7-1: Introduction to Vectors

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Definition: A vector is a quantity that has both magnitude and direction

Ex1: State, whether each quantity described, is a vector quantity or a scalar quantity.

a. A boat traveling at 15 kilometers per hour

Scalar

b. A hiker walking 25 paces due west

Vector

c. A person's weight on a bathroom scale

$W = m \cdot g$ ↓ Vector

$w = mg$

d. A car traveling 60 kilometers per hour $\boxed{15° \text{ east}}$ of south

Vector

e. A parachutist falling straight down at 20.2 kilometers per hour

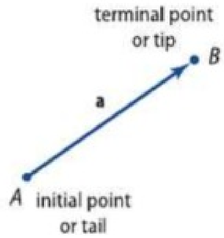
Vector

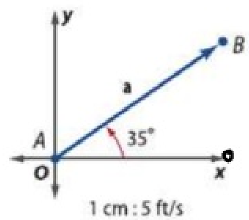
f. A child pulling a sled with a force of 40 newtons

Scalar

Note: A vector can be represented geometrically by an arrow that shows magnitude and direction. The vector is in standard position if the initial point is at the origin. The direction of the vector is the directed angle between the vector and the horizontal line that could be used to represent the positive x-axis.

*** The length of the line segment represents and is proportional to, the magnitude of the vector.**





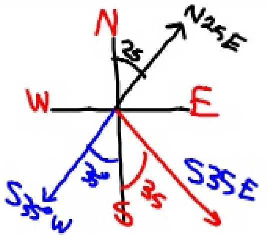
S 35°E N25°E
S 35° W N 25 W

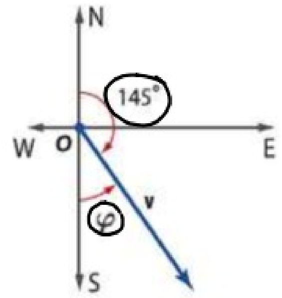
*A quadrant bearing ϕ, or phi, is a directional measurement between 0° and 90° east or west of the north-south line. The quadrant bearing of vector v shown is 35° east of south or southeast, written S35°E.

*A true bearing is a directional measurement where the angle is measured clockwise from north. True bearings are always given using three digits. So, a direction that measures 25° clockwise from north would be written as a true bearing of 025°.

025°

Note: If the type of bearing is not mentioned that means it's a true bearing.

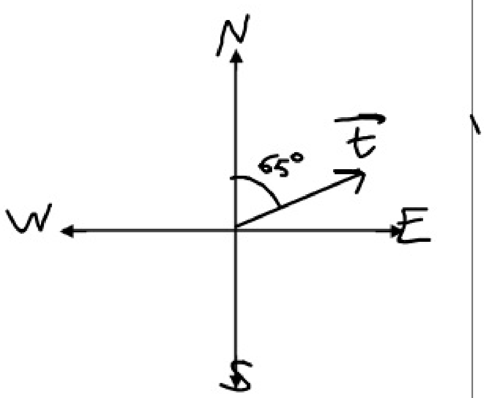




Ex2: Use a ruler and a protractor to draw an arrow diagram for each quantity described. Include a scale on each diagram.

