50	31.6390	1000.0295

For $h = 0.05$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.00	3.0000	8.0000
1	0.05	3.4000	10.5600
2	0.10	3.9280	14.4292
3	0.15	4.6495	20.6175
4	0.20	5.6803	31.2662
10	0.50	218.1215	47576.0009

For $h = 0.01$:

n	x_n	$y(x_n)$	$f(x_n, y_n)$
0	0.00	3.0000	8.0000
1	0.01	3.0800	8.4864
9	0.09	3.9396	14.5203
10	0.10	4.0848	15.6855
20	0.20	6.5184	41.4900
21	0.21	6.9333	47.0711
30	0.30	15.8434	250.0139

x	Exact
0.000	3.0000
0.100	4.1374
0.200	6.8713
0.300	21.4869
0.400	−18.7351
0.500	−6.5688

34. The first part is just a matter of checking:

$$f'(x) = [f(x)]^2 - 1 = \frac{8e^{2x}}{(2 - e^{2x})^2}$$

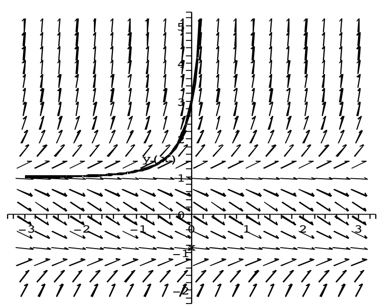
For the second part,
$f(0.1) \approx 4.1374$

$f(0.2) \approx 6.8713$
$f(0.3) \approx 21.4869$
$f(0.4) \approx -18.7351$
$f(0.5) \approx -6.5688$

35. The vertical asymptote in the solution occurs when the denominator vanishes, which is to say when $e^{2x} = 1/k$, or $x = -\ln(k)/2$. In our case, with $y(0) = 3$, we have $k = 1/2$ and the vertical asymptote at $x = \ln(2)/2 = .3466$.

The field diagram cannot give any fore-warning of the vertical asymptote. Dependent as the field equations are only on y, they can only hint at things which likewise depend on y. The location of the vertical asymptote, by its very nature an x-measurement, is instead dependent directly on the solution-parameter k and indirectly on the initial condition.

In this case where the actual x-value does not enter the calculations, the Euler process merely generates the numbers in the recursive sequence $y_n = hy_{n-1}^2 + y_{n-1} - h$ subject to an initial condition of $y_o = 3$. The numbers in such a sequence will increase to infinity, with growth rate depending on h. The simultaneous determination of x_n through the law $x_n = hn$ has nothing to do with the geometry of the solution to the differential equation. "Jumping over the asymptote" is the pseudo-event which happens when n passes from below $\frac{.3466}{h}$ to above, has no special relation to the Euler y-numbers, and no relation whatever to the solution of the differential equation.



36. When x moves across the vertical asymptote, the values for y change from positive to negative.

This means that if y represents force on a robot arm then it doesn't make sense for the force to approach infinity in a finite amount of time (which is what the vertical asymptote represents). The Euler approximation with a small step size probably gives a better idea of the actual force that the robot arm will actually encounter.

37. For $h = 1$

x	*Euler*	*Exact*	*Error*
0	1	4	3
0.1	3.33333333	4.5301951	1.1968618
0.2	3.85185185	5.0420817	1.19022993
0.3	4.38445358	5.5197958	1.1353422
0.4	4.91286010	5.9516999	1.03883983
0.5	5.41841632	6.3311318	0.91271546
0.6	5.88468616	6.6561471	0.77146092
0.7	6.29961809	6.9285769	0.628959
0.8	6.65667665	7.152808	0.49613166
0.9	6.9547456	7.3346184	0.37987281
1	7.19706156	7.4802467	0.28318508
1.1	7.38968814	7.595759	0.20607056
1.2	7.54002195	7.6866742	0.14665227
1.3	7.65563011	7.7577945	0.10216436
1.4	7.74350906	7.8131639	0.06965481
1.5	7.80971372	7.8561103	0.0463966
1.6	7.85924977	7.8893249	0.03007510
1.7	7.89612281	7.9149552	0.01883258
1.8	7.9234637	7.9346994	0.01123572

For $h = 0.1$

x	*Euler*	*Exact*	*Error*
0	1	4	3
0.1	1.23333333	4.53019515	3.2968618
0.2	1.51151852	5.04208178	3.53056327
0.3	1.83843385	5.51979585	3.681362
0.4	2.21602156	5.95169993	3.73567836
0.5	2.64326894	6.33113178	3.68786284
0.6	3.11524497	6.65614708	3.54090211
0.7	3.62248525	6.92857689	3.3060916
0.8	4.15106801	7.15280830	3.0017403
0.9	4.68364062	7.33461843	2.65097781
1	5.20139514	7.48024665	2.27885151
1.1	5.6866168	7.5957587	1.9091419
1.2	6.12512759	7.68667421	1.56154663
1.3	6.50792201	7.75779446	1.24987245
1.4	6.83159959	7.81316387	0.98156428
1.5	7.09766771	7.8561103	0.75844261
1.6	7.31114954	7.88932487	0.57817533
1.7	7.47902583	7.91495539	0.43592956
1.8	7.60890514	7.93469943	0.32579429

The smaller we make h (Time Step) the more accurate approximation at a given point tends to be. As well the smaller the value of h the more steps it takes to reach a given value of x.

38. $f(x) = x^2 - 2, f'(x) = 2x$

Use $f(x)$ and $f'(x)$ in $x'(t) = -\dfrac{f(x(t))}{f'(x(t))}$.

$x'(t) = -\dfrac{\left[(x(t))^2 - 2\right]}{2x(t)}$

Let $x(t) = z$.

$x'(t) = \dfrac{dz}{dt}$

$\dfrac{dz}{dt} = -\dfrac{\left[z^2 - 2\right]}{2z}$

Hence, $\dfrac{2z}{2 - z^2} dz = dt$

Integrate both sides.

$-\ln\left(2 - z^2\right) = t + c_1$

$2 - z^2 = ce^{-t}$ where $c = e^{-c_1}$

$z = \sqrt{2 - ce^{-t}}$

$x(t) = \sqrt{2 - ce^{-t}}$

$x(0) = 1 \Rightarrow 1 = \sqrt{2 - c(1)}$

$c = 1$

Therefore, $x(t) = \sqrt{2 - e^{-t}}$.

$\lim\limits_{x\to\infty} x(t) = \lim\limits_{x\to\infty} \sqrt{2 - e^{-t}}$

$= \sqrt{2 - 0}$ $\left(\text{As } t \to \infty, e^{-t} \to 0\right)$

$= \sqrt{2}$

Euler's Method

$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\, dx$

is used to solve

$\dfrac{dy}{dx} = f(x, y)$

$x_1 = x_0 + h$

$y_1 = y_0 + hf(x_0, y_0)$

Subsequently,

$y_{n+1} = y_n + hf(x_n, y_n)$

$y_{n+1} = y_n + h \left.\dfrac{dy}{dx}\right|_{x=x_n}$

$\dfrac{y_{n+1} - y_n}{\left.\frac{dy}{dx}\right|_{x=x_n}} = h$

While solving for y=0 in Newton's Method, we make an assumption that $y_{n+1} = 0$ Hence ,

$\dfrac{-y_n}{\left.\frac{dy}{dx}\right|_{x=x_n}} = h$

$x_{n+1} = x_n + h = x_n - \dfrac{y_n}{\left.\frac{dy}{dx}\right|_{x=x_n}}$

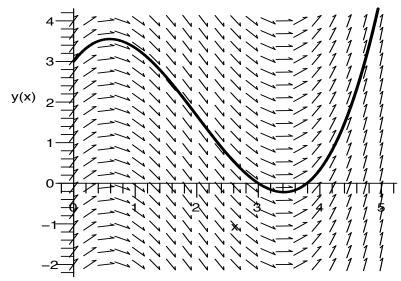
This is Newton's Method.

