# 6-1 Multivariable Linear Systems and Row Operations

**Write each system of equations in triangular form using Gaussian elimination. Then solve the system.**

1. $5x = -3y - 31$
   $2y = -4x - 22$

*SOLUTION:*

Align the variables on the left side of the equal sign.

$$5x + 3y = -31$$
$$4x + 2y = -22$$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by $-4$ and the 2nd equation by 5.

$$(-4)(5x + 3y = -31) \quad \Rightarrow \quad -20x - 12y = 124$$
$$(5)(4x + 2y = -22) \quad \Rightarrow \quad 20x + 10y = -110$$

Add the equations together.

$$\begin{aligned} -20x - 12y &= 124 \\ 20x + 10y &= -110 \\ -2y &= 14 \\ y &= -7 \end{aligned}$$

Replace the 2nd equation with $y = -7$.

$$\begin{aligned} 5x + 3y &= -31 \\ y &= -7 \end{aligned}$$

The system is now in triangular form. Substitute for $y$ and solve for $x$.

$$\begin{aligned} 5x + 3y &= -31 \\ 5x + 3(-7) &= -31 \\ 5x - 21 &= -31 \\ 5x &= -10 \\ x &= -2 \end{aligned}$$

The solution is $(-2, -7)$.

# 6-1 Multivariable Linear Systems and Row Operations

2. $4y + 17 = -7x$
$8x + 5y = -19$

*SOLUTION:*

Align the variables on the left side of the equal sign.

$$7x + 4y = -17$$
$$8x + 5y = -19$$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by $-8$ and the 2nd equation by 7.

$$(-8)(7x + 4y = -17) \quad \Rightarrow \quad -56x - 32y = 136$$
$$(7)(8x + 5y = -19) \quad \Rightarrow \quad 56x + 35y = -133$$

Add the equations together.

$$\begin{aligned} -56x - 32y &= 136 \\ 56x + 35y &= -133 \\ 3y &= 3 \\ y &= 1 \end{aligned}$$

Replace the 2nd equation with $y = 1$.

$$\begin{aligned} 7x + 4y &= -17 \\ y &= 1 \end{aligned}$$

The system is now in triangular form. Substitute for $y$ and solve for $x$.

$$\begin{aligned} 7x + 4y &= -17 \\ 7x + 4(1) &= -17 \\ 7x + 4 &= -17 \\ 7x &= -21 \\ x &= -3 \end{aligned}$$

The solution is $(-3, 1)$.

# 6-1 Multivariable Linear Systems and Row Operations

3. $12x = 21 - 3y$
$2y = 6x + 7$

*SOLUTION:*

Align the variables on the left side of the equal sign.

$$12x + 3y = 21$$
$$-6x + 2y = 7$$

Eliminate the $x$-term from the 2nd equation. Multiply the 2nd equation by 2.

$$(12x + 3y = -21) \quad \Rightarrow \quad 12x + 3y = 21$$
$$(2)(-6x + 2y = 7) \quad \Rightarrow \quad -12x + 4y = 14$$

Add the equations together.

$$\begin{aligned} 12x + 3y &= 21 \\ -12x + 4y &= 14 \\ 7y &= 35 \\ y &= 5 \end{aligned}$$

Replace the 2nd equation with $y = 5$.

$$\begin{aligned} 12x + 3y &= 21 \\ y &= 5 \end{aligned}$$

The system is now in triangular form. Substitute for $y$ and solve for $x$.

$$\begin{aligned} 12x + 3y &= 21 \\ 12x + 3(5) &= 21 \\ 12x + 15 &= 21 \\ 12x &= 6 \\ x &= 0.5 \end{aligned}$$

The solution is $\left(\frac{1}{2}, 5\right)$.

4. $4y = 12x - 3$
   $9x = 20y - 2$

   *SOLUTION:*

   Align the variables on the left side of the equal sign.

$$-12x + 4y = -3$$
$$9x - 20y = -2$$

   Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by 3 and the 2nd equation by 4.

$$(3)(-12x + 4y = -3) \quad \Rightarrow \quad -36x + 12y = -9$$
$$(4)(9x - 20y = -2) \quad \Rightarrow \quad 36x - 80y = -8$$

   Add the equations together.

$$-36x + 12y = -9$$
$$36x - 80y = -8$$
$$-68y = -17$$
$$y = 0.25$$

   Replace the 2nd equation with $y = 0.25$.

$$-12x + 4y = -3$$
$$y = 0.25$$

   The system is now in triangular form. Substitute for $y$ and solve for $x$.

$$-12x + 4y = -3$$
$$-12x + 4(0.25) = -3$$
$$-12x + 1 = -3$$
$$-12x = -4$$
$$x = \frac{1}{3}$$

   The solution is $\left(\frac{1}{3}, \frac{1}{4}\right)$.

$$-3x + y + 6z = 15$$

5. $2x + 2y - 5z = 9$
   $4x - 5y + 2z = -3$

   *SOLUTION:*

   The variables are aligned on the left side of the equal sign.

$$-3x + y + 6z = 15$$
$$2x + 2y - 5z = 9$$
$$4x - 5y + 2z = -3$$

   Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by 2 and the 2nd equation by 3.

$$(2)(-3x + y + 6z = 15) \quad \Rightarrow \quad -6x + 2y + 12z = 30$$
$$(3)(2x + 2y - 5z = 9) \quad \Rightarrow \quad 6x + 6y - 15z = 27$$

   Add the equations together.

$$-6x + 2y + 12z = 30$$
$$(+)\ 6x + 6y - 15z = 27$$
$$8y - 3z = 57$$

   The new set of equations is:

$$-3x+y+6z=15$$
$$8y-3z=57$$
$$4x-5y+2z=-3$$

Eliminate the $x$-term from the 3rd equation. Multiply the 1st equation by 4 and the 3rd equation by 3.

$(4)(-3x+y+6z=15) \quad \Rightarrow \quad -12x+4y+24z=60$

$(3)(4x-5y+2z=-3) \quad \Rightarrow \quad 12x-15y+6z=-9$

Add the equations together.

$$-12x+4y+24z=60$$
$$(+)\,12x-15y+6z=-9$$
$$-11y+30z=51$$

The new set of equations is:

$$-3x+y+6z=15$$
$$8y-3z=57$$
$$-11y+30z=51$$

Eliminate the $y$-term from the 3rd equation. Multiply the 2nd equation by 11 and the 3rd equation by 8.

$(11)(8y-3z=57) \quad \Rightarrow \quad 88y-33z=627$

$(8)(-11y+30z=51) \quad \Rightarrow \quad -88y+240z=408$

Add the equations together.

$$88y-33z=627$$
$$(+)\,-88y+240z=408$$
$$207z=1035$$
$$z=5$$

Replace the 3rd equation with $z = 5$.

$$-3x+y+6z=15$$
$$8y-3z=57$$
$$z=5$$

The system is now in triangular form. Substitute for $z$ and solve for $y$, then substitute for $y$ and $z$ and solve for $x$.

$$8y-3z=57$$
$$8y-3(5)=57$$
$$8y-15=57$$
$$8y=72$$
$$y=9$$
$$-3x+y+6z=15$$
$$-3x+9+6(5)=15$$
$$-3x+9+30=15$$
$$-3x=-24$$
$$x=8$$

The solution is $(8,9,5)$.

6. $8x-24y+16z=-7$
   $40x-9y+2z=10$
   $32x+8y-z=-2$

*SOLUTION:*

The variables are aligned on the left side of the equal sign.

$$8x-24y+16z=-7$$
$$40x-9y+2z=10$$
$$32x+8y-z=-2$$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by $-5$.

$$(-5)(8x-24y+16z=-7) \quad \Rightarrow \quad -40x+120y-80z=35$$
$$(40x-9y+2z=10) \quad \Rightarrow \quad 40x-9y+2z=10$$

Add the equations together.

$$-40x+120y-80z=35$$
$$(+)\,40x-9y+2z=10$$
$$111y-78z=45$$

The new set of equations is:

$$8x-24y+16z=-7$$
$$111y-78z=45$$
$$32x+8y-z=-2$$

Eliminate the $x$-term from the 3rd equation. Multiply the 1st equation by $-4$.

$$(-4)(8x-24y+16z=-7) \quad \Rightarrow \quad -32x+96y-64z=28$$
$$(32x+8y-z=-2) \quad \Rightarrow \quad 32x+8y-z=-2$$

Add the equations together.

$$-32x+96y-64z=28$$
$$(+)\,32x+8y-z=-2$$
$$104y-65z=26$$

The new set of equations is:

$$8x-24y+16z=-7$$
$$111y-78z=45$$
$$104y-65z=26$$

The 2nd and 3rd equations simplify to:

$$8x-24y+16z=-7$$
$$37y-26z=15$$
$$8y-5z=2$$

Eliminate the $y$-term from the 3rd equation. Multiply the 2nd equation by 8 and the 3rd equation by $-37$.

$$(8)(37y-26z=15) \quad \Rightarrow \quad 296y-208z=120$$
$$(-37)(8y-5z=2) \quad \Rightarrow \quad -296y+185z=-74$$

Add the equations together.

$$296y-208z=120$$
$$(+)\,-296y+185z=-74$$
$$-23z=46$$
$$z=-2$$

Replace the 3rd equation with $z=-2$.

$$8x-24y+16z=-7$$
$$37y-26z=15$$
$$z=-2$$

The system is now in triangular form. Substitute for $z$ and solve for $y$, then substitute for $y$ and $z$ and solve for $x$.

$$37y - 26z = 15$$
$$37y - 26(-2) = 15$$
$$37y + 52 = 15$$
$$37y = -37$$
$$y = -1$$
$$8x - 24y + 16z = -7$$
$$8x - 24(-1) + 16(-2) = -7$$
$$8x + 24 - 32 = -7$$
$$8x = 1$$
$$x = \frac{1}{8}$$

The solution is $\left(\frac{1}{8}, -1, -2\right)$.

7. $3x + 9y - 6z = 17$
$-2x - y + 24z = 12$
$2x - 5y + 12z = -30$

*SOLUTION:*

The variables are aligned on the left side of the equal sign.

$$3x + 9y - 6z = 17$$
$$-2x - y + 24z = 12$$
$$2x - 5y + 12z = -30$$

Eliminate the $x$-term from the 2nd equation. Add the 2nd and 3rd equations together.

$$-2x - y + 24z = 12$$
$$(+)2x - 5y + 12z = -30$$
$$-6y + 36z = -18$$

The new set of equations is:

$$3x + 9y - 6z = 17$$
$$-6y + 36z = -18$$
$$2x - 5y + 12z = -30$$

Eliminate the $x$-term from the 3rd equation. Multiply the 1st equation by $-2$ and the 3rd equation by 3.

$(-2)(3x + 9y - 6z = 17) \Rightarrow -6x - 18y + 12z = -34$
$(3)(2x - 5y + 12z = -30) \Rightarrow 6x - 15y + 36z = -90$

Add the equations together.

$$-6x - 18y + 12z = -34$$
$$(+)6x - 15y + 36z = -90$$
$$-33y + 48z = -124$$

The new set of equations is:

$$3x + 9y - 6z = 17$$
$$-6y + 36z = -18$$
$$-33y + 48z = -124$$

This reduces to:

$$\begin{aligned} 3x+9y-6z&=17\\ -y+6z&=-3\\ -33y+48z&=-124 \end{aligned}$$

Eliminate the $y$-term from the 3rd equation. Multiply the 2nd equation by −33.

$$(-33)(-y+6z=-3) \quad \Rightarrow \quad 33y-198z=99$$
$$(-33y+48z=-124) \quad \Rightarrow \quad -33y+48z=-124$$

Add the equations together.

$$\begin{aligned} 33y-198z&=99\\ (+)\,-33y+48z&=-124\\ -150z&=-25\\ z&=\frac{1}{6} \end{aligned}$$

Replace the 3rd equation with $z = \frac{1}{6}$.

$$\begin{aligned} 3x+9y-6z&=17\\ -y+6z&=-3\\ z&=\frac{1}{6} \end{aligned}$$

The system is now in triangular form. Substitute for $z$ and solve for $y$, then substitute for $y$ and $z$ and solve for $x$.

$$\begin{aligned} -y+6z&=-3\\ -y+6\left(\frac{1}{6}\right)&=-3\\ -y+1&=-3\\ -y&=-4\\ y&=4\\ 3x+9y-6z&=17\\ 3x+9(4)-6\left(\frac{1}{6}\right)&=17\\ 3x+36-1&=17\\ 3x&=-18\\ x&=-6 \end{aligned}$$

The solution is $\left(-6, 4, \frac{1}{6}\right)$.

8. $5x - 50y + z = 24$
$2x + 10y + 3z = 23$
$-5x - 20y + 10z = 13$

*SOLUTION:*

The variables are aligned on the left side of the equal sign.

$$\begin{aligned} 5x-50y+z&=24\\ 2x+10y+3z&=23\\ -5x-20y+10z&=13 \end{aligned}$$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by −2 and the 2nd by 5.

$(-2)(5x-50y+z=24) \quad \Rightarrow \quad -10x+100y-2z=-48$

$(5)(2x+10y+3z=23) \quad \Rightarrow \quad 10x+50y+15z=115$

Add the equations together.

$$\begin{aligned} -10x+100y-2z &= -48 \\ (+)10x+50y+15z &= 115 \\ 150y+13z &= 67 \end{aligned}$$

The new set of equations is:

$$\begin{aligned} 5x-50y+z &= 24 \\ 150y+13z &= 67 \\ -5x-20y+10z &= 13 \end{aligned}$$

Eliminate the $x$-term from the 3rd equation. Add the 1st and 3rd equations together.

$$\begin{aligned} 5x-50y+z &= 24 \\ (+)-5x-20y+10z &= 13 \\ -70y+11z &= 37 \end{aligned}$$

The new set of equations is:

$$\begin{aligned} 5x-50y+z &= 24 \\ 150y+13z &= 67 \\ -70y+11z &= 37 \end{aligned}$$

Eliminate the $y$-term from the 3rd equation. Multiply the 2nd equation by 7 and the 3rd equation by 15.

$(7)(150y+13z=67) \quad \Rightarrow \quad 1050y+91z=469$

$(15)(-70y+11z=37) \quad \Rightarrow \quad -1050y+165z=555$

Add the equations together.

$$\begin{aligned} 1050y+91z &= 469 \\ (+)-1050y+165z &= 555 \\ 256z &= 1024 \\ z &= 4 \end{aligned}$$

Replace the 3rd equation with $z = 4$.

$$\begin{aligned} 5x-50y+z &= 24 \\ 150y+13z &= 67 \\ z &= 4 \end{aligned}$$

The system is now in triangular form. Substitute for $z$ and solve for $y$, then substitute for $y$ and $z$ and solve for $x$.

$$\begin{aligned} 150y+13z &= 67 \\ 150y+13(4) &= 67 \\ 150y+52 &= 67 \\ 150y &= 15 \\ y &= \frac{1}{10} \end{aligned}$$

$$\begin{aligned} 5x-50y+z &= 24 \\ 5x-50\left(\frac{1}{10}\right)+4 &= 24 \\ 5x-5 &= 20 \\ 5x &= 25 \\ x &= 5 \end{aligned}$$

The solution is $\left(5, \frac{1}{10}, 4\right)$.

**Write the augmented matrix for each system of linear equations.**

9. $12x - 5y = -9$
$-3x + 8y = 10$

*SOLUTION:*

Each of the variables of the system is represented in each equation and aligned. Write the augmented matrix.

$$\left[\begin{array}{cc|c} 12 & -5 & -9 \\ -3 & 8 & 10 \end{array}\right]$$

10. $-4x - 6y = 25$
$7x + 2y = 16$

*SOLUTION:*

Each of the variables of the system is represented in each equation and aligned. Write the augmented matrix.

$$\left[\begin{array}{cc|c} -4 & -6 & 25 \\ 7 & 2 & 16 \end{array}\right]$$

11. $3x - 5y + 7z = 9$
$-10x + y + 8z = 6$
$4x - 15z = -8$

*SOLUTION:*

Each of the variables of the system is not represented in each equation. Rewrite the system, aligning the like terms and using 0 for missing terms.

$3x - 5y + 7z = 9$
$-10x + y + 8z = 6$
$4x + 0y - 15z = -8$

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 3 & -5 & 7 & 9 \\ -10 & 1 & 8 & 6 \\ 4 & 0 & -15 & -8 \end{array}\right]$$

12. $$\begin{aligned} 4x - z &= 27 \\ -8x + 7y - 6z &= -35 \\ 12x - 3y + 5z &= 20 \end{aligned}$$

*SOLUTION:*

Each of the variables of the system is not represented in each equation. Rewrite the system, aligning the like terms and using 0 for missing terms.

$$\begin{aligned} 4x + 0y - z &= 27 \\ -8x + 7y - 6z &= -35 \\ 12x - 3y + 5z &= 20 \end{aligned}$$

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 4 & 0 & -1 & 27 \\ -8 & 7 & -6 & -35 \\ 12 & -3 & 5 & 20 \end{array}\right]$$

13. $$\begin{aligned} w - 8x + 5y &= 11 \\ 7w + 2x - 3y + 9z &= -5 \\ 6w + 12y - 15z &= 4 \\ 3x + 4y - 8z &= -13 \end{aligned}$$

*SOLUTION:*

Each of the variables of the system is not represented in each equation. Rewrite the system, aligning the like terms and using 0 for missing terms.

$$\begin{aligned} w - 8x + 5y + 0z &= 11 \\ 7w + 2x - 3y + 9z &= -5 \\ 6w + 0x + 12y - 15z &= 4 \\ 0w + 3x + 4y - 8z &= -13 \end{aligned}$$

Write the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & -8 & 5 & 0 & 11 \\ 7 & 2 & -3 & 9 & -5 \\ 6 & 0 & 12 & -15 & 4 \\ 0 & 3 & 4 & -8 & -13 \end{array}\right]$$

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14. $$\begin{aligned} 14x - 2y + 3z &= -22 \\ 5w - 4x + 11z &= -8 \\ 2w - 6y + 3z &= 15 \\ 3w + 7x - y &= 1 \end{aligned}$$

*SOLUTION:*

Each of the variables of the system is not represented in each equation. Rewrite the system, aligning the like terms and using 0 for missing terms.

$$\begin{aligned} 0w + 14x - 2y + 3z &= -22 \\ 5w - 4x + 0y + 11z &= -8 \\ 2w + 0x - 6y + 3z &= 15 \\ 3w + 7x - y + 0z &= 1 \end{aligned}$$

Write the augmented matrix.

$$\left[\begin{array}{cccc|c} 0 & 14 & -2 & 3 & -22 \\ 5 & -4 & 0 & 11 & -8 \\ 2 & 0 & -6 & 3 & 15 \\ 3 & 7 & -1 & 0 & 1 \end{array}\right]$$

15. **BAKE SALE** Members of a youth group held a bake sale to raise money for summer trip. They sold 30 cakes, 40 pies, and 200 giant cookies and raised $684.50. A pie cost $2 less than a cake and cake cost 5 times as much as a giant cookie.

**a.** Let $c$ = cost of one cakes, $p$ = cost of one pies, and $g$ = cost of one giant cookies. Write a system of three linear equations to represent the problem.

**b.** Write the augmented matrix for the system of linear equations that you wrote in part **a**.

**c.** Solve the system of equations. Interpret your solution.

*SOLUTION:*

**a**. The variables are the individual costs of each item. One equation used is the total cost of everything: $30c + 40p + 200g = 684.50$. The second equation is the comparison in price between cake and pie: $p - c = -2$.
The third equation is the comparison in price between cake and the giant cookie: $c - 5g = 0$.

b. Each equation is represented by a row.

$$\left[\begin{array}{ccc|c} 30 & 40 & 200 & 684.50 \\ -1 & 1 & 0 & -2 \\ 1 & 0 & -5 & 0 \end{array}\right]$$

c. The last 2 equations each have a $c$. Find $g$ and $p$ in terms of $c$ in each and then substitute these values into the 1st equation.

$$p - c = -2$$
$$p = -2 + c$$

$$c - 5g = 0$$
$$-5g = -c$$
$$g = 0.2c$$

$$30c + 40p + 200g = 684.5$$
$$30c + 40(-2 + c) + 200(0.2c) = 684.5$$
$$30c - 80 + 40c + 40c = 684.5$$
$$110c = 764.5$$
$$c = 6.95$$

Use the value of $c$ to find $p$ and $g$.

$$g = 0.2c$$
$$g = 0.2(6.95)$$
$$g = 1.39$$

$$p = -2 + c$$
$$p = -2 + 6.95$$
$$p = 4.95$$

(6.95, 4.95, 1.39); A cake cost $6.95, a pie cost $4.95, and a giant cookie cost $1.39.

# 6-1 Multivariable Linear Systems and Row Operations

**Determine whether each matrix is in row-echelon form.**

16. $\left[\begin{array}{cc|c} 1 & 8 & 7 \\ 0 & 1 & 3 \end{array}\right]$

*SOLUTION:*

There is a zero below the leading one in the first row. The matrix is in row-echelon form.

17. $\left[\begin{array}{cc|c} 1 & -2 & 9 \\ 0 & 0 & 1 \end{array}\right]$

*SOLUTION:*

There is a zero below the leading one in the first row. The matrix is in row-echelon form.

18. $\left[\begin{array}{cc|c} 1 & -8 & 12 \\ 1 & 3 & -7 \\ 0 & 1 & 4 \end{array}\right]$

*SOLUTION:*

There is a nonzero number below the leading one in the first row. The matrix is not in row-echelon form.

19. $\left[\begin{array}{cc|c} 1 & -4 & 10 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{array}\right]$

*SOLUTION:*

There is a zero below each of the leading ones in each row. The matrix is in row-echelon form.

20. $\left[\begin{array}{ccc|c} 0 & 1 & -7 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right]$

*SOLUTION:*

There is a nonzero number below the leading one in the second row. The matrix is not in row-echelon form.

21. $\left[\begin{array}{ccc|c} 0 & 1 & -8 & 5 \\ 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

*SOLUTION:*

There is a zero below each of the leading ones in each row. The matrix is in row-echelon form.

# 6-1 Multivariable Linear Systems and Row Operations

**Solve each system of equations using Gauss-Jordan elimination.**

22. $2x = -10y + 11$
$-8y = -9x + 23$

*SOLUTION:*

Rewrite the equations, aligning the terms.

$2x + 10y = 11$
$9x - 8y = 23$

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 2 & 10 & 11 \\ 9 & -8 & 23 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cc|c} 2 & 10 & 11 \\ 9 & -8 & 23 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 10 & 11 \\ 0 & -53 & -26.5 \end{array}\right] \leftarrow R_2 + -4.5R_1$$

$$\left[\begin{array}{cc|c} 2 & 10 & 11 \\ 0 & 1 & 0.5 \end{array}\right] \leftarrow R_2 \div (-53)$$

$$\left[\begin{array}{cc|c} 2 & 0 & 6 \\ 0 & 1 & 0.5 \end{array}\right] \leftarrow R_1 - 10R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0.5 \end{array}\right] \leftarrow 0.5R_1$$

The solution is (3, 0.5).

23. $4y + 17 = -7x$
$8x + 5y = -19$

*SOLUTION:*

Rewrite the equations, aligning the terms.

$7x + 4y = -17$

$8x + 5y = -19$

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 7 & 4 & -17 \\ 8 & 5 & -19 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cc|c} 7 & 4 & -17 \\ 8 & 5 & -19 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 7 & 4 & -17 \\ 0 & 3 & 3 \end{array}\right] \leftarrow -8R_1 + 7R_2$$

$$\left[\begin{array}{cc|c} 7 & 4 & -17 \\ 0 & 1 & 1 \end{array}\right] \leftarrow R_2 \div 3$$

$$\left[\begin{array}{cc|c} 7 & 0 & -21 \\ 0 & 1 & 1 \end{array}\right] \leftarrow R_1 - 4R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \leftarrow R_1 \div 7$$

The solution is (−3, 1).

24. $x + 7y = 10$
$3x + 9y = -6$

*SOLUTION:*

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 7 & 10 \\ 3 & 9 & -6 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cc|c} 1 & 7 & 10 \\ 3 & 9 & -6 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 7 & 10 \\ 0 & -12 & -36 \end{array}\right] \leftarrow R_2 - 3R_1$$

$$\left[\begin{array}{cc|c} 1 & 7 & 10 \\ 0 & 1 & 3 \end{array}\right] \leftarrow R_2 \div (-12)$$

$$\left[\begin{array}{cc|c} 1 & 0 & -11 \\ 0 & 1 & 3 \end{array}\right] \leftarrow R_1 - 7R_2$$

The solution is (−11, 3).

25. $7y = 9 - 5x$
$8x = 2 - 5y$

*SOLUTION:*

Rewrite the equations, aligning the terms.

$5x + 7y = 9$

$8x + 5y = 2$

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 5 & 7 & 9 \\ 8 & 5 & 2 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cc|c} 5 & 7 & 9 \\ 8 & 5 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 5 & 7 & 9 \\ 0 & -31 & -62 \end{array}\right] \leftarrow 5R_2 - 8R_1$$

$$\left[\begin{array}{cc|c} 5 & 7 & 9 \\ 0 & 1 & 2 \end{array}\right] \leftarrow R_2 \div (-31)$$

$$\left[\begin{array}{cc|c} 5 & 0 & -5 \\ 0 & 1 & 2 \end{array}\right] \leftarrow R_1 - 7R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right] \leftarrow R_1 \div 5$$

The solution is $(-1, 2)$.

26. $3x - 4y + 8z = 27$
$9x - y - z = 3$
$x + 8y - 2z = 9$

*SOLUTION:*

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 3 & -4 & 8 & 27 \\ 9 & -1 & -1 & 3 \\ 1 & 8 & -2 & 9 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 3 & -4 & 8 & 27 \\ 9 & -1 & -1 & 3 \\ 1 & 8 & -2 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & -4 & 8 & 27 \\ 0 & 11 & -25 & -78 \\ 1 & 8 & -2 & 9 \end{array}\right] \leftarrow R_2 - 3R_3$$

$$\left[\begin{array}{ccc|c} 3 & -4 & 8 & 27 \\ 0 & 11 & -25 & -78 \\ 0 & -28 & 14 & 0 \end{array}\right] \leftarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 3 & -4 & 8 & 27 \\ 0 & 11 & -25 & -78 \\ 0 & 2 & -1 & 0 \end{array}\right] \leftarrow R_3 + (-14)$$

$$\left[\begin{array}{ccc|c} 3 & -4 & 8 & 27 \\ 0 & 2 & -1 & 0 \\ 0 & 11 & -25 & -78 \end{array}\right] \begin{array}{l} \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & -4 & 8 & 27 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -19.5 & -78 \end{array}\right] \begin{array}{l} \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & -4 & 8 & 27 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right] \leftarrow R_3 + (-19.5)$$

$$\left[\begin{array}{ccc|c} 3 & -4 & 8 & 27 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right] \leftarrow R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 3 & -4 & 8 & 27 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \leftarrow \frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 3 & -4 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \leftarrow R_1 - 8R_3$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \leftarrow R_1 + 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \leftarrow \frac{1}{3}R_1$$

The solution is (1, 2, 4).

27. $x + 9y + 8z = 0$
$5x + 8y + z = 35$
$x - 4y - z = 17$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{rrr|r} 1 & 9 & 8 & 0 \\ 5 & 8 & 1 & 35 \\ 1 & -4 & -1 & 17 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 9 & 8 & 0 \\ 0 & 28 & 6 & -50 \\ 1 & -4 & -1 & 17 \end{array}\right] \leftarrow R_2 - 5R_1$$

$$\left[\begin{array}{rrr|r} 1 & 9 & 8 & 0 \\ 0 & 14 & 3 & -25 \\ 1 & -4 & -1 & 17 \end{array}\right] \leftarrow R_2 \div 2$$

$$\left[\begin{array}{rrr|r} 1 & 9 & 8 & 0 \\ 0 & 14 & 3 & -25 \\ 0 & -13 & -9 & 17 \end{array}\right] \leftarrow R_3 - R_1$$

$$\left[\begin{array}{rrr|r} 1 & 9 & 8 & 0 \\ 0 & 1 & -6 & -8 \\ 0 & -13 & -9 & 17 \end{array}\right] \leftarrow R_2 + R_1$$

$$\left[\begin{array}{rrr|r} 1 & 9 & 8 & 0 \\ 0 & 1 & -6 & -8 \\ 0 & 0 & -87 & -87 \end{array}\right] \leftarrow R_3 + 13R_2$$

$$\left[\begin{array}{rrr|r} 1 & 9 & 8 & 0 \\ 0 & 1 & -6 & -8 \\ 0 & 0 & 1 & 1 \end{array}\right] \leftarrow R_3 \div (-87)$$

$$\left[\begin{array}{rrr|r} 1 & 9 & 8 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \leftarrow R_2 + 6R_3$$

$$\left[\begin{array}{rrr|r} 1 & 9 & 0 & -8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \leftarrow R_1 - 8R_3$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \leftarrow R_1 - 9R_2$$

The solution is (10, −2, 1).

28. $4x + 8y - z = 10$
$3x - 8y + 9z = 14$
$7x + 6y + 5z = 0$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 4 & 8 & -1 & 10 \\ 3 & -8 & 9 & 14 \\ 7 & 6 & 5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -12 & -24 & 3 & -30 \\ 12 & -32 & 36 & 56 \\ 7 & 6 & 5 & 0 \end{array}\right] \begin{array}{l} \leftarrow -3R_1 \\ \leftarrow 4R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 8 & -1 & 10 \\ 0 & -56 & 39 & 26 \\ 7 & 6 & 5 & 0 \end{array}\right] \begin{array}{l} \leftarrow R_1+(-3) \\ \leftarrow R_2+R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} -28 & -56 & 7 & -70 \\ 0 & -56 & 39 & 26 \\ 28 & 24 & 20 & 0 \end{array}\right] \begin{array}{l} \leftarrow -7R_1 \\ \\ \leftarrow 4R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 8 & -1 & 10 \\ 0 & -56 & 39 & 26 \\ 0 & -32 & 27 & -70 \end{array}\right] \begin{array}{l} \leftarrow R_1+(-7) \\ \\ \leftarrow R_3+R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 8 & -1 & 10 \\ 0 & 224 & -156 & -104 \\ 0 & -224 & 189 & -490 \end{array}\right] \begin{array}{l} \leftarrow -4R_2 \\ \leftarrow 7R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 8 & -1 & 10 \\ 0 & -56 & 39 & 26 \\ 0 & 0 & 33 & -594 \end{array}\right] \begin{array}{l} \leftarrow R_2+(-4) \\ \leftarrow R_3+R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 8 & -1 & 10 \\ 0 & -56 & 39 & 26 \\ 0 & 0 & 1 & -18 \end{array}\right] \leftarrow R_3 \div 33$$

$$\left[\begin{array}{ccc|c} 4 & 8 & -1 & 10 \\ 0 & -56 & 0 & 728 \\ 0 & 0 & 1 & -18 \end{array}\right] \leftarrow R_2-39R_3$$

$$\left[\begin{array}{ccc|c} 4 & 8 & -1 & 10 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -18 \end{array}\right] \leftarrow R_2 \div (-56)$$

$$\left[\begin{array}{ccc|c} 4 & 8 & 0 & -8 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -18 \end{array}\right] \leftarrow R_1+R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -18 \end{array}\right] \leftarrow R_1 \div 4$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 24 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -18 \end{array}\right]$$

The solution is (24, −13, −18).

29. $2x - 10y + z = 28$
$-5x + 11y + 7z = 18$
$6x - y - 12z = 14$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

# 6-1 Multivariable Linear Systems and Row Operations

$$\left[\begin{array}{rrr|r} 2 & -10 & 1 & 28 \\ -5 & 11 & 7 & 18 \\ 6 & -1 & -12 & 14 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 2 & -10 & 1 & 28 \\ 1 & 10 & -5 & 32 \\ 6 & -1 & -12 & 14 \end{array}\right] \leftarrow R_2 + R_3$$

$$\left[\begin{array}{rrr|r} 2 & -10 & 1 & 28 \\ 0 & -30 & 11 & -36 \\ 6 & -1 & -12 & 14 \end{array}\right] \leftarrow R_1 - 2R_2$$

$$\left[\begin{array}{rrr|r} 2 & -10 & 1 & 28 \\ 0 & -30 & 11 & -36 \\ 0 & 29 & -15 & -70 \end{array}\right] \leftarrow R_3 - 3R_1$$

$$\left[\begin{array}{rrr|r} 2 & -10 & 1 & 28 \\ 0 & -30 & 11 & -36 \\ 0 & -1 & -4 & -106 \end{array}\right] \leftarrow R_2 + R_3$$

$$\left[\begin{array}{rrr|r} 2 & -10 & 1 & 28 \\ 0 & -30 & 11 & -36 \\ 0 & 30 & 120 & 3180 \end{array}\right] \leftarrow -30R_3$$

$$\left[\begin{array}{rrr|r} 2 & -10 & 1 & 28 \\ 0 & -30 & 11 & -36 \\ 0 & 0 & 131 & 3144 \end{array}\right] \leftarrow R_3 + R_2$$

$$\left[\begin{array}{rrr|r} 2 & -10 & 1 & 28 \\ 0 & -30 & 11 & -36 \\ 0 & 0 & 1 & 24 \end{array}\right] \leftarrow R_3 \div 131$$

$$\left[\begin{array}{rrr|r} 2 & -10 & 1 & 28 \\ 0 & -30 & 0 & -300 \\ 0 & 0 & 1 & 24 \end{array}\right] \leftarrow R_2 - 11R_3$$

$$\left[\begin{array}{rrr|r} 2 & -10 & 1 & 28 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 24 \end{array}\right] \leftarrow R_2 \div (-30)$$

$$\left[\begin{array}{rrr|r} 2 & -10 & 0 & 4 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 24 \end{array}\right] \leftarrow R_1 - R_3$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 0 & 2 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 24 \end{array}\right] \leftarrow R_1 \div 2$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 52 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 24 \end{array}\right] \leftarrow R_1 + 5R_2$$

The solution is (52, 10, 24).

30. **COFFEE** A local coffee shop specializes in espresso drinks. The table shows the cups of each drink sold throughout the day. Write and solve a system of equations to determine the price of each espresso drink. Interpret your solution.

| Hours | Cappuccino | Latte | Macchiato | Earnings ($) |
|---|---|---|---|---|
| 8–11 | 103 | 86 | 79 | 1040.25 |
| 11–2 | 48 | 32 | 26 | 406.50 |
| 2–5 | 45 | 25 | 18 | 334.00 |

*SOLUTION:*

Write the equations with $c$ = cappuccino, $l$ = latte, and $m$ = macchiato.

$$103c + 86l + 79m = 1040.45$$

$$48c + 32l + 26m = 406.50$$

$$45c + 25l + 18m = 334.00$$

Write and solve the system of equations in a matrix.

$$\left[\begin{array}{ccc|c} 103 & 86 & 79 & 1040.25 \\ 48 & 32 & 26 & 406.5 \\ 45 & 25 & 18 & 334 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 103 & 86 & 79 & 1040.25 \\ 720 & 480 & 390 & 6097.5 \\ 720 & 400 & 288 & 5344 \end{array}\right] \begin{array}{l} \\ \leftarrow 15R_2 \\ \leftarrow 16R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 103 & 86 & 79 & 1040.25 \\ 0 & 80 & 102 & 753.5 \\ 45 & 25 & 18 & 334 \end{array}\right] \begin{array}{l} \\ \leftarrow R_2 - R_3 \\ \leftarrow R_3 + 16 \end{array}$$

$$\left[\begin{array}{ccc|c} 4635 & 3870 & 3555 & 46,811.25 \\ 0 & 80 & 102 & 753.5 \\ 4635 & 2575 & 1854 & 34,402 \end{array}\right] \begin{array}{l} \leftarrow 45R_1 \\ \\ \leftarrow 103R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 103 & 86 & 79 & 1040.25 \\ 0 & 80 & 102 & 753.5 \\ 0 & 1295 & 1701 & 12,409.25 \end{array}\right] \begin{array}{l} \leftarrow R_2 + 45 \\ \\ \leftarrow R_1 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 103 & 86 & 79 & 1040.25 \\ 0 & 20,720 & 26,418 & 195,156.5 \\ 0 & 20,720 & 27,216 & 198,548 \end{array}\right] \begin{array}{l} \\ \leftarrow 259R_2 \\ \leftarrow 16R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 103 & 86 & 79 & 1040.25 \\ 0 & 80 & 102 & 753.5 \\ 0 & 0 & 798 & 3391.5 \end{array}\right] \begin{array}{l} \\ \leftarrow R_2 + 259 \\ \leftarrow R_2 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 103 & 86 & 79 & 1040.25 \\ 0 & 80 & 102 & 753.5 \\ 0 & 0 & 1 & 4.25 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow R_3 + 798 \end{array}$$

$$\left[\begin{array}{ccc|c} 103 & 86 & 79 & 1040.25 \\ 0 & 80 & 0 & 320 \\ 0 & 0 & 1 & 4.25 \end{array}\right] \begin{array}{l} \\ \leftarrow R_2 - 102R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 103 & 86 & 79 & 1040.25 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4.25 \end{array}\right] \begin{array}{l} \\ \leftarrow R_2 + 80 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 103 & 0 & 79 & 696.25 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4.25 \end{array}\right] \begin{array}{l} \leftarrow R_1 - 86R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 103 & 0 & 0 & 360.5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4.25 \end{array}\right] \begin{array}{l} \leftarrow R_1 - 79R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3.5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4.25 \end{array}\right] \begin{array}{l} \leftarrow R_1 + 103 \\ \\ \\ \end{array}$$

cappuccino: $3.50, latte: $4.00, macchiato: $4.25

31. **FLORIST** An advertisement for a floral shop shows the price of several flower arrangements and a list of the flowers included in each arrangement, as shown below. Write and solve a system of equations to determine the price of each type of flower. Interpret your solution.




*SOLUTION:*

Write the system of equations with $r$ = roses, $l$ = lilies, and $s$ = irises.

$$4r + 12l + 5s = 35$$

$$6r + 9l + 12s = 50.25$$

$$10r + 15l + 20s = 83.75$$

Write and solve the system of equations in a matrix.

$$\left[\begin{array}{ccc|c} 4 & 12 & 5 & 35 \\ 6 & 9 & 12 & 50.25 \\ 10 & 15 & 20 & 83.75 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 12 & 5 & 35 \\ 2 & 3 & 4 & 16.75 \\ 2 & 3 & 4 & 16.75 \end{array}\right] \begin{array}{l} \leftarrow R_2 + 3 \\ \leftarrow R_3 + 5 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 12 & 5 & 35 \\ 2 & 3 & 4 & 16.75 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow R_2 + 3 \\ \leftarrow R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 12 & 5 & 35 \\ 0 & 6 & -3 & 1.5 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 11 & 32 \\ 0 & 6 & -3 & 1.5 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_1 - 2R_2$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 11 & 32 \\ 0 & 2 & -1 & 0.5 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_2 + 3$$

Sample answer: There are an infinite number of solutions to the system. One of the rows in the matrix is eliminated, so the solution set is $(-2.75s + 8, 0.5s + 0.25, s)$. More information is needed to determine which of these solutions is valid.

# 6-1 Multivariable Linear Systems and Row Operations

**Solve each system of equations.**

32. $2x + y - 3z = 0$
$3x - 4y + 10z = -7$
$5x + 2y + 8z = 23$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} -2 & 1 & -3 & 0 \\ 3 & -4 & 10 & -7 \\ 5 & 2 & 8 & 23 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 1 & -3 & 0 \\ 0 & -5 & 11 & -14 \\ 5 & 2 & 8 & 23 \end{array}\right] \xleftarrow{3R_1+2R_2}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & -3 & 0 \\ 0 & -5 & 11 & -14 \\ 0 & 9 & 1 & 46 \end{array}\right] \xleftarrow{5R_1+2R_3}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & -3 & 0 \\ 0 & -5 & 11 & -14 \\ 0 & 0 & 104 & 104 \end{array}\right] \xleftarrow{5R_3+9R_2}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & -3 & 0 \\ 0 & -5 & 11 & -14 \\ 0 & 0 & 1 & 1 \end{array}\right] \xleftarrow{R_3+104}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & -3 & 0 \\ 0 & -5 & 0 & -25 \\ 0 & 0 & 1 & 1 \end{array}\right] \xleftarrow{R_2-11R_3}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & -3 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \xleftarrow{R_2+(-5)}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \xleftarrow{R_1+3R_3}$$

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \xleftarrow{R_1-R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \xleftarrow{R_1+(-2)}$$

33. $4x - 5y - 9z = -25$
$-6x + y + 7z = -21$
$7x - 3y - 10z = 8$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 4 & -5 & -9 & -25 \\ -6 & 1 & 7 & -21 \\ 7 & -3 & -10 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 12 & -15 & -27 & -75 \\ -12 & 2 & 14 & -42 \\ 7 & -3 & -10 & 8 \end{array}\right] \begin{array}{l} \\ \leftarrow 3R_1 \\ \leftarrow 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 12 & -15 & -27 & -75 \\ 0 & -13 & -13 & -117 \\ 7 & -3 & -10 & 8 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow R_1 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & -5 & -9 & -25 \\ 0 & 1 & 1 & 9 \\ 7 & -3 & -10 & 8 \end{array}\right] \begin{array}{l} \leftarrow R_1 \div 3 \\ \leftarrow R_2 \div (-13) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} -28 & 35 & 63 & 175 \\ 0 & 1 & 1 & 9 \\ 28 & -12 & -40 & 32 \end{array}\right] \begin{array}{l} \leftarrow -7R_1 \\ \\ \leftarrow 4R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & -5 & -9 & -25 \\ 0 & 1 & 1 & 9 \\ 0 & 23 & 23 & 207 \end{array}\right] \begin{array}{l} \leftarrow R_1 \div (-7) \\ \\ \leftarrow R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & -5 & -9 & -25 \\ 0 & 1 & 1 & 9 \\ 0 & 1 & 1 & 9 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow R_3 \div 23 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & -5 & -9 & -25 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow R_3 - R_2 \end{array}$$

# 6-1 Multivariable Linear Systems and Row Operations

34. $-x + 3y + 10z = 8$
$4x - 9y - 34z = -17$
$3x + 5y - 2z = 46$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{rrr|r} -1 & 3 & 10 & 8 \\ 4 & -9 & -34 & -17 \\ 3 & 5 & -2 & 46 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} -1 & 3 & 10 & 8 \\ 0 & 3 & 6 & 15 \\ 3 & 5 & -2 & 46 \end{array}\right] \leftarrow 4R_1 + R_2$$

$$\left[\begin{array}{rrr|r} -1 & 3 & 10 & 8 \\ 0 & 1 & 2 & 5 \\ 3 & 5 & -2 & 46 \end{array}\right] \leftarrow R_2 \div 3$$

$$\left[\begin{array}{rrr|r} -1 & 3 & 10 & 8 \\ 0 & 1 & 2 & 5 \\ 0 & 14 & 28 & 70 \end{array}\right] \leftarrow 3R_1 + R_3$$

$$\left[\begin{array}{rrr|r} -1 & 3 & 10 & 8 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{array}\right] \leftarrow R_3 \div 14$$

$$\left[\begin{array}{rrr|r} -1 & 3 & 10 & 8 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_3 - R_2$$

$$\left[\begin{array}{rrr|r} -1 & 0 & 4 & -7 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_1 - 3R_2$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -4 & 7 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow -R_1$$

We are reduced to two equations with three variables, so there will be infinitely many solutions. Write the equations in terms of $z$.

$$x - 4z = 7$$
$$x = 4z + 7$$

$$y + 2z = 5$$
$$y = 5 - 2z$$

The solutions are $(4z + 7, 5 - 2z, z)$.

## 6-1 Multivariable Linear Systems and Row Operations

35. $5x - 4y - 7z = -31$
$2x + y - 8z = 11$
$-4x + 3y + 6z = 23$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 5 & -4 & -7 & -31 \\ 2 & 1 & -8 & 11 \\ -4 & 3 & 6 & 23 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 5 & -4 & -7 & -31 \\ 0 & 5 & -10 & 45 \\ -4 & 3 & 6 & 23 \end{array}\right] \leftarrow 2R_2 + R_1$$

$$\left[\begin{array}{ccc|c} 5 & -4 & -7 & -31 \\ 0 & 1 & -2 & 9 \\ -4 & 3 & 6 & 23 \end{array}\right] \leftarrow R_2 \div 5$$

$$\left[\begin{array}{ccc|c} 20 & -16 & -28 & -124 \\ 0 & 1 & -2 & 9 \\ -20 & 15 & 30 & 115 \end{array}\right] \begin{array}{l} \leftarrow 4R_1 \\ \\ \leftarrow 5R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 5 & -4 & -7 & -31 \\ 0 & 1 & -2 & 9 \\ 0 & -1 & 2 & -9 \end{array}\right] \begin{array}{l} \leftarrow R_1 \div 4 \\ \\ \leftarrow R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 5 & -4 & -7 & -31 \\ 0 & 1 & -2 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 5 & 0 & -15 & 5 \\ 0 & 1 & -2 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_1 + 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & -2 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_1 \div 5$$

We are reduced to two equations with three variables, so there will be infinitely many solutions. Write the equations in terms of $z$.

$$x - 3z = 1$$
$$x = 3z + 1$$

$$y - 2z = 9$$
$$y = 2z + 9$$

The solutions are $(3z + 1, 2z + 9, z)$.

36. $-3x + 4y - z = -10$
$6x - y - 5z = -29$
$4x - 5y + z = 11$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} -3 & 4 & -1 & -10 \\ 6 & -1 & -5 & -29 \\ 4 & -5 & 1 & 11 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -3 & 4 & -1 & -10 \\ 0 & 7 & -7 & -49 \\ 4 & -5 & 1 & 11 \end{array}\right] \leftarrow R_2 + 2R_1$$

$$\left[\begin{array}{ccc|c} -3 & 4 & -1 & -10 \\ 0 & 1 & -1 & -7 \\ 4 & -5 & 1 & 11 \end{array}\right] \leftarrow R_2 \div 7$$

$$\left[\begin{array}{ccc|c} -3 & 4 & -1 & -10 \\ 0 & 1 & -1 & -7 \\ 0 & 1 & -1 & -7 \end{array}\right] \leftarrow 4R_3 + 3R_1$$

$$\left[\begin{array}{ccc|c} -3 & 4 & -1 & -10 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} -3 & 0 & 3 & 18 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_1 - 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -6 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow R_1 \div (-3)$$

We are reduced to two equations with three variables, so there will be infinitely many solutions. Write the equations in terms of $z$.

$$x - z = -6$$
$$x = z - 6$$

$$y - z = -7$$
$$y = z - 7$$

The solutions are $(z-6, z-7, z)$.

37. $8x - 9y - 4z = -33$
$-2x + 3y - 2z = 9$
$-7x + 6y + 11z = 27$

*SOLUTION:*

## 6-1 Multivariable Linear Systems and Row Operations

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 8 & -9 & -4 & -33 \\ -2 & 3 & -2 & 9 \\ -7 & 6 & 11 & 27 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 8 & -9 & -4 & -33 \\ 0 & 3 & -12 & 3 \\ -7 & 6 & 11 & 27 \end{array}\right] \xleftarrow{4R_2+R_1}$$

$$\left[\begin{array}{ccc|c} 8 & -9 & -4 & -33 \\ 0 & 1 & -4 & 1 \\ -7 & 6 & 11 & 27 \end{array}\right] \xleftarrow{R_2\div 3}$$

$$\left[\begin{array}{ccc|c} 8 & -9 & -4 & -33 \\ 0 & 1 & -4 & 1 \\ 1 & -3 & 7 & -6 \end{array}\right] \xleftarrow{R_3+R_1}$$

$$\left[\begin{array}{ccc|c} 8 & -9 & -4 & -33 \\ 0 & 1 & -4 & 1 \\ 0 & 15 & -60 & 15 \end{array}\right] \xleftarrow{-8R_2+R_1}$$

$$\left[\begin{array}{ccc|c} 8 & -9 & -4 & -33 \\ 0 & 1 & -4 & 1 \\ 0 & 1 & -4 & 1 \end{array}\right] \xleftarrow{R_3\div 15}$$

$$\left[\begin{array}{ccc|c} 8 & -9 & -4 & -33 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_1-R_2}$$

$$\left[\begin{array}{ccc|c} 8 & 0 & -40 & -24 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_1+9R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_1\div 8}$$

We are reduced to two equations with three variables, so there will be infinitely many solutions. Write the equations in terms of $z$.

$$x-5z=-3$$
$$x=5z-3$$

$$y-4z=1$$
$$y=4z+1$$

The solutions are $(5z-3, 4z+1, z)$.

38. $2x - 5y + 4z + 4w = 2$
$-3x + 6y - 2z - 7w = 11$
$5x - 4y + 8z - 5w = 29$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cccc|c} 2 & -5 & 4 & 4 & 2 \\ -3 & 6 & -2 & -7 & 11 \\ 5 & -4 & 8 & -5 & 29 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 2 & -5 & 4 & 4 & 2 \\ 0 & -3 & 8 & -2 & 28 \\ 5 & -4 & 8 & -5 & 29 \end{array}\right] \leftarrow 2R_2+3R_1$$

$$\left[\begin{array}{cccc|c} -10 & 25 & -20 & -20 & -10 \\ 0 & -3 & 8 & -2 & 28 \\ 10 & -8 & 16 & -10 & 58 \end{array}\right] \begin{array}{l} \leftarrow -5R_1 \\ \\ \leftarrow 2R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & -5 & 4 & 4 & 2 \\ 0 & -3 & 8 & -2 & 28 \\ 0 & 17 & -4 & -30 & 48 \end{array}\right] \begin{array}{l} \leftarrow R_1+(-5) \\ \\ \leftarrow R_3+R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & -5 & 4 & 4 & 2 \\ 0 & -3 & 8 & -2 & 28 \\ 0 & 31 & 0 & -62 & 124 \end{array}\right] \leftarrow R_3+2R_3$$

$$\left[\begin{array}{cccc|c} 2 & -5 & 4 & 4 & 2 \\ 0 & -3 & 8 & -2 & 28 \\ 0 & 1 & 0 & -2 & 4 \end{array}\right] \leftarrow R_3+31$$

$$\left[\begin{array}{cccc|c} 2 & -5 & 4 & 4 & 2 \\ 0 & 0 & 8 & -8 & 40 \\ 0 & 1 & 0 & -2 & 4 \end{array}\right] \leftarrow R_2+3R_3$$

$$\left[\begin{array}{cccc|c} 2 & -5 & 4 & 4 & 2 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 1 & 0 & -2 & 4 \end{array}\right] \leftarrow R_2+8$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 4 & -6 & 22 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 1 & 0 & -2 & 4 \end{array}\right] \leftarrow R_1+5R_3$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 1 & 0 & -2 & 4 \end{array}\right] \leftarrow R_1-4R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 1 & 0 & -2 & 4 \end{array}\right] \leftarrow R_1+2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & -1 & 5 \end{array}\right] \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \end{array}$$

We are reduced to two equations with three variables, so there will be infinitely many solutions. Write the equations

in terms of $w$.

$$x - w = 1$$
$$x = w + 1$$

$$y - 2w = 4$$
$$y = 2w + 4$$

$$z - w = 5$$
$$z = w + 5$$

The solutions are $(w+1, 2w+4, w+5, w)$

39. $x - 4y + 4z + 3w = 2$
$-2x - 3y + 7z - 3w = -9$
$3x - 5y + z + 10w = 15$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cccc|c} 1 & -4 & 4 & 3 & 2 \\ -2 & -3 & 7 & -3 & -9 \\ 3 & -5 & 1 & 10 & 15 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -4 & 4 & 3 & 2 \\ 0 & -11 & 15 & 3 & -5 \\ 3 & -5 & 1 & 10 & 15 \end{array}\right] \leftarrow R_2+2R_1$$

$$\left[\begin{array}{cccc|c} 1 & -4 & 4 & 3 & 2 \\ 0 & -11 & 15 & 3 & -5 \\ 0 & 7 & -11 & 1 & 9 \end{array}\right] \leftarrow R_3-3R_1$$

$$\left[\begin{array}{cccc|c} 1 & -4 & 4 & 3 & 2 \\ 0 & -77 & 105 & 21 & -35 \\ 0 & 77 & -121 & 11 & 99 \end{array}\right] \begin{array}{l} \leftarrow 7R_2 \\ \leftarrow 11R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -4 & 4 & 3 & 2 \\ 0 & -11 & 15 & 3 & -5 \\ 0 & 0 & -16 & 32 & 64 \end{array}\right] \begin{array}{l} \leftarrow R_2+7 \\ \leftarrow R_3+R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -4 & 4 & 3 & 2 \\ 0 & -11 & 15 & 3 & -5 \\ 0 & 0 & 1 & -2 & -4 \end{array}\right] \leftarrow R_3+(-16)$$

$$\left[\begin{array}{cccc|c} 11 & -44 & 44 & 33 & 22 \\ 0 & 44 & -60 & -12 & 20 \\ 0 & 0 & 1 & -2 & -4 \end{array}\right] \begin{array}{l} \leftarrow 11R_1 \\ \leftarrow -4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 11 & 0 & -16 & 21 & 42 \\ 0 & 11 & -15 & -3 & 5 \\ 0 & 0 & 1 & -2 & -4 \end{array}\right] \begin{array}{l} \leftarrow R_1+R_2 \\ \leftarrow R_2+4 \end{array}$$

$$\left[\begin{array}{cccc|c} 11 & 0 & 0 & -11 & -22 \\ 0 & 11 & -15 & -3 & 5 \\ 0 & 0 & 1 & -2 & -4 \end{array}\right] \leftarrow R_1+16R_3$$

$$\left[\begin{array}{cccc|c} 11 & 0 & 0 & -11 & -22 \\ 0 & 11 & 0 & -33 & -55 \\ 0 & 0 & 1 & -2 & -4 \end{array}\right] \leftarrow R_2+15R_3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & -5 \\ 0 & 0 & 1 & -2 & -4 \end{array}\right] \begin{array}{l} \leftarrow R_1+11 \\ \leftarrow R_2+11 \end{array}$$

We are reduced to two equations with three variables, so there will be infinitely many solutions. Write the equations in terms of $w$.

$x - w = -2$

$x = w - 2$

$y - 3w = -5$

$y = 3w - 5$

$z - 2w = -4$

$z = 2w - 4$

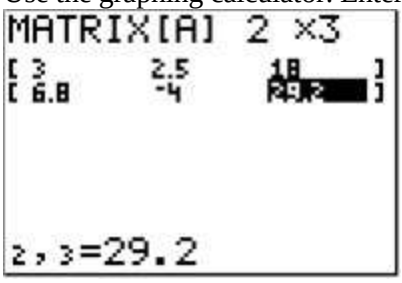
The solutions are $(w - 2, 3w - 5, 2w - 4, w)$.

**GRAPHING CALCULATOR Find the row-echelon form and reduced row-echelon form of each system. Round to the nearest tenth, if necessary.**

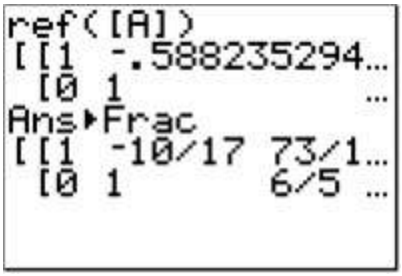
40. $3x + 2.5y = 18$
$6.8x - 4y = 29.2$

*SOLUTION:*

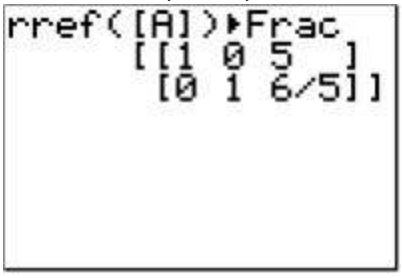
Use the graphing calculator. Enter the elements into the matrix.




Select **matrix**, **math**, **ref** for row-echelon form. Also, when viewing the matrix, you can select **Ans→Frac** to see it in fraction form.




Select **matrix**, **math**, **rref** for reduced row-echelon form.

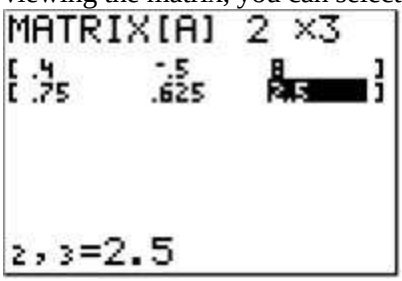


## 6-1 Multivariable Linear Systems and Row Operations

41. $\frac{2}{5}x-\frac{1}{2}y=8$
$\frac{3}{4}x+\frac{5}{8}y=\frac{5}{2}$

*SOLUTION:*

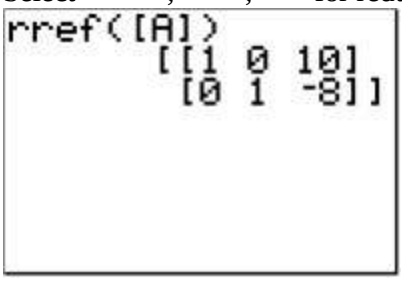
Use the graphing calculator. Enter the elements into the matrix. You can enter 5 ÷ 6 for the fraction. Also, when viewing the matrix, you can select **Ans→Frac** to see it in fraction form.




Select **matrix**, **math**, **ref** for row-echelon form.




Select **matrix**, **math**, **rref** for reduced row-echelon form.



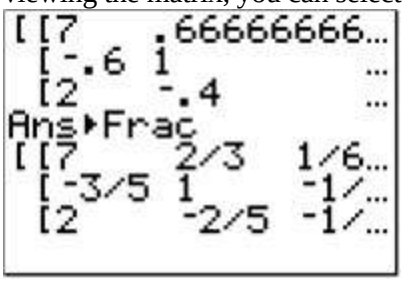
42. $7x+\frac{2}{3}y+\frac{1}{6}z=-\frac{13}{3}$

$-\frac{3}{5}x+y-\frac{1}{3}z=\frac{11}{10}$

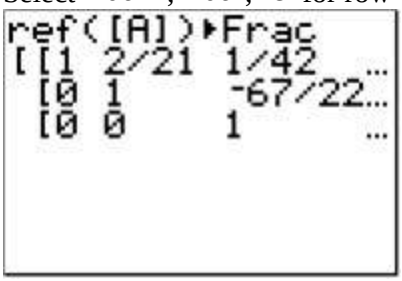
$-2x-\frac{2}{5}y-\frac{1}{2}z=-6$

*SOLUTION:*

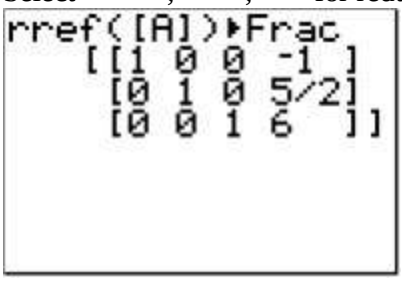
Use the graphing calculator. Enter the elements into the matrix. You can enter 2 ÷ 3 for the fraction. Also, when viewing the matrix, you can select **Ans→Frac** to see it in fraction form.




Select **matrix**, **math**, **ref** for row-echelon form.



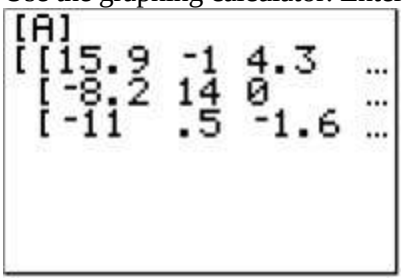

Select **matrix**, **math**, **rref** for reduced row-echelon form.



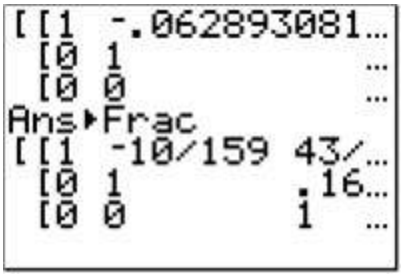
43. $15.9x - y + 4.3z = 14.8$
$-8.2x + 14y = 14.6$
$-11x + 0.5y - 1.6z = -20.4$

*SOLUTION:*

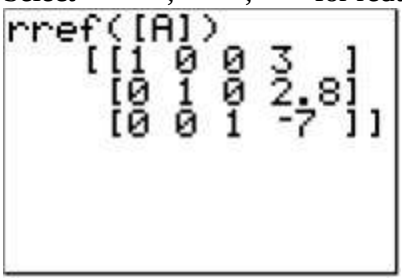
Use the graphing calculator. Enter the elements into the matrix.




Select **matrix**, **math**, **ref** for row-echelon form. Also, when viewing the matrix, you can select **Ans→Frac** to see it in fraction form.




Select **matrix**, **math**, **rref** for reduced row-echelon form.



44. **FINANCIAL LITERACY** A sports equipment company took out three different loans from a bank to buy treadmills. The bank statement after the first year is shown below. The amount borrowed at the 6.5% rate was $50,000 less than the amounts borrowed at the other two rates combined.




**a.** Write a system of three linear equations to represent this situation.
**b.** Use a graphing calculator to solve the system of equations. Interpret the solution.

*SOLUTION:*

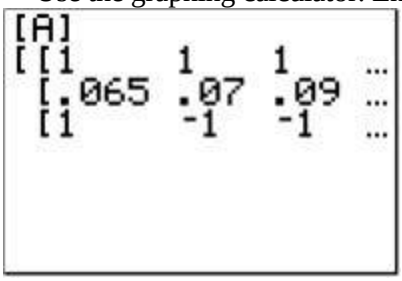
**a.** Write a system of equations. Let $x$ = the amount for the 1st loan, $y$ = the amount for the 2nd loan, and $z$ = the amount for the 3rd loan.

The total amount borrowed was $350,000, so $x + y + z = 350{,}000$. The total interest paid was $24,950, so $0.65x + 0.07y + 0.09z = 24{,}950$. The borrowed for the 1st loan is $50,000 less than the other 2 combined, so $x = y + z - 50{,}000$.

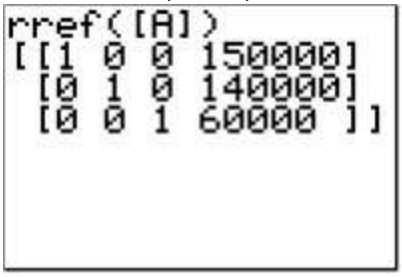
Align the terms.

$$x + y + z = 350{,}000$$
$$0.065x + 0.07y + 0.09z = 24{,}950$$
$$x - y - z = -50{,}000$$

**b.** Use the graphing calculator. Enter the elements into the matrix.




Select **matrix**, **math**, **rref**.




The solution is (150,000, 140,000, 60,000). The company borrowed $150,000 at 6.5%, $140,000 at 7%, and $60,000 at 9%.

**Determine the row operation performed to obtain each matrix.**

45. $\left[\begin{array}{ccc|c} 1 & 5 & -6 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & -1 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 5 & -6 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & -1 & -1 \end{array}\right]$

*SOLUTION:*

The 3rd row is the new row. $-1 + 1 = 0$, so it appears that the 2nd row was added to the third row. Check the other values.

$-3 + 2 = -1$

$-2 + 1 = -1$

$R_2 + R_3$

46. $\left[\begin{array}{ccc|c} 3 & 1 & -5 & 4 \\ 9 & -1 & 4 & -2 \\ 8 & 4 & -3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 1 & -5 & 4 \\ 9 & -1 & 4 & -2 \\ 2 & 2 & 7 & -7 \end{array}\right]$

*SOLUTION:*

The 3rd row is the new row. There are a few ways for the 2 in the 2nd column of the third row to be produced. Dividing the row by 2 does not work with the other values. We can also either subtract twice the 1st row or add twice the 2nd row. Check the other values.

$8 - 2(3) = 2$

$4 - 2(1) = 2$

$-3 - 2(-5) = 7$

$1 - 2(4) = -7$

$R_3 - 2R_1$ or $-2R_1 + R_3$

47. $\left[\begin{array}{cccc|c} 1 & 15 & 2 & 4 & 14 \\ 0 & 8 & 5 & -5 & 15 \\ 2 & 1 & 0 & 16 & 20 \\ -3 & -11 & -1 & 6 & -4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 15 & 2 & 4 & 14 \\ 0 & 8 & 5 & -5 & 15 \\ 2 & 1 & 0 & 16 & 20 \\ 0 & 34 & 5 & 18 & 38 \end{array}\right]$

*SOLUTION:*

The 4th row is the new row. The easiest way to change $-3$ to 0 is to find the sum of the 4th row and the 1st row multiplied by 3. Check the other values.

$3(15) + (-11) = 34$

$3(2) + (-1) = 5$

$3(4) + 6 = 18$

$3(14) + (-4) = 38$

$3R_1 + R_4$

## 6-1 Multivariable Linear Systems and Row Operations

48. $\begin{bmatrix} 8 & -2 & 0 & 2 & 12 & -2 \\ 8 & 5 & -7 & 1 & 6 & 9 \\ -1 & 0 & 9 & 3 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & -2 & 0 & 2 & 12 & -2 \\ 0 & 7 & -7 & -1 & -6 & 11 \\ -1 & 0 & 9 & 3 & 3 & 2 \end{bmatrix}$

*SOLUTION:*

The 2nd row is the new row. $8 - 8 = 0$, so it appears that the $1^{st}$ row was subtracted from the $2^{nd}$ row or vice versa. Check the other values.

$5 - (-2) = 7$

$-7 - 0 = -7$

$1 - 2 = -1$

$6 - 12 = -6$

$9 - (-2) = 11$

$R_2 - R_1$ or $-R_1 + R_2$

49. **MEDICINE** A diluted saline solution is needed for routine procedures in a hospital. The supply room has a large quantity of 20% saline solution and 40% saline solution, but needs 10 liters of 25% saline solution.

**a.** Write a system of equations to represent this situation.

**b.** Solve the system of equations. Interpret the solution.

*SOLUTION:*

a. Set up a system of equations. Let $x$ = the 20% solution and $y$ = the 40% solution.

The total number of liters is 10, so $x + y = 10$. The 10 liter solution needs to be 25% saline, so $0.20x + 0.40y = 0.25(10)$.

The two equations are:

$$x + y = 10$$
$$0.2x + 0.4y = 2.5$$

b. Solve the system.

$\begin{bmatrix} 1 & 1 & 10 \\ 0.2 & 0.4 & 2.5 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 10 \\ 2 & 4 & 25 \end{bmatrix} \leftarrow 10R_2$

$\begin{bmatrix} 1 & 1 & 10 \\ 0 & 2 & 5 \end{bmatrix} \leftarrow R_2 - 2R_1$

$\begin{bmatrix} 1 & 1 & 10 \\ 0 & 1 & 2.5 \end{bmatrix} \leftarrow \frac{1}{2}R_2$

$\begin{bmatrix} 1 & 0 & 7.5 \\ 0 & 1 & 2.5 \end{bmatrix} \leftarrow R_1 - R_2$

The nurses will need to mix 7.5 L of 20% and 2.5 L of 40% saline.

50. **OPEN ENDED** Create a system of 3 variable equations that has infinitely many solutions. Explain your reasoning.

*SOLUTION:*

One method of writing these equations is to write the first equation and then multiply all of the terms in this equation by the same value to make the second equation. This will guarantee that two of the equations will cancel each other out, the reduced row-echelon form of the matrix will have a row of 0s and there will be infinitely many solutions to the system.

Sample answer:

$$x + y + z = 3$$
$$2x + 2y + 2z = 6$$
$$5x + 5y + 5z = 15$$

The system written as an augmented matrix and in reduced row-echelon form yields

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 6 \\ 5 & 5 & 5 & 15 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ respectively.}$$

Any ordered triple of real numbers $(x, y, z)$ such that $x + y + z = 3$ is a solution. This is to be expected since the second and third equations are multiples of the first equation.

51. **CHALLENGE** Consider the following system of equations. What value of $k$ would make the system consistent and independent?

$3x + 2y = 5$
$5y - kz = -22$
$2x + 5z = 26$
$-2x + ky + z = -8$

*SOLUTION:*

For a system to be consistent and independent, there must be only one solution. Set up the augmented matrix, aligning the variables.

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 5 & -k & -22 \\ 2 & 0 & 5 & 26 \\ -2 & k & 1 & -8 \end{array}\right]$$

Apply elementary row operations until you obtain 2 rows with an element in $k$ and 0s in the remaining elements.

# 6-1 Multivariable Linear Systems and Row Operations

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 5 & -k & -22 \\ 2 & 0 & 5 & 26 \\ -2 & k & 1 & -8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 5 & -k & -22 \\ 0 & 4 & -15 & -68 \\ -2 & k & 1 & -8 \end{array}\right] \xleftarrow{2R_1-3R_3}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 5 & -k & -22 \\ 0 & 4 & -15 & -68 \\ 0 & 3k+4 & 3 & -14 \end{array}\right] \xleftarrow{3R_4+2R_1}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 5 & -k & -22 \\ 0 & 4 & -15 & -68 \\ 0 & 15k+24 & 0 & -138 \end{array}\right] \xleftarrow{5R_4+R_2}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 0 & -4k+75 & 252 \\ 0 & 4 & -15 & -68 \\ 0 & 15k+24 & 0 & -138 \end{array}\right] \xleftarrow{4R_2-5R_3}$$

Notice that using the last three rows, we can come up with three equations and 3 variables, $y$, $z$, and $k$. However, these equations are not linear. Therefore, we need to take them out of the matrix.

$$(-4k+75)z = 252$$
$$4y - 15z = -68$$
$$(15k+24)y = -138$$

Get $y$ and $z$ in terms of $k$ in the 1st and 3rd equations.

$$(-4k+75)z = 252$$
$$z = \frac{252}{-4k+75}$$
$$(15k+24)y = -138$$
$$y = \frac{-138}{15k+24}$$

Substitute these values into the 2nd equation.

$$4y - 15z = -68$$
$$4\left(\frac{-138}{15k+24}\right) - 15\left(\frac{252}{-4k+75}\right) = -68$$
$$\frac{-552}{15k+24} - \frac{3780}{-4k+75} = -68$$
$$-552(-4k+75) - 3780(15k+24) = -68(15k+24)(-4k+75)$$
$$2208k - 41,400 - 56,700k - 90,720 = 4080k^2 + 6528k - 76,500k - 122,400$$
$$-54,492k - 132,120 = 4080k^2 - 69,972k - 122,400$$
$$0 = 4080k^2 - 15,480k + 9720$$
$$0 = 102k^2 - 387k + 243$$
$$0 = 34k^2 - 129k + 81$$
$$0 = (k-3)(34k-27)$$
$$k = 3 \text{ or } \frac{27}{34}$$

52. **ERROR ANALYSIS** Ken and Sari are writing the augmented matrix of the system below in row − echelon form.

$2x - y + z = 0$
$x + y - 2z = -7$
$x - 3y + 4z = 9$

Ken

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Sari

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Is either of them correct? Explain your reasoning.

*SOLUTION:*

Two matrices may appear to be different in row-echelon form yet still have the same solutions.

Apply elementary row operations to each matrix to obtain reduced row-echelon form.
Ken:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \xleftarrow{R_1+R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \xleftarrow{R_1+2R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \xleftarrow{R_1-R_2}$$

Sari:

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \xleftarrow{R_2+R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \xleftarrow{R_1+4R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \xleftarrow{R_1-3R_2}$$

Sari's and Ken's matrices produce the same solution. Confirm that this is the correct solution by substituting the values into the original equations.

$$\begin{aligned} 2x-y+z &= 0 \\ 2(-1)-2+4 &= 0 \\ -2+2 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x+y-2z &= -7 \\ -1+2-2(4) &= -7 \\ 1-8 &= -7 \\ -7 &= -7 \end{aligned}$$

$$\begin{aligned} x-3y+4z &= 9 \\ -1-3(2)+4(4) &= 9 \\ -1-6+16 &= 9 \\ 9 &= 9 \end{aligned}$$

53. **REASONING** *True* or *false*: If an augmented square matrix in row-echelon form has a row of zeros as its last row, then the corresponding system of equations has no solution. Explain your reasoning.

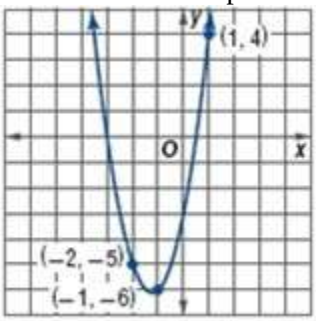
*SOLUTION:*

False; Sample answer: Because the equation in the last row is 0 = 0, the value of one of the variables cannot be determined. Therefore, the corresponding system of equations may have infinitely many solutions.

The system may also have no solution if one of the other rows is 0 = 6.

The system may have one solution if the row of 0s is considered *extra*. For example, if the system includes 4 equations in 3 variables, one of the equations can be eliminated and the remaining equations may be solved with one solution.

54. **CHALLENGE** A parabola passes through the three points shown on the graph below.




**a.** Write a system of equations that can be used to find the equation of the parabola in the form $f(x) = ax^2 + bx + c$.
**b.** Use matrices to solve the system of equations that you wrote in part **a**.
**c.** Use the solution you found in part **b** to write an equation of the parabola. Verify your results using a graphing calculator.

*SOLUTION:*

**a.** Replace $(x, f(x))$ with the coordinates of each point.

$$f(x) = ax^2 + bx + c$$
$$4 = a(1)^2 + b(1) + c$$
$$4 = a + b + c$$
$$f(x) = ax^2 + bx + c$$
$$-6 = a(-1)^2 + b(-1) + c$$
$$-6 = a - b + c$$
$$f(x) = ax^2 + bx + c$$
$$-5 = a(-2)^2 + b(-2) + c$$
$$-5 = 4a - 2b + c$$

**b.** We have a system of three equations with three variables. Solve using Gauss-Jordan elimination.

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & -6 \\ 4 & -2 & 1 & -5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -10 \\ 4 & -2 & 1 & -5 \end{array}\right] \leftarrow R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & -6 & -3 & -21 \end{array}\right] \begin{array}{l} \leftarrow R_2 \cdot (-2) \\ \leftarrow R_3 - 4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & -3 & 9 \end{array}\right] \leftarrow R_3 + 6R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right] \leftarrow R_3 \cdot (-3)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right] \leftarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right] \leftarrow R_1 - R_2$$

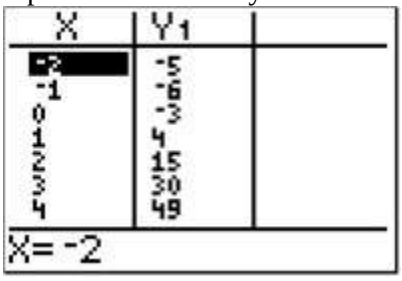
The solution is (2, 5, −3).

**c.** Replace $a$, $b$, and $c$ in $f(x)$.

$$f(x) = ax^2 + bx + c$$

$$f(x) = 2x^2 + 5x - 3$$

Input the function in your calculator and check the values in the table.



55. **WRITING IN MATH** Compare and contrast Gaussian elimination and Gauss-Jordan elimination.

*SOLUTION:*

Sample answer: In Gaussian elimination, matrices are in row-echelon form. In Gauss-Jordan elimination, matrices are in reduced row-echelon form. Matrices in row-echelon form and reduced row-echelon form may have rows consisting entirely of 0s. If they do, these rows appear at the bottom of the matrices.

Row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & -29 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In both forms of matrices, the first entry in any row with nonzero entries is 1. This is called a leading 1. In both forms of matrices, if there are two successive rows with nonzero entries, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

Row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & -29 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

For a system with a unique solution, when the matrix is in reduced row-echelon form, every column that has a leading 1 has 0s above and/or below that 1.

Row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & -29 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

**Verify each identity.**

56. $2\cos^2\frac{x}{2} = 1 + \cos x$

*SOLUTION:*

We need to go from a squared trigonometric term to a trigonometric term that is not squared, so we should use a power-reducing identity.

$$\begin{aligned} 2\cos^2\frac{x}{2} &= 2\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2 \\ &= 2\left(\frac{1+\cos x}{2}\right) \\ &= 1+\cos x \end{aligned}$$

57. $\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x}$

*SOLUTION:*

We need to go from a squared trigonometric term to a trigonometric term that is not squared, so we should use a power-reducing identity.

$$\begin{aligned} \tan^2\frac{x}{2} &= \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} \\ &= \frac{1-\cos\left[2\left(\frac{x}{2}\right)\right]}{1+\cos\left[2\left(\frac{x}{2}\right)\right]} \\ &= \frac{1-\cos x}{1+\cos x} \end{aligned}$$

58. $\frac{1}{\sin x\cos x} - \frac{\cos x}{\sin x} = \tan x$

*SOLUTION:*

Get a common denominator.

$$\frac{1}{\sin x\cos x} - \frac{\cos x}{\sin x} = \frac{1-\cos^2 x}{\sin x\cos x}$$

Use a Pythagorean identity.

$$\begin{aligned} \frac{1-\cos^2 x}{\sin x\cos x} &= \frac{\sin^2 x}{\sin x\cos x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned}$$

**Find the exact value of each trigonometric expression.**

59. cos 105°

*SOLUTION:*

$$\begin{aligned}\cos 105 &= \cos(45+60)\\ &= \cos 45\cos 60 - \sin 45\sin 60\\ &= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

60. sin 165°

*SOLUTION:*

$$\begin{aligned}\sin 165 &= \sin(45+120)\\ &= \sin 45\cos 120 + \cos 45\sin 120\\ &= \frac{\sqrt{2}}{2}\cdot\left(-\frac{1}{2}\right) + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\\ &= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

61. $\cos \frac{7\pi}{12}$

*SOLUTION:*

$$\begin{aligned}\cos\frac{7\pi}{12} &= \cos 105\\ &= \cos(45+60)\\ &= \cos 45\cos 60 - \sin 45\sin 60\\ &= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\\ &= -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

62. $\sin \frac{\pi}{12}$

*SOLUTION:*

$$\begin{aligned}\sin\frac{\pi}{12} &= \sin 15 \\ &= \sin(45-30) \\ &= \sin 45\cos 30 - \cos 45\sin 30 \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

63. $\cot \frac{113\pi}{12}$

*SOLUTION:*

$$\begin{aligned}\cot\frac{113\pi}{12} &= \cot 1695 \\ &= \cot 255 \\ &= \frac{1}{\tan 255} \\ &= \frac{1}{\tan(135+120)} \\ &= \frac{1}{\frac{\tan 135+\tan 120}{1-\tan 135\tan 120}} \\ &= \frac{1-\tan 135\tan 120}{\tan 135+\tan 120} \\ &= \frac{1-(-1)(-\sqrt{3})}{-1-\sqrt{3}} \\ &= \frac{1-\sqrt{3}}{-1-\sqrt{3}} \\ &= \frac{1-\sqrt{3}}{-1-\sqrt{3}}\cdot\frac{-1+\sqrt{3}}{-1+\sqrt{3}} \\ &= \frac{-1+\sqrt{3}+\sqrt{3}-3}{1-3} \\ &= \frac{-4+2\sqrt{3}}{-2} \\ &= 2-\sqrt{3}\end{aligned}$$

64. sec 1275°

*SOLUTION:*

$$\begin{aligned}
\sec 1275 &= \sec 195 \\
&= \frac{1}{\cos 195} \\
&= \frac{1}{\cos(135+60)} \\
&= \frac{1}{\cos 135\cos 60 - \sin 135\sin 60} \\
&= \frac{1}{-\frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}} \\
&= \frac{1}{-\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}} \\
&= \frac{1}{\frac{-\sqrt{2}-\sqrt{6}}{4}} \\
&= \frac{4}{-\sqrt{2}-\sqrt{6}} \\
&= \frac{4}{-\sqrt{2}-\sqrt{6}}\cdot\frac{-\sqrt{2}+\sqrt{6}}{-\sqrt{2}+\sqrt{6}} \\
&= \frac{-4\sqrt{2}+4\sqrt{6}}{2-6} \\
&= \frac{-4\sqrt{2}+4\sqrt{6}}{-4} \\
&= \sqrt{2}-\sqrt{6}
\end{aligned}$$

65. **SOFTBALL** In slow-pitch softball, the diamond is a square that is 65 feet on each side. The distance between the pitcher's mound and home plate is 50 feet. How far does the pitcher have to throw the softball from the pitcher's mound to third base to stop a player who is trying to steal third base?

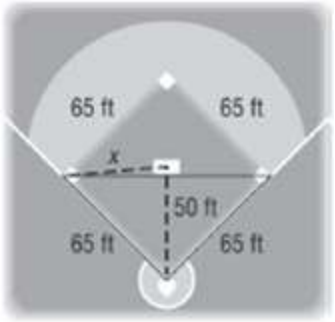



*SOLUTION:*

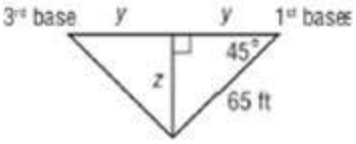
From the diagram, it may appear that we are dealing with a right triangle, but that is not the case. Notice how the angle at the pitcher's mound is not quite a right angle.

# 6-1 Multivariable Linear Systems and Row Operations

Make a triangle from 1st base to 3rd base to home plate as shown below.




The diamond is a square, so we know that the angle at 1st base must be 45 degrees. We can now use trigonometry to identify some missing side lengths.




$$\cos\theta = \frac{adjacent}{hypotenuse}$$

$$\cos 45 = \frac{y}{65}$$

$$\frac{\sqrt{2}}{2} = \frac{y}{65}$$

$$32.5\sqrt{2} = y$$

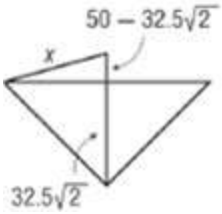
$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\sin 45 = \frac{z}{65}$$

$$\frac{\sqrt{2}}{2} = \frac{z}{65}$$

$$32.5\sqrt{2} = z$$

We can now use the Pythagorean theorem and the small triangle to find $x$.




$$a^2 + b^2 = c^2$$

$$(32.5\sqrt{2})^2 + (50 - 32.5\sqrt{2})^2 = x^2$$

$$\sqrt{(32.5\sqrt{2})^2 + (50 - 32.5\sqrt{2})^2} = x$$

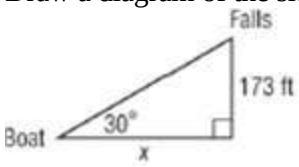
$$46.1 \approx x$$

The distance is about 46.1 feet.

66. **TRAVEL** In a sightseeing boat near the base of the Horseshoe Falls at Niagara Falls, a passenger estimates the angle of elevation to the top of the falls to be 30°. If the Horseshoe Falls are 173 feet high, what is the distance from the boat to the base of the falls?

*SOLUTION:*

Draw a diagram of the situation.




The diagram represents a right triangle. You are given the angle and the opposite side and you need to determine the adjacent side. This can be done using the tangent function.

$$\tan\theta = \frac{opposite}{adjacent}$$

$$\tan 30 = \frac{173}{x}$$

$$x \tan 30 = 173$$

$$x = \frac{173}{\tan 30}$$

$$x \approx 300$$

67. RABBITS Rabbits reproduce at a tremendous rate and their population increases exponentially in the absence of natural enemies. Suppose there were originally 65,000 rabbits in a region, and two years later there were 2,500,000.
a. Write an exponential function that could be used to model the rabbit population $y$ in that region. Write the function in terms of $x$, the number of years since the original year.
b. Assume that the rabbit population continued to grow at that rate. Estimate the rabbit population in that region seven years after the initial year.

*SOLUTION:*

a. The model for exponential growth is $N = N_0(1 + r)^t$. The initial population is 65,000, so $N_0 = 65{,}000$. The population after 2 years is 2,500,000, so $t = 2$ and $N = 2{,}500{,}000$. Solve for $r$.

$$N = N_0(1+r)^t$$
$$2{,}500{,}000 = 65{,}000(1+r)^2$$
$$\frac{2{,}500{,}000}{65{,}000} = (1+r)^2$$
$$\sqrt{\frac{2{,}500{,}000}{65{,}000}} = 1+r$$
$$\sqrt{\frac{2{,}500{,}000}{65{,}000}} - 1 = r$$
$$5.2 \approx r$$

Replace $r$ in the model.

$$N = N_0(1+r)^t$$
$$N = 65{,}000(1+r)^t$$
$$N = 65{,}000(1+5.2)^t$$
$$N = 65{,}000(6.2)^t$$

Rewrite the function using $y$ and $x$.