1) t = 20 meters per second at a bearing of 065°

1cm = 10m/s

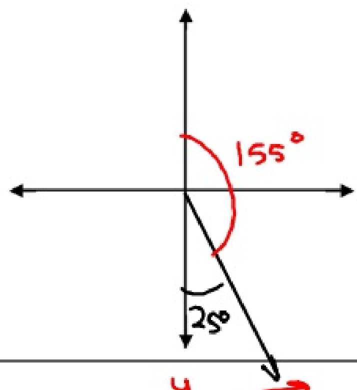


2) u = 15 kilometers per hour at a bearing of S25°E

u = 15 km/h

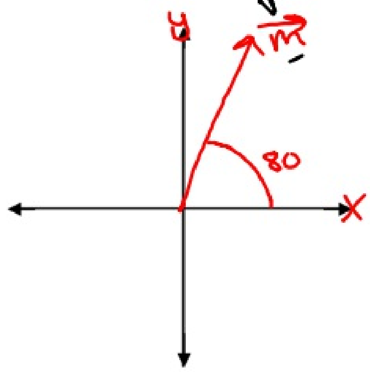
$\phi = 25$

$\theta = 180° - 25 = 155°$



3) m = 60 Newtons of force at 80° to the horizontal

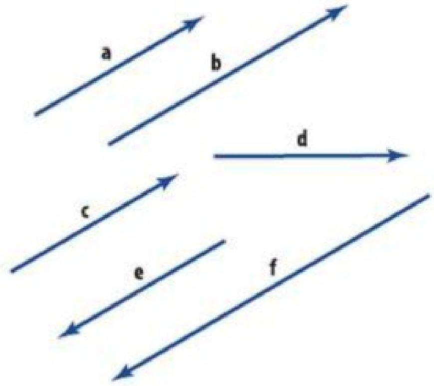
1 cm = 20 N



Parallel vectors have the same or opposite direction but not necessarily the same magnitude. In the figure, $\mathbf{a} \parallel \mathbf{b} \parallel \mathbf{c} \parallel \mathbf{e} \parallel \mathbf{f}$.

Equivalent vectors have the same magnitude and direction. In the figure, $\mathbf{a} = \mathbf{c}$ because they have the same magnitude and direction. Notice that $\mathbf{a} \neq \mathbf{b}$, since $|\mathbf{a}| \neq |\mathbf{b}|$, and $\mathbf{a} \neq \mathbf{d}$, since $\mathbf{a}$ and $\mathbf{d}$ do not have the same direction.

Opposite vectors have the same magnitude but opposite direction. The vector opposite $\mathbf{a}$ is written $-\mathbf{a}$. In the figure, $\mathbf{e} = -\mathbf{a}$.



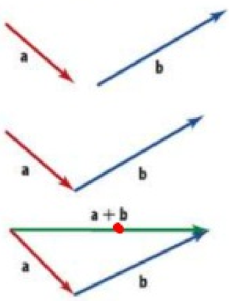
KeyConcept Finding Resultants

Triangle Method (Tip-to-Tail)

To find the resultant of a and b, follow these steps.

Step 1 Translate b so that the tail of b touches the tip of a.

Step 2 The resultant is the vector from the tail of a to the tip of b.



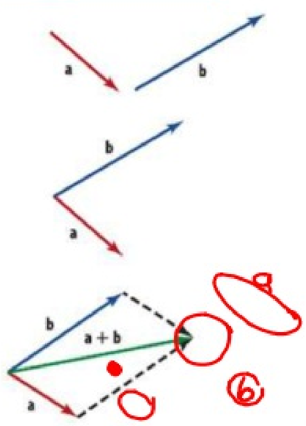
Parallelogram Method (Tail-to-Tail)

To find the resultant of a and b, follow these steps.

Step 1 Translate b so that the tail of b touches the tail of a.

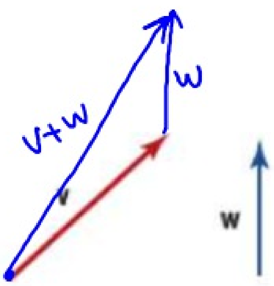
Step 2 Complete the parallelogram that has a and b as two of its sides.

Step 3 The resultant is the vector that forms the indicated diagonal of the parallelogram.

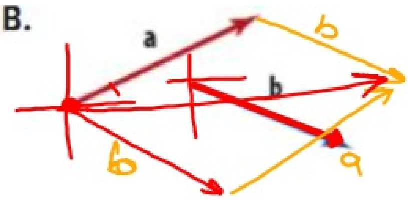


3: Find the resultant of each pair of vectors using either the triangle or parallelogram method. State he magnitude of the resultant to the nearest centimeter and its direction relative to the horizontal.

A.