39. The general solution is

$y = \dfrac{x^3}{3} - 2x^2 + 2x + c.$

Using the initial condition $y(3) = 0$ gives $y(0) = c = 3$ and therefore

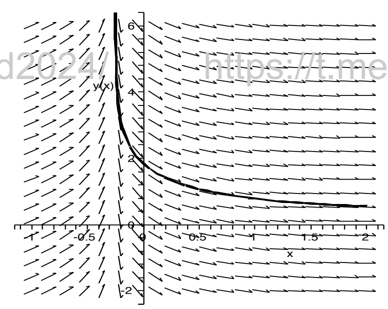
$y = \dfrac{x^3}{3} - 2x^2 + 2x + 3$



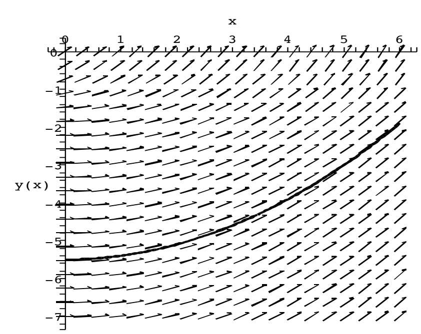
40. Integrating gives

$y = \dfrac{1}{4} - \dfrac{13}{16}\ln(4x + 1) + c$

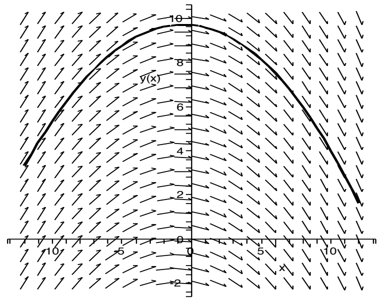
Using the initial condition $y(8) = 1$ gives $c = -1 + \dfrac{13}{16}\ln 33$ and therefore $y = \dfrac{1}{4} - \dfrac{13}{16}\ln(4x + 1) - 1 + \dfrac{13}{16}\ln 33$ $y(0) \approx 1.8409$

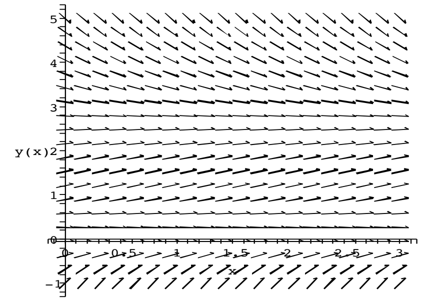


41. Using a CAS gives $y(0) \approx -5.55$.



42. Using a CAS gives $y(0) \approx 9.6832$.





43. The equilibrium solutions are the constant solutions to the DE. If indeed g is a certain constant k, then
$0 = g' = -k + 3k^2/(1+k^2) = -k(k^2 - 3k + 1)/(k^2+1)$. Thus $k = 0$ is clearly one solution, while the two roots of the quadratic in the numerator are also solutions. These are the numbers
$a = \dfrac{3-\sqrt{5}}{2} \approx .3820$ and

$b = \dfrac{3+\sqrt{5}}{2} \approx 2.6810.$

Of the three, 0 and b are stable, while a is unstable. As a result of this stability feature,
$\lim_{t\to\infty} g(t) = 0$ if $0 \le g(0) < a$, while
$\lim_{t\to\infty} g(t) = b$ if $a < g(0)$.

As the problem evolves, g depends not only on time t, but on a certain real parameter x. We could write $g = g_x(t)$, and the dependence on x is through the initial condition:
$g_x(0) = \dfrac{3}{2} + \dfrac{3\sin(x)}{2}.$

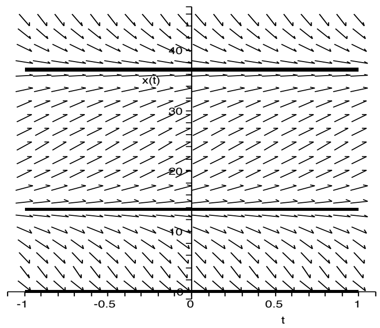
With x restricted to the interval $[0, 4\pi]$ (4π being about 12.5664), the first event ($g_x(0) < a$, equivalently $\lim_{t\to\infty} g_x(t) = 0$, equivalently eventual black - stripe zone) occurs when x lies in one of the two intervals $(3.9827, 5.4421)$ or $(10.2658, 11.7253)$. More precisely, these are the intervals with endpoints
$\dfrac{3\pi}{2} \pm \cos^{-1}\left(\dfrac{\sqrt{5}}{3}\right)$ and $\dfrac{7\pi}{2} \pm \cos^{-1}\left(\dfrac{\sqrt{5}}{3}\right)$.

44. If $k = 10$ then the differential equation becomes
$x' = \dfrac{-0.01x(x^3 - 10x^2 + 101x - 10)}{1+x^2}$

It is clear that $x = 0$ is a solution. The other solution(s) come from solving
$g(x) = x^3 - 10x^2 + 101x - 10 = 0$

Notice that $g'(x) = 3x^2 - 20x + 101$ and $g'(x) = 0$ has no real solutions (use the quadratic formula). This means that $g'(x)$ is always positive and $g(x)$ is always increasing, which means there is exactly one real solution to $g(x) = 0$. By graphing (or using Newton's method, for example), one case see that an equilibrium solution is $x = 0.0999899$ and this must be the only solution.

If $k = 50$, then the differential equation becomes
$x' = \dfrac{-0.002x(x^3 - 50x^2 + 501x - 50)}{1+x^2}$
Again, $x = 0$ is a solution, and the other solutions come from solving
$g(x) = x^3 - 50x^2 + 501x - 50 = 0$
If you graph $g(x)$, you can see that there are now three positive solutions $x \approx 0.10081$, $x \approx 13.7018$, $x \approx 36.1974$. If we look at the direction field, notice that the middle equilibrium solution ($x = 36.2$) is *unstable*—a small decrease in population will send the population to the lower equilibrium solution whereas a small increase in population will send the population towards the higher equilibrium solution.



7.4 Systems of First-Order Differential Equations

1. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.2x - 0.2x^2 - 0.4xy$
$0 = -0.1y + 0.2xy$
$0 = x(0.2 - 0.2x - 0.4y)$
$0 = y(-0.1 + 0.2x)$
$x = 0$ or $0.2 - 0.2x - 0.4y = 0$
$y = 0$ or $x = 0.5$
The equilibrium points are
$(0, 0)$, corresponding to the case where there are no predators or prey
$(1, 0)$, corresponding to the case where there are 200 prey but no predators
$(0.5, 0.25)$, corresponding to the having both populations constant, with two times as many prey as predators.

2. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.4x - 0.1x^2 - 0.2xy$
$0 = -0.2y + 0.1xy$
$0 = x(0.4 - 0.1x - 0.2y)$
$0 = y(-0.2 + 0.1x)$
$x = 0$ or $0.4 - 0.1x - 0.2y = 0$
$y = 0$ or $x = 2$
The equilibrium points are
$(0, 0)$, corresponding to the case where there are no predators or prey
$(4, 0)$, corresponding to the case where there are 400 prey but no predators
$(2, 1)$, corresponding to the having both populations constant, with two times as many prey as predators.

3. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.3x - 0.1x^2 - 0.2xy$
$0 = -0.2y + 0.1xy$
$0 = x(0.3 - 0.1x - 0.2y)$
$0 = y(-0.2 + 0.1x)$
$x = 0$ or $0.3 - 0.1x - 0.2y = 0$
$y = 0$ or $x = 2$
The equilibrium points are
$(0, 0)$, corresponding to the case where there are no predators or prey
$(3, 0)$, corresponding to the case where there are 300 prey but no predators
$(2, 0.5)$, corresponding to the having both populations constant, with four times as many prey as predators.

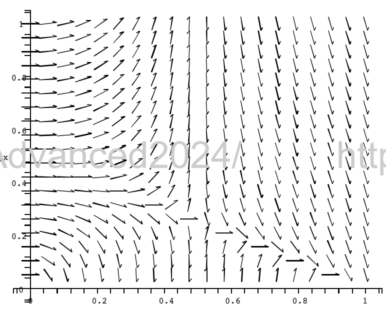
4. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.1x - 0.21x^2 - 0.4xy$
$0 = -0.1y + 0.2xy$
$0 = x(0.1 - 0.1x - 0.4y)$
$0 = y(-0.1 + 0.2x)$
$x = 0$ or $0.1 - 0.1x - 0.4y = 0$
$y = 0$ or $x = 0.5$
The equilibrium points are
$(0, 0)$, corresponding to the case where there are no predators or prey
$(1, 0)$, corresponding to the case where there are 100 prey but no predators
$(0.5, 0.125)$, corresponding to the having both populations constant, with four times as many prey as predators.

5. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.2x - 0.1x^2 - 0.4xy$
$0 = -0.3y + 0.1xy$
$0 = x(0.2 - 0.1x - 0.4y)$
$0 = y(-0.3 + 0.2x)$
$x = 0$ or $0.2 - 0.1x - 0.4y = 0$
$y = 0$ or $x = 1.5$
The equilibrium points are
$(0, 0)$, corresponding to the case where there are no predators or prey
$(2, 0)$, corresponding to the case where there are 200 prey but no predators
$(1.5, 0.125)$, corresponding to the having both populations constant, with twelve times as many prey as predators.

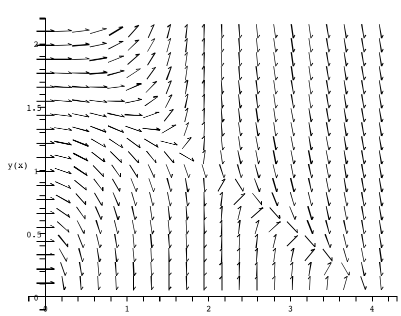
6. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the

equations, we have
$0 = 0.2x - 0.1x^2 - 0.4xy$
$0 = -0.2y + 0.1xy$
$0 = x(0.2 - 0.1x - 0.4y)$
$0 = y(-0.2 + 0.1x)$
$x = 0$ or $0.2 - 0.1x - 0.4y = 0$
$y = 0$ or $x = 2$
The equilibrium points are
$(0, 0)$, corresponding to the case where there are no predators or prey
$(2, 0)$, corresponding to the case where there are 200 prey but no predators.

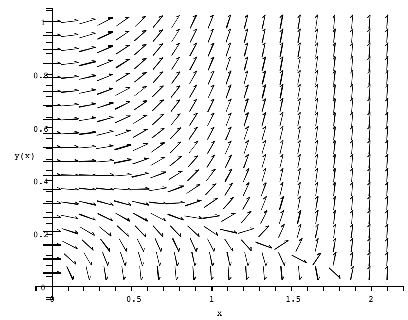
7. In Exercise 1,
$\dfrac{dy}{dx} = \dfrac{0.2x - 0.2x^2 - 0.4xy}{-0.1y + 0.2xy}$
From the following phase portrait, we observe that
$(0, 0)$ is an unstable equilibrium,
$(1, 0)$ is a stable equilibrium,
$(0.5, 0.25)$ is an unstable equilibrium.



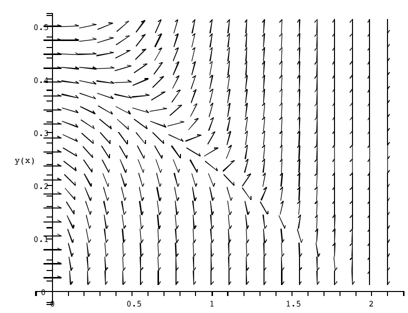
8. In Exercise 2,
$\dfrac{dy}{dx} = \dfrac{0.4x - 0.1x^2 - 0.2xy}{-0.2y + 0.1xy}$
From the following phase portrait, we observe that
$(0, 0)$ is an unstable equilibrium,
$(4, 0)$ is a stable equilibrium,
$(2, 1)$ is an unstable equilibrium.



9. In Exercise 5,
$\dfrac{dy}{dx} = \dfrac{0.2x - 0.1x^2 - 0.4xy}{-0.3y + \quad 0.1xy}$
From the following phase portrait, we observe that
$(0, 0)$ is an unstable equilibrium,
$(2, 0)$ is an unstable equilibrium,
$(1.5, 0.125)$ is a stable equilibrium.



10. In Exercise 6,
$\dfrac{dy}{dx} = \dfrac{0.2x - 0.1x^2 - 0.4xy}{-0.2y + 0.1xy}$
From the following phase portrait, we observe that
$(0, 0)$ is an unstable equilibrium,
$(2, 0)$ is a stable equilibrium.



11. The point $(0, 0)$ is an unstable equilibrium.

12. The point $(0.5, 0.5)$ is a stable equilibrium.

13. The point $(0.5, 0.5)$ is a stable equilibrium.

14. The point $(1, 0)$ is a stable equilibrium.

15. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.3x - 0.2x^2 - 0.1xy$
$0 = 0.2y - 0.1y^2 - 0.1xy$
$0 = x(0.3 - 0.2x - 0.1y)$
$0 = y(0.2 - 0.1y - 0.1x)$
$x = 0$ or $0.3 - 0.2x - 0.1y = 0$
$y = 0$ or $0.2 - 0.1y - 0.1x = 0$

$x = 0$ or $2x + y = 3$
$y = 0$ or $x + y = 2$

The equilibrium points are
$(0, 0)$, corresponding to the case where neither species exists,
$(0, 2)$, corresponding to the case where species Y exists but species X does not,
$(1.5, 0)$, corresponding to the case where species X exists but species Y does not,
$(1, 1)$, corresponding to the have both species exist, with species Y as many as species X.

16. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.4x - 0.1x^2 - 0.2xy$
$0 = 0.5y - 0.4y^2 - 0.1xy$

$0 = x(0.4 - 0.1x - 0.2y)$
$0 = y(0.5 - 0.4y - 0.1x)$

$x = 0$ or $0.4 - 0.1x - 0.2y = 0$
$y = 0$ or $0.5 - 0.4y - 0.1x = 0$

$x = 0$ or $x + 2y = 4$
$y = 0$ or $x + 4y = 5$

The equilibrium points are
$(0, 0)$, corresponding to the case where neither species exists,
$(0, 1.25)$, corresponding to the case where species Y exists but species X does not,
$(4, 0)$, corresponding to the case where species X exists but species Y does not,
$(3, 0.5)$, corresponding to the have both species exist, with species X six times as many as species Y.

17. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.3x - 0.2x^2 - 0.2xy$
$0 = 0.2y - 0.1y^2 - 0.2xy$

$0 = x(0.3 - 0.2x - 0.2y)$
$0 = y(0.2 - 0.1y - 0.2x)$

$x = 0$ or $0.3 - 0.2x - 0.2y = 0$
$y = 0$ or $0.2 - 0.1y - 0.2x = 0$

$x = 0$ or $x + y = 1.5$
$y = 0$ or $2x + y = 2$

The equilibrium points are
$(0, 0)$, corresponding to the case where neither species exists,
$(0, 2)$, corresponding to the case where species Y exists but species X does not,
$(1.5, 0)$, corresponding to the case where species X exists but species Y does not,
$(0.5, 1)$, corresponding to the have both species exist, with species Y twice as many as species X.

18. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.4x - 0.3x^2 - 0.1xy$
$0 = 0.3y - 0.2y^2 - 0.1xy$

$0 = x(0.4 - 0.3x - 0.1y)$
$0 = y(0.3 - 0.2y - 0.1x)$

$x = 0$ or $0.4 - 0.3x - 0.1y = 0$
$y = 0$ or $0.3 - 0.2y - 0.1x = 0$

$x = 0$ or $3x + y = 4$
$y = 0$ or $x + 2y = 3$

The equilibrium points are
$(0, 0)$, corresponding to the case where neither species exists,
$(0, 1.5)$, corresponding to the case where species Y exists but species X does not,
$(4/3, 0)$, corresponding to the case where species X exists but species Y does not,
$(1, 1)$, corresponding to the have both species exist, with species Y as many as species X.

19. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.2x - 0.2x^2 - 0.1xy$
$0 = 0.1y - 0.1y^2 - 0.2xy$

$0 = x(0.2 - 0.2x - 0.1y)$
$0 = y(0.1 - 0.1y - 0.2x)$

$x = 0$ or $0.2 - 0.2x - 0.1y = 0$
$y = 0$ or $0.1 - 0.1y - 0.2x = 0$

$x = 0$ or $2x + y = 2$
$y = 0$ or $2x + y = 1$

The equilibrium points are
$(0, 0)$, corresponding to the case where neither species exists,
$(0, 1)$, corresponding to the case where species Y exists but species X does not,
$(1, 0)$, corresponding to the case where species X exists but species Y does not.

20. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = 0.1x - 0.2x^2 - 0.1xy$
$0 = 0.3y - 0.2y^2 - 0.1xy$

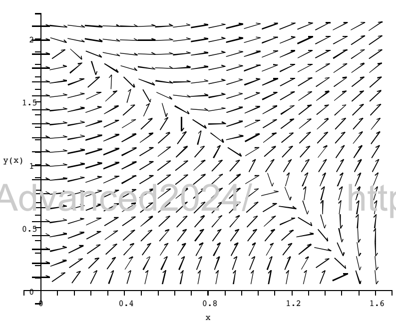
$0 = x(0.1 - 0.2x - 0.1y)$
$0 = y(0.3 - 0.2y - 0.1x)$

$x = 0$ or $0.1 - 0.2x - 0.1y = 0$
$y = 0$ or $0.3 - 0.2y - 0.1x = 0$

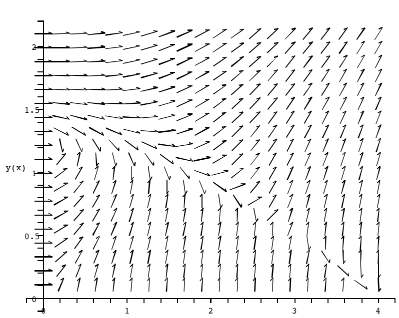
$x = 0$ or $2x + y = 1$
$y = 0$ or $x + 2y = 3$

The equilibrium points are
$(0, 0)$, corresponding to the case where neither species exists,
$(0, 1.5)$, corresponding to the case where species Y exists but species X does not,
$(0.5, 0)$, corresponding to the case where species X exists but species Y does not.

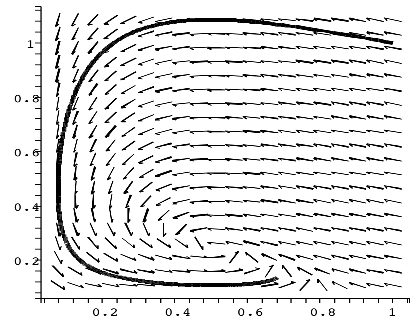
21. In Exercise 15,
$$\frac{dy}{dx} = \frac{0.3x - 0.2x^2 - 0.1xy}{0.2y - 0.1y^2 - 0.1xy}$$
From the following phase portrait, we observe that
$(0, 0)$ is an unstable equilibrium,
$(0, 2)$ is an unstable equilibrium,
$(1.5, 0)$ is an unstable equilibrium,
$(1, 1)$ is a stable equilibrium.



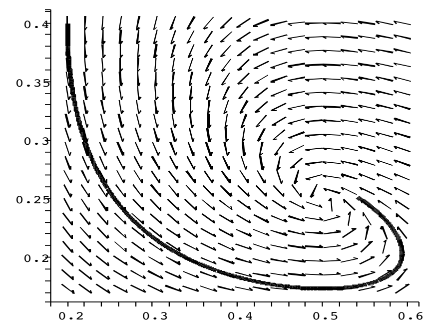
22. In Exercise 16,
$$\frac{dy}{dx} = \frac{0.4x - 0.1x^2 - 0.2xy}{0.5y - 0.4y^2 - 0.1xy}$$
From the following phase portrait, we observe that
$(0, 0)$ is an unstable equilibrium,
$(0, 1.25)$ is an unstable equilibrium,
$(4, 0)$ is a stable equilibrium,
$(3, 0.5)$ is a stable equilibrium.



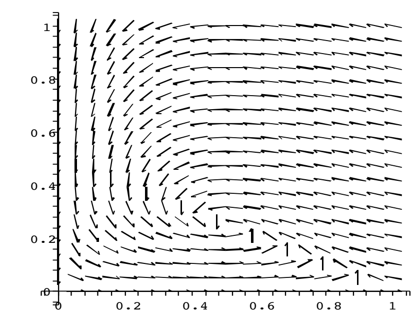
23. **(a)**



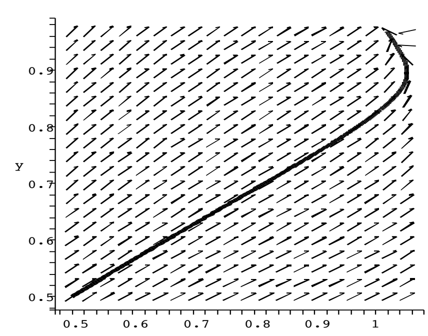
(b)



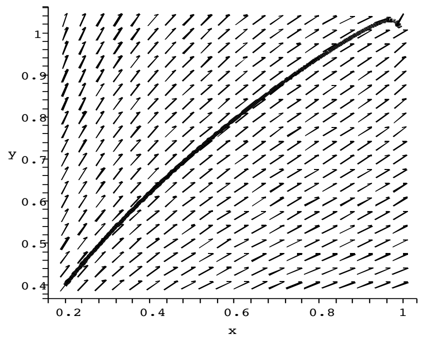
(c)



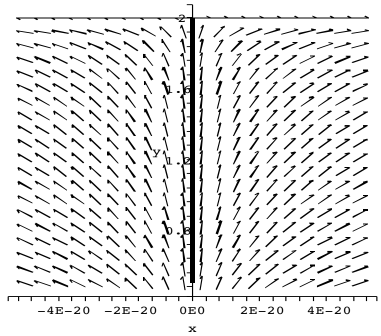
24. **(a)**



(b)



(c)