$y = 65{,}000(6.2)^x$

b. Let $t = 7$ and solve for $y$.

$$y = 65{,}000(6.2)^t$$
$$y = 65{,}000(6.2)^7$$
$$y \approx 22{,}890{,}495{,}000$$

# 6-1 Multivariable Linear Systems and Row Operations

**Solve each equation.**

68. $\frac{3}{x}+\frac{2}{x-1}=\frac{17}{12}$

*SOLUTION:*

$$\frac{3}{x}+\frac{2}{x-1}=\frac{17}{12}$$
$$\frac{3(12)(x-1)}{12(x)(x-1)}+\frac{2(12)(x)}{12(x)(x-1)}=\frac{17(x)(x-1)}{12(x)(x-1)}$$
$$36x-36+24x=17x^2-17x$$
$$0=17x^2-77x+36$$
$$0=(17x-9)(x-4)$$
$$x=4 \text{ or } \frac{9}{17}$$

69. $\frac{4}{x+3}-\frac{2}{x+1}=\frac{2}{15}$

*SOLUTION:*

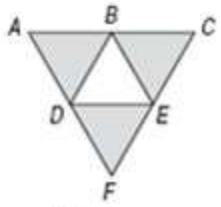
$$\frac{4}{x+3}-\frac{2}{x+1}=\frac{2}{15}$$
$$\frac{4(15)(x+1)}{15(x+1)(x+3)}-\frac{2(15)(x+3)}{15(x+1)(x+3)}=\frac{2(x+1)(x+3)}{15(x+1)(x+3)}$$
$$60x+60-30x-90=2x^2+8x+6$$
$$0=2x^2-22x+36$$
$$0=x^2-11x-18$$
$$0=(x-9)(x-2)$$
$$x=2 \text{ or } 9$$

70. $\frac{4}{3}-\frac{1}{x-2}=\frac{13}{2x}$

*SOLUTION:*

$$\frac{4}{3}-\frac{1}{x-2}=\frac{13}{2x}$$
$$\frac{4(2x)(x-2)}{3(x-2)(2x)}-\frac{1(3)(2x)}{(x-2)(3)(2x)}=\frac{13(3)(x-2)}{2x(3)(x-2)}$$
$$8x^2-16x-6x=39x-78$$
$$8x^2-61x+78=0$$
$$(x-6)(8x-13)=0$$
$$x=6 \text{ or } \frac{13}{8}$$

71. **SAT/ACT** r*ACF* is equilateral with sides of length 4. If *B*, *D*, and *E* are the midpoints of their respective sides, what is the sum of the areas of the shaded regions?

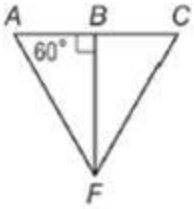



**A** $3\sqrt{2}$
**B** $3\sqrt{3}$
**C** $4\sqrt{2}$
**D** $4\sqrt{3}$
**E** $6\sqrt{3}$

*SOLUTION:*

If $\triangle ACF$ is equilateral with sides 4 units long and *B*, *D*, and *E* are midpoints, then the four small triangles are also equilateral and their sides are 2 units in length. More importantly, these small triangles are congruent. Therefore, the area of the shaded region covers 3 of the 4 equal triangles or $\frac{3}{4}$ of $\triangle ACF$.

The base of $\triangle ACF$ is 4 units. We can also form a 30-60-90 triangle to determine the height of the triangle.




Using the 30-60-90 formula, the height is $2\sqrt{3}$.

The area of the region is:

$$A = \frac{3}{4}\left(\frac{1}{2}\right)(4)(2\sqrt{3})$$
$$= \frac{3}{8}(8\sqrt{3})$$
$$= 3\sqrt{3}$$

The correct choice is B.

72. **REVIEW** The caterer for a lunch bought several pounds of chicken and tuna salad. The chicken salad cost $9 per pound, and the tuna salad cost $6 per pound. He bought a total of 14 pounds of salad and paid a total of $111. How much chicken salad did the caterer buy?

**F** 6 pounds
**G** 7 pounds
**H** 8 pounds
**J** 9 pounds

*SOLUTION:*

Set up a system of equations. Let $c$ = chicken salad and $t$ = tuna salad. He bought a total of 14 ponds, so $c + t = 14$. He spent $111, so $9c + 6t = 111$.
Solve using Gauss-Jordan elimination.

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cc|c} 1 & 1 & 14 \\ 9 & 6 & 111 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 14 \\ 0 & -3 & -15 \end{array}\right] \leftarrow R_2 - 9R_1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 14 \\ 0 & 1 & 5 \end{array}\right] \leftarrow R_2 + (-3)$$

$$\left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 5 \end{array}\right] \leftarrow R_1 - R_2$$

$c$ is the 1st term, so $c = 9$ and the correct choice is J.

73. The Yogurt Shoppe sells small cones in for $0.89, medium cones for $1.19, and large cones for $1.39. One day, Scott sold 52 cones. He sold seven more medium cones than small cones. If he sold $58.98 in cones, how many medium cones did he sell?

**A** 11
**B** 17
**C** 24
**D** 36

*SOLUTION:*

Set up a system of equations. Let $s$ = small, $m$ = medium, and $l$ = large. Fifty-two cones were sold, so $s + m + l = 52$. Seven more medium cones were sold than small cones, so $m = s + 7$. The total cost was $58.98, so $0.89s + 1.19m + 1.39l = 58.98$.

We have a system of three equations with three variables. Rewrite the system, aligning the variables.

$$\begin{aligned} s + m + l &= 52 \\ s - m + 0l &= -7 \\ 0.89s + 1.19m + 1.39l &= 58.98 \end{aligned}$$

Solve using Gauss-Jordan elimination.

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 52 \\ 1 & -1 & 0 & -7 \\ 0.89 & 1.19 & 1.39 & 58.98 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 52 \\ 0 & 2 & 1 & 59 \\ 0.89 & 1.19 & 1.39 & 58.98 \end{array}\right] \xleftarrow{R_1 - R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 52 \\ 0 & 2 & 1 & 59 \\ 0 & -0.3 & -0.5 & -12.7 \end{array}\right] \xleftarrow{0.89R_1 - R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 52 \\ 0 & 2 & 1 & 59 \\ 0 & 0 & -7 & -77 \end{array}\right] \xleftarrow{20R_3 + 3R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 52 \\ 0 & 2 & 1 & 59 \\ 0 & 0 & 1 & 11 \end{array}\right] \xleftarrow{R_3 + (-7)}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 52 \\ 0 & 2 & 0 & 48 \\ 0 & 0 & 1 & 11 \end{array}\right] \xleftarrow{R_2 - R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 52 \\ 0 & 1 & 0 & 24 \\ 0 & 0 & 1 & 11 \end{array}\right] \xleftarrow{R_2 \div 2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 41 \\ 0 & 1 & 0 & 24 \\ 0 & 0 & 1 & 11 \end{array}\right] \xleftarrow{R_1 - R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & 24 \\ 0 & 0 & 1 & 11 \end{array}\right] \xleftarrow{R_1 - R_2}$$

$m$ is the 2nd term, so $m = 24$. The correct choice is C.

74. **REVIEW** To practice at home, Tate purchased a basketball and a volleyball that cost a total of $67, not including tax. If the cost of the basketball $b$ was $4 more than twice the cost of the volleyball $v$, which system of linear equations could be used to determine the cost of each ball?

**F**

$b + v = 67$

$b = 2v - 4$

**G**

$b + v = 67$

$b = 2v + 4$

**H**

$b + v = 4$

$b = 2v - 67$

**J**

$b + v = 4$

$b = 2v + 67$

*SOLUTION:*

Let $b$ equal the price of the basketball and $v$ = the cost of the volleyball. The first statement says that their total cost is $67, so the first equation is $b + v = 67$. The next statement says that the cost of the basketball was $4 more than *twice* the cost of the volleyball. Therefore, the second equation is $2v + 4 = b$. The correct choice is G.

# 6-2 Matrix Multiplication, Inverses and Determinants

**Find $AB$ and $BA$, if possible.**

1. $A = \begin{bmatrix} 8 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 3 & -7 \\ -5 & 2 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 8 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 3 & -7 \\ -5 & 2 \end{bmatrix}$

$A$ is a $1 \times 2$ matrix and $B$ is a $2 \times 2$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$.

$$AB = \begin{bmatrix} 8(3) + 1(-5) & 8(-7) + 1(2) \end{bmatrix}$$
$$= \begin{bmatrix} 24 - 5 & -56 + 2 \end{bmatrix}$$
$$= \begin{bmatrix} 19 & -54 \end{bmatrix}$$

Because the number of columns of $B$ is *not* equal to the number of rows of $A$, $BA$ is undefined.

## 6-2 Matrix Multiplication, Inverses and Determinants

2. $A = \begin{bmatrix} 2 & 9 \\ -7 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 6 & -4 \\ 0 & 3 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 2 & 9 \\ -7 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 6 & -4 \\ 0 & 3 \end{bmatrix}$

$A$ is a $2 \times 2$ matrix and $B$ is a $2 \times 2$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$ and the remaining entries.

$$AB = \begin{bmatrix} 2(6)+9(0) & 2(-4)+9(3) \\ -7(6)+3(0) & -7(-4)+3(3) \end{bmatrix}$$
$$= \begin{bmatrix} 12+0 & -8+27 \\ -42+0 & 28+9 \end{bmatrix}$$
$$= \begin{bmatrix} 12 & 19 \\ -42 & 37 \end{bmatrix}$$

Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

$$BA = \begin{bmatrix} 6(2)+-4(-7) & 6(9)+-4(3) \\ 0(2)+3(-7) & 0(9)+3(3) \end{bmatrix}$$
$$= \begin{bmatrix} 40 & 42 \\ -21 & 9 \end{bmatrix}$$

3. $A = \begin{bmatrix} 3 & -5 \end{bmatrix}$

$B = \begin{bmatrix} 4 & 0 & -2 \\ 1 & -3 & 2 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 3 & -5 \end{bmatrix}$; $B = \begin{bmatrix} 4 & 0 & -2 \\ 1 & -3 & 2 \end{bmatrix}$

$A$ is a $1 \times 2$ matrix and $B$ is a $2 \times 3$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$ and the remaining entry.

$$AB = \begin{bmatrix} 3(4)+(-5)1 & 3(0)+(-5)(-3) & 3(-2)+(-5)2 \end{bmatrix}$$
$$= \begin{bmatrix} 12-5 & 0+15 & -6-10 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 15 & -16 \end{bmatrix}$$

Because the number of columns of $B$ is *not* equal to the number of rows of $A$, $BA$ is undefined.

4. $A = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$B = \begin{bmatrix} 6 & 1 & -10 & 9 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$; $B = \begin{bmatrix} 6 & 1 & -10 & 9 \end{bmatrix}$

$A$ is a $2 \times 1$ matrix and $B$ is a $1 \times 4$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$ and the remaining entries.

$$AB = \begin{bmatrix} 4(6) & 4(1) & 4(-10) & 4(9) \\ 5(6) & 5(1) & 5(-10) & 5(9) \end{bmatrix}$$
$$= \begin{bmatrix} 24 & 4 & -40 & 36 \\ 30 & 5 & -50 & 45 \end{bmatrix}$$

Because the number of columns of $B$ is *not* equal to the number of rows of $A$, $BA$ is undefined.

5. $A = \begin{bmatrix} 2 \\ 5 \\ -6 \end{bmatrix}$

$B = \begin{bmatrix} 6 & 0 & -1 \\ -4 & 9 & 8 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 2 \\ 5 \\ -6 \end{bmatrix}$; $B = \begin{bmatrix} 6 & 0 & -1 \\ -4 & 9 & 8 \end{bmatrix}$

$A$ is a $3 \times 1$ matrix and $B$ is a $2 \times 3$ matrix. Because the number of columns of $A$ is *not* equal to the number of rows of $B$, $AB$ is undefined.

$B$ is a $2 \times 3$ matrix and $A$ is a $3 \times 1$ matrix. Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

To find the first entry of $BA$, find the sum of the products of the entries in row 1 of $B$ and column 1 of $A$. Follow the same procedure for row 2 column 1 of $BA$ and the remaining entries.

$$BA = \begin{bmatrix} 6(2)+0(5)+(-1)(-6) \\ -4(2)+9(5)+8(-6) \end{bmatrix}$$
$$= \begin{bmatrix} 12+0+6 \\ -8+45-48 \end{bmatrix}$$
$$= \begin{bmatrix} 18 \\ -11 \end{bmatrix}$$

# 6-2 Matrix Multiplication, Inverses and Determinants

6. $A = \begin{bmatrix} 2 & 0 \\ -4 & -3 \\ 1 & -2 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 6 & -5 \\ 2 & -7 & 1 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 2 & 0 \\ -4 & -3 \\ 1 & -2 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 6 & -5 \\ 2 & -7 & 1 \end{bmatrix}$

$A$ is a $3 \times 2$ matrix and $B$ is a $2 \times 3$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$ and the remaining entries.

$$AB = \begin{bmatrix} 2(0)+0(2) & 2(6)+0(-7) & 2(-5)+0(1) \\ -4(0)-3(2) & -4(6)-3(-7) & -4(-5)-3(1) \\ 1(0)-2(2) & 1(6)-2(-7) & 1(-5)-2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 12 & -10 \\ -6 & -3 & 17 \\ -4 & 20 & -7 \end{bmatrix}$$

Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

$$BA = \begin{bmatrix} 0(2)+6(-4)-5(1) & 0(0)+6(-3)-5(-2) \\ 2(2)-7(-4)+1(1) & 2(0)-7(-3)+1(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -29 & -8 \\ 33 & 19 \end{bmatrix}$$

7. $A = \begin{bmatrix} 3 & 4 \\ -7 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 5 & 2 & -8 \\ -6 & 0 & 9 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 3 & 4 \\ -7 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 2 & -8 \\ -6 & 0 & 9 \end{bmatrix}$

$A$ is a $2 \times 2$ matrix and $B$ is a $2 \times 3$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$ and the remaining entries.

$$AB = \begin{bmatrix} 3(5)+4(-6) & 3(2)+4(0) & 3(-8)+4(9) \\ -7(5)+1(-6) & -7(2)+1(0) & -7(-8)+1(9) \end{bmatrix}$$

$$= \begin{bmatrix} 15-24 & 6+0 & -24+36 \\ -35-6 & -14+0 & 56+9 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 6 & 12 \\ -41 & -14 & 65 \end{bmatrix}$$

Because the number of columns of $B$ is *not* equal to the number of rows of $A$, $BA$ is undefined.

## 6-2 Matrix Multiplication, Inverses and Determinants

8. $A = \begin{bmatrix} 6 & -9 & 10 \\ 4 & 3 & 8 \end{bmatrix}$

$B = \begin{bmatrix} 6 & -8 \\ 3 & -9 \\ -2 & 5 \\ 4 & 1 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 6 & -9 & 10 \\ 4 & 3 & 8 \end{bmatrix}$; $B = \begin{bmatrix} 6 & -8 \\ 3 & -9 \\ -2 & 5 \\ 4 & 1 \end{bmatrix}$

$A$ is a $2 \times 3$ matrix and $B$ is a $4 \times 2$ matrix. Because the number of columns of $A$ is *not* equal to the number of rows of $B$, $AB$ is undefined.

$B$ is a $4 \times 2$ matrix and $A$ is a $2 \times 3$ matrix. Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

To find the first entry of $BA$, find the sum of the products of the entries in row 1 of $B$ and column 1 of $A$. Follow the same procedure for row 2 column 1 of $BA$ and the remaining entries.

$$BA = \begin{bmatrix} 6(6)+(-8)(4) & 6(-9)+(-8)(3) & 6(10)+(-8)(8) \\ 3(6)+(-9)(4) & 3(-9)+(-9)(3) & 3(10)+(-9)(8) \\ -2(6)+5(4) & -2(-9)+5(3) & -2(10)+5(8) \\ 4(6)+1(4) & 4(-9)+1(3) & 4(10)+1(8) \end{bmatrix}$$

$$= \begin{bmatrix} 36-32 & -54-24 & 60-64 \\ 18-36 & -27-27 & 30-72 \\ -12+20 & 18+15 & -20+40 \\ 24+4 & -36+3 & 40+8 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -78 & -4 \\ -18 & -54 & -42 \\ 8 & 33 & 20 \\ 28 & -33 & 48 \end{bmatrix}$$

9. **BASKETBALL** Different point values are awarded for different shots in basketball. Use the information to determine the total amount of points scored by each player.

| Player | FT | 2-pointer | 3-pointer |
|---|---|---|---|
| Rey | 44 | 32 | 25 |
| Chris | 37 | 24 | 31 |
| Jerry | 35 | 39 | 29 |

| Shots | Points |
|---|---|
| free throw | 1 |
| 2-pointer | 2 |
| 3-pointer | 3 |

*SOLUTION:*

This can be solved by multiplying two matrices. Matrix $A$ is a $3 \times 3$ describing the total number of shots made by each player for each value of shot. Matrix $B$ is a $3 \times 1$ describing the point values of each shot.

$$A = \begin{bmatrix} 44 & 32 & 25 \\ 37 & 24 & 31 \\ 35 & 39 & 29 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Find $AB$.

$$AB = \begin{bmatrix} 44 & 32 & 25 \\ 37 & 24 & 31 \\ 35 & 39 & 29 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 44(1) + 32(2) + 25(3) \\ 37(1) + 24(2) + 31(3) \\ 35(1) + 39(2) + 29(3) \end{bmatrix}$$

$$= \begin{bmatrix} 44 + 64 + 75 \\ 37 + 48 + 93 \\ 35 + 78 + 87 \end{bmatrix}$$

$$= \begin{bmatrix} 183 \\ 178 \\ 200 \end{bmatrix}$$

Rey, 183 points; Chris, 178 points; Jerry, 200 points

10. **CARS** The number of vehicles that a company manufactures each day from two different factories is shown, as well as the price of the vehicle during each sales quarter of the year. Use this information to determine which factory produced the highest sales in the 4th quarter.

| Factory | Model | | | |
|---|---|---|---|---|
| | Coupe | Sedan | SUV | Mini Van |
| 1 | 500 | 600 | 150 | 250 |
| 2 | 250 | 350 | 250 | 400 |

| Model | Quarter | | | |
|---|---|---|---|---|
| | 1st ($) | 2nd ($) | 3rd ($) | 4th ($) |
| Coupe | 18,700 | 17,100 | 16,200 | 15,600 |
| Sedan | 25,400 | 24,600 | 23,900 | 23,400 |
| SUV | 36,300 | 35,500 | 34,900 | 34,500 |
| Mini Van | 38,600 | 37,900 | 37,400 | 36,900 |

*SOLUTION:*

We need to determine the factory which produced the highest sales in the $4^{th}$ quarter only. Therefore, the information we need from the $2^{nd}$ table is the far right column.

For Factory 1, multiply the $1^{st}$ row in the first table by the last column in the $2^{nd}$ table. For Factory 2, multiply the $2^{nd}$ row in the first table by the last column in the $2^{nd}$ table.

Factory 1:

$$\begin{bmatrix}500 & 600 & 150 & 250\end{bmatrix}\cdot\begin{bmatrix}15,600\\23,400\\34,500\\36,900\end{bmatrix}$$

$$=\left[500(15,600)+600(23,400)+150(34,500)+250(36,900)\right]$$

$$=\left[36,240,000\right]$$

Factory 2:

$$\begin{bmatrix}250 & 350 & 250 & 400\end{bmatrix}\cdot\begin{bmatrix}15,600\\23,400\\34,500\\36,900\end{bmatrix}$$

$$=\left[250(15,600)+350(23,400)+250(34,500)+400(36,900)\right]$$

$$=\left[35,475,000\right]$$

Factory 1 which produced the highest sales in the $4^{th}$ quarter.

**Write each system of equations as a matrix equation, $AX = B$. Then use Gauss-Jordan elimination on the augmented matrix to solve for $X$.**

## 6-2 Matrix Multiplication, Inverses and Determinants

11. $$\begin{aligned} 2x_1 - 5x_2 + 3x_3 &= 9 \\ 4x_1 + x_2 - 6x_3 &= 35 \\ -3x_1 + 9x_2 - 7x_3 &= -6 \end{aligned}$$

*SOLUTION:*

Write the system matrix in form $AX = B$. Make sure you align the variables. For the first matrix, the first column should include $x_1$, the second column $x_2$, and the third column $x_3$. The column matrix and the matrix of constant terms should be listed in order.

$$\begin{bmatrix} 2 & -5 & 3 \\ 4 & 1 & -6 \\ -3 & 9 & -7 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 35 \\ -6 \end{bmatrix}$$

Write the augmented matrix $[A \mid B]$. Attach the matrix of constant terms to the end of the 3 × 3 matrix.

$$\left[\begin{array}{ccc|c} 2 & -5 & 3 & 9 \\ 4 & 1 & -6 & 35 \\ -3 & 9 & -7 & -6 \end{array}\right]$$

Use Gauss-Jordan elimination to solve the system. First, use elementary row operations to transform $A$ into $I$.

$$[A \mid B] = \left[\begin{array}{ccc|c} 2 & -5 & 3 & 9 \\ 4 & 1 & -6 & 35 \\ -3 & 9 & -7 & -6 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & -5 & 3 & 9 \\ 0 & 11 & -12 & 17 \\ -3 & 9 & -7 & -6 \end{array}\right] \leftarrow R_2 - 2R_1$$

$$= \left[\begin{array}{ccc|c} 2 & -5 & 3 & 9 \\ 0 & 11 & -12 & 17 \\ 0 & 3 & -5 & 15 \end{array}\right] \leftarrow 3R_3 + 2R_1$$

$$= \left[\begin{array}{ccc|c} 2 & -5 & 3 & 9 \\ 0 & 11 & -12 & 17 \\ 0 & 0 & -19 & 114 \end{array}\right] \leftarrow -3R_2 + 11R_3$$

$$= \left[\begin{array}{ccc|c} 2 & -5 & 3 & 9 \\ 0 & 11 & -12 & 17 \\ 0 & 0 & 1 & -6 \end{array}\right] \leftarrow R_3 \div (-19)$$

$$= \left[\begin{array}{ccc|c} 2 & -5 & 3 & 9 \\ 0 & 11 & 0 & -55 \\ 0 & 0 & 1 & -6 \end{array}\right] \leftarrow R_2 + 12R_3$$

$$= \left[\begin{array}{ccc|c} 2 & -5 & 3 & 9 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -6 \end{array}\right] \leftarrow R_2 \div 11$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 3 & -16 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -6 \end{array}\right] \leftarrow R_1 + 5R_2$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -6 \end{array}\right] \leftarrow R_1 - 3R_3$$

$$[I \mid X] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -6 \end{array}\right] \leftarrow R_1 \div 2$$

The solution of the equation is given by $X$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -6 \end{bmatrix}$$

Therefore, the solution is (1, −5, −6).

12. $$\begin{aligned} 3x_1 - 10x_2 - x_3 &= 6 \\ -5x_1 + 12x_2 + 2x_3 &= -5 \\ -4x_1 - 8x_2 + 3x_3 &= 16 \end{aligned}$$

*SOLUTION:*

Write the system matrix in form $AX = B$. Make sure you align the variables. For the first matrix, the first column should include $x_1$, the second column $x_2$, and the third column $x_3$. The column matrix and the matrix of constant terms should be listed in order.

$$\begin{bmatrix} 3 & -10 & -1 \\ -5 & 12 & 2 \\ -4 & -8 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 16 \end{bmatrix}$$

Write the augmented matrix $[A \mid B]$. Attach the matrix of constant terms to the end of the $3 \times 3$ matrix.

$$\left[\begin{array}{ccc|c} 3 & -10 & -1 & 6 \\ -5 & 12 & 2 & -5 \\ -4 & -8 & 3 & 16 \end{array}\right]$$

Use Gauss-Jordan elimination to solve the system. First, use elementary row operations to transform $A$ into $I$.

$$[A \mid B] = \left[\begin{array}{ccc|c} 3 & -10 & -1 & 6 \\ -5 & 12 & 2 & -5 \\ -4 & -8 & 3 & 16 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 3 & -10 & -1 & 6 \\ 0 & -14 & 1 & 15 \\ -4 & -8 & 3 & 16 \end{array}\right] \leftarrow 5R_1 + 3R_2$$

$$= \left[\begin{array}{ccc|c} 3 & -10 & -1 & 6 \\ 0 & -14 & 1 & 15 \\ 0 & -64 & 5 & 72 \end{array}\right] \leftarrow 4R_3 + 3R_1$$

$$= \left[\begin{array}{ccc|c} 3 & -10 & -1 & 6 \\ 0 & -14 & 1 & 15 \\ 0 & 0 & -3 & -24 \end{array}\right] \leftarrow 32R_3 - 7R_2$$

$$= \left[\begin{array}{ccc|c} 3 & -10 & -1 & 6 \\ 0 & -14 & 1 & 15 \\ 0 & 0 & 1 & 8 \end{array}\right] \leftarrow R_3 \div (-3)$$

$$= \left[\begin{array}{ccc|c} 3 & -10 & -1 & 6 \\ 0 & -14 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}\right] \leftarrow R_2 - R_3$$

$$= \left[\begin{array}{ccc|c} 3 & -10 & -1 & 6 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 8 \end{array}\right] \leftarrow R_2 \div -14$$

$$= \left[\begin{array}{ccc|c} 3 & 0 & -1 & 1 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 8 \end{array}\right] \leftarrow R_1 + 10R_2$$

$$= \left[\begin{array}{ccc|c} 3 & 0 & 0 & 9 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 8 \end{array}\right] \leftarrow R_1 + R_3$$

$$[I \mid X] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 8 \end{array}\right] \leftarrow R_1 \div 3$$

The solution of the equation is given by $X$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -0.5 \\ 8 \end{bmatrix}$$

Therefore, the solution is (3, −0.5, 8).

13.
$$\begin{aligned} 2x_1 - 10x_2 + 7x_3 &= 7 \\ 6x_1 - x_2 + 5x_3 &= -2 \\ -4x_1 + 8x_2 - 3x_3 &= -22 \end{aligned}$$

*SOLUTION:*

Write the system matrix in form $AX = B$. Make sure you align the variables. For the first matrix, the first column should include $x_1$, the second column $x_2$, and the third column $x_3$. The column matrix and the matrix of constant terms should be listed in order.

$$\begin{bmatrix} 2 & -10 & 7 \\ 6 & -1 & 5 \\ -4 & 8 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ -22 \end{bmatrix}$$

Write the augmented matrix $[A \mid B]$. Attach the matrix of constant terms to the end of the $3 \times 3$ matrix.

$$\left[\begin{array}{ccc|c} 2 & -10 & 7 & 7 \\ 6 & -1 & 5 & -2 \\ -4 & 8 & -3 & -22 \end{array}\right]$$

Use Gauss-Jordan elimination to solve the system. First, use elementary row operations to transform $A$ into $I$.

$$[A \mid B] = \left[\begin{array}{ccc|c} 2 & -10 & 7 & 7 \\ 6 & -1 & 5 & -2 \\ -4 & 8 & -3 & -22 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & -10 & 7 & 7 \\ 0 & 29 & -16 & -23 \\ -4 & 8 & -3 & -22 \end{array}\right] \leftarrow R_2 - 3R_1$$

$$= \left[\begin{array}{ccc|c} 2 & -10 & 7 & 7 \\ 0 & 29 & -16 & -23 \\ 0 & -12 & 11 & -8 \end{array}\right] \leftarrow 2R_1 + R_3$$

$$= \left[\begin{array}{ccc|c} 2 & -10 & 7 & 7 \\ 0 & 29 & -16 & -23 \\ 0 & 0 & 127 & -508 \end{array}\right] \leftarrow 12R_3 + 29R_2$$

$$= \left[\begin{array}{ccc|c} 2 & -10 & 7 & 7 \\ 0 & 29 & -16 & -23 \\ 0 & 0 & 1 & -4 \end{array}\right] \leftarrow R_3 \div (127)$$

$$= \left[\begin{array}{ccc|c} 2 & -10 & 7 & 7 \\ 0 & 29 & 0 & -87 \\ 0 & 0 & 1 & -4 \end{array}\right] \leftarrow R_2 + 16R_3$$

$$= \left[\begin{array}{ccc|c} 2 & -10 & 7 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -4 \end{array}\right] \leftarrow R_2 \div 29$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 7 & -23 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -4 \end{array}\right] \leftarrow R_1 + 10R_2$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -4 \end{array}\right] \leftarrow R_1 - 7R_3$$

$$[I \mid X] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2.5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -4 \end{array}\right] \leftarrow R_1 \div 2$$

The solution of the equation is given by $X$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -3 \\ -4 \end{bmatrix}$$

Therefore, the solution is (2.5, −3, −4).

14. $$\begin{aligned} x_1 + 5x_2 + 5x_3 &= -18 \\ -7x_1 - 3x_2 + 2x_3 &= -3 \\ 6x_1 + 7x_2 - x_3 &= 42 \end{aligned}$$

*SOLUTION:*

Write the system matrix in form $AX = B$. Make sure you align the variables. For the first matrix, the first column should include $x_1$, the second column $x_2$, and the third column $x_3$. The column matrix and the matrix of constant terms should be listed in order.

$$\begin{bmatrix} 1 & 5 & 5 \\ -7 & -3 & 2 \\ 6 & 7 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -18 \\ -3 \\ 42 \end{bmatrix}$$

Write the augmented matrix $[A \mid B]$. Attach the matrix of constant terms to the end of the $3 \times 3$ matrix.

$$\left[\begin{array}{ccc|c} 1 & 5 & 5 & -18 \\ -7 & -3 & 2 & -3 \\ 6 & 7 & -1 & 42 \end{array}\right]$$

Use Gauss-Jordan elimination to solve the system. First, use elementary row operations to transform $A$ into $I$.

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 5 & 5 & -18 \\ -7 & -3 & 2 & -3 \\ 6 & 7 & -1 & 42 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 5 & 5 & -18 \\ 0 & 32 & 37 & -129 \\ 6 & 7 & -1 & 42 \end{array}\right] \leftarrow R_2 + 7R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 5 & 5 & -18 \\ 0 & 32 & 37 & -129 \\ 0 & 23 & 31 & -150 \end{array}\right] \leftarrow 6R_3 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 5 & 5 & -18 \\ 0 & 32 & 37 & -129 \\ 0 & 0 & -141 & 1833 \end{array}\right] \leftarrow 23R_2 - 32R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 5 & 5 & -18 \\ 0 & 32 & 37 & -129 \\ 0 & 0 & 1 & -13 \end{array}\right] \leftarrow R_3 + (-141)$$

$$= \left[\begin{array}{ccc|c} 1 & 5 & 5 & -18 \\ 0 & 32 & 0 & 352 \\ 0 & 0 & 1 & -13 \end{array}\right] \leftarrow R_2 - 37R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 5 & 5 & -18 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -13 \end{array}\right] \leftarrow R_2 \div 32$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 5 & -73 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -13 \end{array}\right] \leftarrow R_1 - 5R_2$$

$$[I \mid X] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -13 \end{array}\right] \leftarrow R_1 - 5R_3$$

The solution of the equation is given by $X$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 11 \\ -13 \end{bmatrix}$$

Therefore, the solution is $(-8, 11, -13)$.

15. $$\begin{aligned} 2x_1 + 6x_2 - 5x_3 &= -20 \\ 8x_1 - 12x_2 + 7x_3 &= 28 \\ -4x_1 + 10x_2 - x_3 &= 7 \end{aligned}$$

*SOLUTION:*

Write the system matrix in form $AX = B$. Make sure you align the variables. For the first matrix, the first column should include $x_1$, the second column $x_2$, and the third column $x_3$. The column matrix and the matrix of constant terms should be listed in order.

$$\begin{bmatrix} 2 & 6 & -5 \\ 8 & -12 & 7 \\ -4 & 10 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -20 \\ 28 \\ 7 \end{bmatrix}$$

Write the augmented matrix $[A \mid B]$. Attach the matrix of constant terms to the end of the $3 \times 3$ matrix.

$$\left[\begin{array}{ccc|c} 2 & 6 & -5 & -20 \\ 8 & -12 & 7 & 28 \\ -4 & 10 & -1 & 7 \end{array}\right]$$

Use Gauss-Jordan elimination to solve the system. First, use elementary row operations to transform $A$ into $I$.

$$[A \mid B] = \left[\begin{array}{ccc|c} 2 & 6 & -5 & -20 \\ 8 & -12 & 7 & 28 \\ -4 & 10 & -1 & 7 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 6 & -5 & -20 \\ 0 & -36 & 27 & 108 \\ -4 & 10 & -1 & 7 \end{array}\right] \leftarrow R_2 - 4R_1$$

$$= \left[\begin{array}{ccc|c} 2 & 6 & -5 & -20 \\ 0 & -36 & 27 & 108 \\ 0 & 22 & -11 & -33 \end{array}\right] \leftarrow 2R_3 + R_1$$

$$= \left[\begin{array}{ccc|c} 2 & 6 & -5 & -20 \\ 0 & -36 & 27 & 108 \\ 0 & 0 & 99 & 594 \end{array}\right] \leftarrow 11R_3 + 18R_2$$

$$= \left[\begin{array}{ccc|c} 2 & 6 & -5 & -20 \\ 0 & -36 & 27 & 108 \\ 0 & 0 & 1 & 6 \end{array}\right] \leftarrow R_3 \div (99)$$

$$= \left[\begin{array}{ccc|c} 2 & 6 & -5 & -20 \\ 0 & -36 & 0 & -54 \\ 0 & 0 & 1 & 6 \end{array}\right] \leftarrow R_2 - 27R_3$$

$$= \left[\begin{array}{ccc|c} 2 & 6 & -5 & -20 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 6 \end{array}\right] \leftarrow R_2 \div -36$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & -5 & -29 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 6 \end{array}\right] \leftarrow R_1 - 6R_2$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 6 \end{array}\right] \leftarrow R_1 + 5R_3$$

$$[I \mid X] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 6 \end{array}\right] \leftarrow R_1 \div 2$$

The solution of the equation is given by $X$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \\ 6 \end{bmatrix}$$

Therefore, the solution is (0.5, 1.5, 6).

16. $$\begin{aligned} 3x_1 - 5x_2 + 12x_3 &= 9 \\ 2x_1 + 4x_2 - 11x_3 &= 1 \\ -5x_1 + 7x_2 - 15x_3 &= -28 \end{aligned}$$

*SOLUTION:*

Write the system matrix in form $AX = B$. Make sure you align the variables. For the first matrix, the first column should include $x_1$, the second column $x_2$, and the third column $x_3$. The column matrix and the matrix of constant terms should be listed in order.

$$\begin{bmatrix} 3 & -5 & 12 \\ 2 & 4 & -11 \\ -5 & 7 & -15 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ -28 \end{bmatrix}$$

Write the augmented matrix $[A \mid B]$. Attach the matrix of constant terms to the end of the $3 \times 3$ matrix.

$$\left[\begin{array}{ccc|c} 3 & -5 & 12 & 9 \\ 2 & 4 & -11 & 1 \\ -5 & 7 & -15 & -28 \end{array}\right]$$

Use Gauss-Jordan elimination to solve the system. First, use elementary row operations to transform $A$ into $I$.

## 6-2 Matrix Multiplication, Inverses and Determinants

$$[A \mid B] = \left[\begin{array}{ccc|c} 3 & -5 & 12 & 9 \\ 2 & 4 & -11 & 1 \\ -5 & 7 & -15 & -28 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 3 & -5 & 12 & 9 \\ 0 & -22 & 57 & 15 \\ -5 & 7 & -15 & -28 \end{array}\right] \leftarrow 2R_1 - 3R_2$$

$$= \left[\begin{array}{ccc|c} 3 & -5 & 12 & 9 \\ 0 & -22 & 57 & 15 \\ 0 & -4 & 15 & -39 \end{array}\right] \leftarrow 5R_1 + 3R_3$$

$$= \left[\begin{array}{ccc|c} 3 & -5 & 12 & 9 \\ 0 & -22 & 57 & 15 \\ 0 & 0 & -51 & 459 \end{array}\right] \leftarrow 2R_2 - 11R_3$$

$$= \left[\begin{array}{ccc|c} 3 & -5 & 12 & 9 \\ 0 & -22 & 57 & 15 \\ 0 & 0 & 1 & -9 \end{array}\right] \leftarrow R_3 \div (-51)$$

$$= \left[\begin{array}{ccc|c} 3 & -5 & 12 & 9 \\ 0 & -22 & 0 & 528 \\ 0 & 0 & 1 & -9 \end{array}\right] \leftarrow R_2 - 57R_3$$

$$= \left[\begin{array}{ccc|c} 3 & -5 & 12 & 9 \\ 0 & 1 & 0 & -24 \\ 0 & 0 & 1 & -9 \end{array}\right] \leftarrow R_2 \div -22$$

$$= \left[\begin{array}{ccc|c} 3 & 0 & 12 & -111 \\ 0 & 1 & 0 & -24 \\ 0 & 0 & 1 & -9 \end{array}\right] \leftarrow R_1 + 5R_2$$

$$= \left[\begin{array}{ccc|c} 3 & 0 & 0 & -3 \\ 0 & 1 & 0 & -24 \\ 0 & 0 & 1 & -9 \end{array}\right] \leftarrow R_1 - 12R_3$$

$$[I \mid X] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -24 \\ 0 & 0 & 1 & -9 \end{array}\right] \leftarrow R_1 \div 3$$

The solution of the equation is given by $X$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -24 \\ -9 \end{bmatrix}$$

Therefore, the solution is $(-1, -24, -9)$.

17. $$\begin{aligned} -x_1 - 3x_2 + 9x_3 &= 25 \\ -5x_1 + 11x_2 + 8x_3 &= 33 \\ 2x_1 + x_2 - 13x_3 &= -45 \end{aligned}$$

*SOLUTION:*

Write the system matrix in form $AX = B$. Make sure you align the variables. For the first matrix, the first column should include $x_1$, the second column $x_2$, and the third column $x_3$. The column matrix and the matrix of constant terms should be listed in order.

$$\begin{bmatrix} -1 & -3 & 9 \\ -5 & 11 & 8 \\ 2 & 1 & -13 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 25 \\ 33 \\ -45 \end{bmatrix}$$

Write the augmented matrix $[A \mid B]$. Attach the matrix of constant terms to the end of the $3 \times 3$ matrix.

$$\left[\begin{array}{ccc|c} -1 & -3 & 9 & 25 \\ -5 & 11 & 8 & 33 \\ 2 & 1 & -13 & -45 \end{array}\right]$$

Use Gauss-Jordan elimination to solve the system. First, use elementary row operations to transform $A$ into $I$.

$$[A \mid B]=\left[\begin{array}{ccc|c}-1 & -3 & 9 & 25\\ -5 & 11 & 8 & 33\\ 2 & 1 & -13 & -45\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}-1 & -3 & 9 & 25\\ 0 & 26 & -37 & -92\\ 2 & 1 & -13 & -45\end{array}\right] \leftarrow R_2-5R_1$$

$$=\left[\begin{array}{ccc|c}-1 & -3 & 9 & 25\\ 0 & 26 & -37 & -92\\ 0 & -5 & 5 & 5\end{array}\right] \leftarrow 2R_1+R_3$$

$$=\left[\begin{array}{ccc|c}-1 & -3 & 9 & 25\\ 0 & 26 & -37 & -92\\ 0 & 0 & -55 & -330\end{array}\right] \leftarrow 5R_3+26R_3$$

$$=\left[\begin{array}{ccc|c}-1 & -3 & 9 & 25\\ 0 & 26 & -37 & -92\\ 0 & 0 & 1 & 6\end{array}\right] \leftarrow R_3\div(-55)$$

$$=\left[\begin{array}{ccc|c}-1 & -3 & 9 & 25\\ 0 & 26 & 0 & 130\\ 0 & 0 & 1 & 6\end{array}\right] \leftarrow R_2+37R_3$$

$$=\left[\begin{array}{ccc|c}-1 & -3 & 9 & 25\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & 6\end{array}\right] \leftarrow R_2\div 26$$

$$=\left[\begin{array}{ccc|c}-1 & 0 & 9 & 40\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & 6\end{array}\right] \leftarrow R_1+3R_2$$

$$=\left[\begin{array}{ccc|c}-1 & 0 & 0 & -14\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & 6\end{array}\right] \leftarrow R_1-9R_3$$

$$[I \mid X]=\left[\begin{array}{ccc|c}1 & 0 & 0 & 14\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & 6\end{array}\right] \leftarrow R_1\div -1$$

The solution of the equation is given by $X$.

$$X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}14\\ 5\\ 6\end{bmatrix}$$

Therefore, the solution is (14, 5, 6).

18.
$$\begin{aligned}x_1-8x_2-3x_3&=-4\\ -3x_1+10x_2+5x_3&=-42\\ 2x_1+7x_2+3x_3&=20\end{aligned}$$

*SOLUTION:*

Write the system matrix in form $AX = B$. Make sure you align the variables. For the first matrix, the first column should include $x_1$, the second column $x_2$, and the third column $x_3$. The column matrix and the matrix of constant terms should be listed in order.

$$\begin{bmatrix} 1 & -8 & -3 \\ -3 & 10 & 5 \\ 2 & 7 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -42 \\ 20 \end{bmatrix}$$

Write the augmented matrix $[A \mid B]$. Attach the matrix of constant terms to the end of the $3 \times 3$ matrix.

$$\left[\begin{array}{ccc|c} 1 & -8 & -3 & -4 \\ -3 & 10 & 5 & -42 \\ 2 & 7 & 3 & 20 \end{array}\right]$$

Use Gauss-Jordan elimination to solve the system. First, use elementary row operations to transform $A$ into $I$.

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & -8 & -3 & -4 \\ -3 & 10 & 5 & -42 \\ 2 & 7 & 3 & 20 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -8 & -3 & -4 \\ 0 & -14 & -4 & -54 \\ 2 & 7 & 3 & 20 \end{array}\right] \leftarrow R_2 + 3R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -8 & -3 & -4 \\ 0 & -14 & -4 & -54 \\ 0 & -23 & -9 & -28 \end{array}\right] \leftarrow 2R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -8 & -3 & -4 \\ 0 & -14 & -4 & -54 \\ 0 & 0 & -34 & 850 \end{array}\right] \leftarrow -23R_2 + 14R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -8 & -3 & -4 \\ 0 & -14 & -4 & -54 \\ 0 & 0 & 1 & -25 \end{array}\right] \leftarrow R_3 \div (-34)$$

$$= \left[\begin{array}{ccc|c} 1 & -8 & -3 & -4 \\ 0 & -14 & 0 & -154 \\ 0 & 0 & 1 & -25 \end{array}\right] \leftarrow R_2 + 4R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -8 & -3 & -4 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -25 \end{array}\right] \leftarrow R_2 \div -14$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -3 & 84 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -25 \end{array}\right] \leftarrow R_1 + 8R_2$$

$$[I \mid X] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -25 \end{array}\right] \leftarrow R_1 + 3R_3$$

The solution of the equation is given by $X$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \\ -25 \end{bmatrix}$$

Therefore, the solution is (9, 11, −25).

**Determine whether $A$ and $B$ are inverse matrices.**

19. $A = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$

If $A$ and $B$ are inverse matrices, then $AB = BA = I$.

$$AB = \begin{bmatrix} 12(3)-7(5) & 12(7)-7(12) \\ -5(3)+3(5) & -5(7)+3(12) \end{bmatrix}$$

$$= \begin{bmatrix} 36-35 & 84-84 \\ -15+15 & -35+36 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3(12)+7(-5) & 3(-7)+7(3) \\ 5(12)+12(-5) & 5(-7)+12(3) \end{bmatrix}$$

$$= \begin{bmatrix} 36-35 & -21+21 \\ 60-60 & -35+36 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because $AB = BA = I$, $B = A^{-1}$ and $A = B^{-1}$.

20. $A = \begin{bmatrix} 4 & -5 \\ 5 & -6 \end{bmatrix}$

$B = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 4 & -5 \\ 5 & -6 \end{bmatrix}$; $B = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

If $A$ and $B$ are inverse matrices, then $AB = BA = I$.

$$AB = \begin{bmatrix} 4(-6)-5(-5) & 4(5)-5(4) \\ 5(-6)-6(-5) & 5(5)-6(4) \end{bmatrix}$$
$$= \begin{bmatrix} -24+25 & 20-20 \\ -30+30 & -25+24 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -6(4)+5(5) & -6(-5)+5(-6) \\ -5(4)+4(5) & -5(-5)+4(-6) \end{bmatrix}$$
$$= \begin{bmatrix} -24+25 & 30-30 \\ -20+20 & 25-24 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because $AB = BA = I$, $B = A^{-1}$ and $A = B^{-1}$.

21. $A = \begin{bmatrix} -5 & 3 \\ 6 & -4 \end{bmatrix}$

$B = \begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} -5 & 3 \\ 6 & -4 \end{bmatrix}$; $B = \begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$

If $A$ and $B$ are inverse matrices, then $AB = BA = I$.

$$AB = \begin{bmatrix} -5(4)+3(6) & -5(3)+3(5) \\ 6(4)-4(6) & 6(3)-4(5) \end{bmatrix}$$
$$= \begin{bmatrix} -20+18 & -15+15 \\ 24-24 & 18-20 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$AB \neq I$, so $A$ and $B$ are not inverses.

## 6-2 Matrix Multiplication, Inverses and Determinants

22. $A = \begin{bmatrix} -8 & 4 \\ 6 & -3 \end{bmatrix}$

$B = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} -8 & 4 \\ 6 & -3 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

If $A$ and $B$ are inverse matrices, then $AB = BA = I$.

$$AB = \begin{bmatrix} -8(3)+4(6) & -8(4)+4(8) \\ 6(3)-3(6) & 6(4)-3(8) \end{bmatrix}$$
$$= \begin{bmatrix} -24+24 & -32+32 \\ 18-18 & 24-24 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$AB \neq I$, so $A$ and $B$ are not inverses.

23. $A = \begin{bmatrix} 9 & 2 \\ 5 & 1 \end{bmatrix}$

$B = \begin{bmatrix} -1 & 2 \\ 5 & -9 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 9 & 2 \\ 5 & 1 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 2 \\ 5 & -9 \end{bmatrix}$

If $A$ and $B$ are inverse matrices, then $AB = BA = I$.

$$AB = \begin{bmatrix} 9(-1)+2(5) & 9(2)+2(-9) \\ 5(-1)+1(5) & 5(2)+1(-9) \end{bmatrix}$$
$$= \begin{bmatrix} -9+10 & 18-18 \\ -5+5 & 10-9 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1(9)+2(5) & -1(2)+2(1) \\ 5(9)-9(5) & 5(2)-9(1) \end{bmatrix}$$
$$= \begin{bmatrix} -9+10 & -2+2 \\ 45-45 & 10-9 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because $AB = BA = I$, $B = A^{-1}$ and $A = B^{-1}$.

24. $A = \begin{bmatrix} 7 & 5 \\ -6 & -4 \end{bmatrix}$

$B = \begin{bmatrix} -4 & -5 \\ 6 & 7 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 7 & 5 \\ -6 & -4 \end{bmatrix}; B = \begin{bmatrix} -4 & -5 \\ 6 & 7 \end{bmatrix}$

If $A$ and $B$ are inverse matrices, then $AB = BA = I$.

$$AB = \begin{bmatrix} 7(-4)+5(6) & 7(-5)+5(7) \\ -6(-4)-4(6) & -6(-5)-4(7) \end{bmatrix}$$
$$= \begin{bmatrix} -28+30 & -35+35 \\ 24-24 & 30-28 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$AB \neq I$, so $A$ and $B$ are not inverses.

25. $A = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$

$B = \begin{bmatrix} -4 & -3 \\ -3 & -2 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}; B = \begin{bmatrix} -4 & -3 \\ -3 & -2 \end{bmatrix}$

If $A$ and $B$ are inverse matrices, then $AB = BA = I$.

$$AB = \begin{bmatrix} 2(-4)-3(-3) & 2(-3)-3(-2) \\ -3(-4)+4(-3) & -3(-3)+4(-2) \end{bmatrix}$$
$$= \begin{bmatrix} -8+9 & -6+6 \\ 12-12 & 9-8 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$BA = \begin{bmatrix} -4(2)-3(-3) & -4(-3)-3(4) \\ -3(2)-2(-3) & -3(-3)-2(4) \end{bmatrix}$$
$$= \begin{bmatrix} -8+9 & 12-12 \\ -6+6 & 9+8 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because $AB = BA = I$, $B = A^{-1}$ and $A = B^{-1}$.

## 6-2 Matrix Multiplication, Inverses and Determinants

26. $A = \begin{bmatrix} 9 & -7 \\ 8 & -5 \end{bmatrix}$

$B = \begin{bmatrix} 1 & -6 \\ 4 & 10 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 9 & -7 \\ 8 & -5 \end{bmatrix}; B = \begin{bmatrix} 1 & -6 \\ 4 & 10 \end{bmatrix}$

If $A$ and $B$ are inverse matrices, then $AB = BA = I$.

$$AB = \begin{bmatrix} 9(1)-7(4) & 9(-6)-7(10) \\ 8(1)-5(4) & 8(-6)-5(10) \end{bmatrix}$$

$$= \begin{bmatrix} 9-28 & -54-70 \\ 8-20 & -48-50 \end{bmatrix}$$

$$= \begin{bmatrix} -19 & -124 \\ -12 & -98 \end{bmatrix}$$

$AB \neq I$, so $A$ and $B$ are not inverses.

**Find $A^{-1}$, if it exists. If $A^{-1}$ does not exist, write *singular*.**

27. $A = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{cc|cc} -4 & 2 & 1 & 0 \\ -6 & 3 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{cc|cc} -4 & 2 & 1 & 0 \\ -6 & 3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} -4 & 2 & 1 & 0 \\ 0 & 0 & -3 & 2 \end{array}\right] \xleftarrow{2R_2 - 3R_1}$$

A row of 0s has been formed, so the first 2 columns cannot become the identity matrix. Therefore, $A$ is singular.

## 6-2 Matrix Multiplication, Inverses and Determinants

28. $A = \begin{bmatrix} -4 & 8 \\ 1 & -2 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{cc|cc} -4 & 8 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{cc|cc} -4 & 8 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} -4 & 2 & 1 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right] \xleftarrow{4R_2 + R_1}$$

A row of 0s has been formed, so the first 2 columns cannot become the identity matrix. Therefore, $A$ is singular.

## 6-2 Matrix Multiplication, Inverses and Determinants

29. $A = \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{array}\right] \leftarrow 3R_2 + 2R_1$$

$$\left[\begin{array}{cc|cc} 3 & 0 & -9 & -15 \\ 0 & 1 & 2 & 3 \end{array}\right] \leftarrow R_1 - 5R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -3 & -5 \\ 0 & 1 & 2 & 3 \end{array}\right] \leftarrow R_1 \div 3$$

The first two columns are the identity matrix. Therefore, $A$ is invertible and $A^{-1} = \begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$.

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(-3)+5(2) & 3(-5)+5(3) \\ -2(-3)-3(2) & -2(-5)-3(9) \end{bmatrix}$$

$$= \begin{bmatrix} -9+10 & -15+15 \\ 6-6 & 10-9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -3(3)-5(-2) & -3(5)-5(-3) \\ 2(3)+3(-2) & 2(5)+3(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -9+10 & -15+15 \\ 6-6 & 10-9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

30. $A = \begin{bmatrix} 8 & 5 \\ 6 & 4 \end{bmatrix}$

*SOLUTION:*

## 6-2 Matrix Multiplication, Inverses and Determinants

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{cc|cc} 8 & 5 & 1 & 0 \\ 6 & 4 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{cc|cc} 8 & 5 & 1 & 0 \\ 6 & 4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 8 & 5 & 1 & 0 \\ 0 & 1 & -3 & 4 \end{array}\right] \leftarrow 4R_2 - 3R_1$$

$$\left[\begin{array}{cc|cc} 8 & 0 & 16 & -20 \\ 0 & 1 & -3 & 4 \end{array}\right] \leftarrow R_1 - 5R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -\frac{5}{2} \\ 0 & 1 & -3 & 4 \end{array}\right] \leftarrow \frac{1}{8}R_1$$

The first two columns are the identity matrix. Therefore, $A$ is invertible and $A^{-1} = \begin{bmatrix} 2 & -\frac{5}{2} \\ -3 & 4 \end{bmatrix}$.

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 8 & 5 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} 2 & -\frac{5}{2} \\ -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8(2)+5(-3) & 8\left(-\frac{5}{2}\right)+5(4) \\ 6(2)+4(-3) & 6\left(-\frac{5}{2}\right)+4(4) \end{bmatrix}$$

$$= \begin{bmatrix} 16-15 & -20+20 \\ 12-12 & -15+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 2 & -\frac{5}{2} \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 8 & 5 \\ 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2(8)-\frac{5}{2}(6) & 2(5)-\frac{5}{2}(4) \\ -3(8)+4(6) & -3(5)+4(4) \end{bmatrix}$$

$$= \begin{bmatrix} 16-15 & 10-10 \\ -24+24 & -15+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

31. $A = \begin{bmatrix} -1 & -1 & -3 \\ 3 & 6 & 4 \\ 2 & 1 & 8 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} -1 & -1 & -3 & 1 & 0 & 0 \\ 3 & 6 & 4 & 0 & 1 & 0 \\ 2 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|ccc} -1 & -1 & -3 & 1 & 0 & 0 \\ 3 & 6 & 4 & 0 & 1 & 0 \\ 2 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & 3 & 1 & 0 \\ 2 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \xleftarrow{R_2+3R_1}$$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & 3 & 1 & 0 \\ 0 & -1 & 2 & 2 & 0 & 1 \end{array}\right] \xleftarrow{R_3+2R_1}$$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & 3 & 1 & 0 \\ 0 & 0 & 1 & 9 & 1 & 3 \end{array}\right] \xleftarrow{3R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & -3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 48 & 6 & 15 \\ 0 & 0 & 1 & 9 & 1 & 3 \end{array}\right] \xleftarrow{R_2+5R_3}$$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 16 & 2 & 5 \\ 0 & 0 & 1 & 9 & 1 & 3 \end{array}\right] \xleftarrow{R_2 \div 3}$$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 28 & 3 & 9 \\ 0 & 1 & 0 & 16 & 2 & 5 \\ 0 & 0 & 1 & 9 & 1 & 3 \end{array}\right] \xleftarrow{R_1+3R_3}$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 44 & 5 & 14 \\ 0 & 1 & 0 & 16 & 2 & 5 \\ 0 & 0 & 1 & 9 & 1 & 3 \end{array}\right] \xleftarrow{R_1+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -44 & -5 & -14 \\ 0 & 1 & 0 & 16 & 2 & 5 \\ 0 & 0 & 1 & 9 & 1 & 3 \end{array}\right] \xleftarrow{-R_1}$$

The first three columns are the identity matrix. Therefore, $A$ is invertible and $A^{-1} = \begin{bmatrix} -44 & -5 & -14 \\ 16 & 2 & 5 \\ 9 & 1 & 3 \end{bmatrix}$.

## 6-2 Matrix Multiplication, Inverses and Determinants

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} -1 & -1 & -3 \\ 3 & 6 & 4 \\ 2 & 1 & 8 \end{bmatrix}\begin{bmatrix} -44 & -5 & -14 \\ 16 & 2 & 5 \\ 9 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1(-44)-1(16)-3(9) & -1(-5)-1(2)-3(1) & -1(-14)-1(5)-3(3) \\ 3(-44)+6(16)+4(9) & 3(-5)+6(2)+4(1) & 3(-14)+6(5)+4(3) \\ 2(-44)+1(16)+8(9) & 2(-5)+1(2)+8(1) & 2(-14)+1(5)+8(3) \end{bmatrix}$$

$$= \begin{bmatrix} 44-16-27 & 5-2-3 & 14-5-9 \\ -132+96+36 & -15+12+4 & -42+30+12 \\ -88+16+72 & -10+2+8 & -28+5+24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -44 & -5 & -14 \\ 16 & 2 & 5 \\ 9 & 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & -1 & -3 \\ 3 & 6 & 4 \\ 2 & 1 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -44(-1)-5(3)-14(2) & -44(-1)-5(6)-14(1) & -44(-3)-5(4)-14(8) \\ 16(-1)+2(3)+5(2) & 16(-1)+2(6)+5(1) & 16(-3)+2(4)+5(8) \\ 9(-1)+1(3)+3(2) & 9(-1)+1(6)+3(1) & 9(-3)+1(4)+3(8) \end{bmatrix}$$

$$= \begin{bmatrix} 44-15-28 & 44-30+14 & 132-20-112 \\ -16+6+10 & -16+12+5 & -48+8+40 \\ -9+3+6 & -9+6+3 & -27+4+24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

32. $A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & 3 & 5 \\ 6 & -1 & -4 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ -2 & 3 & 5 & 0 & 1 & 0 \\ 6 & -1 & -4 & 0 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ -2 & 3 & 5 & 0 & 1 & 0 \\ 6 & -1 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 0 & 8 & 11 & 1 & 2 & 0 \\ 6 & -1 & -4 & 0 & 0 & 1 \end{array}\right] \leftarrow 2R_2 + R_1$$

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 0 & 8 & 11 & 1 & 2 & 0 \\ 0 & -8 & -11 & -3 & 0 & 2 \end{array}\right] \leftarrow 2R_3 - 3R_1$$

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 0 & 8 & 11 & 1 & 2 & 0 \\ 0 & 0 & 0 & -2 & 2 & 2 \end{array}\right] \leftarrow R_2 + R_3$$

A row of 0s has been formed, so the first 3 columns cannot become the identity matrix. Therefore, $A$ is singular.

## 6-2 Matrix Multiplication, Inverses and Determinants

33. $A = \begin{bmatrix} 5 & 2 & -1 \\ 4 & 7 & -3 \\ 1 & -5 & 2 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 5 & 2 & -1 & 1 & 0 & 0 \\ 4 & 7 & -3 & 0 & 1 & 0 \\ 1 & -5 & 2 & 0 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|ccc} 5 & 2 & -1 & 1 & 0 & 0 \\ 4 & 7 & -3 & 0 & 1 & 0 \\ 1 & -5 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 5 & 2 & -1 & 1 & 0 & 0 \\ 0 & 27 & -11 & -4 & 5 & 0 \\ 1 & -5 & 2 & 0 & 0 & 1 \end{array}\right] \leftarrow 5R_2 - 4R_1$$

$$\left[\begin{array}{ccc|ccc} 5 & 2 & -1 & 1 & 0 & 0 \\ 0 & 27 & -11 & -4 & 5 & 0 \\ 5 & -25 & 10 & 0 & 0 & 5 \end{array}\right] \leftarrow 5R_3$$

$$\left[\begin{array}{ccc|ccc} 5 & 2 & -1 & 1 & 0 & 0 \\ 0 & 27 & -11 & -4 & 5 & 0 \\ 0 & -27 & 11 & -1 & 0 & 5 \end{array}\right] \leftarrow R_3 - R_1$$

$$\left[\begin{array}{ccc|ccc} 5 & 2 & -1 & 1 & 0 & 0 \\ 0 & 27 & -11 & -4 & 5 & 0 \\ 0 & 0 & 0 & -5 & 5 & 5 \end{array}\right] \leftarrow R_3 + R_2$$

A row of 0s has been formed, so the first 3 columns cannot become the identity matrix. Therefore, $A$ is singular.

34. $A = \begin{bmatrix} 2 & 3 & -4 \\ 3 & 6 & -5 \\ -2 & -8 & 1 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 3 & 6 & -5 & 0 & 1 & 0 \\ -2 & -8 & 1 & 0 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

## 6-2 Matrix Multiplication, Inverses and Determinants

$$\left[\begin{array}{ccc|ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 3 & 6 & -5 & 0 & 1 & 0 \\ -2 & -8 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 0 & 3 & 2 & -3 & 2 & 0 \\ -2 & -8 & 1 & 0 & 0 & 1 \end{array}\right] \leftarrow 2R_2 - 3R_1$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 0 & 3 & 2 & -3 & 2 & 0 \\ 0 & -5 & -3 & 1 & 0 & 1 \end{array}\right] \leftarrow R_3 + R_1$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 0 & 3 & 2 & -3 & 2 & 0 \\ 0 & 0 & 1 & -12 & 10 & 3 \end{array}\right] \leftarrow 3R_3 + 5R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 0 & 3 & 0 & 21 & -18 & -6 \\ 0 & 0 & 1 & -12 & 10 & 3 \end{array}\right] \leftarrow R_2 - 2R_3$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 7 & -6 & -2 \\ 0 & 0 & 1 & -12 & 10 & 3 \end{array}\right] \leftarrow R_2 \div 3$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -4 & -20 & 18 & 6 \\ 0 & 1 & 0 & 7 & -6 & -2 \\ 0 & 0 & 1 & -12 & 10 & 3 \end{array}\right] \leftarrow R_1 - 3R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -68 & 58 & 18 \\ 0 & 1 & 0 & 7 & -6 & -2 \\ 0 & 0 & 1 & -12 & 10 & 3 \end{array}\right] \leftarrow R_1 + 4R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -34 & 29 & 9 \\ 0 & 1 & 0 & 7 & -6 & -2 \\ 0 & 0 & 1 & -12 & 10 & 3 \end{array}\right] \leftarrow R_1 \div 2$$

The first three columns are the identity matrix. Therefore, $A$ is invertible and $A^{-1} = \begin{bmatrix} -34 & 29 & 9 \\ 7 & -6 & -2 \\ -12 & 10 & 3 \end{bmatrix}$.

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 2 & 3 & -4 \\ 3 & 6 & -5 \\ -2 & -8 & 1 \end{bmatrix}\begin{bmatrix} -34 & 29 & 9 \\ 7 & -6 & -2 \\ -12 & 10 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-34)+3(7)-4(-12) & 2(29)+3(-6)-4(10) & 2(9)+3(-2)-4(3) \\ 3(-34)+6(7)-5(-12) & 3(29)+6(-6)-5(10) & 3(9)+6(-2)-5(3) \\ -2(-34)-8(7)+1(-12) & -2(29)-8(-6)+1(10) & -2(9)-8(-2)+1(3) \end{bmatrix}$$

$$= \begin{bmatrix} -68+21+48 & 58-18-40 & 18-6-12 \\ -102+42+60 & 87-36-50 & 27-12-15 \\ 68-56-12 & -58+48+10 & -18+16+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -34 & 29 & 9 \\ 7 & -6 & -2 \\ -12 & 10 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & -4 \\ 3 & 6 & -5 \\ -2 & -8 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -34(2)+29(3)+9(-2) & -34(3)+29(6)+9(-8) & -34(-4)+29(-5)+9(1) \\ 7(2)-6(3)-2(-2) & 7(3)-6(6)-2(-8) & 7(-4)-6(-5)-2(1) \\ -12(2)+10(3)+3(-2) & -12(3)+10(6)+3(-8) & -12(-4)+10(-5)+3(1) \end{bmatrix}$$

$$= \begin{bmatrix} -68+87-18 & -102+174-72 & 136-145+9 \\ 14-18+4 & 21-36+16 & -28+30-2 \\ -24+30-6 & -36+60-24 & 48-50+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Find the determinant of each matrix. Then find the inverse of the matrix, if it exists.**

35. $\begin{bmatrix} 6 & -5 \\ 3 & -2 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} 6 & -5 \\ 3 & -2 \end{vmatrix} = 6(-2) - 3(-5)$$

$$= -12 + 15$$

$$= 3$$

Because the determinant is not 0, the matrix is invertible. Find the inverse matrix.

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} -2 & 5 \\ -3 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{2}{3} & \frac{5}{3} \\ -1 & 2 \end{bmatrix}$$

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 6 & -5 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} -\frac{2}{3} & \frac{5}{3} \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 6\left(-\frac{2}{3}\right) - 5(-1) & 6\left(\frac{5}{3}\right) - 5(2) \\ 3\left(-\frac{2}{3}\right) - 2(-1) & 3\left(\frac{5}{3}\right) - 2(2) \end{bmatrix}$$
$$= \begin{bmatrix} -4+5 & 10-10 \\ -2+2 & 5-4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A^{-1}A = \begin{bmatrix} -\frac{2}{3} & \frac{5}{3} \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & -5 \\ 3 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} \left(-\frac{2}{3}\right)6 + \left(\frac{5}{3}\right)3 & \left(-\frac{2}{3}\right)(-5) + \left(\frac{5}{3}\right)(-2) \\ -1(6) + 2(3) & -1(-5) + 2(-2) \end{bmatrix}$$
$$= \begin{bmatrix} -4+5 & \frac{10}{3} - \frac{10}{3} \\ -6+6 & 5-4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

36. $\begin{bmatrix} -2 & 7 \\ 1 & 8 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} -2 & 7 \\ 1 & 8 \end{vmatrix} = -2(8) - 1(7)$$
$$= -16 - 7$$
$$= -23$$

Because the determinant is not 0, the matrix is invertible. Find the inverse matrix.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= -\frac{1}{23} \begin{bmatrix} 8 & -7 \\ -1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{8}{23} & \frac{7}{23} \\ \frac{1}{23} & \frac{2}{23} \end{bmatrix}$$

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} -2 & 7 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} -\frac{8}{23} & \frac{7}{23} \\ \frac{1}{23} & \frac{2}{23} \end{bmatrix}$$
$$= \begin{bmatrix} -2\left(-\frac{8}{23}\right) + 7\left(\frac{1}{23}\right) & -2\left(\frac{7}{23}\right) + 7\left(\frac{2}{23}\right) \\ 1\left(-\frac{8}{23}\right) + 8\left(\frac{1}{23}\right) & 1\left(\frac{7}{23}\right) + 8\left(\frac{2}{23}\right) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{16}{23} + \frac{7}{23} & -\frac{14}{23} + \frac{14}{23} \\ -\frac{8}{23} + \frac{8}{23} & \frac{7}{23} + \frac{16}{23} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -\frac{8}{23} & \frac{7}{23} \\ \frac{1}{23} & \frac{2}{23} \end{bmatrix} \begin{bmatrix} -2 & 7 \\ 1 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{8}{23}(-2) + \frac{7}{23}(1) & -\frac{8}{23}(7) + \frac{7}{23}(8) \\ \frac{1}{23}(-2) + \frac{2}{23}(1) & \frac{1}{23}(7) + \frac{2}{23}(8) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{16}{23} + \frac{7}{23} & -\frac{56}{23} + \frac{56}{23} \\ -\frac{2}{23} + \frac{2}{23} & \frac{7}{23} + \frac{16}{23} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

## 6-2 Matrix Multiplication, Inverses and Determinants

37. $\begin{bmatrix} -4 & -7 \\ 6 & 9 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} -4 & -7 \\ 6 & 9 \end{vmatrix} = -4(9) - 6(-7)$$
$$= -36 + 42$$
$$= 6$$

Because the determinant is not 0, the matrix is invertible. Find the inverse matrix.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{6} \begin{bmatrix} 9 & 7 \\ -6 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2} & \frac{7}{6} \\ -1 & -\frac{2}{3} \end{bmatrix}$$

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} -4 & -7 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{7}{6} \\ -1 & -\frac{2}{3} \end{bmatrix}$$
$$= \begin{bmatrix} -4\left(\frac{3}{2}\right) - 7(-1) & -4\left(\frac{7}{6}\right) - 7\left(-\frac{2}{3}\right) \\ 6\left(\frac{3}{2}\right) + 9(-1) & 6\left(\frac{7}{6}\right) + 9\left(-\frac{2}{3}\right) \end{bmatrix}$$
$$= \begin{bmatrix} -6 + 7 & -\frac{14}{3} + \frac{14}{3} \\ 9 - 9 & 7 - 6 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{3}{2} & \frac{7}{6} \\ -1 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} -4 & -7 \\ 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}(-4)+\frac{7}{6}(6) & \frac{3}{2}(-7)+\frac{7}{6}(9) \\ -1(-4)-\frac{2}{3}(6) & -1(-7)-\frac{2}{3}(9) \end{bmatrix}$$

$$= \begin{bmatrix} -6+7 & -\frac{21}{2}+\frac{21}{2} \\ 4-4 & 7-6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

38. $\begin{bmatrix} 12 & -9 \\ -4 & 3 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} 12 & -9 \\ -4 & 3 \end{vmatrix} = 12(3) - (-4)(-9)$$

$$= 36 - 36$$

$$= 0$$

Because the determinant is 0, the matrix is singular and the inverse does not exist.

## 6-2 Matrix Multiplication, Inverses and Determinants

39. $\begin{bmatrix} 3 & 1 & -2 \\ 8 & -5 & 2 \\ -4 & 3 & -1 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 3 & 1 & -2 \\ 8 & -5 & 2 \\ -4 & 3 & -1 \end{vmatrix} = 3\begin{vmatrix} -5 & 2 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 8 & 2 \\ -4 & -1 \end{vmatrix} - 2\begin{vmatrix} 8 & -5 \\ -4 & 3 \end{vmatrix}$$

$$= 3[-5(-1) - 3(2)] - 1[8(-1) - (-4)(2)] - 2[8(3) - (-4)(-5)]$$

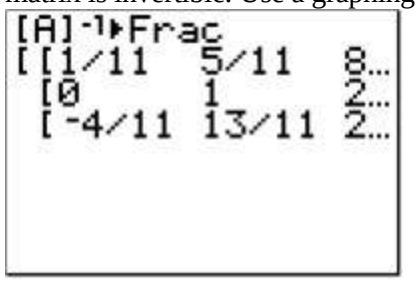
$$= 3(5 - 6) - 1(-8 + 8) - 2(24 - 20)$$

$$= 3(-1) - 1(0) - 2(4)$$

$$= -3 - 8$$

$$= -11$$

Because the determinant is not 0, the matrix is invertible. Use a graphing calculator to find the inverse matrix.




Use the „**Frac** feature under the **MATH** menu to write the inverse using fractions.

Therefore, $A^{-1} = \begin{bmatrix} \frac{1}{11} & \frac{5}{11} & \frac{8}{11} \\ 0 & 1 & 2 \\ -\frac{4}{11} & \frac{13}{11} & \frac{23}{11} \end{bmatrix}$.

## 6-2 Matrix Multiplication, Inverses and Determinants

40. $\begin{bmatrix} 1 & -1 & -2 \\ 5 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix}$

*SOLUTION:*

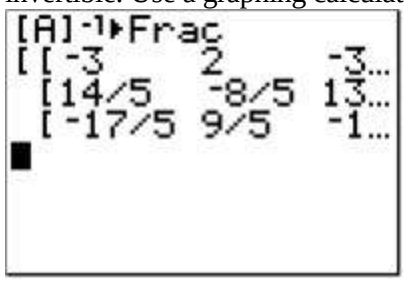
Find the determinant.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 1 & -1 & -2 \\ 5 & 9 & 3 \\ 2 & 7 & 4 \end{vmatrix} = 1\begin{vmatrix} 9 & 3 \\ 7 & 4 \end{vmatrix} + 1\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} 5 & 9 \\ 2 & 7 \end{vmatrix}$$

$$= 1[9(4) - 7(3)] + 1[5(4) - (2)(3)] - 2[5(7) - (2)(9)]$$
$$= 1(36 - 21) + 1(20 - 6) - 2(35 - 18)$$
$$= 1(15) + 1(14) - 2(17)$$
$$= 15 + 14 - 34$$
$$= -5$$

Because the determinant is not 0, the matrix is invertible. Use a graphing calculator to find the inverse matrix.




Use the „**Frac** feature under the **MATH** menu to write the inverse using fractions.

Therefore, $A^{-1} = \begin{bmatrix} -3 & 2 & -3 \\ \frac{14}{5} & -\frac{8}{5} & \frac{13}{5} \\ -\frac{17}{5} & \frac{9}{5} & -\frac{14}{5} \end{bmatrix}$.

## 6-2 Matrix Multiplication, Inverses and Determinants

41. $\begin{bmatrix} 9 & 3 & 7 \\ -6 & -2 & -5 \\ 3 & 1 & 4 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 9 & 3 & 7 \\ -6 & -2 & -5 \\ 3 & 1 & 4 \end{vmatrix} = 9\begin{vmatrix} -2 & -5 \\ 1 & 4 \end{vmatrix} - 3\begin{vmatrix} -6 & -5 \\ 3 & 4 \end{vmatrix} + 7\begin{vmatrix} -6 & -2 \\ 3 & 1 \end{vmatrix}$$

$$= 9[-2(4) - 1(-5)] - 3[-6(4) - 3(-5)] + 7[-6(1) - 3(-2)]$$
$$= 9(-8 + 5) - 3(-24 + 15) + 7(-6 + 6)$$
$$= 9(-3) - 3(-9) + 7(0)$$
$$= -27 + 27$$
$$= 0$$

Because the determinant is 0, the matrix is singular and the inverse does not exist.

## 6-2 Matrix Multiplication, Inverses and Determinants

42. $\begin{bmatrix} 2 & 3 & -1 \\ -4 & -5 & 2 \\ 6 & 1 & 3 \end{bmatrix}$

*SOLUTION:*

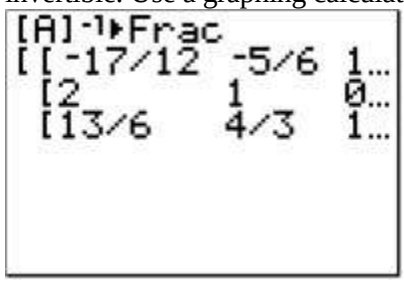
Find the determinant.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & -1 \\ -4 & -5 & 2 \\ 6 & 1 & 3 \end{vmatrix} = 2\begin{vmatrix} -5 & 2 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} -4 & 2 \\ 6 & 3 \end{vmatrix} - 1\begin{vmatrix} -4 & -5 \\ 6 & 1 \end{vmatrix}$$

$$= 2[-5(3) - 1(2)] - 3[-4(3) - 6(2)] - 1[-4(1) - 6(-5)]$$
$$= 2(-15 - 2) - 3(-12 - 12) - 1(-4 + 30)$$
$$= 2(-17) - 3(-24) - 1(26)$$
$$= -34 + 72 - 26$$
$$= 12$$

Because the determinant is not 0, the matrix is invertible. Use a graphing calculator to find the inverse matrix.




Use the „**Frac** feature under the **MATH** menu to write the inverse using fractions.

Therefore, $A^{-1} = \begin{bmatrix} -\frac{17}{12} & -\frac{5}{6} & \frac{1}{12} \\ 2 & 1 & 0 \\ \frac{13}{6} & \frac{4}{3} & \frac{1}{6} \end{bmatrix}$.

## 6-2 Matrix Multiplication, Inverses and Determinants

43. $\begin{bmatrix} -1 & 3 & 2 \\ 3 & -5 & -3 \\ 4 & 2 & 6 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} -1 & 3 & 2 \\ 3 & -5 & -3 \\ 4 & 2 & 6 \end{vmatrix} = -1\begin{vmatrix} -5 & -3 \\ 2 & 6 \end{vmatrix} - 3\begin{vmatrix} 3 & -3 \\ 4 & 6 \end{vmatrix} + 2\begin{vmatrix} 3 & -5 \\ 4 & 2 \end{vmatrix}$$

$$= -1[-5(6) - 2(-3)] - 3[3(6) - 4(-3)] + 2[3(2) - 4(-5)]$$

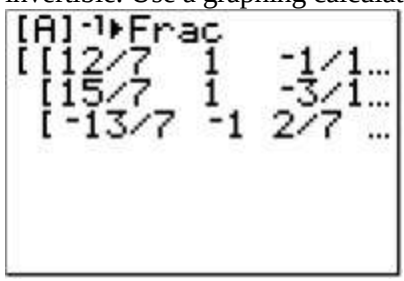
$$= -1(-30 + 6) - 3(18 + 12) + 2(6 + 20)$$

$$= -1(-24) - 3(30) + 2(26)$$

$$= 24 - 90 + 52$$

$$= -14$$

Because the determinant is not 0, the matrix is invertible. Use a graphing calculator to find the inverse matrix.




Use the „**Frac** feature under the **MATH** menu to write the inverse using fractions.

Therefore, $A^{-1} = \begin{bmatrix} \frac{12}{7} & 1 & -\frac{1}{14} \\ \frac{15}{7} & 1 & -\frac{3}{14} \\ -\frac{13}{7} & -1 & \frac{2}{7} \end{bmatrix}$.

## 6-2 Matrix Multiplication, Inverses and Determinants

44. $\begin{bmatrix} 6 & -1 & 2 \\ 1 & -2 & -4 \\ -3 & 1 & -5 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 6 & -1 & 2 \\ 1 & -2 & -4 \\ -3 & 1 & -5 \end{vmatrix} = 6\begin{vmatrix} -2 & -4 \\ 1 & -5 \end{vmatrix} + 1\begin{vmatrix} 1 & -4 \\ -3 & -5 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix}$$

$$= 6[-2(-5) - 1(-4)] + 1[1(-5) - (-3)(-4)] + 2[1(1) - (-3)(-2)]$$

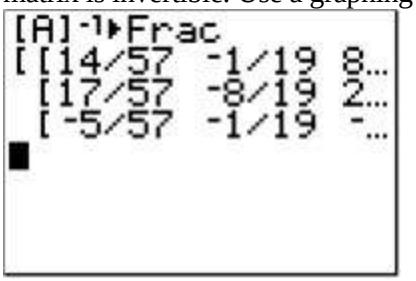
$$= 6(10 + 4) + 1(-5 - 12) + 2(1 - 6)$$

$$= 6(14) + 1(-17) + 2(-5)$$

$$= 84 - 17 - 10$$

$$= 57$$

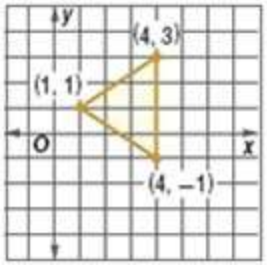
Because the determinant is not 0, the matrix is invertible. Use a graphing calculator to find the inverse matrix.




Use the „**Frac** feature under the **MATH** menu to write the inverse using fractions.

Therefore, $A^{-1} = \begin{bmatrix} \frac{14}{57} & -\frac{1}{19} & \frac{8}{57} \\ \frac{17}{57} & -\frac{8}{19} & \frac{26}{57} \\ -\frac{5}{57} & -\frac{1}{19} & -\frac{11}{57} \end{bmatrix}$.

**Find the area $A$ of each triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$, by using $A = \frac{1}{2}|\det(X)|$, where $X$ is $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$.**

45. 


*SOLUTION:*

$$A = \frac{1}{2}\left|\det\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 1 \\ 4 & -1 & 1 \end{bmatrix}\right|$$

$$= \frac{1}{2}\left|1\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 3 \\ 4 & -1 \end{vmatrix}\right|$$

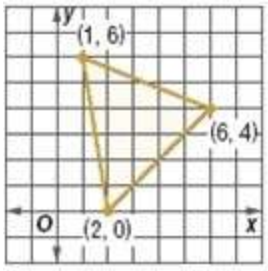
$$= \frac{1}{2}|(3)(1) - (-1)(1) - [4(1) - 4(1)] + 4(-1) - 4(3)|$$

$$= \frac{1}{2}|3 + 1 - 0 + -4 - 12|$$

$$= \frac{1}{2}|-12|$$

$$= \frac{1}{2}(12)$$

$$= 6$$

46. 


*SOLUTION:*

$$A = \frac{1}{2}\left|\det\begin{bmatrix}1 & 6 & 1\\ 2 & 0 & 1\\ 6 & 4 & 1\end{bmatrix}\right|$$

$$= \frac{1}{2}\left|1\begin{vmatrix}0 & 1\\ 4 & 1\end{vmatrix} - 6\begin{vmatrix}2 & 1\\ 6 & 1\end{vmatrix} + 1\begin{vmatrix}2 & 0\\ 6 & 4\end{vmatrix}\right|$$

$$= \frac{1}{2}\left|(0)(1) - (4)(1) - 6[2(1) - 6(1)] + 2(4) - 6(0)\right|$$

$$= \frac{1}{2}\left|0 - 4 - 6(-4) + 8 - 0\right|$$

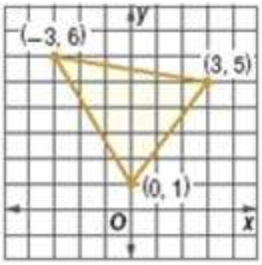
$$= \frac{1}{2}\left|-4 + 24 + 8\right|$$

$$= \frac{1}{2}\left|28\right|$$

$$= \frac{1}{2}(28)$$

$$= 14$$

## 6-2 Matrix Multiplication, Inverses and Determinants

47. 


*SOLUTION:*

$$A = \frac{1}{2}\left|\det\begin{bmatrix} -3 & 6 & 1 \\ 3 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right|$$

$$= \frac{1}{2}\left|-3\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} - 6\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix}\right|$$

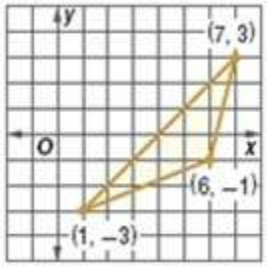
$$= \frac{1}{2}\left|-3[(5)(1)-(11)(1)] - 6[3(1)-0(1)] + 3(1) - 5(0)\right|$$

$$= \frac{1}{2}\left|-3(4) - 6(3) + 3 - 0\right|$$

$$= \frac{1}{2}\left|-27\right|$$

$$= \frac{1}{2}(27)$$

$$= 13.5$$

48. 


*SOLUTION:*

$$A = \frac{1}{2}\left|\det\begin{bmatrix} 7 & 3 & 1 \\ 6 & -1 & 1 \\ 1 & -3 & 1 \end{bmatrix}\right|$$

$$= \frac{1}{2}\left|7\begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix} - 3\begin{vmatrix} 6 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 6 & -1 \\ 1 & -3 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|7[(-1)(1) - (-3)(1)] - 3[6(1) - 1(1)] + 6(-3) - 1(-1)\right|$$

$$= \frac{1}{2}\left|7(2) - 3(5) - 18 + 1\right|$$

$$= \frac{1}{2}\left|-18\right|$$

$$= \frac{1}{2}(18)$$

$$= 9$$

**Given $A$ and $AB$, find $B$.**

49. $A = \begin{bmatrix} 8 & -4 \\ 3 & 6 \end{bmatrix}, AB = \begin{bmatrix} 36 & 48 \\ -24 & 48 \end{bmatrix}$

*SOLUTION:*

Let $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

Set up two systems of equations.

$$AB = A \cdot B$$

$$\begin{bmatrix} 36 & 48 \\ -24 & 48 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$8w - 4y = 36$$
$$3w + 6y = -24$$
$$8x - 4z = 48$$
$$3x + 6z = 48$$

Solve for $w, y, x$, and $z$.

$$(3)(8w - 4y = 36)$$
$$(+)(-8)(3w + 6y = -24)$$

$$24w - 12y = 108$$
$$(+) -24w - 48y = 192$$
$$-60y = 300$$
$$y = -5$$
$$8w - 4(-5) = 36$$
$$8w + 20 = 36$$
$$8w = 16$$
$$w = 2$$

Solve the next system.

$$(3)(8x - 4z = 48)$$
$$(+)(-8)(3x + 6z = 48)$$

$$24x - 12z = 144$$
$$(+) -24x - 48z = -384$$
$$-60z = -240$$
$$z = 4$$
$$8x - 4(4) = 48$$
$$8x - 16 = 48$$
$$8x = 64$$
$$x = 8$$

$$B = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -5 & 4 \end{bmatrix}$$

50. $A = \begin{bmatrix} 5 & 0 & 1 \\ 2 & -3 & 2 \\ 1 & -1 & 4 \end{bmatrix}, AB = \begin{bmatrix} 1 & 4 \\ -16 & -6 \\ -2 & -5 \end{bmatrix}$

*SOLUTION:*

Matrix $B$ must be a $3 \times 2$ matrix in order for $AB$ to exist.

Let $B = \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$.