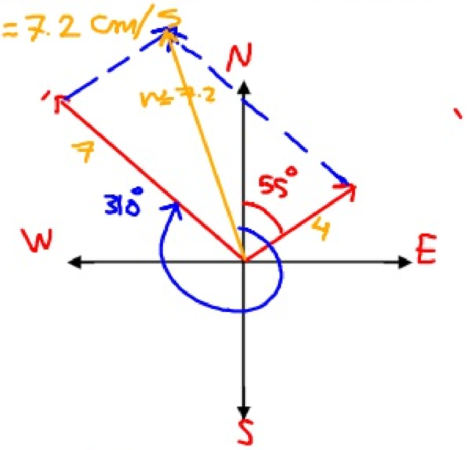
B.



1cm = 2 cm/s

C. PINBALL **A pinball is struck by flipper and is sent 310° at a velocity of 7 centimeters per second.The ball then bounces off of a bumper and heads 055° at a velocity of 4 centimeters per second. Find the resulting direction and velocity of the pinball.** $r = 3.6 \times 2 = 7.2$ cm/s

direction = 343°



Notes:

***The sum of two opposite vectors the result is the zero vector denoted by** $\vec{0}$

*** To find** $p-q$**, add the opposite of q to p That is** $p-q=p+(-q)$

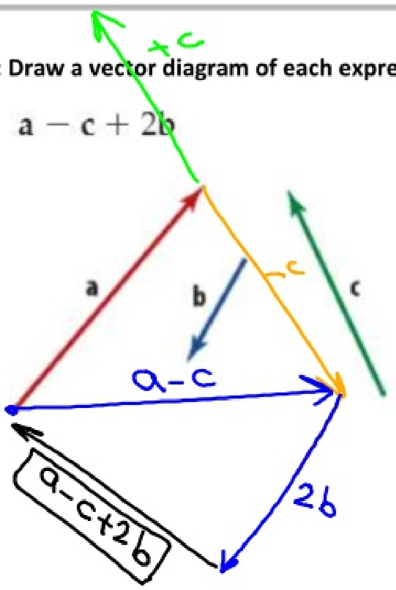
KeyConcept Multiplying Vectors by a Scalar

If a vector **v** is multiplied by a real number scalar k, the scalar multiple $k\mathbf{v}$ has a magnitude of $|k|\ |\mathbf{v}|$. Its direction is determined by the sign of k.

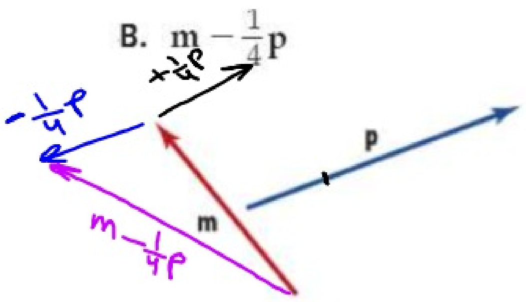
- If $k > 0$, $k\mathbf{v}$ has the same direction as **v**.
- If $k < 0$, $k\mathbf{v}$ has the opposite direction as **v**.

Ex4: Draw a vector diagram of each expression.

A. $\mathbf{a} - \mathbf{c} + 2\mathbf{b}$



B. $\mathbf{m} - \frac{1}{4}\mathbf{p}$

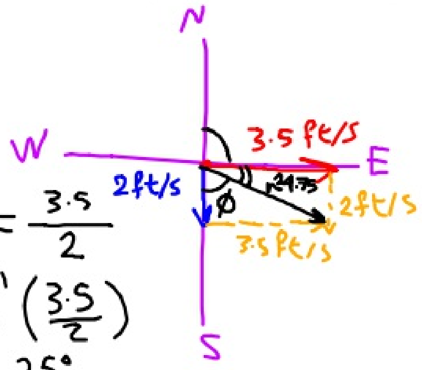


Ex5: Ali rows due east at a speed of 3.5 feet per second across a river directly toward the opposite bank. At the same time, the current of the river is carrying him due south at a rate of 2 feet per second. Find Ali's speed and direction relative to the shore.