25. Write $u = y, v = y'$. We then have
$u' = v$
$v' = -2xv - 4u + 4x^2$

26. Write $u = y, v = y'$. We then have
$u' = v$
$v' = 3v - 3\sqrt{x}u + 4$

27. Write $u = y, v = y'$. We then have
$u' = v$
$v' = \cos x\, v - xu^2 + 2x$

28. Write $u = y, v = y'$. We then have
$u' = v$
$v' = \dfrac{-3v^2 + u + 2x}{x}$

29. Write $u_1 = y$, $u_2 = y'$, and $u_3 = y''$,
$u_1' = u_2$
$u_2' = u_3$
$u_3' = -2xu_3 + 4u_2 - 2u_1 + x^2$

30. Write $u_1 = y$, $u_2 = y'$, and $u_3 = y''$,
$u_1' = u_2$
$u_2' = u_3$
$u_3' = 2x^2u_2 - u_1^2 + 2$

31. Write $u_1 = y$, $u_2 = y', u_3 = y'', u_4 = y'''$,
$u_1' = u_2$
$u_2' = u_3$
$u_3' = u_4$
$u_4' = 2u_4 - xu_2 + 2 - e^x$

32. Write $u_1 = y$, $u_2 = y', u_3 = y'', u_4 = y'''$,
$u_1' = u_2$
$u_2' = u_3$
$u_3' = u_4$
$u_4' = 2u_3u_2 - (\cos x)u_1^2$

33. An approximate solution is
$x(1) \approx 0.253718$, $y(1) \approx 0.167173$.

n	x_n	y_n
0	0.2	0.2
1	0.2048	0.1964
2	0.2097201152	0.19287422728
3	0.2147629013	0.1894212388
5	0.2252268589	0.1827279868
10	0.2537179001	0.1671729953

34. An approximate solution is
$x(1) \approx .252044$, $y(1) \approx .23354$.

n	x_n	y_n
0	0.2	0.2
1	.2048	.2032
2	.2096889856	.2064349440
3	.2146673944	.2097046177
5	.2248939443	.2163471744
10	.2520442475	.2335415381

35. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = (x^2 - 4)(y^2 - 9)$
$0 = x^2 - 2xy$

$0 = (x + 2)(x - 2)(y + 3)(y - 3)$
$0 = x(x - 2y)$

$x = 2, x = -2, y = 3, y = -3$
$x = 0, x = 2y$

The equilibrium points are
$(2, 1), (-2, -1), (6, 3), (-6, -3), (0, 3), (0, -3)$.

36. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = (x - y)(1 - x - y)$
$0 = 2x - xy = x(2 - y)$

$x = y$ or $x + y = 1$
$x = 0$ or $y = 2$

The equilibrium points are
$(0, 0), (2, 2), (0, 1), (-1, 2)$.

37. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = (2 + x)(y - x)$
$0 = (4 - x)(x + y)$

$x = -2$ or $x = y$
$x = 4$ or $x = -y$

The equilibrium points are
$(0, 0), (-2, 2), (4, 4)$.

38. Equilibrium points are those that satisfy $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have
$0 = -x + y$
$0 = y + x^2$

$0 = 0 - 0 = (y + x^2) - (-x + y) = x^2 - x$

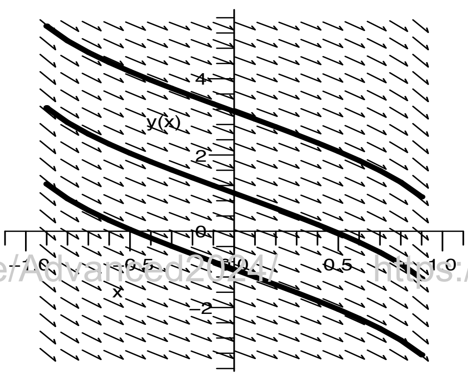
$x(x - 1) = 0, x = 0$ or $x = 1$

When $x = 0, y = 0$

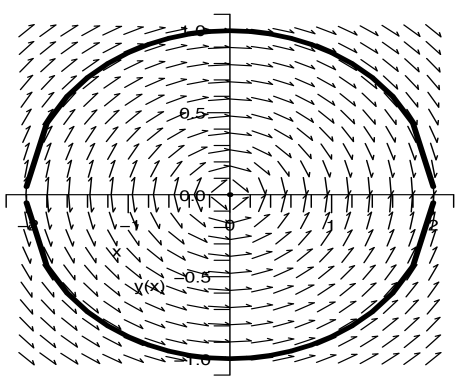
and when $x = 1, y = -1$

The equilibrium points are $(0, 0), (1, -1)$.

39.



40.



41. For equilibrium solutions, set $x' = y' = 0$ to get
$0 = 0.4x - 0.1x^2 - 0.2xy$
$0 = -0.5y + 0.1xy$

$0 = 0.1x(4 - x - 2y)$
$0 = 0.1y(-5 + x)$

Equilibrium points are $(0, 0)$, $(4, 0)$.
Neither of these solutions has non-zero values for both populations, so the species cannot co-exist. Now suppose that the death rate of species Y is D instead of 0.5, and let us search for equilibrium solutions where both population values are non-zero. The equations are now
$x' = 0.4x - 0.1x^2 - 0.2xy$
$y' = -Dy + 0.1xy$

where $D > 0$. $0 = 0.1x(4 - x - 2y)$
$0 = 0.1y(x - 10D)$

Since we are searching for non-zero solutions,
$0 = 4 - x - 2y$
$0 = x - 10D$

Solving the second equation gives $x = 10D$, and substituting this expression into the first equation gives
$0 = 4 - 10D - 2y = 2 - 5D - y$
$y = 2 - 5D$

The equilibrium solution for y will be positive provided that $2 - 5D > 0$, which means that $D < 0.4$.

42. Continuing the computation in the solution to Exercise 43. Suppose that the birth rate of species X is B instead of 0.4, and let us search for equilibrium solutions where both population values are non-zero. The equations are now
$x' = Bx - 0.1x^2 - 0.2xy$
$y' = -0.5y + 0.1xy$

where $B > 0$. $0 = x(B - 0.1x - 0.2y)$
$0 = y(-0.5 + 0.1x)$

Since we are searching for non-zero solutions,
$0 = B - 0.1x - 0.2y$
$0 = -5 + x$

Solving the second equation gives $x = 5$, and substituting this expression into the first equation gives
$0 = B - 0.5 - 0.2y$
$y = 5B - 2.5$

The equilibrium solution for y will be positive provided that $5B - 2.5 > 0$, which means that $B > 0.5$.

43. Assume that all coefficients are positive. The equations that define equilibrium are
$0 = x(b - cx - k_1 y)$
$0 = y(-d + k_2 y)$

For the species to coexist, both x and y must be nonzero, and so the equations reduce to
$0 = b - cx - k_1 y$
$0 = -d + k_2 y$

Solving the second equation, we get $y = \dfrac{d}{k_2}$.
Substituting the result into the first equation,

$0 = b - cx - k_1 \frac{d}{k_2}$
$cx = b - \frac{dk_1}{k_2} = \frac{bk_2 - dk_1}{k_2}$
$x = \frac{bk_2 - dk_1}{ck_2}$

Thus, $x > 0$ if and only if $bk_2 - dk_1 > 0$, which is equivalent to $bk_2 > dk_1$.