Set up two systems of equations.

$$AB = A \cdot B$$

$$\begin{bmatrix} 1 & 4 \\ -16 & -6 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 1 \\ 2 & -3 & 2 \\ 1 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

$$\begin{aligned} 5u + 0w + y &= 1 \\ 2u - 3w + 2y &= -16 \\ u - w + 4y &= -2 \\ 5v + 0x + z &= 4 \\ 2v - 3x + 2z &= -6 \\ v - x + 4z &= -5 \end{aligned}$$

Solve the system.

$$\begin{aligned} 2u - 3w + 2y &= -16 \\ (+)(-3)(u - w + 4y &= -2) \end{aligned}$$

$$\begin{aligned} 2u - 3w + 2y &= -16 \\ (+)\ -3u + 3w - 12y &= 6 \\ -u - 10y &= -10 \end{aligned}$$

$$\begin{aligned} 5u + y &= 1 \\ y &= 1 - 5u \\ -u - 10y &= -10 \\ -u - 10(1 - 5u) &= -10 \\ -u - 10 + 50u &= -10 \\ 49u &= 0 \\ u &= 0 \\ y &= 1 - 5u \\ y &= 1 - 5(0) \\ y &= 1 \end{aligned}$$

$$\begin{aligned} u - w + 4y &= -2 \\ 0 - w + 4(1) &= -2 \\ -w &= -6 \\ w &= 6 \end{aligned}$$

Now solve the next system.

$$\begin{aligned} 2v - 3x + 2z &= -6 \\ (+)(-3)(v - x + 4z &= -5) \end{aligned}$$

$$\begin{aligned} 2v - 3x + 2z &= -6 \\ (+)\ -3v + 3x - 12z &= 15 \\ -v - 10z &= 9 \end{aligned}$$

$$\begin{aligned} 5v + z &= 4 \\ z &= 4 - 5v \\ -v - 10z &= 9 \\ -v - 10(4 - 5v) &= 9 \\ -v - 40 + 50v &= 9 \\ 49v &= 49 \\ v &= 1 \\ z &= 4 - 5v \\ z &= 4 - 5(1) \\ z &= -1 \end{aligned}$$

$$\begin{aligned} v - x + 4z &= -5 \\ 1 - x + 4(-1) &= -5 \\ -x - 3 &= -5 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

$$B = \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 2 \\ 1 & -1 \end{bmatrix}$$

## 6-2 Matrix Multiplication, Inverses and Determinants

**Find $x$ and $y$.**

51. $A = \begin{bmatrix} 2x & -y \\ -3y & 5x \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, and $AB = \begin{bmatrix} -2 \\ 31 \end{bmatrix}$

*SOLUTION:*

$$AB = A \cdot B$$

$$\begin{bmatrix} -2 \\ 31 \end{bmatrix} = \begin{bmatrix} 2x & -y \\ -3y & 5x \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$-2 = 2x(4) + (-y)(-2)$$
$$-2 = 8x + 2y$$
$$-1 = 4x + y$$
$$-4x - 1 = y$$

$$31 = -3y(4) + 5x(-2)$$
$$31 = -12y - 10x$$
$$31 = -12(-4x - 1) - 10x$$
$$31 = 48x + 12 - 10x$$
$$19 = 38x$$
$$\frac{1}{2} = x$$

$$-4\left(\frac{1}{2}\right) - 1 = y$$
$$-2 - 1 = y$$
$$-3 = y$$

**Find the determinant of each matrix.**

52. $\begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{bmatrix}$

*SOLUTION:*

$$\begin{vmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{vmatrix} = r\begin{vmatrix} s & 0 \\ 0 & t \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & t \end{vmatrix} + 0\begin{vmatrix} 0 & s \\ 0 & 0 \end{vmatrix}$$

$$= r[st - 0(0)] - 0 + 0$$
$$= rst$$

## 6-2 Matrix Multiplication, Inverses and Determinants

53. $\begin{bmatrix} c & c & c \\ 0 & c & c \\ 0 & 0 & c \end{bmatrix}$

*SOLUTION:*

$$\begin{vmatrix} c & c & c \\ 0 & c & c \\ 0 & 0 & c \end{vmatrix} = c\begin{vmatrix} c & c \\ 0 & c \end{vmatrix} - c\begin{vmatrix} 0 & c \\ 0 & c \end{vmatrix} + c\begin{vmatrix} 0 & c \\ 0 & 0 \end{vmatrix}$$

$$= c(c^2 - 0) - c[0c - 0c] + c[0 - 0c]$$

$$= c^3 - c(0) + c(0)$$

$$= c^3$$

54. **FUNDRAISER** Hawthorne High School had a fundraiser selling popcorn. The school bought the four flavors of popcorn by the case. The prices paid for the different types of popcorn and the selling prices are shown.

| Class | Cases of Popcorn | | | |
|---|---|---|---|---|
| | Butter | Kettle | Cheese | Caramel |
| freshman | 152 | 80 | 125 | 136 |
| sophomore | 112 | 92 | 112 | 150 |
| junior | 176 | 90 | 118 | 122 |
| senior | 140 | 102 | 106 | 143 |

| Flavor | Price Paid per Case ($) | Selling Price per Case ($) | Profit per Case |
|---|---|---|---|
| butter | 18.90 | 42.00 | |
| kettle | 21.00 | 45.00 | |
| cheese | 23.10 | 48.00 | |
| caramel | 25.20 | 51.00 | |

**a.** Complete the last column of the second table.
**b.** Which class had the highest total sales?
**c.** How much more profit did the seniors earn than the sophomores?

*SOLUTION:*

**a.** The profit is the selling price minus the price paid, or the 2nd column minus the 1st column in the 2nd table.

$42.00 - 18.90 = 23.10$

$45.00 - 21.00 = 24.00$

$48.00 - 23.10 = 24.90$

$51.00 - 25.20 = 25.80$

**b.** The total sales per class can be found by finding the sum of each row in the 1st table.

$152 + 80 + 125 + 136 = 493$

$112 + 92 + 112 + 150 = 466$

$176 + 90 + 118 + 122 = 506$

$140 + 102 + 106 + 143 = 491$

The junior class had the highest total sales.

**c.** To determine the profit per class, multiply the 2nd and 4th rows of the 1st table by the last column of the 2nd table.

Sophomores:

$$\begin{bmatrix}112 & 92 & 112 & 150\end{bmatrix} \cdot \begin{bmatrix}23.10\\24.00\\24.90\\25.80\end{bmatrix}$$

$$= \begin{bmatrix}112(23.1) + 92(24) + 112(24.9) + 150(25.8)\end{bmatrix}$$

$$= \begin{bmatrix}11,454\end{bmatrix}$$

Seniors:

$$\begin{bmatrix}140 & 102 & 106 & 143\end{bmatrix} \cdot \begin{bmatrix}23.10\\24.00\\24.90\\25.80\end{bmatrix}$$

$$= \begin{bmatrix}140(23.1) + 102(24) + 106(24.9) + 143(25.8)\end{bmatrix}$$

$$= \begin{bmatrix}12,010.8\end{bmatrix}$$

$12{,}010.8 - 11{,}454 = 556.80$

## 6-2 Matrix Multiplication, Inverses and Determinants

55. Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$.

**a.** Find $A^2$, $A^3$, and $A^4$. Then use the pattern to write a matrix for $A^n$.

**b.** Find $B^2$, $B^3$, $B^4$, and $B^5$. Then use the pattern to write a general formula for $B^n$.

**c.** Find $C^2$, $C^3$, $C^4$, $C^5$, … until you notice a pattern. Then use the pattern to write a general formula for $C^n$.

**d.** Use the formula that you wrote in part **c** to find $C^7$.

*SOLUTION:*

**a.**

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = AA$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+1(0) & 1(1)+1(1) \\ 0(1)+1(0) & 0(1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2A$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+2(0) & 1(1)+2(1) \\ 0(1)+1(0) & 0(1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^4 = A^3A$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+3(0) & 1(1)+3(1) \\ 0(1)+1(0) & 0(1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

Assuming the pattern continues, $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

**b.**

$$B=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$B^2 = BB$$

$$=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1(1)+1(1) & 1(1)+1(1)\\ 1(1)+1(1) & 1(1)+1(1)\end{bmatrix}$$

$$=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$$

$$B^3 = B^2B$$

$$=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2(1)+2(1) & 2(1)+2(1)\\ 2(1)+2(1) & 2(1)+2(1)\end{bmatrix}$$

$$=\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix}$$

$$B^4 = B^3B$$

$$=\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}4(1)+4(1) & 4(1)+4(1)\\ 4(1)+4(1) & 4(1)+4(1)\end{bmatrix}$$

$$=\begin{bmatrix}8 & 8\\ 8 & 8\end{bmatrix}$$

$$B^5 = B^4B$$

$$=\begin{bmatrix}8 & 8\\ 8 & 8\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}8(1)+8(1) & 8(1)+8(1)\\ 8(1)+8(1) & 8(1)+8(1)\end{bmatrix}$$

$$=\begin{bmatrix}16 & 16\\ 16 & 16\end{bmatrix}$$

It appears that each entry is 2 to the $n-1$ power.

Assuming the pattern continues, $B^n=\begin{bmatrix}2^{n-1} & 2^{n-1}\\ 2^{n-1} & 2^{n-1}\end{bmatrix}$.

**c.**

$$C = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$C^2 = CC$$

$$= \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2)+0(2) & 2(0)+0(2) \\ 2(2)+2(2) & 2(0)+2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 8 & 4 \end{bmatrix}$$

$$C^3 = C^2C$$

$$= \begin{bmatrix} 4 & 0 \\ 8 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4(2)+0(2) & 4(0)+0(2) \\ 8(2)+4(2) & 8(0)+4(2) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 24 & 8 \end{bmatrix}$$

$$C^4 = C^3C$$

$$= \begin{bmatrix} 8 & 0 \\ 24 & 8 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8(2)+0(2) & 8(0)+0(2) \\ 24(2)+8(2) & 24(0)+8(2) \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 \\ 64 & 16 \end{bmatrix}$$

$$C^5 = C^4C$$

$$= \begin{bmatrix} 16 & 0 \\ 64 & 16 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 16(2)+0(2) & 16(0)+0(2) \\ 64(2)+16(2) & 64(0)+16(2) \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 0 \\ 160 & 32 \end{bmatrix}$$

It appears that $a_{11}$ and $a_{22}$ are 2 to the $n$th power. The $a_{21}$ term looks like $n \cdot 2^n$. Assuming the pattern continues, $C^n = \begin{bmatrix} 2^n & 0 \\ n2^n & 2^n \end{bmatrix}$.

**d.**

$$C^n = \begin{bmatrix} 2^n & 0 \\ n2^n & 2^n \end{bmatrix}$$

$$C^7 = \begin{bmatrix} 2^7 & 0 \\ 7 \cdot 2^7 & 2^7 \end{bmatrix}$$

$$= \begin{bmatrix} 128 & 0 \\ 896 & 128 \end{bmatrix}$$

56. **HORSES** The owner of each horse stable buys bales of hay and bags of feed each month. In May, hay cost $2.50 per bale and feed cost $7.95 per bag. In June, the cost per bale of hay was $3.00 and the cost per bag of feed was $6.75.

| Stables | Bales of Hay | Bags of Feed |
|---|---|---|
| Galloping Hills | 45 | 5 |
| Amazing Acres | 75 | 9 |
| Fairwind Farms | 135 | 16 |
| Saddle-Up Stables | 90 | 11 |

**a.** Write a matrix $X$ to represent the bales of hay $i$ and bags of feed $j$ that are bought monthly by each stable.
**b.** Write a matrix $Y$ to represent the costs per bale of hay and bags of feed for May and June.
**c.** Find the product $YX$ and label its rows and columns.
**d.** How much more were the total costs in June for Fairwind Farms than the total costs in May for Galloping Hills?

*SOLUTION:*

**a**. Each column will represent a stable.

$$X = \begin{bmatrix} i & 45 & 75 & 135 & 90 \\ j & 5 & 9 & 16 & 11 \end{bmatrix}$$

**b.** Let May be the 1st row and June be the 2nd row.

$$Y = \begin{bmatrix} 2.50 & 7.95 \\ 3.00 & 6.75 \end{bmatrix}$$

**c.**

$$YX = \begin{bmatrix} 2.50 & 7.95 \\ 3.00 & 6.75 \end{bmatrix} \cdot \begin{bmatrix} 45 & 75 & 135 & 90 \\ 5 & 9 & 16 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5(45) + 7.95(5) & 2.5(75) + 7.95(9) & 2.5(135) + 7.95(16) & 2.5(90) + 7.95(11) \\ 3(45) + 6.75(5) & 3(75) + 6.75(9) & 3(135) + 6.75(16) & 3(90) + 6.75(11) \end{bmatrix}$$

**d.**

$$= \begin{bmatrix} 152.25 & 259.05 & 464.7 & 312.45 \\ 168.75 & 285.75 & 513 & 344.25 \end{bmatrix}$$

Fairwind Farms (June):
135(3.00) + 16(6.75) = $513

Galloping Hills (May):
45(2.50) + 5(7.95) = $152.25

513 − 152.25 = $360.75

# 6-2 Matrix Multiplication, Inverses and Determinants

**Evaluate each expression.**

$$A=\begin{bmatrix}4 & 1 & -3\\ 0 & 2 & 8\end{bmatrix} \qquad C=\begin{bmatrix}-1 & 9 & -6\\ 7 & 5 & 0\end{bmatrix}$$

$$B=\begin{bmatrix}2 & 1\\ 0 & 1\\ 3 & -2\end{bmatrix} \qquad D=\begin{bmatrix}7 & 2\\ -4 & -1\end{bmatrix}$$

57. $BD + B$

*SOLUTION:*

$$\begin{aligned}BD+B&=\begin{bmatrix}2 & 1\\ 0 & 1\\ 3 & -2\end{bmatrix}\cdot\begin{bmatrix}7 & 2\\ -4 & -1\end{bmatrix}+\begin{bmatrix}2 & 1\\ 0 & 1\\ 3 & -2\end{bmatrix}\\ &=\begin{bmatrix}2(7)+1(-4) & 2(2)+1(-1)\\ 0(7)+1(-4) & 0(2)+1(-1)\\ 3(7)+(-2)(-4) & 3(2)+(-2)(-1)\end{bmatrix}+\begin{bmatrix}2 & 1\\ 0 & 1\\ 3 & -2\end{bmatrix}\\ &=\begin{bmatrix}10 & 3\\ -4 & -1\\ 29 & 8\end{bmatrix}+\begin{bmatrix}2 & 1\\ 0 & 1\\ 3 & -2\end{bmatrix}\\ &=\begin{bmatrix}12 & 4\\ -4 & 0\\ 32 & 6\end{bmatrix}\end{aligned}$$

58. $DC - A$

*SOLUTION:*

$$\begin{aligned}DC-A&=\begin{bmatrix}7 & 2\\ -4 & -1\end{bmatrix}\cdot\begin{bmatrix}-1 & 9 & -6\\ 7 & 5 & 0\end{bmatrix}-\begin{bmatrix}4 & 1 & -3\\ 0 & 2 & 8\end{bmatrix}\\ &=\begin{bmatrix}7(-1)+2(7) & 7(9)+2(5) & 7(-6)+2(0)\\ -4(-1)-1(7) & -4(9)-1(5) & -4(-6)-1(0)\end{bmatrix}-\begin{bmatrix}4 & 1 & -3\\ 0 & 2 & 8\end{bmatrix}\\ &=\begin{bmatrix}7 & 73 & -42\\ -3 & -41 & 24\end{bmatrix}-\begin{bmatrix}4 & 1 & -3\\ 0 & 2 & 8\end{bmatrix}\\ &=\begin{bmatrix}3 & 72 & -39\\ -3 & -43 & 16\end{bmatrix}\end{aligned}$$

## 6-2 Matrix Multiplication, Inverses and Determinants

59. $B(A + C)$

*SOLUTION:*

$$B(A+C)=\begin{bmatrix}2 & 1\\ 0 & 1\\ 3 & -2\end{bmatrix}\cdot\left(\begin{bmatrix}4 & 1 & -3\\ 0 & 2 & 8\end{bmatrix}+\begin{bmatrix}-1 & 9 & -6\\ 7 & 5 & 0\end{bmatrix}\right)$$

$$=\begin{bmatrix}2 & 1\\ 0 & 1\\ 3 & -2\end{bmatrix}\cdot\begin{bmatrix}3 & 10 & -9\\ 7 & 7 & 8\end{bmatrix}$$

$$=\begin{bmatrix}2(3)+1(7) & 2(10)+1(7) & 2(-9)+1(8)\\ 0(3)+1(7) & 0(10)+1(7) & 0(-9)+1(8)\\ 3(3)-2(7) & 3(10)-2(7) & 3(-9)-2(8)\end{bmatrix}$$

$$=\begin{bmatrix}13 & 27 & -10\\ 7 & 7 & 8\\ -5 & 16 & -43\end{bmatrix}$$

60. $AB + CB$

*SOLUTION:*

$$AB=\begin{bmatrix}4 & 1 & -3\\ 0 & 2 & 8\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 1\\ 3 & -2\end{bmatrix}$$

$$=\begin{bmatrix}4(2)+1(0)-3(3) & 4(1)+1(1)-3(-2)\\ 0(2)+2(0)+8(3) & 0(1)+2(1)+8(-2)\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 11\\ 24 & -14\end{bmatrix}$$

$$CB=\begin{bmatrix}-1 & 9 & -6\\ 7 & 5 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 1\\ 3 & -2\end{bmatrix}$$

$$=\begin{bmatrix}-1(2)+9(0)-6(3) & -1(1)+9(1)-6(-2)\\ 7(2)+5(0)+0(3) & 7(1)+5(1)+0(-2)\end{bmatrix}$$

$$=\begin{bmatrix}-20 & 20\\ 14 & 12\end{bmatrix}$$

$$AB+CB=\begin{bmatrix}-1 & 11\\ 24 & -14\end{bmatrix}+\begin{bmatrix}-20 & 20\\ 14 & 12\end{bmatrix}$$

$$=\begin{bmatrix}-21 & 31\\ 38 & -2\end{bmatrix}$$

## 6-2 Matrix Multiplication, Inverses and Determinants

**Solve each equation for $X$, if possible.**

$$A=\begin{bmatrix}1 & 7\\ 2 & 0\end{bmatrix} \qquad C=\begin{bmatrix}5 & 3\\ -6 & -2\end{bmatrix}$$

$$B=\begin{bmatrix}8 & 4\\ 1 & -3\\ 6 & 7\end{bmatrix} \qquad D=\begin{bmatrix}2 & 1\\ 0 & 1\\ 3 & -2\end{bmatrix}$$

61. $A + C = 2X$

*SOLUTION:*

$$A+C=2X$$
$$0.5(A+C)=X$$
$$X=0.5(A+C)$$
$$X=0.5\left(\begin{bmatrix}1 & 7\\ 2 & 0\end{bmatrix}+\begin{bmatrix}5 & 3\\ -6 & -2\end{bmatrix}\right)$$
$$X=0.5\begin{bmatrix}6 & 10\\ -4 & -2\end{bmatrix}$$
$$X=\begin{bmatrix}3 & 5\\ -2 & -1\end{bmatrix}$$

62. $4X + A = C$

*SOLUTION:*

$$4X+A=C$$
$$4X=C-A$$
$$X=\frac{1}{4}(C-A)$$
$$X=\frac{1}{4}\left(\begin{bmatrix}5 & 3\\ -6 & -2\end{bmatrix}-\begin{bmatrix}1 & 7\\ 2 & 0\end{bmatrix}\right)$$
$$X=\frac{1}{4}\begin{bmatrix}4 & -4\\ -8 & -2\end{bmatrix}$$
$$X=\begin{bmatrix}1 & -1\\ -2 & -\frac{1}{2}\end{bmatrix}$$

## 6-2 Matrix Multiplication, Inverses and Determinants

63. $B - 3X = D$

*SOLUTION:*

$$B - 3X = D$$
$$-3X = D - B$$
$$X = \frac{1}{3}(B - D)$$
$$X = \frac{1}{3}\left(\begin{bmatrix} 8 & 4 \\ 1 & -3 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 3 & -2 \end{bmatrix}\right)$$
$$X = \frac{1}{3}\begin{bmatrix} 6 & 3 \\ 1 & -4 \\ 3 & 9 \end{bmatrix}$$
$$X = \begin{bmatrix} 2 & 1 \\ \frac{1}{3} & -\frac{4}{3} \\ 1 & 3 \end{bmatrix}$$

64. $DA = 7X$

*SOLUTION:*

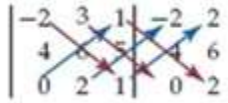
$$DA = 7X$$
$$7X = DA$$
$$X = \frac{1}{7}DA$$
$$X = \frac{1}{7}\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 7 \\ 2 & 0 \end{bmatrix}\right)$$
$$X = \frac{1}{7}\left(\begin{bmatrix} 2(1)+1(2) & 2(7)+1(0) \\ 0(1)+1(2) & 0(7)+1(0) \\ 3(1)-2(2) & 3(7)-2(0) \end{bmatrix}\right)$$
$$X = \frac{1}{7}\begin{bmatrix} 4 & 14 \\ 2 & 0 \\ -1 & 21 \end{bmatrix}$$
$$X = \begin{bmatrix} \frac{4}{7} & 2 \\ \frac{2}{7} & 0 \\ -\frac{1}{7} & 3 \end{bmatrix}$$

65. **3 × 3 DETERMINANTS** In this problem, you will investigate an alternative method for calculating the determinant of a 3 × 3 matrix.

## 6-2 Matrix Multiplication, Inverses and Determinants

**a.** Calculate $\det(A) = \begin{vmatrix} -2 & 3 & 1 \\ 4 & 6 & 5 \\ 0 & 2 & 1 \end{vmatrix}$ using the method shown in this lesson.

**b.** Adjoin the first two columns to the right of $\det(A)$ as shown. Then find the difference between the sum of the products along the indicated downward diagonals and the sum of the products along the indicated upward diagonals.




**c.** Compare your answers in parts **a** and **b.**

**d.** Show that, in general, the determinant of a $3 \times 3$ matrix can be found using the procedure described above.

**e.** Does this method work for a $4 \times 4$ matrix? If so, explain your reasoning. If not, provide a counterexample.

*SOLUTION:*

**a.**

$$\begin{vmatrix} -2 & 3 & 1 \\ 4 & 6 & 5 \\ 0 & 2 & 1 \end{vmatrix} = -2\begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 6 \\ 0 & 2 \end{vmatrix}$$

$$= -2[6(1) - 2(5)] - 3[4(1) - 0(5)] + 1[4(2) - 0(6)]$$
$$= -2(-4) - 3(4) + 8$$
$$= 8 - 12 + 8$$
$$= 4$$

**b.**

$$\begin{vmatrix} -2 & 3 & 1 \\ 4 & 6 & 5 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} -2 & 3 \\ 4 & 6 \\ 0 & 2 \end{matrix}$$

$$-2(6)(1) + 3(5)(0) + 1(4)(2) - [1(6)(0) - 2(5)(2) + 3(4)(1)]$$
$$= -12 + 0 + 8 - (0 - 20 + 12)$$
$$= -4 - (-8)$$
$$= 4$$

**c**. They are the same.

**d**. Let *a*-*i* represent each element in the matrix. Multiply the values out and show that the output is equal to the method taught in the text.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

$$= (aei + bfg + cdh) - (gec + hfa + idb)$$
$$= aei + bfg + cdh - gec - hfa - idb$$
$$= a(ei - hf) + b(fg - id) + c(dh - ge)$$
$$= a(ei - hf) - b(id - fg) + c(dh - ge)$$
$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$
$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

**e**. This method does not work for a 4 × 4 matrix.

If $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{bmatrix}$, det($A$) = –60.

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 \\ 2 & 3 & 1 & 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 1 & 3 & 1 & 2 \end{vmatrix}$$

$$= 1(1)(1)(1) + 2(2)(1)(3) + 3(3)(2)(1) + 0(0)(3)(2) - [3(3)(2)(0) + 1(1)(3)(1) + 2(1)(0)(2) + 1(2)(1)(3)]$$

$$= 1 + 12 + 18 + 0 - (0 + 3 + 0 + 6)$$

$$= 11$$

Since 11 ≠ –60, this method does not work for a 4 × 4 matrix.

66. **PROOF** Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$. Use the matrix equation $AA^{-1} = I_2$ to derive the formula for the inverse of a 2 × 2 matrix.

*SOLUTION:*

First, find $AA^{-1}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ cx_1 + dx_2 & cy_1 + dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Set up a system of equations for each element in the resulting matrix equality.

$ax_1 + bx_2 = 1$

$ay_1 + by_2 = 0$

$cx_1 + dx_2 = 0$

$cy_1 + dy_2 = 1$

Solving each system of equations, $x_1 = \frac{d}{adcb}$, $x_2 = \frac{c}{adcb}$, $y_1 = \frac{b}{adcb}$, and $y_2 = \frac{a}{adcb}$.

Substitute these values back into the original matrix.

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-cb} & \frac{-c}{ad-cb} \\ \frac{-b}{ad-cb} & \frac{a}{ad-cb} \end{bmatrix} = \frac{1}{adcb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

67. **PROOF** Write a paragraph proof to show that if a square matrix has an inverse, that inverse is unique. (*Hint*: Assume that a square matrix $A$ has two different inverses, $B$ and $C$. Then show that $B = C$.)

*SOLUTION:*

Suppose $B$ and $C$ are both inverses of square matrix $A$. Then by definition, $AB = BA = I$ and $AC = CA = I$. Then $B(AC) = BI = B$ and $(BA)C = IC = C$. But $B(AC) = (BA)C$ by the Associative Property of Matrix Multiplication. Therefore, $B = C$.

68. **MULTIPLE REPRESENTATIONS** In this problem, you will explore square matrices. A square matrix is called main diagonal are 0, and *lower triangular* if all elements above the main diagonal are 0. If all elements not on the di called *diagonal*. In this problem you will investigate the determinants of 3 × 3 upper triangular, lower triangular, and
**a. ANALYTICAL** Write one upper triangular, one lower triangular, and one diagonal 2 × 2 matrix. Then find the de
**b. ANALYTICAL** Write one upper triangular, one lower triangular, and one diagonal 3 × 3 matrix. Then find the de
**c. VERBAL** Make a conjecture as to the value of the determinant of any 3 × 3 upper triangular, lower triangular, or
**d. ANALYTICAL** Find the inverse of each of the diagonal matrices you wrote in part **a** and **b**.
**e. VERBAL** Make a conjecture about the inverse of any 3 × 3 diagonal matrix.

*SOLUTION:*

**a**. Sample answer:

upper triangular: $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$;

lower triangular: $B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$;

diagonal: $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$;

$$\begin{aligned} |A| &= ad - bc \\ &= 1(3) - 0(2) \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

$$\begin{aligned} |B| &= ad - bc \\ &= 1(3) - 2(0) \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

$$\begin{aligned} |C| &= ad - bc \\ &= 1(2) - 0(0) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

**b**. Sample answer:

upper triangular: $D = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$;

lower triangular: $E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$;

diagonal: $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$;

$$|D| = 1\begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} - 2\begin{vmatrix} 0 & 5 \\ 0 & 6 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix}$$
$$= 1[4(6) - 0(5)] - 2[(0)(6) - (0)(5)] + 3[0(0) - 0(4)]$$
$$= 1(24) - 2(0) + 3(0)$$
$$= 24$$

$$|E| = 1\begin{vmatrix} 3 & 0 \\ 5 & 6 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 \\ 4 & 6 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$$
$$= 1[3(6) - 0(5)] - 2[(2)(6) - (0)(4)] + 0[2(5) - 4(3)]$$
$$= 1(18) - 2(12) + 0$$
$$= -6$$

$$|F| = 1\begin{vmatrix} 2 & \\ 0 & \end{vmatrix}$$
$$= 1[2($$
$$= 1(6)$$
$$= 6$$

**c.** The formula used to find the determinant is called *expansion by cofactors*. For the purposes of this answer, in th considered a cofactor. Looking at the pattern, it appears that for *E* and *F*, the first cofactor is all that is needed. Noti by 0.

Now consider the upper triangular matrix $D = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$. If we interchange rows 1 and 3, we get $\begin{bmatrix} 0 & 0 & 6 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$. No

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

If $A$ is lower triangular or diagonal, the value of the determinant is $a\begin{vmatrix} e & f \\ h & i \end{vmatrix}$. If $A$ is upper triangular, the value of the

**d.** $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$;

$$C^{-1} = \frac{1}{1(2) - 0(0)}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$;

```
[F]^-1▸Frac
   [[1  0   0  ]
    [0  1/2 0  ]
    [0  0   1/3]]
```

**e.** The pattern appears to indicate that the inverse matrix will have the same 0s as the original matrix. The nonzero e actually be the reciprocals of the original elements.

Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. The inverse of $A$ will also be a diagonal matrix of the form $\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$.

69. **CHALLENGE** Given $A$ and $AB$, find $B$. $A = \begin{bmatrix} 3 & -1 & 5 \\ 1 & 0 & 2 \\ 6 & 4 & 1 \end{bmatrix}$, $AB = \begin{bmatrix} 14 & 6 & 33 \\ 4 & 4 & 13 \\ 1 & 18 & 12 \end{bmatrix}$

*SOLUTION:*

Let $B = \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}$.

Set up three systems of equations.

$$AB = A \cdot B$$

$$\begin{bmatrix} 14 & 6 & 33 \\ 4 & 4 & 13 \\ 1 & 18 & 12 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 5 \\ 1 & 0 & 2 \\ 6 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}$$

$$3r - u + 5x = 14$$
$$r + 0u + 2x = 4$$
$$6r + 4u + x = 1$$

$$3s - v + 5y = 6$$
$$s + 0v + 2y = 4$$
$$6s + 4v + y = 18$$

$$3t - w + 5z = 33$$
$$t + 0w + 2z = 13$$
$$6t + 4w + z = 12$$

Solve the first system.

$$r + 2x = 4$$
$$r = 4 - 2x$$

$$(4)(3r - u + 5x = 14)$$
$$(+)6r + 4u + x = 1$$

$$\begin{array}{r} 12r - 4u + 20x = 56 \\ (+)6r + 4u + x = 1 \\ \hline 18r + 21x = 57 \end{array}$$

$$\begin{aligned} 18(4 - 2x) + 21x &= 57 \\ 72 - 36x + 21x &= 57 \\ -15x &= -15 \\ x &= 1 \\ r &= 4 - 2(1) \\ r &= 2 \end{aligned}$$

$$\begin{aligned} 3r - u + 5x &= 14 \\ 3(2) - u + 5(1) &= 14 \\ 6 - u + 5 &= 14 \\ -u &= 3 \\ u &= -3 \end{aligned}$$

Solve the next system.

$$s + 2y = 4$$
$$s = 4 - 2y$$

$$(4)(3s - v + 5y = 6)$$
$$(+)6s + 4v + y = 18$$

$$\begin{array}{r} 12s - 4v + 20y = 24 \\ (+)6s + 4v + y = 18 \\ \hline 18s + 21y = 42 \end{array}$$

$$\begin{aligned} 18s + 21y &= 42 \\ 18(4 - 2y) + 21y &= 42 \\ 72 - 36y + 21y &= 42 \\ -15y &= -30 \\ y &= 2 \\ s &= 4 - 2(2) \\ s &= 0 \end{aligned}$$

$$\begin{aligned} 3s - v + 5y &= 6 \\ 3(0) - v + 5(2) &= 6 \\ -v + 10 &= 6 \\ v &= -4 \\ v &= 4 \end{aligned}$$

Solve the last system.

$$\begin{aligned} t + 2z &= 13 \\ t &= 13 - 2z \end{aligned}$$

$$\begin{aligned} 4(3t - w + 5z &= 33) \\ (+)6t + 4w + z &= 12 \end{aligned}$$

$$\begin{aligned} 12t - 4w + 20z &= 132 \\ (+)6t + 4w + z &= 12 \\ \hline 18t + 21z &= 144 \end{aligned}$$

$$\begin{aligned} 18t + 21z &= 144 \\ 18(13 - 2z) + 21z &= 144 \\ 234 - 36z + 21z &= 144 \\ -15z &= -90 \\ z &= 6 \\ t &= 13 - 2(6) \\ t &= 1 \end{aligned}$$

$$\begin{aligned} 3t - w + 5z &= 33 \\ 3(1) - w + 5(6) &= 33 \\ 3 - w + 30 &= 33 \\ -w &= 0 \\ w &= 0 \end{aligned}$$

$$B = \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & 0 \\ 1 & 2 & 6 \end{bmatrix}$$

70. **REASONING** Explain why a nonsquare matrix cannot have an inverse.

*SOLUTION:*

One way to prove this statement is to assume that everything is true and find a contradiction.

Sample answer:

First, assume that $A$ is a nonsquare matrix. If this is the case, then $A$ has dimensions $m \times n$ where $m \neq n$. Next, assume that $A$ has an inverse. If this is the case, then $AA^{-1} = A^{-1}A = I$.

However, we have a contradiction. If $A$ has dimensions $m \times n$ where $m \neq n$, then $AA^{-1} = I_m$ and $A^{-1}A = I_n$. Both of these multiplications result in $I$, however, they are in different dimensions. Thus, since $I_m \neq I_n, AA^{-1} \neq A^{-1}A$.

Therefore, $A$ cannot be nonsquare *and* also have an inverse.

71. **OPEN ENDED** Write two matrices $A$ and $B$ such that $AB = BA$, but neither $A$ nor $B$ is the identity matrix.

*SOLUTION:*

While other options are also true, the simplest way to make $AB = BA$ is to have the identical elements in $A$ and identical elements in $B$. This way, each element of $AB$ or $BA$ will also be identical.

Sample answer: $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$

**PROOF Show that each property is true for all 2 × 2 matrices.**

72. Right Distributive Property

*SOLUTION:*

$$(A + B)C = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right)\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}+b_{11})c_{11}+(a_{12}+b_{12})c_{21} & (a_{11}+b_{11})c_{12}+(a_{12}+b_{12})c_{22} \\ (a_{21}+b_{21})c_{11}+(a_{21}+b_{21})c_{21} & (a_{21}+b_{21})c_{12}+(a_{21}+b_{21})c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_{11}+b_{11}c_{11}+a_{12}c_{21}+b_{12}c_{21} & a_{11}c_{12}+b_{11}c_{12}+a_{12}c_{22}+b_{12}c_{22} \\ a_{21}c_{11}+b_{21}c_{11}+a_{21}c_{21}+b_{21}c_{21} & a_{21}c_{12}+b_{21}c_{12}+a_{21}c_{22}+b_{21}c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_{11}+a_{12}c_{21}+b_{11}c_{11}+b_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22}+b_{11}c_{12}+b_{12}c_{22} \\ a_{21}c_{11}+a_{21}c_{21}+b_{21}c_{11}+b_{21}c_{21} & a_{21}c_{12}+a_{21}c_{22}+b_{21}c_{12}+b_{21}c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{21}c_{21} & a_{21}c_{12}+a_{21}c_{22} \end{bmatrix} + \begin{bmatrix} b_{11}c_{11}+b_{12}c_{21} & b_{11}c_{12}+b_{12}c_{22} \\ b_{21}c_{11}+b_{21}c_{21} & b_{21}c_{12}+b_{21}c_{22} \end{bmatrix}$$

$$= AC + BC$$

73. Left Distributive Property

*SOLUTION:*

$$
\begin{aligned}
C(A+B) &= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) \\
&= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \\
&= \begin{bmatrix} c_{11}(a_{11}+b_{11})+c_{12}(a_{21}+b_{21}) & c_{11}(a_{12}+b_{12})+c_{12}(a_{22}+b_{22}) \\ c_{21}(a_{11}+b_{11})+c_{22}(a_{21}+b_{21}) & c_{21}(a_{12}+b_{12})+c_{22}(a_{22}+b_{22}) \end{bmatrix} \\
&= \begin{bmatrix} c_{11}a_{11}+c_{11}b_{11}+c_{12}a_{21}+c_{12}b_{21} & c_{11}a_{12}+c_{11}b_{12}+c_{12}a_{22}+c_{12}b_{22} \\ c_{21}a_{11}+c_{21}b_{11}+c_{22}a_{21}+c_{22}b_{21} & c_{21}a_{12}+c_{21}b_{12}+c_{22}a_{22}+c_{22}b_{22} \end{bmatrix} \\
&= \begin{bmatrix} c_{11}a_{11}+c_{12}a_{21}+c_{11}b_{11}+c_{12}b_{21} & c_{11}a_{12}+c_{12}a_{22}+c_{11}b_{12}+c_{12}b_{22} \\ c_{21}a_{11}+c_{22}a_{21}+c_{21}b_{11}+c_{22}b_{21} & c_{21}a_{12}+c_{22}a_{22}+c_{21}b_{12}+c_{22}b_{22} \end{bmatrix} \\
&= \begin{bmatrix} c_{11}a_{11}+c_{12}a_{21} & c_{11}a_{12}+c_{12}a_{22} \\ c_{21}a_{11}+c_{22}a_{21} & c_{21}a_{12}+c_{22}a_{22} \end{bmatrix}+\begin{bmatrix} c_{11}b_{11}+c_{12}b_{21} & c_{11}b_{12}+c_{12}b_{22} \\ c_{21}b_{11}+c_{22}b_{21} & c_{21}b_{12}+c_{22}b_{22} \end{bmatrix} \\
&= CA+CB
\end{aligned}
$$

74. Associative Property of Matrix Multiplication

*SOLUTION:*

$$
\begin{aligned}
(AB)C &= \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right)\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \\
&= \begin{bmatrix} (a_{11}b_{11}+a_{12}b_{21})c_{11}+(a_{11}b_{12}+a_{12}b_{22})c_{21} & (a_{11}b_{11}+a_{12}b_{21})c_{12}+(a_{11}b_{12}+a_{12}b_{22})c_{22} \\ (a_{21}b_{11}+a_{22}b_{21})c_{11}+(a_{21}b_{12}+a_{22}b_{22})c_{21} & (a_{21}b_{11}+a_{22}b_{21})c_{12}+(a_{21}b_{12}+a_{22}b_{22})c_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11}+a_{11}b_{12}c_{21}+a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12}+a_{12}b_{21}c_{12}+a_{11}b_{12}c_{22}+a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11}+a_{22}b_{21}c_{11}+a_{21}b_{12}c_{21}+a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12}+a_{22}b_{21}c_{12}+a_{21}b_{12}c_{22}+a_{22}b_{22}c_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11}c_{11}+a_{11}b_{12}c_{21}+a_{12}b_{21}c_{11}+a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12}+a_{11}b_{12}c_{22}+a_{12}b_{21}c_{12}+a_{12}b_{22}c_{22} \\ a_{21}b_{11}c_{11}+a_{21}b_{12}c_{21}+a_{22}b_{21}c_{11}+a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12}+a_{21}b_{12}c_{22}+a_{22}b_{21}c_{12}+a_{22}b_{22}c_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}(b_{11}c_{11}+b_{12}c_{21})+a_{12}(b_{21}c_{11}+b_{22}c_{21}) & a_{11}(b_{11}c_{12}+b_{12}c_{22})+a_{12}(b_{21}c_{12}+b_{22}c_{22}) \\ a_{21}(b_{11}c_{11}+b_{12}c_{21})+a_{22}(b_{21}c_{11}+b_{22}c_{21}) & a_{21}(b_{11}c_{12}+b_{12}c_{22})+a_{22}(b_{21}c_{12}+b_{22}c_{22}) \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11}c_{11}+b_{12}c_{21} & b_{11}c_{12}+b_{12}c_{22} \\ b_{21}c_{11}+b_{22}c_{21} & b_{21}c_{12}+b_{22}c_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\right) \\
&= A(BC)
\end{aligned}
$$

75. Associative Property of Scalar Multiplication

*SOLUTION:*

$$
\begin{aligned}
c(AB) &= c\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) \\
&= c\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \\
&= \begin{bmatrix} c(a_{11}b_{11} + a_{12}b_{21}) & c(a_{11}b_{12} + a_{12}b_{22}) \\ c(a_{21}b_{11} + a_{22}b_{21}) & c(a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix} \\
&= \begin{bmatrix} ca_{11}b_{11} + ca_{12}b_{21} & ca_{11}b_{12} + ca_{12}b_{22} \\ ca_{21}b_{11} + ca_{22}b_{21} & ca_{21}b_{12} + ca_{22}b_{22} \end{bmatrix} \\
&= \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\
&= \left(c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\
&= (cA)B
\end{aligned}
$$

$$
\begin{aligned}
c(AB) &= c\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) \\
&= c\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \\
&= \begin{bmatrix} c(a_{11}b_{11} + a_{12}b_{21}) & c(a_{11}b_{12} + a_{12}b_{22}) \\ c(a_{21}b_{11} + a_{22}b_{21}) & c(a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix} \\
&= \begin{bmatrix} ca_{11}b_{11} + ca_{12}b_{21} & ca_{11}b_{12} + ca_{12}b_{22} \\ ca_{21}b_{11} + ca_{22}b_{21} & ca_{21}b_{12} + ca_{22}b_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}cb_{11} + a_{12}cb_{21} & a_{11}cb_{12} + a_{12}cb_{22} \\ a_{21}cb_{11} + a_{22}cb_{21} & a_{21}cb_{12} + a_{22}cb_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} cb_{11} & cb_{12} \\ cb_{21} & cb_{22} \end{bmatrix} \\
&= A(cB)
\end{aligned}
$$

76. **ERROR ANALYSIS** Alexis and Paul are discussing determinants. Alexis theorizes that the determinant of a $2 \times 2$ matrix $A$ remains unchanged if two rows of the matrix are interchanged. Paul theorizes that the determinant of the new matrix will have the same absolute value, but will be different in sign. Is either of them correct? Explain your reasoning.

*SOLUTION:*

Paul; the determinant of a matrix involves finding the difference of the product of the diagonals. If two rows are interchanged, then the sign of the difference changes.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad - bc$. If we interchange the rows… For $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$, $\det(B) = bc - ad$.

$ad - bc = -(bc - ad)$

77. **REASONING** If $AB$ has dimensions $5 \times 8$, with $A$ having dimensions $5 \times 6$, what are the dimensions of $B$?

*SOLUTION:*

$6 \times 8$; sample answer: Since the number of columns of matrix $A$ must be equal to the number of rows of matrix $B$, matrix $B$ must have 6 rows. The product matrix $AB$ has the same number of columns as matrix $B$. So, matrix $B$ must have 8 columns.

78. **Writing in Math** Explain why order is important when finding the product of two matrices $A$ and $B$. Give some general examples to support your answer.

*SOLUTION:*

Sample answer: Order is important because the Commutative Property does not hold for matrix multiplication. For example, if $A$ is a $2 \times 3$ matrix and $B$ is a $3 \times 2$ matrix, then $AB$ is a $2 \times 2$ matrix and $BA$ is a $3 \times 3$ matrix. Also, sometimes $AB$ will exist, but $BA$ will not. For example, if $A$ is a $2 \times 3$ matrix and $B$ is a $3 \times 3$ matrix, then $AB$ is a $2 \times 3$ matrix, but $BA$ does not exist. Even when both products are defined, $AB$ and $BA$ are generally not equal.

**Write the augmented matrix for each system of linear equations.**

79. $10x - 3y = -12$
$-6x + 4y = 20$

*SOLUTION:*

Each of the variables of the system is represented in each equation and aligned. Write the augmented matrix.

$$\left[\begin{array}{cc|c} 10 & -3 & -12 \\ -6 & 4 & 20 \end{array}\right]$$

80. $15x + 7y - 2z = 41$
$9x - 8y + z = 32$
$5x + y - 11z = 36$

*SOLUTION:*

Each of the variables of the system is represented in each equation and aligned. Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 15 & 7 & -2 & 41 \\ 9 & -8 & 1 & 32 \\ 5 & 1 & -11 & 36 \end{array}\right]$$

81.
$$\begin{aligned} w-6x+14y&=19\\ 3w+2x-4y+8z&=-2\\ 9w+18y-12z&=3\\ 5x+10y-16z&=-26 \end{aligned}$$

*SOLUTION:*

Each of the variables of the system is not represented in each equation. Rewrite the system, aligning the like terms and using 0 for missing terms.

$$\begin{aligned} w-6x+14y+0z&=19\\ 3w+2x-4y+8z&=-2\\ 9w+0x+18y-12z&=3\\ 0w+5x+10y-16z&=-26 \end{aligned}$$

Write the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & -6 & 14 & 0 & 19 \\ 3 & 2 & -4 & 8 & -2 \\ 9 & 0 & 18 & -12 & 3 \\ 0 & 5 & 10 & -16 & -26 \end{array}\right]$$

82. **PHYSICS** The work done to move an object is given by $W = Fd \cos \theta$, where $\theta$ is the angle between the displacement $d$ and the force exerted $F$. If Lisa does 2400 joules of work while exerting a force of 120 newtons over 40 meters, at what angle was she exerting the force?

*SOLUTION:*

$$\begin{aligned} W &= Fd\cos\theta\\ 2400 &= 120(40)\cos\theta\\ 2400 &= 4800\cos\theta\\ 0.5 &= \cos\theta\\ \cos^{-1}(0.5) &= \theta\\ 60^\circ &= \theta \end{aligned}$$

**Write each degree measure in radians as a multiple of π and each radian measure in degrees.**

83. −10°

*SOLUTION:*

$$\begin{aligned} -10^\circ &= -10^\circ\left(\frac{\pi \text{ radians}}{180^\circ}\right)\\ &= -\frac{10}{180}\pi \text{ radians}\\ &= -\frac{\pi}{18} \text{ radians} \end{aligned}$$

84. 485°

*SOLUTION:*

$$485^\circ = 485^\circ\left(\frac{\pi \text{ radians}}{180^\circ}\right)$$
$$= \frac{485}{180}\pi \text{ radians}$$
$$= \frac{97\pi}{36} \text{ radians}$$

85. $\frac{3\pi}{4}$

*SOLUTION:*

$$\frac{3\pi}{4} \text{ radians} = \frac{3\pi}{4} \text{ radians}\left(\frac{180^\circ}{\pi \text{ radians}}\right)$$
$$= \frac{3(180^\circ)}{4}$$
$$= 135^\circ$$

**Solve each equation.**

86. $\log_{10} \sqrt[3]{10} = x\sqrt{\ }$

*SOLUTION:*

$$\log_{10} \sqrt[3]{10} = x$$
$$10^{\log_{10} \sqrt[3]{10}} = 10^x$$
$$\sqrt[3]{10} = 10^x$$
$$10^{\frac{1}{3}} = 10^x$$
$$\frac{1}{3} = x$$

87. $2 \log_5(x - 2) = \log_5 36$

*SOLUTION:*

$$2\log_5(x-2) = \log_5 36$$
$$\log_5(x-2)^2 = \log_5 36$$
$$(x-2)^2 = 36$$
$$x^2 - 4x + 4 = 36$$
$$x^2 - 4x - 32 = 0$$
$$(x-8)(x+4) = 0$$
$$x = 8 \text{ or } -4$$

The logarithm of a negative number provides no real solution, so $x$ cannot equal −4.

88. $\log_5 (x + 4) + \log_5 8 = \log_5 64$

*SOLUTION:*

$$\log_5(x+4)+\log_5 8=\log_5 64$$
$$\log_5[(x+4)\cdot 8]=\log_5 64$$
$$\log_5(8x+32)=\log_5 64$$
$$8x+32=64$$
$$8x=32$$
$$x=4$$

89. $\log_4 (x - 3) + \log_4 (x + 3) = 2$

*SOLUTION:*

$$\log_4(x-3)+\log_4(x+3)=2$$
$$\log_4[(x-3)(x+3)]=2$$
$$\log_4(x^2-9)=\log_4 16$$
$$x^2-9=16$$
$$x^2=25$$
$$x=\pm 5$$

The logarithm of a negative number provides no real solution, so $x$ cannot equal $-5$.

90. $\frac{1}{2}(\log_7 x + \log_7 8) = \log_7 16$

*SOLUTION:*

$$\frac{1}{2}(\log_7 x+\log_7 8)=\log_7 16$$
$$\frac{1}{2}\log_7(8x)=\log_7 16$$
$$\log_7(8x)^{\frac{1}{2}}=\log_7 16$$
$$(8x)^{\frac{1}{2}}=16$$
$$8x=256$$
$$x=32$$

91. $\log_{12} x = \frac{1}{2} \log_{12} 9 + \frac{1}{3} \log_{12} 27$

*SOLUTION:*

$$\log_{12} x = \frac{1}{2}\log_{12} 9 + \frac{1}{3}\log_{12} 27$$
$$\log_{12} x = \log_{12}(9)^{\frac{1}{2}} + \log_{12}(27)^{\frac{1}{3}}$$
$$\log_{12} x = \log_{12} 3 + \log_{12} 3$$
$$\log_{12} x = \log_{12} 9$$
$$x = 9$$

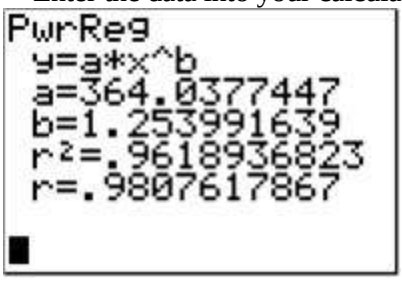
92. **AERONAUTICS** The data below represent the lift of a jet model's wing in a wind tunnel at certain angles of attack. The angle of attack $\alpha$ of the wing is the angle between the wing and the flow of the wind.

| Angle of Attack $\alpha$ | 0.1 | 0.5 | 1.0 | 1.5 | 2.0 | 3.0 | 5.0 | 10.0 |
|---|---|---|---|---|---|---|---|---|
| Lift (lbs) | 11.7 | 236.0 | 476.2 | 711.6 | 950.3 | 1782.6 | 2384.4 | 4049.3 |

**a.** Determine a power function to model the data.
**b.** Use the function to predict the lift of the wing at 4.0 degrees.

*SOLUTION:*

**a.** Enter the data into your calculator. Select the power regression.




After rounding, the power function is approximately $y = 364.04x^{1.254}$.

**b**. Select **Y=**, **VARS**, **Statistics**, **EQ**, **RegEQ**, to graph the entire equation. After graphing the equation, select **CALC**, **Value**, **4**, to find $y$ when $x = 4$.

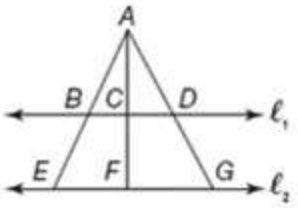



[0, 20] scl: 2 by [0, 5000] scl: 500

The lift is approximately 2071 lbs.

## 6-2 Matrix Multiplication, Inverses and Determinants

93. **SAT/ACT** In the figure, $\ell_1 \parallel \ell_2$. If $EF = x$, and $EG = y$, which of the following represents the ratio of $CD$ to $BC$?




A $1-\frac{y}{x}$

B $1+\frac{y}{x}$

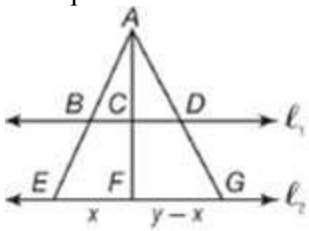
C $\frac{y}{x}-1$

D $1-\frac{x}{y}$

E $1+\frac{x}{y}$

*SOLUTION:*

This question involves similar triangles.




From the figure, if $EF = x$ and $EG = y$, then $FG = y - x$. The question asks for a comparison of $CD$ to $BC$. By similarity, $CD$ to $BC$ is equivalent to $FG$ to $EF$. Therefore, the ratio of $CD$ to $BC$ is $\frac{y-x}{x}=\frac{y}{x}-1$. The correct choice is A.

94. What are the dimensions of the matrix that results from the multiplication shown?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

F $1 \times 3$
G $3 \times 1$
H $3 \times 3$
J $4 \times 3$

*SOLUTION:*

The first matrix is 3 rows by 3 columns, or $3 \times 3$. The second matrix is 3 rows by 1 column, or $3 \times 1$. The multiplication of these will produce a matrix with the rows of the first matrix (3) and the columns of the second matrix (1). Therefore, the correct choice is G.

95. **REVIEW** Shenae spent $42 on 1 can of primer and 2 cans of paint for her room. If the price of one can of paint $p$ is 150% of the price of one can of primer $r$, which system of equations can be used to find the price of paint and primer?

A $p = r + \frac{1}{2}r, r + 2p = 42$

B $p = r + 2r, r + \frac{1}{2}p = 42$

C $r = p + \frac{1}{2}p, r + 2r = 42$

D $r = p + 2p, r + \frac{1}{2} = 42$

*SOLUTION:*

She spent $42 on one can of primer $r$ and two cans of paint $p$, so $r + 2p = 42$. The cost of one can of paint $p$ equals 150% the cost of one can of primer $r$, so $p = 1.5r$. This can also be written as $p = r + \frac{1}{2}r$. Therefore, the correct choice is A.

96. **REVIEW** To join the football team, a student must have a GPA of at least 2.0 and must have attended at least five after-school practices. Which system of inequalities best represents this situation if $x$ represents a student's GPA, and $y$ represents the amount of after-school practices the student attended.

F $x \geq 2, y \geq 5$
G $x \leq 2, y \leq 5$
H $x < 2, y < 5$
J $x > 2, y > 5$

*SOLUTION:*

The phrase *at least* is represented by $\geq$. The correct choice is F.

# 6-3 Solving Linear Systems using Inverses and Cramer's Rule

**Use an inverse matrix to solve each system of equations, if possible.**

1. $5x - 2y = 11$
$-4x + 7y = 2$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 5 & -2 \\ -4 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$$

Use the formula for the inverse of a 2 × 2 matrix to find $A^{-1}$.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{5(7) - (-4)(-2)} \begin{bmatrix} 7 & 2 \\ 4 & 5 \end{bmatrix}$$

$$= \frac{1}{27} \begin{bmatrix} 7 & 2 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{27} & \frac{2}{27} \\ \frac{4}{27} & \frac{5}{27} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} \frac{7}{27} & \frac{2}{27} \\ \frac{4}{27} & \frac{5}{27} \end{bmatrix} \begin{bmatrix} 11 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{27}(11) + \frac{2}{27}(2) \\ \frac{4}{27}(11) + \frac{5}{27}(2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{77}{27} + \frac{4}{27} \\ \frac{44}{27} + \frac{10}{27} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{81}{27} \\ \frac{54}{27} \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The solution is (3, 2)

# 6-3 Solving Linear Systems using Inverses and Cramer's Rule

2. $2x+3y=2$
$x-4y=-21$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix}2 & 3\\ 1 & -4\end{bmatrix}\cdot\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ -21\end{bmatrix}$$

Use the formula for the inverse of a $2 \times 2$ matrix to find $A^{-1}$.

$$A^{-1}=\frac{1}{\det(A)}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$
$$=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$
$$=\frac{1}{2(-4)-(3)(1)}\begin{bmatrix}-4 & -3\\ -1 & 2\end{bmatrix}$$
$$=-\frac{1}{11}\begin{bmatrix}-4 & -3\\ -1 & 2\end{bmatrix}$$
$$=\begin{bmatrix}\frac{4}{11} & \frac{3}{11}\\ \frac{1}{11} & -\frac{2}{11}\end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B=\begin{bmatrix}\frac{4}{11} & \frac{3}{11}\\ \frac{1}{11} & -\frac{2}{11}\end{bmatrix}\begin{bmatrix}2\\ -21\end{bmatrix}$$
$$=\begin{bmatrix}\frac{4}{11}(2)+\frac{3}{11}(-21)\\ \frac{1}{11}(2)+-\frac{2}{11}(-21)\end{bmatrix}$$
$$=\begin{bmatrix}\frac{8}{11}-\frac{63}{11}\\ \frac{2}{11}+\frac{42}{11}\end{bmatrix}$$
$$=\begin{bmatrix}-\frac{55}{11}\\ \frac{44}{11}\end{bmatrix}$$
$$=\begin{bmatrix}-5\\ 4\end{bmatrix}$$

The solution is $(-5, 4)$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

3. $-3x+5y=33$
$2x-4y=-26$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix}-3 & 5\\ 2 & -4\end{bmatrix}\cdot\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}33\\ -26\end{bmatrix}$$

Use the formula for the inverse of a $2 \times 2$ matrix to find $A^{-1}$.

$$A^{-1}=\frac{1}{\det(A)}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$
$$=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$
$$=\frac{1}{-3(-4)-(5)(2)}\begin{bmatrix}-4 & -5\\ -2 & -3\end{bmatrix}$$
$$=\frac{1}{2}\begin{bmatrix}-4 & -5\\ -2 & -3\end{bmatrix}$$
$$=\begin{bmatrix}-\frac{4}{2} & \frac{-5}{2}\\ -\frac{2}{2} & \frac{-3}{2}\end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B=\begin{bmatrix}-\frac{4}{2} & \frac{-5}{2}\\ -\frac{2}{2} & \frac{-3}{2}\end{bmatrix}\begin{bmatrix}33\\ -26\end{bmatrix}$$
$$=\begin{bmatrix}-\frac{4}{2}(33)+\frac{-5}{2}(-26)\\ -\frac{2}{2}(33)+\frac{-3}{2}(-26)\end{bmatrix}$$
$$=\begin{bmatrix}-\frac{132}{2}+\frac{130}{2}\\ -\frac{66}{2}+\frac{78}{2}\end{bmatrix}$$
$$=\begin{bmatrix}\frac{-2}{2}\\ \frac{12}{2}\end{bmatrix}$$
$$=\begin{bmatrix}-1\\ 6\end{bmatrix}$$

The solution is $(-1, 6)$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

4. $-4x + y = 19$
$3x - 2y = -18$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} -4 & 1 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ -18 \end{bmatrix}$$

Use the formula for the inverse of a $2 \times 2$ matrix to find $A^{-1}$.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-4(-2) - (1)(3)} \begin{bmatrix} -2 & -1 \\ -3 & -4 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -2 & -1 \\ -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{5} & -\frac{1}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} -\frac{2}{5} & -\frac{1}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix} \begin{bmatrix} 19 \\ -18 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{5}(19) + -\frac{1}{5}(-18) \\ -\frac{3}{5}(19) + -\frac{4}{5}(-18) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{38}{5} + \frac{18}{5} \\ -\frac{57}{5} + \frac{72}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{20}{5} \\ \frac{15}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

The solution is $(-4, 3)$

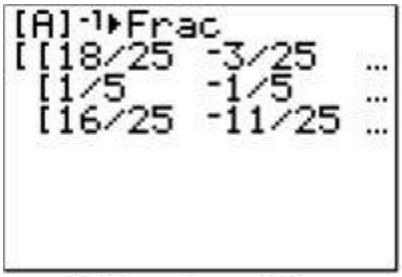
5. $\begin{aligned} 2x + y - z &= -13 \\ 3x + 2y - 4z &= -36 \\ x + 6y - 3z &= 12 \end{aligned}$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -4 \\ 1 & 6 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -13 \\ -36 \\ 12 \end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$. Enter the values for the matrix $A$. Enter $[A]^{-1}$. Select $\rightarrow Frac$ to get the values in reduced fraction form.




$$A^{-1} = \begin{bmatrix} \frac{18}{25} & -\frac{3}{25} & -\frac{2}{25} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \\ \frac{16}{25} & -\frac{11}{25} & \frac{1}{25} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} \frac{18}{25} & -\frac{3}{25} & -\frac{2}{25} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \\ \frac{16}{25} & -\frac{11}{25} & \frac{1}{25} \end{bmatrix} \cdot \begin{bmatrix} -13 \\ -36 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{18}{25}(-13) - \frac{3}{25}(-36) - \frac{2}{25}(12) \\ \frac{1}{5}(-13) - \frac{1}{5}(-36) + \frac{1}{5}(12) \\ \frac{16}{25}(-13) - \frac{11}{25}(-36) + \frac{1}{25}(12) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{234}{25} + \frac{108}{25} - \frac{24}{25} \\ -\frac{13}{5} + \frac{36}{5} + \frac{12}{5} \\ -\frac{208}{25} + \frac{396}{25} + \frac{12}{25} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{150}{25} \\ \frac{35}{5} \\ \frac{200}{25} \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 7 \\ 8 \end{bmatrix}$$

The solution is (−6, 7, 8).

6. $3x - 2y + 8z = 38$
$6x + 3y - 9z = -12$
$4x + 4y + 20z = 0$

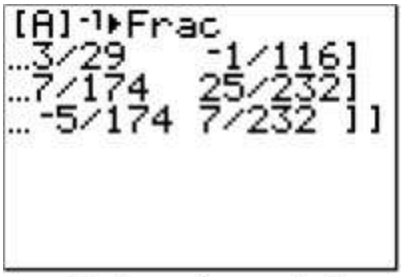
*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 3 & -2 & 8 \\ 6 & 3 & -9 \\ 4 & 4 & 20 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 38 \\ -12 \\ 0 \end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$. Enter the values for the matrix $A$. Enter $[A]^{-1}$. Select $\rightarrow Frac$ to get the values in reduced fraction form.

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule




$$A^{-1} = \begin{bmatrix} \frac{4}{29} & \frac{3}{29} & -\frac{1}{116} \\ -\frac{13}{58} & \frac{7}{174} & \frac{25}{232} \\ \frac{1}{58} & -\frac{5}{174} & \frac{7}{232} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} \frac{4}{29} & \frac{3}{29} & -\frac{1}{116} \\ -\frac{13}{58} & \frac{7}{174} & \frac{25}{232} \\ \frac{1}{58} & -\frac{5}{174} & \frac{7}{232} \end{bmatrix} \cdot \begin{bmatrix} 38 \\ -12 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{29}(38) + \frac{3}{29}(-12) - \frac{1}{116}(0) \\ -\frac{13}{58}(38) + \frac{7}{174}(-12) + \frac{25}{232}(0) \\ \frac{1}{58}(38) - \frac{5}{174}(-12) + \frac{7}{232}(0) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{152}{29} - \frac{36}{29} - 0 \\ -\frac{494}{58} - \frac{84}{174} + 0 \\ \frac{38}{58} + \frac{60}{174} + 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{116}{29} \\ -\frac{1482}{174} - \frac{84}{174} \\ \frac{114}{174} + \frac{60}{174} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{116}{29} \\ -9 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -9 \\ 1 \end{bmatrix}$$

The solution is (4, −9, 1).

7. $x+2y-z=2$
$2x-y+3z=4$
$3x+y+2z=6$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix}1 & 2 & -1\\ 2 & -1 & 3\\ 3 & 1 & 2\end{bmatrix}\cdot\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2\\ 4\\ 6\end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$. Enter the values for the matrix $A$. Enter $[A]^{-1}$. Select $\rightarrow Frac$ to get the values in reduced fraction form.

Because $A$ is a singular matrix, its determinant is zero and $A^{-1}$ does not exist.

There is no unique solution.

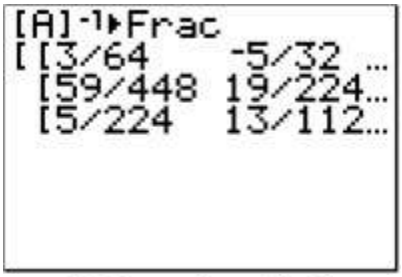
8. $4x+6y+z=-1$
$-x-y+8z=8$
$6x-4y+11z=21$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix}4 & 6 & 1\\ -1 & -1 & 8\\ 6 & -4 & 11\end{bmatrix}\cdot\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1\\ 8\\ 21\end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$. Enter the values for the matrix $A$. Enter $[A]^{-1}$. Select $\rightarrow Frac$ to get the values in reduced fraction form.




$$A^{-1}=\begin{bmatrix}\frac{3}{64} & -\frac{5}{32} & \frac{7}{64}\\ \frac{59}{448} & \frac{19}{224} & -\frac{33}{448}\\ \frac{5}{224} & \frac{13}{112} & \frac{1}{224}\end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} \frac{3}{64} & -\frac{5}{32} & \frac{7}{64} \\ \frac{59}{448} & \frac{19}{224} & -\frac{33}{448} \\ \frac{5}{224} & \frac{13}{112} & \frac{1}{224} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 8 \\ 21 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{64}(-1) - \frac{5}{32}(8) + \frac{7}{64}(21) \\ \frac{59}{448}(-1) + \frac{19}{224}(8) - \frac{33}{448}(21) \\ \frac{5}{224}(-1) + \frac{13}{112}(8) + \frac{1}{224}(21) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{64} - \frac{40}{32} + \frac{147}{64} \\ -\frac{59}{448} + \frac{152}{224} - \frac{693}{448} \\ -\frac{5}{224} + \frac{104}{112} + \frac{21}{224} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{64} - \frac{80}{64} + \frac{147}{64} \\ -\frac{59}{448} + \frac{304}{448} - \frac{693}{448} \\ -\frac{5}{224} + \frac{208}{224} + \frac{21}{224} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{64}{64} \\ -\frac{448}{448} \\ \frac{224}{224} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The solution is $(1, -1, 1)$.

9. **DOWNLOADING** Marcela downloaded some programs on her portable media player. In general, a 30 minute sitcom uses 0.3 gigabyte of memory, a 1 hour talk show uses 0.6 gigabyte, and a 2 hour movie uses 1.2 gigabytes. She downloaded 9 programs totaling 5.4 gigabytes. If she downloaded two more sitcoms than movies, what number of each type of show did Marcela download?

*SOLUTION:*

Let $s$ = the number of sitcoms, $t$ = the number of talk shows, and $m$ = the number of movies. She downloaded 9 programs, so $s + t + m = 9$. They totaled 5.4 gigabytes, so $0.3s + 0.6t + 1.2m = 5.4$. She also downloaded 2 more sitcoms than movies, so $s = m + 2$.
Write the system, aligning the variables.

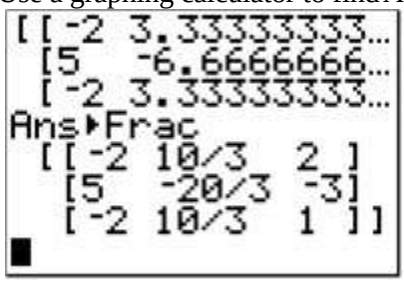
$$\begin{aligned} s + t + m &= 9 \\ 0.3s + 0.6t + 1.2m &= 5.4 \\ s + 0t - m &= 2 \end{aligned}$$

Write the system in matrix form $AX = B$.

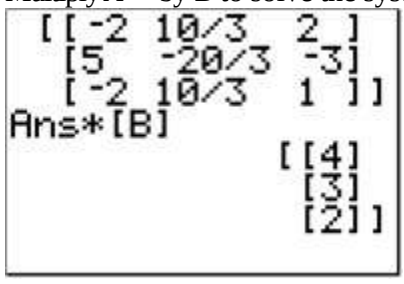
$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.3 & 0.6 & 1.2 \\ 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} s \\ t \\ m \end{bmatrix} = \begin{bmatrix} 9 \\ 5.4 \\ 2 \end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$.




Multiply $A^{-1}$ by $B$ to solve the system.




4 sitcoms, 3 talk shows, and 2 movies

10. **BASKETBALL** Trevor knows that he has scored a total of 70 points thus far this basketball season. His coach told him that he has scored 37 times. Trevor wants to know how many free throws, 2-point field goals, and 3-point field goals he has made. The sum of his 2- and 3-point field goals equals twice the number of free throws minus two. How many free throws, 2-point field goals, and 3-point field goals has Trevor made?

*SOLUTION:*

Let $x_1$ = free throws, $x_2$ = field goals, and $x_3$ = three-point field goals. Trevor scored 37 times, so $x_1 + x_2 + x_3 = 37$.

He scored 70 points, so $x_1 + 2x_2 + 3x_3 = 70$. Also, $x_2 + x_3 = 2x_1 - 2$.

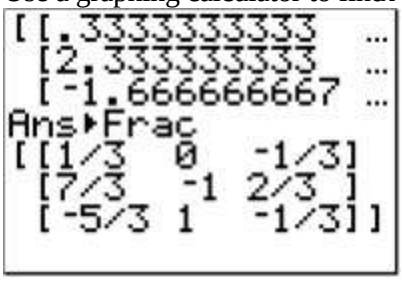
Write the system, aligning the variables.

$$x_1 + x_2 + x_3 = 37$$
$$x_1 + 2x_2 + 3x_3 = 70$$
$$-2x_1 + x_2 + x_3 = -2$$

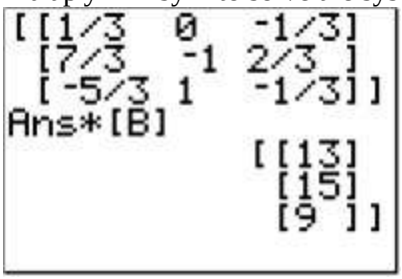
Write the system in matrix form $AX = B$.

$$AX = B$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 37 \\ 70 \\ -2 \end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$.




Multiply $A^{-1}$ by $B$ to solve the system.




13 free throws, 15 field goals, and 9 three-point field goals

# 6-3 Solving Linear Systems using Inverses and Cramer's Rule

**Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists.**

11. $-3x + y = 4$
$2x + y = -6$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$. Calculate the determinant of $A$.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix} = -3(1) - 2(1)$$
$$= -3 - 2$$
$$= -5$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$
$$= \frac{\begin{vmatrix} 4 & 1 \\ -6 & 1 \end{vmatrix}}{-5}$$
$$= \frac{4(1) - (-6)(1)}{-5}$$
$$= \frac{4 + 6}{-5}$$
$$= -2$$
$$y = \frac{|A_y|}{|A|}$$
$$= \frac{\begin{vmatrix} -3 & 4 \\ 2 & -6 \end{vmatrix}}{-5}$$
$$= \frac{-3(-6) - 2(4)}{-5}$$
$$= \frac{18 - 8}{-5}$$
$$= -2$$

The solution is $(-2, -2)$.

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

12. $2x+3y=4$
$5x+6y=5$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$. Calculate the determinant of $A$.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 2(6) - 5(3)$$
$$= 12 - 15$$
$$= -3$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$
$$= \frac{\begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix}}{-3}$$
$$= \frac{4(6) - (5)(3)}{-3}$$
$$= \frac{24 - 15}{-3}$$
$$= -3$$
$$y = \frac{|A_y|}{|A|}$$
$$= \frac{\begin{vmatrix} 2 & 4 \\ 5 & 5 \end{vmatrix}}{-3}$$
$$= \frac{2(5) - 5(4)}{-3}$$
$$= \frac{10 - 20}{-3}$$
$$= \frac{10}{3}$$

The solution is $\left(-3, 3\frac{1}{3}\right)$.

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

13. $5x + 4y = 7$
$-x - 4y = -3$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 5 & 4 \\ -1 & -4 \end{bmatrix}$. Calculate the determinant of $A$.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} 5 & 4 \\ -1 & -4 \end{vmatrix} = 5(-4) - (-1)(4)$$
$$= -20 + 4$$
$$= -16$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$
$$= \frac{\begin{vmatrix} 7 & 4 \\ -3 & -4 \end{vmatrix}}{-16}$$
$$= \frac{7(-4) - (-3)(4)}{-16}$$
$$= \frac{-28 + 12}{-16}$$
$$= 1$$
$$y = \frac{|A_y|}{|A|}$$
$$= \frac{\begin{vmatrix} 5 & 7 \\ -1 & -3 \end{vmatrix}}{-16}$$
$$= \frac{5(-3) - (-1)(7)}{-16}$$
$$= \frac{-15 + 7}{-16}$$
$$= \frac{1}{2}$$

The solution is $\left(1, \frac{1}{2}\right)$.

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

14. $4x + \frac{1}{3}y = 8$
$3x + y = 6$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 4 & \frac{1}{3} \\ 3 & 1 \end{bmatrix}$. Calculate the determinant of $A$.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} 4 & \frac{1}{3} \\ 3 & 1 \end{vmatrix} = 4(1) - 3\left(\frac{1}{3}\right)$$
$$= 4 - 1$$
$$= 3$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$
$$= \frac{\begin{vmatrix} 8 & \frac{1}{3} \\ 6 & 1 \end{vmatrix}}{3}$$
$$= \frac{8(1) - 6\left(\frac{1}{3}\right)}{3}$$
$$= \frac{8 - 2}{3}$$
$$= 2$$
$$y = \frac{|A_y|}{|A|}$$
$$= \frac{\begin{vmatrix} 4 & 8 \\ 3 & 6 \end{vmatrix}}{3}$$
$$= \frac{4(6) - (3)(8)}{3}$$
$$= \frac{24 - 24}{3}$$
$$= 0$$

The solution is (2, 0).

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

15. $\begin{aligned} 2x - y + z &= 1 \\ x + 2y - 4z &= 3 \\ 4x + 3y - 7z &= -8 \end{aligned}$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -4 \\ 4 & 3 & -7 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -4 \\ 4 & 3 & -7 \end{vmatrix} = 2\begin{vmatrix} 2 & -4 \\ 3 & -7 \end{vmatrix} + 1\begin{vmatrix} 1 & -4 \\ 4 & -7 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$$

$$\begin{aligned} &= 2[2(-7) - 3(-4)] + 1[1(-7) - 4(-4)] + 1[1(3) - 4(2)] \\ &= 2(-14 + 12) + 1(-7 + 16) + 1(3 - 8) \\ &= -4 + 9 - 5 \\ &= 0 \end{aligned}$$

Because the determinant of $A$ is 0, you cannot apply Cramer's Rule; therefore, no unique solution exists.

16. $\begin{aligned} x + y + z &= 12 \\ 6x - 2y - z &= 16 \\ 3x + 4y + 2z &= 28 \end{aligned}$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -2 & -1 \\ 3 & 4 & 2 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 6 & -2 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} -2 & -1 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 6 & -1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 6 & -2 \\ 3 & 4 \end{vmatrix}$$

$$\begin{aligned} &= 1[-2(2) - 4(-1)] - 1[6(2) - 3(-1)] + 1[6(4) - 3(-2)] \\ &= 1(-4 + 4) - 1(12 + 3) + 1(24 + 6) \\ &= 0 - 15 + 30 \\ &= 15 \end{aligned}$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$

$$= \frac{\begin{vmatrix} 12 & 1 & 1 \\ 16 & -2 & -1 \\ 28 & 4 & 2 \end{vmatrix}}{15}$$

$$= \frac{12\begin{vmatrix} -2 & -1 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 16 & -1 \\ 28 & 2 \end{vmatrix} + 1\begin{vmatrix} 16 & -2 \\ 28 & 4 \end{vmatrix}}{15}$$

$$= \frac{12[-2(2) - 4(-1)] - 1[16(2) - 28(-1)] + 1[16(4) - 28(-2)]}{15}$$

$$= \frac{0 - 60 + 120}{15}$$

$$= \frac{60}{15}$$

$$= 4$$

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 12 & 1 \\ 6 & 16 & -1 \\ 3 & 28 & 2 \end{vmatrix}}{15}$$

$$= \frac{1\begin{vmatrix} 16 & -1 \\ 28 & 2 \end{vmatrix} - 12\begin{vmatrix} 6 & -1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 6 & 16 \\ 3 & 28 \end{vmatrix}}{15}$$

$$= \frac{1[16(2) - 28(-1)] - 12[6(2) - 3(-1)] + 1[6(28) - 3(16)]}{15}$$

$$= \frac{1(32 + 28) - 12(12 + 3) + 1(168 - 48)}{15}$$

$$= \frac{60 - 180 + 120}{15}$$

$$= \frac{0}{15}$$

$$= 0$$

$$Z = \frac{|A_z|}{|A|}$$
$$= \frac{\begin{vmatrix} 1 & 1 & 12 \\ 6 & -2 & 16 \\ 3 & 4 & 28 \end{vmatrix}}{15}$$
$$= \frac{1\begin{vmatrix} -2 & 16 \\ 4 & 28 \end{vmatrix} - 1\begin{vmatrix} 6 & 16 \\ 3 & 28 \end{vmatrix} + 12\begin{vmatrix} 6 & -2 \\ 3 & 4 \end{vmatrix}}{15}$$
$$= \frac{1[-2(28) - 4(16)] - 1[6(28) - 3(16)] + 12[6(4) - 3(-2)]}{15}$$
$$= \frac{-120 - 120 + 360}{15}$$
$$= \frac{120}{15}$$
$$= 8$$

The solution is (4, 0, 8).

17. $$\begin{aligned} x + 2y &= 12 \\ 3y - 4z &= 25 \\ x + 6y + z &= 20 \end{aligned}$$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -4 \\ 1 & 6 & 1 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & -4 \\ 1 & 6 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & -4 \\ 6 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & -4 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 1 & 6 \end{vmatrix}$$
$$= 1[3(1) - 6(-4)] - 2[0(1) - 1(-4)] + 0[0(6) - 1(3)]$$
$$= 1(3 + 24) - 2(0 + 4) + 0(0 - 3)$$
$$= 27 - 8 + 0$$
$$= 19$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$

$$= \frac{\begin{vmatrix} 12 & 2 & 0 \\ 25 & 3 & -4 \\ 20 & 6 & 1 \end{vmatrix}}{19}$$

$$= \frac{12\begin{vmatrix} 3 & -4 \\ 6 & 1 \end{vmatrix} - 2\begin{vmatrix} 25 & -4 \\ 20 & 1 \end{vmatrix} + 0\begin{vmatrix} 25 & 3 \\ 20 & 6 \end{vmatrix}}{19}$$

$$= \frac{12[3(1) - 6(-4)] - 2[25(1) - 20(-4)] + 0}{19}$$

$$= \frac{12(27) - 2(105)}{19}$$

$$= \frac{114}{19}$$

$$= 6$$

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 12 & 0 \\ 0 & 25 & -4 \\ 1 & 20 & 1 \end{vmatrix}}{19}$$

$$= \frac{1\begin{vmatrix} 25 & -4 \\ 20 & 1 \end{vmatrix} - 12\begin{vmatrix} 0 & -4 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 25 \\ 1 & 20 \end{vmatrix}}{19}$$

$$= \frac{1[25(1) - 20(-4)] - 12[0(1) - 1(-4)] + 0}{19}$$

$$= \frac{1(25 + 80) - 12(0 + 4)}{19}$$

$$= \frac{105 - 48}{19}$$

$$= \frac{57}{19}$$

$$= 3$$

$$Z = \frac{|A_z|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 2 & 12 \\ 0 & 3 & 25 \\ 1 & 6 & 20 \end{vmatrix}}{19}$$

$$= \frac{1\begin{vmatrix} 3 & 25 \\ 6 & 20 \end{vmatrix} - 2\begin{vmatrix} 0 & 25 \\ 1 & 20 \end{vmatrix} + 12\begin{vmatrix} 0 & 3 \\ 1 & 6 \end{vmatrix}}{19}$$

$$= \frac{1[3(20) - 6(25)] - 2[0(20) - 1(25)] + 12[0(6) - 1(3)]}{19}$$

$$= \frac{-90 + 50 - 36}{19}$$

$$= \frac{-76}{19}$$

$$= -4$$

The solution is (6, 3, −4).

18. $\begin{aligned} 9x + 7y &= -30 \\ 8y + 5z &= 11 \\ -3x + 10y &= 73 \end{aligned}$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 9 & 7 & 0 \\ 0 & 8 & 5 \\ -3 & 0 & 10 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 9 & 7 & 0 \\ 0 & 8 & 5 \\ -3 & 0 & 10 \end{vmatrix} = 9\begin{vmatrix} 8 & 5 \\ 0 & 10 \end{vmatrix} - 7\begin{vmatrix} 0 & 5 \\ -3 & 10 \end{vmatrix} + 0\begin{vmatrix} -6 & -2 \\ 3 & 1 \end{vmatrix}$$

$$= 9[8(10) - 0(5)] - 7[0(10) - (-3)(5)] + 0$$

$$= 9(80 - 0) - 7(0 + 15)$$

$$= 720 - 105$$

$$= 615$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x=\frac{|A_x|}{|A|}$$

$$=\frac{\begin{vmatrix}-30 & 7 & 0\\ 11 & 8 & 5\\ 73 & 0 & 10\end{vmatrix}}{615}$$

$$=\frac{-30\begin{vmatrix}8 & 5\\ 0 & 10\end{vmatrix}-7\begin{vmatrix}11 & 5\\ 73 & 10\end{vmatrix}+0\begin{vmatrix}11 & 8\\ 73 & 0\end{vmatrix}}{615}$$

$$=\frac{-30[8(10)-0(5)]-7[11(10)-73(5)]+0}{615}$$

$$=\frac{-30(80)-7[110-365]}{615}$$

$$=\frac{-2400-7[-255]}{615}$$

$$=\frac{-2400+1785}{615}$$

$$=-1$$

$$y=\frac{|A_y|}{|A|}$$

$$=\frac{\begin{vmatrix}9 & -30 & 0\\ 0 & 11 & 5\\ -3 & 73 & 10\end{vmatrix}}{615}$$

$$=\frac{9\begin{vmatrix}11 & 5\\ 73 & 10\end{vmatrix}+30\begin{vmatrix}0 & 5\\ -3 & 10\end{vmatrix}+0\begin{vmatrix}0 & 11\\ -3 & 73\end{vmatrix}}{615}$$

$$=\frac{9[11(10)-73(5)]+30[0(10)-(-3)(5)]+0}{615}$$

$$=\frac{9(110-365)+30(15)}{615}$$

$$=\frac{9(-255)+450}{615}$$

$$=\frac{-2295+450}{615}$$

$$=-3$$

$$Z = \frac{|A_z|}{|A|}$$

$$= \frac{\begin{vmatrix} 9 & 7 & -30 \\ 0 & 8 & 11 \\ -3 & 0 & 73 \end{vmatrix}}{615}$$

$$= \frac{9\begin{vmatrix} 8 & 11 \\ 0 & 73 \end{vmatrix} - 7\begin{vmatrix} 0 & 11 \\ -3 & 73 \end{vmatrix} - 30\begin{vmatrix} 0 & 8 \\ -3 & 0 \end{vmatrix}}{615}$$

$$= \frac{9[8(73) - 0(11)] - 7[0(73) - (-3)(11)] - 30[0(0) - (-3)(8)]}{615}$$

$$= \frac{9(584) - 7(33) - 30(24)}{615}$$

$$= \frac{5256 - 231 - 720}{615}$$

$$= \frac{4305}{615}$$

$$= 7$$

The solution is $(-1, -3, 7)$.

19. **ROAD TRIP** Dena stopped for gasoline twice during a road trip. The price of gasoline at each station is shown below. She bought a total of 33.5 gallons and spent $134.28. Use Cramer's Rule to determine the number of gallons of gasoline Dena bought for $3.96 a gallon.




*SOLUTION:*

Let $f$ = the number of gallons purchased at the first station and let $s$ = the second station. She purchased a total of 33.5, so $f + s = 33.5$. She spent $134.28, so $3.96f + 4.05s = 134.28$.

Find the coefficient matrix $A$.

$$A = \begin{bmatrix} 1 & 1 \\ 3.96 & 4.05 \end{bmatrix}$$

Find the determinant.

$\det(A) = 1(4.05) - 3.96(1) = 0.09$

The determinant does not equal 0 so we can apply Cramer's Rule.

$$f = \frac{|A_f|}{|A|}$$

$$= \frac{\begin{vmatrix} 33.5 & 1 \\ 134.28 & 4.05 \end{vmatrix}}{0.09}$$

$$= \frac{33.5(4.05) - 134.28(1)}{0.09}$$

$$= \frac{1.395}{0.09}$$

$$= 15.5$$

20. **GROUP PLANNING** A class reunion committee is planning for 400 guests for its 10-year reunion. The guests can choose one of the three options for dessert that are shown below. The chef preparing the desserts must spend 5 minutes on each pie, 8 minutes on each trifle, and 12 minutes on each cheesecake. The total cost of the desserts was $1170 and the chef spends exactly 45 hours preparing them. Use Cramer's Rule to determine how many servings of each dessert were prepared.




*SOLUTION:*

Let $p$ = the number of pies, let $t$ = the number of trifles, and let $c$ = the number of cheesecakes. There are 400 guests, so $p + t + c = 400$. The total cost was $1170, so $3p + 2.5t + 4c = 1170$. The chef spent 45 hours or 2700 minutes preparing them, so $5p + 8t + 12c = 2700$.