$r = \sqrt{(3.5)^2 + (2)^2}$

$= \sqrt{16.25}$

$r \approx 4.03$ ft/s



$\tan\phi = \frac{3.5}{2}$

$\phi = \tan^{-1}\left(\frac{3.5}{2}\right)$

$= 60.25°$

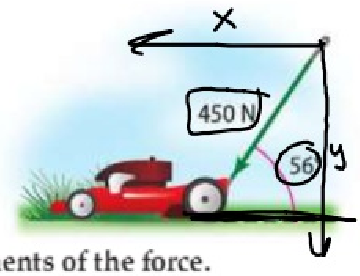
direction = S 60.25 E

or True direction = $180° - 60.25 = 119.75°$

or → $90 + 29.75 = 119.75$

Ex6: **LAWN CARE** Hala is pushing the handle of a lawn mower with a force of 450 newtons at an angle of 56° with the ground.

a. Draw a diagram that shows the resolution of the force that Hala exerts into its rectangular components.



b. Find the magnitudes of the horizontal and vertical components of the force.

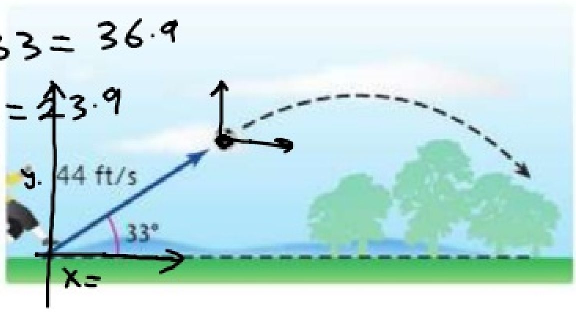
$x = 450 \cos 56 = 251.6$ N

$y = 450 \sin 56 = 373.1$ N

FOOTBALL A player kicks a football so that it leaves the ground with a velocity of 44 feet per second at an angle of 33° with the ground.

$x = 44 \cos 33 = 36.9$

$y = 44 \sin 33 = 23.9$



A. Draw a diagram that shows the resolution of this force into its rectangular components.

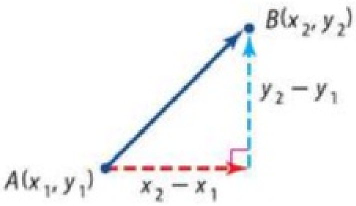
B. Find the magnitude of the horizontal and vertical components of the velocity.

7-2: Vectors in the Coordinate Plane

KeyConcept Component Form of a Vector

The component form of a vector $\overrightarrow{AB}$ with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$ is given by

$$\langle x_2 - x_1, y_2 - y_1 \rangle.$$



Ex1: Find the component form of $\overrightarrow{AB}$ with the given initial and terminal points.

A. $A(-2, -7), B(6, 1)$

$\overrightarrow{AB} = \langle 6-(-2), 1-(-7) \rangle$

$\overrightarrow{AB} = \langle 8, 8 \rangle \Rightarrow |\overrightarrow{AB}| = \sqrt{8^2+8^2}$
$= \sqrt{128}$
$= 8\sqrt{2}$

$|\overrightarrow{AB}| = \sqrt{(6-(-2))^2 + (1-(-7))^2}$
$= \sqrt{64+64}$
$= \sqrt{128}$
$= 8\sqrt{2}$
$= 11.3$

B. $A(0, 8), B(-9, -3)$

$\overrightarrow{AB} = \langle -9-0, -3-8 \rangle$
$= \langle -9, -11 \rangle$

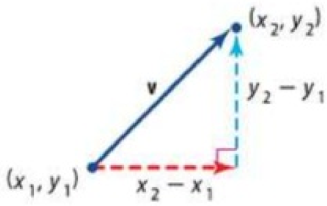
$|\overrightarrow{AB}| = \sqrt{(-9)^2 + (-11)^2}$
$= \sqrt{202}$
$= 14.2$

KeyConcept Magnitude of a Vector in the Coordinate Plane

If **v** is a vector with initial point (x_1, y_1) and terminal point (x_2, y_2), then the magnitude of **v** is given by

$$|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

If **v** has a component form of $\langle a, b \rangle$, then $|\mathbf{v}| = \sqrt{a^2 + b^2}$.