44. Assume that $c = 0$, the model becomes
$x' = bx - k_1xy$
$y' = -dy + k_2xy$

For the species to coexist, we look for nonzero equilibrium solutions to
$0 = bx - k_1xy = x(b - k_1y)$
$0 = -dy + k_2xy = y(-d + k_2x)$

which gives
$0 = b - k_1y, 0 = -d + k_2x$
$y = \frac{b}{k_1}, x = \frac{d}{k_2}.$

If the pesticide is used, b will be reduced and d will be increased. This means that the equilibrium population for the pest will be increases, while that for the predator will be decreases. This is **not** a desired effect of the pesticide.

Ch. 7 Review Exercises

1. We separate variables and integrate.
$\frac{1}{y}y' = 2$
$\int \frac{1}{y}\,dy = \int 2\,dx$
$\ln|y| = 2x + c$
$y = ke^{2x}$

The initial condition gives $3 = k$ so the solution is $y = 3e^{2x}$

2. We separate variables and integrate.
$\frac{1}{y}y' = -3$
$\int \frac{1}{y}\,dy = \int -3\,dx$
$\ln|y| = -3x + c$
$y = Ae^{-3x}$

The initial condition gives us $2 = A$ so the solution is $y = 2e^{-3x}$

3. We separate variables and integrate.
$yy' = 2x$
$\int y\,dy = \int 2x\,dx$
$\frac{y^2}{2} = x^2 + c$
$y = \sqrt{2x^2 + c}$

The initial condition gives us $2 = \sqrt{c}, c = 4$ so the solution is $y = \sqrt{2x^2 + 4}$

4. We separate variables and integrate.
$-\frac{1}{y^2}y' = 3x$
$-\int \frac{1}{y^2}\,dy = \int 3x\,dx$
$\frac{1}{y} = \frac{3x^2}{2} + c$
$y = \frac{2}{3x^2 + 2c}$

The initial condition gives us $4 = \frac{2}{2c}$

so the solution is $y = \frac{2}{3x^2 + 1/2}$

5. We separate variables and integrate.
$\frac{1}{\sqrt{y}}y' = \sqrt{x}$
$\int y^{-1/2}\,dy = \int x^{1/2}\,dx$
$2y^{1/2} = \frac{2}{3}x^{3/2} + c$
$y = \left(\frac{x^{3/2}}{3} + c\right)^2$

The initial condition gives us
$4 = \left(\frac{1}{3} + c\right)^2, c = \frac{5}{3}$

so the solution is $y = \left(\frac{x^{3/2}}{3} + \frac{5}{3}\right)^2$

6. We separate variables and integrate.
$\frac{1}{1 + y^2}y' = x$
$\int \frac{1}{1 + y^2}\,dy = \int x\,dx$
$\tan^{-1} y = \frac{x^2}{2} + c$
$y = \tan\left(\frac{x^2}{2} + c\right)$

The initial condition gives us $1 = \tan c$

so the solution is $y = \tan\left(\frac{x^2}{2} + \frac{\pi}{4}\right)$

7. With t measured in hours, we have
$y = Ae^{kt}$, $A = y(0) = 10^4$.

If the doubling time is 2, then
$2 = e^{2k}$, $k = \ln(2)/2$, and
$y = 10^4e^{t\ln(2)/2} = 10^4 2^{t/2}$.

To reach $y = 10^6$ at a certain unknown time t, we need
$2^{t/2} = 100$,
$t = \frac{2\ln(100)}{\ln(2)} \approx 13.3$ hours.

8. Assuming that the growth is exponential, we have $y(t) = Ae^{kt}$.

Since the population at $t = 0$ is 100, we have $A = 100$.

$y(2) = 140$ allows us to solve for k:
$140 = 100e^{2k}$, or $k = \frac{1}{2}\ln(7/5)$.
Population at $t = 6$ is
$y(6) = 100e^{6(1/2)\ln(7/5)} = 274.4$

9. With t measured in hours, x in milligrams, we get
$x = 2\left(\frac{1}{2}\right)^{t/2} = \frac{2}{2^{t/2}}$.

To get to $x = .1$ at a certain unknown time t, we need
$2^{t/2} = \frac{2}{.1} = 20$,
$t = \frac{2\ln(20)}{\ln(2)} \approx 8.64$ hours.

10. The relationship between half-life (τ) and the growth constant (r) is
$\tau = -\frac{\ln 2}{r}$ (see Exercise 18 of Section 6.1).
Therefore, our growth constant is
$r = -\frac{\ln 2}{3}$

The proportion of material left after 9 hours is
$e^{9r} = \frac{1}{8}$ (= 12.5%).
The proportion of material left after 11 hours is $e^{11r} \approx 0.07874$ ($\approx 7.874\%$).

11. The equation for the doubling time t_d in this case is
$2 = e^{.08t_d}$, hence
$t_d = \frac{\ln(2)}{.08} \approx 8.66$ years.

12. With continuous compounding, this investment will be worth
$\$4000e^{(0.06)(10)} \approx \$7288.48$

13. For temperature T at time t, and ambient temperature T_a, we have
$\frac{T - T_a}{T(0) - T_a} = e^{kt}$.

In this case with $T_a = 68, T(0) = 180$ and $T(1) = 176$, we have
$\frac{108}{112} = \frac{176 - 68}{180 - 68} = e^k$,
$k = \ln\left(\frac{108}{112}\right) = \ln\left(\frac{27}{28}\right)$,
$\frac{T - 68}{112} = e^{tk} = e^{t\ln(27/28)} = \left(\frac{27}{28}\right)^t$,
$T = 68 + 112\left(\frac{27}{28}\right)^t$.

To reach $T = 120$ at unknown time t, we need
$t = \frac{\ln(52/112)}{\ln(27/28)} \approx 21.1$ minutes.

14. Let $y(t)$ represent the temperature of the drink.
We start with the differential equation
$y'(t) = k[y(t) - 70]$
This has solution $y(t) = Ae^{kt} + 70$.
We now find constants A and k. The initial condition is $y(0) = 46$ and we also know $y(4) = 48$.
$46 = y(0) = A + 70$
$48 = y(4) = Ae^{4k} + 70$
This tells us that $A = -24$ and
$k = [\ln(11/12)]/4$.
To determine when $y(t) = 58$ we solve
$58 = -24e^{kt} + 70$
Solving gives
$t = -\frac{\ln 2}{k} \approx 127$ minutes.
So, just over 2 hours.

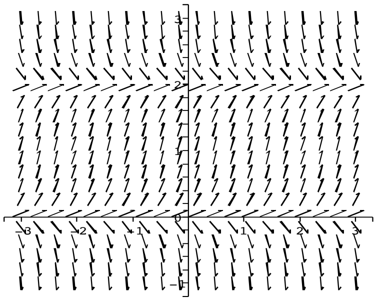
15. $\frac{y'}{y} = 2x^3$
$\ln|y| = \frac{x^4}{2} + c$
$y = Ae^{x^4/2}$

16. $\frac{1}{y}y' = \frac{1}{\sqrt{1 - x^2}}$
$\int \frac{1}{y}\,dy = \int \frac{1}{\sqrt{1 - x^2}}\,dx$
$\ln|y| = \sin^{-1} x + c$
$y = Ae^{\sin^{-1} x}$

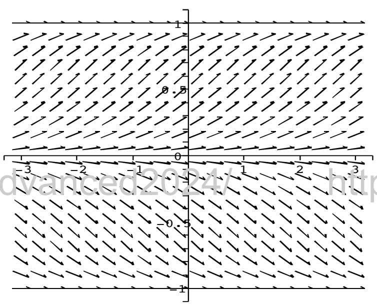
17. $(y^2 + y)y' = \frac{4}{1 + x^2}$
$\int (y^2 + y)\,dy = \int 41 + x^2\,dx$
$\frac{y^3}{3} + \frac{y^2}{2} = 4\tan^{-1} x + c$
It is impossible (without using a CAS) to write out the explicit formula of y in terms of x.