# 6-3 Solving Linear Systems using Inverses and Cramer's Rule

Find the coefficient matrix $A$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2.5 & 4 \\ 5 & 8 & 12 \end{bmatrix}$$

Find the determinant.

$$\begin{aligned} \det(A) &= 1\begin{vmatrix} 2.5 & 4 \\ 8 & 12 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 \\ 5 & 12 \end{vmatrix} + 1\begin{vmatrix} 3 & 2.5 \\ 5 & 8 \end{vmatrix} \\ &= 2.5(12) - 8(4) - [3(12) - 5(4)] + 3(8) - 5(2.5) \\ &= 30 - 32 - (36 - 20) + 24 - 12.5 \\ &= -2 - 16 + 11.5 \\ &= -6.5 \end{aligned}$$

The determinant does not equal 0 so we can apply Cramer's Rule.

$$\begin{aligned} p &= \frac{|A_p|}{|A|} \\ &= \frac{\begin{vmatrix} 400 & 1 & 1 \\ 1170 & 2.5 & 4 \\ 2700 & 8 & 12 \end{vmatrix}}{-6.5} \\ &= \frac{400\begin{vmatrix} 2.5 & 4 \\ 8 & 12 \end{vmatrix} - 1\begin{vmatrix} 1170 & 4 \\ 2700 & 12 \end{vmatrix} + 1\begin{vmatrix} 1170 & 2.5 \\ 2700 & 8 \end{vmatrix}}{-6.5} \\ &= \frac{400[2.5(12) - 8(4)] - [1170(12) - 2700(4)] + 1170(8) - 2700(2.5)}{-6.5} \\ &= \frac{400(30 - 32) - (14,040 - 10,800) + 9360 - 6750}{-6.5} \\ &= \frac{-800 - 3240 + 2610}{-6.5} \\ &= 220 \end{aligned}$$

$$t = \frac{|A_t|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 400 & 1 \\ 3 & 1170 & 4 \\ 5 & 2700 & 12 \end{vmatrix}}{-6.5}$$

$$= \frac{1\begin{vmatrix} 1170 & 4 \\ 2700 & 12 \end{vmatrix} - 400\begin{vmatrix} 3 & 4 \\ 5 & 12 \end{vmatrix} + 1\begin{vmatrix} 3 & 1170 \\ 5 & 2700 \end{vmatrix}}{-6.5}$$

$$= \frac{1170(12) - 2700(4) - 400[3(12) - 5(4)] + 3(2700) - 5(1170)}{-6.5}$$

$$= \frac{14,040 - 10,800 - 400(16) + 8100 - 5850}{-6.5}$$

$$= \frac{3240 - 6400 + 2250}{-6.5}$$

$$= 140$$

$p = 220$ and $t = 140$, so $c = 400 - (220 + 140) = 40$

220 blueberry pies, 140 chocolate trifles, and 40 cherry cheesecakes

21. **PHONES** Megan, Emma, and Mora all went over their allotted phone plans. For an extra 30 minutes of gaming, 12 minutes of calls, and 40 text messages, Megan paid $52.90. Emma paid $48.07 for 18 minutes of gaming, 15 minutes of calls, and 55 text messages. Mora only paid $13.64 for 6 minutes of gaming and 7 minutes of calls. If they all have the same plan, find the cost of each service.

*SOLUTION:*

Let $g$ = the cost for gaming, let $c$ = the cost for calls, and let $t$ = the cost for text messages. Use each girl's individual cost to set up a system of equations.

$$30g + 12c + 40t = 52.9$$
$$18g + 15c + 55t = 48.07$$
$$6g + 7c + 0t = 13.64$$

Find the coefficient matrix $A$.

$$A = \begin{bmatrix} 30 & 12 & 40 \\ 18 & 15 & 55 \\ 6 & 7 & 0 \end{bmatrix}$$

Find the determinant.

$$\det(A) = 30\begin{vmatrix} 15 & 55 \\ 7 & 0 \end{vmatrix} - 12\begin{vmatrix} 18 & 55 \\ 6 & 0 \end{vmatrix} + 40\begin{vmatrix} 18 & 15 \\ 6 & 7 \end{vmatrix}$$

$$= 30[15(0) - 7(55)] - 12[18(0) - 6(55)] + 40[18(7) - 6(15)]$$

$$= 30(-385) - 12(-330) + 40(36)$$

$$= -11,550 + 3960 + 1440$$

$$= -6150$$

The determinant does not equal 0 so we can apply Cramer's Rule.

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

$$g = \frac{|A_g|}{|A|}$$

$$= \frac{\begin{vmatrix} 52.9 & 12 & 40 \\ 48.07 & 15 & 55 \\ 13.64 & 7 & 0 \end{vmatrix}}{-6150}$$

$$= \frac{52.9\begin{vmatrix} 15 & 55 \\ 7 & 0 \end{vmatrix} - 12\begin{vmatrix} 48.07 & 55 \\ 13.64 & 0 \end{vmatrix} + 40\begin{vmatrix} 48.07 & 15 \\ 13.64 & 7 \end{vmatrix}}{-6150}$$

$$= \frac{52.9[15(0) - 7(55)] - 12[48.07(0) - 13.64(55)] + 40[48.07(7) - 13.64(15)]}{-6150}$$

$$= \frac{52.9(-385) - 12(-750.2) + 40(131.89)}{-6150}$$

$$= \frac{-20,366.5 + 9002.4 + 5275.6}{-6150}$$

$$= 0.99$$

$$c = \frac{|A_c|}{|A|}$$

$$= \frac{\begin{vmatrix} 30 & 52.9 & 40 \\ 18 & 48.07 & 55 \\ 6 & 13.64 & 0 \end{vmatrix}}{-6150}$$

$$= \frac{30\begin{vmatrix} 48.07 & 55 \\ 13.64 & 0 \end{vmatrix} - 52.9\begin{vmatrix} 18 & 55 \\ 6 & 0 \end{vmatrix} + 40\begin{vmatrix} 18 & 48.07 \\ 6 & 13.64 \end{vmatrix}}{-6150}$$

$$= \frac{30[48.07(0) - 13.64(55)] - 52.9[18(0) - 6(55)] + 40[18(13.64) - 6(48.07)]}{-6150}$$

$$= \frac{30(-750.2) - 52.9(-330) + 40(-42.9)}{-6150}$$

$$= \frac{-22,506 + 17,457 - 1716}{-6150}$$

$$= 1.1$$

$$\begin{aligned} 30g + 12c + 40t &= 52.9 \\ 30(0.99) + 12(1.1) + 40t &= 52.9 \\ 29.7 + 13.2 + 40t &= 52.9 \\ 40t &= 10 \\ t &= 0.25 \end{aligned}$$

$0.99/min for gaming; $1.10/min. for calls; $0.25/text message

# 6-3 Solving Linear Systems using Inverses and Cramer's Rule

**Find the solution to each matrix equation.**

22. $\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix}$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix}$$

Find $A^{-1}$.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{(-2)(-1)-3(1)}\begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$$

$$= -1\begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$$

Find $A^{-1}B$.

$$A^{-1}B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\cdot\begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1(4)+1(-2) & 1(0)+1(3) \\ 3(4)+2(-2) & 3(0)+2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 8 & 6 \end{bmatrix}$$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

23. $\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 9 & -8 \\ 0 & 5 \end{bmatrix}$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 9 & -8 \\ 0 & 5 \end{bmatrix}$$

Find $A^{-1}$.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{5(1)-(-2)(2)}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{9} & -\frac{2}{9} \\ \frac{2}{9} & \frac{5}{9} \end{bmatrix}$$

Find $A^{-1}B$.

$$A^{-1}B = \begin{bmatrix} \frac{1}{9} & -\frac{2}{9} \\ \frac{2}{9} & \frac{5}{9} \end{bmatrix} \cdot \begin{bmatrix} 9 & -8 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{9}(9)+\left(-\frac{2}{9}\right)0 & \frac{2}{9}(9)+\left(\frac{5}{9}\right)0 \\ \frac{1}{9}(-8)+\left(-\frac{2}{9}\right)5 & \frac{2}{9}(-8)+\left(\frac{5}{9}\right)5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

24. $\begin{bmatrix} 6 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 12 & 6 \end{bmatrix}$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 6 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 12 & 6 \end{bmatrix}$$

Find $A^{-1}$.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{6(3)-(0)(4)}\begin{bmatrix} 3 & -4 \\ 0 & 6 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 3 & -4 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} & -\frac{2}{9} \\ 0 & \frac{1}{3} \end{bmatrix}$$

Find $A^{-1}B$.

$$A^{-1}B = \begin{bmatrix} \frac{1}{6} & -\frac{2}{9} \\ 0 & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 10 & 8 \\ 12 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6}(10)+\left(-\frac{2}{9}\right)(12) & \frac{1}{6}(8)+\left(-\frac{2}{9}\right)(6) \\ 0(10)+\left(\frac{1}{3}\right)(12) & 0(8)+\left(\frac{1}{3}\right)(6) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{6}-\frac{24}{9} & \frac{4}{3}-\frac{12}{9} \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

25. $\begin{bmatrix} -2 & 1 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 17 & -9 \end{bmatrix}$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} -2 & 1 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 17 & -9 \end{bmatrix}$$

Find $A^{-1}$.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{(-2)(-5)-3(1)}\begin{bmatrix} -5 & -1 \\ -3 & -2 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} -5 & -1 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{7} & -\frac{1}{7} \\ -\frac{3}{7} & -\frac{2}{7} \end{bmatrix}$$

Find $A^{-1}B$.

$$A^{-1}B = \begin{bmatrix} -\frac{5}{7} & -\frac{1}{7} \\ -\frac{3}{7} & -\frac{2}{7} \end{bmatrix} \cdot \begin{bmatrix} -2 & -1 \\ 17 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{7}(-2)+\left(-\frac{1}{7}\right)(17) & -\frac{5}{7}(-1)+\left(-\frac{1}{7}\right)(-9) \\ -\frac{3}{7}(-2)+\left(-\frac{2}{7}\right)(17) & -\frac{3}{7}(-1)+\left(-\frac{2}{7}\right)(-9) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{7}-\frac{17}{7} & \frac{5}{7}+\frac{9}{7} \\ \frac{6}{7}-\frac{34}{7} & \frac{3}{7}+\frac{18}{7} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}$$

26. **FITNESS** Eva is training for a half-marathon and consumes energy gels, bars, and drinks every week. This week, she consumed 12 energy items for a total of 1450 Calories and 310 grams of carbohydrates. The nutritional content of each item is shown in the table below. How many energy gels, bars, and drinks did Eva consume this week?

| Energy Item | gel | bar | drink |
|---|---|---|---|
| Calories | 100 | 250 | 50 |
| Carbohydrates (g) | 25 | 43 | 14 |

*SOLUTION:*

Let $g$ = gels, $b$ = bars, and $d$ = drinks. She consumed 12 items, so $g + b + d = 12$. She consumed 310 grams of carbohydrates, so $25g + 43b + 14d = 310$. She consumed 1450 calories, so $100g + 250b + 50d = 1450$.

Write the system in matrix form $AX = B$.

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 25 & 43 & 14 \\ 100 & 250 & 50 \end{bmatrix} \cdot \begin{bmatrix} g \\ b \\ d \end{bmatrix} = \begin{bmatrix} 12 \\ 310 \\ 1450 \end{bmatrix}$$

Use a calculator to find $A^{-1}$.

```
[[-1.8 .2666666...
 [.2   -.066666...
 [2.6  -.2      ...
Ans▸Frac
[[-9/5 4/15  -2...
 [1/5  -1/15 11...
 [13/5 -1/5  3/...
```

Find $A^{-1}B$.

```
[[-9/5 4/15  -2...
 [1/5  -1/15 11...
 [13/5 -1/5  3/...
Ans*[B]
          [[5]
           [3]
           [4]]
```

5 gels, 3 bars, 4 drinks

**GRAPHING CALCULATOR Solve each system of equations using inverse matrices.**

27. $2a - b + 4c = 6$
$a + 5b - 2c = -6$
$3a - 2b + 6c = 8$

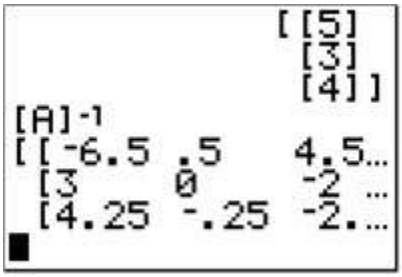
*SOLUTION:*

Write the system in matrix form $AX = B$.

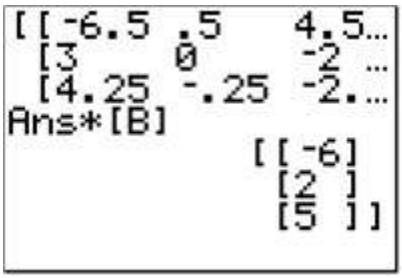
$$AX = B$$

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 5 & -2 \\ 3 & -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 8 \end{bmatrix}$$

Use a calculator to find $A^{-1}$.




Find $A^{-1}B$.



28. $3x - 5y + 2z = 22$
$2x + 3y - z = -9$
$4x + 3y + 3z = 1$

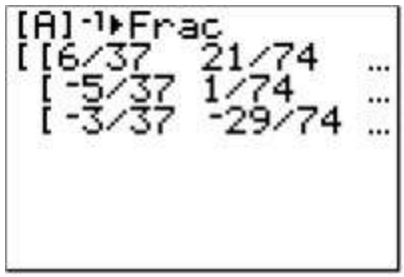
*SOLUTION:*

Write the system in matrix form $AX = B$.

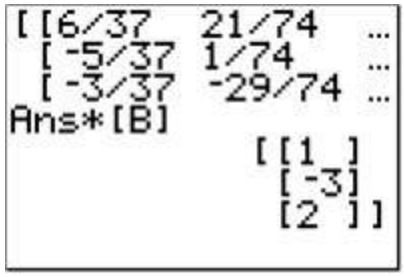
$$AX = B$$

$$\begin{bmatrix} 3 & -5 & 2 \\ 2 & 3 & -1 \\ 4 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 22 \\ -9 \\ 1 \end{bmatrix}$$

Use a calculator to find $A^{-1}$.




Find $A^{-1}B$.



29. $r + 5s - 2t = 16$
$-2r - s + 3t = 3$
$3r + 2s - 4t = -2$

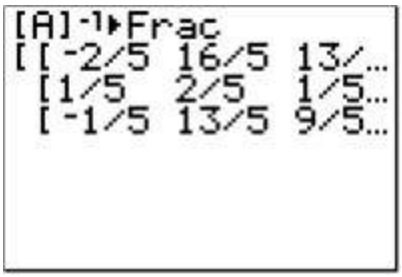
*SOLUTION:*

Write the system in matrix form $AX = B$.

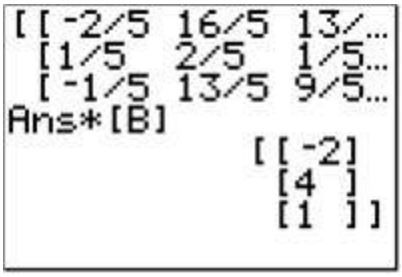
$$AX = B$$

$$\begin{bmatrix} 1 & 5 & -2 \\ -2 & -1 & 3 \\ 3 & 2 & -4 \end{bmatrix} \cdot \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 16 \\ 3 \\ -2 \end{bmatrix}$$

Use a calculator to find $A^{-1}$.




Find $A^{-1}B$.



## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

30. $\begin{aligned} -4m+n+6p&=17\\ 3m-n-p&=5\\ -5m-2n+3p&=2 \end{aligned}$

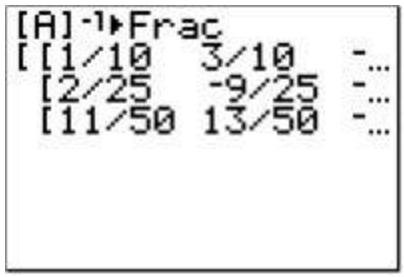
*SOLUTION:*

Write the system in matrix form $AX = B$.

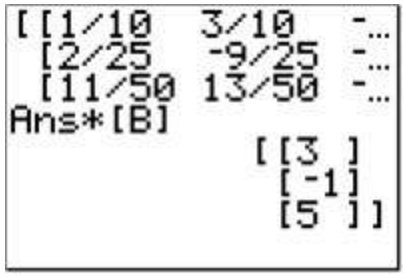
$$AX = B$$

$$\begin{bmatrix} -4 & 1 & 6 \\ 3 & -1 & -1 \\ -5 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \\ p \end{bmatrix} = \begin{bmatrix} 17 \\ 5 \\ 2 \end{bmatrix}$$

Use a calculator to find $A^{-1}$.




Find $A^{-1}B$.




**Find the values of *n* such that the system represented by the given augmented matrix cannot be solved using an inverse matrix.**

31. $\left[\begin{array}{cc|c} n & -8 & 6 \\ 1 & 2 & 3 \end{array}\right]$

*SOLUTION:*

When the determinant of the coefficient matrix is 0, the augmented matrix cannot be solved using an inverse matrix.

$$\begin{vmatrix} n & -8 \\ 1 & 2 \end{vmatrix} = 0$$

$$\begin{aligned} 2n-1(-8)&=0\\ 2n+8&=0\\ 2n&=-8\\ n&=-4 \end{aligned}$$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

32. $\left[\begin{array}{cc|c} 3 & n & 4 \\ n & 2 & -5 \end{array}\right]$

*SOLUTION:*

When the determinant of the coefficient matrix is 0, the augmented matrix cannot be solved using an inverse matrix.

$$\begin{vmatrix} 3 & n \\ n & 2 \end{vmatrix} = 0$$

$$6-(n)(n)=0$$

$$6-n^2=0$$

$$-n^2=-6$$

$$n^2=6$$

$$n=\pm\sqrt{6}$$

33. $\left[\begin{array}{cc|c} -5 & -9 & 3 \\ n & n & 11 \end{array}\right]$

*SOLUTION:*

When the determinant of the coefficient matrix is 0, the augmented matrix cannot be solved using an inverse matrix.

$$\begin{vmatrix} -5 & -9 \\ n & n \end{vmatrix} = 0$$

$$-5n-(-9n)=0$$

$$4n=0$$

$$n=0$$

34. $\left[\begin{array}{cc|c} n & -n & 0 \\ 7 & n & -8 \end{array}\right]$

*SOLUTION:*

When the determinant of the coefficient matrix is 0, the augmented matrix cannot be solved using an inverse matrix.

$$\begin{vmatrix} n & -n \\ 7 & n \end{vmatrix} = 0$$

$$n^2-(-7n)=0$$

$$n^2+7n=0$$

$$n(n+7)=0$$

$$n=0 \text{ or } -7$$

35. **CHEMICALS** Three alloys of copper and silver contain 35% pure silver, 55% pure silver, and 60% pure silver, respectively. How much of each type should be mixed to produce 2.5 kilograms of an alloy containing 54.4% silver if there is to be 0.5 kilogram more of the 60% alloy than the 55% alloy?

*SOLUTION:*

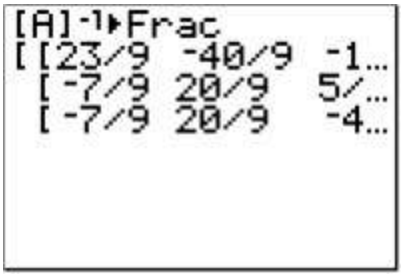
Let $x$ = the 35% alloy, let $y$ = the 55% alloy, and let $z$ = the 60% alloy. There are 2.5 total kilograms, so $x + y + z = 2.5$. The new alloy contains 54.4% silver and 54.4% of 2.5 is 1.36, so $0.35x + 0.55y + 0.6z = 1.36$. Also, $z = y + 0.5$. Write the system, aligning the variables.

$$\begin{aligned} x + y + z &= 2.5 \\ 0.35x + 0.55y + 0.6z &= 1.36 \\ 0x + y - z &= -0.5 \end{aligned}$$

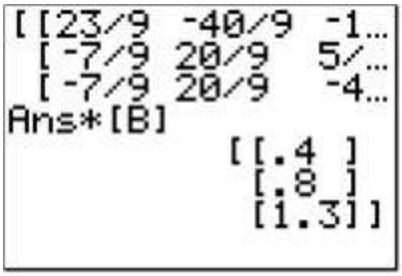
Write the matrix in the form $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.35 & 0.55 & 0.6 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1.36 \\ -0.5 \end{bmatrix}$$

Use a calculator to find $A^{-1}$.




Find $A^{-1}B$.




0.4 kg of 35% alloy, 0.8 kg of 55% alloy, and 1.3 kg of 60% alloy

36. **DELI** A Greek deli sells the gyros shown below. During one lunch, the deli sold a total of 74 gyros and earned $320.50. The total amount of meat used for the small, large, and jumbo gyros was 274 ounces. The number of large gyros sold was one more than twice the number of jumbo gyros sold. How many of each type of gyro did the deli sell during lunch?




*SOLUTION:*

Let $s$ = small, $l$ = large, $j$ = jumbo, and $c$ = chicken. A total of 74 were sold, so $s + l + j + c = 74$. The gyros sold for $320.50, so $3.5s + 4.25l + 5.25j + 5c = 320.5$. The total amount of meat for the non-chicken gyros was 274 ounces, so $3s + 4l + 6j = 274$. Also, $2j + 1 = l$.

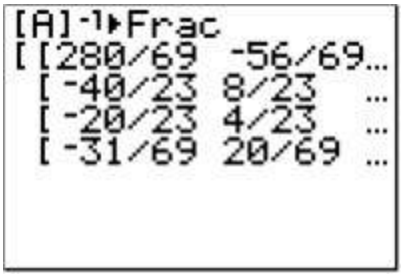
Write the system, aligning the variables.

$$\begin{aligned} s + l + j + c &= 74 \\ 3.5s + 4.25l + 5.25j + 5c &= 320.5 \\ 3s + 4l + 6j + 0c &= 274 \\ 0s - l + 2j + 0c &= -1 \end{aligned}$$

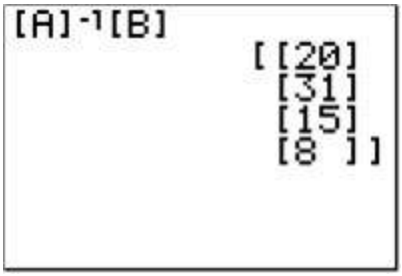
Write the matrix in the form $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3.5 & 4.25 & 5.25 & 5 \\ 3 & 4 & 6 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} s \\ l \\ j \\ c \end{bmatrix} = \begin{bmatrix} 74 \\ 320.5 \\ 274 \\ -1 \end{bmatrix}$$

Use a calculator to find $A^{-1}$.

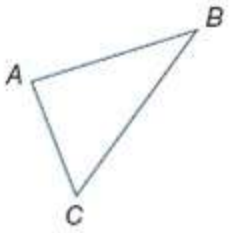



Find $A^{-1}B$.




20 small, 31 large, 15 jumbo, 8 chicken

37. **GEOMETRY** The perimeter of r*ABC* is 89 millimeters. The length of $\overline{AC}$ is 47 millimeters less than the sum of the lengths of the other two sides. The length of $\overline{BC}$ is 20 millimeters more than half the length of $\overline{AB}$. Use a system of equations to find the length of each side.




*SOLUTION:*

Let $x$ = the length of $AB$, let $y$ = the length of $BC$, and let $z$ = the length of $AC$.
The perimeter is 89, so $x + y + z = 89$. Also, $z = x + y - 47$ and $y = 0.5x + 20$.

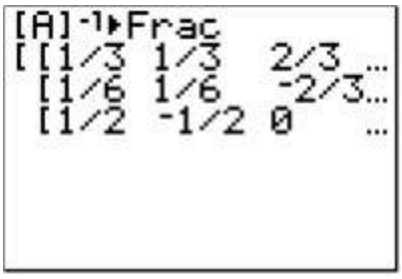
Write the system, aligning the variables.

$$x + y + z = 89$$
$$x + y - z = 47$$
$$0.5x - y + 0z = -20$$

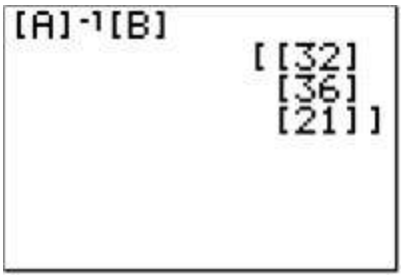
Write the matrix in the form $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0.5 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 89 \\ 47 \\ -20 \end{bmatrix}$$

Use a calculator to find $A^{-1}$.




Find $A^{-1}B$.



# 6-3 Solving Linear Systems using Inverses and Cramer's Rule

**Find the inverse of each matrix, if possible.**

38. $\begin{bmatrix} e^{2x} & e^{-x} \\ e^{x} & e^{-3x} \end{bmatrix}$

*SOLUTION:*

$$A = \begin{bmatrix} e^{2x} & e^{-x} \\ e^{x} & e^{-3x} \end{bmatrix}$$

$$A^{-1} = \frac{1}{e^{2x}e^{-3x} - e^{x}e^{-x}} \begin{bmatrix} e^{-3x} & -e^{-x} \\ -e^{x} & e^{2x} \end{bmatrix}$$

$$= \frac{1}{e^{-x} - 1} \begin{bmatrix} e^{-3x} & -e^{-x} \\ -e^{x} & e^{2x} \end{bmatrix}$$

$$= \frac{1}{\frac{1}{e^{x}} - 1} \begin{bmatrix} e^{-3x} & -e^{-x} \\ -e^{x} & e^{2x} \end{bmatrix}$$

$$= \frac{1}{\frac{1 - e^{x}}{e^{x}}} \begin{bmatrix} e^{-3x} & -e^{-x} \\ -e^{x} & e^{2x} \end{bmatrix}$$

$$= \frac{e^{x}}{1 - e^{x}} \begin{bmatrix} e^{-3x} & -e^{-x} \\ -e^{x} & e^{2x} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{-2x}}{1 - e^{x}} & \frac{-1}{1 - e^{x}} \\ \frac{-e^{2x}}{1 - e^{x}} & \frac{e^{3x}}{1 - e^{x}} \end{bmatrix}$$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

39. $\begin{bmatrix} \frac{1}{x} & \frac{3}{x} \\ x & 2 \end{bmatrix}$

*SOLUTION:*

$$A = \begin{bmatrix} \frac{1}{x} & \frac{3}{x} \\ x & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\frac{1}{x}(2) - x\left(\frac{3}{x}\right)} \begin{bmatrix} 2 & -\frac{3}{x} \\ -x & \frac{1}{x} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\frac{2}{x} - 3} \begin{bmatrix} 2 & -\frac{3}{x} \\ -x & \frac{1}{x} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\frac{2-3x}{x}} \begin{bmatrix} 2 & -\frac{3}{x} \\ -x & \frac{1}{x} \end{bmatrix}$$

$$A^{-1} = \frac{x}{2-3x} \begin{bmatrix} 2 & -\frac{3}{x} \\ -x & \frac{1}{x} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{x}{2-3x}(2) & \frac{x}{2-3x}\left(-\frac{3}{x}\right) \\ \frac{x}{2-3x}(-x) & \frac{x}{2-3x}\left(\frac{1}{x}\right) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{2x}{2-3x} & \frac{-3}{2-3x} \\ \frac{x^2}{2-3x} & \frac{1}{2-3x} \end{bmatrix}$$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

40. $\begin{bmatrix} \pi^x & 1 \\ 0 & \pi^{-2x} \end{bmatrix}$

*SOLUTION:*

$$A = \begin{bmatrix} \pi^x & 1 \\ 0 & \pi^{-2x} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\pi^x(\pi^{-2x})}\begin{bmatrix} \pi^{-2x} & -1 \\ 0 & \pi^x \end{bmatrix}$$

$$A^{-1} = \frac{1}{\pi^{-x}}\begin{bmatrix} \pi^{-2x} & -1 \\ 0 & \pi^x \end{bmatrix}$$

$$A^{-1} = \pi^x\begin{bmatrix} \pi^{-2x} & -1 \\ 0 & \pi^x \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \pi^{-x} & -\pi^x \\ 0 & \pi^{2x} \end{bmatrix}$$

41. $\begin{bmatrix} -i & -3 \\ i^2 & 2i \end{bmatrix}$

*SOLUTION:*

$$A = \begin{bmatrix} -i & -3 \\ i^2 & 2i \end{bmatrix}$$

$$A^{-1} = \frac{1}{-i(2i) - i^2(-3)}\begin{bmatrix} 2i & 3 \\ -i^2 & -i \end{bmatrix}$$

$$A^{-1} = \frac{1}{-2i^2 + 3i^2}\begin{bmatrix} 2i & 3 \\ -i^2 & -i \end{bmatrix}$$

$$A^{-1} = \frac{1}{i^2}\begin{bmatrix} 2i & 3 \\ -i^2 & -i \end{bmatrix}$$

$$A^{-1} = -\begin{bmatrix} 2i & 3 \\ -i^2 & -i \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2i & -3 \\ i^2 & i \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2i & -3 \\ -1 & i \end{bmatrix}$$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

**Let $A$ and $B$ be $n \times n$ matrices and let $C$, $D$, and $X$ be $n \times 1$ matrices. Solve each equation for $X$. Assume that all inverses exist.**

42. $AX = BX - C$

*SOLUTION:*

$$\begin{aligned} AX &= BX - C \\ AX - BX &= -C \\ (A - B)X &= -C \\ (A - B)^{-1}(A - B)X &= (A - B)^{-1}(-C) \\ X &= (A - B)^{-1}(-C) \end{aligned}$$

43. $D = AX + BX$

*SOLUTION:*

$$\begin{aligned} D &= AX + BX \\ D &= (A + B)X \\ (A + B)^{-1}D &= (A + B)^{-1}(A + B)X \\ (A + B)^{-1}(D) &= X \end{aligned}$$

44. $AX + BX = 2C - X$

*SOLUTION:*

$$\begin{aligned} AX + BX &= 2C - X \\ AX + BX + X &= 2C \\ AX + BX + IX &= 2C \\ (A + B + I)X &= 2C \\ (A + B + I)^{-1}(A + B + I)X &= (A + B + I)^{-1}(2C) \\ X &= (A + B + I)^{-1}(2C) \end{aligned}$$

45. $X + C = AX - D$

*SOLUTION:*

$$\begin{aligned} X + C &= AX - D \\ X - AX &= -C - D \\ IX - AX &= -C - D \\ X(I - A) &= -C - D \\ X(I - A)(I - A)^{-1} &= (-C - D)(I - A)^{-1} \\ X &= (-C - D)(I - A)^{-1} \end{aligned}$$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

46. $3X - D = C - BX$

*SOLUTION:*

$$\begin{aligned} 3X - D &= C - BX \\ 3X + BX &= C + D \\ 3IX + BX &= C + D \\ (3I + B)X &= C + D \\ (3I + B)^{-1}(3I + B)X &= (3I + B)^{-1}(C + D) \\ X &= (3I + B)^{-1}(C + D) \end{aligned}$$

47. $BX = AD + AX$

*SOLUTION:*

$$\begin{aligned} BX &= AD + AX \\ BX - AX &= AD \\ (B - A)X &= AD \\ (B - A)^{-1}(B - A)X &= (B - A)^{-1}AD \\ X &= (B - A)^{-1}(AD) \end{aligned}$$

**CALCULUS In calculus, systems of equations can be obtained using *partial derivatives*. These equations contain $\lambda$, which is called a *Lagrange multiplier*. Find the values of $x$ and $y$ to satisfy the system.**

48.
$$x + \lambda + 1 = 0$$
$$2y + \lambda = 0$$
$$x + y + 7 = 0$$

*SOLUTION:*

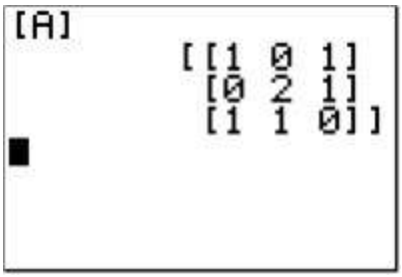
We need to find the values of $x$ and $y$ to satisfy the system. Replace λ with $z$.

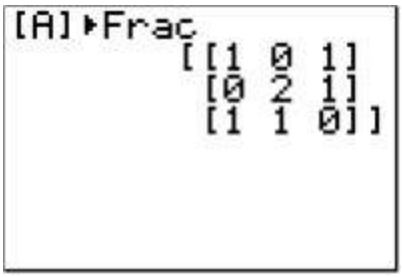
$$x + z + 1 = 0$$
$$2y + z = 0$$
$$x + y + 7 = 0$$

Align the variables.

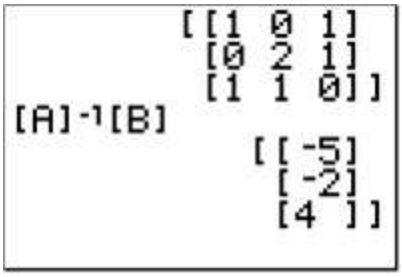
$$x + 0y + z = -1$$
$$0x + 2y + z = 0$$
$$x + y + 0z = -7$$

Let $A$ be the coefficient matrix and $B$ be the matrix of constant terms. Find $A^{-1}$.




Find $A^{-1}B$.




The value of $x$ is −5 and the value of $y$ is −2.

49. **ERROR ANALYSIS** Trent and Kate are trying to solve the system below using Cramer's Rule.

$2x + 7y = 10$

$6x + 21y = 30$

| Trent | Kate |
|---|---|
| The system has no solution because the determinant of the coefficient matrix is 0. | The system has one solution but cannot be found by using Cramer's Rule. |

Is either of them correct? Explain your reasoning.

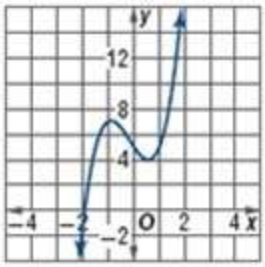
*SOLUTION:*

$$A = \begin{bmatrix} 2 & 7 \\ 6 & 21 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 2(21) - 6(7) \\ &= 42 - 42 \\ &= 0 \end{aligned}$$

Neither; if the determinant of the coefficient matrix is 0, then there is no unique solution. The system may have no solution, or it may have an infinite number of solutions. The system of equations shown has an infinite number of solutions.

50. **CHALLENGE** The graph shown below goes through points at (–2, –1), (–1, 7), (1, 5), and (2, 19). The equation of the graph is of the form $f(x) = ax^3 + bx^2 + cx + d$.




Find the equation of the graph by solving a system of equations using an inverse matrix.

*SOLUTION:*

Rewrite $f(x) = ax^3 + bx^2 + cx + d$, replacing $x$ with the $x$-coordinate and $f(x)$ with the $y$-coordinate of every point on the graph.

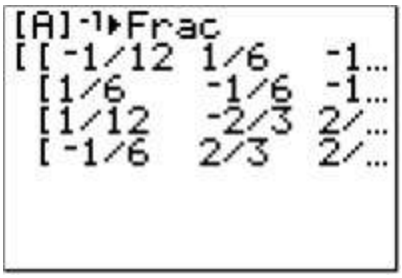
$$\begin{aligned} f(x) &= ax^3 + bx^2 + cx + d \\ -1 &= a(-2)^3 + b(-2)^2 + c(-2) + d \\ -1 &= -8a + 4b - 2c + d \\ 7 &= a(-1)^3 + b(-1)^2 + c(-1) + d \\ 7 &= -a + b - c + d \\ 5 &= a(1)^3 + b(1)^2 + c(1) + d \\ 5 &= a + b + c + d \\ 19 &= a(2)^3 + b(2)^2 + c(2) + d \\ 19 &= 8a + 4b + 2c + d \end{aligned}$$

Write the equations in the form $AX = B$

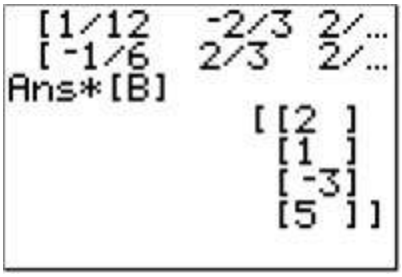
## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

$$\begin{bmatrix} -8 & 4 & -2 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 5 \\ 19 \end{bmatrix}$$

Use a calculator to find $A^{-1}$.




Find $A^{-1}B$.




Rewrite $f(x) = ax^3 + bx^2 + cx + d$, replacing $a$, $b$, $c$, and $d$ with their values.

$f(x) = 2x^3 + x^2 - 3x + 5$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

51. **REASONING** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A$ is nonsingular , does $(A^2)^{-1} = (A^{-1})^2$? Explain your reasoning.

*SOLUTION:*

Yes; Sample answer:

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} \cdot$$

$$(A^2)^{-1} = \frac{1}{(a^2 + bc)(bc + d^2) - (ac + cd)(ab + bd)} \begin{bmatrix} bc + d^2 & -ab - bd \\ -ac - cd & a^2 + bc \end{bmatrix}$$

$$= \frac{1}{(a^2bc + a^2d^2 + b^2c^2 + bcd^2) - (a^2bc + abcd + abcd + bcd^2)} \begin{bmatrix} bc + d^2 & -ab - bd \\ -ac - cd & a^2 + bc \end{bmatrix}$$

$$= \frac{1}{a^2d^2 - 2abcd + b^2c^2} \begin{bmatrix} bc + d^2 & -ab - bd \\ -ac - cd & a^2 + bc \end{bmatrix}$$

$$= \frac{1}{(ad - bc)^2} \begin{bmatrix} bc + d^2 & -ab - bd \\ -ac - cd & a^2 + bc \end{bmatrix}$$

$$= \begin{bmatrix} \frac{bc + d^2}{(ad - bc)^2} & \frac{-ab - bd}{(ad - bc)^2} \\ \frac{-ac - cd}{(ad - bc)^2} & \frac{a^2 + bc}{(ad - bc)^2} \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus, $(A^2)^{-1} = (A^{-1})^2$.

52. **OPEN ENDED** Give an example of a system of equations of two variables that has no solution and demonstrate how the system expressed as a matrix equation would have no solution.

*SOLUTION:*

Any system in which the coefficient matrix has a determinant of 0 cannot have a unique solution. Therefore, these systems cannot be considered *consistent and independent*. One method of creating such a system is to have one set of coefficients (for one equation) be a multiple of another set of coefficients (for a different equation). For example, if one equation is $ax + by = c$, another equation should be $3ax + 3ay = d$.

Sample answer: $x + y = 4$; $x + y = 5$. Expressed as a matrix equation, this becomes $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. To solve, multiply by the inverse of the coefficient matrix. However, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{(1)(1) - (1)(1)} = \frac{1}{0}$ which is undefined.

Therefore, there is no solution to the system.

53. **WRITING IN MATH** Describe what types of systems can be solved using each method. Explain your reasoning.
**a.** Gauss-Jordan elimination
**b.** inverse matrices
**c.** Cramer's Rule

*SOLUTION:*

**a.** Sample answer: Gauss-Jordan elimination can be used to solve any system of linear equations. It is possible to perform row operations on any matrix.
**b**. Sample answer: Inverse matrices can only be used to solve systems with square coefficient matrices because matrix multiplication can only be done if the number of columns of the first matrix is equal to the number of rows of the second matrix. They also cannot be performed when the determinant of the coefficient matrix is 0.
**c.** Sample answer: Cramer's Rule uses determinants to solve systems, and because it is only possible to find the determinant of a square matrix, this method can only be used to solve square systems. They also cannot be performed when the determinant of the coefficient matrix is 0.

**Find *AB* and *BA*, if possible.**

54. $A=\begin{bmatrix}3 & -2\\ 5 & 1\end{bmatrix}, B=\begin{bmatrix}4 & 1\\ 2 & 7\end{bmatrix}$

*SOLUTION:*

$A=\begin{bmatrix}3 & -2\\ 5 & 1\end{bmatrix}, B=\begin{bmatrix}4 & 1\\ 2 & 7\end{bmatrix}$

$A$ is a $2 \times 2$ matrix and $B$ is a $2 \times 2$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$ and the remaining entries.

$$AB=\begin{bmatrix}3(4)+(-2)(2) & 3(1)+(-2)(7)\\ 5(4)+1(2) & 5(1)+1(7)\end{bmatrix}$$
$$=\begin{bmatrix}8 & -11\\ 22 & 12\end{bmatrix}$$

Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

$$BA=\begin{bmatrix}4(3)+1(5) & 4(-2)+1(1)\\ 2(3)+7(5) & 2(-2)+7(1)\end{bmatrix}$$
$$=\begin{bmatrix}17 & -7\\ 41 & 3\end{bmatrix}$$

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

55. $A=\begin{bmatrix}4&-2&6\\7&8&3\\11&-5&-1\end{bmatrix}, B=\begin{bmatrix}17&2&-4\\10&-9&6\\1&0&-8\end{bmatrix}$

*SOLUTION:*

$A=\begin{bmatrix}4&-2&6\\7&8&3\\11&-5&-1\end{bmatrix}, B=\begin{bmatrix}17&2&-4\\10&-9&6\\1&0&-8\end{bmatrix}$

$A$ is a $3 \times 3$ matrix and $B$ is a $3 \times 3$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$ and the remaining entries.

$$AB=\begin{bmatrix}4(17)+(-2)(10)+6(1) & 4(2)+(-2)(-9)+6(0) & 4(-4)+(-2)(6)+6(-8)\\ 7(17)+8(10)+3(1) & 7(2)+8(-9)+3(0) & 7(-4)+8(6)+3(-8)\\ 11(17)+(-5)(10)+1(-1) & 11(2)-5(-9)+0(-1) & 11(-4)-5(6)+(-1)(-8)\end{bmatrix}$$

$$=\begin{bmatrix}54&26&-76\\202&-58&-4\\136&67&-66\end{bmatrix}$$

Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

$$BA=\begin{bmatrix}17(4)+2(7)-4(11) & 17(-2)+2(8)-4(-5) & 17(6)+2(3)-4(-1)\\ 10(4)-9(7)+6(11) & 10(-2)-9(8)+6(-5) & 10(6)-9(3)+6(-1)\\ 1(4)+0(7)-8(11) & 1(-2)+0(8)-8(-5) & 1(6)+0(3)-8(-1)\end{bmatrix}$$

$$=\begin{bmatrix}38&2&112\\43&-122&27\\-84&38&14\end{bmatrix}$$

**Determine whether each matrix is in row-echelon form.**

56. $\left[\begin{array}{ccc|c}0&-3&-6&4\\9&-1&-2&-1\\3&1&-2&-3\\0&3&-1&1\end{array}\right]$

*SOLUTION:*

There are no leading 1s. The matrix is not in row-echelon form.

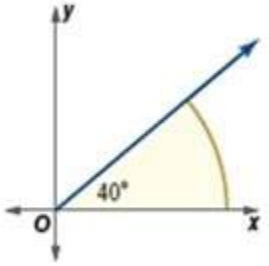
## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

57. $\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$

*SOLUTION:*

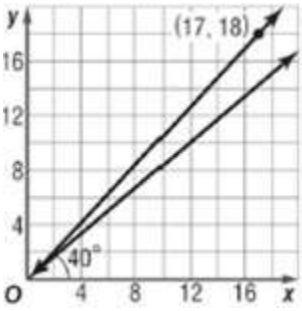
There is a zero below each of the leading ones in each row. The matrix is in row-echelon form.

58. **TRACK AND FIELD** A shot put must land in a 40° sector. The vertex of the sector is at the origin, and one side lies along the *x*-axis. If an athlete puts the shot at a point with coordinates (18, 17), did the shot land in the required region? Explain your reasoning.




*SOLUTION:*

Sample answer: No; with this point on the terminal side of the throwing angle $\theta$, the measure of $\theta$ is found by solving $\tan \theta = \frac{17}{18}$. Thus $\theta = \tan^{-1} \frac{17}{18}$ or about 43.4°, which is greater than the 40° requirement. In the graph shown below, it is clear that the plotted point of (17, 18) falls outside of the 40° line.



59. **STARS** Some stars appear bright only because they are very close to us. Absolute magnitude $M$ is a measure of how bright a star would appear if it were 10 parsecs, or about 32 light years, away from Earth. A lower magnitude indicates a brighter star. Absolute magnitude is given by $M = m + 5 - 5 \log d$, where $d$ is the star's distance from Earth measured in parsecs and $m$ is its apparent magnitude.

| Star | Apparent Magnitude | Distance (parsecs) |
|---|---|---|
| Sirius | −1.44 | 2.64 |
| Vega | 0.03 | 7.76 |

a. Sirius and Vega are two of the brightest stars. Which star appears brighter?
b. Find the absolute magnitudes of Sirius and Vega.
c. Which star is actually brighter? That is, which has a lower absolute magnitude?

*SOLUTION:*

**a**. The apparent magnitude of Sirius is less than that of Vega, so Sirius must appear brighter.
**b.**
Sirius:

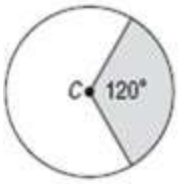
$$M = m + 5 - 5 \log d$$
$$M = -1.44 + 5 - 5 \log 2.64$$
$$M \approx 1.45$$

Vega:

$$M = m + 5 - 5 \log d$$
$$M = 0.03 + 5 - 5 \log 7.76$$
$$M \approx 0.58$$

**c**. 1.45 > 0.58, so Vega has a lower absolute magnitude and is actually brighter.

60. **SAT/ACT** Point *C* is the center of the circle on the figure below. The shaded region has an area of 3π square centimeters. What is the perimeter of the shaded region in centimeters?




A $2\pi + 6$
B $2\pi + 9$
C $2\pi + 12$
D $3\pi + 6$
E $3\pi + 12$

*SOLUTION:*

The shaded region is one-third of the entire circle. Solve for the radius.

$$3\pi = \frac{1}{3}\pi r^2$$
$$9 = r^2$$
$$3 = r$$

The perimeter of the circle is the circumference. The arc of the shaded region represents one-third of the circumference of the entire circle.

$$C = 2\pi r$$
$$= 2\pi(3)$$
$$= 6\pi$$
$$\frac{1}{3}C = \frac{1}{3}(6\pi)$$
$$= 2\pi$$

Including the 2 radii, the perimeter of the shaded region is $2\pi + 6$. The correct choice is A.

## 6-3 Solving Linear Systems using Inverses and Cramer's Rule

61. In March, Claudia bought 2 standard and 2 premium ring tones from her cell phone provider for $8.96. In May, she paid $9.46 for 1 standard and 3 premium ring tones. What are the prices for standard and premium ring tones?
F $1.99, $2.49
G $2.29, $2.79
H $1.99, $2.79
J $2.49, $2.99

*SOLUTION:*

Let $s$ = standard and let $p$ = premium.

$$\begin{aligned} 2s + 2p &= 8.96 \\ s + 3p &= 9.46 \\ s &= 9.46 - 3p \\ 2(9.46 - 3p) + 2p &= 8.96 \\ 18.92 - 6p + 2p &= 8.96 \\ -4p &= -9.96 \\ p &= 2.49 \\ s &= 9.46 - 3(2.49) \\ s &= 9.46 - 7.47 \\ s &= 1.99 \end{aligned}$$

The correct choice is F.

62. **REVIEW** Each year, the students at Capital High School vote to choose the theme of that year's homecoming dance. The theme "A Night Under the Stars" received 225 votes, and "The Time of My Life" received 480 votes. If 40% of girls voted for the star theme, 75% of boys voted for the life theme and all of the students voted, how many girls and boys are there at Capital High School?
A 854 boys and 176 girls
B 705 boys and 325 girls
C 395 boys and 310 girls
D 380 boys and 325 girls

*SOLUTION:*

There were 225 + 480 = 705 votes, so there are 705 students, and $b + g = 705$. If 75% of the boys voted for the life theme, then 25% of the boys voted for the star theme. Therefore, $0.4g + 0.25b = 225$.
Solve the system.

$$\begin{aligned} g &= 705 - b \\ 0.4(705 - b) + 0.25b &= 225 \\ 282 - 0.4b + 0.25b &= 225 \\ -0.15b &= -57 \\ b &= 380 \\ g &= 705 - 380 \\ g &= 325 \end{aligned}$$

The correct choice is D.

63. **REVIEW** What is the solution of the following system?

$$\frac{1}{8}x-\frac{2}{3}y+\frac{5}{6}z=-8$$
$$\frac{3}{4}x+\frac{1}{6}y-\frac{1}{3}z=-12$$
$$\frac{3}{16}x-\frac{5}{8}y-\frac{7}{12}z=-25$$

F (−4, 6, 3)
G (−8, 12, 6)
H (−16, 24, 12)
J no solution

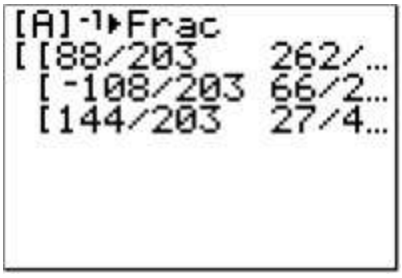
*SOLUTION:*

$$\frac{1}{8}x-\frac{2}{3}y+\frac{5}{6}z=-8$$
$$\frac{3}{4}x+\frac{1}{6}y-\frac{1}{3}z=-12$$
$$\frac{3}{16}x-\frac{5}{8}y-\frac{7}{12}z=-25$$

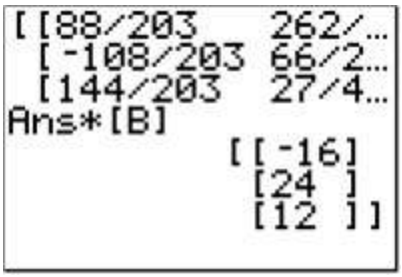
Write the equations in the form $AX = B$

$$\begin{bmatrix} \frac{1}{8} & -\frac{2}{3} & \frac{5}{6} \\ \frac{3}{4} & \frac{1}{6} & -\frac{1}{3} \\ \frac{3}{16} & -\frac{5}{8} & -\frac{7}{12} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ -12 \\ -25 \end{bmatrix}$$

Use a calculator to find $A^{-1}$.




Find $A^{-1}B$.




The correct choice is H.

# Mid-Chapter Quiz: Lessons 6-1 through 6-3

**Write each system of equations in triangular form using Gaussian elimination. Then solve the system.**

1. $2x - y = 13$
$2x + y = 23$

*SOLUTION:*

Align the variables on the left side of the equal sign.

$2x - y = 13$
$2x + y = 23$

Eliminate the $x$-term from the 2nd equation. Multiply the 2nd equation by −1.

$(-1)(2x + y = 23) \Rightarrow -2x - y = -23$

Add the equations together.

$$\begin{array}{r} 2x - y = 13 \\ -2x - y = -23 \\ \hline -2y = -10 \\ y = 5 \end{array}$$

Replace the 2nd equation with $y = 5$.

$2x - y = 13$
$y = 5$

The system is now in triangular form. Substitute for $y$ and solve for $x$.

$2x - y = 13$
$2x - 5 = 13$
$2x = 18$
$x = 9$

The solution is $(9, 5)$.

2. $x + y + z = 6$
$2x - y - z = -3$
$3x - 5y + 7z = 14$

*SOLUTION:*

The variables are aligned on the left side of the equal sign.

$x + y + z = 6$
$2x - y - z = -3$
$3x - 5y + 7z = 14$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by −2.

$(-2)(x + y + z = 6) \Rightarrow -2x - 2y - 2z = -12$

Add the equations together.

$$\begin{array}{rl} -2x - 2y - 2z & = -12 \\ (+)\ 2x - y - z & = -3 \\ \hline -3y - 3z & = -15 \end{array}$$

The new set of equations is:

$x + y + z = 6$
$-3y - 3z = -15$
$3x - 5y + 7z = 14$

Eliminate the $x$-term from the 3rd equation. Multiply the 1st equation by −3.

$(-3)(x+y+z=6) \quad \Rightarrow \quad -3x-3y-3z=-18$

Add the equations together.

$$\begin{aligned} -3x-3y-3z &= -18 \\ (+)\,3x-5y+7z &= 14 \\ -8y+4z &= -4 \end{aligned}$$

The new set of equations is:

$$\begin{aligned} x+y+z &= 6 \\ -3y-3z &= -15 \\ -8y+4z &= -4 \end{aligned}$$

Eliminate the $y$-term from the 3rd equation. Multiply the 2nd equation by −8 and the 3rd equation by 3.

$(-8)(-3y-3z=-15) \quad \Rightarrow \quad 24y+24z=120$

$(3)(-8y+4z=-4) \quad \Rightarrow \quad -24y+12z=-12$

Add the equations together.

$$\begin{aligned} 24y+24z &= 120 \\ (+)\,-24y+12z &= -12 \\ 36z &= 108 \\ z &= 3 \end{aligned}$$

Replace the 3rd equation with $z = 3$.

$$\begin{aligned} x+y+z &= 6 \\ -3y-3z &= -15 \\ z &= 3 \end{aligned}$$

The system is now in triangular form. Substitute for $z$ and solve for $y$, then substitute for $y$ and $z$ and solve for $x$.

$$\begin{aligned} -3y-3z &= -15 \\ -3y-3(3) &= -15 \\ -3y-9 &= -15 \\ -3y &= -6 \\ y &= 2 \\ x+y+z &= 6 \\ x+2+3 &= 6 \\ x+5 &= 6 \\ x &= 1 \end{aligned}$$

The solution is $(1,2,3)$.

# Mid-Chapter Quiz: Lessons 6-1 through 6-3

**Solve each system of equations.**

3. $3x+3y=-8$
   $6x-5y=28$

*SOLUTION:*

Each of the variables of the system is represented in each equation and aligned. Write the augmented matrix.

$$\left[\begin{array}{cc|c} 3 & 3 & -8 \\ 6 & -5 & 28 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cc|c} 3 & 3 & -8 \\ 6 & -5 & 28 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & 3 & -8 \\ 0 & -11 & 44 \end{array}\right] \xleftarrow{R_2-2R_1}$$

$$\left[\begin{array}{cc|c} 3 & 3 & -8 \\ 0 & 1 & -4 \end{array}\right] \xleftarrow{R_2\div(-11)}$$

$$\left[\begin{array}{cc|c} 3 & 0 & 4 \\ 0 & 1 & -4 \end{array}\right] \xleftarrow{R_1-3R_2}$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{4}{3} \\ 0 & 1 & -4 \end{array}\right] \xleftarrow{R_1\div 3}$$

The solution is $\left(\frac{4}{3},-4\right)$.

4. $-x+8y-2z=-37$
   $2x+5y-11z=-7$
   $4x-7y+6z=4$

*SOLUTION:*

Each of the variables of the system is represented in each equation and aligned. Write the augmented matrix.

$$\left[\begin{array}{ccc|c} -1 & 8 & -2 & -37 \\ 2 & 5 & -11 & -7 \\ 4 & -7 & 6 & 4 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

$$\left[\begin{array}{ccc|c} -1 & 8 & -2 & -37 \\ 2 & 5 & -11 & -7 \\ 4 & -7 & 6 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 8 & -2 & -37 \\ 0 & 21 & -15 & -81 \\ 4 & -7 & 6 & 4 \end{array}\right] \leftarrow R_2+2R_1$$

$$\left[\begin{array}{ccc|c} -1 & 8 & -2 & -37 \\ 0 & 7 & -5 & -27 \\ 0 & 25 & -2 & -144 \end{array}\right] \begin{array}{l} \leftarrow R_2+3 \\ \leftarrow R_3+4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 8 & -2 & -37 \\ 0 & 175 & -125 & -675 \\ 0 & 175 & -14 & -1008 \end{array}\right] \begin{array}{l} \leftarrow 25R_2 \\ \leftarrow 7R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 8 & -2 & -37 \\ 0 & 7 & -5 & -27 \\ 0 & 0 & 111 & -333 \end{array}\right] \begin{array}{l} \leftarrow R_2+25 \\ \leftarrow R_3-R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 8 & -2 & -37 \\ 0 & 7 & -5 & -27 \\ 0 & 0 & 1 & -3 \end{array}\right] \leftarrow R_3+111$$

$$\left[\begin{array}{ccc|c} -1 & 8 & -2 & -37 \\ 0 & 7 & 0 & -42 \\ 0 & 0 & 1 & -3 \end{array}\right] \leftarrow R_2+5R_3$$

$$\left[\begin{array}{ccc|c} -1 & 8 & -2 & -37 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{array}\right] \leftarrow R_2+7$$

$$\left[\begin{array}{ccc|c} -1 & 8 & 0 & -43 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{array}\right] \leftarrow R_1+2R_3$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{array}\right] \leftarrow R_1-8R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{array}\right] \leftarrow -R_1$$

The solution is $(-5, -6, -3)$.

5. $\begin{array}{l} -2x+2y+z=5 \\ 3x-2y+2z=7 \\ 5x-y+4z=8 \end{array}$

*SOLUTION:*

Each of the variables of the system is represented in each equation and aligned. Write the augmented matrix.

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 5 \\ 3 & -2 & 2 & 7 \\ 5 & -1 & 4 & 8 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 5 \\ 3 & -2 & 2 & 7 \\ 5 & -1 & 4 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 5 \\ 0 & 2 & 7 & 29 \\ 5 & -1 & 4 & 8 \end{array}\right] \leftarrow 2R_2+3R_3$$

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 5 \\ 0 & 2 & 7 & 29 \\ 0 & 8 & 13 & 41 \end{array}\right] \leftarrow 2R_3+5R_1$$

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 5 \\ 0 & 2 & 7 & 29 \\ 0 & 0 & -15 & -75 \end{array}\right] \leftarrow R_3-4R_2$$

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 5 \\ 0 & 2 & 7 & 29 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_3\div(-15)$$

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 5 \\ 0 & 2 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_2-7R_3$$

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_2\div 2$$

$$\left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1-R_3$$

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1-2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1\div(-2)$$

The solution is $(-3, -3, 5)$.

6. $$\begin{aligned} x-5y+8z&=7\\ -8x+3y+12z&=-9\\ 5x-4y-3z&=9 \end{aligned}$$

*SOLUTION:*

Each of the variables of the system is represented in each equation and aligned. Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & -5 & 8 & 7 \\ -8 & 3 & 12 & -9 \\ 5 & -4 & -3 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 8 & 7 \\ 0 & -37 & 76 & 47 \\ 5 & -4 & -3 & 9 \end{array}\right] \leftarrow R_2+8R_1$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 8 & 7 \\ 0 & -37 & 76 & 47 \\ 0 & 21 & -43 & -26 \end{array}\right] \leftarrow R_3-5R_1$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 8 & 7 \\ 0 & -777 & 1596 & 987 \\ 0 & 777 & -1591 & -962 \end{array}\right] \begin{array}{l} \leftarrow 21R_2 \\ \leftarrow 37R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 8 & 7 \\ 0 & -37 & 76 & 47 \\ 0 & 0 & 5 & 25 \end{array}\right] \begin{array}{l} \leftarrow R_2+21 \\ \leftarrow R_3+R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 8 & 7 \\ 0 & -37 & 76 & 47 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_3+5$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 8 & 7 \\ 0 & -37 & 0 & -333 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_2-76R_3$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 8 & 7 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_2-76R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 8 & 52 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1+5R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1-8R_3$$

The solution is (12, 9, 5).

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

7. **PET CARE** Amelia purchased 25 total pounds of dog food, bird seed, and cat food for $100. She purchased 10 pounds more dog food than bird seed. The cost per pound of each type of food is shown.




**a**. Write a set of linear equations for this situation.

**b**. Determine the number of pounds of each type of food Amelia purchased.

*SOLUTION:*

**a.** Let $d$ equal the number of pounds of dog food, $b$ equal the number of pounds of bird seed, and $c$ equal the number of pounds of cat food.

Use the total number of pounds to make the first equation: $b + c + d = 25$. Use the comparison of bird seed to dog food for the second equation: $d = b + 10$.

Use the total cost for the third equation: $4d + 3c + 7b = 100$.

Align the variables in the equations.

$$b + c + d = 25$$
$$b + 0c - d = -10$$
$$7b + 3c + 4d = 100$$

**b**. Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 1 & 0 & -1 & -10 \\ 7 & 3 & 4 & 100 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 0 & 1 & 2 & 35 \\ 7 & 3 & 4 & 100 \end{array}\right] \leftarrow R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 0 & 1 & 2 & 35 \\ 0 & -4 & -3 & -75 \end{array}\right] \leftarrow R_3 - 7R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 0 & 1 & 2 & 35 \\ 0 & 0 & 5 & 65 \end{array}\right] \leftarrow R_3 + 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 0 & 1 & 2 & 35 \\ 0 & 0 & 1 & 13 \end{array}\right] \leftarrow R_3 \div 5$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 13 \end{array}\right] \leftarrow R_2 - 2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 12 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 13 \end{array}\right] \leftarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 13 \end{array}\right] \leftarrow R_1 - R_2$$

13 pounds of dog food, 9 pounds of cat food, and 3 pounds of bird seed

8. **MULTIPLE CHOICE** Which of the following is non-singular?

A $\begin{bmatrix} 2 & 3 & -5 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 6 & -5 & 0 \\ 4 & 4 & 3 & 4 \end{bmatrix}$

B $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C $\begin{bmatrix} 3 & 2 & 1 & 4 \\ 2 & 0 & 0 & 5 \\ 2 & 0 & 0 & 5 \\ 5 & 1 & -7 & 8 \end{bmatrix}$

D $\begin{bmatrix} 5 & 3 & 1 & 0 \\ 10 & 6 & 2 & 0 \\ 4 & 3 & -1 & 5 \\ 7 & 7 & 3 & 9 \end{bmatrix}$

*SOLUTION:*

When a square matrix has a nonzero determinant, then the matrix is non-singular. Use a calculator to find the determinant of each matrix.




The determinant of *B* is 1, so the correct choice is B.

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

**Find *AB* and *BA*, if possible.**

9. $A = \begin{bmatrix} 1 & -3 & 4 \\ -2 & 5 & 1 \\ 0 & -4 & -6 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 1 & -3 & 4 \\ -2 & 5 & 1 \\ 0 & -4 & -6 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

$A$ is a 3 × 3 matrix and $B$ is a 3 × 4 matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$ and the remaining entries.

$$AB = \begin{bmatrix} 1(1)-3(1)+4(1) & 1(1)-3(0)+4(1) & 1(0)-3(1)+4(-1) & 1(1)-3(1)+4(0) \\ -2(1)+5(1)+1(1) & -2(1)+5(0)+1(1) & -2(0)+5(1)+1(-1) & -2(1)+5(1)+1(0) \\ 0(1)-4(1)-6(1) & 0(1)-4(0)-6(1) & 0(0)-4(1)-6(-1) & 0(1)-4(1)-6(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+4 & 1-0+4 & 0-3-4 & 1-3+0 \\ -2+5+1 & -2+0+1 & 0+5-1 & -2+5+0 \\ 0-4-6 & 0-0-6 & 0-4+6 & 0-4-0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & -7 & -2 \\ 4 & -1 & 4 & 3 \\ -10 & -6 & 2 & -4 \end{bmatrix}$$

Because the number of columns of $B$ is *not* equal to the number of rows of $A$, $BA$ is undefined.

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

10. $A = \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}$; $B = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & 0 \\ 5 & -8 & 2 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}$; $B = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & 0 \\ 5 & -8 & 2 \end{bmatrix}$

$A$ is a $3 \times 1$ matrix and $B$ is a $3 \times 3$ matrix. Because the number of columns of $A$ is *not* equal to the number of rows of $B$, $AB$ is undefined.

$B$ is a $3 \times 3$ matrix and $A$ is a $3 \times 1$ matrix. Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

To find the first entry of $BA$, find the sum of the products of the entries in row 1 of $B$ and column 1 of $A$. Follow the same procedure for row 2 column 1 of $BA$ and the remaining entries.

$$BA = \begin{bmatrix} 2(3)-3(8)+4(9) \\ 1(3)+1(8)+0(9) \\ 5(3)-8(8)+2(9) \end{bmatrix}$$

$$= \begin{bmatrix} 6-24+36 \\ 3+8+0 \\ 15-64+18 \end{bmatrix}$$

$$= \begin{bmatrix} 18 \\ 11 \\ -31 \end{bmatrix}$$

**Find the determinant of each matrix. Then find the inverse of the matrix, if it exists.**

11. $\begin{bmatrix} 3 & 8 \\ -1 & -2 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} 3 & 8 \\ -1 & -2 \end{vmatrix} = 3(-2) - (-1)(8)$$

$$= -6 + 8$$

$$= 2$$

Because the determinant is not 0, the matrix is invertible. Find the inverse matrix.

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} -2 & -8 \\ 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -4 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 3 & 8 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -1 & -4 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 3(-1)+8\left(\frac{1}{2}\right) & 3(-4)+8\left(\frac{3}{2}\right) \\ -1(-1)-2\left(\frac{1}{2}\right) & -1(-4)-2\left(\frac{3}{2}\right) \end{bmatrix}$$
$$= \begin{bmatrix} -3+4 & -12+12 \\ 1-1 & 4-3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A^{-1}A = \begin{bmatrix} -1 & -4 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & 8 \\ -1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} -1(3)-4(-1) & -1(8)-4(-2) \\ \frac{1}{2}(3)+\frac{3}{2}(-1) & \frac{1}{2}(8)+\frac{3}{2}(-2) \end{bmatrix}$$
$$= \begin{bmatrix} -3+4 & -8+8 \\ \frac{3}{2}-\frac{3}{2} & 4-3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

12. $\begin{bmatrix} -9 & -5 \\ -7 & -4 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} -9 & -5 \\ -7 & -4 \end{vmatrix} = -9(-4) - (-7)(-5)$$
$$= 36 - 35$$
$$= 1$$

Because the determinant is not 0, the matrix is invertible. Find the inverse matrix.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{1} \begin{bmatrix} -4 & 5 \\ 7 & -9 \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 5 \\ 7 & -9 \end{bmatrix}$$

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} -9 & -5 \\ -7 & -4 \end{bmatrix} \begin{bmatrix} -4 & 5 \\ 7 & -9 \end{bmatrix}$$
$$= \begin{bmatrix} -9(-4) + (-5)(7) & -9(5) + (-5)(-9) \\ -7(-4) + (-4)(7) & -7(5) + (-4)(-9) \end{bmatrix}$$
$$= \begin{bmatrix} 36 - 35 & -45 + 45 \\ 28 - 28 & -35 + 36 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -4 & 5 \\ 7 & -9 \end{bmatrix} \begin{bmatrix} -9 & -5 \\ -7 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} -4(-9) + 5(-7) & -4(-5) + 5(-4) \\ 7(-9) + (-9)(-7) & 7(-5) + (-9)(-4) \end{bmatrix}$$
$$= \begin{bmatrix} 36 - 35 & 20 - 20 \\ -63 + 63 & -35 + 36 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

13. $\begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} -4 & 3 \\ 7 & -5 \end{vmatrix} = -4(-5) - 7(3)$$
$$= 20 - 21$$
$$= -1$$

Because the determinant is not 0, the matrix is invertible. Find the inverse matrix.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= -\frac{1}{1} \begin{bmatrix} -5 & -3 \\ -7 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$$

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} -4(5) + (3)(7) & -4(3) + (3)(4) \\ 7(5) + (-5)(7) & 7(3) + (-5)(4) \end{bmatrix}$$
$$= \begin{bmatrix} -20 + 21 & -12 + 12 \\ 35 - 35 & 21 - 20 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A^{-1}A = \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$$
$$= \begin{bmatrix} 5(-4) + 3(7) & 5(3) + 3(-5) \\ 7(-4) + 4(7) & 7(3) + 4(-5) \end{bmatrix}$$
$$= \begin{bmatrix} -20 + 21 & 15 - 15 \\ -28 + 28 & 21 - 20 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

14. $\begin{bmatrix} 5 & -10 \\ 4 & -6 \end{bmatrix}$

*SOLUTION:*

Find the determinant.

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} 5 & -10 \\ 4 & -6 \end{vmatrix} = 5(-6) - 4(-10)$$
$$= -30 + 40$$
$$= 10$$

Because the determinant is not 0, the matrix is invertible. Find the inverse matrix.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{10} \begin{bmatrix} -6 & 10 \\ -4 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{3}{5} & 1 \\ -\frac{2}{5} & \frac{1}{2} \end{bmatrix}$$

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 5 & -10 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & 1 \\ -\frac{2}{5} & \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 5\left(-\frac{3}{5}\right) + (-10)\left(-\frac{2}{5}\right) & 5(1) + (-10)\left(\frac{1}{2}\right) \\ 4\left(-\frac{3}{5}\right) + (-6)\left(-\frac{2}{5}\right) & 4(1) + (-6)\left(\frac{1}{2}\right) \end{bmatrix}$$
$$= \begin{bmatrix} -3 + 4 & 5 - 5 \\ -\frac{12}{5} + \frac{12}{5} & 4 - 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -\frac{3}{5} & 1 \\ -\frac{2}{5} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 & -10 \\ 4 & -6 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{3}{5}(5) + 1(4) & -\frac{3}{5}(-10) + 1(-6) \\ -\frac{2}{5}(5) + \frac{1}{2}(4) & -\frac{2}{5}(-10) + \frac{1}{2}(-6) \end{bmatrix}$$
$$= \begin{bmatrix} -3 + 4 & 6 - 6 \\ -2 + 2 & 4 - 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

15. **NURSING** Troy is an Emergency Room nurse. He earns $24 per hour during regular shifts and $30 per hour when working overtime. The table shows the hours Troy worked during the past three weeks.

| Week | Regular Hours | Overtime Hours |
|---|---|---|
| 1 | 35 | 7 |
| 2 | 38 | 0 |
| 3 | 40 | 9 |

**a.** Use matrices to determine how much Troy earned during each week.

**b.** During week 4, Troy worked four times more regular hours than overtime hours. Determine the number of hours he worked if he earned $1008.

*SOLUTION:*

Multiply a 3 × 2 matrix (hours) by a 2 × 1 matrix (rate).

**a.**

$$\begin{bmatrix} 35 & 7 \\ 38 & 0 \\ 40 & 9 \end{bmatrix} \cdot \begin{bmatrix} 24 \\ 30 \end{bmatrix} = \begin{bmatrix} 35(24)+7(30) \\ 38(24)+0(30) \\ 40(24)+9(30) \end{bmatrix}$$

$$= \begin{bmatrix} 1050 \\ 912 \\ 1230 \end{bmatrix}$$

Week 1 = $1050, Week 2 = $912, Week 3 = $1230

**b.** Let $x$ = the number of overtime hours.

$24(4x) + 30x = 1008$

$$\begin{aligned} 24(4x)+30x &= 1008 \\ 96x+30x &= 1008 \\ 126x &= 1008 \\ x &= 8 \end{aligned}$$

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

**Use an inverse matrix to solve each system of equations, if possible.**

16. $2x - y = 6$
$3x + 2y = 37$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 37 \end{bmatrix}$$

Use the formula for the inverse of a 2 × 2 matrix to find $A^{-1}$.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{2(2) - (-1)(3)} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 6 \\ 37 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{7}(6) + \frac{1}{7}(37) \\ -\frac{3}{7}(6) + \frac{2}{7}(37) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{7} + \frac{37}{7} \\ -\frac{18}{7} + \frac{74}{7} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{49}{7} \\ \frac{56}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

The solution is (7, 8)

$2x + y + z = 19$

17. $3x - 2y + 3z = 2$

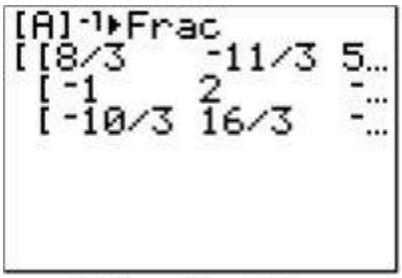
$4x - 6y + 5z = -26$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 3 \\ 4 & -6 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 19 \\ 2 \\ -26 \end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$. Enter the values for the matrix $A$. Enter $[A]^{-1}$. Select $\rightarrow Frac$ to get the values in reduced fraction form.




$$A^{-1} = \begin{bmatrix} \frac{8}{3} & -\frac{11}{3} & \frac{5}{3} \\ -1 & 2 & -1 \\ -\frac{10}{3} & \frac{16}{3} & -\frac{7}{3} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

$$A^{-1}B = \begin{bmatrix} \frac{8}{3} & -\frac{11}{3} & \frac{5}{3} \\ -1 & 2 & -1 \\ -\frac{10}{3} & \frac{16}{3} & -\frac{7}{3} \end{bmatrix} \cdot \begin{bmatrix} 19 \\ 2 \\ -26 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{3}(19) - \frac{11}{3}(2) + \frac{5}{3}(-26) \\ -1(19) + 2(2) - 1(-26) \\ -\frac{10}{3}(19) + \frac{16}{3}(2) - \frac{7}{3}(-26) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{152}{3} - \frac{22}{3} - \frac{130}{3} \\ -19 + 4 + 26 \\ -\frac{190}{3} + \frac{32}{3} + \frac{182}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 11 \\ \frac{24}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 11 \\ 8 \end{bmatrix}$$

The solution is (0, 11, 8).

18. **MULTIPLE CHOICE** Which of the augmented matrices represents the solutions of the system of equations?

$$\begin{aligned} x + y &= 13 \\ 2x - 3y &= -9 \end{aligned}$$

F $\left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$

G $\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -3 \end{array}\right]$

H $\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 7 \end{array}\right]$

J $\left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 5 \end{array}\right]$

*SOLUTION:*

The first two options, F and G, are doubly augmented matrices and cannot represent the solution to the system. One method of checking the other two choices is to plug the values back into the original equations.

Choice H:

$$\begin{aligned} x + y &= 13 \\ 2x - 3y &= -9 \end{aligned}$$

$$\begin{aligned} 6 + 7 &= 13 \\ 13 &= 13 \\ 2(6) - 3(7) &= -9 \\ 12 - 21 &= -9 \\ -9 &= -9 \end{aligned}$$

The correct choice is H.

**Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists.**

19. $\begin{aligned} 2x - y &= 6 \\ 4x - 2y &= 12 \end{aligned}$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$. Calculate the determinant of $A$.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{aligned} \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} &= 2(-2) - 4(-1) \\ &= -4 + 4 \\ &= 0 \end{aligned}$$

Because the determinant of $A$ is 0, you cannot apply Cramer's Rule; therefore, no unique solution exists.

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

20. $\begin{aligned} 3x - y - z &= 13 \\ 3x - 2y + 3z &= 16 \\ -x + 4y - 8z &= -9 \end{aligned}$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 3 & -1 & -1 \\ 3 & -2 & 3 \\ -1 & 4 & -8 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 3 & -1 & -1 \\ 3 & -2 & 3 \\ -1 & 4 & -8 \end{vmatrix} = 3\begin{vmatrix} -2 & 3 \\ 4 & -8 \end{vmatrix} + 1\begin{vmatrix} 3 & 3 \\ -1 & -8 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix}$$

$$\begin{aligned} &= 3\left[-2(-8) - 4(3)\right] + 1\left[3(-8) - (-1)(3)\right] - 1\left[3(4) - (-1)(-2)\right] \\ &= 3(16 - 12) + 1(-24 + 3) - 1(12 - 2) \\ &= 12 - 21 - 10 \\ &= -19 \end{aligned}$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$\begin{aligned} x &= \frac{|A_x|}{|A|} \\ &= \frac{\begin{vmatrix} 13 & -1 & -1 \\ 16 & -2 & 3 \\ -9 & 4 & -8 \end{vmatrix}}{-19} \\ &= \frac{13\begin{vmatrix} -2 & 3 \\ 4 & -8 \end{vmatrix} + 1\begin{vmatrix} 16 & 3 \\ -9 & -8 \end{vmatrix} - 1\begin{vmatrix} 16 & -2 \\ -9 & 4 \end{vmatrix}}{-19} \\ &= \frac{13\left[-2(-8) - 4(3)\right] + 1\left[16(-8) - (-9)(3)\right] - 1\left[16(4) - (-9)(-2)\right]}{-19} \\ &= \frac{52 - 101 - 46}{-19} \\ &= \frac{-95}{-19} \\ &= 5 \end{aligned}$$

## Mid-Chapter Quiz: Lessons 6-1 through 6-3

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 3 & 13 & -1 \\ 3 & 16 & 3 \\ -1 & -9 & -8 \end{vmatrix}}{-19}$$

$$= \frac{3\begin{vmatrix} 16 & 3 \\ -9 & -8 \end{vmatrix} - 13\begin{vmatrix} 3 & 3 \\ -1 & -8 \end{vmatrix} - 1\begin{vmatrix} 3 & 16 \\ -1 & -9 \end{vmatrix}}{-19}$$

$$= \frac{3[16(-8) - (-9)(3)] - 13[3(-8) - (-1)(3)] - 1[3(-9) - (-1)(16)]}{-19}$$

$$= \frac{3(-128 + 27) - 13(-24 + 3) - 1(-27 + 16)}{-19}$$

$$= \frac{-303 + 273 + 11}{-19}$$

$$= \frac{-19}{-19}$$

$$= 1$$

$$Z = \frac{|A_z|}{|A|}$$

$$= \frac{\begin{vmatrix} 3 & -1 & 13 \\ 3 & -2 & 16 \\ -1 & 4 & -9 \end{vmatrix}}{-19}$$

$$= \frac{3\begin{vmatrix} -2 & 16 \\ 4 & -9 \end{vmatrix} + 1\begin{vmatrix} 3 & 16 \\ -1 & -9 \end{vmatrix} + 13\begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix}}{-19}$$

$$= \frac{3[-2(-9) - 4(16)] + 1[3(-9) - (-1)(16)] + 13[3(4) - (-1)(-2)]}{-19}$$

$$= \frac{-138 - 11 + 130}{-19}$$

$$= \frac{-19}{-19}$$

$$= 1$$

The solution is (5, 1, 1).

# 6-4 Partial Fractions

**Find the partial fraction decomposition of each rational expression.**

1. $\frac{x+1}{x^2+5x+6}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{x+1}{x^2+5x+6}=\frac{A}{x+3}+\frac{B}{x+2}$$

Multiply each side by the LCD, $x^2 + 5x + 6$.

$$x+1=A(x+2)+B(x+3)$$

Group the like terms.

$$x+1=A(x+2)+B(x+3)$$
$$x+1=Ax+2A+Bx+3B$$
$$x+1=Ax+Bx+2A+3B$$
$$x+1=(A+B)x+2A+3B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=1$$
$$2A+3B=1$$

Use any method to solve the new system.

$$A+B=1$$
$$A=1-B$$
$$2A+3B=1$$
$$2(1-B)+3B=1$$
$$2-2B+3B=1$$
$$2+B=1$$
$$B=-1$$
$$A=1-(-1)$$
$$A=2$$

Replace $A$ and $B$ with 2 and −1 in the partial fraction decomposition.

$$\frac{x+1}{x^2+5x+6}=\frac{A}{x+3}+\frac{B}{x+2}$$
$$\frac{x+1}{x^2+5x+6}=\frac{2}{x+3}+\frac{-1}{x+2}$$

# 6-4 Partial Fractions

2. $\dfrac{x-18}{x^2-13x+42}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{x-18}{x^2-13x+42}=\frac{A}{x-6}+\frac{B}{x-7}$$

Multiply each side by the LCD, $x^2 - 13x + 42$.

$$x-18=A(x-7)+B(x-6)$$

Group the like terms.

$$x-18=A(x-7)+B(x-6)$$
$$x-18=Ax-7A+Bx-6B$$
$$x-18=Ax+Bx-7A-6B$$
$$x-18=(A+B)x-7A-6B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=1$$
$$-7A-6B=-18$$

Use any method to solve the new system.

$$A+B=1$$
$$A=1-B$$
$$-7A-6B=-18$$
$$-7(1-B)-6B=-18$$
$$-7+7B-6B=-18$$
$$-7+B=-18$$
$$B=-11$$
$$A=1-(-11)$$
$$A=12$$

Replace $A$ and $B$ with 12 and −11 in the partial fraction decomposition.

$$\frac{x-18}{x^2-13x+42}=\frac{A}{x-6}+\frac{B}{x-7}$$
$$\frac{x-18}{x^2-13x+42}=\frac{12}{x-6}+\frac{-11}{x-7}$$

# 6-4 Partial Fractions

3. $\dfrac{x+13}{x^2+7x+12}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{x+13}{x^2+7x+12}=\frac{A}{x+3}+\frac{B}{x+4}$$

Multiply each side by the LCD, $x^2 + 7x + 12$.

$$x+13=A(x+4)+B(x+3)$$

Group the like terms.

$$x+13=A(x+4)+B(x+3)$$
$$x+13=Ax+4A+Bx+3B$$
$$x+13=Ax+Bx+4A+3B$$
$$x+13=(A+B)x+4A+3B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=1$$
$$4A+3B=13$$

Use any method to solve the new system.

$$A+B=1$$
$$A=1-B$$
$$4A+3B=13$$
$$4(1-B)+3B=13$$
$$4-4B+3B=13$$
$$4-B=13$$
$$B=-9$$
$$A=1-(-9)$$
$$A=10$$

Replace $A$ and $B$ with 10 and −9 in the partial fraction decomposition.

$$\frac{x+13}{x^2+7x+12}=\frac{A}{x+3}+\frac{B}{x+4}$$
$$\frac{x+13}{x^2+7x+12}=\frac{10}{x+3}+\frac{-9}{x+4}$$

# 6-4 Partial Fractions

4. $\dfrac{x+12}{x^2+14x+48}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{x+12}{x^2+14x+48}=\frac{A}{x+6}+\frac{B}{x+8}$$

Multiply each side by the LCD, $x^2 + 14x + 48$.

$$x+12=A(x+8)+B(x+6)$$

Group the like terms.

$$x+12=A(x+8)+B(x+6)$$
$$x+12=Ax+8A+Bx+6B$$
$$x+12=Ax+Bx+8A+6B$$
$$x+12=(A+B)x+8A+6B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=1$$
$$8A+6B=12$$

Use any method to solve the new system.

$$A+B=1$$
$$A=1-B$$
$$8A+6B=12$$
$$8(1-B)+6B=12$$
$$8-8B+6B=12$$
$$8-2B=12$$
$$2B=-4$$
$$B=-2$$
$$A=1-(-2)$$
$$A=3$$

Replace $A$ and $B$ with 3 and −2 in the partial fraction decomposition.

$$\frac{x+12}{x^2+14x+48}=\frac{A}{x+6}+\frac{B}{x+8}$$
$$\frac{x+12}{x^2+14x+48}=\frac{3}{x+6}+\frac{-2}{x+8}$$

# 6-4 Partial Fractions

5. $\dfrac{x+6}{-2x^2-19x-45}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{x+6}{-2x^2-19x-45}=\frac{A}{-2x-9}+\frac{B}{x+5}$$

Multiply each side by the LCD, $-2x^2-19x-45$.

$$x+6=A(x+5)+B(-2x-9)$$

Group the like terms.

$$x+6=A(x+5)+B(-2x-9)$$
$$x+6=Ax+5A-2Bx-9B$$
$$x+6=Ax-2Bx+5A-9B$$
$$x+6=(A-2B)x+5A-9B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A-2B=1$$
$$5A-9B=6$$

Use any method to solve the new system.

$$A-2B=1$$
$$A=1+2B$$
$$5A-9B=6$$
$$5(1+2B)-9B=6$$
$$5+10B-9B=6$$
$$5+B=6$$
$$B=1$$
$$A=1+2(1)$$
$$A=3$$

Replace $A$ and $B$ with 3 and 1 in the partial fraction decomposition.

$$\frac{x+6}{-2x^2-19x-45}=\frac{A}{-2x-9}+\frac{B}{x+5}$$
$$\frac{x+6}{-2x^2-19x-45}=\frac{3}{-2x-9}+\frac{1}{x+5}$$

# 6-4 Partial Fractions

6. $\dfrac{x+7}{2x^2+15x+28}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{x+7}{2x^2+15x+28}=\frac{A}{2x+7}+\frac{B}{x+4}$$

Multiply each side by the LCD, $2x^2 + 15x + 28$.

$$x+7=A(x+4)+B(2x+7)$$

Group the like terms.

$$x+7=A(x+4)+B(2x+7)$$
$$x+7=Ax+4A+2Bx+7B$$
$$x+7=Ax+2Bx+4A+7B$$
$$x+7=(A+2B)x+4A+7B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+2B=1$$
$$4A+7B=7$$

Use any method to solve the new system.

$$A+2B=1$$
$$A=1-2B$$
$$4A+7B=7$$
$$4(1-2B)+7B=7$$
$$4-8B+7B=7$$
$$4-B=7$$
$$B=-3$$
$$A=1-2(-3)$$
$$A=7$$

Replace $A$ and $B$ with 7 and −3 in the partial fraction decomposition.

$$\frac{x+7}{2x^2+15x+28}=\frac{A}{2x+7}+\frac{B}{x+4}$$
$$\frac{x+7}{2x^2+15x+28}=\frac{7}{2x+7}+\frac{-3}{x+4}$$

# 6-4 Partial Fractions

**Find the partial fraction decomposition of each improper rational expression.**

7. $\frac{3x^2+x-4}{x^2-2x}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$\begin{array}{r} 3 \\ x^2-2x\overline{)\,3x^2+x-4} \\ (-)\,(3x^2-6x) \\ \hline 7x-4 \end{array}$$

Therefore, the proper rational expression is $3+\frac{7x-4}{x^2-2x}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$, and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{7x-4}{x^2-2x}=\frac{A}{x}+\frac{B}{x-2}$$

Multiply each side by the LCD, $x^2-2x$.

$$7x-4=A(x-2)+B(x)$$

Group like terms.

$$7x-4=A(x-2)+B(x)$$
$$7x-4=Ax-2A+Bx$$
$$7x-4=Ax+Bx-2A$$
$$7x-4=(A+B)x-2A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=7$$
$$-2A=-4$$

Use any method to solve the new system.

$$-2A=-4$$
$$A=2$$
$$A+B=7$$
$$2+B=7$$
$$B=5$$

Replace $A$, and $B$ with 2, and 5 in the partial fraction decomposition.

$$\frac{7x-4}{x^2-2x}=\frac{A}{x}+\frac{B}{x-2}$$
$$\frac{7x-4}{x^2-2x}=\frac{2}{x}+\frac{5}{x-2}$$

The partial fraction decomposition of the improper rational expression is $3+\frac{2}{x}+\frac{5}{x-2}$.

