

❖ ID Resources: Topic 3 – Application of Differentiation

Subtopic 3.2 The Indeterminate Forms and Hospital's Rule

1. Find the limit of each of the following:

i. $\lim_{x \rightarrow 0} \frac{8x^2}{\cos x - 1} = \frac{0}{0}$

$$\lim_{x \rightarrow 0} \frac{8x^2}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{16x}{-\sin x} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{8x^2}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{16}{-\cos x} = \frac{16}{-\cos 0} = -16$$

ii. $\lim_{x \rightarrow 0} \frac{x \cdot 2^x}{2^x - 1} = \frac{0}{0}$

$$\lim_{x \rightarrow 0} \frac{x \cdot 2^x}{2^x - 1} = \lim_{x \rightarrow 0} \frac{(1)(2^x) + (2^x \ln 2)(x)}{2^x \ln 2} = \lim_{x \rightarrow 0} \frac{2^x(1 + x \ln 2)}{2^x \ln 2} = \lim_{x \rightarrow 0} \frac{(1 + x \ln 2)}{\ln 2} = \frac{1 + 0}{\ln 2} = \frac{1}{\ln 2}$$

iii. $\lim_{x \rightarrow 0^+} \frac{\ln(x^2 + 2x)}{\ln x} = \frac{\infty}{\infty}$

$$\lim_{x \rightarrow 0^+} \frac{\ln(x^2 + 2x)}{\ln x} = \lim_{x \rightarrow 0^+} \frac{\frac{2x + 2}{x^2 + 2x}}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{x(2x + 2)}{x^2 + 2x} = \lim_{x \rightarrow 0^+} \frac{x(2x + 2)}{x(x + 2)} = \lim_{x \rightarrow 0^+} \frac{(2x + 2)}{(x + 2)} =$$

$$\lim_{x \rightarrow 0^+} \frac{\ln(x^2 + 2x)}{\ln x} = \frac{2(0) + 2}{(0) + 2} = 1$$

$$\text{iv. } \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{1 + \cos 2x} = \frac{0}{0}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{1 + \cos 2x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\cos x}{-2 \sin 2x} = \frac{0}{0}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{1 + \cos 2x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{-4 \cos 2x} = \frac{1}{-4(-1)} = \frac{1}{4}$$

$$\text{v. } \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 + 2 \sec x}{1 + \tan x} = \frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 + 2 \sec x}{1 + \tan x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{2 \sec x \tan x}{\sec^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{2 \tan x}{\sec x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{2 \sin x}{\cos x}}{\frac{1}{\cos x}} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{2 \sin x \cos x}{\cos x} =$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 + 2 \sec x}{1 + \tan x} = \lim_{x \rightarrow \frac{\pi}{2}} 2 \sin x = 2 \sin \left(\frac{\pi}{2} \right) = 2$$

$$\text{vi. } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = \left(1 + \frac{1}{\infty} \right)^{\infty} = 1^{\infty}$$

$$y = \left(1 + \frac{1}{x} \right)^x$$

$$\ln y = x \ln \left(1 + \frac{1}{x} \right)$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x} \right)$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x} \right)}{\left(\frac{1}{x} \right)} = \frac{0}{0}$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\left[\frac{\left(\frac{-1}{x^2} \right)}{1 + \frac{1}{x}} \right]}{\left(\frac{-1}{x^2} \right)}$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{x} \right)} = 1$$

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} e^{\ln y} = e$$

$$\text{vii. } \lim_{x \rightarrow 0^+} \frac{\tan x}{x^2} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0^+} \frac{\tan x}{x^2} = \lim_{x \rightarrow 0^+} \frac{\sec^2 x}{2x} = \frac{\sec^2(0)}{2(0)} = \frac{1}{0} = \infty$$

$$\text{viii. } \lim_{x \rightarrow \pi} \frac{\csc x}{1 + \cot x} = \frac{\csc \pi}{1 + \cot \pi} = \frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \pi} \frac{\csc x}{1 + \cot x} = \lim_{x \rightarrow \pi} \frac{-\csc x \cot x}{-\csc^2 x} = \lim_{x \rightarrow \pi} \frac{\cot x}{\csc x} = \lim_{x \rightarrow \pi} \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)} = \lim_{x \rightarrow \pi} \cos x = -1$$

$$\text{ix. } \lim_{x \rightarrow 0} \frac{2\cos x - 2\cos 2x}{1 - \cos x}$$

$$\lim_{x \rightarrow 0} \frac{2\cos x - 2\cos 2x}{1 - \cos x} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{2\cos x - 2\cos 2x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{-2\sin x + 4\sin 2x}{\sin x} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{2\cos x - 2\cos 2x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{-2\cos x + 8\cos 2x}{\cos x}$$

$$= \frac{-2+8}{1} = 6$$

$$\text{x. } \lim_{x \rightarrow 0} \frac{e^x - 1}{x^2 + x}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - 1}{x^2 + x} &= \frac{e^0 - 1}{(0)^2 + 0} \\ &= \frac{0}{0} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - 1}{x^2 + x} &= \lim_{x \rightarrow 0} \frac{e^x}{2x + 1} \\ &= \frac{e^0}{2(0) + 1} \\ &= 1 \end{aligned}$$