18. $e^{-y}y' = e^x$
$-\int e^{-y}\,dy = \int -e^x\,dx$
$e^{-y} = -e^x + c$
$y = \ln(c - e^x)$

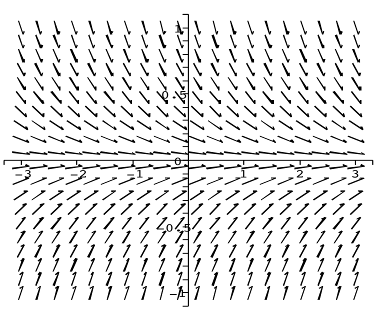
19. Equilibrium solutions occur where $y' = 0$ which occurs when $y = 0$ and $y = 2$. $y = 0$ is unstable and $y = 2$ is stable which can be seen by drawing the direction field.



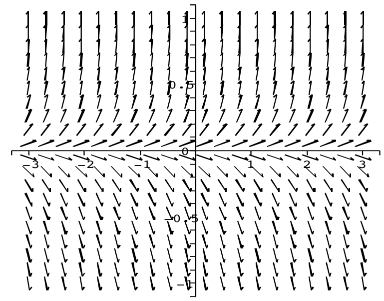
20. Equilibrium solutions occur where $y' = 0$ which occurs when $y = 0$, $y = 1$ and $y = -1$. $y = 0$ is unstable and $y = -1$ and $y = 1$ are both stable which can be seen by drawing the direction field.



21. Equilibrium solutions occur where $y' = 0$ which occurs when $y = 0$, and it is stable.



22. We have $y' = \dfrac{y(y-3)}{y-1}$.
Equilibrium solutions occur where $y' = 0$ which occurs when $y = 0$ and $y = 3$, both unstable.



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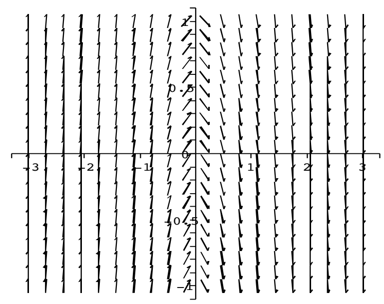

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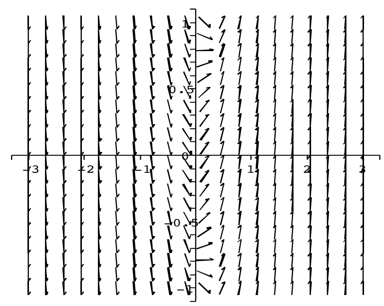

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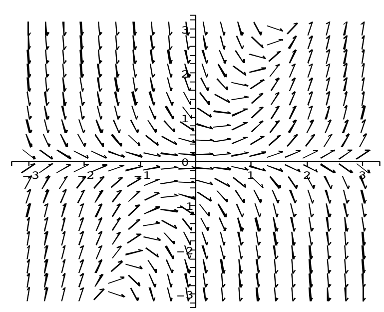

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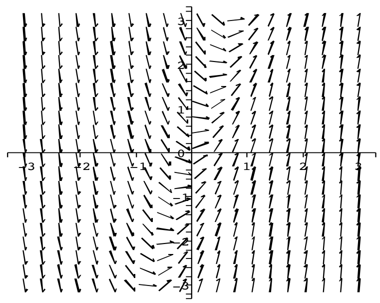
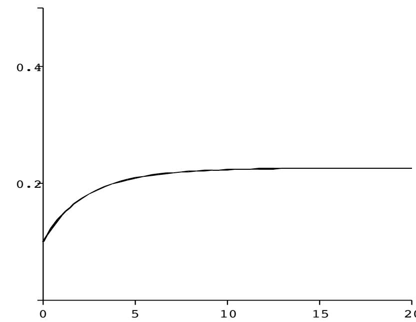
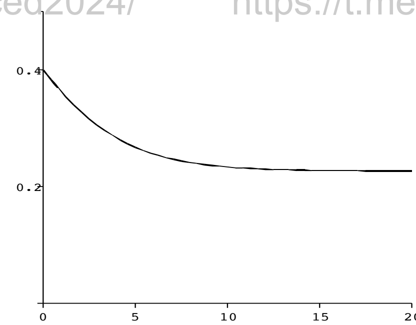


27. The differential equation is
$x' = (.3 - x)(.4 - x) - .25x^2$
$= .12 - .7x + .75x^2$
$= \frac{3}{4}(x^2 - \frac{14}{15}x + \frac{4}{25})$
$= \frac{3}{4}(x - r)(x - s)$. in which
$r = \frac{7 + \sqrt{13}}{15}, s = \frac{7 - \sqrt{13}}{15},$
$r - s = \frac{2\sqrt{13}}{15}$.
When separated it takes the form
$\frac{x'}{(x - r)(x - s)} = k$

in which $k = 3/4$.
By partial fractions we find
$\frac{1}{(x - r)(x - s)}$
$= \frac{1}{(r - s)}\left\{\frac{1}{(x - r)}\frac{1}{(x - s)}\right\}$
and after integration we find
$\frac{1}{(r - s)}\ln\left|\frac{x - r}{x - s}\right| = kt + c_1$ or in this case
$\ln\left|\frac{x - r}{x - s}\right| = \frac{2\sqrt{13}}{15}\left(\frac{3}{4}t + c_1\right) = wt + c_2$
$\left(w = \frac{\sqrt{13}}{10} \approx .36056, c_2 = \frac{2\sqrt{13}}{15}c_1\right)$.
Using the initial condition $x(0) = c$, we find
$c2 = \ln|(c - r)/(c - s)|$,
$\ln\left|\frac{(c - s)(x - r)}{(c - r)(x - s)}\right| = wt$ and
$\frac{x - r}{x - s} = \pm\frac{c - r}{c - s}e^{wt} = \frac{c - r}{c - s}e^{wt},$
$x = \frac{s(r - c)e^{wt} + r(c - s)}{(r - c)e^{wt} + (c - s)}$
$= \frac{r\left(\frac{c - s}{r - c}\right)e^{-wt} + s}{\left(\frac{c - s}{r - c}\right)e^{-wt} + 1}$
The choice of sign is + since the left side of the middle equation is $(c - r)/(c - s)$ when $t = 0$ and $x = c$. The last expression is one of many possible ways to normalize. It is apparent that $x \to s \approx .22630$ as $t \to \infty$ Numerically, when $c = 0.1$, this comes to
$x = \frac{.22630 - .14710e^{-.36056t}}{1 - .20806e^{-.36056t}}$. and the graph looks like



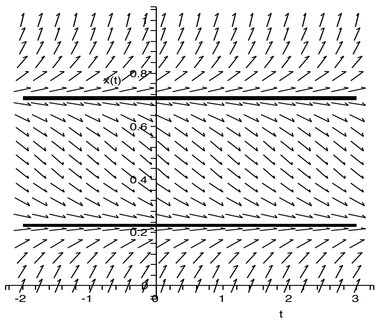
When $c = 0.4$, this comes to
$x = \frac{.22630 + .39999e^{-.36056t}}{1 + .56574e^{-.36056t}}$. and the graph looks like



28. The equilibrium solutions are where $x'(t) = 0$, or when
$0 = (0.3 - x)(0.4 - x) - 0.25x^2$
$= 0.75x^2 - 0.7x + 0.12$

We can solve this using the quadratic formula:
$x = \frac{0.7 \pm \sqrt{(-0.7)^2 - 4(0.75)(0.12)}}{2(0.75)}$
$\approx 0.22630, 0.70704$

These are our equilibrium solutions.
$x = 0.22630$ is a stable equilibrium.
$x = 0.70703$ is an unstable equilibrium.