8. $\dfrac{-5x^2-30x-21}{x^2+7x}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$\begin{array}{r} -5 \\ x^2+7x\overline{)-5x^2-30x-21} \\ (-)\underline{(-5x^2-35x)} \\ 5x-21 \end{array}$$

Therefore, the proper rational expression is $-5+\dfrac{5x-21}{x^2+7x}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, *A*, and *B*, and denominators that are the linear factors of the original denominator.

$$\frac{5x-21}{x^2+7x}=\frac{A}{x}+\frac{B}{x+7}$$

Multiply each side by the LCD, $x^2 + 7x$.

$$5x-21=A(x+7)+B(x)$$

Group like terms.

$$5x-21=A(x+7)+B(x)$$
$$5x-21=Ax+7A+Bx$$
$$5x-21=Ax+Bx+7A$$
$$5x-21=(A+B)x+7A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$$A+B=5$$
$$7A=-21$$

Use any method to solve the new system.

$$7A=-21$$
$$A=-3$$
$$A+B=5$$
$$-3+B=5$$
$$B=8$$

Replace *A*, and *B* with −3, and 8 in the partial fraction decomposition.

$$\frac{5x-21}{x^2+7x}=\frac{A}{x}+\frac{B}{x+7}$$
$$\frac{5x-21}{x^2+7x}=\frac{-3}{x}+\frac{8}{x+7}$$

The partial fraction decomposition of the improper rational expression is $-5+\dfrac{-3}{x}+\dfrac{8}{x+7}$.

9. $\dfrac{-2x^3+4x^2+22x-32}{x^3+2x^2-8x}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$\begin{array}{r} -2 \\ x^3+2x^2-8x\overline{)-2x^3+4x^2+22x-32} \\ (-)\,(-2x^3-4x^2+16x) \\ \hline 8x^2+6x-32 \end{array}$$

Therefore, the proper rational expression is $-2+\dfrac{8x^2+6x-32}{x^3+2x^2-8x}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, *A*, *B*, and *C*, and denominators that are the linear factors of the original denominator.

$$\frac{8x^2+6x-32}{x^3+2x^2-8x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+4}$$

Multiply each side by the LCD, $x^3 + 2x^2 - 8x$.

$$8x^2+6x-32=A(x-2)(x+4)+B(x)(x+4)+C(x)(x-2)$$

Group like terms.

$$8x^2+6x-32=A(x-2)(x+4)+B(x)(x+4)+C(x)(x-2)$$
$$8x^2+6x-32=A(x^2+2x-8)+B(x^2+4x)+C(x^2-2x)$$
$$8x^2+6x-32=Ax^2+2Ax-8A+Bx^2+4Bx+Cx^2-2Cx$$
$$8x^2+6x-32=Ax^2+Bx^2+Cx^2+2Ax+4Bx-2Cx-8A$$
$$8x^2+6x-32=(A+B+C)x^2+(2A+4B-2C)x-8A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$$\begin{aligned} A+B+C&=8\\ 2A+4B-2C&=6\\ -8A&=-32 \end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned} -8A&=-32\\ A&=4\\ 4+B+C&=8\\ B+C&=4\\ 2(4)+4B-2C&=6\\ 8+4B-2C&=6\\ 4B-2C&=-2 \end{aligned}$$

We now have a system of two equations and two variables.

$$\begin{aligned} B + C &= 4 \\ B &= -C + 4 \\ 4B - 2C &= -2 \\ 4(-C + 4) - 2C &= -2 \\ -4C + 16 - 2C &= -2 \\ -6C &= -18 \\ C &= 3 \\ B &= -(3) + 4 \\ B &= 1 \end{aligned}$$

Replace $A$, $B$, and $C$ with 4, 1, and 3 in the partial fraction decomposition.

$$\frac{8x^2 + 6x - 32}{x^3 + 2x^2 - 8x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+4}$$

$$\frac{8x^2 + 6x - 32}{x^3 + 2x^2 - 8x} = \frac{4}{x} + \frac{1}{x-2} + \frac{3}{x+4}$$

The partial fraction decomposition of the improper rational expression is $-2 + \frac{4}{x} + \frac{1}{x-2} + \frac{3}{x+4}$.

## 6-4 Partial Fractions

10. $\dfrac{x^4-2x^3-2x^2+8x-6}{x^2-2x}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$\begin{array}{r} x^2-2 \\ x^2-2x\overline{)\,x^4-2x^3-2x^2+8x-6} \\ (-)\,\underline{(x^4-2x^3)} \\ -2x^2+8x-6 \\ (-)\,\underline{(-2x^2+4x)} \\ 4x-6 \end{array}$$

Therefore, the proper rational expression is $x^2-2+\dfrac{4x-6}{x^2-2x}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{4x-6}{x^2-2x}=\frac{A}{x}+\frac{B}{x-2}$$

Multiply each side by the LCD, $x^2 - 2x$.

$$4x-6=A(x-2)+B(x)$$

Group like terms.

$$4x-6=A(x-2)+B(x)$$

$$4x-6=A(x-2)+B(x)$$

$$4x-6=Ax-2A+Bx$$

$$4x-6=Ax+Bx-2A$$

$$4x-6=(A+B)x-2A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=4$$

$$-2A=-6$$

Use any method to solve the new system.

$$-2A=-6$$

$$A=3$$

$$A+B=4$$

$$3+B=4$$

$$B=1$$

Replace $A$ and $B$ with 3 and 1 in the partial fraction decomposition.

$$\frac{4x-6}{x^2-2x}=\frac{A}{x}+\frac{B}{x-2}$$

$$\frac{4x-6}{x^2-2x}=\frac{3}{x}+\frac{1}{x-2}$$

The partial fraction decomposition of the improper rational expression is $x^2-2+\dfrac{3}{x}+\dfrac{1}{x-2}$.

## 6-4 Partial Fractions

11. $\dfrac{x^3+12x^2+33x+2}{x^2+8x+15}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$\begin{array}{r} x+4 \\ x^2+8x+15\overline{)\,x^3+12x^2+33x+2} \\ (-)\,(x^3+8x^2+15x) \\ \hline 4x^2+18x+2 \\ (-)\,(4x^2+32x+60) \\ \hline -14x-58 \end{array}$$

Therefore, the proper rational expression is $x+4+\dfrac{-14x-58}{x^2+8x+15}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$, and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{-14x-58}{x^2+8x+15}=\frac{A}{x+3}+\frac{B}{x+5}$$

Multiply each side by the LCD, $x^2 + 8x + 15$.

$$-14x-58=A(x+5)+B(x+3)$$

Group like terms.

$$-14x-58=A(x+5)+B(x+3)$$
$$-14x-58=Ax+5A+Bx+3B$$
$$-14x-58=Ax+Bx+5A+3B$$
$$-14x-58=(A+B)x+5A+3B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=-14$$
$$5A+3B=-58$$

Use any method to solve the new system.

$$A+B=-14$$
$$A=-B-14$$
$$5A+3B=-58$$
$$5(-B-14)+3B=-58$$
$$-5B-70+3B=-58$$
$$-2B=12$$
$$B=-6$$
$$A+(-6)=-14$$
$$A=-8$$

Replace $A$, and $B$ with $-8$ and $-6$ in the partial fraction decomposition.

$$\frac{-14x-58}{x^2+8x+15}=\frac{A}{x+3}+\frac{B}{x+5}$$

$$\frac{-14x-58}{x^2+8x+15}=\frac{-8}{x+3}+\frac{-6}{x+5}$$

The partial fraction decomposition of the improper rational expression is $x+4+\frac{-8}{x+3}+\frac{-6}{x+5}$.

12. $\frac{x^4-9x^3+24x^2-4x-12}{x^3-6x^2+8x}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$\begin{array}{r} x-3 \\ x^3-6x^2+8x\overline{)\,x^4-9x^3+24x^2-4x-12} \\ (-)\,\underline{(x^4-6x^3+8x^2)} \\ -3x^3+16x^2-4x \\ (-)\,\underline{(-3x^3+18x^2-24x)} \\ -2x^2+20x-12 \end{array}$$

Therefore, the proper rational expression is $x-3+\frac{-2x^2+20x-12}{x^3-6x^2+8x}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, *A*, *B*, and *C*, and denominators that are the linear factors of the original denominator.

$$\frac{-2x^2+20x-12}{x^3-6x^2+8x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x-4}$$

Multiply each side by the LCD, $x^2 + 5x + 6$.

$$-2x^2+20x-12=A(x-2)(x-4)+B(x)(x-4)+C(x)(x-2)$$

Group like terms.

$$-2x^2+20x-12=A(x-2)(x-4)+B(x)(x-4)+C(x)(x-2)$$
$$-2x^2+20x-12=A(x^2-6x+8)+B(x^2-4x)+C(x^2-2x)$$
$$-2x^2+20x-12=Ax^2-6Ax+8A+Bx^2-4Bx+Cx^2-2Cx$$
$$-2x^2+20x-12=Ax^2+Bx^2+Cx^2-6Ax-4Bx-2Cx+8A$$
$$-2x^2+20x-12=(A+B+C)x^2+(-6A-4B-2C)x+8A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$$\begin{aligned} A+B+C&=-2 \\ -6A-4B-2C&=20 \\ 8A&=-12 \end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned} 8A &= -12 \\ A &= -1.5 \\ -1.5 + B + C &= -2 \\ B + C &= -0.5 \\ -6(-1.5) - 4B - 2C &= 20 \\ 9 - 4B - 2C &= 20 \\ -4B - 2C &= 29 \end{aligned}$$

We now have a system of two equations and two variables.

$$\begin{aligned} B + C &= -0.5 \\ B &= -C - 0.5 \\ -4B - 2C &= 29 \\ -4(-C - 0.5) - 2C &= 29 \\ 4C + 2 - 2C &= 29 \\ 2C &= 27 \\ C &= 13.5 \\ B &= -(13.5) - 0.5 \\ B &= -14 \end{aligned}$$

Replace $A$, $B$, and $C$ with $-1.5$, $-14$, and 13.5 in the partial fraction decomposition.

$$\frac{-2x^2 + 20x - 12}{x^3 - 6x^2 + 8x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4}$$

$$\frac{-2x^2 + 20x - 12}{x^3 - 6x^2 + 8x} = \frac{-1.5}{x} + \frac{-14}{x-2} + \frac{13.5}{x-4}$$

The partial fraction decomposition of the improper rational expression is $x - 3 + \frac{-1.5}{x} + \frac{-14}{x-2} + \frac{13.5}{x-4}$.

# 6-4 Partial Fractions

**Find the partial fraction decomposition of each rational expression with repeated factors in the denominator.**

13. $\dfrac{x^2-3}{x^3+2x^2+x}$

*SOLUTION:*

The expression is proper, so we can rewrite the expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator.
This time, however, we have repeated factors in the denominator.

$x^3 + 2x^2 + x = x(x + 1)(x + 1)$

Because the factor $x + 1$ has a multiplicity of 2, include partial fractions with denominators of $x$, $x + 1$, and $(x + 1)^2$.

$$\frac{x^2-3}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Multiply each side by the LCD, $x(x + 1)^2$.

$$x^2-3 = A(x+1)^2 + B(x)(x+1) + Cx$$

Group like terms.

$$x^2-3 = A(x+1)^2 + B(x)(x+1) + Cx$$
$$x^2-3 = A(x^2+2x+1) + B(x^2+x) + Cx$$
$$x^2-3 = Ax^2+2Ax+A+Bx^2+Bx+Cx$$
$$x^2-3 = Ax^2+Bx^2+2Ax+Bx+Cx+A$$
$$x^2-3 = (A+B)x^2+(2A+B+C)x+A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=1$$
$$2A+B+C=0$$
$$A=-3$$

Use any method to solve the new system.

$$A=-3$$
$$-3+B=1$$
$$B=4$$
$$2A+B+C=0$$
$$2(-3)+4+C=0$$
$$-6+4+C=0$$
$$C=2$$

Replace $A$, $B$, and $C$ with −3, 4, and 2 in the partial fraction decomposition.

$$\frac{x^2-3}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$
$$\frac{x^2-3}{x^3+2x^2+x} = \frac{-3}{x} + \frac{4}{x+1} + \frac{2}{(x+1)^2}$$

## 6-4 Partial Fractions

14. $\dfrac{5x^2-18x+24}{x^3-4x^2+4x}$

*SOLUTION:*

The expression is proper, so we can rewrite the expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator.
This time, however, we have repeated factors in the denominator.

$x^3 - 4x^2 + 4x = x(x - 2)(x - 2)$

Because the factor $x - 2$ has a multiplicity of 2, include partial fractions with denominators of $x$, $x - 2$, and $(x - 2)^2$.

$$\frac{5x^2-18x+24}{x^3-4x^2+4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

Multiply each side by the LCD, $x(x - 2)^2$.

$$5x^2-18x+24=A(x-2)^2+B(x)(x-2)+Cx$$

Group like terms.

$$5x^2-18x+24=A(x-2)^2+B(x)(x-2)+Cx$$
$$5x^2-18x+24=A(x^2-4x+4)+B(x^2-2x)+Cx$$
$$5x^2-18x+24=Ax^2-4Ax+4A+Bx^2-2Bx+Cx$$
$$5x^2-18x+24=Ax^2+Bx^2-4Ax-2Bx+Cx+4A$$
$$5x^2-18x+24=(A+B)x^2+(-4A-2B+C)x+4A$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned} A+B&=5\\ -4A-2B+C&=-18\\ 4A&=24 \end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned} 4A&=24\\ A&=6\\ A+B&=5\\ 6+B&=5\\ B&=-1\\ -4A-2B+C&=-18\\ -4(6)-2(-1)+C&=-18\\ -24+2+C&=-18\\ C&=4 \end{aligned}$$

Replace $A$, $B$, and $C$ with 6, −1, and 4 in the partial fraction decomposition.

$$\frac{5x^2-18x+24}{x^3-4x^2+4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$
$$\frac{5x^2-18x+24}{x^3-4x^2+4x}=\frac{6}{x}+\frac{-1}{x-2}+\frac{4}{(x-2)^2}$$

## 6-4 Partial Fractions

15. $\dfrac{-x^2-22x-50}{x^3+10x^2+25x}$

*SOLUTION:*

The expression is proper, so we can rewrite the expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator.
This time, however, we have repeated factors in the denominator.
$x^3 + 10x^2 + 25x = x(x + 5)(x + 5)$
Because the factor $x + 5$ has a multiplicity of 2, include partial fractions with denominators of $x$, $x + 5$, and $(x + 5)^2$.

$$\frac{-x^2-22x-50}{x^3+10x^2+25x}=\frac{A}{x}+\frac{B}{x+5}+\frac{C}{(x+5)^2}$$

Multiply each side by the LCD, $x(x + 5)^2$.

$$-x^2-22x-50=A(x+5)^2+B(x)(x+5)+Cx$$

Group like terms.

$$-x^2-22x-50=A(x+5)^2+B(x)(x+5)+Cx$$
$$-x^2-22x-50=A(x^2+10x+25)+B(x^2+5x)+Cx$$
$$-x^2-22x-50=Ax^2+10Ax+25A+Bx^2+5Bx+Cx$$
$$-x^2-22x-50=Ax^2+Bx^2+10Ax+5Bx+Cx+25A$$
$$-x^2-22x-50=(A+B)x^2+(10A+5B+C)x+25A$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned} A+B&=-1\\ 10A+5B+C&=-22\\ 25A&=-50 \end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned} 25A&=-50\\ A&=-2\\ A+B&=-1\\ -2+B&=-1\\ B&=1\\ 10A+5B+C&=-22\\ 10(-2)+5(1)+C&=-22\\ -20+5+C&=-22\\ C&=-7 \end{aligned}$$

Replace $A$, $B$, and $C$ with $-2$, 1, and $-7$ in the partial fraction decomposition.

$$\frac{-x^2-22x-50}{x^3+10x^2+25x}=\frac{A}{x}+\frac{B}{x+5}+\frac{C}{(x+5)^2}$$
$$\frac{-x^2-22x-50}{x^3+10x^2+25x}=\frac{-2}{x}+\frac{1}{x+5}+\frac{-7}{(x+5)^2}$$

## 6-4 Partial Fractions

16. $\frac{-5x^2-6x+16}{x^3+8x^2+16x}$

*SOLUTION:*

The expression is proper, so we can rewrite the expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator.

This time, however, we have repeated factors in the denominator.

$x^3 + 8x^2 + 16x = x(x + 4)(x + 4)$

Because the factor $x + 4$ has a multiplicity of 2, include partial fractions with denominators of $x$, $x + 4$, and $(x + 4)^2$.

$$\frac{-5x^2-6x+16}{x^3+8x^2+16x} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{(x+4)^2}$$

Multiply each side by the LCD, $x(x + 4)^2$.

$$-5x^2-6x+16 = A(x+4)^2 + B(x)(x+4) + Cx$$

Group like terms.

$$-5x^2-6x+16 = A(x+4)^2 + B(x)(x+4) + Cx$$

$$-5x^2-6x+16 = A(x^2+8x+16) + B(x^2+4x) + Cx$$

$$-5x^2-6x+16 = Ax^2+8Ax+16A + Bx^2+4Bx + Cx$$

$$-5x^2-6x+16 = Ax^2+Bx^2+8Ax+4Bx+Cx+16A$$

$$-5x^2-6x+16 = (A+B)x^2+(8A+4B+C)x+16A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned} A+B &= -5 \\ 8A+4B+C &= -6 \\ 16A &= 16 \end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned} 16A &= 16 \\ A &= 1 \\ A+B &= -5 \\ 1+B &= -5 \\ B &= -6 \\ 8A+4B+C &= -6 \\ 8(1)+4(-6)+C &= -6 \\ 8-24+C &= -6 \\ C &= 10 \end{aligned}$$

Replace $A$, $B$, and $C$ with 1, −6, and 10 in the partial fraction decomposition.

$$\frac{-5x^2-6x+16}{x^3+8x^2+16x} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{(x+4)^2}$$

$$\frac{-5x^2-6x+16}{x^3+8x^2+16x} = \frac{1}{x} + \frac{-6}{x+4} + \frac{10}{(x+4)^2}$$

## 6-4 Partial Fractions

17. $\dfrac{17x+256}{x^3-16x^2+64x}$

*SOLUTION:*

The expression is proper, so we can rewrite the expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator.
This time, however, we have repeated factors in the denominator.

$x^3 - 16x^2 + 64x = x(x - 8)(x - 8)$

Because the factor $x - 8$ has a multiplicity of 2, include partial fractions with denominators of $x$, $x - 8$, and $(x - 8)^2$.

$$\frac{17x+256}{x^3-16x^2+64x} = \frac{A}{x} + \frac{B}{x-8} + \frac{C}{(x-8)^2}$$

Multiply each side by the LCD, $x(x - 8)^2$.

$$17x+256 = A(x-8)^2 + B(x)(x-8) + Cx$$

Group like terms.

$$17x+256 = A(x-8)^2 + B(x)(x-8) + Cx$$
$$17x+256 = A(x^2-16x+64) + B(x^2-8x) + Cx$$
$$17x+256 = Ax^2-16Ax+64A + Bx^2-8Bx + Cx$$
$$17x+256 = Ax^2 + Bx^2-16Ax-8Bx + Cx + 64A$$
$$17x+256 = (A+B)x^2 + (-16A-8B+C)x + 64A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned} A + B &= 0 \\ -16A - 8B + C &= 17 \\ 64A &= 256 \end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned} 64A &= 256 \\ A &= 4 \\ A + B &= 0 \\ 4 + B &= 0 \\ B &= -4 \\ -16A - 8B + C &= 17 \\ -16(4) - 8(-4) + C &= 17 \\ -64 + 32 + C &= 17 \\ C &= 49 \end{aligned}$$

Replace $A$, $B$, and $C$ with 4, −4, and 49 in the partial fraction decomposition.

$$\frac{17x+256}{x^3-16x^2+64x} = \frac{A}{x} + \frac{B}{x-8} + \frac{C}{(x-8)^2}$$
$$\frac{17x+256}{x^3-16x^2+64x} = \frac{4}{x} + \frac{-4}{x-8} + \frac{49}{(x-8)^2}$$

## 6-4 Partial Fractions

18. $\dfrac{-10x-108}{x^3+12x^2+36x}$

*SOLUTION:*

The expression is proper, so we can rewrite the expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator.
This time, however, we have repeated factors in the denominator.

$x^3 + 12x^2 + 36x = x(x + 6)(x + 6)$

Because the factor $x + 6$ has a multiplicity of 2, include partial fractions with denominators of $x$, $x + 6$, and $(x + 6)^2$.

$$\frac{-10x-108}{x^3+12x^2+36x} = \frac{A}{x} + \frac{B}{x+6} + \frac{C}{(x+6)^2}$$

Multiply each side by the LCD, $x(x + 6)^2$.

$$-10x-108 = A(x+6)^2 + B(x)(x+6) + Cx$$

Group like terms.

$$-10x-108 = A(x+6)^2 + B(x)(x+6) + Cx$$
$$-10x-108 = A(x^2+12x+36) + B(x^2+6x) + Cx$$
$$-10x-108 = Ax^2+12Ax+36A + Bx^2+6Bx + Cx$$
$$-10x-108 = Ax^2 + Bx^2 + 12Ax + 6Bx + Cx + 36A$$
$$-10x-108 = (A+B)x^2 + (12A+6B+C)x + 36A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A + B = 0$$
$$12A + 6B + C = -10$$
$$36A = -108$$

Use any method to solve the new system.

$$36A = -108$$
$$A = -3$$
$$A + B = 0$$
$$-3 + B = 0$$
$$B = 3$$
$$12A + 6B + C = -10$$
$$12(-3) + 6(3) + C = -10$$
$$-36 + 18 + C = -10$$
$$C = 8$$

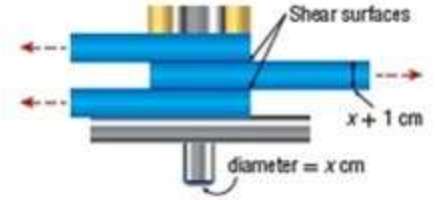
Replace $A$, $B$, and $C$ with $-3$, 3, and 8 in the partial fraction decomposition.

$$\frac{-10x-108}{x^3+12x^2+36x} = \frac{A}{x} + \frac{B}{x+6} + \frac{C}{(x+6)^2}$$
$$\frac{-10x-108}{x^3+12x^2+36x} = \frac{-3}{x} + \frac{3}{x+6} + \frac{8}{(x+6)^2}$$

19. **ENGINEERING** The sum of the average tensile and shear stresses in the bar shown below can be approximated by $s(x) = \dfrac{20x + 10\pi x + 20}{\pi x^3 + \pi x^2}$, where $x$ is the diameter of the pin.

# 6-4 Partial Fractions

**Art Change**




**a.** Find the partial fraction decomposition.
**b**. Graph $s(x)$ and the answer to part **a** in the same viewing window.

*SOLUTION:*

**a.** Decompose $s(x) = \dfrac{20x + 10\pi x + 20}{\pi x^3 + \pi x^2}$. First, divide the numerator by $\pi$.

$$s(x) = \frac{20x + 10\pi x + 20}{\pi x^3 + \pi x^2}$$
$$= \frac{\frac{20}{\pi}x + 10x + \frac{20}{\pi}}{x^3 + x^2}$$

The fraction is proper, so we can rewrite the expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator.
This time, however, we have repeated factors in the denominator.
$x^3 + x^2 = (x^2)(x + 1)$
Because the factor $x^2$ has a multiplicity of 2, include partial fractions with denominators of $x$, $x^2$, and $x + 1$.

$$\frac{\frac{20}{\pi}x + 10x + \frac{20}{\pi}}{x^3 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

Multiply each side by the LCD, $x^3 + x^2$.

$$\frac{\frac{20}{\pi}x + 10x + \frac{20}{\pi}}{x^3 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$
$$\frac{\frac{20}{\pi}x + 10x + \frac{20}{\pi}}{x^3 + x^2} = \frac{A(x)(x+1)}{x^3 + x^2} + \frac{B(x+1)}{x^3 + x^2} + \frac{Cx^2}{x^3 + x^2}$$
$$\frac{20}{\pi}x + 10x + \frac{20}{\pi} = A(x)(x+1) + B(x+1) + Cx^2$$
$$\left(\frac{20}{\pi} + 10\right)x + \frac{20}{\pi} = Ax^2 + Ax + Bx + B + Cx^2$$
$$\left(\frac{20}{\pi} + 10\right)x + \frac{20}{\pi} = Ax^2 + Cx^2 + Ax + Bx + B$$
$$\left(\frac{20}{\pi} + 10\right)x + \frac{20}{\pi} = (A + C)x^2 + (A + B)x + B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A + C = 0$$

$$A + B = \frac{20}{\pi} + 10$$

$$B = \frac{20}{\pi}$$

Use any method to solve the system.

$$B = \frac{20}{\pi}$$

$$A + B = \frac{20}{\pi} + 10$$

$$A + \frac{20}{\pi} = \frac{20}{\pi} + 10$$

$$A = 10$$

$$A + C = 0$$

$$10 + C = 0$$

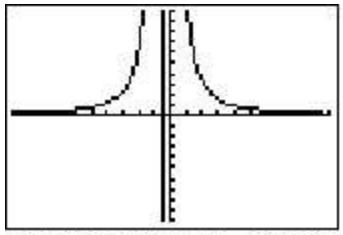
$$C = -10$$

Replace $A$, $B$, and $C$ with 10, $\frac{20}{\delta}$, and −10 in the partial fraction decomposition.

$$\frac{\frac{20}{\pi}x + 10x + \frac{20}{\pi}}{x^3 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$= \frac{10}{x} + \frac{\frac{20}{\pi}}{x^2} + \frac{-10}{x+1}$$

**b.**



[−10, 10] scl: 1 by [−10, 10] scl: 1

## 6-4 Partial Fractions

**Find the partial fraction decomposition of each rational expression with prime quadratic factors in the denominator.**

20. $\dfrac{x^3+5x-5}{(x^2+4)^2}$

*SOLUTION:*

This rational expression is proper. The denominator has one prime quadratic factor of multiplicity 2.
Rewrite the expression as partial fractions. For denominators which are quadratic factors, use $Ax + B$ and $Cx + D$ in the numerators.

$$\frac{x^3+5x-5}{(x^2+4)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{(x^2+4)^2}$$

Multiply each side by the LCD, $(x^2 + 4)^2$.

$$\begin{aligned}x^3+5x-5&=(Ax+B)(x^2+4)+(Cx+D)\\&=Ax^3+4Ax+Bx^2+4B+Cx+D\\&=Ax^3+Bx^2+4Ax+Cx+4B+D\\&=(A)x^3+(B)x^2+(4A+C)x+4B+D\end{aligned}$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned}A&=1\\B&=0\\4A+C&=5\\4B+D&=-5\end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned}A&=1\\B&=0\\4A+C&=5\\4(1)+C&=5\\C&=1\\4B+D&=-5\\4(0)+D&=-5\\D&=-5\end{aligned}$$

Replace $A$, $B$, $C$, and $D$ with 1, 0, 1, and −5 in the partial fraction decomposition.

$$\frac{x^3+5x-5}{(x^2+4)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{(x^2+4)^2}$$

$$\frac{x^3+5x-5}{(x^2+4)^2}=\frac{x}{x^2+4}+\frac{x-5}{(x^2+4)^2}$$

21. $\dfrac{3x^4+4x^2+8x+18}{x(x^2+3)^2}$

*SOLUTION:*

This rational expression is proper. The denominator has one linear factor and one prime quadratic factor of

multiplicity 2.
Rewrite the expression as partial fractions with constant numerator, $A$, and a denominator that is the linear factor. For denominators which are quadratic factors, use $Bx + C$ and $Dx + E$ in the numerators.

$$\frac{3x^4+4x^2+8x+18}{x(x^2+3)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+3}+\frac{Dx+E}{(x^2+3)^2}$$

Multiply each side by the LCD, $x(x^2 + 3)^2$.

$$\begin{aligned}3x^4+4x^2+8x+18&=A(x^2+3)^2+(Bx+C)(x)(x^2+3)+(Dx+E)(x)\\&=A(x^4+6x^2+9)+(Bx+C)(x^3+3x)+Dx^2+Ex\\&=Ax^4+6Ax^2+9A+Bx^4+3Bx^2+Cx^3+3Cx+Dx^2+Ex\\&=Ax^4+Bx^4+Cx^3+6Ax^2+3Bx^2+Dx^2+3Cx+Ex+9A\\&=(A+B)x^4+(C)x^3+(6A+3B+D)x^2+(3C+E)x+9A\end{aligned}$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned}A+B&=3\\C&=0\\6A+3B+D&=4\\3C+E&=8\\9A&=18\end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned}C&=0\\9A&=18\\A&=2\\A+B&=3\\2+B&=3\\B&=1\\6A+3B+D&=4\\6(2)+3(1)+D&=4\\12+3+D&=4\\D&=-11\\3C+E&=8\\3(0)+E&=8\\E&=8\end{aligned}$$

Replace $A$, $B$, $C$, $D$, and $E$ with 2, 1, 0, −11, and 8 in the partial fraction decomposition.

$$\frac{3x^4+4x^2+8x+18}{x(x^2+3)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+3}+\frac{Dx+E}{(x^2+3)^2}$$

$$\frac{3x^4+4x^2+8x+18}{x(x^2+3)^2}=\frac{2}{x}+\frac{x}{x^2+3}+\frac{-11x+8}{(x^2+3)^2}$$

22. $\dfrac{4x^4+x^2-25x+32}{x^5-4x^3+4x}$

*SOLUTION:*

This rational expression is proper. The denominator has one linear factor and one prime quadratic factor of multiplicity 2.

Rewrite the expression as partial fractions with constant numerator, $A$, and a denominator that is the linear factor. For denominators which are quadratic factors, use $Bx + C$ and $Dx + E$ in the numerators.

$$\frac{4x^4+x^2-25x+32}{x(x^2-2)^2}=\frac{A}{x}+\frac{Bx+C}{x^2-2}+\frac{Dx+E}{(x^2-2)^2}$$

Multiply each side by the LCD, $x(x^2 - 2)^2$.

$$\begin{aligned}4x^4+x^2-25x+32&=A(x^2-2)^2+(Bx+C)(x)(x^2-2)+(Dx+E)(x)\\&=A(x^4-4x^2+4)+(Bx+C)(x^3-2x)+Dx^2+Ex\\&=Ax^4-4Ax^2+4A+Bx^4-2Bx^2+Cx^3-2Cx+Dx^2+Ex\\&=Ax^4+Bx^4+Cx^3-4Ax^2-2Bx^2+Dx^2-2Cx+Ex+4A\\&=(A+B)x^4+(C)x^3+(-4A-2B+D)x^2+(-2C+E)x+4A\end{aligned}$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned}A+B&=4\\C&=0\\-4A-2B+D&=1\\-2C+E&=-25\\4A&=32\end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned}C&=0\\4A&=32\\A&=8\\A+B&=4\\8+B&=4\\B&=-4\\-4A-2B+D&=1\\-4(8)-2(-4)+D&=1\\-32+8+D&=1\\D&=25\\-2C+E&=-25\\-2(0)+E&=-25\\E&=-25\end{aligned}$$

Replace $A$, $B$, $C$, $D$, and $E$ with 8, −4, 0, 25, and −25 in the partial fraction decomposition.

$$\frac{4x^4+x^2-25x+32}{x(x^2-2)^2}=\frac{A}{x}+\frac{Bx+C}{x^2-2}+\frac{Dx+E}{(x^2-2)^2}$$

$$\frac{4x^4+x^2-25x+32}{x(x^2-2)^2}=\frac{8}{x}-\frac{4x}{x^2-2}+\frac{25x-25}{(x^2-2)^2}$$

## 6-4 Partial Fractions

23. $\dfrac{8x^3 - 48x + 7}{(x^2 - 6)^2}$

*SOLUTION:*

This rational expression is proper. The denominator has one prime quadratic factor of multiplicity 2.
Rewrite the expression as partial fractions. For denominators which are quadratic factors, use $Ax + B$ and $Cx + D$ in the numerators.

$$\frac{8x^3 - 48x + 7}{(x^2 - 6)^2} = \frac{Ax + B}{x^2 - 6} + \frac{Cx + D}{(x^2 - 6)^2}$$

Multiply each side by the LCD, $(x^2 - 6)^2$.

$$\begin{aligned} 8x^3 - 48x + 7 &= (Ax + B)(x^2 - 6) + (Cx + D) \\ &= Ax^3 - 6Ax + Bx^2 - 6B + Cx + D \\ &= Ax^3 + Bx^2 - 6Ax + Cx - 6B + D \\ &= (A)x^3 + (B)x^2 + (-6A + C)x - 6B + D \end{aligned}$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned} A &= 8 \\ B &= 0 \\ -6A + C &= -48 \\ -6B + D &= 7 \end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned} A &= 8 \\ B &= 0 \\ -6A + C &= -48 \\ -6(8) + C &= -48 \\ C &= 0 \\ -6B + D &= 7 \\ -6(0) + D &= 7 \\ D &= 7 \end{aligned}$$

Replace $A$, $B$, $C$, and $D$ with 8, 0, 0, and 7 in the partial fraction decomposition.

$$\frac{8x^3 - 48x + 7}{(x^2 - 6)^2} = \frac{Ax + B}{x^2 - 6} + \frac{Cx + D}{(x^2 - 6)^2}$$

$$\frac{8x^3 - 48x + 7}{(x^2 - 6)^2} = \frac{8x}{x^2 - 6} + \frac{7}{(x^2 - 6)^2}$$

## 6-4 Partial Fractions

24. $\dfrac{-5x^3-10x^2-6x+4}{(x^2+2x+3)^2}$

*SOLUTION:*

This rational expression is proper. The denominator has one prime quadratic factor of multiplicity 2.
Rewrite the expression as partial fractions. For denominators which are quadratic factors, use $Ax + B$ and $Cx + D$ in the numerators.

$$\frac{-5x^3-10x^2-6x+4}{(x^2+2x+3)^2}=\frac{Ax+B}{x^2+2x+3}+\frac{Cx+D}{(x^2+2x+3)^2}$$

Multiply each side by the LCD, $(x^2 + 2x + 3)^2$.

$$\begin{aligned}-5x^3-10x^2-6x+4&=(Ax+B)(x^2+2x+3)+(Cx+D)\\&=Ax^3+2Ax^2+3Ax+Bx^2+2Bx+3B+Cx+D\\&=Ax^3+2Ax^2+Bx^2+3Ax+2Bx+Cx+3B+D\\&=(A)x^3+(2A+B)x^2+(3A+2B+C)x+3B+D\end{aligned}$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned}A&=-5\\2A+B&=-10\\3A+2B+C&=-6\\3B+D&=4\end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned}A&=-5\\2A+B&=-10\\2(-5)+B&=-10\\B&=0\\3A+2B+C&=-6\\3(-5)+2(0)+C&=-6\\C&=9\\3B+D&=4\\3(0)+D&=4\\D&=4\end{aligned}$$

Replace $A$, $B$, $C$, and $D$ with $-5$, 0, 9, and 4 in the partial fraction decomposition.

$$\frac{-5x^3-10x^2-6x+4}{(x^2+2x+3)^2}=\frac{Ax+B}{x^2+2x+3}+\frac{Cx+D}{(x^2+2x+3)^2}$$

$$\frac{-5x^3-10x^2-6x+4}{(x^2+2x+3)^2}=\frac{-5x}{x^2+2x+3}+\frac{9x+4}{(x^2+2x+3)^2}$$

25. $\dfrac{4x^3-12x^2-5x+20}{(x^2-3x+3)^2}$

*SOLUTION:*

This rational expression is proper. The denominator has one prime quadratic factor of multiplicity 2.
Rewrite the expression as partial fractions. For denominators which are quadratic factors, use $Ax + B$ and $Cx + D$ in the numerators.

$$\frac{4x^3-12x^2-5x+20}{(x^2-3x+3)^2}=\frac{Ax+B}{x^2-3x+3}+\frac{Cx+D}{(x^2-3x+3)^2}$$

Multiply each side by the LCD, $(x^2 - 2x + 3)^2$.

$$\begin{aligned}4x^3-12x^2-5x+20&=(Ax+B)(x^2-3x+3)+(Cx+D)\\&=Ax^3-3Ax^2+3Ax+Bx^2-3Bx+3B+Cx+D\\&=Ax^3-3Ax^2+Bx^2+3Ax-3Bx+Cx+3B+D\\&=(A)x^3+(-3A+B)x^2+(3A-3B+C)x+3B+D\end{aligned}$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned}A&=4\\-3A+B&=-12\\3A-3B+C&=-5\\3B+D&=20\end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned}A&=4\\-3A+B&=-12\\-3(4)+B&=-12\\B&=0\\3A-3B+C&=-5\\3(4)-3(0)+C&=-5\\C&=-17\\3B+D&=20\\3(0)+D&=20\\D&=20\end{aligned}$$

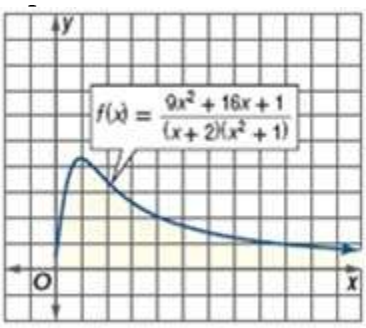
Replace $A$, $B$, $C$, and $D$ with 4, 0, −17, and 20 in the partial fraction decomposition.

$$\frac{4x^3-12x^2-5x+20}{(x^2-3x+3)^2}=\frac{Ax+B}{x^2-3x+3}+\frac{Cx+D}{(x^2-3x+3)^2}$$

$$\frac{4x^3-12x^2-5x+20}{(x^2-3x+3)^2}=\frac{4x}{x^2-3x+3}+\frac{-17x+20}{(x^2-3x+3)^2}$$

**CALCULUS In calculus, you can find the area of the region between the graph of a rational function and the $x$-axis on a restricted domain. The first step in this process is to write the partial fraction decomposition of the rational expression. Find the partial fraction decomposition of each rational expression.**

## 6-4 Partial Fractions

26. 


*SOLUTION:*

This rational expression is proper. The denominator has one linear factor and one prime quadratic factor. Rewrite the expression as partial fractions with constant numerator, $A$, and a denominator that is the linear factor. For the denominator which is a quadratic factor, use $Bx + C$ in the numerator.

$$\frac{9x^2+16x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

Multiply each side by the LCD, $(x + 2)(x^2 + 1)$.

$$\frac{9x^2+16x+1}{(x+2)(x^2+1)} = \frac{A(x^2+1)}{(x+2)(x^2+1)} + \frac{(Bx+C)(x+2)}{(x+2)(x^2+1)}$$

$$9x^2+16x+1 = A(x^2+1)+(Bx+C)(x+2)$$

$$9x^2+16x+1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$$

$$9x^2+16x+1 = Ax^2 + Bx^2 + 2Bx + Cx + A + 2C$$

$$9x^2+16x+1 = (A+B)x^2 + (2B+C)x + A + 2C$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A + B = 9$$

$$2B + C = 16$$

$$A + 2C = 1$$

Use any method to solve the new system.

$$A + 2C = 1$$
$$A = 1 - 2C$$

$$A + B = 9$$
$$(1 - 2C) + B = 9$$
$$-2C + B = 8$$

$$2B + C = 16$$
$$C = 16 - 2B$$

$$-2(16 - 2B) + B = 8$$
$$-32 + 4B + B = 8$$
$$5B = 40$$
$$B = 8$$

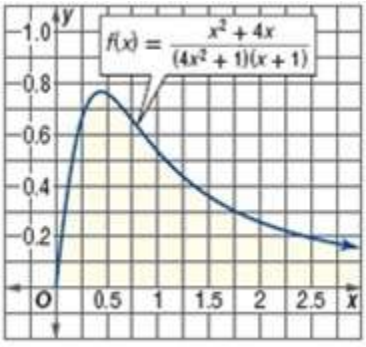
$$C = 16 - 2(8)$$
$$C = 0$$

$$A + 2(0) = 1$$
$$A = 1$$

Replace $A$, $B$, and $C$ with 1, 8, and 0 in the partial fraction decomposition.

$$\frac{9x^2 + 16x + 1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx + C}{x^2+1}$$
$$= \frac{1}{x+2} + \frac{8x}{x^2+1}$$

27.




*SOLUTION:*

This rational expression is proper. The denominator has one linear factor and one prime quadratic factor.
Rewrite the expression as partial fractions with constant numerator, $C$, and a denominator that is the linear factor.
For the denominator which is a quadratic factor, use $Ax + B$ in the numerator.

$$\frac{x^2 + 4x}{(4x^2+1)(x+1)} = \frac{Ax + B}{4x^2+1} + \frac{C}{x+1}$$

Multiply each side by the LCD, $(x + 1)(4x^2 + 1)$.

$$\frac{x^2+4x}{(4x^2+1)(x+1)} = \frac{(Ax+B)(x+1)}{(4x^2+1)(x+1)} + \frac{C(4x^2+1)}{(4x^2+1)(x+1)}$$

$$x^2+4x = Ax^2+Ax+Bx+B+4Cx^2+C$$

$$x^2+4x = Ax^2+4Cx^2+Ax+Bx+B+C$$

$$x^2+4x = (A+4C)x^2+(A+B)x+B+C$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$$A+4C=1$$

$$A+B=4$$

$$B+C=0$$

Use any method to solve the new system.

$$B+C=0$$

$$B=-C$$

$$A+B=4$$

$$A-C=4$$

$$A=C+4$$

$$A+4C=1$$

$$C+4+4C=1$$

$$5C=-3$$

$$C=-0.6$$

$$A=-0.6+4$$

$$A=3.4$$

$$B=-(-0.6)$$

$$B=0.6$$

Replace *A*, *B*, and *C* with 3.4, 0.6, and −0.6 in the partial fraction decomposition.

$$\frac{x^2+4x}{(4x^2+1)(x+1)} = \frac{Ax+B}{4x^2+1} + \frac{C}{x+1}$$

$$= \frac{3.4x+0.6}{4x^2+1} + \frac{-0.6}{x+1}$$

**Find the partial fraction decomposition of each rational expression. Then use a graphing calculator to check your answer.**

28. $\frac{x+4}{3x^2-x-2}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, *A* and *B*, and denominators that are the linear factors of the original denominator.

## 6-4 Partial Fractions

$$\frac{x+4}{3x^2-x-2}=\frac{A}{3x+2}+\frac{B}{x-1}$$

Multiply each side by the LCD, $3x^2 - x - 2$.

$$x+4=A(x-1)+B(3x+2)$$

Group the like terms.

$$x+4=A(x-1)+B(3x+2)$$
$$x+4=Ax-A+3Bx+2B$$
$$x+4=Ax+3Bx-A+2B$$
$$x+4=(A+3B)x-A+2B$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

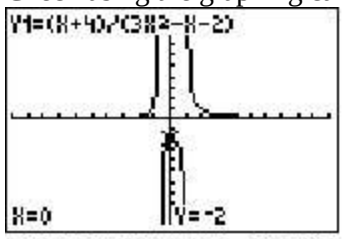
Use any method to solve the new system.

$$\begin{aligned} A+3B&=1\\ A&=1-3B\\ -A+2B&=4\\ -(1-3B)+2B&=4\\ -1+3B+2B&=4\\ 5B&=5\\ B&=1\\ A&=1-3(1)\\ A&=-2 \end{aligned}$$

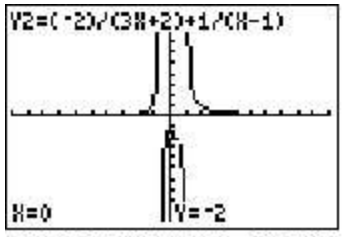
Replace $A$ and $B$ with $-2$ and 1 in the partial fraction decomposition.

$$\frac{x+4}{3x^2-x-2}=\frac{A}{3x+2}+\frac{B}{x-1}$$
$$=\frac{-2}{3x+2}+\frac{1}{x-1}$$

Check using the graphing calculator.




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[−10, 10] sd: 1 by [−10, 10] scl: 1

As indicated in the graphs the two equations represent the same curve.

29. $\frac{5x^2-2x+8}{x^3-4x}$

*SOLUTION:*

$x^3 - 4x = x(x + 2)(x - 2)$

Rewrite the expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator.

$$\frac{5x^2-2x+8}{x^3-4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

Multiply each side by the LCD, $x^3 - 4x$.

$$\frac{5x^2-2x+8}{x^3-4x} = \frac{A(x-2)(x+2)}{x^3-4x} + \frac{B(x)(x+2)}{x^3-4x} + \frac{C(x)(x-2)}{x^3-4x}$$

Group the like terms.

$$5x^2-2x+8 = A(x-2)(x+2) + B(x)(x+2) + C(x)(x-2)$$
$$5x^2-2x+8 = Ax^2 - 4A + Bx^2 + 2Bx + Cx^2 - 2Cx$$
$$5x^2-2x+8 = Ax^2 + Bx^2 + Cx^2 + 2Bx - 2Cx - 4A$$
$$5x^2-2x+8 = (A+B+C)x^2 + (2B-2C)x - 4A$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A + B + C = 5$$
$$2B - 2C = -2$$
$$-4A = 8$$

Solve the system.

$$-4A = 8$$
$$A = -2$$

$$A + B + C = 5$$
$$-2 + B + C = 5$$
$$B + C = 7$$
$$B = 7 - C$$

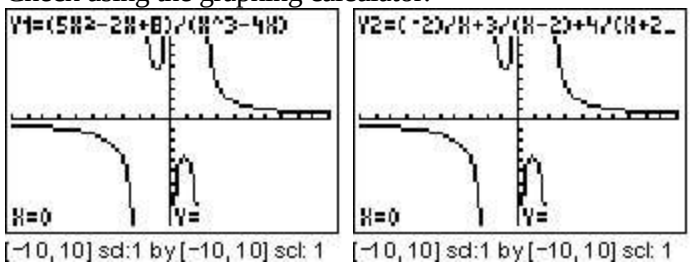
$$2B - 2C = -2$$
$$2(7 - C) - 2C = -2$$
$$14 - 2C - 2C = -2$$
$$-4C = -16$$
$$C = 4$$
$$B = 7 - 4$$
$$B = 3$$

Replace $A$, $B$, and $C$ with $-2$, 3, and 4 in the partial fraction decomposition.

## 6-4 Partial Fractions

$$\frac{5x^2-2x+8}{x^3-4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}$$
$$=\frac{-2}{x}+\frac{3}{x-2}+\frac{4}{x+2}$$

Check using the graphing calculator.




As indicated in the graphs the two equations represent the same curve.

30. $\frac{4x^2-3x+3}{4x(x-1)^2}$

*SOLUTION:*

The expression is proper, so we can rewrite the expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator.
This time, however, we have repeated factors in the denominator.

$4x(x-1)^2 = 4x(x-1)(x-1)$

Because the factor $x-1$ has a multiplicity of 2, include partial fractions with denominators of $4x$, $x-1$, and $(x-1)^2$.

$$\frac{4x^2-3x+3}{4x(x-1)^2}=\frac{A}{4x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Multiply each side by the LCD, $4x(x-1)^2$.

$$\frac{4x^2-3x+3}{4x(x-1)^2}=\frac{A}{4x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$
$$\frac{4x^2-3x+3}{4x(x-1)^2}=\frac{A(x-1)^2}{4x(x-1)^2}+\frac{B(4x)(x-1)}{4x(x-1)^2}+\frac{C(4x)}{4x(x-1)^2}$$

Group like terms.

$$4x^2-3x+3=A(x-1)^2+B(4x)(x-1)+C(4x)$$
$$4x^2-3x+3=A(x^2-2x+1)+4Bx^2-4Bx+4Cx$$
$$4x^2-3x+3=Ax^2-2Ax+A+4Bx^2-4Bx+4Cx$$
$$4x^2-3x+3=Ax^2+4Bx^2-2Ax-4Bx+4Cx+A$$
$$4x^2-3x+3=(A+4B)x^2+(-2A-4B+4C)x+A$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+4B=4$$
$$-2A-4B+4C=-3$$
$$A=3$$

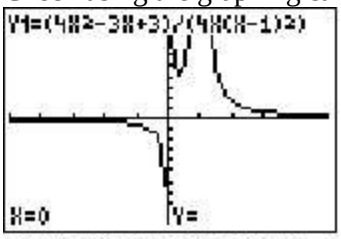
Solve the system.

$$A = 3$$
$$3 + 4B = 4$$
$$4B = 1$$
$$B = 0.25$$
$$-2A - 4B + 4C = -3$$
$$-2(3) - 4(0.25) + 4C = -3$$
$$-6 - 1 + 4C = -3$$
$$4C = 4$$
$$C = 1$$

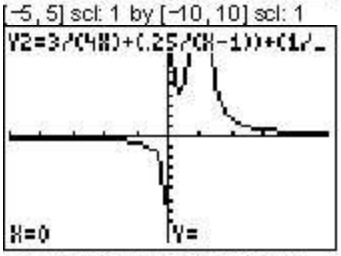
Replace $A$, $B$, and $C$ with 3, 0.25, and 1 in the partial fraction decomposition.

$$\frac{4x^2 - 3x + 3}{4x(x-1)^2} = \frac{A}{4x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$= \frac{3}{4x} + \frac{0.25}{x-1} + \frac{1}{(x-1)^2}$$

Check using the graphing calculator.




[−5, 5] scl: 1 by [−10, 10] scl: 1




[−5, 5] scl: 1 by [−10, 10] scl: 1

As indicated in the graphs the two equations represent the same curve.

31. $\frac{x^2 + x + 5}{(x^2 + 3)^2}$

*SOLUTION:*

This rational expression is proper. The denominator has one prime quadratic factor of multiplicity 2. Rewrite the expression as partial fractions with $Ax + B$ and $Cx + D$ in the numerators.

$$\frac{x^2 + x + 5}{(x^2 + 3)^2} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{(x^2 + 3)^2}$$

Multiply each side by the LCD, $(x^2 + 3)^2$.

$$\frac{x^2 + x + 5}{(x^2 + 3)^2} = \frac{(Ax + B)(x^2 + 3)}{(x^2 + 3)^2} + \frac{Cx + D}{(x^2 + 3)^2}$$

Group like terms.

$$x^2 + x + 5 = (Ax + B)(x^2 + 3) + Cx + D$$
$$x^2 + x + 5 = Ax^3 + 3Ax + Bx^2 + 3B + Cx + D$$
$$x^2 + x + 5 = Ax^3 + Bx^2 + 3Ax + Cx + 3B + D$$
$$x^2 + x + 5 = Ax^3 + Bx^2 + (3A + C)x + 3B + D$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$$A = 0$$
$$B = 1$$
$$3A + C = 1$$
$$3B + D = 5$$

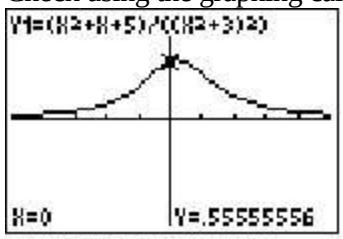
Solve the system.

$$A = 0$$
$$B = 1$$
$$3(0) + C = 1$$
$$C = 1$$
$$3(1) + D = 5$$
$$D = 2$$

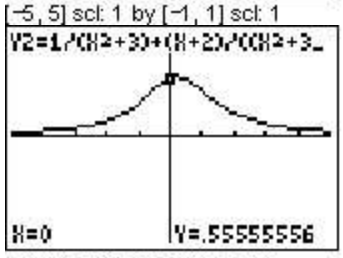
Replace *A*, *B*, *C*, and *D* with 0, 1, 1 and 2 in the partial fraction decomposition.

$$\frac{x^2 + x + 5}{(x^2 + 3)^2} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{(x^2 + 3)^2}$$
$$= \frac{1}{x^2 + 3} + \frac{x + 2}{(x^2 + 3)^2}$$

Check using the graphing calculator.




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[−5, 5] scl: 1 by [−1, 1] scl: 1

As indicated in the graphs the two equations represent the same curve.

32. $$\frac{2x^3}{(x-1)^2(x+1)^2}$$

*SOLUTION:*

This rational expression is proper. The denominator has two linear factors of multiplicity 2.

## 6-4 Partial Fractions

Rewrite the expression as partial fractions with $A$, $B$, $C$, and $D$ in the numerators.

$$\frac{2x^3}{(x-1)^2(x+1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

Multiply each side by the LCD, $(x-1)^2(x+1)^2$.

$$\frac{2x^3}{(x-1)^2(x+1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

$$\frac{2x^3}{(x-1)^2(x+1)^2}=\frac{A(x-1)(x+1)^2}{(x-1)^2(x+1)^2}+\frac{B(x+1)^2}{(x-1)^2(x+1)^2}+\frac{C(x-1)^2(x+1)}{(x-1)^2(x+1)^2}+\frac{D(x-1)^2}{(x-1)^2(x+1)^2}$$

Group like terms.

$$2x^3=A(x-1)(x+1)^2+B(x+1)^2+C(x-1)^2(x+1)+D(x-1)^2$$

$$2x^3=A(x-1)(x+1)^2+B(x+1)^2+C(x-1)^2(x+1)+D(x-1)^2$$

$$2x^3=A(x-1)(x^2+2x+1)+B(x^2+2x+1)+C(x^2-2x+1)(x+1)+D(x^2-2x+1)$$

$$2x^3=A(x^3+x^2-x-1)+Bx^2+2Bx+B+C(x^3-x^2-x+1)+Dx^2-2Dx+D$$

$$2x^3=Ax^3+Ax^2-Ax-A+Bx^2+2Bx+B+Cx^3-Cx^2-Cx+C+Dx^2-2Dx+D$$

$$2x^3=Ax^3+Cx^3+Ax^2+Bx^2-Cx^2+Dx^2-Ax+2Bx-Cx-2Dx-A+B+C+D$$

$$2x^3=(A+C)x^3+(A+B-C+D)x^2+(-A+2B-C-2D)x-A+B+C+D$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned}A+C&=2\\A+B-C+D&=0\\-A+2B-C-2D&=0\\-A+B+C+D&=0\end{aligned}$$

Solve the system.

$$\begin{aligned}A+C&=2\\A&=2-C\\A+B+C+D&=0\\(2-C)+B-C+D&=0\\B-2C+D&=-2\\-A+2B-C-2D&=0\\-(2-C)+2B-C-2D&=0\\2B-2D&=2\\-A+B+C+D&=0\\-(2-C)+B+C+D&=0\\B+2C+D&=2\end{aligned}$$

Use elimination to get 2 equations with 2 variables.

$$\begin{array}{r}B-2C+D=-2\\(+)\,B+2C+D=2\\\hline 2B+2D=0\end{array}$$

Solve the system of 2 equations.

$$2B + 2D = 0$$
$$(+)\,2B - 2D = 2$$
$$4B = 2$$
$$B = 0.5$$
$$2(0.5) + 2D = 0$$
$$1 + 2D = 0$$
$$D = -0.5$$

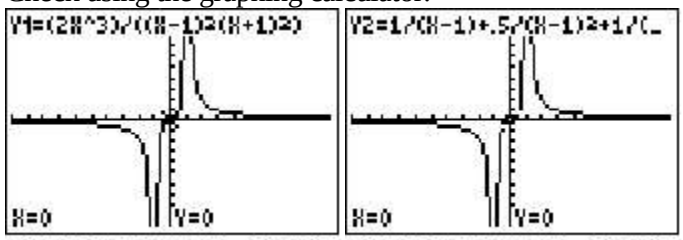
Solve for $C$ and $A$.

$$B + 2C + D = 2$$
$$0.5 + 2C - 0.5 = 2$$
$$C = 1$$
$$A = 2 - C$$
$$A = 1$$

Replace $A$, $B$, $C$, and $D$ with 1, 0.5, 1 and −0.5 in the partial fraction decomposition.

$$\frac{2x^3}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$
$$\frac{2x^3}{(x-1)^2(x+1)^2} = \frac{1}{x-1} + \frac{0.5}{(x-1)^2} + \frac{1}{x+1} + \frac{-0.5}{(x+1)^2}$$

Check using the graphing calculator.




[−10, 10] scl: 1 by [−10, 10] scl: 1 [−10, 10] scl: 1 by [−10, 10] scl: 1

As indicated in the graphs the two equations represent the same curve.

33. $\dfrac{2x^3 + 12x^2 - 3x + 3}{x^2 + 6x + 5}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$\begin{array}{r} 2x \\ x^2 + 6x + 5 \overline{)\,2x^3 + 12x^2 - 3x + 3} \\ (-)\,\underline{(2x^3 + 12x^2 + 10x)} \\ -13x + 3 \end{array}$$

Therefore, the proper rational expression is $2x + \dfrac{-13x + 3}{x^2 + 6x + 5}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{-13x + 3}{x^2 + 6x + 5} = \frac{A}{x+5} + \frac{B}{x+1}$$

Multiply each side by the LCD, $x^2 + 6x + 5$.

$$\frac{-13x+3}{x^2+6x+5} = \frac{A}{x+5} + \frac{B}{x+1}$$

$$\frac{-13x+3}{x^2+6x+5} = \frac{A(x+1)}{x^2+6x+5} + \frac{B(x+5)}{x^2+6x+5}$$

$$-13x+3 = A(x+1) + B(x+5)$$

Group like terms.

$$-13x+3 = A(x+1) + B(x+5)$$

$$-13x+3 = Ax + A + Bx + 5B$$

$$-13x+3 = Ax + Bx + A + 5B$$

$$-13x+3 = (A+B)x + A + 5B$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$$A + B = -13$$

$$A + 5B = 3$$

Solve the system.

$$A + B = -13$$

$$A = -13 - B$$

$$A + 5B = 3$$

$$-13 - B + 5B = 3$$

$$4B = 16$$

$$B = 4$$

$$A = -13 - 4$$

$$A = -17$$

Replace $A$ and $B$ with −17 and 4in the partial fraction decomposition.

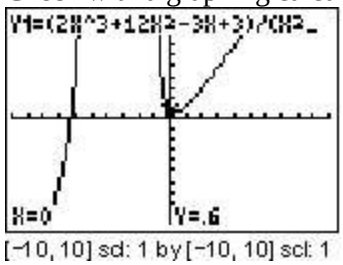
$$\frac{-13x+3}{x^2+6x+5} = \frac{A}{x+5} + \frac{B}{x+1}$$

$$= \frac{-17}{x+5} + \frac{4}{x+1}$$

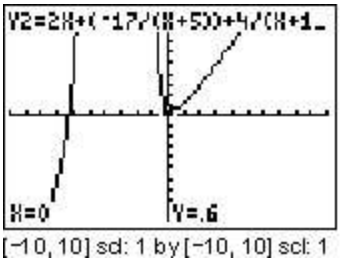
Rewrite the entire partial fraction decomposition.

$$2x + \frac{-17}{x+5} + \frac{4}{x+1}$$

Check with a graphing calculator.




[−10, 10] sd: 1 by [−10, 10] scl: 1




[−10, 10] sd: 1 by [−10, 10] scl: 1

As indicated in the graphs the two equations represent the same curve.

## 6-4 Partial Fractions

34. Find two rational expressions that have the sum $\frac{x+4}{3x^2-x-2}$.

*SOLUTION:*

This problem is another way of asking for the decomposition.

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{x+4}{3x^2-x-2}=\frac{A}{3x+2}+\frac{B}{x-1}$$

Multiply each side by the LCD, $3x^2 - x - 2$.

$$x+4=A(x-1)+B(3x+2)$$

Group the like terms.

$$x+4=A(x-1)+B(3x+2)$$
$$x+4=Ax-A+3Bx+2B$$
$$x+4=Ax+3Bx-A+2B$$
$$x+4=(A+3B)x-A+2B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+3B=1$$
$$-A+2B=4$$

Use any method to solve the system.

$$A+3B=1$$
$$A=1-3B$$
$$-A+2B=4$$
$$-(1-3B)+2B=4$$
$$-1+3B+2B=4$$
$$5B=5$$
$$B=1$$
$$A=1-3(1)$$
$$A=-2$$

Replace $A$ and $B$ with $-2$ and 1 in the partial fraction decomposition.

$$\frac{x+4}{3x^2-x-2}=\frac{A}{3x+2}+\frac{B}{x-1}$$
$$\frac{x+4}{3x^2-x-2}=\frac{-2}{3x+2}+\frac{1}{x-1}$$

The rational expressions are $\frac{-2}{3x+2}, \frac{1}{x-1}$.

35. Find three rational expressions that have the sum $\frac{6-x}{x^3+2x^2+x}$.

*SOLUTION:*

This problem is another way of asking for the decomposition.

## 6-4 Partial Fractions

Rewrite the expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator. This time, however, we have repeated factors in the denominator. Because the factor $x + 1$ has a multiplicity of 2, include partial fractions with denominators of $x$, $x + 1$, and $(x + 1)^2$.

$$\frac{6-x}{x^3+2x^2+x}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

Multiply each side by the LCD, $x^3 + 2x^2 + x$.

$$\frac{6-x}{x^3+2x^2+x}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

Group the like terms.

$$\frac{6-x}{x^3+2x^2+x}=\frac{A(x+1)^2}{x^3+2x^2+x}+\frac{Bx(x+1)}{x^3+2x^2+x}+\frac{Cx}{x^3+2x^2+x}$$

$$6-x=A(x+1)^2+Bx(x+1)+Cx$$

$$6-x=A(x^2+2x+1)+Bx^2+Bx+Cx$$

$$6-x=Ax^2+2Ax+A+Bx^2+Bx+Cx$$

$$6-x=Ax^2+Bx^2+2Ax+Bx+Cx+A$$

$$6-x=(A+B)x^2+(2A+B+C)x+A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned} A+B&=0\\ 2A+B+C&=-1\\ A&=6 \end{aligned}$$

Use any method to solve the system.

$$\begin{aligned} A&=6\\ A+B&=0\\ 6+B&=0\\ B&=-6\\ 2A+B+C&=-1\\ 2(6)+(-6)+C&=-1\\ 12-6+C&=-1\\ C&=-7 \end{aligned}$$

Replace $A$, $B$, and $C$ with 6, −6, and −7 in the partial fraction decomposition.

$$\frac{6-x}{x^3+2x^2+x}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$\frac{6-x}{x^3+2x^2+x}=\frac{6}{x}+\frac{-6}{x+1}+\frac{-7}{(x+1)^2}$$

The rational expressions are $\frac{6}{x}, \frac{-6}{x+1}, \frac{-7}{(x+1)^2}$.

**Find *A*, *B*, *C*, and *D* in terms of *r* and *t*.**

36. $\frac{rx-t}{x^2-x-2}=\frac{A}{x-2}+\frac{B}{x+1}$

*SOLUTION:*

$\frac{rx-t}{x^2-x-2}=\frac{A}{x-2}+\frac{B}{x+1}$

Multiply by the GCD, $x^2 - x - 2$.

$\frac{rx-t}{x^2-x-2}=\frac{A}{x-2}+\frac{B}{x+1}$

$\frac{rx-t}{x^2-x-2}=\frac{A(x+1)}{x^2-x-2}+\frac{B(x-2)}{x^2-x-2}$

$rx-t=Ax+A+Bx-2B$

$rx-t=Ax+Bx+A-2B$

$rx-t=(A+B)x+A-2B$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$A+B=r$

$A-2B=-t$

Solve the system.

$A+B=r$

$A=r-B$

$A-2B=-t$

$r-B-2B=-t$

$-3B=-t-r$

$B=\frac{t+r}{3}$

$A=r-\frac{t+r}{3}$

$A=\frac{3r}{3}-\frac{t+r}{3}$

$A=\frac{2r-t}{3}$

## 6-4 Partial Fractions

37. $\dfrac{4x^2+rx+2t}{x^2+3x}=4+\dfrac{A}{x}+\dfrac{B}{x+3}$

*SOLUTION:*

$\dfrac{4x^2+rx+2t}{x^2+3x}=4+\dfrac{A}{x}+\dfrac{B}{x+3}$

Multiply by the GCD, $x^2 - x - 2$.

$\dfrac{4x^2+rx+2t}{x^2+3x}=4+\dfrac{A}{x}+\dfrac{B}{x+3}$

$\dfrac{4x^2+rx+2t}{x^2+3x}=\dfrac{4(x^2+3x)}{x^2+3x}+\dfrac{A(x+3)}{x^2+3x}+\dfrac{Bx}{x^2+3x}$

$4x^2+rx+2t=4x^2+12x+Ax+3A+Bx$

$4x^2+rx+2t=4x^2+(12+A+B)x+3A$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$r=12+A+B$

$2t=3A$

Solve the system.

$2t=3A$

$\dfrac{2}{3}t=A$

$r=12+A+B$

$r=12+\dfrac{2}{3}t+B$

$r-12-\dfrac{2}{3}t=B$

## 6-4 Partial Fractions

38. $\frac{rx+t}{x^3+x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$

*SOLUTION:*

$\frac{rx+t}{x^3+x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$

Multiply by the GCD, $x^3 + x^2$.

$\frac{rx+t}{x^3+x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$

$\frac{rx+t}{x^3+x^2}=\frac{A(x)(x+1)}{x^3+x^2}+\frac{B(x+1)}{x^3+x^2}+\frac{Cx^2}{x^3+x^2}$

$rx+t=Ax^2+Ax+Bx+B+Cx^2$

$rx+t=Ax^2+Cx^2+Ax+Bx+B$

$rx+t=(A+C)x^2+(A+B)x+B$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$A+C=0$

$A+B=r$

$B=t$

Solve the system.

$B=t$

$A+B=r$

$A+t=r$

$A=r-t$

$A+C=0$

$r-t+C=0$

$C=t-r$

## 6-4 Partial Fractions

39. $\dfrac{3x^3+5rx^2-16tx+32}{x^2(x^2+16)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+16}$

*SOLUTION:*

$\dfrac{3x^3+5rx^2-16tx+32}{x^2(x^2+16)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+16}$

Multiply by the GCD, $x^2(x^2 + 16)$.

$\dfrac{3x^3+5rx^2-16tx+32}{x^2(x^2+16)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+16}$

$\dfrac{3x^3+5rx^2-16tx+32}{x^2(x^2+16)}=\dfrac{Ax(x^2+16)}{x^2(x^2+16)}+\dfrac{B(x^2+16)}{x^2(x^2+16)}+\dfrac{(Cx+D)x^2}{x^2(x^2+16)}$

$3x^3+5rx^2-16tx+32=Ax^3+16Ax+Bx^2+16B+Cx^3+Dx^2$ Equate the coefficients on the left and right side of

$3x^3+5rx^2-16tx+32=Ax^3+Cx^3+Bx^2+Dx^2+16Ax+16B$

$3x^3+5rx^2-16tx+32=(A+C)x^3+(B+D)x^2+16Ax+16B$

the equation to form two equations with 2 variables. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$A+C=3$

$B+D=5r$

$16A=-16r$

$16B=32$

Solve the system.

$16B=32$

$B=2$

$B+D=5r$

$D+2=5r$

$D=5r-2$

$16A=-16t$

$A=-t$

$A+C=3$

$-t+C=3$

$C=3+t$

**Find the partial fraction decomposition of each rational expression.**

40. $\dfrac{x^3+2x-1}{(x^2-x-2)^2}$

*SOLUTION:*

$x^2 - x - 2 = (x - 2)(x + 1)$

This rational expression is proper. The denominator has two linear factors of multiplicity 2.
Rewrite the expression as partial fractions with *A*, *B*, *C*, and *D* in the numerators.

$\dfrac{x^3+2x-1}{(x^2-x-2)^2}=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{C}{x+1}+\dfrac{D}{(x+1)^2}$

Multiply each side by the LCD, $(x^2 - x - 2)^2$.

## 6-4 Partial Fractions

$$\frac{x^3+2x-1}{(x^2-x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

$$\frac{x^3+2x-1}{(x^2-x-2)^2}=\frac{A(x+1)^2(x-2)}{(x^2-x-2)^2}+\frac{B(x+1)^2}{(x^2-x-2)^2}+\frac{C(x+1)(x-2)^2}{(x^2-x-2)^2}+\frac{D(x-2)^2}{(x^2-x-2)^2}$$

$$\begin{aligned}
&x^3+2x-1\\
&=A(x^2+2x+1)(x-2)+B(x^2+2x+1)+C(x+1)(x^2-4x+4)+D(x^2-4x+4)\\
&=A(x^3-3x-2)+Bx^2+2Bx+B+C(x^3-3x^2+4)+Dx^2-4Dx+4D\\
&=Ax^3-3Ax-2A+Bx^2+2Bx+B+Cx^3-3Cx^2+4C+Dx^2-4Dx+4D\\
&=Ax^3+Cx^3+Bx^2-3Cx^2+Dx^2-3Ax+2Bx-4Dx-2A+B+4C+4D\\
&=(A+C)x^3+(B-3C+D)x^2+(-3A+2B-4D)x-2A+B+4C+4D
\end{aligned}$$

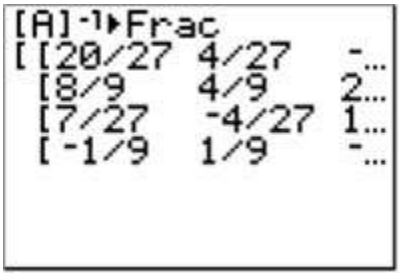
Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$$\begin{aligned}
A+C&=1\\
B-3C+D&=0\\
-3A+2B-4D&=2\\
-2A+B+4C+4D&=-1
\end{aligned}$$

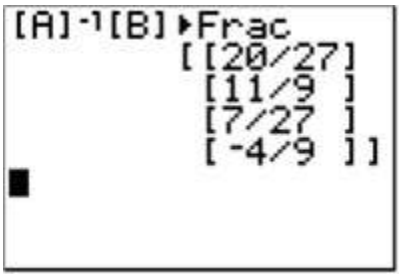
Solve the system using inverse matrices. Write the system in the matrix form $AX = B$.

$$\begin{bmatrix}1&0&1&0\\0&1&-3&1\\-3&2&0&-4\\-2&1&4&4\end{bmatrix}\cdot\begin{bmatrix}A\\B\\C\\D\end{bmatrix}=\begin{bmatrix}1\\0\\2\\-1\end{bmatrix}$$

Find $A^{-1}$.




Find $A^{-1}B$.




Replace $A$, $B$, $C$, and $D$ with $\frac{20}{27}, \frac{11}{9}, \frac{7}{27}$, and $-\frac{4}{9}$ in the partial fraction decomposition.

$$\frac{x^3+2x-1}{(x^2-x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

$$\frac{x^3+2x-1}{(x^2-x-2)^2}=\frac{\frac{20}{27}}{x-2}+\frac{\frac{11}{9}}{(x-2)^2}+\frac{\frac{7}{27}}{x+1}+\frac{-\frac{4}{9}}{(x+1)^2}$$

$$=\frac{\frac{7}{27}}{x+1}+\frac{-\frac{4}{9}}{(x+1)^2}+\frac{\frac{20}{27}}{x-2}+\frac{\frac{11}{9}}{(x-2)^2}$$

41. $\dfrac{x^3+4}{(x^2-1)(x^2+3x+2)}$

*SOLUTION:*

This rational expression is proper. The denominator has four linear factors. Notice, however, that one of the factors, $x + 1$, is repeated.

Rewrite the expression as partial fractions with $A$, $B$, $C$, and $D$ in the numerators.

$$\frac{x^3+4}{(x^2-1)(x^2+3x+2)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x+2}+\frac{D}{(x+1)^2}$$

Multiply each side by the LCD, $(x + 1)^2(x - 1)(x + 2)$.

$$\frac{x^3+4}{(x^2-1)(x^2+3x+2)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x+2}+\frac{D}{(x+1)^2}$$

$$\frac{x^3+4}{(x^2-1)(x^2+3x+2)}=\frac{A(x^2-1)(x+2)}{(x^2-1)(x^2+3x+2)}+\frac{B(x+1)(x^2+3x+2)}{(x^2-1)(x^2+3x+2)}+\frac{C(x^2-1)(x+1)}{(x^2-1)(x^2+3x+2)}+\frac{D(x-1)(x+2)}{(x^2-1)(x^2+3x+2)}$$

$$x^3+4=A(x^3+2x^2-x-2)+B(x^3+4x^2+5x+2)+C(x^3+x^2-x-1)+D(x^2+x-2)$$

$$x^3+4=Ax^3+2Ax^2-Ax-2A+Bx^3+4Bx^2+5Bx+2B+Cx^3+Cx^2-Cx-C+Dx^2+Dx-2D$$

$$x^3+4=Ax^3+Bx^3+Cx^3+2Ax^2+4Bx^2+Cx^2+Dx^2-Ax+5Bx-Cx+Dx-2A+2B-C-2D$$

$$x^3+4=(A+B+C)x^3+(2A+4B+C+D)x^2+(-A+5B-C+D)x-2A+2B-C-2D$$

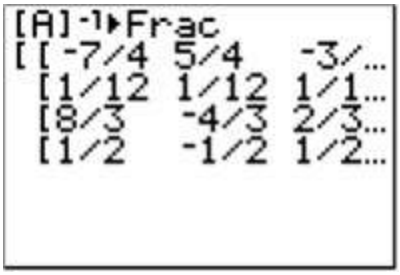
Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned}A+B+C&=1\\2A+4B+C+D&=0\\-A+5B-C+D&=0\\-2A+2B-C-2D&=4\end{aligned}$$

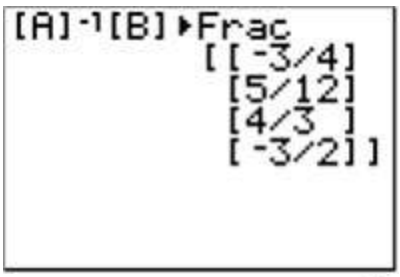
Solve the system using inverse matrices. Write the system in the matrix form $AX = B$.

$$\begin{bmatrix}1&1&1&0\\2&4&1&1\\-1&5&-1&1\\-2&2&-1&-2\end{bmatrix}\cdot\begin{bmatrix}A\\B\\C\\D\end{bmatrix}=\begin{bmatrix}1\\0\\0\\4\end{bmatrix}$$

Find $A^{-1}$.




Find $A^{-1}B$.




Replace $A$, $B$, $C$, and $D$ with $-\frac{3}{4}, \frac{5}{12}, \frac{4}{3}$, and $-\frac{3}{2}$ in the partial fraction decomposition.

$$\frac{x^3+4}{(x^2-1)(x^2+3x+2)} = \frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x+2}+\frac{D}{(x+1)^2}$$
$$= \frac{-\frac{3}{4}}{x+1}+\frac{\frac{5}{12}}{x-1}+\frac{\frac{4}{3}}{x+2}+\frac{-\frac{3}{2}}{(x+1)^2}$$

42. $\dfrac{4x^3+x^2-3x+3}{x(x-1)^2}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$x(x-1)^2 = x^3-2x^2+1$$

$$\begin{array}{r} 4 \\ x^3-2x^2+x \overline{)\,4x^3+x^2-3x+3} \\ (-)\,(4x^3-8x^2+4x) \\ \hline 9x^2-7x+3 \end{array}$$

Therefore, the proper rational expression is $4+\dfrac{9x^2-7x+3}{x(x-1)^2}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$, $B$, and $C$, and denominators that are the linear factors of the original denominator. Remember the repeated linear factor.

$$\frac{9x^2-7x+3}{x(x-1)^2} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Multiply each side by the LCD, $x(x-1)^2$.

$$\frac{9x^2-7x+3}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$\frac{9x^2-7x+3}{x(x-1)^2}=\frac{A(x-1)^2}{x(x-1)^2}+\frac{Bx(x-1)}{x(x-1)^2}+\frac{Cx}{x(x-1)^2}$$

$$9x^2-7x+3=A(x^2-2x+1)+Bx^2-Bx+Cx$$

$$9x^2-7x+3=Ax^2-2Ax+A+Bx^2-Bx+Cx$$

$$9x^2-7x+3=Ax^2+Bx^2-2Ax-Bx+Cx+A$$

$$9x^2-7x+3=(A+B)x^2+(-2A-B+C)x+A$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$$A+B=9$$

$$-2A-B+C=-7$$

$$A=3$$

Solve the system.

$$A=3$$

$$3+B=9$$

$$B=6$$

$$-2A-B+C=-7$$

$$-2(3)-6+C=-7$$

$$-6-6+C=-7$$

$$C=5$$

Replace *A*, *B*, and *C* with 3, 6, and 5 in the partial fraction decomposition.

$$\frac{9x^2-7x+3}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$=\frac{3}{x}+\frac{6}{x-1}+\frac{5}{(x-1)^2}$$

The complete partial fraction decomposition is $4+\frac{3}{x}+\frac{6}{x-1}+\frac{5}{(x-1)^2}$.

43. $\dfrac{7x^7+2x^6-13x^5+32x^4-19x^3+8x^2-7x+2}{x(x-1)^2(x+2)(x^2+1)}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expressi numerator by the denominator.

$$x(x-1)^2(x+2)(x^2+1)$$

$$=(x^3-2x^2+x)(x^3+2x^2+x+2)$$

$$=x^6+2x^5+x^4+2x^3-2x^5-4x^4-2x^3-4x^2+x^4+2x^3+x^2+2x$$

$$=x^6+2x^5-2x^5+x^4-4x^4+x^4+2x^3-2x^3+2x^3-4x^2+x^2+2x$$

$$=x^6-2x^4+2x^3-3x^2+2x$$

$$
\begin{array}{r}
7x+2 \\
x^6-2x^4+2x^3-3x^2+2x\overline{)\,7x^7+2x^6-13x^5+32x^4-19x^3+8x^2-7x+2} \\
(-)\,\underline{(7x^7\qquad -14x^5+14x^4-21x^3+14x^2)} \\
2x^6\quad +x^5+18x^4+2x^3-6x^2-7x \\
(-)\,\underline{(2x^6\qquad -4x^4+4x^3-6x^2+4x)} \\
x^5+22x^4-2x^3\qquad -11x+2
\end{array}
$$

Therefore, the proper rational expression is $7x+2+\dfrac{x^5+22x^4-2x^3-11x+2}{x(x-1)^2(x+2)(x^2+1)}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, *A*, *B*, *C*, and D and den denominator. Remember the repeated linear factor and the prime quadratic factor.

$$\frac{x^5+22x^4-2x^3-11x+2}{x(x-1)^2(x+2)(x^2+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{x+2}+\frac{Ex+F}{x^2+1}$$

Multiply each side by the LCD, $x(x-1)^2(x+2)(x^2+1)$.

## 6-4 Partial Fractions

$$\begin{aligned}
&x^5+22x^4-2x^3-11x+2\\
&\quad=A(x-1)^2(x+2)(x^2+1)+Bx\\
&\quad+Dx(x-1)^2(x^2+1)+(Ex+F\\
&\quad=A(x^2-2x+1)(x^3+2x^2+x\\
&\quad+D(x^2-2x+1)(x^3+x)+(Ex\\
&\quad=A(x^5+2x^4+x^3+2x^2-2x^4\\
&\quad+B(x^5+x^3+x^4+x^2-2x^3-2\\
&\quad+C(x^4+x^2+2x^3+2x)\\
&\quad+D(x^5+x^3-2x^4-2x^2+x^3+\\
&\quad+(Ex+F)(x^4+2x^3-2x^3-4x\\
&\quad=A(x^5-2x^3+2x^2-3x+2)\\
&\quad+B(x^5+x^4-x^3+x^2-2x)\\
&\quad+C(x^4+2x^3+x^2+2x)\\
&\quad+D(x^5-2x^4+2x^3-2x^2+x)\\
&\quad+(Ex+F)(x^4-3x^2+2x)\\
&\quad=Ax^5-2Ax^3+2Ax^2-3Ax+\\
&\quad+Bx^5+Bx^4-Bx^3+Bx^2-2Bx\\
&\quad+Cx^4+2Cx^3+Cx^2+2Cx\\
&\quad+Dx^5-2Dx^4+2Dx^3-2Dx^2\\
&\quad+Ex^5-3Ex^3+2Ex^2+Fx^4-3\\
&\quad=Ax^5+Bx^5+Dx^5+Ex^5\\
&\quad+Bx^4+Cx^4-2Dx^4+Fx^4\\
&\quad-2Ax^3-Bx^3+2Cx^3+2Dx^3-\\
&\quad+2Ax^2+Bx^2+Cx^2-2Dx^2+\\
&\quad-3Ax-2Bx+2Cx+Dx+2Fx\\
&\quad+2A\\
&\quad=(A+B+D+E)x^5\\
&\quad+(B+C-2D+F)x^4\\
&\quad+(-2A-B+2C+2D-3E)x^3\\
&\quad+(2A+B+C-2D+2E-3F\\
&\quad+(-3A-2B+2C+D+2F)x\\
&\quad+2A
\end{aligned}$$

$$\begin{aligned}
&\frac{x^5+22x^4-2x^3-11x+2}{x(x-1)^2(x+2)(x^2+1)}\\
&\quad=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{x+2}+\frac{Ex+F}{x^2+1}\\
&\quad=\frac{A(x-1)^2(x+2)(x^2+1)+Bx(x-1)(x+2)(x^2+1)+Cx(x+2)(x^2+1)}{x(x-1)^2(x+2)(x^2+1)}\\
&\quad+\frac{Dx(x-1)^2(x^2+1)+(Ex+F)x(x-1)^2(x+2)}{x(x-1)^2(x+2)(x^2+1)}
\end{aligned}$$

the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the equation must equal the coefficients of the *x*-terms on the right side.

$$\begin{aligned} A+B+D+E&=1\\ B+C-2D+F&=22\\ -2A-B+2C+2D-3E&=-2\\ 2A+B+C-2D+2E-3F&=0\\ -3A-2B+2C+D+2F&=-11\\ 2A&=2 \end{aligned}$$

Solve the system using inverse matrices. Write the system in the matrix form $AX = B$.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0\\ 0 & 1 & 1 & -2 & 0 & 1\\ -2 & -1 & 2 & 2 & -3 & 0\\ 2 & 1 & 1 & -2 & 2 & -3\\ -3 & -2 & 2 & 1 & 0 & 2\\ 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} A\\ B\\ C\\ D\\ E\\ F \end{bmatrix} = \begin{bmatrix} 1\\ 22\\ -2\\ 0\\ -11\\ 2 \end{bmatrix}$$

Find $A^{-1}$.

```
[A]^-1▸Frac
[[0        0        0...
 [4/9      5/18     1...
 [1/6      1/6      1...
 [16/45    -8/45    4...
 [1/5      -1/10    -...
 [1/10     1/5      -...
```

Find $A^{-1}B$.

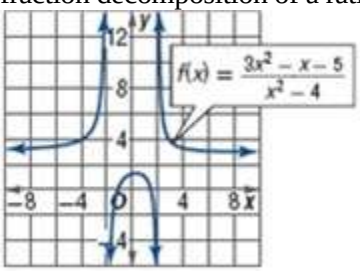
```
[A]^-1[B]
          [[1  ]
           [8  ]
           [2  ]
           [-4 ]
           [-4 ]
           [4  ]]
```

Replace $A$, $B$, $C$, $D$, $E$, and $F$ with 1, 8, 2, −4, −4, and 4 in the partial fraction decomposition.

$$\frac{x^5+22x^4-2x^3-11x+2}{x(x-1)^2(x+2)(x^2+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{x+2}+\frac{Ex+F}{x^2+1}$$

$$\frac{x^5+22x^4-2x^3-11x+2}{x(x-1)^2(x+2)(x^2+1)}=\frac{1}{x}+\frac{8}{x-1}+\frac{2}{(x-1)^2}+\frac{-4}{x+2}+\frac{-4x+4}{x^2+1}$$

44. **MULTIPLE REPRESENTATIONS** In this problem, you will discover the relationship between the partial fraction decomposition of a rational function and its graph. Consider the rational function shown below.