Ex2: Find the magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

$\overrightarrow{BA}$

A. $A(-2, -7), B(6, 1)$

$\overrightarrow{BA} = \langle -2-6, -7-1 \rangle$
$= \langle -8, -8 \rangle$

$|\overrightarrow{BA}| = \sqrt{(-8)^2 + (-8)^2}$
$= \sqrt{128}$
$= 8\sqrt{2} = 11.3$

B. $A(0, 8), B(-9, -3)$

KeyConcept Vector Operations

If $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$ are vectors and k is a scalar, then the following are true.

Vector Addition	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2\rangle$
Vector Subtraction	$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2\rangle$
Scalar Multiplication	$k\mathbf{a} = \langle ka_1, ka_2\rangle$

Ex3: Find each of the following for $\mathbf{w} = \langle -4, 1\rangle$, $\mathbf{y} = \langle 2, 5\rangle$, and $\mathbf{z} = \langle -3, 0\rangle$.

A. $4\mathbf{w} + \mathbf{z}$

$= 4\langle -4, 1\rangle + \langle -3, 0\rangle$

$= \langle -16, 4\rangle + \langle -3, 0\rangle$

$= \langle -19, 4\rangle$.

B. $-3\mathbf{w}$

$= -3\langle -4, 1\rangle$

$= \langle -12, -3\rangle$

C. $2\mathbf{w} + 4\mathbf{y} - \mathbf{z}$

$= 2\langle -4, 1\rangle + 4\langle 2, 5\rangle - \langle -3, 0\rangle$

$= \langle -8, 2\rangle + \langle 8, 20\rangle + \langle 3, 0\rangle$

$= \langle 3, 22\rangle$

Note: A vector that has a magnitude of $\boxed{1 \text{ unit}}$ is called a unit vector $\boxed{\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}}$ or $\boxed{\dfrac{1}{|\mathbf{v}|}\mathbf{v}}$

Ex4: Find a unit vector with the same direction as the given vector.

A. $\mathbf{w} = \langle 6, -2\rangle$

$u = \dfrac{w}{|w|}$

$|w| = \sqrt{(6)^2 + (-2)^2}$

$= \sqrt{40} = 2\sqrt{10}$

$\vec{u} = \dfrac{\langle 6, -2\rangle}{2\sqrt{10}} = \left\langle \dfrac{\cancel{6}^{3}}{\cancel{2}_{1}\sqrt{10}}, \dfrac{-\cancel{2}^{1}}{\cancel{2}_{1}\sqrt{10}} \right\rangle = \left\langle \dfrac{3\sqrt{10}}{10}, \dfrac{-\sqrt{10}}{10} \right\rangle$

B. $\mathbf{x} = \langle -4, -8\rangle$

$u = \left\langle \dfrac{-4}{4\sqrt{5}}, \dfrac{-8}{4\sqrt{5}} \right\rangle$

$= \left\langle \dfrac{-\sqrt{5}}{5}, \dfrac{-2\sqrt{5}}{5} \right\rangle$

Note2: The unit vector in the positive direction of the x-axis is $\boxed{i=\langle 1,0\rangle}$ and the unit vector in the positive direction of the y-axis is $\boxed{j=\langle 0,1\rangle}$. Vectors i and j are called standard unit vectors. These vectors can be used to express any vecror $\boxed{v=\langle a,b\rangle \text{ as } ai+bj}$

Ex5: Let $\overrightarrow{DE}$ be the vector with the given initial and terminal points. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j.

A. $D(-6, 0), E(2, 5)$

$\overrightarrow{DE}=\langle 2-(-6), 5-0\rangle$

$\overrightarrow{DE}=\langle 8,5\rangle = 8i+5j$

B. $D(-3, -8), E(-7, 1)$

$\overrightarrow{DE}=\langle -7-(-3), 1-(-8)\rangle$

$=\langle -4, 9\rangle$

$=-4i+9j$

Note3: The component form of the vector $\mathbf{v}$ with magnitude $|v|$ and direction angle θ is

$$\boxed{v=\langle |v|\cos\theta, |v|\sin\theta\rangle}$$

Ex6: Find the component form of $\mathbf{v}$ with the given magnitude and direction angle