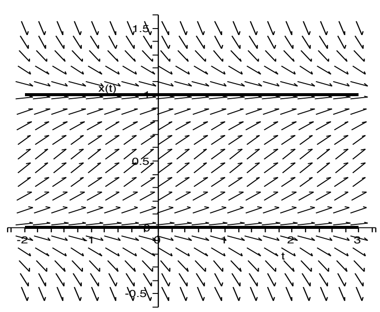
29. The DE $\dfrac{x'}{(x-a)^2} = r$ integrates to
$\dfrac{-1}{x-a} = rt + c.$ and then
$c = \dfrac{1}{a},$
$x - a = \dfrac{-1}{c+rt} = \dfrac{-a}{1+art},$
$x = a\left(1 - \dfrac{1}{1+art}\right) = \dfrac{a^2rt}{1+art}.$

One can see that all values of x lie between 0 and a, and that $\lim_{t\to\infty} x(t) = a$ All the initial amounts of the A,B substances (both a in this case) will eventually be converted to the X substance which ultimately will have the same concentration as the original concentrations of the other two substances.

30. Using partial fractions gives

$$rt + c = \int \frac{dx}{x(1-x)} = \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx$$
$$= \ln|x| - \ln|1-x| = \ln\left|\frac{x}{1-x}\right|$$

Solving for x,
$\dfrac{x}{1-x} = ke^{rt} \quad (k = e^c)$
$x(t) = \dfrac{ke^{rt}}{1+ke^{rt}}$
We clearly have $\lim_{t\to\infty} x(t) = 1$



31. With A the amount in the account at time t the DE is
$A'(t) = .10A + 20{,}000$ with an IC of
$A(0) = 100{,}000$.

The DE separates and integrates easily, yielding
$10\ln|.10A + 20{,}000| = t + c$
$c = 10\ln(30{,}000),$
$.10A + 20{,}000 = 30{,}000e^{t/10}.$

If the fortune is to reach 1,000,000 at unknown time t, we must have
$120{,}000 = 30{,}000e^{t/10}$
$\dfrac{t}{10} = \ln\dfrac{12}{3} = \ln(4),$
$t = 10\ln(4) \approx 13.86$ years.

32. If the payments are made at the end of each year instead of continuously, we will have a sequence of differential equations with exponential growth solutions. For the first year, we will have $A_1(t) = 100,000e^{0.1t} \quad 0 \le t < 1$
When $t = 1$, we have $A_1(1) \approx \$110,517$. At the beginning of the second year, we deposit \$20,000 so we start with a total of \$130,517. We can now use this for the second year and see that for the second year $A_2(t) = \$130,517e^{0.1t}$. At the end of the second year we have
$A_2(1) \approx \$144,244$. If we continue this process, one can see that at the end of the 14th year, or when $t = 14$, (including the \$20,000 deposit), there will be \$986,517 in the account. At the end of the 15th year (including the \$20,000 deposit), there will be \$1,110,270 in the account. There will be exactly \$1,000,000 in the account when $t \approx 14.14$

33. It is a predator-prey model. For equilibrium solutions, set $x' = y' = 0$ to get
$0 = 0.1x - 0.1x^2 - 0.2xy$
$0 = -0.1y + 0.1xy$
which are equivalent to
$0 = 0.1x(1 - x - 2y)$
$0 = 0.1y(-1 + x)$ $x = 0$ or $x = 1 - 2y$
$y = 0$ or $x = 1$.

The equilibrium solutions are $(0, 0)$ (no prey or predators) and $(1, 0)$ (prey but no predators).

34. It is a competing species model. For equilibrium solutions, set $x' = y' = 0$ to get
$0 = 0.2x - 0.1x^2 - 0.2xy$
$0 = 0.1y - 0.1y^2 - 0.1xy$
which are equivalent to
$0 = x(0.2 - 0.1x - 0.2y)$
$0 = y(0.1 - 0.1y - 0.1x)$ $x = 0$ or $x + 2y = 2$
$y = 0$ or $x + y = 1$.

The equilibrium solutions are $(0,0)$ (neither species), $(0,1)$ (species Y but no species X) and $(2,0)$ (species X but no species Y).

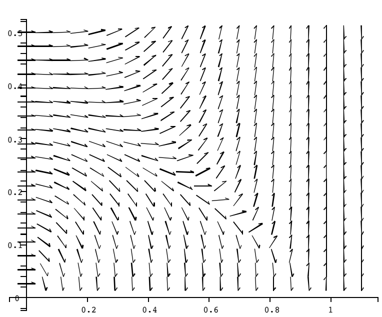
35. It is a competing species model. For equilibrium solutions, set $x' = y' = 0$ to get
$0 = 0.5x - 0.1x^2 - 0.2xy$
$0 = 0.4y - 0.1y^2 - 0.2xy$
which are equivalent to
$0 = 0.1x(5 - x - 2y)$
$0 = 0.1y(4 - y - 2x)$ $x = 0$ or $x + 2y = 5$
$y = 0$ or $2x + y = 4$.

The equilibrium solutions are $(0,0)$ (none of either species), $(0,4)$ (none of first species, some of second), $(5,0)$ (some of first species, none of second), $(1,2)$ (twice as many of second species as first species)

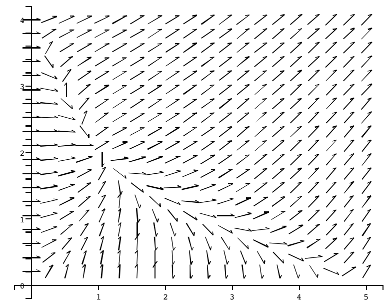
36. It is a predator-prey model. For equilibrium solutions, set $x' = y' = 0$ to get
$0 = 0.4x - 0.1x^2 - 0.2xy$
$0 = -0.2y + 0.1xy$
which are equivalent to
$0 = 0.1x(4 - x - 2y)$
$0 = 0.1y(-2 + x)$ $x = 0$ or $x = 4 - 2y$
$y = 0$ or $x = 2$.

The equilibrium solutions are $(0,0)$ (no prey or predators) and $(4,0)$ (prey but no predators).

37. In Exercise 33,
$$\frac{dy}{dx} = \frac{0.1x - 0.1x^2 - 0.2xy}{-0.1y + 0.1xy}$$
From the direction field, we see that $(0,0)$ is unstable and $(1,0)$ is stable.

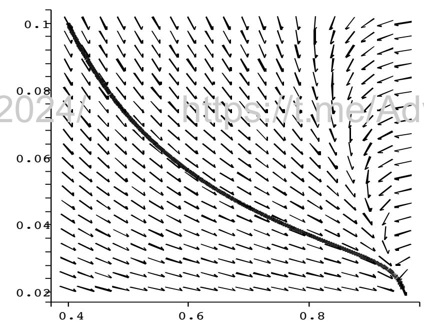


38. In Exercise 33,
$$\frac{dy}{dx} = \frac{0.5x - 0.1x^2 - 0.2xy}{0.4y - 0.1y^2 - 0.2xy}$$
From the direction field, $(0,0)$, $(0,4)$ and $(5,0)$ are unstable, $(1,2)$ is stable.



39. Write $u = y, v = y'$, then
$u' = v$
$v' = 4x^2v - 2u + 4xu - 1$

40. (a)



(b)