**a. VERBAL** Identify the end behavior and vertical and horizontal asymptotes of the function.
**b. ANALYTICAL** Write the partial fraction decomposition of $f(x)$.
**c. GRAPHICAL** Graph each addend of the partial fraction decomposition you wrote in part **b** as a separate function.
**d. VERBAL** Compare the graphs from part **c** with the graph of $f(x)$ and the analysis you wrote in part **b**.
**e. ANALYTICAL** Make a conjecture as to how the partial fraction decomposition of a function can be used to graph a rational function.

*SOLUTION:*

**a.** It appears that the graph will never hit $y = 3$. Also, as $f(x)$ approaches 3 as $x$ approaches both negative infinity and positive infinity. This also indicates that a horizontal asymptote is present at $y = 3$. It also appears that the graph will never cross $x$-values of 2 and −2. This indicates the presence of vertical asymptotes at $x = 2$ and $x = –2$.

**b.** Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$\begin{array}{r} 3 \\ x^2-4\overline{)\,3x^2-x-5} \\ (-)\,(3x^2\quad -12) \\ \hline -x+7 \end{array}$$

Therefore, the proper rational expression is $3+\frac{-x+7}{x^2-4}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{-x+7}{x^2-4}=\frac{A}{x+2}+\frac{B}{x-2}$$

Multiply each side by the LCD, $x^2 - 4$.

$$-x+7=A(x-2)+B(x+2)$$

Group like terms.

$$-x+7=A(x-2)+B(x+2)$$
$$-x+7=Ax-2A+Bx+2B$$
$$-x+7=Ax+Bx-2A+2B$$
$$-x+7=(A+B)x-2A+2B$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=-1$$
$$-2A+2B=7$$

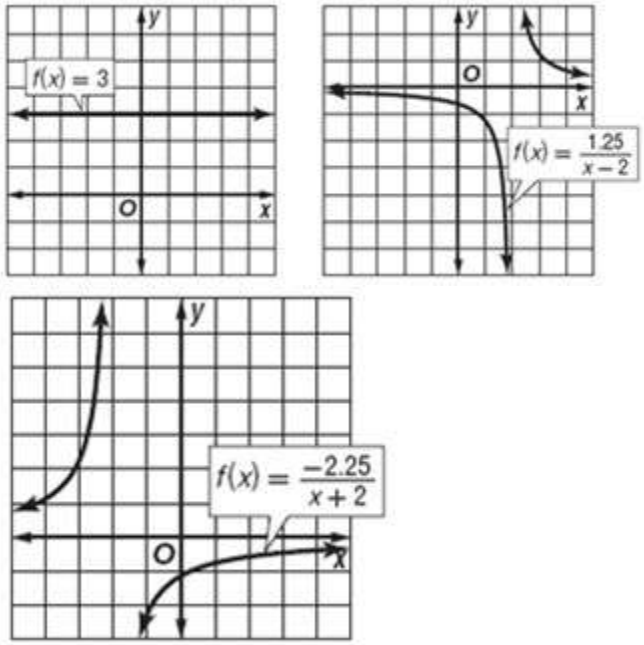
Solve the system using any method.

$$A + B = -1$$
$$A = -1 - B$$
$$-2A + 2B = 7$$
$$-2(-1 - B) + 2B = 7$$
$$2 + 2B + 2B = 7$$
$$4B = 5$$
$$B = 1.25$$
$$A = -1 - 1.25$$
$$A = -2.25$$

Replace $A$ and $B$ with −2.25 and 1.25 in the partial fraction decomposition.

$$3 + \frac{1.25}{x-2} + \frac{-2.25}{x+2}$$

**c.**




**d.** Sample answer: The graph of $f(x)$ has a horizontal asymptote at $y = 3$, which is one of the terms in the decomposition. The graph of $y = \frac{1.25}{x2}$ is similar to the graph of the original function near $x = 2$. The graph of the term $y = \frac{2.25}{x+2}$ is similar to the graph of the original function near $x = -2$.

e. Sample answer: The graph of $f(x)$ has a horizontal asymptote at $y = 3$, could be represented by one of the terms in the decomposition. Possibly there is a direct relationship between the $y = 3$ graph and the horizontal asymptote. Further exploration into other improper rational functions where the degree of the numerator is equal to the degree of the denominator will prove this to be true.

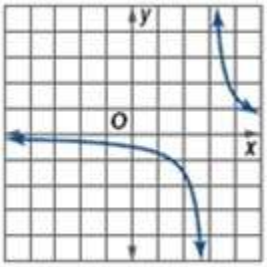
Therefore, for a rational function $f(x) = \frac{ax^n + \ldots}{bx^n + \ldots}$, where $n$ is the degree, the graph of the function will have a horizontal asymptote at $y = \frac{a}{b}$ and the decomposition will have a constant term of $\frac{a}{b}$, the graph of which will also

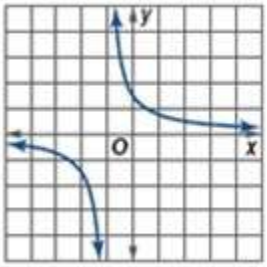
be $y = \frac{a}{b}$.

The graph of $y = \frac{1.25}{x2}$ is similar to the graph of the original function near $x = 2$. The graph of the term $y = \frac{2.25}{x+2}$ is similar to the graph of the original function near $x = -2$.

There also may be a relationship between the other terms of the decomposition and the original function. Each denominator of the decomposition is also a factor of the original denominator. Therefore, they will share vertical asymptotes.

45. **GRAPH ANALYSIS** The rational functions shown make up the partial fraction decomposition of $f(x)$.




Determine which of the four functions listed below could be the original function $f(x)$.

I. $f(x) = \frac{6}{x^2 - 2x - 3}$

II. $f(x) = \frac{6}{x^2 + 2x - 3}$

III. $f(x) = \frac{6}{x^2 + 4x + 3}$

IV. $f(x) = \frac{6}{x^2 - 4x + 3}$

*SOLUTION:*

The graphs show $x$-intercepts at $x = 3$ and $x = -1$. Thus, the two partial fractions will be of the form $\frac{A}{x-3} + \frac{B}{x+1}$ and the original function could be of the form $\frac{C}{x^2 - 2x - 3}$. Therefore, the correct choice is choice I.

**REASONING Use the partial fraction decompositions of $f(x)$ to explain each of the following.**

46. If $f(x) = \frac{-2x^3 - 7x^2 + 13x + 43}{(x-2)(x+3)^2}$, explain why $\lim_{x \to \infty} f(x) = -2$.

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$(x-2)(x+3)^2 = (x-2)(x^2 + 6x + 9)$$
$$= x^3 + 4x^2 - 3x - 18$$

## 6-4 Partial Fractions

$$x^3+4x^2-3x-18\overline{)-2x^3-7x^2+13x+43}$$

$$\begin{array}{r} -2 \\ (-)\,(-2x^3-8x^2+6x+36) \\ \hline x^2+7x+7 \end{array}$$

Therefore, the proper rational expression is $-2+\dfrac{x^2+7x+7}{x^3+4x^2-3x-18}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, *A*, *B*, and C and denominators that are the linear factors of the original denominator. Remember the repeated linear factor.

$$\frac{x^2+7x+7}{x^3+4x^2-3x-18}=\frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$$

Multiply each side by the LCD, $(x-2)(x+3)^2$. Group like terms.

$$\frac{x^2+7x+7}{x^3+4x^2-3x-18}=\frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$$

$$\frac{x^2+7x+7}{x^3+4x^2-3x-18}=\frac{A(x+3)^2}{(x-2)(x+3)^2}+\frac{B(x-2)(x+3)}{(x-2)(x+3)^2}+\frac{C(x-2)}{(x-2)(x+3)^2}$$

$$x^2+7x+7=A(x+3)^2+B(x-2)(x+3)+C(x-2)$$

$$x^2+7x+7=A(x^2+6x+9)+B(x^2+x-6)+Cx-2C$$

Equate the coefficients on the left and right side

$$x^2+7x+7=Ax^2+6Ax+9A+Bx^2+Bx-6B+Cx-2C$$

$$x^2+7x+7=Ax^2+Bx^2+6Ax+Bx+Cx+9A-6B-2C$$

$$x^2+7x+7=(A+B)x^2+(6A+B+C)x+9A-6B-2C$$

of the equation to form two equations with 2 variables. In other words, the coefficients of the *x*-terms on the left side of the equation must equal the coefficients of the *x*-terms on the right side.

$$A+B=1$$

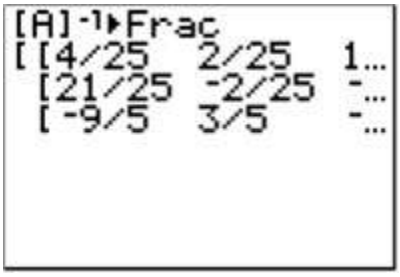
$$6A+B+C=7$$

$$9A-6B-2C=7$$

Solve the system using inverse matrices. Write the system in the matrix form $AX = B$.

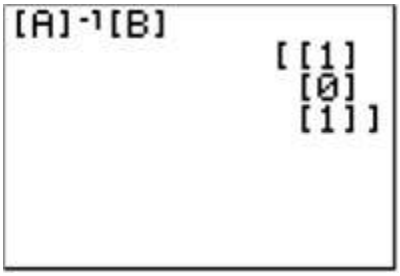
$$\begin{bmatrix}1 & 1 & 0\\ 6 & 1 & 1\\ 9 & -6 & -2\end{bmatrix}\cdot\begin{bmatrix}A\\ B\\ C\end{bmatrix}=\begin{bmatrix}1\\ 7\\ 7\end{bmatrix}$$

Find $A^{-1}$.




Find $A^{-1}B$.

## 6-4 Partial Fractions




Replace $A$, $B$, and $C$ with 1, 0, and 1 in the partial fraction decomposition.

$$\frac{x^2+7x+7}{x^3+4x^2-3x-18} = \frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$$
$$= \frac{1}{x-2}+\frac{0}{x+3}+\frac{1}{(x+3)^2}$$
$$= \frac{1}{x-2}+\frac{1}{(x+3)^2}$$

Therefore, $\frac{-2x^3-7x^2+13x+43}{(x-2)(x+3)^2} = -2+\frac{1}{x-2}+\frac{1}{(x+3)^2}$.

As $x$ approaches infinity, the values of $\frac{1}{x-2}$ and $\frac{1}{(x+3)^2}$ both approach 0. So the limit of the expression is $-2$.

## 6-4 Partial Fractions

47. If $f(x) = \dfrac{x^2 - x + 1}{(x+1)^3}$, then what is $\lim_{x \to \infty} f(x)$?

*SOLUTION:*

This rational expression is proper. The denominator has four linear factors. Notice, however, that the factor $x + 1$ is repeated twice.

Rewrite the expression as partial fractions with $A$, $B$, and $C$ in the numerators.

$$\frac{x^2 - x + 1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Multiply each side by the LCD, $(x + 1)^3$.

$$\frac{x^2 - x + 1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$\frac{x^2 - x + 1}{(x+1)^3} = \frac{A(x+1)^2}{(x+1)^3} + \frac{B(x+1)}{(x+1)^3} + \frac{C}{(x+1)^3}$$

Group like terms.

$$x^2 - x + 1 = A(x+1)^2 + B(x+1) + C$$

$$x^2 - x + 1 = A(x^2 + 2x + 1) + Bx + B + C$$

$$x^2 - x + 1 = Ax^2 + 2Ax + A + Bx + B + C$$

$$x^2 - x + 1 = Ax^2 + 2Ax + Bx + A + B + C$$

$$x^2 - x + 1 = Ax^2 + (2A + B)x + A + B + C$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A = 1$$

$$2A + B = -1$$

$$A + B + C = 1$$

Solve the system.

$$A = 1$$

$$2A + B = -1$$

$$2(1) + B = -1$$

$$B = -3$$

$$A + B + C = 1$$

$$1 + (-3) + C = 1$$

$$C = 3$$

Replace $A$, $B$, and $C$ with 1, −3, and 3 in the partial fraction decomposition.

$$\frac{x^2 - x + 1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$\frac{x^2 - x + 1}{(x+1)^3} = \frac{1}{x+1} + \frac{-3}{(x+1)^2} + \frac{3}{(x+1)^3}$$

As $x$ approaches infinity, the values of the three terms approach 0. So the limit of the expression is 0.

# 6-4 Partial Fractions

**CHALLENGE Match the graph of each rational function with its equation.**

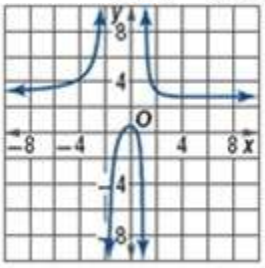
**a.** $y = x + 2 + \frac{1}{x-1} + \frac{-3}{x+2}$

**b.** $y = x + 2 + \frac{-2}{x-1} + \frac{3}{x+2}$

**c.** $y = 3 + \frac{1}{x-1} + \frac{-3}{x+2}$

**d.** $y = 3 + \frac{-2}{x-1} + \frac{3}{x+2}$

48.




*SOLUTION:*

The graphs of these rational equations can be differentiated by their asymptotes. The graphs of #48 and #51 both appear to have vertical asymptotes at $x = -2$ and $x = 1$, and a horizontal asymptote at $y = 3$.

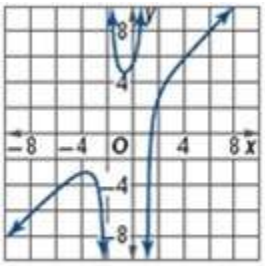
Notice how choices **c** and **d** have a constant of 3. It can be deduced that as $x$ approaches positive or negative infinity, the $y$-values of **c** and **d** will approach 3. This is consistent with the horizontal asymptotes in #48 and #51.

Therefore, we know that **c** and **d** are the answers for #48 and #51. We just do not know which is which. Further analysis of the graph must be done.

Notice in #51 that as $x$ approaches 1 from the positive side, $y$ approaches negative infinity. In #48, the opposite occurs: as $x$ approaches 1 from the positive side, $y$ approaches positive infinity. Let $x$ approach 1 from the positive side in choices **c** and **d**.

For choice **c**, as $x$ approaches 1 from the positive side, $\frac{1}{x-1}$ approaches positive infinity. Therefore, the equation in choice **c** also approaches positive infinity and **c** is the correct choice for #48.

## 6-4 Partial Fractions

49. 


*SOLUTION:*

The graphs of these rational equations can be differentiated by their asymptotes.

The graphs of #49 and #50 appear to have vertical asymptotes at $x = -2$ and $x = 1$, as well as an oblique asymptote at $y = x + 2$.

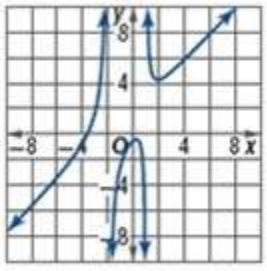
Notice how choices **a** and **b** have a linear value of $x + 2$ outside of the fractions. It can be deduced that as $x$ approaches positive or negative infinity, the $y$-values of **a** and **b** will approach $x + 2$. This is consistent with the oblique asymptotes in #49 and #50.

Therefore, we know that **a** and **b** are the answers for #49 and #50. We just do not know which is which. Further analysis of the graph must be done.

Notice in #49 that as $x$ approaches $-2$ from the negative side, $y$ approaches negative infinity. In #50, the opposite occurs: as $x$ approaches $-2$ from the negative side, $y$ approaches positive infinity. Let $x$ approach $-2$ from the negative side in choices **a** and **b**.

For choice **b**, as $x$ approaches $-2$ from the negative side, $\frac{3}{x+2}$ approaches negative infinity. Therefore, the equation in choice **b** also approaches negative infinity and **b** is the correct choice for #49.

# 6-4 Partial Fractions

50. 


*SOLUTION:*

The graphs of these rational equations can be differentiated by their asymptotes.

The graphs of #49 and #50 appear to have vertical asymptotes at $x = -2$ and $x = 1$, as well as an oblique asymptote at $y = x + 2$.

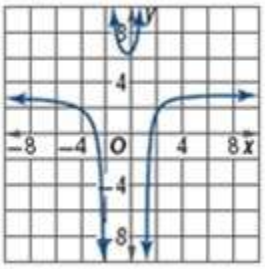
Notice how choices **a** and **b** have a linear value of $x + 2$ outside of the fractions. It can be deduced that as $x$ approaches positive or negative infinity, the $y$-values of **a** and **b** will approach $x + 2$. This is consistent with the oblique asymptotes in #49 and #50.

Therefore, we know that **a** and **b** are the answers for #49 and #50. We just do not know which is which. Further analysis of the graph must be done.

Notice in #49 that as $x$ approaches $-2$ from the negative side, $y$ approaches negative infinity. In #50, the opposite occurs: as $x$ approaches $-2$ from the negative side, $y$ approaches positive infinity. Let $x$ approach $-2$ from the negative side in choices **a** and **b**.

For choice **a**, as $x$ approaches $-2$ from the negative side, $\frac{-3}{x+2}$ approaches positive infinity. Therefore, the equation in choice **a** also approaches positive infinity and **a** is the correct choice for #50.

# 6-4 Partial Fractions

51. 


*SOLUTION:*

The graphs of these rational equations can be differentiated by their asymptotes. The graphs of #48 and #51 both appear to have vertical asymptotes at $x = -2$ and $x = 1$, and a horizontal asymptote at $y = 3$.

Notice how choices **c** and **d** have a constant of 3. It can be deduced that as $x$ approaches positive or negative infinity, the $y$-values of **c** and **d** will approach 3. This is consistent with the horizontal asymptotes in #48 and #51.

Therefore, we know that **c** and **d** are the answers for #48 and #51. We just do not know which is which. Further analysis of the graph must be done.

Notice in #51 that as $x$ approaches 1 from the positive side, $y$ approaches negative infinity. In #48, the opposite occurs: as $x$ approaches 1 from the positive side, $y$ approaches positive infinity. Let $x$ approach 1 from the positive side in choices **c** and **d**.

For choice **d**, as $x$ approaches 1 from the positive side, $\frac{-2}{x-1}$ approaches negative infinity. Therefore, the equation in choice **d** also approaches positive infinity and **d** is the correct choice for #51.

**REASONING Determine whether each of the following statements is *true* or *false* . Explain your reasoning**

52. If $f(x) = \frac{x^3+8}{(x^2-1)(x-2)}$, then $\lim_{x \to \infty} f(x) = 8$.

*SOLUTION:*

One method of solving this would be to find the partial fraction decomposition. Another method is to use what we learned in Chapter 2.

As we learned in Lesson 2-5, with a rational function of the form $f(x) = \frac{a(x)}{b(x)} = \frac{a_n x^n + \ldots}{b_m x^m + \ldots}$ where $n = m$, the horizontal asymptote is located at $y = \frac{a_n}{b_m}$. This also determines the end behavior of the graph, which in this case approaches $\frac{a_n}{b_m}$ as $x$ approaches positive and negative infinity.

In this problem, $f(x) = \frac{x^3+8}{(x^2-1)(x-2)} = \frac{x^3+8}{x^3-2x^2-x+2}$, and $\frac{a_n}{b_m} = \frac{1}{1} = 1$. Therefore, the graph approaches 1 and the limit of the expression is 1.

## 6-4 Partial Fractions

53. The partial fraction decomposition of $f(x) = \dfrac{-4x^4+5x^3+27x^2-11x-45}{x(x^2-3)^2}$ is $\dfrac{-5}{x}+\dfrac{4+x}{(x^2-3)}+\dfrac{x^2+1}{(x^2-3)^2}$.

*SOLUTION:*

To determine whether this is true, combine the fractions by finding a common denominator.

$$\frac{-5}{x}+\frac{4+x}{x^2-3}+\frac{x^2+1}{(x^2-3)^2}$$

$$=\frac{-5(x^2-3)^2+(4+x)(x)(x^2-3)+x(x^2+1)}{x(x^2-3)^2}$$

$$=\frac{-5x^4+30x^2-45+4x^3-12x+x^4-3x^2+x^3+x}{x(x^2-3)^2}$$

$$=\frac{-4x^4+5x^3+27x^2-11x-45}{x(x^2-3)^2}$$

The statement is true.

54. **OPEN ENDED** Write a rational expression of the form $\frac{P(x)}{Q(x)}$ in which the partial fraction decomposition contains each of the following in the denominator.

**a**. nonrepeated linear factors only
**b.** at least one repeated linear factor

*SOLUTION:*

**a.** The best way to do this is to start with the factors and then combine them to form the rational expression. Choose two arbitrary factors, $\frac{1}{x-2}+\frac{1}{x+3}$. Now combine them by finding the LCD.

$$\frac{1}{x-2}+\frac{1}{x+3}=\frac{x+3}{(x-2)(x+3)}+\frac{x-2}{(x-2)(x+3)}$$
$$=\frac{x+3+x-2}{(x-2)(x+3)}$$
$$=\frac{2x+1}{(x-2)(x+3)}$$
$$=\frac{2x+1}{x^2+x-6}$$

**b.** The best way to do this is to start with the factors and then combine them to form the rational expression. Choose some arbitrary factors, at least one of them repeated: $\frac{2}{x}+\frac{3}{x^2}+\frac{-1}{x-2}+\frac{4}{(x-2)^2}$. Now combine them by finding the LCD.

$$\frac{2}{x}+\frac{3}{x^2}+\frac{-1}{x-2}+\frac{4}{(x-2)^2}$$
$$=\frac{2x(x-2)^2}{x^2(x-2)^2}+\frac{3(x-2)^2}{x^2(x-2)^2}+\frac{-1(x^2)(x-2)}{x^2(x-2)^2}+\frac{4x^2}{x^2(x-2)^2}$$
$$=\frac{2x(x^2-4x+4)}{x^2(x-2)^2}+\frac{3(x^2-4x+4)}{x^2(x-2)^2}+\frac{-x^2(x-2)}{x^2(x-2)^2}+\frac{4x^2}{x^2(x-2)^2}$$
$$=\frac{2x^3-8x^2+8x}{x^2(x-2)^2}+\frac{3x^2-12x+12}{x^2(x-2)^2}+\frac{-x^3+2x^2}{x^2(x-2)^2}+\frac{4x^2}{x^2(x-2)^2}$$
$$=\frac{2x^3-8x^2+8x+3x^2-12x+12-x^3+2x^2+4x^2}{x^2(x-2)^2}$$
$$=\frac{2x^3-x^3-8x^2+3x^2+2x^2+4x^2+8x-12x+12}{x^2(x-2)^2}$$
$$=\frac{x^3+x^2-4x+12}{x^2(x-2)^2}$$

55. **WRITING IN MATH** Describe the steps used to obtain the partial fraction decomposition of a rational expression.

*SOLUTION:*

Sample answer:

You are given a rational expression of the form $\frac{a_n x^n + a_{n-1}x^{n-1} + \ldots}{b_m x^m + b_{m-1}x^{m-1} + \ldots}$.

First, factor the denominator, if necessary.

$$\frac{a_n x^n + a_{n-1}x^{n-1} + \ldots}{b_m x^m + b_{m-1}x^{m-1} + \ldots} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots}{(X_1)^{m_1}(X_2)^{m_2}\ldots}$$

Notice that the $X_n$ expressions could be linear, prime quadratic, or even prime cubic expressions. The exponents $m_1$, $m_2$, … represent the possibility of repeated factors.

Rewrite the equation with constant numerators, $A$, $B$, $C$, and $D$, and denominators that are the linear or quadratic factors of the original denominator.

$$\frac{a_n x^n + a_{n-1}x^{n-1} + \ldots}{(X_1)^{m_1}(X_2)^{m_2}\ldots} = \frac{A}{(X_1)^1} + \frac{B}{(X_1)^2} + \frac{Cx + D}{(X_2)^1}$$

In this example, we show a repeated linear factor, $X_1$, and a prime quadratic factor, $X_2$.

Write the original expression equal to the new expression and multiply each side by the LCD, in this case, $(X_1)^2(X_2)$.

$$= \frac{A(X_1)(X_2)}{(X_1)^2(X_2)^1} + \frac{B(X_2)}{(X_1)^2(X_2)^1} + \frac{(Cx + D)(X_1)^2}{(X_1)^2(X_2)^1}$$

Use the Distributive Property to group like terms. Equate the coefficients on both sides of the equation to get a new system of equations.

Then solve the system by writing it in matrix form $CX = D$ and solving for $X$. Finally, use substitution to find the partial fraction decomposition.

**Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists.**

56. $x + y + z = 6$
$2x + y - 4z = -15$
$5x - 3y + z = -10$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -4 \\ 5 & -3 & 1 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -4 \\ 5 & -3 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -4 \\ -3 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -4 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 5 & -3 \end{vmatrix}$$

$$= 1[1(1) - (-3)(-4)] - 1[2(1) - 5(-4)] + 1[2(-3) - 5(1)]$$

$$= 1(1 - 12) - 1(2 + 20) + 1(-6 - 5)$$

$$= -11 - 22 - 11$$

$$= -44$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$

$$= \frac{\begin{vmatrix} 6 & 1 & 1 \\ -15 & 1 & -4 \\ -10 & -3 & 1 \end{vmatrix}}{-44}$$

$$= \frac{6\begin{vmatrix} 1 & -4 \\ -3 & 1 \end{vmatrix} - 1\begin{vmatrix} -15 & -4 \\ -10 & 1 \end{vmatrix} + 1\begin{vmatrix} -15 & 1 \\ -10 & -3 \end{vmatrix}}{-44}$$

$$= \frac{6[1(1) - (-3)(-4)] - 1[-15(1) - (-10)(-4)] + 1[-15(-3) - (-10)(1)]}{-44}$$

$$= \frac{-66 + 55 + 55}{-44}$$

$$= \frac{44}{-44}$$

$$= -1$$

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 6 & 1 \\ 2 & -15 & -4 \\ 5 & -10 & 1 \end{vmatrix}}{-44}$$

$$= \frac{1\begin{vmatrix} -15 & -4 \\ -10 & 1 \end{vmatrix} - 6\begin{vmatrix} 2 & -4 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & -15 \\ 5 & -10 \end{vmatrix}}{-44}$$

$$= \frac{1[-15(1)-(-10)(-4)] - 6[2(1)-5(-4)] + 1[2(-10)-5(-15)]}{-44}$$

$$= \frac{1(-15-40) - 6(2+20) + 1(-20+75)}{-44}$$

$$= \frac{-55-132+55}{-44}$$

$$= \frac{-132}{-44}$$

$$= 3$$

$$Z = \frac{|A_z|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & -15 \\ 5 & -3 & -10 \end{vmatrix}}{-44}$$

$$= \frac{1\begin{vmatrix} 1 & -15 \\ -3 & -10 \end{vmatrix} - 1\begin{vmatrix} 2 & -15 \\ 5 & -10 \end{vmatrix} + 6\begin{vmatrix} 2 & 1 \\ 5 & -3 \end{vmatrix}}{-44}$$

$$= \frac{1[1(-10)-(-3)(-15)] - 1[2(-10)-5(-15)] + 6[2(-3)-5(1)]}{-44}$$

$$= \frac{-55-55-66}{-44}$$

$$= \frac{-176}{-44}$$

$$= 4$$

The solution is (−1, 3, 4).

57. $a - 2b + c = 7$
$6a + 2b - 2c = 4$
$4a + 6b + 4c = 14$

*SOLUTION:*

## 6-4 Partial Fractions

The coefficient matrix is $A = \begin{bmatrix} 1 & -2 & 1 \\ 6 & 2 & -2 \\ 4 & 6 & 4 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 1 & -2 & 1 \\ 6 & 2 & -2 \\ 4 & 6 & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & -2 \\ 6 & 4 \end{vmatrix} + 2\begin{vmatrix} 6 & -2 \\ 4 & 4 \end{vmatrix} + 1\begin{vmatrix} 6 & 2 \\ 4 & 6 \end{vmatrix}$$

$$= 1[2(4) - 6(-2)] + 2[6(4) - 4(-2)] + 1[6(6) - 4(2)]$$
$$= 1(8 + 12) + 2(24 + 8) + 1(36 - 8)$$
$$= 20 + 64 + 28$$
$$= 112$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$
$$= \frac{\begin{vmatrix} 7 & -2 & 1 \\ 4 & 2 & -2 \\ 14 & 6 & 4 \end{vmatrix}}{112}$$
$$= \frac{7\begin{vmatrix} 2 & -2 \\ 6 & 4 \end{vmatrix} + 2\begin{vmatrix} 4 & -2 \\ 14 & 4 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 14 & 6 \end{vmatrix}}{112}$$
$$= \frac{7[2(4) - 6(-2)] + 2[4(4) - 14(-2)] + 1[4(6) - 14(2)]}{112}$$
$$= \frac{140 + 88 - 4}{112}$$
$$= \frac{224}{112}$$
$$= 2$$

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 7 & 1 \\ 6 & 4 & -2 \\ 4 & 14 & 4 \end{vmatrix}}{112}$$

$$= \frac{1\begin{vmatrix} 4 & -2 \\ 14 & 4 \end{vmatrix} - 7\begin{vmatrix} 6 & -2 \\ 4 & 4 \end{vmatrix} + 1\begin{vmatrix} 6 & 4 \\ 4 & 14 \end{vmatrix}}{112}$$

$$= \frac{1[4(4) - 14(-2)] - 7[6(4) - 4(-2)] + 1[6(14) - 4(4)]}{112}$$

$$= \frac{1(16 + 28) - 7(24 + 8) + 1(84 - 16)}{112}$$

$$= \frac{44 - 224 + 68}{112}$$

$$= \frac{-112}{112}$$

$$= -1$$

$$Z = \frac{|A_z|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & -2 & 7 \\ 6 & 2 & 4 \\ 4 & 6 & 14 \end{vmatrix}}{112}$$

$$= \frac{1\begin{vmatrix} 2 & 4 \\ 6 & 14 \end{vmatrix} + 2\begin{vmatrix} 6 & 4 \\ 4 & 14 \end{vmatrix} + 7\begin{vmatrix} 6 & 2 \\ 4 & 6 \end{vmatrix}}{112}$$

$$= \frac{1[2(14) - 6(4)] + 2[6(14) - 4(4)] + 7[6(6) - 4(2)]}{112}$$

$$= \frac{4 + 136 + 196}{112}$$

$$= \frac{336}{112}$$

$$= 3$$

The solution is (2, −1, 3).

58. $p - 2r - 5t = -1$
$p + 2r - 2t = 5$
$4p + r + t = -1$

*SOLUTION:*

## 6-4 Partial Fractions

The coefficient matrix is $A = \begin{bmatrix} 1 & -2 & -5 \\ 1 & 2 & -2 \\ 4 & 1 & 1 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 1 & -2 & -5 \\ 1 & 2 & -2 \\ 4 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$

$$= 1[2(1) - 1(-2)] + 2[1(1) - 4(-2)] - 5[1(1) - 4(2)]$$
$$= 1(2 + 2) + 2(1 + 8) - 5(1 - 8)$$
$$= 4 + 18 + 35$$
$$= 57$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$
$$= \frac{\begin{vmatrix} -1 & -2 & -5 \\ 5 & 2 & -2 \\ -1 & 1 & 1 \end{vmatrix}}{57}$$
$$= \frac{-1\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 5 & -2 \\ -1 & 1 \end{vmatrix} - 5\begin{vmatrix} 5 & 2 \\ -1 & 1 \end{vmatrix}}{57}$$
$$= \frac{-1[2(1) - 1(-2)] + 2[5(1) - (-1)(-2)] - 5[5(1) - (-1)(2)]}{57}$$
$$= \frac{-4 + 6 - 35}{57}$$
$$= \frac{-33}{57}$$
$$= -\frac{11}{19}$$

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & -1 & -5 \\ 1 & 5 & -2 \\ 4 & -1 & 1 \end{vmatrix}}{57}$$

$$= \frac{1\begin{vmatrix} 5 & -2 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & 5 \\ 4 & -1 \end{vmatrix}}{57}$$

$$= \frac{1[5(1)-(-1)(-2)]+1[1(1)-4(-2)]-5[1(-1)-4(5)]}{57}$$

$$= \frac{1(5-2)+1(1+8)-5(-1-20)}{57}$$

$$= \frac{3+9+105}{57}$$

$$= \frac{117}{57}$$

$$= \frac{39}{19}$$

$$Z = \frac{|A_z|}{|A|}$$

$$= \frac{\begin{vmatrix} -1 & 1 & -2 \\ 5 & 1 & 2 \\ -1 & 4 & 1 \end{vmatrix}}{57}$$

$$= \frac{-1\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 5 & 2 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 5 & 1 \\ -1 & 4 \end{vmatrix}}{57}$$

$$= \frac{-1[1(1)-4(2)]-1[5(1)-(-1)(2)]-2[5(4)-(-1)(1)]}{57}$$

$$= \frac{7-7-42}{57}$$

$$= -\frac{42}{57}$$

$$= -\frac{14}{19}$$

The solution is $\left(-\frac{11}{19}, \frac{39}{19}, -\frac{14}{19}\right)$.

59. **FINANCE** For a class project, Jane "bought" shares of stock in three companies. She bought 150 shares of a utility company, 100 shares of a computer company, and 200 shares of a food company. At the end of the project, she "sold" all of her stock.

| Company | Purchase Price per share ($) | Selling Price per share ($) |
|---|---|---|
| utility | 54.00 | 55.20 |
| computer | 48.00 | 58.60 |
| food | 60.00 | 61.10 |

**a.** Organize the data in two matrices and use matrix multiplication to find the total amount that Jane spent for the stock.
**b.** Write two matrices and use matrix multiplication to find the total amount she received for selling the stock.
**c**. Use matrix operations to find how much money Jane "made" or "lost" in her project.

*SOLUTION:*

**a**. The purchase price can be placed into a $3 \times 1$ matrix $A = \begin{bmatrix} 54 \\ 48 \\ 60 \end{bmatrix}$. The number of shares purchased can be placed into a $1 \times 3$ matrix $B = \begin{bmatrix} 150 & 100 & 200 \end{bmatrix}$.
In order to find the total amount spent, find $BA$.

$$BA = \begin{bmatrix} 150 & 100 & 200 \end{bmatrix} \cdot \begin{bmatrix} 54 \\ 48 \\ 60 \end{bmatrix}$$
$$= \begin{bmatrix} 150(54) + 100(48) + 200(60) \end{bmatrix}$$
$$= \begin{bmatrix} 8100 + 4800 + 12,000 \end{bmatrix}$$
$$= \begin{bmatrix} 24,900 \end{bmatrix}$$

**b**. The sale price can be placed into a $3 \times 1$ matrix $C = \begin{bmatrix} 55.2 \\ 58.6 \\ 61.1 \end{bmatrix}$. In order to find the total amount earned from selling the stock, find $BC$.

$$BC = \begin{bmatrix} 150 & 100 & 200 \end{bmatrix} \cdot \begin{bmatrix} 55.2 \\ 58.6 \\ 61.1 \end{bmatrix}$$
$$= \begin{bmatrix} 150(55.2) + 100(58.6) + 200(61.1) \end{bmatrix}$$
$$= \begin{bmatrix} 8280 + 5860 + 12,220 \end{bmatrix}$$
$$= \begin{bmatrix} 26,360 \end{bmatrix}$$

**c.** Find $BC - BA$.

$$BC - BA = \begin{bmatrix} 26,360 \end{bmatrix} - \begin{bmatrix} 24,900 \end{bmatrix}$$
$$= \begin{bmatrix} 1460 \end{bmatrix}$$

**Simplify each expression.**

60. $\csc \theta \cos \theta \tan \theta$

*SOLUTION:*

$$\begin{aligned}\csc\theta\cos\theta\tan\theta &= \frac{1}{\sin\theta}\cdot\cos\theta\cdot\frac{\sin\theta}{\cos\theta}\\ &= \frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}\\ &= 1\end{aligned}$$

61. $\sec^2 \theta - 1$

*SOLUTION:*

Use the Pythagorean identity.

$$\begin{aligned}\sec^2\theta - 1 &= (\tan^2\theta + 1) - 1\\ &= \tan^2\theta\end{aligned}$$

62. $\dfrac{\tan \theta}{\sin \theta}$

*SOLUTION:*

$$\begin{aligned}\frac{\tan\theta}{\sin\theta} &= \tan\theta\cdot\frac{1}{\sin\theta}\\ &= \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ &= \frac{1}{\cos\theta}\\ &= \sec\theta\end{aligned}$$

## 6-4 Partial Fractions

63. **MEDICINE** Doctors may use a tuning fork that resonates at a given frequency as an aid to diagnose hearing problems. The sound wave produced by a tuning fork can be modeled using a sine function.
a. If the amplitude of the sine function is 0.25, write the equations for tuning forks that resonate with a frequency of 64, 256, and 512 Hertz.
b. How do the periods of the tuning forks compare?

*SOLUTION:*

**a.** For the sine function, the frequency equals the reciprocal of the period, or $\frac{|b|}{2\delta}$. Find $b$ for each of the frequencies in the problem.

$$64\ Hertz: \frac{b}{2\pi} = 64$$
$$b = 128\pi$$

$$256\ Hertz: \frac{b}{2\pi} = 256$$
$$b = 512\pi$$

$$512\ Hertz: \frac{b}{2\pi} = 512$$
$$b = 1024\pi$$

When the amplitude $a$ = 0.25, the equations for the tuning forks are:

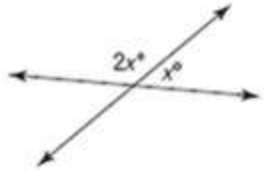
$$64\ Hertz: y = 0.25\sin 128\pi t$$

$$256\ Hertz: y = 0.25\sin 512\pi t$$

$$512\ Hertz: y = 0.25\sin 1024\pi t$$

**b**. For the sine function, the frequency equals the reciprocal of the period. Therefore, the period and frequency have an inverse relationship, so as the frequency increases, the period decreases.

64. **SAT/ACT** In the figure, what is the value of $x$?




A 40
B 45
C 60
D 75
E 90

*SOLUTION:*

The two angles in the diagram form a linear pair, which totals 180°. This is also the angle measure of a straight line.

$$2x + x = 180$$
$$3x = 180$$
$$x = 60$$

The correct choice is C.

65. Decompose $\frac{3p-1}{p^2-1}$ into partial fractions.

F $\frac{2}{p+1} + \frac{1}{p-1}$

G $\frac{2}{p-1} + \frac{1}{p+1}$

H $\frac{2}{p+1} - \frac{1}{p-1}$

J $\frac{2}{p-1} - \frac{1}{p+1}$

*SOLUTION:*

This rational expression is proper. The denominator has four linear factors. Notice, however, that the factor $x + 1$ is repeated twice.

Rewrite the expression as partial fractions with $A$, $B$, and $C$ in the numerators.

$$\frac{3p-1}{p^2-1} = \frac{A}{p+1} + \frac{B}{p-1}$$

Multiply each side by the LCD, $p^2 - 1$.

$$\frac{3p-1}{p^2-1} = \frac{A}{p+1} + \frac{B}{p-1}$$
$$\frac{3p-1}{p^2-1} = \frac{A(p-1)}{p^2-1} + \frac{B(p+1)}{p^2-1}$$

Group like terms.

$$3p-1 = A(p-1) + B(p+1)$$
$$3p-1 = Ap - A + Bp + B$$
$$3p-1 = Ap + Bp - A + B$$
$$3p-1 = (A+B)p - A + B$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A + B = 3$$
$$-A + B = -1$$

Solve the system.

$$\begin{aligned} A + B &= 3 \\ (+)\ -A + B &= -1 \\ \hline 2B &= 2 \\ B &= 1 \\ A + 1 &= 3 \\ A &= 2 \end{aligned}$$

Replace $A$ and $B$ with 2 and 1 in the partial fraction decomposition.

$$\frac{3p-1}{p^2-1} = \frac{A}{p+1} + \frac{B}{p-1}$$
$$\frac{3p-1}{p^2-1} = \frac{2}{p+1} + \frac{1}{p-1}$$

The correct choice is F.

66. **REVIEW** A sprinkler waters a circular section of lawn about 20 feet in diameter. The homeowner decides that placing the sprinkler at (7, 5) will maximize the area of grass being watered. Which equation represents the boundary of the areas that the sprinkler waters?

A $(x - 7)^2 + (y - 5)^2 = 100$
B $(x + 7)^2 - (y + 5)^2 = 100$
C $(x - 7)^2 - (y + 5)^2 = 100$
D $(x + 7)^2 + (y - 5)^2 = 100$

*SOLUTION:*

The region is circular and the equation for a circle in the coordinate plane is $(x - h)^2 + (y - k)^2 = r^2$, where $(h, k)$ is the center of the circle and $r$ is the radius. If the diameter is 20 feet, then the radius is 10 feet and $r^2$ is $10^2$ or 100. The sprinkler is located at (7, 5), so $h = 7$ and $k = 5$.

Therefore, the equation is $(x - 7)^2 + (y - 5)^2 = 100$ and the correct choice is A.

67. **REVIEW** Which of the following is the sum of $\frac{x+2}{x+3}$ and $\frac{4}{x^2+x-6}$?

F $\frac{-3x-9}{x^2+x-6}$

G $\frac{x^2-3x-24}{x^2+x-6}$

H $\frac{x^2}{x^2+x-6}$

J $\frac{x^2+x-1}{x^2+x-6}$

*SOLUTION:*

To find the sum, multiply by the LCD.

$$\frac{x+2}{x+3}+\frac{4}{x^2+x-6}=\frac{x+2}{x+3}+\frac{4}{(x+3)(x-2)}$$

$$=\frac{(x+2)(x-2)}{(x+3)(x-2)}+\frac{4}{(x+3)(x-2)}$$

$$=\frac{x^2-4}{(x+3)(x-2)}+\frac{4}{(x+3)(x-2)}$$

$$=\frac{x^2}{(x+3)(x-2)}$$

$$=\frac{x^2}{x^2+x-6}$$

The correct choice is H.

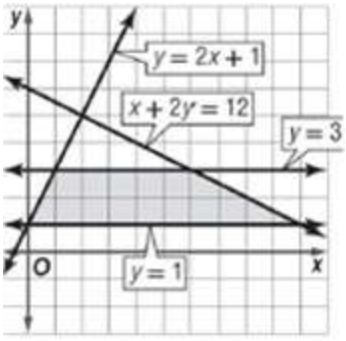
# 6-5 Linear Optimization

**Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of $x$ and $y$ they occur, subject to the given constraints.**

1. $f(x, y) = 3x + y$
$y \le 2x + 1$
$x + 2y \le 12$
$1 \le y \le 3$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has four vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$y = 1$
$y = 2x + 1$
$1 = 2x + 1$
$0 = 2x$
$0 = x$

vertex at (0, 1)

$y = 1$
$x + 2y = 12$
$x + 2(1) = 12$
$x + 2 = 12$
$x = 10$

vertex at (10, 1)

$y = 3$
$x + 2y = 12$
$x + 2(3) = 12$
$x + 6 = 12$
$x = 6$

vertex at (6, 3)

$y = 3$
$y = 2x + 1$
$3 = 2x + 1$
$2 = 2x$
$1 = x$

vertex at (1, 3)

Find the value of the objective function for each of the vertices.

$f(x, y) = 3x + y$

$f(0,1) = 3(0) + 1$

$f(0,1) = 1$

$f(10,1) = 3(10) + 1$

$f(1,10) = 31$

$f(6,3) = 3(6) + 3$

$f(6,3) = 21$

$f(1,3) = 3(1) + 3$

$f(1,3) = 6$

The maximum value of $f$ is 31 at (10, 1). The minimum value of $f$ is 1 at (0, 1).

2. $f(x, y) = -x + 4y$

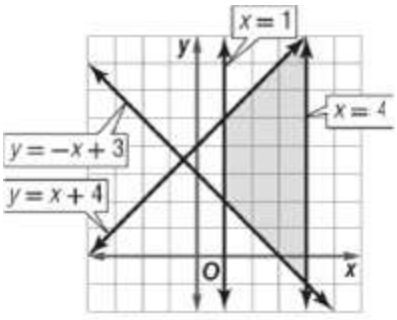
$y \le x + 4$

$y \ge -x + 3$

$1 \le x \le 4$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has four vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$x = 1$

$y = -x + 3$

$y = -1 + 3$

$y = 2$

vertex at (1, 2)

$x = 1$

$y = x + 4$

$y = 1 + 4$

$y = 5$

vertex at (1, 5)

$x = 4$

$y = -x + 3$

$y = -4 + 3$

$y = -1$

vertex at (4, -1)

$x = 4$

$y = x + 4$

$y = 4 + 4$

$y = 8$

vertex at (4, 8)

Find the value of the objective function for each of the vertices.

$f(x, y) = -x + 4y$

$f(1, 2) = -(1) + 4(2)$

$f(1, 2) = 7$

$f(1, 5) = -(1) + 4(5)$

$f(1, 5) = 19$

$f(4, -1) = -(4) + 4(-1)$

$f(4, -1) = -8$

$f(4, 8) = -(4) + 4(8)$

$f(4, 8) = 28$

The maximum value of $f$ is 28 at (4, 8). The minimum value of $f$ is -8 at (4, -1).

$f(x, y) = x - y$

$x + 2y \le 6$

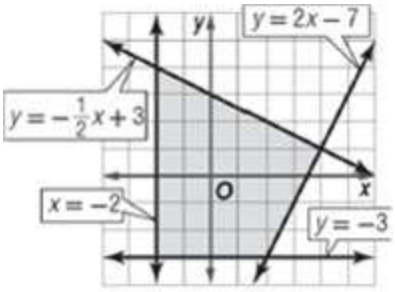
3. $2x - y \le 7$

$x \ge -2$

$y \ge -3$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has four vertices. Each vertex is located at an intersection of two of the

boundaries. Solve a system of equations to identify each vertex.

$x = -2$

$y = -3$

vertex at (-2, -3)

$$y = 2x - 7$$

$$y = -\frac{1}{2}x + 3$$

$$2x - 7 = -\frac{1}{2}x + 3$$

$$4x - 14 = -x + 6$$

$$5x = 20$$

$$x = 4$$

$$y = 2x - 7$$

$$y = 2(4) - 7 = 1$$

vertex at (4, 1)

$$y = 2x - 7$$

$$y = -3$$

$$-3 = 2(x) - 7$$

$$4 = 2x$$

$$x = 2$$

vertex at (2, $-3$)

$$y = -\frac{1}{2}x + 3$$

$$x = -2$$

$$y = -\frac{1}{2}(-2) + 3$$

$$y = 1 + 3$$

$$y = 4$$

vertex at ($-2$ ,4)

Find the value of the objective function for each of the vertices.

$$f(x, y) = x - y$$

$$f(-2, -3) = -2 - (-3)$$

$$f(-2, -3) = 1$$

$$f(4, 1) = 4 - 1$$

$$f(4, 1) = 3$$

$$f(2, -3) = 2 - (-3)$$

$$f(2, -3) = 5$$

$$f(-2, 4) = -2 - 4$$

$$f(-2, 4) = -6$$

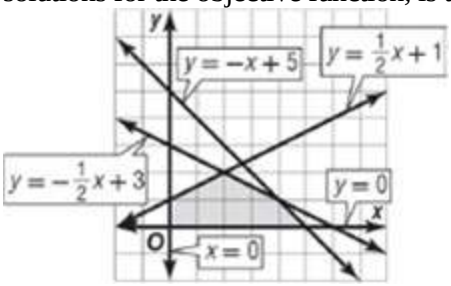
The maximum value of $f$ is 5 at (2, $-3$). The minimum value of $f$ is $-6$ at ($-2$,4).

# 6-5 Linear Optimization

4. $f(x,y)=3x-5y$
$x\geq 0$
$y\geq 0$
$x+2y\leq 6$
$2y-x\leq 2$
$x+y\leq 5$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has five vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$y=-x+5$
$y=0$
$0=-x+5$
$x=5$

vertex at (5, 0)

$x=0$
$y=\frac{1}{2}x+1$
$y=\frac{1}{2}(0)+1$
$y=1$

vertex at (0, 1)

$y=0$
$x=0$

vertex at (0, 0)

$y=-\frac{1}{2}x+3$
$y=\frac{1}{2}x+1$
$\frac{1}{2}x+1=-\frac{1}{2}x+3$
$x=2$
$y=\frac{1}{2}(2)+1$
$y=2$

vertex at (2 ,2)

$$y = -\frac{1}{2}x + 3$$
$$y = -x + 5$$
$$-\frac{1}{2}x + 3 = -x + 5$$
$$-x + 6 = -2x + 10$$
$$x = 4$$
$$y = -\frac{1}{2}(4) + 3$$
$$y = 1$$

vertex at (4 ,1)

Find the value of the objective function for each of the vertices.

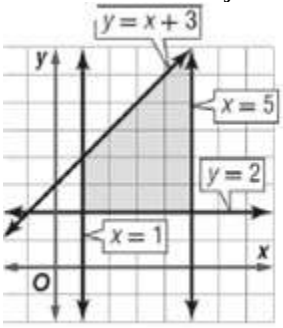
$f(x, y) = 3x - 5y$
$f(0,0) = 3(0) - 5(0)$
$f(0,0) = 0$
$f(0,1) = 3(0) - 5(1)$
$f(0,1) = -5$
$f(5,0) = 3(5) - 5(0)$
$f(5,0) = 15$
$f(2,2) = 3(2) - 5(2)$
$f(2,2) = 6 - 10 = -4$
$f(4,1) = 3(4) - 5(1)$
$f(4,1) = 12 - 5 = 7$

The maximum value of $f$ is 15 at (5, 0). The minimum value of $f$ is $-5$ at (0, 1).

5. $f(x, y) = 3x - 2y$
$y \le x + 3$
$1 \le x \le 5$
$y \ge 2$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.



The polygonal region of feasible solutions has four vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$y = 2$

$x = 1$

vertex at (1, 2)

$x = 5$

$y = 2$

vertex at (5, 2)

$y = x + 3$

$x = 1$

$y = 1 + 3 = 4$

vertex at (1, 4)

$y = x + 3$

$x = 5$

$y = x + 3$

$y = 5 + 3 = 8$

vertex at (5 ,8)

Find the value of the objective function for each of the vertices.

$f(x, y) = 3x - 2y$

$f(1, 2) = 3(1) - 2(2)$

$f(1, 2) = 3 - 4 = 1$

$f(1, 4) = 3(1) - 2(4)$

$f(1, 4) = 3 - 8 = -5$

$f(5, 2) = 3(5) - 2(2)$

$f(5, 2) = 15 - 4 = 11$

$f(5, 8) = 3(5) - 2(8)$

$f(5, 8) = 15 - 16 = -1$

The maximum value of $f$ is 11 at (5, 2). The minimum value of $f$ is $-5$ at (1, 4).

6. $f(x, y) = 3y + x$

$4y \le x + 8$

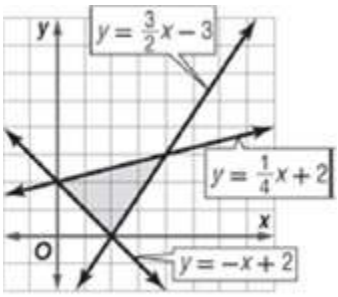
$2y \ge 3x - 6$

$2x + 2y \ge 4$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.

## 6-5 Linear Optimization




The polygonal region of feasible solutions has three vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$$y = \frac{1}{4}x + 2$$
$$y = \frac{3}{2}x - 3$$
$$\frac{1}{4}x + 2 = \frac{3}{2}x - 3$$
$$x + 8 = 6x - 12$$
$$20 = 5x$$
$$x = 4$$
$$y = \frac{1}{4}(4) + 2 = 3$$

vertex at (4, 3)

$$y = -x + 2$$
$$y = \frac{3}{2}x - 3$$
$$-x + 2 = \frac{3}{2}x - 3$$
$$-2x + 4 = 3x - 6$$
$$10 = 5x$$
$$x = 2$$
$$y = -(2) + 2 = 0$$

vertex at (2, 0)

$$y = \frac{1}{4}x + 2$$
$$y = -x + 2$$
$$\frac{1}{4}x + 2 = -x + 2$$
$$x + 8 = -4x + 8$$
$$5x = 0$$
$$x = 0$$
$$y = \frac{1}{4}(0) + 2 = 2$$

vertex at (0, 2)

Find the value of the objective function for each of the vertices.

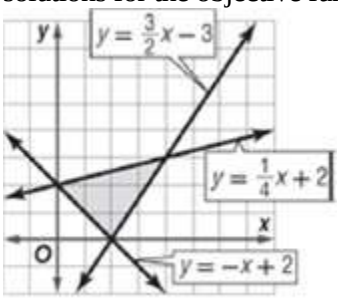
$f(x,y)=3y+x$
$f(0,2)=3(2)+0$
$f(0,2)=6$
$f(2,0)=3(0)+2$
$f(2,0)=2$
$f(4,3)=3(3)+4$
$f(4,3)=9+4=13$

The maximum value of $f$ is 13 at (4, 3). The minimum value of $f$ is 2 at (2, 0).

7.
$f(x,y)=x-4y$
$x\geq 2$
$y\geq 1$
$x-2y\geq -4$
$x+y\leq 8$
$2x-y\leq 7$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has five vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$y=1$
$x=2$
vertex at (2,1)
$y=1$
$y=2x-7$
$1=2x-7$
$8=2x$
$x=4$
vertex at (4, 1)
$y=2x-7$
$y=-x+8$
$2x-7=-x+8$
$3x=15$
$x=5$
$y=-5+8=3$

vertex at (5, 3)

$$y = \frac{1}{2}x + 2$$
$$y = -x + 8$$
$$\frac{1}{2}x + 2 = -x + 8$$
$$x + 4 = -2x + 16$$
$$3x = 12$$
$$x = 4$$
$$y = -4 + 8 = 4$$

vertex at (4, 4)

$$y = \frac{1}{2}x + 2$$
$$x = 2$$
$$y = \frac{1}{2}(2) + 2$$
$$y = 1 + 2 = 3$$

vertex at (2, 3)

Find the value of the objective function for each of the vertices.

$$f(x, y) = x - 4y$$
$$f(2,1) = 2 - 4(1)$$
$$f(2,1) = 2 - 4 = -2$$
$$f(2,3) = 2 - 4(3)$$
$$f(2,3) = 2 - 12 = -10$$
$$f(4,1) = 4 - 4(1)$$
$$f(4,1) = 4 - 4 = 0$$
$$f(4,4) = 4 - 4(4)$$
$$f(4,4) = 4 - 16 = -12$$
$$f(5,3) = 5 - 4(3)$$
$$f(5,3) = 5 - 12 = -7$$

The maximum value of $f$ is 0 at (4, 1). The minimum value of $f$ is $-12$ at (4, 4).

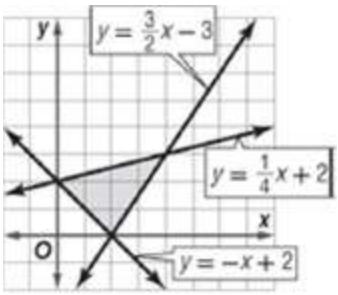
$$f(x, y) = x - y$$
$$3x - 2y \geq -7$$

8. $$x + 6y \geq -9$$
$$5x + y \leq 13$$
$$x - 3y \geq -7$$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.

# 6-5 Linear Optimization




The polygonal region of feasible solutions has fort vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$$y = \frac{3}{2}x + \frac{7}{2}$$
$$y = -\frac{1}{6}x - \frac{3}{2}$$
$$\frac{3}{2}x + \frac{7}{2} = -\frac{1}{6}x - \frac{3}{2}$$
$$9x + 21 = -x - 9$$
$$10x = -30$$
$$x = -3$$
$$y = -\frac{1}{6}(-3) - \frac{3}{2}$$
$$y = \frac{1}{2} - \frac{3}{2} = -\frac{2}{2} = -1$$

vertex at $(-3, -1)$

$$y = -\frac{1}{6}x - \frac{3}{2}$$
$$y = -5x + 13$$
$$-\frac{1}{6}x - \frac{3}{2} = -5x + 13$$
$$-x - 9 = -30x + 78$$
$$29x = 87$$
$$x = 3$$
$$y = -5(3) + 13$$
$$y = -15 + 13 = -2$$

vertex at $(3, -2)$

$$y = -5x + 13$$
$$y = \frac{1}{3}x + \frac{7}{3}$$
$$-5x + 13 = \frac{1}{3}x + \frac{7}{3}$$
$$-15x + 39 = x + 7$$
$$-16x = -32$$
$$x = 2$$
$$y = -5(2) + 13$$
$$y = -10 + 13 = 3$$

vertex at (2, 3)

$$y = \frac{3}{2}x + \frac{7}{2}$$
$$y = \frac{1}{3}x + \frac{7}{3}$$
$$\frac{3}{2}x + \frac{7}{2} = \frac{1}{3}x + \frac{7}{3}$$
$$9x + 21 = 2x + 14$$
$$7x = -7$$
$$x = -1$$
$$y = \frac{1}{3}(-1) + \frac{7}{3}$$
$$y = -\frac{1}{3} + \frac{7}{3} = \frac{6}{3} = 2$$

vertex at $(-1, 2)$

Find the value of the objective function for each of the vertices.

$$f(x, y) = x - y$$
$$f(-1, 2) = -1 - 2 = -3$$
$$f(2, 3) = 2 - 3 = -1$$
$$f(3, -2) = 3 - (-2) = 5$$
$$f(-3, -1) = -3 - (-1) = -2$$

The maximum value of $f$ is 5 at (3, $-2$). The minimum value of $f$ is $-3$ at $(-1, 2)$.

9. **MEDICAL OFFICE** Olivia is a receptionist for a medical clinic. One of her tasks is to schedule appointments. She allots 20 minutes for a checkup and 40 minutes for a physical. The doctor can do no more than 6 physicals per day, and the clinic has 7 hours available for appointments. A checkup costs $55, and a physical costs $125.

**a.** Write an objective function and list the constraints that model the given situation.

**b.** Sketch a graph of the region determined by the constraints from part **a** to find the set of feasible solutions for the objective function.

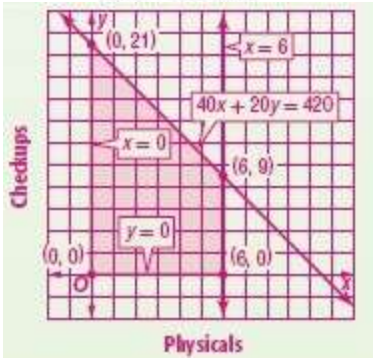
**c.** How many of each appointment should Olivia make to maximize income? What is the maximum income?

*SOLUTION:*

**a.** Let $x$ = physicals and let $y$ = checkups.
There are no more than 6 physicals per day, so $x \leq 6$. There are 420 minutes available, so $40x + 20y \leq 420$. The number of physicals and checkups must be positive, so $x \geq 0$ and $y \geq 0$. A checkup costs $55 and a physical costs $125, so the objective function is $f(x, y) = 125x + 55y$.

**b**. Graph each constraint and locate the feasible region.

$$40x + 20y \leq 420$$
$$2x + y \leq 21$$
$$y \leq -2x + 21$$




The vertices of the feasible region are (0, 0), (6, 0), (6, 9), and (0, 21). Find the value of the objective function for each of these vertices.

$$f(x, y) = 125x + 55y$$
$$f(0,0) = 125(0) + 55(0) = 0$$
$$f(6,0) = 125(6) + 55(0) = 750$$
$$f(0,21) = 125(0) + 55(21) = 1165$$
$$f(6,9) = 125(6) + 55(9) = 1245$$

The maximum income is $1245 when Olivia schedules 6 checkups and 9 physicals.

10. **INCOME** Josh is working part-time to pay for some of his college expenses. Josh delivers pizza for $5 per hour plus tips, which run about $8 per hour, and he also tutors in the math lab for $15 per hour. The math lab is only open 2 hours daily, Monday through Friday, when Josh is available to tutor. Josh can work no more than 20 hours per week due to his class schedule.

**a.** Write an objective function and list the constraints that model the given situation.

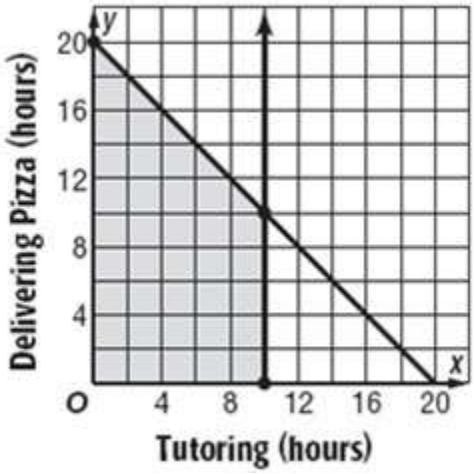
**b.** Sketch a graph of the region determined by the constraints from part **a** to find the set of feasible solutions for the objective function.

**c.** How can Josh make the most money, and how much is it?

*SOLUTION:*

**a.** Let $l$ = tutoring hours in lab and let $p$ = hours delivering pizza.
The number of hours worked must be positive, so $p \geq 0$ and $l \geq 0$. He cannot work more than 10 hours in the lab, so $l \leq 10$. He can only work up to 20 hours total, so $l + p \leq 20$. He earns $15 per hour tutoring and $13 per hour delivering pizza, so the objective function is $f(l, p) = 15l + 13p$.

**b.** Graph each constraint and locate the feasible region.




**c.** The vertices of the feasible region are (0, 0), (10, 0), (0, 20), and (10, 10). Find the value of the objective function for each of these vertices.

$$f(l, p) = 15l + 13p$$
$$f(0,0) = 15(0) + 13(0) = 0$$
$$f(10,0) = 15(10) + 13(0) = 150$$
$$f(0,20) = 15(0) + 13(20) = 260$$
$$f(10,10) = 15(10) + 13(10) = 280$$

Josh should work 10 hours delivering pizza and 10 hours tutoring. He will make $280.

11. **SMALL BUSINESS** A design company creates Web sites and E-albums. Each Web site requires 10 hours of planning and 4.5 hours of page design. Each family E-album requires 15 hours of planning and 9 hours of page design. There are 70 hours available each week for the staff to plan and 36 hours for page design.

**a.** If the profit is $600 for each Web site and $700 for each family E-album, write an objective function and list the constraints that model the given situation.

**b.** Sketch a graph of the region determined by the constraints from part **a** to find the set of feasible solutions for the

objective function.

**c.** How much of each product should be produced to achieve maximum profit? What is the profit?

*SOLUTION:*

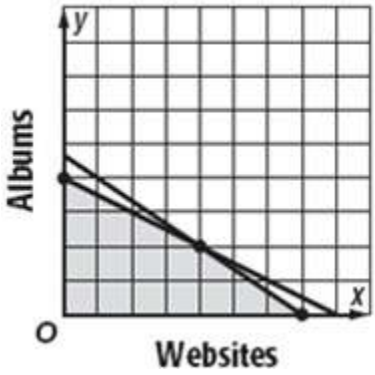
**a.** Let $w$ = websites and $a$ = albums. There are 70 hours available to plan, websites require 10 hours, and albums require 15 hours, so $10w + 15a \leq 70$. There are 36 hours available for design, websites require 4.5 hours, and albums require 9 hours, so $4.5w + 9a \leq 36$. The profit is \$600 per website and \$700 per albums, so the objective function is $f(w, a) = 600w + 700a$.

**b.** Solve the inequalities for $a$.

$$10w + 15a \leq 70$$
$$15a \leq -10w + 70$$
$$a \leq -\frac{2}{3}w + \frac{14}{3}$$

$$4.5w + 9a \leq 36$$
$$9a \leq -4.5w + 36$$
$$a \leq -\frac{1}{2}w + 4$$

Graph the constraints.




**c**. Three of the vertices of the feasible region are (0, 0), (0, 4), and (7, 0). Find the other vertex by locating the intersection of the two constraints.

$$-\frac{1}{2}w + 4 = -\frac{2}{3}w + \frac{14}{3}$$
$$-3w + 24 = -4w + 28$$
$$w = 4$$
$$a = -\frac{1}{2}w + 4$$
$$a = -\frac{1}{2}(4) + 4$$
$$a = 2$$

The last vertex is (4, 2).

Find the value of the objective function for each of these vertices.

$$f(w,a) = 600w + 700a$$
$$f(0,0) = 600(0) + 700(0)$$
$$= 0$$
$$f(0,4) = 600(0) + 700(4)$$
$$= 2800$$
$$f(7,0) = 600(7) + 700(0)$$
$$= 4200$$
$$f(4,2) = 600(4) + 700(2)$$
$$= 3800$$

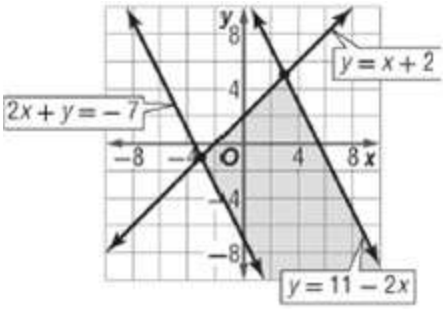
The company should create 7 Web sites and no albums for a maximum profit of $4200.

**Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of $x$ and $y$ they occur, subject to the given constraints.**

12. $f(x,y) = 4x - 4y$
$2x + y \geq -7$
$y \leq x + 2$
$y \leq 11 - 2x$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has two vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$$2x + y = -7$$
$$y = x + 2$$
$$2x + (x + 2) = -7$$
$$3x = -9$$
$$x = -3$$
$$y = -3 + 2$$
$$y = -1$$
$$(-3, -1)$$

$$
\begin{aligned}
y &= 11 - 2x \\
y &= x + 2 \\
11 - 2x &= x + 2 \\
9 &= 3x \\
3 &= x \\
y &= 3 + 2 \\
y &= 5 \\
(3, 5)
\end{aligned}
$$

Find the value of the objective function for each vertex.

$$
\begin{aligned}
f(x, y) &= 4x - 4y \\
f(-3, -1) &= 4(-3) - 4(-1) \\
&= -12 + 4 \\
&= -8 \\
f(3, 5) &= 4(3) - 4(5) \\
&= 12 - 20 \\
&= -8
\end{aligned}
$$

The problem has multiple solutions from (−3, −1) to (3, 5), or along the line $y = x + 2$ in [−3, 3]. The feasible region is unbounded, so check another point in the region to determine if the line represents maxima or minima.

$$
\begin{aligned}
f(x, y) &= 4x - 4y \\
f(0, 0) &= 4(0) - 4(0) \\
&= 0
\end{aligned}
$$

There is no maximum and the minimum of −8 falls on every point along the line $y = x + 2$ in [−3, 3].

## 6-5 Linear Optimization

13. $f(x, y) = 3x + 6y$

$y \le -\frac{1}{2}x + \frac{5}{2}$

$y \le 2$

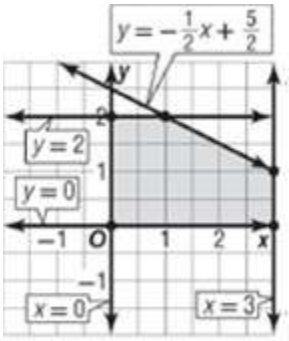
$y \ge 0$

$x \le 3$

$x \ge 0$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has five vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

Find the value of the objective function for each vertex.

$f(x, y) = 3x + 6y$

$f(0, 0) = 3(0) + 6(0)$

$= 0$

$f(0, 2) = 3(0) + 6(2)$

$= 12$

$f(1, 2) = 3(1) + 6(2)$

$= 15$

$f(3, 1) = 3(3) + 6(1)$

$= 15$

$f(3, 0) = 3(3) + 6(0)$

$= 9$

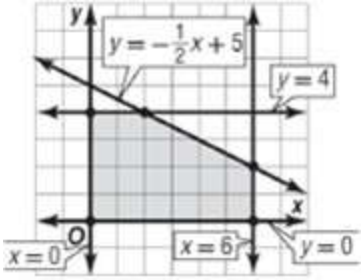
The problem has multiple maxima of 15 from (1, 2) to (3, 1), or along the line $y = -0.5x + 2.5$ in $[-1, 3]$. There is a minimum of 0 at (0, 0).

14. $f(x, y) = -3x - 6y$

$$y \le -\frac{1}{2}x + 5$$
$$y \le 4$$
$$y \ge 0$$
$$x \le 6$$
$$x \ge 0$$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has five vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.
Find the value of the objective function for each vertex.

$f(x, y) = -3x - 6y$

$f(0, 0) = -3(0) - 6(0)$
$= 0$

$f(0, 4) = -3(0) - 6(4)$
$= -24$

$f(2, 4) = -3(2) - 6(4)$
$= -30$

$f(6, 2) = -3(6) - 6(2)$
$= -30$

$f(6, 0) = -3(6) - 6(0)$
$= -18$

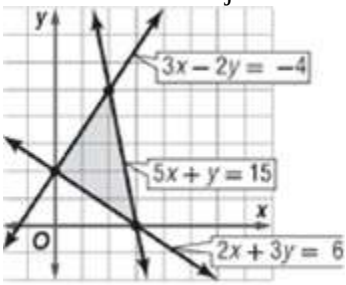
The problem has multiple minima of −30 from (2, 4) to (6, 2), or along the line $y = -0.5x + 5$ in [2, 6]. There is a maximum of 0 at (0, 0).

## 6-5 Linear Optimization

15. $f(x, y) = 6x - 4y$
$2x + 3y \geq 6$
$3x - 2y \geq -4$
$5x + y \geq 15$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has three vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.
Find the value of the objective function for each vertex.

$f(x, y) = 6x - 4y$
$f(0, 2) = 6(0) - 4(2)$
$= -8$
$f(2, 5) = 6(2) - 4(5)$
$= -8$
$f(3, 0) = 6(3) - 4(0)$
$= 18$

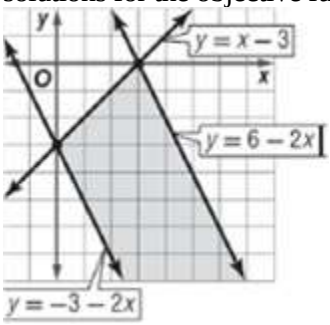
The problem has multiple minima of −8 from (0, 2) to (2, 5), or along the line $y = 1.5x + 2$ in [0, 2]. There is a maximum of 18 at (3, 0).

## 6-5 Linear Optimization

16. $f(x, y) = 3x + 4y$
$y \le x - 3$
$y \le 6 - 2x$
$2x + y \ge -3$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has two vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

Find the value of the objective function for each vertex.

$f(x, y) = 3x + 4y$

$f(0, -3) = 3(0) + 4(-3)$
$= -12$

$f(3, 0) = 3(3) + 4(0)$
$= 9$

The feasible region is unbounded, so check other points in the region.

$f(x, y) = 3x + 4y$

$f(4, -5) = 3(4) + 4(-5)$
$= -8$

$f(2, -6) = 3(2) + 4(-6)$
$= -18$

There is a maximum of 9 at (3, 0), and no minimum.

17. $f(x, y) = 8x + 10y$

$y \le -\frac{4}{5}x + 4$

$y \le 4$

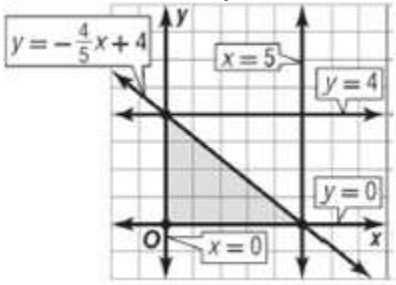
$y \ge 0$

$x \le 5$

$x \ge 0$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has three vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

Find the value of the objective function for each vertex.

$f(x, y) = 8x + 10y$

$f(0, 0) = 8(0) + 10(0)$

$= 0$

$f(0, 4) = 8(0) + 10(4)$

$= 40$

$f(5, 0) = 8(5) + 10(0)$

$= 40$

The problem has multiple maxima of 40 from (0, 4) to (5, 0), or along the line $y = -0.8x + 4$ in [0, 5]. There is a minimum of 0 at (0, 0).

18. **NUTRITION** Michelle wants to consume more nutrients. She wants to receive at least 40 milligrams of calcium, 600 milligrams of potassium, and 50 milligrams of vitamin C. Michelle's two favorite fruits are apples and bananas. The average nutritional content of both are given.

| Fruit | Calcium | Potassium | Vitamin C |
|---|---|---|---|
| apple | 9.5 mg | 158 mg | 9 mg |
| banana | 7.0 mg | 467 mg | 11 mg |

**a.** If each apple costs \$0.55 and each banana costs \$0.35, write an objective function. List the constraints that model the given situation.

**b.** Sketch a graph of the region determined by the constraints from part **a** to find the set of feasible solutions for the objective function.

**c.** Determine the number of each type of fruit that Michelle should eat to minimize his cost while still obtaining his desired nutritional intake.

*SOLUTION:*

**a**. Let $a$ = apples and let $b$ = bananas. Apples cost 55 cents and bananas cost 35 cents, so the objective function is $f$

$(b, a) = 0.35b + 0.55a$. The number of apples and bananas must be positive, so $a \geq 0$ and $b \geq 0$. She wants at least 40 mg of calcium, so $7b + 9.5a \geq 40$. She wants at least 600 mg of potassium, so $467b + 158a \geq 600$. She wants at least 50 mg of vitamin C, so $11b + 9a \geq 50$.

**b.** Solve the equations for $a$ and sketch the graphs. Another method of graphing these linear equations would be to locate the $x$- and $y$-intercepts and then connect the points.

$$\begin{aligned} 11b + 9a &= 50 \\ 9a &= -11b + 50 \\ a &= -\frac{11}{9}b + \frac{50}{9} \end{aligned}$$

$$\begin{aligned} 7b + 9.5a &= 40 \\ 9.5a &= -7b + 40 \\ a &= -\frac{14}{19}b + \frac{80}{19} \end{aligned}$$

$$\begin{aligned} 467b + 158a &= 600 \\ 158a &= -467b + 600 \\ a &= -\frac{467}{158}b + \frac{600}{158} \end{aligned}$$

Two of the vertices of the feasible region are intercepts. Set $a$ and $b$ equal to 0 to find these points.

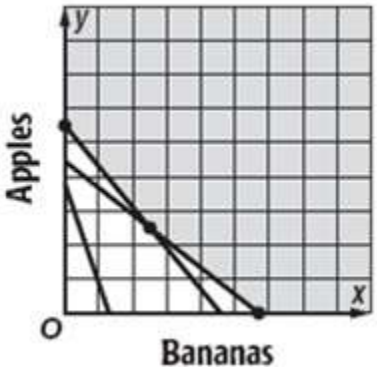
$$\begin{aligned} 11b + 9a &= 50 \\ 11(0) + 9a &= 50 \\ 9a &= 50 \\ a &= \frac{50}{9} \end{aligned}$$

$$\begin{aligned} 7b + 9.5a &= 40 \\ 7b &= 40 \\ b &= \frac{40}{7} \end{aligned}$$

$$\begin{aligned} 9.5a &= 40 \\ a &= \frac{80}{19} \end{aligned}$$

$$\begin{aligned} 467b + 158a &= 600 \\ 467b &= 600 \\ b &= \frac{600}{467} \end{aligned}$$

$$\begin{aligned} 158a &= 600 \\ a &= \frac{600}{158} \end{aligned}$$




**c.**

Two of the vertices are $\left(0,\frac{50}{9}\right)$ and $\left(\frac{40}{7},0\right)$. The last vertex can be found by setting the two constraints that intersect to form the vertex equal to each other.

$$-\frac{11}{9}b+\frac{50}{9}=-\frac{14}{19}b+\frac{80}{19}$$
$$-11(19)b+50(19)=-14(9)b+80(9)$$
$$-209b+950=-126b+720$$
$$230=83b$$
$$\frac{230}{83}=b$$
$$2.77\approx b$$
$$a=-\frac{11}{9}\left(\frac{230}{83}\right)+\frac{50}{9}$$
$$a=\frac{180}{83}$$
$$a\approx 2.17$$

Please note that these points can also be located using the intersection feature of your graphing calculator.

Find the value of the objective function for each of these points.

$$f(b,a)=0.35b+0.55a$$
$$f\left(0,\frac{50}{9}\right)=0.35(0)+0.55\left(\frac{50}{9}\right)$$
$$=\frac{27.5}{9}$$
$$\approx 3.06$$
$$f\left(\frac{40}{7},0\right)=0.35\left(\frac{40}{7}\right)+0.55(0)$$
$$=\frac{14}{7}$$
$$=2$$
$$f(2.77,2.17)=0.35(2.77)+0.55(2.17)$$
$$=0.9695+1.1935$$
$$=2.163$$

Her cost will be minimized by consuming $\frac{40}{7}$ bananas. Rounding up, we get 6 bananas. Check to see that this cost is still lower than the other two vertices.

$$f(6,0) = 0.35(6) + 0.4(0)$$
$$= 2.1$$

**Find a value of *a* so that each objective function has maximum values at the indicated vertex, subject to the following constraints.**

$$x \geq 0$$
$$y \geq 2$$
$$x + y \leq 9$$
$$-4x + 3y \leq 6$$

19. $f(x, y) = -4x + ay$, (0, 2)

*SOLUTION:*

Find the feasible region by graphing the constraints.

Equation 1

$$x \geq 0$$

Equation 2

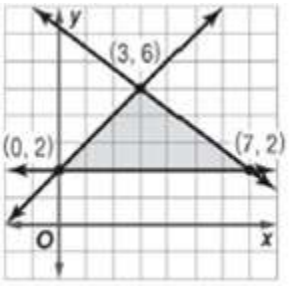
$$y \geq 2$$

Equation 3

$$x + y \leq 9$$
$$y \leq -x + 9$$

Equation 4

$$-4x + 3y \leq 6$$
$$3y \leq 4x + 6$$
$$y \leq \frac{4}{3}x + 2$$




The vertices of the region are (0, 2), (3, 6), and (7, 2).

We need to maximize $f(x, y) = -4x + ay$ at (0, 2). With the $-4$ coefficient in front of the $x$, the (3, 6) and (7, 2) values will be far more negative than the (0, 2) value. A small nonnegative value for $a$ will ensure that (0, 2) is the maximum. If $a \geq 3$, then (3, 6) will become the maximum.

20. $f(x, y) = -4x + ay$, (3, 6)

*SOLUTION:*

Find the feasible region by graphing the constraints.

Equation 1

$x \geq 0$

Equation 2

$y \geq 2$

Equation 3

$x + y \leq 9$

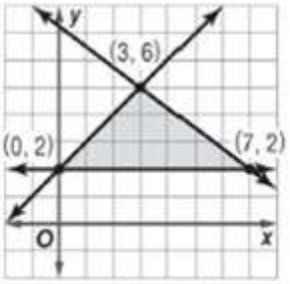
$y \leq -x + 9$

Equation 4

$-4x + 3y \leq 6$

$3y \leq 4x + 6$

$y \leq \frac{4}{3}x + 2$




The vertices of the region are (0, 2), (3, 6), and (7, 2).

We need to maximize $f(x, y) = -4x + ay$ at (3, 6). With the −4 coefficient in front of the $x$, the (3, 6) and (7, 2) values will be far more negative than the (0, 2) value. A small nonnegative value for $a$ will ensure that (0, 2) is the maximum. If $a \geq 3$, then (3, 6) will become the maximum.

## 6-5 Linear Optimization

21. $f(x, y) = x - ay$, (3, 6)

*SOLUTION:*

Find the feasible region by graphing the constraints.

Equation 1

$x \geq 0$

Equation 2

$y \geq 2$

Equation 3

$x + y \leq 9$

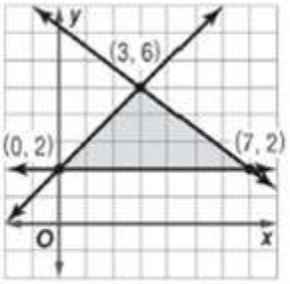
$y \leq -x + 9$

Equation 4

$-4x + 3y \leq 6$

$3y \leq 4x + 6$

$y \leq \frac{4}{3}x + 2$




The vertices of the region are (0, 2), (3, 6), and (7, 2).

We need to maximize $f(x, y) = x - ay$ at (3, 6). With the $-a$ coefficient in front of the $y$, the (3, 6) value will be far more positive than the other values as long as $a$ is negative. If $a \leq -2$, then (3, 6) will become the maximum.

22. $f(x, y) = x - ay$, (7, 2)

*SOLUTION:*

Find the feasible region by graphing the constraints.

Equation 1

$x \geq 0$

Equation 2

$y \geq 2$

Equation 3

$x + y \leq 9$

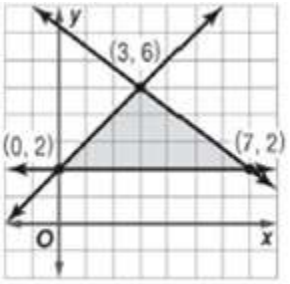
$y \leq -x + 9$

Equation 4

$-4x + 3y \leq 6$

$3y \leq 4x + 6$

$y \leq \frac{4}{3}x + 2$




The vertices of the region are (0, 2), (3, 6), and (7, 2).

We need to maximize $f(x, y) = x - ay$ at (7, 2). With the $-a$ coefficient in front of the $y$, the (3, 6) value will be far more negative than the other values as long as $a$ is positive. Since 7 > 0, the (7, 20) value will be greater than the (0, 2) value as long as $a$ is positive.

23. $f(x, y) = ax + 4y$, (7, 2)

*SOLUTION:*

Find the feasible region by graphing the constraints.

Equation 1

$x \geq 0$

Equation 2

$y \geq 2$

Equation 3

$x + y \leq 9$

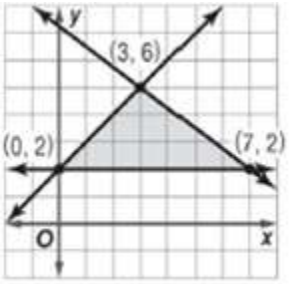
$y \leq -x + 9$

Equation 4

$-4x + 3y \leq 6$

$3y \leq 4x + 6$

$y \leq \frac{4}{3}x + 2$




The vertices of the region are (0, 2), (3, 6), and (7, 2).

We need to maximize $f(x, y) = ax + 4y$ at (7, 2). With the coefficient of 4 in front of the $y$, the (3, 6) value will be far more positive than the other values unless $a$ is large enough so that $7a + 4(2) > 3a + 4(6)$. Therefore, $a$ must be greater than 4.

24. $f(x, y) = ax - 3y$, (0, 2)

*SOLUTION:*

Find the feasible region by graphing the constraints.

Equation 1

$x \geq 0$

Equation 2

$y \geq 2$

Equation 3

$x + y \leq 9$

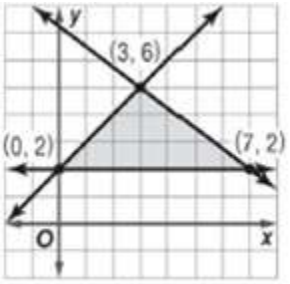
$y \leq -x + 9$

Equation 4

$-4x + 3y \leq 6$

$3y \leq 4x + 6$

$y \leq \frac{4}{3}x + 2$



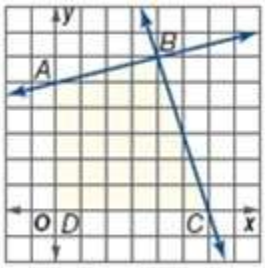

The vertices of the region are (0, 2), (3, 6), and (7, 2).

We need to maximize $f(x, y) = ax - 3y$ at (0, 2). With the coefficient of −3 in front of the $y$, the (3, 6) value will be far more negative than the (0, 2) value unless $a$ is small enough so that $0 - 3(2) > 3a - 3(6)$. Therefore, $a$ must be less than 4. The (7, 2) value will be greater than the (0, 2) value unless $a$ is small enough such that $0 - 3(2) > 7a - 3(2)$. Therefore, $a$ must be less than 0.

# 6-5 Linear Optimization

**Find an objective function that has a maximum or minimum value at each indicated vertex.**

25. minimum at $A$



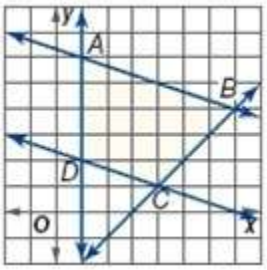

*SOLUTION:*

The vertices are $A(0, 5)$, $B(4, 6)$, $C(6, 0)$, and $D(0, 0)$. We want the objective function to have a minimum at $A$. If we make the coefficient of $y$ negative, then we only have to worry about vertex $B$. We must also make the coefficient of $x$ positive to ensure that $A$ will be less than $B$. Thus, we can use $f(x, y) = 4x - y$.