A. $\boxed{|\mathbf{v}|=8}$ $\boxed{\theta=45^\circ}$

$V=\langle 8\cos 45, 8\sin 45\rangle$

$=\langle \frac{8\sqrt{2}}{2}, \frac{8\sqrt{2}}{2}\rangle$

$=\langle 4\sqrt{2}, 4\sqrt{2}\rangle$

B. $|\mathbf{v}|=24, \theta=210^\circ$

$V=\langle 24\cos 210, 24\sin 210\rangle$

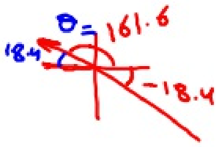
$=\langle -12\sqrt{3}, -12\rangle$

Note4: The direction angle of a vector $v=\langle a,b\rangle$ can be found by solving the trigonometric equation $\boxed{\tan\theta=\frac{b}{a}}$

Ex7: Find the direction angle of each vector to the nearest tenth of a degree.

$V=ai+bj$

A. $-6i+2j$



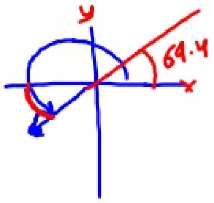
$a=-6$

$b=2$

$\tan\theta=\frac{\cancel{2}^{1}}{\cancel{-6}_{3}}$

$\theta=\tan^{-1}\left(-\frac{1}{3}\right)=$

B. $\langle -3, -8\rangle$



$\tan\theta=\frac{-8}{-3}$

$\theta=\tan^{-1}\left(\frac{8}{3}\right)$

$=69.4$

$\Rightarrow \theta=69.4+180=249.4$

$-, +$ | $+$
$-, -$ | $+, -$

C. $p = 3i + 7j$

$\tan\theta = \frac{7}{3}$

$\theta = \tan^{-1}\left(\frac{7}{3}\right)$

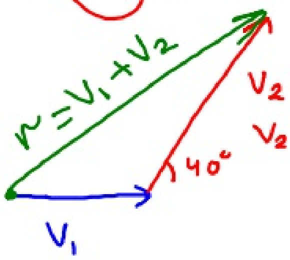
$\theta = 66.8^\circ$

D. $r = \langle 4, -5 \rangle$

$\tan\theta = \frac{-5}{4}$

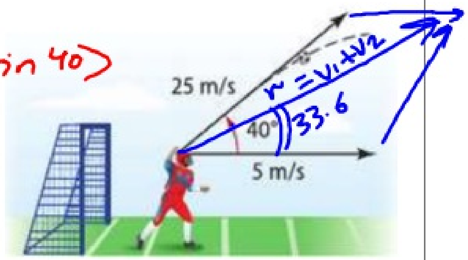
$\theta = \tan^{-1}\left(\frac{-5}{4}\right) = 308.7$

Ex8: A goalkeeper running forward at 5 meters per second throws a ball with a velocity of 25 meters per second at an angle of 40° with the horizontal. What is the resultant speed and direction of the pass?



$V_2 = \langle 25\cos 40, 25\sin 40 \rangle$

$V_2 = \langle 19.2, 16.1 \rangle$



$V_1 = \langle 5\cos 0, 5\sin 0 \rangle$

$\boxed{V_1 = \langle 5, 0 \rangle}$

$r = V_1 + V_2$

$r = \langle 5, 0 \rangle + \langle 19.2, 16.1 \rangle$

$r = \langle 24.2, 16.1 \rangle$

$|r| = \sqrt{(24.2)^2 + (16.1)^2} = 29.1 \text{ m/s}$

$\theta = \tan^{-1}\left(\frac{16.1}{24.2}\right) = 33.6^\circ$

7-3: Dot Products and Vector Projections

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KeyConcept Dot Product of Vectors in a Plane

The dot product of $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$.

KeyConcept Orthogonal Vectors

The vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Note: the zero vector $\langle 0,0 \rangle$ is orthogonal to any vector a.

Ex1: Find the dot product of u and v. Then determine if u and v are orthogonal.