Test:

$$f(x, y) = 4x - y$$
$$f(0, 5) = 4(0) - 5$$
$$= -5$$
$$f(4, 6) = 4(4) - 6$$
$$= 10$$
$$f(6, 0) = 4(6) - 0$$
$$= 24$$
$$f(0, 0) = 4(0) - 0$$
$$= 0$$

26. maximum at $C$



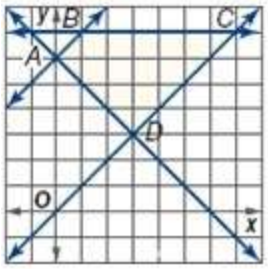

*SOLUTION:*

The vertices are $A(1, 6)$, $B(7, 4)$, $C(4, 1)$, and $D(1, 2)$. We want the objective function to have a maximum at $C$. The problem here is that $B$ has greater coordinates. To offset this, we should make the coefficient of $y$ negative and the coefficient of $y$ greater than the coefficient of $x$. This way, the negative value of the $y$-coordinate of $B$ lowers its value. Also, the negative coefficient of the $y$-coordinate does not lower the value of $C$ that much. Thus, we can use $f(x, y) = x - 2y$.

Test:

$$f(x, y) = x - 2y$$
$$f(1, 6) = 1 - 2(6)$$
$$= -11$$
$$f(7, 4) = 7 - 2(4)$$
$$= -1$$
$$f(4, 1) = 4 - 2(1)$$
$$= 2$$
$$f(1, 2) = 1 - 2(2)$$
$$= -3$$

27. maximum at *B*

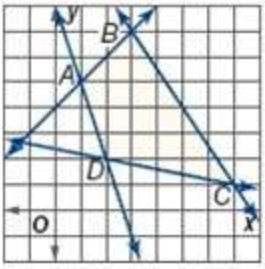



*SOLUTION:*

The vertices are $A(0, 6)$, $B(1, 7)$, $C(7, 7)$, and $D(3, 3)$. We want the objective function to have a maximum at $B$. To do this, we should make the coefficient of $y$ positive. In order to make $B$ at (1, 7) be greater then $C$ at (7, 7), we should make the coefficient of $x$ negative. Therefore, we can use $f(x, y) = -2x + 5y$.

$$f(x,y) = -2x + 5y$$
$$f(0,6) = -2(0) + 5(6)$$
$$= 30$$
$$f(1,7) = -2(1) + 5(7)$$
$$= 33$$
$$f(7,7) = -2(7) + 5(7)$$
$$= 21$$
$$f(3,3) = -2(3) + 5(3)$$
$$= 9$$

28. minimum at *D*




*SOLUTION:*

The vertices are $A(1, 5)$, $B(3, 7)$, $C(7, 1)$, and $D(2, 2)$. We want the objective function to have a minimum at $D$. The value of 2 + 2 is less than the sum of any of the other vertices, so we can use $f(x, y) = x + y$.

Test:

$$f(x,y) = x + y$$
$$f(1,5) = 1 + 5$$
$$= 6$$
$$f(3,7) = 3 + 7$$
$$= 10$$
$$f(7,1) = 7 + 1$$
$$= 8$$
$$f(2,2) = 2 + 2$$
$$= 4$$

29. **BUSINESS** The Seaside Juice Bar makes two types of mango drinks. A batch of Mango Sunrise uses 3 liters of mango juice and 1 liter of strawberry juice. A batch of Oasis Dream uses 2 liters of mango juice and 1 liter of strawberry juice. The store has 40 liters of mango juice and 15 liters of strawberry juice that it wants to use up before the end of the day. The profit on Mango Sunrise is $16 per batch, and the profit on the Oasis Dream is $12 per batch.

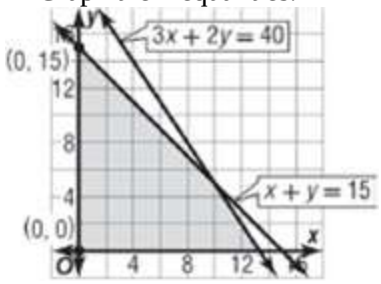
**a.** Write an objective function, and list the constraints that model the given situation.

**b.** In order to maximize profits, how many batches of each drink should the juice store make?

*SOLUTION:*

**a.** Let $x$ = Mango Dream and let $y$ = Oasis Dream. There are 40 liters of mango juice, so $3x + 2y \leq 40$. There are 15 liters of strawberry juice, so $x + y \leq 15$. The number of drinks must be positive, so $x \geq 0$ and $y \geq 0$. Use the profit per batch to write the objective function. $f(x, y) = 16x + 12y$.

**b.** Graph the inequalities.




Two of the vertices of the feasible region are (0, 0) and (0, 15). Find the other vertices by locating the intersection of the two constraints.

$$\begin{aligned} 3x + 2y &= 40 \\ 3x + 2(0) &= 40 \\ 3x &= 40 \\ x &= \frac{40}{3} \\ &\left(\frac{40}{3}, 0\right) \end{aligned}$$

$$\begin{aligned} 3x + 2y &= 40 \\ x + y &= 15 \\ y &= 15 - x \\ 3x + 2(15 - x) &= 40 \\ 3x + 30 - 2x &= 40 \\ x &= 10 \\ y &= 15 - 10 \\ y &= 5 \\ &(10, 5) \end{aligned}$$

Find the value of the objective function for each vertex.

$$f(x,y)=16x+12y$$
$$f(0,0)=16(0)+12(0)$$
$$=0$$
$$f(0,15)=16(0)+12(15)$$
$$=180$$
$$f\left(\frac{40}{3},0\right)=16\left(\frac{40}{3}\right)+12(0)$$
$$=\frac{640}{3}$$
$$=213\frac{1}{3}$$
$$f(10,5)=16(10)+12(5)$$
$$=160+60$$
$$=220$$

10 batches of Mango Sunrise and 5 batches of Oasis Dream; $220

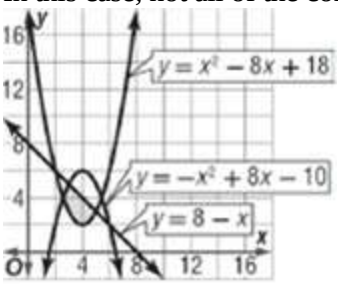
**Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of $x$ and $y$ they occur, subject to the given constraints.**

30. $f(x,y)=4x-8y$

$$y \geq x^2-8x+18$$
$$y \leq -x^2+8x-10$$
$$y \leq 8-x$$

*SOLUTION:*

In this case, not all of the constraints are linear. Graph the constraints and locate the vertices of the feasible region.




Set the equations equal to each other to locate the points of intersection.

$$8-x=x^2-8x+18$$
$$0=x^2-7x+10$$
$$0=(x-5)(x-2)$$
$$x=5 \text{ or } 2$$

The point of intersection is near $x = 5$, so choose 5.

$$y=8-5$$
$$y=3$$
$$(5,3)$$

$8 - x = -x^2 + 8x - 10$

$x^2 - 9x + 18 = 0$

$(x-3)(x-6) = 0$

$x = 3 \text{ or } 6$

The point of intersection is near $x = 3$, so choose 3.

$y = 8 - 3$

$y = 5$

$(3,5)$

$x^2 - 8x + 18 = -x^2 + 8x - 10$

$2x^2 - 16x + 28 = 0$

$x^2 - 8x + 14 = 0$

$(x-4)^2 - 2 = 0$

$(x-4)^2 = 2$

$x - 4 = \pm\sqrt{2}$

$x = 4 + \sqrt{2} \text{ or } 4 - \sqrt{2}$

The point of intersection is near $x = 3$, so choose $4 - \sqrt{2}$.

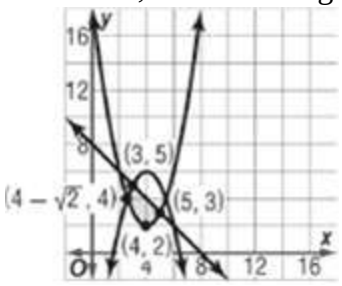
$y = (4-\sqrt{2})^2 - 8(4-\sqrt{2}) + 18$

$y = 16 - 8\sqrt{2} + 2 - 32 + 8\sqrt{2} + 18$

$y = 4$

$(4-\sqrt{2}, 4)$

From the graph, it looks like another point should be checked. This is the minimum of the first constraint at (4, 2). Remember, the feasible region in this case is not a polygonal convex set.




Substitute each of these vertices into the objective function.

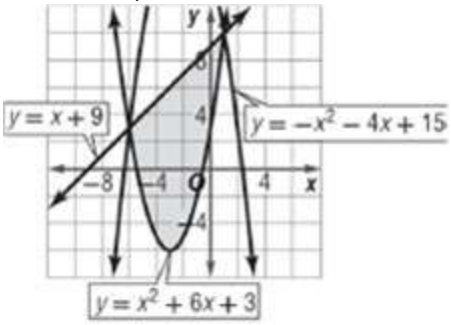
$$f(x,y) = 8x - 4y$$
$$f(3,5) = 8(3) - 4(5)$$
$$= 4$$
$$f(5,3) = 8(5) - 4(3)$$
$$= 28$$
$$f(4-\sqrt{2},4) = 8(4-\sqrt{2}) - 4(4)$$
$$= 32 - 8\sqrt{2} - 16$$
$$= 16 - 8\sqrt{2}$$
$$\approx 4.7$$
$$f(4,2) = 8(4) - 4(2)$$
$$= 24$$

The maximum of 28 is at (5, 3) while the minimum of 4 is at (3, 5).

31.
$$f(x,y) = -2x + 5y$$
$$y \geq x^2 + 6x + 3$$
$$y \leq -x^2 - 4x + 15$$
$$y \leq x + 9$$

*SOLUTION:*

In this case, not all of the constraints are linear. Graph the constraints and locate the vertices of the feasible region.




Set the equations equal to each other to locate the points of intersection.

$$y = x + 9$$
$$x + 9 = x^2 + 6x + 3$$
$$0 = x^2 + 5x - 6$$
$$0 = (x+6)(x-1)$$
$$x = -6 \text{ or } 1$$

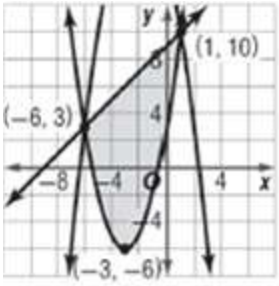
The intersection is near $x = -6$.

$$y = -6 + 9$$
$$y = 3$$
$$(-6, 3)$$

$y = x + 9$
$x + 9 = -x^2 - 4x + 15$
$x^2 + 5x - 6 = 0$
$(x + 6)(x - 1) = 0$
$x = -6 \text{ or } 1$

$y = -6 + 9$
$y = 3$
$(-6, 3)$
All three constraints meet at (−6, 3). They also all meet at (1, 10).

From the graph, it looks like another point should be checked. This is the minimum of the first constraint at (−3, −6). Remember, the feasible region in this case is not a polygonal convex set.




Substitute each of these vertices into the objective function.

$$\begin{aligned} f(x, y) &= -2x + 5y \\ f(-6, 3) &= -2(-6) + 5(3) \\ &= 12 + 15 \\ &= 27 \\ f(-3, -6) &= -2(-3) + 5(-6) \\ &= -24 \\ f(1, 10) &= -2(1) + 5(10) \\ &= 48 \end{aligned}$$

The minimum of −24 is at (−3, −6) while the maximum of 48 is at (1, 10).

**Find the area enclosed by the polygonal convex set defined by each system of inequalities.**

32. $x \geq 0$
$x \leq 12$
$2x + 6y \leq 84$
$2x - 3y \leq -3$
$8x + 3y \geq 33$

*SOLUTION:*

Graph each constraint to form the polygonal convex set. Either solve the equations for $y$ or use intercepts to graph the constraints.

## 6-5 Linear Optimization

Equation 1:

$x \geq 0$

Equation 2:

$x \leq 12$

Equation 3:

$2x + 6y \leq 84$

$6y \leq -2x + 84$

$y \leq -\frac{1}{3}x + 14$

Equation 4:

$2x - 3y \leq -3$

$-3y \leq -2x - 3$

$y \geq \frac{2}{3}x + 1$

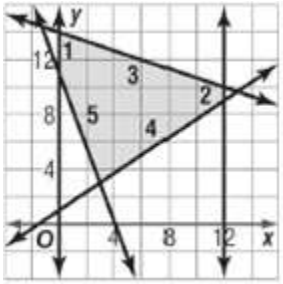
Equation 5:

$8x + 3y \geq 33$

$3y \geq -8x + 33$

$y \geq -\frac{8}{3}x + 11$

Graph the inequalities.




One way to break up this region would be to split it into two trapezoids. First, confirm the intersection of equations 4 and 5.

$$-\frac{8}{3}x + 11 = \frac{2}{3}x + 1$$

$$-8x + 33 = 2x + 3$$

$$30 = 10x$$

$$3 = x$$

$$y = \frac{2}{3}(3) + 1$$

$$y = 2 + 1$$

$$y = 3$$

$$(3,3)$$

Substitute $x = 3$ into equation 3 to find the other endpoint of the vertical line.

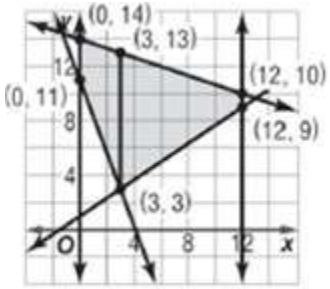
$y = -\frac{1}{3}(3) + 14$

$y = -1 + 14$

$y = 13$

$(3,13)$

Identify the other points on the trapezoid.




The trapezoid on the left has bases of 3 and 10 units and a height of 3 units.

$A = h\left(\frac{b_1 + b_2}{2}\right)$

$A = 3\left(\frac{3+10}{2}\right)$

$A = 19.5$

The trapezoid on the right has bases of 1 and 10 units and a height of 9 units.

$A = h\left(\frac{b_1 + b_2}{2}\right)$

$A = 9\left(\frac{1+10}{2}\right)$

$A = 49.5$

The total area is 19.5 + 49.5 = 69 units$^2$.

33.
$$y \le 9$$
$$x \ge 2$$
$$x - y \le 5$$
$$2x + y \le 25$$
$$3x + 2y \le 20$$

*SOLUTION:*

Graph each constraint to form the polygonal convex set. Either solve the equations for $y$ or use intercepts to graph the constraints.

## 6-5 Linear Optimization

Equation 1:

$y \le 9$

Equation 2:

$x \ge 2$

Equation 3:

$x - y \le 5$

$-y \le -x + 5$

$y \ge x - 5$

Equation 4:

$2x + y \le 25$

$y \le -2x + 25$

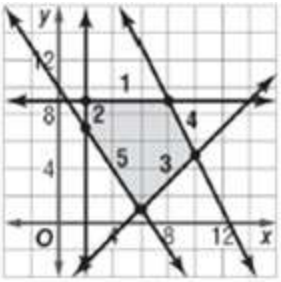
Equation 5:

$3x + 2y \le 20$

$2y \le -3x + 20$

$y \le -\frac{3}{2}x + 10$

Graph the inequalities.



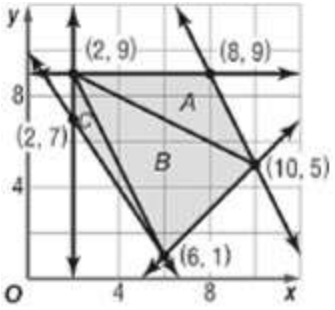

One way to find the area of this region would be to split it into triangles and use determinants to find the area of each triangle. First, confirm the intersection of equations 4 and 5 and equations 3 and 4.

$$y \geq x - 5$$
$$y \leq -2x + 25$$

$$x - 5 = -2x + 25$$
$$3x = 30$$
$$x = 10$$
$$10 - 5 = 5$$
$$(10, 5)$$

$$y \geq x - 5$$
$$y \leq -\frac{3}{2}x + 10$$
$$x - 5 = -\frac{3}{2}x + 10$$
$$2x - 10 = -3x + 20$$
$$5x = 30$$
$$x = 6$$
$$6 - 5 = 1$$
$$(6, 1)$$

Form triangles with the vertices of the figure.




Create a matrix for each triangle. The area of each triangle is equal to one-half the absolute value of the determinant.

$$\frac{1}{2}|\det(A)| = \frac{1}{2}\begin{vmatrix} 2 & 9 & 1 \\ 8 & 9 & 1 \\ 10 & 5 & 1 \end{vmatrix}$$
$$= \frac{1}{2}|2[9(1) - 5(1)] - 9[8(1) - 10(1)] + 8(5) - 10(9)|$$
$$= \frac{1}{2}|2(9 - 5) - 9(8 - 10) + 40 - 90|$$
$$= \frac{1}{2}|2(4) - 9(-2) - 50|$$
$$= \frac{1}{2}|8 + 18 - 50|$$
$$= \frac{1}{2}|-24|$$
$$= 12$$

$$\frac{1}{2}|\det(B)| = \frac{1}{2}\begin{vmatrix} 2 & 9 & 1 \\ 6 & 1 & 1 \\ 10 & 5 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|2[1(1)-5(1)]-9[6(1)-10(1)]+6(5)-10(1)|$$

$$= \frac{1}{2}|-8+36+30-10|$$

$$= 24$$

$$\frac{1}{2}|\det(C)| = \frac{1}{2}\begin{vmatrix} 2 & 9 & 1 \\ 6 & 1 & 1 \\ 2 & 7 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|2[1(1)-7(1)]-9[6(1)-2(1)]+6(7)-2(1)|$$

$$= \frac{1}{2}|-12-36+42-2|$$

$$= 4$$

The area of the region is 12 + 24 + 4 or 40 units$^2$.

$$x \geq 0$$

$$y \geq \frac{1}{2}x$$

34. $3x + 2y \geq 8$

$-3x + 4y \leq 28$

$x + 2y \leq 24$

*SOLUTION:*

Graph each constraint.

## 6-5 Linear Optimization

Equation 1:

$$x \geq 0$$

Equation 2:

$$y \geq \frac{1}{2}x$$

Equation 3:

$$3x + 2y \geq 8$$

$$2y \geq -3x + 8$$

$$y \geq -\frac{3}{2}x + 4$$

Equation 4:

$$-3x + 4y \leq 28$$

$$-4y \leq 3x + 28$$

$$y \geq -\frac{3}{4} - 7$$

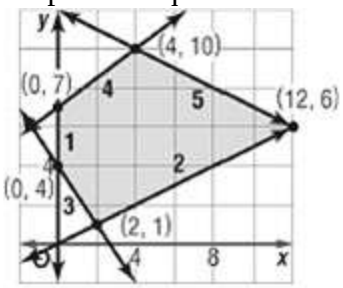
Equation 5:

$$x + 2y \leq 24$$

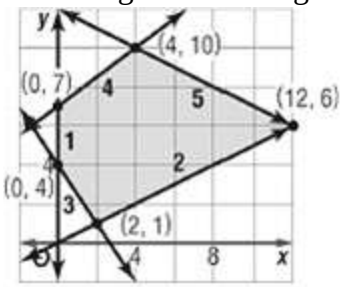
$$2y \leq -x + 24$$

$$y \leq -\frac{1}{2}x + 12$$

Graph the inequalities.




One way to find the area of this region would be to split it into triangles and use determinants to find the area of each triangle. Form triangles with the vertices of the figure.




Create a matrix for each triangle. The area of each triangle is equal to one-half the absolute value of the determinant.

$$\frac{1}{2}|\det(A)| = \frac{1}{2}\begin{vmatrix} 0 & 4 & 1 \\ 0 & 7 & 1 \\ 4 & 10 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|0 - 4[0(1) - 4(1)] + 0(10) - 4(7)|$$

$$= \frac{1}{2}|-4(-4) - 28|$$

$$= \frac{1}{2}|-12|$$

$$= 6$$

$$\frac{1}{2}|\det(B)| = \frac{1}{2}\begin{vmatrix} 0 & 4 & 1 \\ 12 & 6 & 1 \\ 4 & 10 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|0 - 4[12(1) - 4(1)] + 12(10) - 4(6)|$$

$$= \frac{1}{2}|-4(8) + 120 - 24|$$

$$= \frac{1}{2}|-32 + 96|$$

$$= 32$$

$$\frac{1}{2}|\det(C)| = \frac{1}{2}\begin{vmatrix} 0 & 4 & 1 \\ 12 & 6 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|0 - 4[12(1) - 2(1)] + 12(1) - 2(6)|$$

$$= \frac{1}{2}|-4(10) + 12 - 12|$$

$$= \frac{1}{2}|-40|$$

$$= 20$$

The area of the figure is 6 + 32 + 20 or 58 units$^2$.

35.
$$\begin{aligned} x &\le 10 \\ x + y &\le 14 \\ x + 3y &\ge 13 \\ -x + 5y &\le 40 \\ 4x + y &\ge 8 \end{aligned}$$

*SOLUTION:*

Graph each constraint to form the polygonal convex set. Either solve the equations for $y$ or use intercepts to graph the constraints.

## 6-5 Linear Optimization

Equation 1:

$$x \le 10$$

Equation 2:

$$x + y \le 14$$

$$y \le -x + 14$$

Equation 3:

$$x + 3y \ge 13$$

$$3y \ge -x + 13$$

$$y \ge -\frac{1}{3}x + \frac{13}{3}$$

Equation 4:

$$-x + 5y \le 40$$

$$5y \le x + 40$$

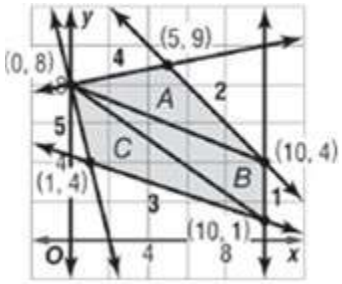
$$y \le \frac{1}{5}x + 8$$

Equation 5:

$$4x + y \ge 8$$

$$y \ge -4x + 8$$

Graph the inequalities. One way to find the area of this region would be to split it into triangles and use determinants to find the area of each triangle. Form triangles with the vertices of the figure.




Create a matrix for each triangle. The area of each triangle is equal to one-half the absolute value of the determinant.

$$\frac{1}{2}|\det(A)| = \frac{1}{2}\begin{vmatrix} 0 & 8 & 1 \\ 5 & 9 & 1 \\ 10 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|0 - 8[5(1) - 10(1)] + (5)(4) - 10(9)|$$

$$= \frac{1}{2}|-8(-5) + 20 - 90|$$

$$= \frac{1}{2}|40 - 70|$$

$$= 15$$

$$\frac{1}{2}|\det(B)| = \frac{1}{2}\begin{vmatrix} 0 & 8 & 1 \\ 10 & 1 & 1 \\ 10 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|0 - 8[10(1) - 10(1)] + (10)(4) - 10(1)|$$

$$= \frac{1}{2}|-8(0) + 40 - 10|$$

$$= \frac{1}{2}|30|$$

$$= 15$$

$$\frac{1}{2}|\det(C)| = \frac{1}{2}\begin{vmatrix} 0 & 8 & 1 \\ 10 & 1 & 1 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|0 - 8[10(1) - 1(1)] + (10)(4) - 1(1)|$$

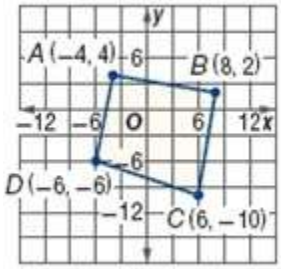
$$= \frac{1}{2}|-8(9) + 40 - 1|$$

$$= \frac{1}{2}|-72 + 39|$$

$$= 16.5$$

The area of the figure is 15 + 15 + 16.5 or 46.5 units$^2$.

36. **CHALLENGE** Provide the system of inequalities that forms the polygonal convex set shown below.




*SOLUTION:*

Use the vertices to find the linear equation associated with each boundary of the region. Then use the shaded area as a guide for which inequality symbol to use.

## 6-5 Linear Optimization

$$AB:(-4,4)\ to\ (8,2)$$

$$m=\frac{4-2}{-4-8}$$

$$=\frac{2}{-12}$$

$$=-\frac{1}{6}$$

$$y-y_1=m(x-x_1)$$

$$y-2=-\frac{1}{6}(x-8)$$

$$y-2=-\frac{1}{6}x+\frac{8}{6}$$

$$6y-12=-x+8$$

$$6y=-x+20$$

$$6y\le -x+20$$

$$BC:(8,2)\ to\ (6,-10)$$

$$m=\frac{2-(-10)}{8-6}$$

$$=\frac{12}{2}$$

$$=6$$

$$y-y_1=m(x-x_1)$$

$$y-2=6(x-8)$$

$$y-2=6x-48$$

$$y=6x-46$$

$$y\ge 6x-46$$

$$CD: (6, -10) to (-6, -6)$$

$$m = \frac{-10 - (-6)}{6 - (-6)}$$

$$= \frac{-4}{12}$$

$$= -\frac{1}{3}$$

$$y - y_1 = m(x - x_1)$$

$$y - (-6) = -\frac{1}{3}(x - (-6))$$

$$y + 6 = -\frac{1}{3}x - 2$$

$$3y + 18 = -x - 6$$

$$3y = -x - 24$$

$$3y \geq -x - 24$$

$$DA: (-6, -6) to (-4, 4)$$

$$m = \frac{-6 - 4}{-6 - (-4)}$$

$$= \frac{-10}{-2}$$

$$= 5$$

$$y - y_1 = m(x - x_1)$$

$$y - 4 = 5(x - (-4))$$

$$y - 4 = 5x - 20$$

$$y = 5x + 24$$

$$y \leq 5x + 24$$

37. **REASONING** Consider the profit function $P(x, y) = ax + by$. Will $P$ always have a positive maximum value if the feasible region lies entirely within the first quadrant? Explain your reasoning.

*SOLUTION:*

No; sample answer: If $a$ or $b$ is negative, then it will be possible for the objective function to have a negative maximum value. If we further restricted $a$ and $b$ to be positive, then $P$ would always have a positive value.

38. **CHALLENGE** Find the maximum and minimum values of the objective function $f(x, y) = -6x + 3y$ and for what values of $x$ and $y$ they occur, subject to the given constraints.

$$y \geq 4$$
$$3x + 2y \geq 14$$
$$-2x + 5y \leq 60$$
$$-x + y \geq -3$$
$$-7x + 5y \leq 35$$
$$2x + y \leq 36$$

*SOLUTION:*

Graph each constraint to identify the feasible region. Either solve the equations for $y$, or use intercepts to graph the constraints.

Equation 1

$$y \geq 4$$

Equation 2

$$3x + 2y \geq 14$$
$$2y \geq -3x + 14$$
$$y \geq -\frac{3}{2}x + 7$$

Equation 3

$$-2x + 5y \leq 60$$
$$5y \leq 2x + 60$$
$$y \leq \frac{2}{5}x + 12$$

Equation 4

$$-x + y \geq -3$$
$$y \geq x - 3$$

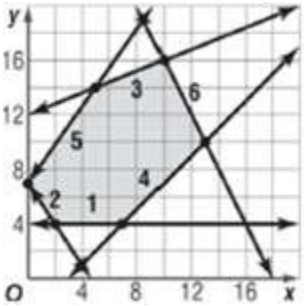
Equation 5

$$-7x + 5y \leq 35$$
$$5y \leq 7x + 35$$
$$y \leq \frac{7}{5}x + 7$$

Equation 6

$$2x + y \leq 36$$
$$y \leq -2x + 36$$

Graph the constraints.




The graph pretty clearly identifies the coordinates of all of the vertices. You can also locate them by using the

*intersection* feature of your calculator. Yet another method is to set the intersecting lines equal to each other.

$$\frac{7}{5}x+7=\frac{2}{5}x+12$$
$$7x+35=2x+60$$
$$5x=25$$
$$x=5$$
$$y=\frac{7}{5}(5)+7$$
$$y=7+7$$
$$y=14$$
$$(5,14)$$

$$-2x+36=\frac{2}{5}x+12$$
$$-10x+180=2x+60$$
$$120=12x$$
$$10=x$$
$$y=\frac{2}{5}(10)+12$$
$$y=4+12$$
$$y=16$$
$$(10,16)$$

$$-2x+36=x-3$$
$$39=3x$$
$$13=x$$
$$y=13-3$$
$$y=10$$
$$(13,10)$$

$$4=x-3$$
$$7=x$$
$$y=4$$
$$(7,4)$$

$$4=-\frac{3}{2}x+7$$
$$8=-3x+14$$
$$-6=-3x$$
$$2=x$$
$$y=4$$
$$(2,4)$$

$$\frac{7}{5}x+7=-\frac{3}{2}x+7$$
$$14x+70=-15x+70$$
$$29x=0$$
$$x=0$$
$$y=\frac{7}{5}(0)+7$$
$$y=7$$
$$(0,7)$$

Substitute these values into the objective function.

$$f(x,y)=-6x+3y$$
$$f(5,14)=-6(5)+3(14)$$
$$=-30+42$$
$$=12$$
$$f(10,16)=-6(10)+3(16)$$
$$=-60+48$$
$$=-12$$
$$f(13,10)=-6(13)+3(10)$$
$$=-78+30$$
$$=-48$$
$$f(7,4)=-6(7)+3(4)$$
$$=-42+12$$
$$=-30$$
$$f(2,4)=-6(2)+3(4)$$
$$=-12+12$$
$$=0$$
$$f(0,7)=-6(0)+3(7)$$
$$=0+21$$
$$=21$$

max of 21at (0, 7), min of −48 at (13, 10)

39. **OPEN ENDED** Linear programming has numerous real-world applications.
**a.** Write a real-world problem that could be solved using linear programming.
**b.** Using at least 4 constraints, write an objective function to be maximized or minimized.
**c.** Sketch a graph of the region determined by the constraints from part **a** to find the set of feasible solutions.
**d.** Find the solution to the problem.

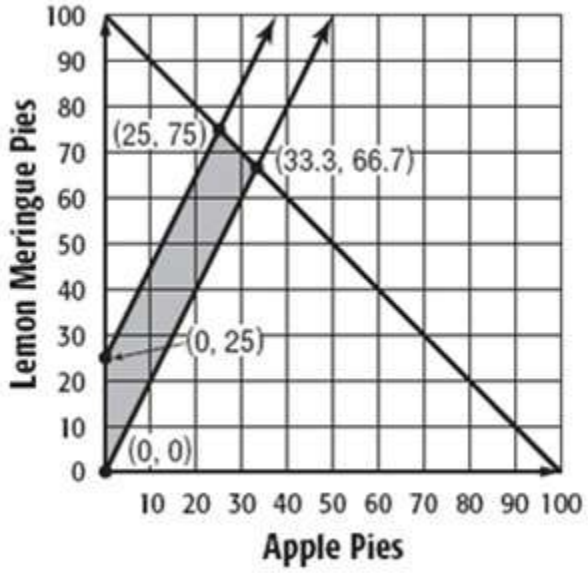
*SOLUTION:*

**a.** Set up an objective using two variables. Put restrictions on the variables using constraints. Set up the constraints to intersect at integral values to make the solution easier. Make sure the problem is logical.

Sample answer: A bakery makes up to 100 apple and lemon meringue pies each week. Due to labor costs, the number of lemon meringue pies that are made needs to be less than or equal to 25 plus two times the number of apple pies. The customer demand for lemon meringue pies is at least twice that of applies pies. The bakery makes a profit of $8 per apple pie and $10 per lemon meringue pie. How many of each type of pie should be made to maximize profit?

**b.** Some good constraints are $x \geq 0$ and $y \geq 0$, since most applications involve positive values.

$f(x, y) = 8x + 10y$
$y \leq 25 + 2x$
$y \geq 2x$
$x + y \leq 100$
$x \geq 0$
$y \geq 0$




c.
The graph of the system of inequalities forms a polygonal region with vertices at (0, 0), (0, 25), (25, 75), and (33.3, 66.7). Locate the intersections of the constraints by solving the system of equations or by using the *intersection* feature of your calculator.

d. Substitute the values of each vertex into the objective function. Since $f$ is greatest at (25, 75), the bakery should make 25 apple pies and 75 lemon meringue pies to earn a maximum profit of $950.

40. **WRITING IN MATH** Is it possible for a linear programming problem to have no maximum solution and no minimum solution? Explain your reasoning.

*SOLUTION:*

Yes; this situation can occur when the constraints of the problem do not form a feasible region. When this happens, there will be no points to test and, thus, no maximum or minimum.

For example, consider the following objective function and set of constraints.

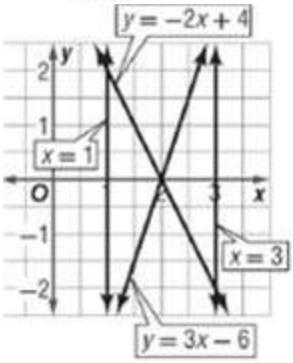
$f(x, y) = 8x + 10y$

$y \geq 3x - 6$

$y \leq -2x + 4$

$x \geq 3$

$x \leq 1$




No feasible region exists, so there is no maximum or minimum.

**Find the partial fraction decomposition of each rational expression.**

41. $\frac{8y+7}{y^2+y-2}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{8y+7}{y^2+y-2}=\frac{A}{y-1}+\frac{B}{y+2}$$

Multiply each side by the LCD, $y^2 + y - 2$.

$$8y+7=A(y+2)+B(y-1)$$

Group the like terms.

$$8y+7=A(y+2)+B(y-1)$$
$$8y+7=Ay+2A+By-B$$
$$8y+7=Ay+By+2A-B$$
$$8y+7=(A+B)y+2A-B$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=8$$
$$2A-B=7$$

Use any method to solve the new system.

$$A+B=8$$
$$A=8-B$$
$$2A-B=7$$
$$2(8-B)-B=7$$
$$16-2B-B=7$$
$$16-3B=7$$
$$3B=9$$
$$B=3$$
$$A=8-(3)$$
$$A=5$$

Replace $A$ and $B$ with 5 and 3 in the partial fraction decomposition.

$$\frac{8y+7}{y^2+y-2}=\frac{A}{y-1}+\frac{B}{y+2}$$
$$\frac{8y+7}{y^2+y-2}=\frac{5}{y-1}+\frac{3}{y+2}$$

42. $\frac{x-6}{x^2-2x}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{x-6}{x^2-2x}=\frac{A}{x}+\frac{B}{x-2}$$

Multiply each side by the LCD, $x^2 - 2x$.

$$x-6=A(x-2)+Bx$$

Group the like terms.

$$x-6=A(x-2)+Bx$$

$$x-6=Ax-2A+Bx$$

$$x-6=Ax+Bx-2A$$

$$x-6=(A+B)x-2A$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=1$$

$$-2A=-6$$

Use any method to solve the new system.

$$-2A=-6$$

$$A=3$$

$$A+B=1$$

$$3+B=1$$

$$B=-2$$

Replace $A$ and $B$ with 3 and −2 in the partial fraction decomposition.

$$\frac{x-6}{x^2-2x}=\frac{A}{x}+\frac{B}{x-2}$$

$$\frac{x-6}{x^2-2x}=\frac{3}{x}+\frac{-2}{x-2}$$

## 6-5 Linear Optimization

43. $\frac{5m-4}{m^2-4}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{5m-4}{m^2-4}=\frac{A}{m+2}+\frac{B}{m-2}$$

Multiply each side by the LCD, $m^2-4$ .

$$5m-4=A(m-2)+B(m+2)$$

Group the like terms.

$$5m-4=A(m-2)+B(m+2)$$
$$5m-4=Am-2A+Bm+2B$$
$$5m-4=Am+Bm-2A+2B$$
$$5m-4=(A+B)m-2A+2B$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $m$-terms on the left side of the equation must equal the coefficients of the $m$ -terms on the right side.

$$A+B=5$$
$$-2A+2B=-4$$

Use any method to solve the new system.

$$A+B=5$$
$$A=5-B$$
$$-2A+2B=-4$$
$$-2(5-B)+2B=-4$$
$$-10+2B+2B=-4$$
$$-10+4B=-4$$
$$4B=6$$
$$B=1.5$$
$$A=5-(1.5)$$
$$A=3.5$$

Replace $A$ and $B$ with 3.5 and 1.5 in the partial fraction decomposition.

$$\frac{5m-4}{m^2-4}=\frac{A}{m+2}+\frac{B}{m-2}$$
$$\frac{5m-4}{m^2-4}=\frac{3.5}{m+2}+\frac{1.5}{m-2}$$

44. $\dfrac{-4y}{3y^2-4y+1}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{-4y}{3y^2-4y+1}=\frac{A}{3y-1}+\frac{B}{y-1}$$

Multiply each side by the LCD, $3y^2-4y+1$.

$$-4y=A(y-1)+B(3y-1)$$

Group the like terms.

$$-4y=A(y-1)+B(3y-1)$$
$$-4y=Ay-A+3By-B$$
$$-4y=Ay+3By-A-B$$
$$-4y=(A+3B)y-A-B$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $y$-terms on the left side of the equation must equal the coefficients of the $y$-terms on the right side.

$$A+3B=-4$$
$$-A-B=0$$

Use any method to solve the new system.

$$-A-B=0$$
$$A=-B$$
$$A+3B=-4$$
$$-B+3B=-4$$
$$2B=-4$$
$$B=-2$$
$$A=-(-2)$$
$$A=2$$

Replace $A$ and $B$ with 2 and −2 in the partial fraction decomposition.

$$\frac{-4y}{3y^2-4y+1}=\frac{A}{3y-1}+\frac{B}{y-1}$$
$$\frac{-4y}{3y^2-4y+1}=\frac{2}{3y-1}+\frac{-2}{y-1}$$

45. **ARCADE GAMES** Marcus and Cody purchased game cards to play virtual games at the arcade. Marcus used 47 points from his game card to drive the race car and snowboard simulators four times each. Cody used 48.25 points from his game card to drive the race car simulator five times and the snowboard simulator three times. How many points did each game require per play?

*SOLUTION:*

Let $r$ = the race car and let $s$ = the snowboard. Set up a system of equations. $4r + 4s = 47$ and $5r + 3s = 48.25$. Solve the system.

$$4r + 4s = 47$$
$$5r + 3s = 48.25$$

$$(-3)(4r + 4s = 47)$$
$$(4)(5r + 3s = 48.25)$$

$$-12r - 12s = 141$$
$$(+)\,20r + 12s = 193$$
$$8r = 52$$
$$r = 6.5$$
$$4(6.5) + 4s = 47$$
$$26 + 4s = 47$$
$$4s = 21$$
$$s = 5.25$$

race car, 6.5 points; snowboard, 5.25 points

46. **WAVES** After a wave is created by a boat, the height of the wave can be modeled using $y = \frac{1}{2}h + \frac{1}{2}h \sin \frac{2\pi t}{P}$, where $h$ is the maximum height of the wave in feet, $P$ is the period in seconds, and $t$ is the propagation of the wave in seconds.
**a**. If $h = 3$ and $P = 2$, write the equation for the wave. Draw its graph over a 10-second interval.
**b**. How many times over the first 10 seconds does the graph predict the wave to be one foot high?

*SOLUTION:*

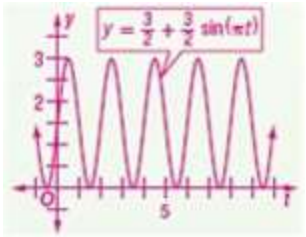
**a.** Substitute the values into the equation and simplify.

$$y = \frac{1}{2}h + \frac{1}{2}h \sin \frac{2\pi t}{P}$$

$$y = \frac{1}{2}(3) + \frac{1}{2}(3) \sin \frac{2\pi t}{2}$$

$$y = \frac{3}{2} + \frac{3}{2} \sin \pi t$$

Draw the graph.




**b.** The first tick mark above the origin represents 1 foot. From $0 < t < 10$, the graph crosses this point 10 times.

47. Verify that $\tan \theta \sin \theta \cos \theta \csc^2 \theta = 1$ is an identity.

*SOLUTION:*

$$\begin{aligned}\tan\theta \sin\theta \cos\theta \csc^2\theta &= \frac{\sin\theta}{\cos\theta} \sin\theta \cos\theta \frac{1}{\sin^2\theta} \\ &= \frac{\sin^2\theta}{\cos\theta} \frac{\cos\theta}{\sin^2\theta} \\ &= 1\end{aligned}$$

**Find the area of each triangle to the nearest tenth.**

48. $\triangle ABC$, if $A = 127°$, $b = 12$m, and $c = 9$m

*SOLUTION:*

The formula for area given SAS is $Area = \frac{1}{2}bc \sin A$. Substitute 127 for $A$, 12 for $b$, and 9 for $c$.

$$Area = \frac{1}{2}bc \sin A$$

$$Area = \frac{1}{2}(12)(9) \sin 127$$

$$Area \approx 43.1$$

49. $\triangle ABC$, if $a = 7$ yd, $b = 8$ yd, and $C = 44°$

*SOLUTION:*

The formula for area given SAS is $Area = \frac{1}{2}ab\sin C$. Substitute 44 for $C$, 7 for $a$, and 8 for $b$.

$$Area = \frac{1}{2}ab\sin C$$

$$Area = \frac{1}{2}(7)(8)\sin 44$$

$$Area \approx 19.5$$

50. $\triangle ABC$, if $A = 50°$, $b = 15$ in., and $c = 10$ in.

*SOLUTION:*

The formula for area given SAS is $Area = \frac{1}{2}bc\sin A$. Substitute 50 for $A$, 15 for $b$, and 10 for $c$.

$$Area = \frac{1}{2}bc\sin A$$

$$Area = \frac{1}{2}(15)(10)\sin 50$$

$$Area \approx 57.5$$

51. $\triangle ABC$, if $a = 6$ cm, $B = 135°$, and $c = 3$ cm

*SOLUTION:*

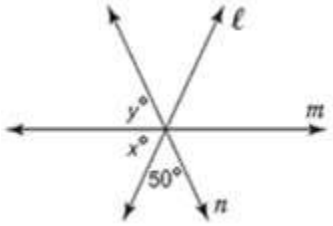
The formula for area given SAS is $Area = \frac{1}{2}ac\sin B$. Substitute 135 for $B$, 6 for $a$, and 3 for $c$.

$$Area = \frac{1}{2}ac\sin B$$

$$Area = \frac{1}{2}(6)(3)\sin 135$$

$$Area \approx 6.4$$

52. **SAT/ACT** In the figure below, lines *l*, *m*, and *n* intersect in a single point. What is the value of $x + y$?




A 40
B 70
C 90
D 130
E 260

*SOLUTION:*

Line *n* is a straight line and, therefore, is 180°. Angles *x*, *y*, and the 50° angle add up to equal line *n*. Thus, $x + y + 50 = 180$ and $x + y = 130$. The correct choice is D.

53. The area of a parking lot is 600 square meters. A car requires 6 square meters of space, and a bus requires 30 square meters of space. The attendant can handle no more than 60 vehicles. If it cost $3 to park a car and $8 to park a bus, how many of each should the attendant accept to maximize income?

F 20 buses and 0 cars
G 10 buses and 50 cars
H 5 buses and 55 cars
J 0 buses and 60 cars

*SOLUTION:*

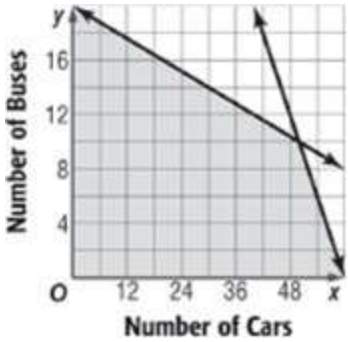
Set up a system of inequalities using the constraints in the problem. Let $c$ = the number of cars and let $b$ = the number of buses. There is 600 square meters of space available, so $30b + 6c \leq 600$. There is a maximum of 60 vehicles, so $b + c \leq 60$.

Solve the inequalities for $b$.

$$b + c \leq 60$$
$$b \leq 60 - c$$

$$30b + 6c \leq 600$$
$$30b \leq 600 - 6c$$
$$b \leq \frac{600 - 6c}{30}$$
$$b \leq \frac{100 - c}{5}$$

Graph the constraints.




Three of the endpoints of the shaded region are (0, 0), (0, 20), and (60, 0). Find the fourth point by locating the intersection of the two lines.

$$60 - c = \frac{100 - c}{5}$$
$$300 - 5c = 100 - c$$
$$200 = 4c$$
$$50 = c$$

$$b = 60 - (50)$$
$$= 10$$

Find the value of the objective function for each of the points.

$$f(c,b) = 3c + 8b$$
$$f(0,0) = 3c + 8b$$
$$= 3(0) + 8(0)$$
$$= 0$$
$$f(0,20) = 3(0) + 8(20)$$
$$= 160$$
$$f(60,0) = 3(60) + 8(0)$$
$$= 180$$
$$f(50,10) = 3(50) + 8(10)$$
$$= 230$$

The correct choice is G.

54. **FREE RESPONSE** Use the two systems of equations to answer each of the following.

| $A$ | $B$ |
|---|---|
| $-5x + 2y + 11z = 31$ | $x + 2y + 2z = 3$ |
| $2y + 6z = 26$ | $3x + 7y + 9z = 30$ |
| $2x - y - 5z = -15$ | $-x - 4y - 7z = -37$ |

**a**. Write the coefficient matrix for each system. Label the matrices $A$ and $B$.

**b.** Find $AB$ and $BA$ if possible.

**c.** Write the augmented matrix for system $A$ in reduced row-echelon form.

**d.** Find the determinant of each coefficient matrix. Which matrices are invertible? Explain your reasoning.

**e.** Find the inverse of matrix $B$.

**f.** Use the inverse of $B$ to solve the system.

**g.** Which systems could you use Cramer's Rule to solve? Explain your reasoning.

*SOLUTION:*

**a**. The coefficients are the values in front of each variable. When the coefficient matrix is written, the variables are all aligned.

$$A = \begin{bmatrix} -5 & 2 & 11 \\ 0 & 2 & 6 \\ 2 & -1 & -5 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

**b.** The matrices are both $3 \times 3$, so $AB$ and $BA$ both exist.

$$AB = \begin{bmatrix} -5 & 2 & 11 \\ 0 & 2 & 6 \\ 2 & -1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} -5(1)+2(3)+11(-1) & -5(2)+2(7)+11(-4) & -5(2)+2(9)+11(-7) \\ 0(1)+2(3)+6(-1) & 0(2)+2(7)+6(-4) & 0(2)+2(9)+6(-7) \\ 2(1)-1(3)-5(-1) & 2(2)-1(7)-5(-4) & 2(2)-1(9)-5(-7) \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -40 & -69 \\ 0 & -10 & -24 \\ 4 & 17 & 30 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} \cdot \begin{bmatrix} -5 & 2 & 11 \\ 0 & 2 & 6 \\ 2 & -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-5)+2(0)+2(2) & 1(2)+2(2)+2(-1) & 1(11)+2(6)+2(-5) \\ 3(-5)+7(0)+9(2) & 3(2)+7(2)+9(-1) & 3(11)+7(6)+9(-5) \\ -1(-5)-4(0)-7(2) & -1(2)-4(2)-7(-1) & -1(11)-4(6)-7(-5) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 & 19 \\ 3 & 11 & 30 \\ -9 & -3 & 0 \end{bmatrix}$$

**c**. The augmented matrix includes the constant terms.

$$\left[\begin{array}{ccc|c} -5 & 2 & 11 & 31 \\ 0 & 2 & 6 & 26 \\ 2 & -1 & -5 & -15 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -5 & 2 & 11 & 31 \\ 0 & 2 & 6 & 26 \\ 0 & -1 & -3 & -13 \end{array}\right] \xleftarrow{2R_3+5R_1}$$

$$\left[\begin{array}{ccc|c} -5 & 2 & 11 & 31 \\ 0 & 2 & 6 & 26 \\ 0 & -1 & -3 & -13 \end{array}\right] \xleftarrow{2R_3+5R_1}$$

$$\left[\begin{array}{ccc|c} -5 & 2 & 11 & 31 \\ 0 & 1 & 3 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \xleftarrow{R_2 \div 2} \\ \xleftarrow{2R_3+R_2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -0.4 & 2.2 & 6.2 \\ 0 & 1 & 3 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_1+(-5)}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3.4 & 11.4 \\ 0 & 1 & 3 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_1+0.4R_2}$$

**d.**

$$\det(A) = \begin{vmatrix} -5 & 2 & 11 \\ 0 & 2 & 6 \\ 2 & -1 & -5 \end{vmatrix}$$

$$= -5\begin{vmatrix} 2 & 6 \\ -1 & -5 \end{vmatrix} - 2\begin{vmatrix} 0 & 6 \\ 2 & -5 \end{vmatrix} + 11\begin{vmatrix} 0 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= -5[2(-5) - (-1)(6)] - 2[0(-5) - 2(6)] + 11[0(-1) - 2(2)]$$

$$= -5(-4) - 2(-12) + 11(-4)$$

$$= 20 + 24 - 44$$

$$= 0$$

The det($A$) equals 0, which is consistent with the infinite solution answer for part **c**.

$$\det(B) = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{vmatrix}$$

$$= 1\begin{vmatrix} 7 & 9 \\ -4 & -7 \end{vmatrix} - 2\begin{vmatrix} 3 & 9 \\ -1 & -7 \end{vmatrix} + 2\begin{vmatrix} 3 & 7 \\ -1 & -4 \end{vmatrix}$$

$$= 1[7(-7) - (-4)(9)] - 2[3(-7) - (-1)(9)] + 2[3(-4) - (-1)(7)]$$

$$= -13 - 2(-12) + 2(-5)$$

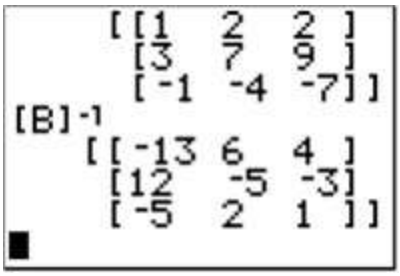
$$= -13 + 24 - 10$$

$$= 1$$

The det($B$) = 1.

$A$ is not invertible because $\det(A) = 0$. $B$ is invertible because $\det(B) \neq 0$.

**e**. Use a calculator.




**f.** Multiply $B^{-1}$ by the matrix of constant terms.

$$\begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 30 \\ -37 \end{bmatrix} = \begin{bmatrix} -13(3) + 6(30) + 4(-37) \\ 12(3) - 5(30) - 3(-37) \\ -5(3) + 2(30) + 1(-37) \end{bmatrix}$$

$$= \begin{bmatrix} -39 + 180 - 148 \\ 36 - 150 + 111 \\ -15 + 60 - 37 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ -3 \\ 8 \end{bmatrix}$$

g. Sample answer: Cramer's Rule can be used to solve system $B$ because $\det(B) \neq 0$. Since $\det(A) = 0$, Cramer's Rule cannot be used to solve system $A$.

# Study Guide and Review - Chapter 6

**Choose the word or phrase that best completes each sentence.**

1. A(n) (augmented matrix, coefficient matrix) is a matrix made up of all the coefficients and constant term, that make up a linear system.

*SOLUTION:*

An augmented matrix, is a matrix made up of all the coefficients and constant terms that make up a linear system. For example, the augmented matrix is shown to the right of the system of equations below.

$$\begin{aligned} 2x-3y&=2\\ -3x+2y&=-8 \end{aligned} \qquad \left[\begin{array}{cc|c} 2 & -3 & 2\\ -3 & 2 & -8 \end{array}\right]$$

The coefficient matrix for this system is $\begin{bmatrix} 2 & -3\\ -3 & 2 \end{bmatrix}$.

2. (Gaussian elimination, Elementary row operations) reduce(s) a system of equations to an equivalent, simpler system, making it easier to solve.

*SOLUTION:*

Gaussian elimination reduces a system of equations to an equivalent, simpler system, making it easier to solve. Gaussian elimination can be performed with or without matrices. Elementary row operations are the method by which Gaussian elimination is performed with matrices.

3. The result of Gauss-Jordan elimination is a matrix that is in (reduced row-echelon, invertible) form.

*SOLUTION:*

The result of Gauss-Jordan elimination is a matrix that is in reduced row-echelon form. Two examples of reduced row-echelon form are shown below.

$\left[\begin{array}{cc|c} 1 & 0 & 2\\ 0 & 1 & -4 \end{array}\right]$ and $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 6\\ 0 & 1 & 2 & -4\\ 0 & 0 & 0 & 0 \end{array}\right]$

4. The product of an $n \times n$ matrix $A$ with the (inverse matrix, identity matrix) is $A$.

*SOLUTION:*

The product of an $n \times n$ matrix $A$ with the identity matrix is $A$.

$A \cdot I = A$

$$\begin{bmatrix} 3 & -4\\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4\\ 3 & 5 \end{bmatrix}$$

The product of an $n \times n$ matrix $A$ with the inverse matrix is $I$, the identity matrix.

$A \cdot A^{-1} = I$

$$\begin{bmatrix} 3 & -4\\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} \frac{5}{27} & \frac{4}{27}\\ -\frac{1}{9} & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$$

5. The identity matrix $I$ is its own (augmented matrix, inverse matrix).

*SOLUTION:*

The identity matrix $I$ is its own inverse matrix.

$I \cdot I = I$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

6. A square matrix that has no inverse is (nonsingular, singular).

*SOLUTION:*

A square matrix that has no inverse is singular. When a square matrix has an inverse, it is said to be invertible, or nonsingular.

7. The (determinant, square system) of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

*SOLUTION:*

The determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. The determinant *determines* the type of solutions to a system, whether a matrix is invertible, and the inverse of a matrix.

8. When solving a square linear system, an alternative to Gaussian elimination is (Cramer's Rule, partial fraction decomposition).

*SOLUTION:*

When solving a square linear system, an alternative to Gaussian elimination is Cramer's Rule, which can be systems of any size as long as there is a unique solution. Partial fraction decomposition does not involve solving systems of equations.

9. A two-dimensional linear programming problem contains (constraints, feasible solutions), which are linear inequalities.

*SOLUTION:*

A two-dimensional linear programming problem contains constraints, which are linear inequalities. It also may contain a region of feasible solutions which is formed by the boundaries of the constraints.

10. If the graph of the objective function to be optimized, $f$, is coincident with one side of the region of feasible solutions, then there may be (multiple optimal, unbounded) solutions.

*SOLUTION:*

If the graph of the objective function to be optimized, $f$, is coincident with one side of the region of feasible solutions, *the feasible region* may be unbounded and then there may be multiple optimal solutions.

**Write each system of equations in triangular form using Gaussian elimination. Then solve the system.**

11. $3x + 4y = 7$
$2y = -5x + 7$

*SOLUTION:*

Align the variables on the left side of the equal sign.

$3x + 4y = 7$

$5x + 2y = 7$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by −5 and the 2nd equation by 3.

$(-5)(3x + 4y = 7) \Rightarrow -15x - 20y = -35$

$(3)(5x + 2y = 7) \Rightarrow 15x + 6y = 21$

Add the equations together.

$-15x - 20y = -35$
$15x + 6y = 21$
$-14y = -14$
$y = 1$

Replace the 2nd equation with $y = 1$.

$3x + 4y = 7$
$y = 1$

The system is now in triangular form. Substitute for $y$ and solve for $x$.

$3x + 4y = 7$
$3x + 4(1) = 7$
$3x + 4 = 7$
$3x = 3$
$x = 1$

The solution is (1, 1).

12. $5x - 3y = 16$
$x + 3y = -4$

*SOLUTION:*

Align the variables on the left side of the equal sign.

$5x - 3y = 16$
$x + 3y = -4$

Eliminate the $x$-term from the 2nd equation. Multiply the 2nd equation by −5.

$(-5)(x + 3y = -4) \Rightarrow -5x - 15y = 20$

Add the equations together.

$5x - 3y = 16$
$-5x - 15y = 20$
$-18y = 36$
$y = -2$

Replace the 2nd equation with $y = -2$.

$5x - 3y = 16$
$y = -2$

The system is now in triangular form. Substitute for $y$ and solve for $x$.

$5x - 3y = 16$
$5x - 3(-2) = 16$
$5x + 6 = 16$
$5x = 10$
$x = 2$

The solution is (2, −2).

13. $x + y + z = 4$
$2x - y - 3z = 4$
$-3x - 4y - 5z = -13$

*SOLUTION:*

The variables are aligned on the left side of the equal sign.

$x + y + z = 4$
$2x - y - 3z = 4$
$-3x - 4y - 5z = -13$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by −2.

$(-2)(x + y + z = 4) \Rightarrow -2x - 2y - 2z = -8$
$2x - y - 3z = 4 \Rightarrow 2x - y - 3z = 4$

Add the equations together.

$-2x - 2y - 2z = -8$
$(+)\ 2x - y - 3z = 4$
$-3y - 5z = -4$

The new set of equations is:

$$\begin{aligned} x+y+z&=4\\ -3y-5z&=-4\\ -3x-4y-5z&=-13 \end{aligned}$$

Eliminate the $x$-term from the 3rd equation. Multiply the 1st equation by 3.

$$(3)(x+y+z=4) \quad \Rightarrow \quad 3x+3y+3z=12$$
$$-3x-4y-5z=-13 \quad \Rightarrow \quad -3x-4y-5z=-13$$

Add the equations together.

$$\begin{aligned} 3x+3y+3z&=12\\ (+)\,-3x-4y-5z&=-13\\ -y-2z&=-1 \end{aligned}$$

The new set of equations is:

$$\begin{aligned} x+y+z&=4\\ -3y-5z&=-4\\ -y-2z&=-1 \end{aligned}$$

Eliminate the $y$-term from the 3rd equation. Multiply the 3rd equation by $-3$.

$$(-3y-5z=-4) \quad \Rightarrow \quad -3y-5z=-4$$
$$(-3)(-y-2z=-1) \quad \Rightarrow \quad 3y+6z=3$$

Add the equations together.

$$\begin{aligned} -3y-5z&=-4\\ (+)\,3y+6z&=3\\ z&=-1 \end{aligned}$$

Replace the 3rd equation with $z = -1$.

$$\begin{aligned} x+y+z&=4\\ -3y-5z&=-4\\ z&=-1 \end{aligned}$$

The system is now in triangular form. Substitute for $z$ and solve for $y$, then substitute for $y$ and $z$ and solve for $x$.

$$\begin{aligned} -3y-5z&=-4\\ -3y-5(-1)&=-4\\ -3y+5&=-4\\ -3y&=-9\\ y&=3\\ x+y+z&=4\\ x+3+(-1)&=4\\ x+3-1&=4\\ x+2&=4\\ x&=2 \end{aligned}$$

The solution is $(2,3,-1)$.

14. $$\begin{aligned} x+y-z&=5\\ 2x-3y+5z&=-1\\ 3x-y+2z&=10 \end{aligned}$$

*SOLUTION:*

The variables are aligned on the left side of the equal sign.

$$x + y - z = 5$$
$$2x - 3y + 5z = -1$$
$$3x - y + 2z = 10$$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by $-2$.

$$(-2)(x + y - z = 5) \quad \Rightarrow \quad -2x - 2y + 2z = -10$$
$$2x - 3y + 5z = -1 \quad \Rightarrow \quad 2x - 3y + 5z = -1$$

Add the equations together.

$$-2x - 2y + 2z = -10$$
$$(+)\ 2x - 3y + 5z = -1$$
$$-5y + 7z = -11$$

The new set of equations is:

$$x + y - z = 5$$
$$-5y + 7z = -11$$
$$3x - y + 2z = 10$$

Eliminate the $x$-term from the 3rd equation. Multiply the 1st equation by $-3$.

$$(-3)(x + y - z = 5) \Rightarrow -3x - 3y + 3z = -15$$
$$3x - y + 2z = 10 \Rightarrow 3x - y + 2z = 10$$

Add the equations together.

$$-3x - 3y + 3z = -15$$
$$(+)\ 3x - y + 2z = 10$$
$$-4y + 5z = -5$$

The new set of equations is:

$$x + y - z = 5$$
$$-5y + 7z = -11$$
$$-4y + 5z = -5$$

Eliminate the $y$-term from the 3rd equation. Multiply the 2nd equation by $-4$ and the 3rd equation by 5.

$$(-4)(-5y + 7z = -11) \Rightarrow 20y - 28z = 44$$
$$(5)(-4y + 5z = -5) \Rightarrow -20y + 25z = -25$$

Add the equations together.

$$20y - 28z = 44$$
$$(+)\ -20y + 25z = -25$$
$$-3z = 19$$
$$z = -\frac{19}{3}$$

Replace the 3rd equation with $z = -\frac{19}{3}$.

$$x + y - z = 5$$
$$-5y + 7z = -11$$
$$z = -\frac{19}{3}$$

The system is now in triangular form. Substitute for $z$ and solve for $y$, then substitute for $y$ and $z$ and solve for $x$.

$$-5y+7z=-11$$
$$-5y+7\left(-\frac{19}{3}\right)=-11$$
$$-5y-\frac{133}{3}=-11$$
$$-5y=\frac{100}{3}$$
$$y=-\frac{20}{3}$$
$$x+y-z=5$$
$$x+\left(-\frac{20}{3}\right)-\left(-\frac{19}{3}\right)=5$$
$$x-\frac{1}{3}=5$$
$$x=\frac{16}{3}$$

The solution is $\left(\frac{16}{3},-\frac{20}{3},-\frac{19}{3}\right)$.

15. $2x-5y=2z+11$
$3y+4z=x-28$
$3z-x=-18-3y$

*SOLUTION:*

Align the variables on the left side of the equal sign.
$$2x-5y-2z=11$$
$$-x+3y+4z=-28$$
$$-x+3y+3z=-18$$

Eliminate the $x$-term from the 2nd equation. Multiply the 2nd equation by 2.

$2x-5y-2z=11 \quad \Rightarrow \quad 2x-5y-2z=11$

$(2)(-x+3y+4z)=-28 \Rightarrow -2x+6y+8z=-56$

Add the equations together.

$$2x-5y-2z=11$$
$$(+)\ -2x+6y+8z=-56$$
$$y+6z=-45$$

The new set of equations is:

$$2x-5y-2z=11$$
$$y+6z=-45$$
$$-x+3y+3z=-18$$

Eliminate the $x$-term from the 3rd equation. Multiply the 3rd equation by 2.

$2x-5y-2z=25 \quad \Rightarrow \quad 2x-5y-2z=11$

$(2)(-x+3y+3z=-18) \quad \Rightarrow \quad -2x+6y+6z=-36$

Add the equations together.

$$\begin{aligned} 2x-5y-2z &= 11 \\ (+)\ -2x+6y+6z &= -36 \\ y+4z &= -25 \end{aligned}$$

The new set of equations is:

$$\begin{aligned} 2x-5y-2z &= 11 \\ y+6z &= -45 \\ y+4z &= -25 \end{aligned}$$

Eliminate the $y$-term from the 3rd equation. Multiply the 2nd equation by $-1$.

$$(-1)(y+6z=-45) \quad \Rightarrow \quad -y-6z=45$$
$$y+4z=-25 \quad \Rightarrow \quad y+4z=-25$$

Add the equations together.

$$\begin{aligned} -y-6z &= 45 \\ (+)\ y+4z &= -25 \\ -2z &= 20 \\ z &= -10 \end{aligned}$$

Replace the 3rd equation with $z = -10$.

$$\begin{aligned} 2x-5y-2z &= 11 \\ y+6z &= -45 \\ z &= -10 \end{aligned}$$

The system is now in triangular form. Substitute for $z$ and solve for $y$, then substitute for $y$ and $z$ and solve for $x$.

$$\begin{aligned} y+6z &= -45 \\ y+6(-10) &= -45 \\ y-60 &= -45 \\ y &= 15 \\ 2x-5y-2z &= 11 \\ 2x-5(15)-2(-10) &= 11 \\ 2x-75+20 &= 11 \\ 2x-55 &= 11 \\ 2x &= 66 \\ x &= 33 \end{aligned}$$

The solution is $(33, 15, -10)$.

16. $\begin{aligned} 2x-3z &= y-1 \\ 5z-8 &= 3x+4y \\ x+y+z &= 3 \end{aligned}$

*SOLUTION:*

Align the variables on the left side of the equal sign.

$$\begin{aligned} 2x-y-3z &= -1 \\ 3x+4y-5z &= -8 \\ x+y+z &= 3 \end{aligned}$$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by $-3$ and the 2nd equation by 2.

$$(-3)(2x-y-3z=-1) \Rightarrow -6x+3y+9z=3$$
$$(2)(3x+4y-5z=-8) \Rightarrow 6x+8y-10z=-16$$

Add the equations together.

$$\begin{aligned} -6x+3y+9z &= 3 \\ (+)\,6x+8y-10z &= -16 \\ 11y-z &= -13 \end{aligned}$$

The new set of equations is:

$$\begin{aligned} 2x-y-3z &= -1 \\ 11y-z &= -13 \\ x+y+z &= 3 \end{aligned}$$

Eliminate the $x$-term from the 3rd equation. Multiply the 3rd equation by −2.

$$\begin{aligned} 2x-y-3z &= -1 \quad \Rightarrow \quad 2x-y-3z = -1 \\ (-2)(-2x-2y-2z &= -6) \quad \Rightarrow \quad -2x+6y+6z = -36 \end{aligned}$$

Add the equations together.

$$\begin{aligned} 2x-y-3z &= -1 \\ (+)\,-2x-2y-2z &= -6 \\ -3y-5z &= -7 \end{aligned}$$

The new set of equations is:

$$\begin{aligned} 2x-y-3z &= -1 \\ 11y-z &= -13 \\ -3y-5z &= -7 \end{aligned}$$

Eliminate the $y$-term from the 3rd equation. Multiply the 2nd equation by 3 and the 3rd equation by 11.

$$\begin{aligned} (3)(11y-z &= -13) \quad \Rightarrow \quad 33y-3z = -39 \\ (11)(-3y-5z &= -7) \quad \Rightarrow \quad -33y-55z = -77 \end{aligned}$$

Add the equations together.

$$\begin{aligned} 33y-3z &= -39 \\ (+)\,-33y-55z &= -77 \\ -58z &= -116 \\ z &= 2 \end{aligned}$$

Replace the 3rd equation with $z = 2$.

$$\begin{aligned} 2x-y-3z &= -1 \\ 11y-z &= -13 \\ z &= 2 \end{aligned}$$

The system is now in triangular form. Substitute for $z$ and solve for $y$, then substitute for $y$ and $z$ and solve for $x$.

$$\begin{aligned} 11y-z &= -13 \\ 11y-(2) &= -13 \\ 11y &= -11 \\ y &= -1 \\ 2x-y-3z &= -1 \\ 2x-(-1)-3(2) &= -1 \\ 2x+1-6 &= -1 \\ 2x-5 &= -1 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

The solution is $(2, -1, 2)$.

**Solve each system of equations.**

17. $2x + 2y = 8$
$3x - 8y = -21$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cc|c} 2 & 2 & 8 \\ 3 & -8 & -21 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & -8 & -21 \end{array}\right] \leftarrow R_1 + 2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -11 & -33 \end{array}\right] \leftarrow R_2 - 3R_1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 3 \end{array}\right] \leftarrow R_2 + 11$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right] \leftarrow R_1 - R_2$$

The solution is (1, 3).

18. $x - 2y = 13$
$-5x - 6y = 15$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cc|c} 1 & -2 & 13 \\ -5 & -6 & 15 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 13 \\ 0 & -16 & 80 \end{array}\right] \leftarrow R_2 + 5R_1$$

$$\left[\begin{array}{cc|c} 1 & -2 & 13 \\ 0 & 1 & -5 \end{array}\right] \leftarrow R_2 + (-16)$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -5 \end{array}\right] \leftarrow R_1 + 2R_2$$

The solution is (3, −5).

19. $\begin{aligned} x+y&=4\\ x+y+z&=7\\ x-z&=-1 \end{aligned}$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 1 & 1 & 1 & 7 \\ 1 & 0 & -1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 2 & 8 \\ 1 & 0 & -1 & -1 \end{array}\right] \leftarrow R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & -1 & -1 & -5 \end{array}\right] \leftarrow R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \leftarrow R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \leftarrow R_2 - 2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \leftarrow R_1 - R_2$$

The solution is (2, 2, 3).

20. $\begin{aligned} x+y&=1\\ 3x-7y+z&=-7\\ 4x+8y+3z&=-9 \end{aligned}$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 3 & -7 & 1 & -7 \\ 4 & 8 & 3 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -10 & 1 & -10 \\ 4 & 8 & 3 & -9 \end{array}\right] \leftarrow R_2 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -10 & 1 & -10 \\ 0 & 4 & 3 & -13 \end{array}\right] \leftarrow R_3 - 4R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -10 & 1 & -10 \\ 0 & 0 & 17 & -85 \end{array}\right] \leftarrow 5R_3 + 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -10 & 1 & -10 \\ 0 & 0 & 1 & -5 \end{array}\right] \leftarrow R_3 \div 17$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -10 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{array}\right] \leftarrow R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & -5 \end{array}\right] \leftarrow R_2 \div (-10)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & -5 \end{array}\right] \leftarrow R_1 - R_2$$

The solution is $\left(\frac{1}{2}, \frac{1}{2}, -5\right)$.

21. $\begin{aligned} 3x - y + z &= 8 \\ 2x - 3y &= 3z - 13 \\ x + z &= 6 - y \end{aligned}$

*SOLUTION:*

Rewrite the equations, aligning the variables.

$$\begin{aligned} 3x - y + z &= 8 \\ 2x - 3y - 3z &= -13 \\ x + y + z &= 6 \end{aligned}$$

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 2 & -3 & -3 & -13 \\ 1 & 1 & 1 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 0 & -5 & -5 & -25 \\ 1 & 1 & 1 & 6 \end{array}\right] \leftarrow R_2 - 2R_3$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 0 & 1 & 1 & 5 \\ 0 & 4 & 2 & 10 \end{array}\right] \begin{array}{l} \leftarrow R_2 + (-5) \\ \leftarrow 3R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -10 \end{array}\right] \leftarrow R_3 - 4R_2$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_3 + (-2)$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1 \div 3$$

22. $x + y = z - 1$
$2x + 2y + z = 13$
$3x - 5y + 4z = 8$

*SOLUTION:*

Rewrite the equations, aligning the variables.

$x + y - z = -1$
$2x + 2y + z = 13$
$3x - 5y + 4z = 8$

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 2 & 2 & 1 & 13 \\ 3 & -5 & 4 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 0 & 3 & 15 \\ 3 & -5 & 4 & 8 \end{array}\right] \leftarrow R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & -8 & 7 & 11 \end{array}\right] \begin{array}{l} \leftarrow R_2 \div 3 \\ \leftarrow R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & -8 & 7 & 11 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & -8 & 0 & -24 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_2 - 7R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_2 \div (-8)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1 - R_2$$

The solution is (1, 3, 5).

**Find *AB* and *BA*, if possible.**

23. $A = \begin{bmatrix} 1 & -3 & 7 \\ 2 & 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} -5 & -4 \\ 1 & 7 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 1 & -3 & 7 \\ 2 & 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} -5 & -4 \\ 1 & 7 \end{bmatrix}$

$A$ is a $2 \times 3$ matrix and $B$ is a $2 \times 2$ matrix. Because the number of columns of $A$ is *not* equal to the number of rows of $B$, $AB$ is undefined.

$B$ is a $2 \times 2$ matrix and $A$ is a $2 \times 3$ matrix. Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

To find the first entry of $BA$, find the sum of the products of the entries in row 1 of $B$ and column 1 of $A$. Follow the same procedure for row 2 column 1 of $BA$ and the remaining entries.

$$BA = \begin{bmatrix} -5(1)-4(2) & -5(-3)-4(0) & -5(7)-4(1) \\ 1(1)+7(2) & 1(-3)+7(0) & 1(7)+7(1) \end{bmatrix}$$

$$= \begin{bmatrix} -5-8 & 15-0 & -35-4 \\ 1+14 & -3+0 & 7+7 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 15 & -39 \\ 15 & -3 & 14 \end{bmatrix}$$

24. $A = \begin{bmatrix} -2 & 3 \\ -4 & 7 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} -2 & 3 \\ -4 & 7 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$A$ is a $2 \times 2$ matrix and $B$ is a $2 \times 2$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$ and the remaining entries.

$$AB = \begin{bmatrix} -2(1)+3(0) & -2(1)+3(1) \\ -4(1)+7(0) & -4(1)+7(1) \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$$

Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

$$BA = \begin{bmatrix} 1(-2)+1(-4) & 1(3)+1(7) \\ 0(-2)+1(-4) & 0(3)+1(7) \end{bmatrix}$$
$$= \begin{bmatrix} -6 & 10 \\ -4 & 7 \end{bmatrix}$$

25. $A = \begin{bmatrix} 4 & -3 \end{bmatrix}$

$B = \begin{bmatrix} -8 & 5 \\ -7 & -1 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 4 & -3 \end{bmatrix}$; $B = \begin{bmatrix} -8 & 5 \\ -7 & -1 \end{bmatrix}$

$A$ is a $1 \times 2$ matrix and $B$ is a $2 \times 2$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$.

$$AB = \begin{bmatrix} 4(-8)-3(-7) & 4(5)-3(-1) \end{bmatrix}$$
$$= \begin{bmatrix} -11 & 23 \end{bmatrix}$$

Because the number of columns of $B$ is *not* equal to the number of rows of $A$, $BA$ is undefined.

26. $A = \begin{bmatrix} 1 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 1 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$

$A$ is a $1 \times 2$ matrix and $B$ is a $2 \times 1$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$.

$$AB = \begin{bmatrix} 1(5) + 3(-4) \end{bmatrix}$$
$$= \begin{bmatrix} -7 \end{bmatrix}$$

Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

$$BA = \begin{bmatrix} 5(1) & 5(3) \\ -4(1) & -4(3) \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 15 \\ -4 & -12 \end{bmatrix}$$

**Find $A^{-1}$, if it exists. If $A^{-1}$ does not exist, write *singular*.**

27. $A = \begin{bmatrix} 2 & 1 \\ -3 & 8 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $\left[ A \mid I \right]$.

$$\left[ A \mid I \right] = \left[ \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -3 & 8 & 0 & 1 \end{array} \right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -3 & 8 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 19 & 3 & 2 \end{array}\right] \leftarrow 3R_1 + 2R_2$$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & \frac{3}{19} & \frac{2}{19} \end{array}\right] \leftarrow R_2 \div 19$$

$$\left[\begin{array}{cc|cc} 2 & 0 & \frac{16}{19} & -\frac{2}{19} \\ 0 & 1 & \frac{3}{19} & \frac{2}{19} \end{array}\right] \leftarrow R_1 - R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{8}{19} & -\frac{1}{19} \\ 0 & 1 & \frac{3}{19} & \frac{2}{19} \end{array}\right] \leftarrow R_1 \div 2$$

The first two columns are the identity matrix. Therefore, $A$ is invertible and $A^{-1} = \begin{bmatrix} \frac{8}{19} & -\frac{1}{19} \\ \frac{3}{19} & \frac{2}{19} \end{bmatrix}$.

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 2 & 1 \\ -3 & 8 \end{bmatrix} \begin{bmatrix} \frac{8}{19} & -\frac{1}{19} \\ \frac{3}{19} & \frac{2}{19} \end{bmatrix}$$

$$= \begin{bmatrix} 2\left(\frac{8}{19}\right)+1\left(\frac{3}{19}\right) & 2\left(-\frac{1}{19}\right)+1\left(\frac{2}{19}\right) \\ -3\left(\frac{8}{19}\right)+8\left(\frac{3}{19}\right) & -3\left(-\frac{1}{19}\right)+8\left(\frac{2}{19}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{19}+\frac{3}{19} & -\frac{2}{19}+\frac{2}{19} \\ -\frac{24}{19}+\frac{24}{19} & \frac{3}{19}+\frac{16}{19} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{8}{19} & -\frac{1}{19} \\ \frac{3}{19} & \frac{2}{19} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{19}(2)-\frac{1}{19}(-3) & \frac{8}{19}(1)-\frac{1}{19}(8) \\ \frac{3}{19}(2)+\frac{2}{19}(-3) & \frac{3}{19}(1)+\frac{2}{19}(8) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{19}+\frac{3}{19} & \frac{8}{19}-\frac{8}{19} \\ \frac{6}{19}-\frac{6}{19} & \frac{3}{19}+\frac{16}{19} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

28. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \leftarrow \text{Switch } R_2 \text{ and } R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 5 & 3 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \leftarrow 2R_1 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 5 & 0 & 2 & -3 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \leftarrow R_2 - 3R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \leftarrow R_2 \div 5$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{1}{5} & \frac{6}{5} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \leftarrow R_1 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & -\frac{9}{5} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \leftarrow R_1 - 3R_3$$

The first three columns are the identity matrix. Therefore, $A$ is invertible and $A^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{9}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & 0 \end{bmatrix}$.

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & -\frac{9}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\left(\frac{1}{5}\right)+2\left(\frac{2}{5}\right)+3(0) & 1\left(-\frac{9}{5}\right)+2\left(-\frac{3}{5}\right)+3(1) & 1\left(\frac{2}{5}\right)+2\left(-\frac{1}{5}\right)+3(0) \\ 0\left(\frac{1}{5}\right)+0\left(\frac{2}{5}\right)+1(0) & 0\left(-\frac{9}{5}\right)+0\left(-\frac{3}{5}\right)+1(1) & 0\left(\frac{2}{5}\right)+0\left(-\frac{1}{5}\right)+1(0) \\ 2\left(\frac{1}{5}\right)-1\left(\frac{2}{5}\right)+3(0) & 2\left(-\frac{9}{5}\right)-1\left(-\frac{3}{5}\right)+3(1) & 2\left(\frac{2}{5}\right)-1\left(-\frac{1}{5}\right)+3(0) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5}+\frac{4}{5}+0 & -\frac{9}{5}-\frac{6}{5}+3 & \frac{2}{5}-\frac{2}{5}+0 \\ 0+0+0 & 0+0+1 & 0+0+0 \\ \frac{2}{5}-\frac{2}{5}+0 & -\frac{18}{5}+\frac{3}{5}+3 & \frac{4}{5}+\frac{1}{5}+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{1}{5} & -\frac{9}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5}(1)-\frac{9}{5}(0)+\frac{2}{5}(2) & \frac{1}{5}(2)-\frac{9}{5}(0)+\frac{2}{5}(-1) & \frac{1}{5}(3)-\frac{9}{5}(1)+\frac{2}{5}(3) \\ \frac{2}{5}(1)-\frac{3}{5}(0)-\frac{1}{5}(2) & \frac{2}{5}(2)-\frac{3}{5}(0)-\frac{1}{5}(-1) & \frac{2}{5}(3)-\frac{3}{5}(1)-\frac{1}{5}(3) \\ 0(1)+1(0)+0(2) & 0(2)+1(0)+0(-1) & 0(3)+1(1)+0(3) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5}-0+\frac{4}{5} & \frac{2}{5}-0-\frac{2}{5} & \frac{3}{5}-\frac{9}{5}+\frac{6}{5} \\ \frac{2}{5}-0-\frac{2}{5} & \frac{4}{5}-0+\frac{1}{5} & \frac{6}{5}-\frac{3}{5}-\frac{3}{5} \\ 0+0+0 & 0+0+0 & 0+1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

29. $A = \begin{bmatrix} 2 & 3 \\ -8 & -12 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -8 & -12 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -8 & -12 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{array}\right] \leftarrow R_2 + 4R_1$$

A row of 0s has been formed, so the first 2 columns cannot become the identity matrix. Therefore, $A$ is singular.

30. $A = \begin{bmatrix} -5 & 2 \\ 3 & 8 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{cc|cc} -5 & 2 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{cc|cc} -5 & 2 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} -5 & 2 & 1 & 0 \\ 0 & 46 & 3 & 5 \end{array}\right] \leftarrow 5R_2 + 3R_1$$

$$\left[\begin{array}{cc|cc} -5 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{46} & \frac{5}{46} \end{array}\right] \leftarrow R_2 \div 46$$

$$\left[\begin{array}{cc|cc} -5 & 0 & \frac{20}{23} & -\frac{5}{23} \\ 0 & 1 & \frac{3}{46} & \frac{5}{46} \end{array}\right] \leftarrow R_1 - 2R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{4}{23} & \frac{1}{23} \\ 0 & 1 & \frac{3}{46} & \frac{5}{46} \end{array}\right] \leftarrow R_1 \div -5$$

The first two columns are the identity matrix. Therefore, $A$ is invertible and $A^{-1} = \begin{bmatrix} -\frac{4}{23} & \frac{1}{23} \\ \frac{3}{46} & \frac{5}{46} \end{bmatrix}$.

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} -\frac{4}{23} & \frac{1}{23} \\ \frac{3}{46} & \frac{5}{46} \end{bmatrix}$$

$$= \begin{bmatrix} -5\left(-\frac{4}{23}\right)+2\left(\frac{3}{46}\right) & -5\left(\frac{1}{23}\right)+2\left(\frac{5}{46}\right) \\ 3\left(-\frac{4}{23}\right)+8\left(\frac{3}{46}\right) & 3\left(\frac{1}{23}\right)+8\left(\frac{5}{46}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{20}{23}+\frac{6}{46} & -\frac{5}{23}+\frac{10}{46} \\ -\frac{12}{23}+\frac{24}{46} & \frac{3}{23}+\frac{40}{46} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -\frac{4}{23} & \frac{1}{23} \\ \frac{3}{46} & \frac{5}{46} \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{4}{23}(-5)+\frac{1}{23}(3) & -\frac{4}{23}(2)+\frac{1}{23}(8) \\ \frac{3}{46}(-5)+\frac{5}{46}(3) & \frac{3}{46}(2)+\frac{5}{46}(8) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{20}{23}+\frac{3}{23} & -\frac{8}{23}+\frac{8}{23} \\ -\frac{15}{46}+\frac{15}{46} & \frac{6}{46}+\frac{40}{46} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Use an inverse matrix to solve each system of equations, if possible.**

31. $2x - 3y = -23$
$3x + 7y = 23$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 2 & -3 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -23 \\ 23 \end{bmatrix}$$

Use the formula for the inverse of a $2 \times 2$ matrix to find $A^{-1}$.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{2(7) - (-3)(3)} \begin{bmatrix} 7 & 3 \\ -3 & 2 \end{bmatrix}$$
$$= \frac{1}{23} \begin{bmatrix} 7 & 3 \\ -3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{7}{23} & \frac{3}{23} \\ \frac{-3}{23} & \frac{2}{23} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} \frac{7}{23} & \frac{3}{23} \\ \frac{-3}{23} & \frac{2}{23} \end{bmatrix} \begin{bmatrix} -23 \\ 23 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{7}{23}(-23) + \frac{3}{23}(23) \\ -\frac{3}{23}(-23) + \frac{2}{23}(23) \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{161}{23} + \frac{69}{23} \\ \frac{69}{23} + \frac{46}{23} \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{92}{23} \\ \frac{115}{23} \end{bmatrix}$$
$$= \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

The solution is $(-4, 5)$

32. $\begin{aligned} 3x - 6y &= 9 \\ -5x - 8y &= -6 \end{aligned}$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 3 & -6 \\ -5 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix}$$

Use the formula for the inverse of a 2 × 2 matrix to find $A^{-1}$.

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3(-8) - (-6)(-5)} \begin{bmatrix} -8 & 6 \\ 5 & 3 \end{bmatrix} \\ &= -\frac{1}{54} \begin{bmatrix} -8 & 6 \\ 5 & 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{8}{54} & -\frac{6}{54} \\ -\frac{5}{54} & -\frac{3}{54} \end{bmatrix} \end{aligned}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$\begin{aligned} A^{-1}B &= \begin{bmatrix} \frac{8}{54} & -\frac{6}{54} \\ -\frac{5}{54} & -\frac{3}{54} \end{bmatrix} \begin{bmatrix} 9 \\ -6 \end{bmatrix} \\ &= \begin{bmatrix} \frac{8}{54}(9) - \frac{6}{54}(-6) \\ -\frac{5}{54}(9) - \frac{3}{54}(-6) \end{bmatrix} \\ &= \begin{bmatrix} \frac{72}{54} + \frac{36}{54} \\ -\frac{45}{54} + \frac{18}{54} \end{bmatrix} \\ &= \begin{bmatrix} \frac{108}{54} \\ -\frac{27}{54} \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix} \end{aligned}$$

The solution is $\left(2, -\frac{1}{2}\right)$.