A. $\mathbf{u} = \langle \overset{a_1}{3}, \overset{a_2}{-2} \rangle, \mathbf{v} = \langle \overset{b_1}{-5}, \overset{b_2}{1} \rangle$

$u \cdot v = (3)(-5) + (-2)(1)$

$= -15 - 2$

$u \cdot v = -17$

B. $\mathbf{u} = \langle -2, -3 \rangle, \mathbf{v} = \langle 9, -6 \rangle$

$u \cdot v = (-2)(9) + (-3)(-6)$

$= 0$

$\boxed{u \perp v}$

KeyConcept Properties of the Dot Product

If **u**, **v**, and **w** are vectors and *k* is a scalar, then the following properties hold.

Commutative Property	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ ✓
Distributive Property	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ ✓
Scalar Multiplication Property	$k(\mathbf{u} \cdot \mathbf{v}) = k\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot k\mathbf{v}$ ✓
Zero Vector Dot Product Property	$\mathbf{0} \cdot \mathbf{u} = 0$ ✓
Dot Product and Vector Magnitude Relationship	$\mathbf{u} \cdot \mathbf{u} = \lvert\mathbf{u}\rvert^2$

Proof

Proof $\quad \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

Let $\mathbf{u} = \langle u_1, u_2 \rangle$.

$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2$ — Dot product

$= \left(\sqrt{(u_1^2 + u_2^2)}\right)^2$ — Write as the square of the square root of $u_1^2 + u_2^2$

$= |\mathbf{u}|^2$ — $\sqrt{u_1^2 + u_2^2} = |\mathbf{u}|$

Ex2: Use the dot product to find the magnitude of the given vector.

A. $\mathbf{b} = \langle 12, 16 \rangle$

$b \cdot b = (12)^2 + (16)^2$

$b \cdot b = 400$

$|b|^2 = 400$

$\boxed{|b| = 20}$

B. $\mathbf{c} = \langle -1, -7 \rangle$

$c \cdot c = (-1)^2 + (-7)^2$

$= 50$

$|c| = \sqrt{50} = 5\sqrt{2}$

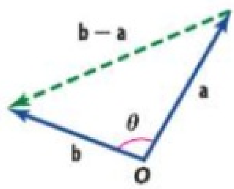
$= 7.07$

KeyConcept Angle Between Two Vectors

If θ is the angle between nonzero vectors **a** and **b**, then

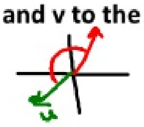
$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$

$$\theta = \cos^{-1}\left(\frac{a \cdot b}{|a||b|}\right)$$



Ex3: Find the angle θ between vectors u and v to the nearest tenth of a degree

A. $\mathbf{u} = \langle -5, -2 \rangle$ and $\mathbf{v} = \langle 4, 4 \rangle$



$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

$u \cdot v = (-5)(4) + (-2)(4) = -28$

$|u| = \sqrt{(-5)^2 + (-2)^2}$

$= \sqrt{29}$

$|v| = \sqrt{(4)^2 + (4)^2}$

$= \sqrt{32} = 4\sqrt{2}$

$$\cos\theta = \frac{-28}{(\sqrt{29})(4\sqrt{2})}$$

$$\theta = \cos^{-1}\left(\frac{-28}{(\sqrt{29})(4\sqrt{2})}\right) = 156.8^\circ$$

B. $\mathbf{u} = \langle 9, 5 \rangle$ and $\mathbf{v} = \langle -6, 7 \rangle$

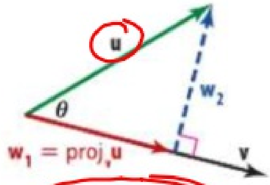
$\theta = 101.6^\circ$

KeyConcept Projection of u onto v

Let **u** and **v** be nonzero vectors, and let $\mathbf{w}_1$ and $\mathbf{w}_2$ be vector components of **u** such that $\mathbf{w}_1$ is parallel to **v** as shown. Then vector $\mathbf{w}_1$ is called the **vector projection** of **u** onto **v**, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, and

$$w_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.$$

$$W_1 = \text{proj}_u v = \left(\frac{u\cdot v}{|u|^2}\right)u$$



$$\boxed{\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2}$$

$$W_2 = u - w_1$$

Ex4: Find the projection of $\mathbf{u} = \langle 1, 2\rangle$ onto $\mathbf{v} = \langle 8, 5\rangle$. Then write **u** as the sum of two orthogonal vectors, one of which is the projection of **u** onto **v**.

$$w_1 = \text{proj}_v u = \left(\frac{u\cdot v}{|v|^2}\right)v$$

$$u\cdot v = (1)(8) + (2)(5) = 18$$

$$|v|^2 = (8)^2 + (5)^2 = 89$$

$$w_1 = \text{proj}_v u = \left(\frac{18}{89}\right)\langle 8, 5\rangle$$

$$W_1 = \left\langle \frac{144}{89}, \frac{90}{89}\right\rangle$$

$$W_2 = u - w_1$$

$$= \langle 1, 2\rangle - \left\langle \frac{144}{89}, \frac{90}{89}\right\rangle$$

$$W_2 = \left\langle \frac{-55}{89}, \frac{88}{89}\right\rangle$$

Ex5: Find the projection of $\mathbf{u} = \langle -3, 4\rangle$ onto $\mathbf{v} = \langle 6, 1\rangle$. Then write **u** as the sum of two orthogonal vectors, one of which is the projection of **u** onto **v**.

$$\boxed{U = W_1 + W_2} \quad \boxed{W_2 = U - W_1}$$

$$W_1 = \text{proj}_v u = \left(\frac{u\cdot v}{|v|^2}\right)v$$

$$u\cdot v = (-3)(6) + (4)(1) = -14$$

$$|v|^2 = (6)^2 + (1)^2 = 37$$

$$= \left(\frac{-14}{37}\right)\langle 6, 1\rangle$$

$$W_1 = \left\langle \frac{-84}{37}, \frac{-14}{37}\right\rangle$$

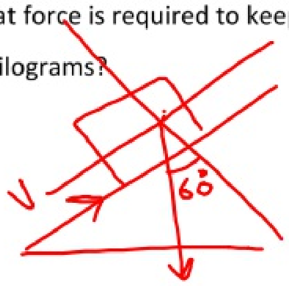
$$W_2 = \langle -3, 4\rangle - \left\langle -\frac{84}{37}, \frac{-14}{37}\right\rangle$$

$$= \left\langle \frac{-27}{37}, \frac{162}{37}\right\rangle$$

Ex6: Nisreen sits on a sled on the side of a hill inclined at 60°. What force is required to keep the sled from sliding down the hill if the weight of Nisreen and the sled is 125 kilograms?

$V = \langle 1\cos 60, 1\sin 60 \rangle$

$= \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$



$F = \langle 0, -125 \rangle$

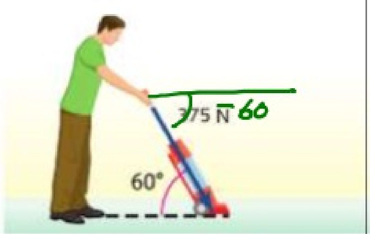
$w_1 = proj_V F = \left(\frac{F \cdot V}{|1|^2}\right) V = -108.25\, V$

$-w_1 = 108.25\, V$

Note: **Consider a constant force F acting on an object to move it from point A to point B , then the work W done by F is the magnitude of the force times cosϴ times the distance from A to B or** $\boxed{W = |\mathbf{F}|\,(\cos\theta)\,|\overrightarrow{AB}|}$

Or simply W= F×d×cosϴ where ϴ is the angle between the force doing the work and the direction of the object movement.

Ex7 **CLEANING** Faris is pushing a vacuum cleaner with a force of 375 Newtons. The handle of the vacuum cleaner makes a 60° angle with the floor. How much work in Newton-meters does he do if he pushes the vacuum cleaner 2 meters?



$W = F\cos\theta\, |\overrightarrow{AB}|$

$= 375 \cos 60\, (2)$

$= 375\ J$