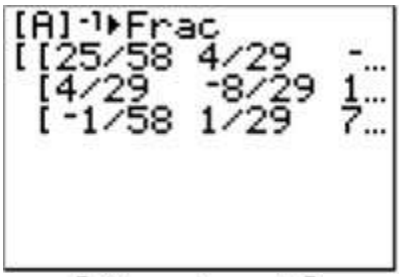
33. $\begin{aligned} 2x + y &= 1 \\ x - 3y + z &= -4 \\ y + 8z &= -7 \end{aligned}$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & 8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -7 \end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$. Enter the values for the matrix $A$. Enter $[A]^{-1}$. Select $\rightarrow Frac$ to get the values in reduced fraction form.




$$A^{-1} = \begin{bmatrix} \frac{25}{58} & \frac{4}{29} & -\frac{1}{58} \\ \frac{4}{29} & -\frac{8}{29} & \frac{1}{29} \\ -\frac{1}{58} & \frac{1}{29} & \frac{7}{58} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} \frac{25}{58} & \frac{4}{29} & -\frac{1}{58} \\ \frac{4}{29} & -\frac{8}{29} & \frac{1}{29} \\ -\frac{1}{58} & \frac{1}{29} & \frac{7}{58} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -4 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{25}{58}(1) + \frac{4}{29}(-4) - \frac{1}{58}(-7) \\ \frac{4}{29}(1) - \frac{8}{29}(-4) + \frac{1}{29}(-7) \\ -\frac{1}{58}(1) + \frac{1}{29}(-4) + \frac{7}{58}(-7) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{25}{58} - \frac{16}{29} + \frac{7}{58} \\ \frac{4}{29} + \frac{32}{29} - \frac{7}{29} \\ -\frac{1}{58} - \frac{4}{29} - \frac{49}{58} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{25}{58} - \frac{32}{58} + \frac{7}{58} \\ \frac{8}{58} + \frac{64}{58} - \frac{14}{58} \\ -\frac{1}{58} - \frac{8}{58} - \frac{49}{58} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0}{58} \\ \frac{58}{58} \\ -\frac{58}{58} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

The solution is (0, 1, −1).

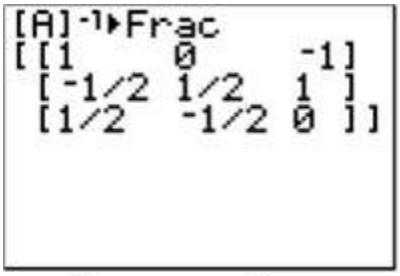
34. $x + y + z = 1$
$x + y - z = -7$
$y + z = -1$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -1 \end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$. Enter the values for the matrix $A$. Enter $[A]^{-1}$. Select $\rightarrow Frac$ to get the values in reduced fraction form.




$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} 1 & 0 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -7 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1) + 0(-7) - 1(-1) \\ -\frac{1}{2}(1) + \frac{1}{2}(-7) + 1(-1) \\ \frac{1}{2}(1) - \frac{1}{2}(-7) + 0(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 0 + 1 \\ -\frac{1}{2} - \frac{7}{2} - 1 \\ \frac{1}{2} + \frac{7}{2} + 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{2} + \frac{0}{2} + \frac{2}{2} \\ -\frac{1}{2} - \frac{7}{2} - \frac{2}{2} \\ \frac{1}{2} + \frac{7}{2} + \frac{0}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{2} \\ -\frac{10}{2} \\ \frac{8}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$$

The solution is (2, −5, 4).

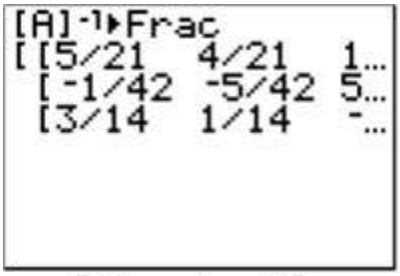
35. $\begin{aligned} 3y + 5z &= 25 \\ 2x - 7y - 3z &= 15 \\ x + y - z &= -11 \end{aligned}$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 0 & 3 & 5 \\ 2 & -7 & -3 \\ 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ 15 \\ -11 \end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$. Enter the values for the matrix $A$. Enter $[A]^{-1}$. Select $\rightarrow Frac$ to get the values in reduced fraction form.




$$A^{-1} = \begin{bmatrix} \frac{5}{21} & \frac{4}{21} & \frac{13}{21} \\ -\frac{1}{42} & -\frac{5}{42} & \frac{5}{21} \\ \frac{3}{14} & \frac{1}{14} & -\frac{1}{7} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} \frac{5}{21} & \frac{4}{21} & \frac{13}{21} \\ -\frac{1}{42} & -\frac{5}{42} & \frac{5}{21} \\ \frac{3}{14} & \frac{1}{14} & -\frac{1}{7} \end{bmatrix} \cdot \begin{bmatrix} 25 \\ 15 \\ -11 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{21}(25) + \frac{4}{21}(15) + \frac{13}{21}(-11) \\ -\frac{1}{42}(25) - \frac{5}{42}(15) + \frac{5}{21}(-11) \\ \frac{3}{14}(25) + \frac{1}{14}(15) - \frac{1}{7}(-11) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{125}{21} + \frac{60}{21} - \frac{143}{21} \\ -\frac{25}{42} - \frac{75}{42} - \frac{55}{21} \\ \frac{75}{14} + \frac{15}{14} + \frac{11}{7} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{250}{42} + \frac{120}{42} - \frac{286}{42} \\ -\frac{25}{42} - \frac{75}{42} - \frac{110}{42} \\ \frac{225}{42} + \frac{45}{42} + \frac{66}{42} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{84}{42} \\ -\frac{210}{42} \\ \frac{336}{42} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -5 \\ 8 \end{bmatrix}$$

The solution is (2, −5, 8).

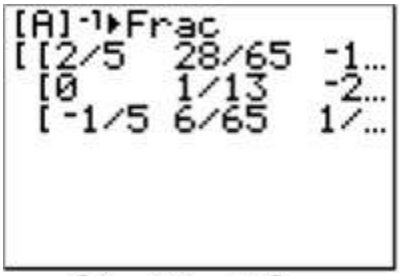
36. $x - 2y - 3z = 0$
$2x - 3y + 4z = 11$
$x - 8y + 2z = -1$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 1 & -2 & -3 \\ 2 & -3 & 4 \\ 1 & -8 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \\ -1 \end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$. Enter the values for the matrix $A$. Enter $[A]^{-1}$. Select $\rightarrow Frac$ to get the values in reduced fraction form.




$$A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{28}{65} & -\frac{17}{65} \\ 0 & \frac{1}{13} & -\frac{2}{13} \\ -\frac{1}{5} & \frac{6}{65} & \frac{1}{65} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} \frac{2}{5} & \frac{28}{65} & -\frac{17}{65} \\ 0 & \frac{1}{13} & -\frac{2}{13} \\ -\frac{1}{5} & \frac{6}{65} & \frac{1}{65} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 11 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5}(0) + \frac{28}{65}(11) - \frac{17}{65}(-1) \\ 0(0) + \frac{1}{13}(11) - \frac{2}{13}(-1) \\ -\frac{1}{5}(0) + \frac{6}{65}(11) + \frac{1}{65}(-1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0}{5} + \frac{308}{65} + \frac{17}{65} \\ 0 + \frac{11}{13} + \frac{2}{13} \\ \frac{0}{65} + \frac{66}{65} - \frac{1}{65} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0}{65} + \frac{308}{65} + \frac{17}{65} \\ \frac{0}{65} + \frac{55}{65} + \frac{10}{65} \\ \frac{0}{65} + \frac{66}{65} - \frac{1}{65} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{325}{65} \\ \frac{65}{65} \\ \frac{65}{65} \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

The solution is (5, 1, 1).

**Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists.**

37. $2x - 4y = 30$
$3x + 5y = 12$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 2 & -4 \\ 3 & 5 \end{bmatrix}$. Calculate the determinant of $A$.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} 2 & -4 \\ 3 & 5 \end{vmatrix} = 2(5) - 3(-4)$$
$$= 10 + 12$$
$$= 22$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$
$$= \frac{\begin{vmatrix} 30 & -4 \\ 12 & 5 \end{vmatrix}}{22}$$
$$= \frac{30(5) - 12(-4)}{22}$$
$$= \frac{150 + 48}{22}$$
$$= 9$$
$$y = \frac{|A_y|}{|A|}$$
$$= \frac{\begin{vmatrix} 2 & 30 \\ 3 & 12 \end{vmatrix}}{22}$$
$$= \frac{2(12) - 3(30)}{22}$$
$$= \frac{-66}{22}$$
$$= -3$$

The solution is $(9, -3)$.

38. $2x + 6y = 14$
$x - 3y = 1$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}$. Calculate the determinant of $A$.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} 2 & 6 \\ 1 & -3 \end{vmatrix} = 2(-3) - 1(6)$$
$$= -6 - 6$$
$$= -12$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$
$$= \frac{\begin{vmatrix} 14 & 6 \\ 1 & -3 \end{vmatrix}}{-12}$$
$$= \frac{14(-3) - 1(6)}{-12}$$
$$= \frac{-42 - 6}{-12}$$
$$= 4$$
$$y = \frac{|A_y|}{|A|}$$
$$= \frac{\begin{vmatrix} 2 & 14 \\ 1 & 1 \end{vmatrix}}{-12}$$
$$= \frac{2(1) - 1(14)}{-12}$$
$$= \frac{2 - 14}{-12}$$
$$= 1$$

The solution is (4, 1).

39. $2x + 3y - z = 1$
$x + y - 3z = 12$
$5x - 7y + 2z = 28$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & -3 \\ 5 & -7 & 2 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the

determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & -3 \\ 5 & -7 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -3 \\ -7 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 5 & -7 \end{vmatrix}$$

$$= 2[1(2) - (-7)(-3)] - 3[1(2) - 5(-3)] - 1[1(-7) - 5(1)]$$
$$= 2(2 - 21) - 3(2 + 15) - 1(-7 - 5)$$
$$= -38 - 51 + 12$$
$$= -77$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 3 & -1 \\ 12 & 1 & -3 \\ 28 & -7 & 2 \end{vmatrix}}{-77}$$

$$= \frac{1\begin{vmatrix} 1 & -3 \\ -7 & 2 \end{vmatrix} - 3\begin{vmatrix} 12 & -3 \\ 28 & 2 \end{vmatrix} - 1\begin{vmatrix} 12 & 1 \\ 28 & -7 \end{vmatrix}}{-77}$$

$$= \frac{1[1(2) - (-7)(-3)] - 3[12(2) - 28(-3)] - 1[12(-7) - 28(1)]}{-77}$$

$$= \frac{-19 - 324 + 112}{-77}$$

$$= \frac{-231}{-77}$$

$$= 3$$

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 2 & 1 & -1 \\ 1 & 12 & -3 \\ 5 & 28 & 2 \end{vmatrix}}{-77}$$

$$= \frac{2\begin{vmatrix} 12 & -3 \\ 28 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 12 \\ 5 & 28 \end{vmatrix}}{-77}$$

$$= \frac{2[12(2) - 28(-3)] - 1[1(2) - 5(-3)] - 1[1(28) - 5(12)]}{-77}$$

$$= \frac{2(24 + 84) - 1(2 + 15) - 1(28 - 60)}{-77}$$

$$= \frac{216 - 17 + 32}{-77}$$

$$= \frac{231}{-77}$$

$$= -3$$

$$Z = \frac{|A_z|}{|A|}$$

$$= \frac{\begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 12 \\ 5 & -7 & 28 \end{vmatrix}}{-77}$$

$$= \frac{2\begin{vmatrix} 1 & 12 \\ -7 & 28 \end{vmatrix} - 3\begin{vmatrix} 1 & 12 \\ 5 & 28 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 5 & -7 \end{vmatrix}}{-77}$$

$$= \frac{2[1(28) - (-7)(12)] - 3[1(28) - 5(12)] + 1[1(-7) - 5(1)]}{-77}$$

$$= \frac{224 + 96 - 12}{-77}$$

$$= \frac{308}{-77}$$

$$= -4$$

The solution is (3, −3, −4).

40. $x + 2y + z = -2$
$2x + 2y - 5z = -19$
$3x - 4y + 8z = -1$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -5 \\ 3 & -4 & 8 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & -5 \\ 3 & -4 & 8 \end{vmatrix} = 1\begin{vmatrix} 2 & -5 \\ -4 & 8 \end{vmatrix} - 2\begin{vmatrix} 2 & -5 \\ 3 & 8 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & -4 \end{vmatrix}$$

$$= 1[2(8) - (-4)(-5)] - 2[2(8) - 3(-5)] + 1[2(-4) - 3(2)]$$
$$= 1(16 - 20) - 2(16 + 15) + 1(-8 - 6)$$
$$= -4 - 62 - 14$$
$$= -80$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$

$$= \frac{\begin{vmatrix} -2 & 2 & 1 \\ -19 & 2 & -5 \\ -1 & -4 & 8 \end{vmatrix}}{-80}$$

$$= \frac{-2\begin{vmatrix} 2 & -5 \\ -4 & 8 \end{vmatrix} - 2\begin{vmatrix} -19 & -5 \\ -1 & 8 \end{vmatrix} + 1\begin{vmatrix} -19 & 2 \\ -1 & -4 \end{vmatrix}}{-80}$$

$$= \frac{-2[2(8) - (-4)(-5)] - 2[-19(8) - (-1)(-5)] + 1[-19(-4) - (-1)(2)]}{-80}$$

$$= \frac{8 + 314 + 78}{-80}$$

$$= \frac{400}{-80}$$

$$= -5$$

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & -2 & 1 \\ 2 & -19 & -5 \\ 3 & -1 & 8 \end{vmatrix}}{-80}$$

$$= \frac{1\begin{vmatrix} -19 & -5 \\ -1 & 8 \end{vmatrix} + 2\begin{vmatrix} 2 & -5 \\ 3 & 8 \end{vmatrix} + 1\begin{vmatrix} 2 & -19 \\ 3 & -1 \end{vmatrix}}{-80}$$

$$= \frac{1[-19(8)-(-1)(-5)] + 2[2(8)-3(-5)] + 1[2(-1)-3(-19)]}{-80}$$

$$= \frac{1(-152-5) + 2(16+15) + 1(-2+57)}{-80}$$

$$= \frac{-157+62+55}{-80}$$

$$= \frac{-40}{-80}$$

$$= \frac{1}{2}$$

$$Z = \frac{|A_z|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 2 & -2 \\ 2 & 2 & -19 \\ 3 & -4 & -1 \end{vmatrix}}{-80}$$

$$= \frac{1\begin{vmatrix} 2 & -19 \\ -4 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & -19 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & -4 \end{vmatrix}}{-80}$$

$$= \frac{1[2(-1)-(-4)(-19)] - 2[2(-1)-3(-19)] - 2[2(-4)-3(2)]}{-80}$$

$$= \frac{1(-2-76) - 2(-2+57) - 2(-8-6)}{-80}$$

$$= \frac{-78-110+28}{-80}$$

$$= \frac{-160}{-80}$$

$$= 2$$

The solution is $\left(-5, \frac{1}{2}, 2\right)$.

41. $-3x - 4y + z = 15$
$x - 5y - z = 3$
$4x - 3y - 2z = -8$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} -3 & -4 & 1 \\ 1 & -5 & -1 \\ 4 & -3 & -2 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} -3 & -4 & 1 \\ 1 & -5 & -1 \\ 4 & -3 & -2 \end{vmatrix} = -3\begin{vmatrix} -5 & -1 \\ -3 & -2 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -5 \\ 4 & -3 \end{vmatrix}$$

$$= -3[-5(-2) - (-3)(-1)] + 4[1(-2) - 4(-1)] + 1[1(-3) - 4(-5)]$$

$$= -3(10 - 3) + 4(-2 + 4) + 1(-3 + 20)$$

$$= -21 + 8 + 17$$

$$= 4$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$

$$= \frac{\begin{vmatrix} 15 & -4 & 1 \\ 3 & -5 & -1 \\ -8 & -3 & -2 \end{vmatrix}}{4}$$

$$= \frac{15\begin{vmatrix} -5 & -1 \\ -3 & -2 \end{vmatrix} + 4\begin{vmatrix} 3 & -1 \\ -8 & -2 \end{vmatrix} + 1\begin{vmatrix} 3 & -5 \\ -8 & -3 \end{vmatrix}}{4}$$

$$= \frac{15[-5(-2) - (-3)(-1)] + 4[3(-2) - (-8)(-1)] + 1[3(-3) - (-8)(-5)]}{4}$$

$$= \frac{105 - 56 - 49}{4}$$

$$= \frac{0}{4}$$

$$= 0$$

$$y = \frac{|A_y|}{|A|}$$
$$= \frac{\begin{vmatrix} -3 & 15 & 1 \\ 1 & 3 & -1 \\ 4 & -8 & -2 \end{vmatrix}}{4}$$
$$= \frac{-3\begin{vmatrix} 3 & -1 \\ -8 & -2 \end{vmatrix} - 15\begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 4 & -8 \end{vmatrix}}{4}$$
$$= \frac{-3[3(-2) - (-8)(-1)] - 15[1(-2) - 4(-1)] + 1[1(-8) - 4(3)]}{4}$$
$$= \frac{-3(-6 - 8) - 15(-2 + 4) + 1(-8 - 12)}{4}$$
$$= \frac{42 - 30 - 20}{4}$$
$$= \frac{-8}{4}$$
$$= -2$$
$$Z = \frac{|A_z|}{|A|}$$
$$= \frac{\begin{vmatrix} -3 & -4 & 15 \\ 1 & -5 & 3 \\ 4 & -3 & -8 \end{vmatrix}}{4}$$
$$= \frac{-3\begin{vmatrix} -5 & 3 \\ -3 & -8 \end{vmatrix} + 4\begin{vmatrix} 1 & 3 \\ 4 & -8 \end{vmatrix} + 15\begin{vmatrix} 1 & -5 \\ 4 & -3 \end{vmatrix}}{4}$$
$$= \frac{-3[-5(-8) - (-3)(3)] + 4[1(-8) - 4(3)] + 15[1(-3) - 4(-5)]}{4}$$
$$= \frac{-147 - 80 + 255}{4}$$
$$= \frac{28}{4}$$
$$= 7$$

The solution is (0, −2, 7).

42. $$\begin{aligned} 2x + 3y + 4z &= 29 \\ x - 8y - z &= -3 \\ 2x + y + z &= 4 \end{aligned}$$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -8 & -1 \\ 2 & 1 & 1 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & -8 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} -8 & -1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -8 \\ 2 & 1 \end{vmatrix}$$

$$= 2[-8(1) - 1(-1)] - 3[1(1) - 2(-1)] + 4[1(1) - 2(-8)]$$
$$= 2(-8 + 1) - 3(1 + 2) + 4(1 + 16)$$
$$= -14 - 9 + 68$$
$$= 45$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$

$$= \frac{\begin{vmatrix} 29 & 3 & 4 \\ -3 & -8 & -1 \\ 4 & 1 & 1 \end{vmatrix}}{45}$$

$$= \frac{29\begin{vmatrix} -8 & -1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} -3 & -1 \\ 4 & 1 \end{vmatrix} + 4\begin{vmatrix} -3 & -8 \\ 4 & 1 \end{vmatrix}}{45}$$

$$= \frac{29[-8(1) - 1(-1)] - 3[-3(1) - 4(-1)] + 4[-3(1) - 4(-8)]}{45}$$

$$= \frac{-203 - 3 + 116}{45}$$

$$= \frac{-90}{45}$$

$$= -2$$

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 2 & 29 & 4 \\ 1 & -3 & -1 \\ 2 & 4 & 1 \end{vmatrix}}{45}$$

$$= \frac{2\begin{vmatrix} -3 & -1 \\ 4 & 1 \end{vmatrix} - 29\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix}}{45}$$

$$= \frac{2[-3(1) - 4(-1)] - 29[1(1) - 2(-1)] + 4[1(4) - 2(-3)]}{45}$$

$$= \frac{2(-3 + 4) - 29(1 + 2) + 4(4 + 6)}{45}$$

$$= \frac{2 - 87 + 40}{45}$$

$$= \frac{-45}{45}$$

$$= -1$$

$$Z = \frac{|A_z|}{|A|}$$

$$= \frac{\begin{vmatrix} 2 & 3 & 29 \\ 1 & -8 & -3 \\ 2 & 1 & 4 \end{vmatrix}}{45}$$

$$= \frac{2\begin{vmatrix} -8 & -3 \\ 1 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} + 29\begin{vmatrix} 1 & -8 \\ 2 & 1 \end{vmatrix}}{45}$$

$$= \frac{2[-8(4) - 1(-3)] - 3[1(4) - 2(-3)] + 29[1(1) - 2(-8)]}{45}$$

$$= \frac{-58 - 30 + 493}{45}$$

$$= \frac{405}{45}$$

$$= 9$$

The solution is (−2, −1, 9).

**Find the partial fraction decomposition of each rational expression.**

43. $\frac{2x}{x2-4}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{2x}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}$$

Multiply each side by the LCD, $x^2-4$.

$$2x=A(x+2)+B(x-2)$$

Group the like terms.

$$2x=A(x+2)+B(x-2)$$
$$2x=Ax+2A+Bx-2B$$
$$2x=Ax+Bx+2A-2B$$
$$2x=(A+B)x+2A-2B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=2$$
$$2A-2B=0$$

Use any method to solve the new system.

$$A+B=2$$
$$B=2-A$$
$$2A-2B=0$$
$$2A-2(2-A)=0$$
$$2A-4+2A=0$$
$$-4+4A=0$$
$$4A=4$$
$$A=1$$
$$B=2-(1)$$
$$B=1$$

Replace $A$ and $B$ with 1 and 1 in the partial fraction decomposition.

$$\frac{2x}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}$$
$$\frac{2x}{x^2-4}=\frac{1}{x-2}+\frac{1}{x+2}$$

44. $\frac{7x-6}{x2-x-6}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{7x-6}{x^2-x-6}=\frac{A}{x-3}+\frac{B}{x+2}$$

Multiply each side by the LCD, $x^2-x-6$.

$$7x-6=A(x+2)+B(x-3)$$

Group the like terms.

$$7x-6=A(x+2)+B(x-3)$$
$$7x-6=Ax+2A+Bx-3B$$
$$7x-6=Ax+Bx+2A-3B$$
$$7x-6=(A+B)x+2A-3B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=7$$
$$2A-3B=-6$$

Use any method to solve the new system.

$$A+B=7$$
$$B=7-A$$
$$2A-3B=-6$$
$$2A-3(7-A)=-6$$
$$2A-21+3A=-6$$
$$-21+5A=-6$$
$$5A=15$$
$$A=3$$
$$B=7-(3)$$
$$B=4$$

Replace $A$ and $B$ with 3 and 4 in the partial fraction decomposition.

$$\frac{7x-6}{x^2-x-6}=\frac{A}{x-3}+\frac{B}{x+2}$$
$$\frac{7x-6}{x^2-x-6}=\frac{3}{x-3}+\frac{4}{x+2}$$

45. $\frac{-2x+9}{x2-11x+30}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{-2x+9}{x^2-11x+30}=\frac{A}{x-5}+\frac{B}{x-6}$$

Multiply each side by the LCD, $x^2 - 11x + 30$.

$$-2x+9=A(x-6)+B(x-5)$$

Group the like terms.

$$-2x+9=A(x-6)+B(x-5)$$
$$-2x+9=Ax-6A+Bx-5B$$
$$-2x+9=Ax+Bx-6A-5B$$
$$-2x+9=(A+B)x-6A-5B$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the *m*-terms on the left side of the equation must equal the coefficients of the *m*-terms on the right side.

$$A+B=-2$$
$$-6A-5B=9$$

Use any method to solve the new system.

$$A+B=-2$$
$$B=-2-A$$
$$-6A-5B=9$$
$$-6A-5(-2-A)=9$$
$$-6A+10+5A=9$$
$$10-A=9$$
$$A=1$$
$$B=-2-(1)$$
$$B=-3$$

Replace $A$ and $B$ with 1 and $-3$ in the partial fraction decomposition.

$$\frac{-2x+9}{x^2-11x+30}=\frac{A}{x-5}+\frac{B}{x-6}$$
$$\frac{-2x+9}{x^2-11x+30}=\frac{1}{x-5}-\frac{3}{x-6}$$

46. $\frac{6x24x-6}{x32x23x}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{6x^2-4x-6}{x^3-2x^2-3x}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}$$

Multiply each side by the LCD, $x^2-4$.

$$6x^2-4x-6=A(x+1)(x-3)+Bx(x-3)+Cx(x+1)$$

Group the like terms.

$$6x^2-4x-6=A(x+1)(x-3)+Bx(x-3)+Cx(x+1)$$
$$6x^2-4x-6=A(x^2-2x-3)+B(x^2-3x)+C(x^2+x)$$
$$6x^2-4x-6=Ax^2-2Ax-3A+Bx^2-3Bx+Cx^2+Cx$$
$$6x^2-4x-6=Ax^2+Bx^2+Cx^2-2Ax-3Bx+Cx-3A$$
$$6x^2-4x-6=(A+B+C)x^2+(-2A-3B+C)x-3A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B+C=6$$
$$-2A-3B+C=-4$$
$$-3A=-6$$

Use any method to solve the new system.

$$-3A=-6$$
$$A=2$$
$$-2A-3B+C=-4$$
$$-2(2)-3B+C=-4$$
$$-4-3B+C=-4$$
$$-3B+C=0$$
$$C=3B$$
$$A+B+C=6$$
$$2+B+3B=6$$
$$2+4B=6$$
$$4B=4$$
$$B=1$$
$$C=3(1)$$
$$C=3$$

Replace $A$, $B$ and $C$ with 2, 1 and 3 in the partial fraction decomposition.

$$\frac{6x^2-4x-6}{x^3-2x^2-3x}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}$$
$$\frac{6x^2-4x-6}{x^3-2x^2-3x}=\frac{2}{x}+\frac{1}{x+1}+\frac{3}{x-3}$$

47. $\frac{2x3-14x2+2x+7}{x2-7x}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$x^2-7x\overline{)2x^3-14x^2+2x+7}\quad 2x$$
$$(-)\,(2x^3-14x^2)$$
$$2x+7$$

Therefore, the proper rational expression is $2x+\frac{2x+7}{x^2-7x}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{2x+7}{x^2-7x}=\frac{A}{x}+\frac{B}{x-7}$$

Multiply each side by the LCD, $x^2 - 7x$.

$$2x+7=A(x-7)+B(x)$$

Group like terms.

$$2x+7=A(x-7)+B(x)$$
$$2x+7=A(x-7)+B(x)$$
$$2x+7=Ax-7A+Bx$$
$$2x+7=(A+B)x-7A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=2$$
$$-7A=7$$

Use any method to solve the new system.

$$-7A=7$$
$$A=-1$$
$$A+B=2$$
$$-1+B=2$$
$$B=3$$

Replace $A$ and $B$ with $-1$ and 3 in the partial fraction decomposition.

$$\frac{2x+7}{x^2-7x}=\frac{A}{x}+\frac{B}{x-7}$$
$$\frac{2x+7}{x^2-7x}=\frac{-1}{x}+\frac{3}{x-7}$$

The partial fraction decomposition of the improper rational expression is $2x-\frac{1}{x}+\frac{3}{x-7}$

48. $\frac{2x4+3x3+5x2+3x+2}{x(x2+1)2}$

*SOLUTION:*

This rational expression is proper. The denominator has one linear factor and one prime quadratic factor of multiplicity 2.

Rewrite the expression as partial fractions with constant numerator, $A$, and a denominator that is the linear factor. For denominators which are quadratic factors, use $Bx + C$ and $Dx + E$ in the numerators.

$$\frac{2x^4+3x^3+5x^2+3x+2}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

Multiply each side by the LCD, $x(x^2 + 1)^2$.

$$\begin{aligned}2x^4+3x^3+5x^2+3x+2&=A(x^2+1)^2+(Bx+C)(x)(x^2+1)+(Dx+E)(x)\\&=A(x^4+2x^2+1)+(Bx+C)(x^3+x)+Dx^2+Ex\\&=Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex\\&=Ax^4+Bx^4+Cx^3+2Ax^2+Bx^2+Dx^2+Cx+Ex+A\\&=(A+B)x^4+(C)x^3+(2A+B+D)x^2+(C+E)x+A\end{aligned}$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$\begin{aligned}A+B&=2\\C&=3\\2A+B+D&=5\\C+E&=3\\A&=2\end{aligned}$$

Use any method to solve the new system.

$$\begin{aligned}A&=2\\A+B&=2\\2+B&=2\\B&=0\\C&=3\\2A+B+D&=5\\2(2)+0+D&=5\\D&=1\\C+E&=3\\3+E&=3\\E&=0\end{aligned}$$

Replace $A$, $B$, $C$, $D$, and $E$ with 2, 0, 3, 1, and 0 in the partial fraction decomposition.

$$\frac{2x^4+3x^3+5x^2+3x+2}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

$$\frac{2x^4+3x^3+5x^2+3x+2}{x(x^2+1)^2}=\frac{2}{x}+\frac{3}{x^2+1}+\frac{x}{(x^2+1)^2}$$

**Find the partial fraction decomposition of each improper rational expression.**

49. $\frac{2x2+4}{x2-2x}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$x^2-2x\overline{)2x^2+4}\quad 2$$
$$(-)\,(2x^2-4x)$$
$$4x+4$$

Therefore, the proper rational expression is $2+\frac{4x+4}{x^2-2x}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{4x+4}{x^2-2x}=\frac{A}{x}+\frac{B}{x-2}$$

Multiply each side by the LCD, $x^2 - 2x$.

$$4x+4=A(x-2)+B(x)$$

Group like terms.

$$4x+4=A(x-2)+B(x)$$
$$4x+4=Ax-2A+Bx$$
$$4x+4=Ax+Bx-2A$$
$$4x+4=(A+B)x-2A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=4$$
$$-2A=4$$

Use any method to solve the new system.

$$-2A=4$$
$$A=-2$$
$$A+B=4$$
$$-2+B=4$$
$$B=6$$

Replace $A$ and $B$ with $-2$ and 6 in the partial fraction decomposition.

$$\frac{4x+4}{x^2-2x}=\frac{A}{x}+\frac{B}{x-2}$$
$$\frac{4x+4}{x^2-2x}=\frac{-2}{x}+\frac{6}{x-2}$$

The partial fraction decomposition of the improper rational expression is $2+\frac{-2}{x}+\frac{6}{x-2}$.

50. $\frac{2x212x20}{x2+4x}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$x^2+4x\overline{)2x^2-12x-20}\quad 2$$
$$(-)\,(2x^2+8x)$$
$$-20x-20$$

Therefore, the proper rational expression is $2+\frac{-20x-20}{x^2+4x}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{-20x-20}{x^2+4x}=\frac{A}{x}+\frac{B}{x+4}$$

Multiply each side by the LCD, $x^2 + 4x$.

$$-20x-20=A(x+4)+B(x)$$

Group like terms.

$$-20x-20=A(x+4)+B(x)$$
$$-20x-20=Ax+4A+Bx$$
$$-20x-20=Ax+Bx+4A$$
$$-20x-20=(A+B)x+4A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=-20$$
$$4A=-20$$

Use any method to solve the new system.

$$4A=-20$$
$$A=-5$$
$$A+B=-20$$
$$-5+B=-20$$
$$B=-15$$

Replace $A$, $B$, and $C$ with $-5$ and $-15$ in the partial fraction decomposition.

$$\frac{-20x-20}{x^2+4x}=\frac{A}{x}+\frac{B}{x+4}$$
$$\frac{-20x-20}{x^2+4x}=\frac{-5}{x}+\frac{-15}{x+4}$$

The partial fraction decomposition of the improper rational expression is $2-\frac{5}{x}-\frac{15}{x+4}$.

51. $\frac{x2+x6}{2x23x}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$\begin{array}{r} 0.5 \\ 2x^2-3x\overline{)\,x^2+x-6} \\ (-)\,(x^2-1.5x) \\ \hline 2.5x-6 \end{array}$$

Therefore, the proper rational expression is $\frac{1}{2}+\frac{2.5x-6}{2x^2-3x}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{2.5x-6}{2x^2-3x}=\frac{A}{x}+\frac{B}{2x-3}$$

Multiply each side by the LCD, $2x^2 - 3x$.

$$2.5x-6=A(2x-3)+B(x)$$

Group like terms.

$$2.5x-6=A(2x-3)+B(x)$$
$$2.5x-6=2Ax-3A+Bx$$
$$2.5x-6=2Ax+Bx-3A$$
$$2.5x-6=(2A+B)x-3A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$2A+B=2.5$$
$$-3A=-6$$

Use any method to solve the new system.

$$-3A=-6$$
$$A=2$$
$$2A+B=2.5$$
$$2(2)+B=2.5$$
$$4+B=2.5$$
$$B=-1.5$$

Replace $A$ and $B$ with 2 and −1.5 in the partial fraction decomposition.

$$\frac{2.5x-6}{2x^2-3x}=\frac{A}{x}+\frac{B}{2x-3}$$
$$\frac{2.5x-6}{2x^2-3x}=\frac{2}{x}+\frac{-1.5}{2x-3}$$

The partial fraction decomposition of the improper rational expression is $\frac{1}{2}+\frac{2}{x}-\frac{1.5}{2x-3}$.

52. $\frac{3x210x20}{2x2+5x}$

*SOLUTION:*

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator.

$$\begin{array}{r} 1.5 \\ 2x^2+5x\overline{)3x^2-10x-20} \\ (-)\,(3x^2+7.5x) \\ \hline -17.5x-20 \end{array}$$

Therefore, the proper rational expression is $1.5+\frac{-17.5x-20}{2x^2+5x}$.

Now we can rewrite the remaining expression as partial fractions with constant numerators, $A$, and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{-17.5x-20}{2x^2+5x}=\frac{A}{x}+\frac{B}{2x+5}$$

Multiply each side by the LCD, $2x^2 + 5x$.

$$-17.5x-20=A(2x+5)+B(x)$$

Group like terms.

$$-17.5x-20=A(2x+5)+B(x)$$
$$-17.5x-20=2Ax+5A+Bx$$
$$-17.5x-20=2Ax+Bx+5A$$
$$-17.5x-20=(2A+B)x+5A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$2A+B=-17.5$$
$$5A=-20$$

Use any method to solve the new system.

$$5A=-20$$
$$A=-4$$
$$2A+B=-17.5$$
$$2(-4)+B=-17.5$$
$$-8+B=-17.5$$
$$B=-9.5$$

Replace $A$ and $B$ with $-4$ and $-9.5$ in the partial fraction decomposition.

$$\frac{-17.5x-20}{2x^2+5x}=\frac{A}{x}+\frac{B}{2x+5}$$
$$\frac{-17.5x-20}{2x^2+5x}=\frac{-4}{x}+\frac{-9.5}{2x+5}$$

The partial fraction decomposition of the improper rational expression is $1.5-\frac{4}{x}-\frac{9.5}{2x+5}$.

**Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of $x$ and $y$**

**they occur, subject to the given constraints.**

53. $f(x, y) = 2x - y$

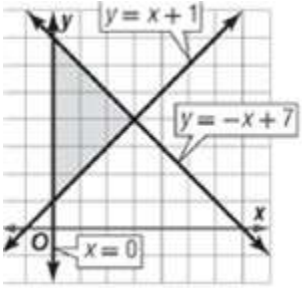
$x \geq 0$

$y \leq -x + 7$

$y \geq x + 1$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has three vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$x = 0$

$y = x + 1$

$y = 0 + 1$

$y = 1$

vertex at (0, 1)

$x = 0$

$y = -x + 7$

$y = 0 + 7$

$y = 7$

vertex at (0, 7)

$y = x + 1$

$y = -x + 7$

$x + 1 = -x + 7$

$2x = 6$

$x = 3$

$y = x + 1$

$y = 3 + 1$

$y = 4$

vertex at (3, 4)

Find the value of the objective function for each of the vertices.

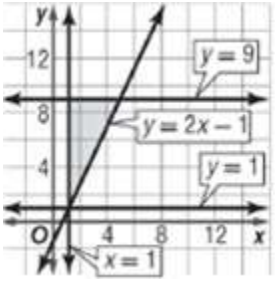
$$f(x, y) = 2x - y$$
$$f(0,1) = 2(0) - 1$$
$$= -1$$
$$f(0,7) = 2(0) - 7$$
$$= -7$$
$$f(3,4) = 2(3) - 4$$
$$= 2$$

The maximum value of $f$ is 2 at (3, 4). The minimum value of $f$ is $-7$ at (0, 7).

54.
$$f(x, y) = 3x + y$$
$$2x - y \le 1$$
$$1 \le y$$
$$y \le 9$$
$$x \ge 1$$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has three vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$$x = 1$$
$$y = 1$$

vertex at (1, 1)

$$x = 1$$
$$y = 9$$

vertex at (1, 9)

$$y = 9$$
$$y = 2x - 1$$
$$9 = 2x - 1$$
$$10 = 2x$$
$$x = 5$$

vertex at (5, 9)

Find the value of the objective function for each of the vertices.

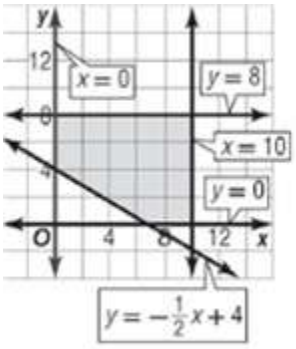
$$f(x, y) = 3x + y$$
$$f(1,1) = 3(1) + 1 = 4$$
$$f(1,9) = 3(1) + 9 = 12$$
$$f(5,9) = 3(5) + 9 = 24$$

The maximum value of $f$ is 24 at (5, 9). The minimum value of $f$ is 4 at (1, 1).

55. $f(x, y) = x + y$
$0 \le x \le 10$
$x + 2y \ge 8$
$0 \le y \le 8$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has five vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$x = 0$

$y = 8$

vertex at (0, 8)

$x = 10$

$y = 8$

vertex at (10, 8)

$x = 10$

$y = 0$

vertex at (10, 0)

$y = 0$

$y = -\frac{1}{2}x + 4$

$0 = -\frac{1}{2}x + 4$

$\frac{1}{2}x = 4$

$x = 8$

vertex at (8, 0)

$x = 0$

$y = -\frac{1}{2}x + 4$

$y = -\frac{1}{2}(0) + 4$

$y = 0 + 4$

$y = 4$

vertex at (0, 4)

Find the value of the objective function for each of the vertices.

$f(x, y) = x + y$

$f(0,8) = 0 + 8$

$= 8$

$f(8,0) = 8 + 0$

$= 8$

$f(10,8) = 10 + 8$

$= 18$

$f(10,0) = 10 + 0$

$= 10$

$f(0,4) = 0 + 4$

$= 4$

The maximum value of $f$ is 18 at (10, 8). The minimum value of $f$ is 4 at (0, 4).

56.

$f(x, y) = 2x - 4y$

$x \geq 3$

$y \geq 3$

$4x + 5y \leq 47$

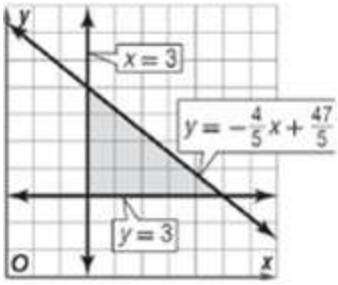
*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.

$4x + 5y \leq 47$

$5y \leq -4x + 47$

$y \leq -\frac{4}{5}x + \frac{47}{5}$




The polygonal region of feasible solutions has three vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$x = 3$

$y = 3$

vertex at (3, 3)

$x = 3$

$y = -\frac{4}{5}x + \frac{47}{5}$

$y = -\frac{4}{5}(3) + \frac{47}{5}$

$y = -\frac{12}{5} + \frac{47}{5}$

$y = \frac{35}{5} = 7$

vertex at (3, 7)

$y = 3$

$y = -\frac{4}{5}x + \frac{47}{5}$

$3 = -\frac{4}{5}x + \frac{47}{5}$

$\frac{15}{5} - \frac{47}{5} = -\frac{4}{5}x$

$-\frac{32}{5} = -\frac{4}{5}x$

$\frac{32}{4} = x$

$x = 8$

vertex at (8, 3)

$x = 0$

$y = -\frac{1}{2}x + 4$

$y = -\frac{1}{2}(0) + 4$

$y = 0 + 4$

$y = 4$

vertex at (0, 4)

Find the value of the objective function for each of the vertices.

$f(x, y) = 2x - 4y$

$f(3,3) = 2(3) - 4(3)$
$= 6 - 12$
$= -6$

$f(3,7) = 2(3) - 4(7)$
$= 6 - 28$
$= -22$

$f(8,3) = 2(8) - 4(3)$
$= 16 - 12$
$= 4$

The maximum value of $f$ is 4 at (8, 3). The minimum value of $f$ is $-22$ at (3, 7).

57.

$f(x, y) = 4x + 3y$

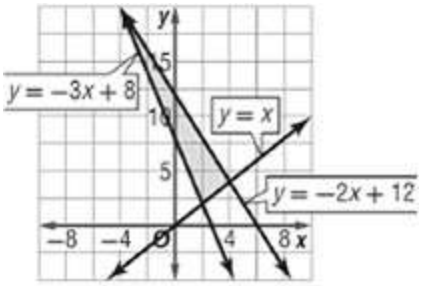
$3x + y \geq 8$

$2x + y \leq 12$

$y \geq x$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has three vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$y = x$

$y = -2x + 12$

$x = -2x + 12$

$3x = 12$

$x = 4$

$y = 4$

vertex at (4, 4)

$$y = x$$
$$y = -3x + 8$$
$$x = -3x + 8$$
$$4x = 8$$
$$x = 2$$
$$y = 2$$

vertex at (2, 2)

$$y = -3x + 8$$
$$y = -2x + 12$$
$$-3x + 8 = -2x + 12$$
$$-4 = x$$
$$y = -2(-4) + 12$$
$$y = 8 + 12$$
$$y = 20$$

vertex at $(-4, 20)$

Find the value of the objective function for each of the vertices.

$$f(x, y) = 4x + 3y$$
$$f(4, 4) = 4(4) + 3(4)$$
$$f(4, 4) = 16 + 16 = 32$$
$$f(2, 2) = 4(2) + 3(2)$$
$$f(2, 2) = 8 + 6 = 14$$
$$f(-4, 20) = 4(-4) + 3(20)$$
$$f(-4, 20) = -16 + 60 = 44$$

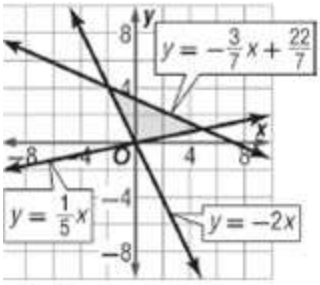
The maximum value of $f$ is 44 at $(-4, 20)$. The minimum value of $f$ is 14 at (2, 2).

58.

$$f(x, y) = 2y - 5x$$
$$2x + y \geq 0$$
$$x - 5y \leq 0$$
$$3x + 7y \leq 22$$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has three vertices. Each vertex is located at an intersection of two of the boundaries. Solve a system of equations to identify each vertex.

$$y = \frac{1}{5}x$$
$$y = -2x$$
$$\frac{1}{5}x = -2x$$
$$x = 0$$
$$y = 0$$

vertex at (0, 0)

$$y = \frac{1}{5}x$$
$$y = -\frac{3}{7}x + \frac{22}{7}$$
$$\frac{1}{5}x = -\frac{3}{7}x + \frac{22}{7}$$
$$7x = -15x + 110$$
$$22x = 110$$
$$x = 5$$
$$y = 1$$

vertex at (5, 1)

$$y = -\frac{3}{7}x + \frac{22}{7}$$
$$y = -2x$$
$$-\frac{3}{7}x + \frac{22}{7} = -2x$$
$$-3x + 22 = -14x$$
$$11x = -22$$
$$x = -2$$
$$y = -2x$$
$$y = -2(-2)$$
$$y = 4$$

vertex at $(-2, 4)$

Find the value of the objective function for each of the vertices.

$f(x, y) = 2y - 5x$

$f(0,0) = 2(0) - 5(0)$

$f(0,0) = 0$

$f(5,1) = 2(1) - 5(5)$

$f(5,1) = 2 - 25 = -23$

$f(-2,4) = 2(4) - 5(-2)$

$f(-2,4) = 8 + 10 = 18$

The maximum value of $f$ is 18 at $(-2, 4)$. The minimum value of $f$ is $-23$ at (5, 1).

59. **HAMBURGERS** The table shows the number of hamburgers, cheeseburgers, and veggie burgers sold at a diner over a 3-hour lunch period. Find the price for each type of burger.

| Hours | Plain | Cheese | Veggie | Total Sales ($) |
|---|---|---|---|---|
| 11 A.M.–12 P.M. | 2 | 8 | 2 | 53 |
| 12–1 P.M. | 7 | 12 | 8 | 119 |
| 1–2 P.M. | 1 | 5 | 7 | 64 |

*SOLUTION:*

Set up a system of equations. Let $p$ = plain, $c$ = cheese, and $v$ = veggie.

$2p + 8c + 2v = 53$

$7p + 12c + 8v = 119$

$1p + 5c + 7v = 64$

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 8 & 2 & 53 \\ 7 & 12 & 8 & 119 \\ 1 & 5 & 7 & 64 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 2 & 8 & 2 & 53 \\ 7 & 12 & 8 & 119 \\ 1 & 5 & 7 & 64 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 2 & 53 \\ 0 & -23 & -41 & -329 \\ 1 & 5 & 7 & 64 \end{array}\right] \leftarrow R_2 - 7R_3$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 2 & 53 \\ 0 & -23 & -41 & -329 \\ 0 & 2 & 12 & 75 \end{array}\right] \leftarrow 2R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 2 & 53 \\ 0 & 46 & 276 & 1725 \\ 0 & -46 & -82 & -658 \end{array}\right] \begin{array}{l} \leftarrow 23R_2 \\ \leftarrow 2R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 2 & 53 \\ 0 & 2 & 12 & 75 \\ 0 & 0 & 194 & 1067 \end{array}\right] \begin{array}{l} \leftarrow R_2 + (23) \\ \leftarrow R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 2 & 53 \\ 0 & 2 & 12 & 75 \\ 0 & 0 & 1 & 5.5 \end{array}\right] \leftarrow R_3 + 194$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 2 & 53 \\ 0 & 2 & 0 & 9 \\ 0 & 0 & 1 & 5.5 \end{array}\right] \leftarrow R_2 - 12R_3$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 2 & 53 \\ 0 & 1 & 0 & 4.5 \\ 0 & 0 & 1 & 5.5 \end{array}\right] \leftarrow R_2 + 2$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 0 & 42 \\ 0 & 1 & 0 & 4.5 \\ 0 & 0 & 1 & 5.5 \end{array}\right] \leftarrow R_1 - 2R_3$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & 4.5 \\ 0 & 0 & 1 & 5.5 \end{array}\right] \leftarrow R_1 - 8R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4.5 \\ 0 & 0 & 1 & 5.5 \end{array}\right] \leftarrow R_1 + 2$$

plain hamburgers = \$3.00, cheeseburgers = \$4.50, veggie burgers = \$5.50

60. **GRADING** Ms. Hebert decides to base grades on tests, homework, projects, and class participation. She assigns a different percentage weight for each category, as shown. Find the final grade for each student to the nearest percent.

| Category | tests | HW | projects | participation |
|---|---|---|---|---|
| Weight | 40% | 30% | 20% | 10% |

| Category | Serena | Andrew | Corey | Shannon |
|---|---|---|---|---|
| tests | 88 | 72 | 78 | 91 |
| HW | 95 | 90 | 68 | 71 |
| projects | 80 | 73 | 75 | 85 |
| participation | 100 | 95 | 100 | 80 |

*SOLUTION:*

Set up the percentages as a 1 × 4 matrix and the individual scores as a 4 × 4 matrix. Remember that when you multiply matrices, the columns of the first matrix must equal the rows of the second matrix. Multiply the matrices.

$$\begin{bmatrix} .4 & .3 & .2 & .1 \end{bmatrix} \cdot \begin{bmatrix} 88 & 72 & 78 & 91 \\ 95 & 90 & 68 & 71 \\ 80 & 73 & 75 & 85 \\ 100 & 95 & 100 & 80 \end{bmatrix}$$

$$= [.4(88) + .3(95) + .2(80) + .1(100) \quad .4(72) + .3(90) + .2(73) + .1(95) \quad .4(78) + .3(68) + .2(75) + .1(100) \quad .4(91) + .3(71) + .2(85) + .1(80)]$$

Round the scores to the nearest

$$= [35.2 + 28.5 + 16 + 10 \quad 28.8 + 27 + 14.6 + 9.5 \quad 31.2 + 20.4 + 15 + 10 \quad 36.4 + 21.3 + 17 + 8]$$

$$= \begin{bmatrix} 89.7 & 79.9 & 76.6 & 82.7 \end{bmatrix}$$

percent.

Serena = 90%, Andrew = 80%, Corey = 77%, Shannon = 83%

61. **SHAVED ICE** A shaved ice stand sells 3 flavors: strawberry, pineapple, and cherry. Each flavor sells for $1.25. O day, the stand had $60 in total sales. The stand made $13.75 more in cherry sales than pineapple sales and $16.25 m than strawberry sales. Use Cramer's Rule to determine how many of each flavor was sold.

*SOLUTION:*

Set up a system of equations. Let $s$ = strawberry, $p$ = pineapple, and $c$ = cherry. If the stand had $60 in total sales a each unit cost $1.25, then 60 ÷ 1.25 or 48 units were sold. Therefore, $s + p + c = 48$. The stand made $13.75 more f cherry sales than from pineapple sales. 13.75 ÷ 1.25 = 11, so $c = 11 + p$. The stand made $16.25 more from cherry s than from strawberry sales. 16.25 ÷ 1.25 = 13, so $c = 13 + s$.

We are asked to solve using Cramer's Rule, so rewrite the equations, aligning the variables, and then set up the coefficient matrix $A$.

$$s + p + c = 48$$
$$0s - p + c = 11$$
$$-s + 0p + c = 13$$

$$A=\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & 1\\ -1 & 0 & 1\end{bmatrix}$$

Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = a\begin{vmatrix}e & f\\ h & i\end{vmatrix} - b\begin{vmatrix}d & f\\ g & i\end{vmatrix} + c\begin{vmatrix}d & e\\ g & h\end{vmatrix}$$

$$\begin{vmatrix}1 & 1 & 1\\ 0 & -1 & 1\\ -1 & 0 & 1\end{vmatrix} = 1\begin{vmatrix}-1 & 1\\ 0 & 1\end{vmatrix} - 1\begin{vmatrix}0 & 1\\ -1 & 1\end{vmatrix} + 1\begin{vmatrix}0 & -1\\ -1 & 0\end{vmatrix}$$

$$= -1(1) - 0(1) - [0(1) - (-1)1] + [0(0) - (-1)(-1)]$$

$$= -1 - 1 - 1$$

$$= -3$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated w the variable with the column of constant terms.

$$s = \frac{|A_s|}{|A|}$$

$$= \frac{\begin{vmatrix}48 & 1 & 1\\ 11 & -1 & 1\\ 13 & 0 & 1\end{vmatrix}}{-3}$$

$$= \frac{48\begin{vmatrix}-1 & 1\\ 0 & 1\end{vmatrix} - 1\begin{vmatrix}11 & 1\\ 13 & 1\end{vmatrix} + 1\begin{vmatrix}11 & -1\\ 13 & 0\end{vmatrix}}{-3}$$

$$= \frac{48[-1(1) - 0(1)] - [11(1) - 13(1)] + [11(0) - 13(-1)]}{-3}$$

$$= \frac{-48 + 2 + 13}{-3}$$

$$= \frac{-33}{-3}$$

$$= 11$$

$$p = \frac{|A_p|}{|A|} = \frac{\begin{vmatrix} 1 & 48 & 1 \\ 0 & 11 & 1 \\ -1 & 13 & 1 \end{vmatrix}}{-3} = \frac{1\begin{vmatrix} 11 & 1 \\ 13 & 1 \end{vmatrix} - 48\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 11 \\ -1 & 13 \end{vmatrix}}{-3}$$

$$= \frac{1[11(1) - 13(1)] - 48[0(1) - (-1)(1)] + [0(13) - (-1)(11)]}{-3} = \frac{-2 - 48 + 11}{-3} = \frac{-39}{-3} = 13$$

$$c = \frac{|A_c|}{|A|} = \frac{\begin{vmatrix} 1 & 1 & 48 \\ 0 & -1 & 11 \\ -1 & 0 & 13 \end{vmatrix}}{-3} = \frac{1\begin{vmatrix} -1 & 11 \\ 0 & 13 \end{vmatrix} - 1\begin{vmatrix} 0 & 11 \\ -1 & 13 \end{vmatrix} + 48\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}}{-3}$$

$$= \frac{1[-1(13) - 0(11)] - 1[0(13) - (-1)(11)] + 48[0(0) - (-1)(-1}{-3} = \frac{-13 - 11 - 48}{-3} = \frac{-72}{-3} = 24$$

strawberry, 13 pineapple, 24 cherry

62. **BIKING** On a biking trip, a couple traveled 240 miles on Day 1 and 270 miles on Day 2. The average rate traveled during Day 1 is 5 miles per hour faster than the average rate traveled during Day 2. The total number of hours spent biking is $T = \frac{510r - 1200}{r(r-5)}$.

a. Find the partial fraction decomposition of $T$.
b. Each fraction represents the time spent biking each day. If the couple biked 6 hours longer on Day 2, how many total hours did they bike?

*SOLUTION:*

**a.**
Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{510r - 1200}{r(r-5)} = \frac{A}{r} + \frac{B}{r-5}$$

Multiply each side by the LCD, $r(r - 5)$.

$$\frac{510r - 1200}{r(r-5)} = \frac{A}{r} + \frac{B}{r-5}$$

$$\frac{510r - 1200}{r(r-5)} = \frac{A(r-5)}{r(r-5)} + \frac{Br}{r(r-5)}$$

$$510r - 1200 = Ar - 5A + Br$$

$$510r - 1200 = Ar + Br - 5A$$

$$510r - 1200 = (A + B)r - 5A$$

Equate the coefficients on the left and right side of the equation to form a system of equations. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A + B = 510$$

$$-5A = -1200$$

Solve the system.

$$-5A = -1200$$
$$A = 240$$
$$A + B = 510$$
$$240 + B = 510$$
$$B = 270$$

Replace $A$ and $B$ with 2 and −1 in the partial fraction decomposition.

$$\frac{510r - 1200}{r(r-5)} = \frac{A}{r} + \frac{B}{r-5}$$
$$= \frac{240}{r} + \frac{270}{r-5}$$

**b**. The second fraction is greater than the first fraction by 6. Set them equal to each other and solve for $r$.

$$\frac{240}{r} + 6 = \frac{270}{r-5}$$
$$\frac{240(r-5)}{r(r-5)} + \frac{6r(r-5)}{r(r-5)} = \frac{270r}{r(r-5)}$$
$$240(r-5) + 6r(r-5) = 270r$$
$$240r - 1200 + 6r^2 - 30r = 270r$$
$$6r^2 - 60r - 1200 = 0$$
$$r^2 - 10r - 200 = 0$$
$$(r-20)(r+10) = 0$$
$$r = 20 \text{ or } -10$$

The rate must be positive, so $r = 20$. Substitute 20 for $r$ into the partial fraction to find the total time traveled.

$$\frac{240}{20} + \frac{270}{20-5} = 12 + 18 = 30$$

63. **RECYCLING** A recycling company will collect from private sites if the site produces at least 60 pounds of recycling a week. The company can collect at most 50 pounds of paper and 30 pounds of glass from each site. The company earns $20 for each pound of glass and $25 for each pound of paper.

**a.** Write an objective function, and list the constraints that model the given situation.

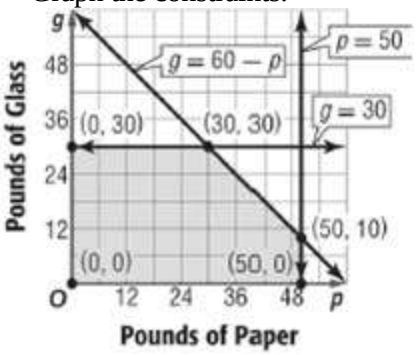
**b.** Determine the number of pounds of glass and paper needed to produce the maximum profit.

**c.** What is the maximum profit?

*SOLUTION:*

**a.** Let $p$ = paper and let $g$ = glass. At least 60 pounds of recycling is produced, so $p + g \geq 60$. At most, 50 pounds of paper can be collected, so $p \leq 50$. At most, 30 pounds of glass can be collected, so $g \leq 30$.The company earns $25 per pound for paper and $20 per pound for glass, so $f(p, g) = 25p + 20g$. The amounts cannot be negative, so $p \geq 0$ and $g \geq 0$.

**b.** Graph the constraints.




Substitute the vertices into the objective function.

$$f(p,g) = 25p + 20g$$
$$f(0,0) = 25(0) + 20(0) = 0$$
$$f(0,30) = 25(0) + 20(30) = 600$$
$$f(30,30) = 25(30) + 20(30) = 1350$$
$$f(50,10) = 25(50) + 20(10) = 1450$$
$$f(50,0) = 25(50) + 20(0) = 1250$$

50 pounds of paper and 10 pounds of glass.

**c**. $1450.

# Practice Test - Chapter 6

**Write each system of equations in triangular form using Gaussian elimination. Then solve the system.**

1. $-3x + y = 4$
   $5x - 7y = 20$

*SOLUTION:*

Align the variables on the left side of the equal sign.

$-3x + y = 4$

$5x - 7y = 20$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by 5 and the 2nd equation by 3.

$(5)(-3x + y = 4) \Rightarrow -15x + 5y = 20$

$(3)(5x - 7y = 20) \Rightarrow 15x - 21y = 60$

Add the equations together.

$$\begin{aligned} -15x + 5y &= 20 \\ 15x - 21y &= 60 \\ -16y &= 80 \\ y &= -5 \end{aligned}$$

Replace the 2nd equation with $y = -5$.

$$\begin{aligned} -3x + y &= 4 \\ y &= -5 \end{aligned}$$

The system is now in triangular form. Substitute for $y$ and solve for $x$.

$$\begin{aligned} -3x + y &= 4 \\ -3x + (-5) &= 4 \\ -3x - 5 &= 4 \\ -3x &= 9 \\ x &= -3 \end{aligned}$$

The solution is $(-3, -5)$.

2. $4y - 3z = -8$
   $5x - 7y + 3z = -4$
   $3x - 2y + 4z = 24$

*SOLUTION:*

Align the variables on the left side of the equal sign.

$$\begin{aligned} x + 4y - 3z &= -8 \\ 5x - 7y + 3z &= -4 \\ 3x - 2y + 4z &= 24 \end{aligned}$$

Eliminate the $x$-term from the 2nd equation. Multiply the 1st equation by $-5$.

$(-5)(x + 4y - 3z = -8) \Rightarrow -5x - 20y + 15z = 40$

$5x - 7y + 3z = -4 \Rightarrow 5x - 7y + 3z = -4$

Add the equations together.

$$\begin{aligned} -5x - 20y + 15z &= 40 \\ (+)\ 5x - 7y + 3z &= -4 \\ -27y + 18z &= 36 \end{aligned}$$

The new set of equations is:

$$x+4y-3z=-8$$
$$-27y+18z=36$$
$$3x-2y+4z=24$$

Eliminate the $x$-term from the 3rd equation. Multiply the 1st equation by $-3$.

$$(-3)(x+4y-3z=-8) \Rightarrow -3x-12y+9z=24$$
$$3x-2y+4z=24 \quad \Rightarrow \quad 3x-2y+4z=24$$

Add the equations together.

$$-3x-12y+9z=24$$
$$(+)\,3x-2y+4z=24$$
$$-14y+13z=48$$

The new set of equations is:

$$x+4y-3z=-8$$
$$-27y+18z=36$$
$$-14y+13z=48$$

We can simplify the 2nd equation by dividing by 9.

$$x+4y-3z=-8$$
$$-3y+2z=4$$
$$-14y+13z=48$$

Eliminate the $y$-term from the 3rd equation. Multiply the 2nd equation by $-14$ and the third equation by 3.

$$(-14)(-3y+2z=4) \quad \Rightarrow \quad 42y-28z=-56$$
$$(3)(-14y+13z=48) \quad \Rightarrow \quad -42y+39z=144$$

Add the equations together.

$$42y-28z=-56$$
$$(+)\,-42y+39z=144$$
$$11z=88$$
$$z=8$$

Replace the 3rd equation with $z = 8$.

$$x+4y-3z=-8$$
$$-3y+2z=4$$
$$z=8$$

The system is now in triangular form. Substitute for $z$ and solve for $y$, then substitute for $y$ and $z$ and solve for $x$.

$$-3y+2z=4$$
$$-3y+2(8)=4$$
$$-3y+16=4$$
$$-3y=-12$$
$$y=4$$
$$x+4y-3z=-8$$
$$x+4(4)-3(8)=-8$$
$$x+16-24=-8$$
$$x-8=-8$$
$$x=0$$

The solution is $(0, 4, 8)$.

**Solve the system of equations.**

3. $5x - 6y = 28$
   $6x + 5y = -3$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{cc|c} 5 & -6 & 28 \\ 6 & 5 & -3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 5 & -6 & 28 \\ 0 & 61 & -183 \end{array}\right] \leftarrow 5R_2 - 6R_1$$

$$\left[\begin{array}{cc|c} 5 & -6 & 28 \\ 0 & 1 & -3 \end{array}\right] \leftarrow \frac{1}{61}R_2$$

$$\left[\begin{array}{cc|c} 5 & 0 & 10 \\ 0 & 1 & -3 \end{array}\right] \leftarrow R_1 + 6R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \leftarrow \frac{1}{5}R_1$$

The solution is (2, −3).

4. $2x - 4y + z = 8$
   $3x + 3y + 4y = 20$
   $5x + y - 3z = -13$

*SOLUTION:*

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 8 \\ 3 & 3 & 4 & 20 \\ 5 & 1 & -3 & -13 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 8 \\ 0 & 18 & 5 & 16 \\ 5 & 1 & -3 & -13 \end{array}\right] \leftarrow 2R_2 - 3R_1$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 8 \\ 0 & 18 & 5 & 16 \\ 0 & 22 & -11 & -66 \end{array}\right] \leftarrow 2R_3 - 5R_1$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 8 \\ 0 & 18 & 5 & 16 \\ 0 & 2 & -1 & -6 \end{array}\right] \leftarrow R_3 + 11$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 8 \\ 0 & 18 & 5 & 16 \\ 0 & 0 & -14 & -70 \end{array}\right] \leftarrow 9R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 8 \\ 0 & 18 & 5 & 16 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_3 \div (-14)$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 8 \\ 0 & 18 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_2 - 5R_3$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 8 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_2 \div 18$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 0 & 3 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1 + 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1 \div 2$$

The solution is $\left(\frac{1}{2}, -\frac{1}{2}, 5\right)$.

5. **LIBRARY** Kristen checked out books, CDs, and DVDs from the library. She checked out a total of 16 items. The number of CDs and DVDs equaled the number of books. She checked out two more CDs than DVDs.
**a.** Let $b$ = number of books, $c$ = number of CDs, and $d$ = number of DVDs. Write a system of three linear equations to represent the problem.
**b.** Solve the system of equations. Interpret your solution.

*SOLUTION:*

**a**. Use the total number of items for the first equation: $b + c + d = 16$. The total number of books equaled the sum of the number of CDs and DVDs: $b = c + d$. She checked out 2 more CDs than DVDs: $c = d + 2$.
Rewrite the equations, aligning the variables.

$$\begin{aligned} b+c+d &= 16 \\ -b+c+d &= 0 \\ c-d &= 2 \end{aligned}$$

**b.** Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 2 & 2 & 16 \\ 0 & 1 & -1 & 2 \end{array}\right] \leftarrow R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 1 & 8 \\ 0 & 1 & -1 & 2 \end{array}\right] \leftarrow \frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 2 & 6 \end{array}\right] \leftarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \leftarrow \frac{1}{2}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \leftarrow R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \leftarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \leftarrow R_1 - R_2$$

Kristen checked out 8 books, 5 CDs, and 3 DVDs.

**Find *AB* and *BA*, if possible.**

6. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 5 & -1 \\ -1 & 6 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}; B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 5 & -1 \\ -1 & 6 \end{bmatrix}$

$A$ is a $4 \times 4$ matrix and $B$ is a $4 \times 2$ matrix. Because the number of columns of $A$ is equal to the number of rows of $B$, $AB$ exists.

To find the first entry of $AB$, find the sum of the products of the entries in row 1 of $A$ and column 1 of $B$. Follow the same procedure for row 2 column 1 of $AB$ and the remaining entries.

$$AB = \begin{bmatrix} 1(1)+0(2)+0(5)+0(-1) & 1(1)+0(3)+0(-1)+0(6) \\ 0(1)+1(2)+3(5)+3(-1) & 0(1)+1(3)+3(-1)+3(6) \\ 0(1)+0(2)+1(5)+2(-1) & 0(1)+0(3)+1(-1)+2(6) \\ 0(1)+0(2)+0(5)+6(-1) & 0(1)+0(3)+0(-1)+6(6) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 14 & 18 \\ 3 & 11 \\ -6 & 36 \end{bmatrix}$$

Because the number of columns of $B$ is *not* equal to the number of rows of $A$, $BA$ is undefined.

7. $A = \begin{bmatrix} 1 & -5 & 4 \\ -2 & 3 & 5 \\ 6 & -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & -8 \end{bmatrix}$

*SOLUTION:*

$A = \begin{bmatrix} 1 & -5 & 4 \\ -2 & 3 & 5 \\ 6 & -3 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 2 & -1 & -8 \end{bmatrix}$

$A$ is a $3 \times 3$ matrix and $B$ is a $1 \times 3$ matrix. Because the number of columns of $A$ is *not* equal to the number of rows of $B$, $AB$ is undefined.

$B$ is a $1 \times 3$ matrix and $A$ is a $3 \times 3$ matrix. Because the number of columns of $B$ is equal to the number of rows of $A$, $BA$ exists.

To find the first entry of $BA$, find the sum of the products of the entries in row 1 of $B$ and column 1 of $A$. Follow the same procedure for row 2 column 1 of $BA$ and the remaining entries.

$$\begin{aligned} BA &= \begin{bmatrix} 2(1)-1(-2)-8(6) & 2(-5)-1(3)-8(-3) & 2(4)-1(5)-8(1) \end{bmatrix} \\ &= \begin{bmatrix} 2+2-48 & -10-3+24 & 8-5-8 \end{bmatrix} \\ &= \begin{bmatrix} -44 & 11 & -5 \end{bmatrix} \end{aligned}$$

8. **GEOMETRY** The coordinates of a point $(x, y)$ can be written as a $2 \times 1$ matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

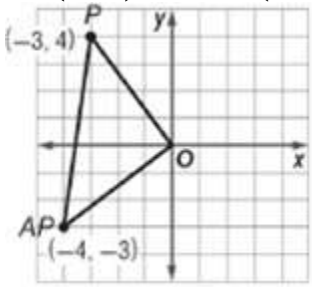
**a.** Let $P$ be the point $(-3, 4)$. Discuss what effect multiplying $A$ by $P$ has on $P$.
**b.** A triangle contains vertices (0, 0), (2, 6), and (8, 3). Create $B$, a $2 \times 3$ matrix to represent the triangle. Find $AB$. What is the effect on the triangle? Does it agree with your answer to part **a**?

*SOLUTION:*

**a.** $P = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

$$\begin{aligned} AP &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 0(-3)-1(4) \\ 1(-3)+0(4) \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -3 \end{bmatrix} \end{aligned}$$

$P = (-3, 4)$ and $AP = (-4, -3)$. Plot these points on a graph.




It appears that $AP$ is $P$ rotated 90 degrees counterclockwise. This angle can be confirmed in a variety of ways.

The distance from $P$ to the origin is 5. The distance from $AP$ to the origin is also 5. Find the distance from $P$ to $AP$.

$$d = \sqrt{(-3-(-4))^2 + (4-(-3))^2}$$
$$= \sqrt{1+49}$$
$$= 5\sqrt{2}$$

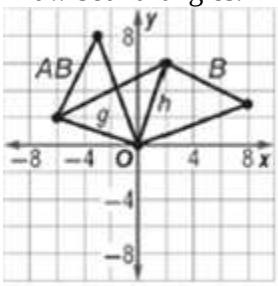
The sides 5, 5, and $5\sqrt{2}$ form a right triangle because $5^2 + 5^2 = (5\sqrt{2})^2$. Therefore, multiplying $P$ by $A$ rotates the point 90° counterclockwise.

**b**. Let each point be a column. $B = \begin{bmatrix} 0 & 2 & 8 \\ 0 & 6 & 3 \end{bmatrix}$

Find $AB$.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 8 \\ 0 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 0(0)+(-1)(0) & 0(2)+(-1)6 & 0(8)+(-1)3 \\ 1(0)+0(0) & 1(2)+0(6) & 1(8)+0(3) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -6 & -3 \\ 0 & 2 & 8 \end{bmatrix}$$

Draw both triangles.




It appears that $AB$ is $B$ rotated 90 degrees counterclockwise. This angle can be confirmed by checking the rotation of one side of the triangle. In this case, we will check the rotation of side $h$ (side $g$ in the rotation).
Sides $g$ and $h$ form a right triangle. Find the length of each side.

$$\sqrt{(0-(-6))^2 + (0-2)^2}$$
$$= \sqrt{36+4}$$
$$= \sqrt{40}$$
$$\sqrt{(0-2)^2 + (0-6)^2}$$
$$= \sqrt{4+36}$$
$$= \sqrt{40}$$
$$\sqrt{(2-(-6))^2 + (6-2)^2}$$
$$= \sqrt{64+16}$$
$$= \sqrt{80}$$

$$\left(\sqrt{40}\right)^2 + \left(\sqrt{40}\right)^2 = \left(\sqrt{80}\right)^2$$
$$40 + 40 = 80$$

These sides form a right triangle, so triangle $A$ is rotated 90° counterclockwise. This agrees with answer **a**.

**Find $A^{-1}$, if it exists. If $A^{-1}$ does not exist, write *singular*.**

9. $\begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & -4 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & -4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right] \leftarrow R_2 + 3R_1$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \end{array}\right] \leftarrow \frac{1}{2}R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \end{array}\right] \leftarrow R_1 - 2R_2$$

The first two columns are the identity matrix. Therefore, $A$ is invertible and $A^{-1} = \begin{bmatrix} -2 & -1 \\ \frac{3}{3} & \frac{1}{2} \end{bmatrix}$.

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 1(-2)+2\left(\frac{3}{2}\right) & 1(-1)+2\left(\frac{1}{2}\right) \\ -3(-2)-4\left(\frac{3}{2}\right) & -3(-1)-4\left(\frac{1}{2}\right) \end{bmatrix}$$
$$= \begin{bmatrix} -2+3 & -1+1 \\ 6-6 & 3-2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A^{-1}A = \begin{bmatrix} -2 & -1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} -2(1)-1(-3) & -2(2)-1(-4) \\ \frac{3}{2}(1)+\frac{1}{2}(-3) & \frac{3}{2}(2)+\frac{1}{2}(-4) \end{bmatrix}$$
$$= \begin{bmatrix} -2+3 & -4+4 \\ \frac{3}{2}-\frac{3}{2} & 3-2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

10. $\begin{bmatrix} -3 & -5 \\ -6 & 8 \end{bmatrix}$

*SOLUTION:*

Create the doubly augmented matrix $[A \mid I]$.

$$[A \mid I] = \left[\begin{array}{cc:cc} -3 & -5 & 1 & 0 \\ -6 & 8 & 0 & 1 \end{array}\right]$$

Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{cc|cc} -3 & -5 & 1 & 0 \\ -6 & 8 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} -3 & -5 & 1 & 0 \\ 0 & 18 & -2 & 1 \end{array}\right] \leftarrow R_2 - 2R_1$$

$$\left[\begin{array}{cc|cc} -3 & -5 & 1 & 0 \\ 0 & 1 & -\frac{1}{9} & \frac{1}{18} \end{array}\right] \leftarrow R_2 \div 18$$

$$\left[\begin{array}{cc|cc} -3 & 0 & \frac{4}{9} & \frac{5}{18} \\ 0 & 1 & -\frac{1}{9} & \frac{1}{18} \end{array}\right] \leftarrow R_1 + 5R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{4}{27} & -\frac{5}{54} \\ 0 & 1 & -\frac{1}{9} & \frac{1}{18} \end{array}\right] \leftarrow R_1 \div -3$$

The first two columns are the identity matrix. Therefore, $A$ is invertible and $A^{-1} = \begin{bmatrix} -\frac{4}{27} & -\frac{5}{54} \\ -\frac{1}{9} & \frac{1}{18} \end{bmatrix}$.

Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} -3 & -5 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} -\frac{4}{27} & -\frac{5}{54} \\ -\frac{1}{9} & \frac{1}{18} \end{bmatrix}$$

$$= \begin{bmatrix} -3\left(-\frac{4}{27}\right) - 5\left(-\frac{1}{9}\right) & -3\left(-\frac{5}{54}\right) - 5\left(\frac{1}{18}\right) \\ -6\left(-\frac{4}{27}\right) + 8\left(-\frac{1}{9}\right) & -6\left(-\frac{5}{54}\right) + 8\left(\frac{1}{18}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{27} + \frac{5}{9} & \frac{15}{54} - \frac{5}{18} \\ \frac{24}{27} - \frac{8}{9} & \frac{30}{54} + \frac{8}{18} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -\frac{4}{27} & -\frac{5}{54} \\ -\frac{1}{9} & \frac{1}{18} \end{bmatrix} \begin{bmatrix} -3 & -5 \\ -6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{4}{27}(-3) - \frac{5}{54}(-6) & -\frac{4}{27}(-5) - \frac{5}{54}(8) \\ -\frac{1}{9}(-3) + \frac{1}{18}(-6) & -\frac{1}{9}(-5) + \frac{1}{18}(8) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{27} + \frac{30}{54} & \frac{20}{27} - \frac{40}{54} \\ \frac{3}{9} - \frac{6}{18} & \frac{5}{9} + \frac{8}{18} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Use an inverse matrix to solve each system of equations, if possible.**

11. $2x - 3y = -7$
$5x + 2y = 11$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 2 & -3 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 11 \end{bmatrix}$$

Use the formula for the inverse of a 2 × 2 matrix to find $A^{-1}$.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{2(2) - (-3)(5)} \begin{bmatrix} 2 & 3 \\ -5 & 2 \end{bmatrix}$$

$$= \frac{1}{19} \begin{bmatrix} 2 & 3 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{19} & \frac{3}{19} \\ -\frac{5}{19} & \frac{2}{19} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} \frac{2}{19} & \frac{3}{19} \\ -\frac{5}{19} & \frac{2}{19} \end{bmatrix} \begin{bmatrix} -7 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{19}(-7) + \frac{3}{19}(11) \\ -\frac{5}{19}(-7) + \frac{2}{19}(11) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{14}{19} + \frac{33}{19} \\ \frac{35}{19} + \frac{22}{19} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{19}{19} \\ \frac{57}{19} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The solution is (1, 3)

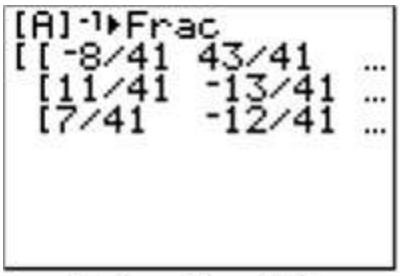
12. $2x + 2y + 5z = -6$
$2x - 3y + 7z = -7$
$x - 5y + 9z = 4$

*SOLUTION:*

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 2 & 2 & 5 \\ 2 & -3 & 7 \\ 1 & -5 & 9 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ -7 \\ 4 \end{bmatrix}$$

Use a graphing calculator to find $A^{-1}$. Enter the values for the matrix $A$. Enter $[A]^{-1}$. Select $\rightarrow Frac$ to get the values in reduced fraction form.




$$A^{-1} = \begin{bmatrix} -\frac{8}{41} & \frac{43}{41} & -\frac{29}{41} \\ \frac{11}{41} & -\frac{13}{41} & \frac{4}{41} \\ \frac{7}{41} & -\frac{12}{41} & \frac{10}{41} \end{bmatrix}$$

Multiply $A^{-1}$ by $B$ to solve the system.

$$A^{-1}B = \begin{bmatrix} -\frac{8}{41} & \frac{43}{41} & -\frac{29}{41} \\ \frac{11}{41} & -\frac{13}{41} & \frac{4}{41} \\ \frac{7}{41} & -\frac{12}{41} & \frac{10}{41} \end{bmatrix} \cdot \begin{bmatrix} -6 \\ -7 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8}{41}(-6) + \frac{43}{41}(-7) - \frac{29}{41}(4) \\ \frac{11}{41}(-6) - \frac{13}{41}(-7) + \frac{4}{41}(4) \\ \frac{7}{41}(-6) - \frac{12}{41}(-7) + \frac{10}{41}(4) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{48}{41} - \frac{301}{41} - \frac{116}{41} \\ -\frac{66}{41} + \frac{91}{41} + \frac{16}{41} \\ -\frac{42}{41} + \frac{84}{41} + \frac{40}{41} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{369}{41} \\ \frac{41}{41} \\ \frac{82}{41} \end{bmatrix}$$

$$= \begin{bmatrix} -9 \\ 1 \\ 2 \end{bmatrix}$$

The solution is $(-9, 1, 2)$.

**Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists.**

13. $3x-2y=-2$
$4x-2y=2$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$. Calculate the determinant of $A$.

$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} 3 & -2 \\ 4 & -2 \end{vmatrix} = 3(-2) - 4(-2)$$
$$= -6 + 8$$
$$= 2$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$
$$= \frac{\begin{vmatrix} -2 & -2 \\ 2 & -2 \end{vmatrix}}{2}$$
$$= \frac{-2(-2) - 2(-2)}{2}$$
$$= \frac{4+4}{2}$$
$$= 4$$
$$y = \frac{|A_y|}{|A|}$$
$$= \frac{\begin{vmatrix} 3 & -2 \\ 4 & 2 \end{vmatrix}}{2}$$
$$= \frac{3(2) - 4(-2)}{2}$$
$$= \frac{6+8}{2}$$
$$= 7$$

The solution is (4, 7).

14. $3x-2y-3z=-24$
$3x+5y+2z=7$
$-x+5y+3z=25$

*SOLUTION:*

The coefficient matrix is $A = \begin{bmatrix} 3 & -2 & -3 \\ 3 & 5 & 2 \\ -1 & 5 & 3 \end{bmatrix}$. Remember to align the terms, using 0s if needed. Calculate the determinant of $A$.

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} 3 & -2 & -3 \\ 3 & 5 & 2 \\ -1 & 5 & 3 \end{vmatrix} = 3\begin{vmatrix} 5 & 2 \\ 5 & 3 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix} - 3\begin{vmatrix} 3 & 5 \\ -1 & 5 \end{vmatrix}$$

$$= 3[5(3) - 5(2)] + 2[3(3) - (-1)(2)] - 3[3(5) - (-1)(5)]$$
$$= 3(15 - 10) + 2(9 + 2) - 3(15 + 5)$$
$$= 15 + 22 - 60$$
$$= -23$$

Because the determinant of $A$ is not 0, you can apply Cramer's Rule. Remember to replace the column associated with the variable with the column of constant terms.

$$x = \frac{|A_x|}{|A|}$$
$$= \frac{\begin{vmatrix} -24 & -2 & -3 \\ 7 & 5 & 2 \\ 25 & 5 & 3 \end{vmatrix}}{-23}$$
$$= \frac{-24\begin{vmatrix} 5 & 2 \\ 5 & 3 \end{vmatrix} + 2\begin{vmatrix} 7 & 2 \\ 25 & 3 \end{vmatrix} - 3\begin{vmatrix} 7 & 5 \\ 25 & 5 \end{vmatrix}}{-23}$$
$$= \frac{-24[5(3) - 5(2)] + 2[7(3) - 25(2)] - 3[7(5) - 25(5)]}{-23}$$
$$= \frac{-120 - 58 + 270}{-23}$$
$$= \frac{92}{-23}$$
$$= -4$$

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 3 & -24 & -3 \\ 3 & 7 & 2 \\ -1 & 25 & 3 \end{vmatrix}}{-23}$$

$$= \frac{3\begin{vmatrix} 7 & 2 \\ 25 & 3 \end{vmatrix} + 24\begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix} - 3\begin{vmatrix} 3 & 7 \\ -1 & 25 \end{vmatrix}}{-23}$$

$$= \frac{3[7(3) - 25(2)] + 24[3(3) - (-1)(2)] - 3[3(25) - (-1)(7)]}{-23}$$

$$= \frac{3(21 - 50) + 24(9 + 2) - 3(75 + 7)}{-23}$$

$$= \frac{-87 + 264 - 246}{-23}$$

$$= \frac{-69}{-23}$$

$$= 3$$

$$Z = \frac{|A_z|}{|A|}$$

$$= \frac{\begin{vmatrix} 3 & -2 & -24 \\ 3 & 5 & 7 \\ -1 & 5 & 25 \end{vmatrix}}{-23}$$

$$= \frac{3\begin{vmatrix} 5 & 7 \\ 5 & 25 \end{vmatrix} + 2\begin{vmatrix} 3 & 7 \\ -1 & 25 \end{vmatrix} - 24\begin{vmatrix} 3 & 5 \\ -1 & 5 \end{vmatrix}}{-23}$$

$$= \frac{3[5(25) - 5(7)] + 2[3(25) - (-1)(7)] - 24[3(5) - (-1)(5)]}{-23}$$

$$= \frac{270 + 164 - 480}{-23}$$

$$= \frac{-46}{-23}$$

$$= 2$$

The solution is (−4, 3, 2).

**Find the partial fraction decomposition of each rational expression.**

15. $\frac{4x}{x2-9}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{4x}{x^2-9}=\frac{A}{x-3}+\frac{B}{x+3}$$

Multiply each side by the LCD, $x^2-9$.

$$4x=A(x+3)+B(x-3)$$

Group the like terms.

$$4x=A(x+3)+B(x-3)$$

$$4x=Ax+3A+Bx-3B$$

$$4x=Ax+Bx+3A-3B$$

$$4x=(A+B)x+3A-3B$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=4$$

$$3A-3B=0$$

Use any method to solve the new system.

$$A+B=4$$

$$A=4-B$$

$$3A-3B=0$$

$$3(4-B)-3B=0$$

$$12-3B-3B=0$$

$$12-6B=0$$

$$6B=12$$

$$B=2$$

$$A=4-(2)$$

$$A=2$$

Replace $A$ and $B$ with 2 and 2 in the partial fraction decomposition.

$$\frac{4x}{x^2-9}=\frac{A}{x-3}+\frac{B}{x+3}$$

$$\frac{4x}{x^2-9}=\frac{2}{x-3}+\frac{2}{x+3}$$

16. $\frac{2x+10}{x2-4x+3}$

*SOLUTION:*

Rewrite the expression as partial fractions with constant numerators, $A$ and $B$, and denominators that are the linear factors of the original denominator.

$$\frac{2x+10}{x^2-4x+3}=\frac{A}{x-3}+\frac{B}{x-1}$$

Multiply each side by the LCD, $x^2 - 4x + 3$.

$$4x = A(x+3)+B(x-3)$$

Group the like terms.

$$2x+10=A(x-1)+B(x-3)$$
$$2x+10=Ax-A+Bx-3B$$
$$2x+10=Ax+Bx-A-3B$$
$$2x+10=(A+B)x-A-3B$$

Equate the coefficients on the left and right side of the equation to form two equations with 2 variables. In other words, the coefficients of the $x$-terms on the left side of the equation must equal the coefficients of the $x$-terms on the right side.

$$A+B=12$$
$$-A-3B=10$$

Use any method to solve the new system.

$$A+B=2$$
$$A=2-B$$
$$-A-3B=10$$
$$-(2-B)-3B=10$$
$$-2+B-3B=10$$
$$-2-2B=10$$
$$2B=-12$$
$$B=-6$$
$$A=2-(-6)$$
$$A=8$$

Replace $A$ and $B$ with 8 and $-6$ in the partial fraction decomposition.

$$\frac{2x+10}{x^2-4x+3}=\frac{A}{x-3}+\frac{B}{x-1}$$
$$\frac{2x+10}{x^2-4x+3}=\frac{8}{x-3}-\frac{6}{x-1}$$

**Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of $x$ and $y$ they occur, subject to the given constraints.**

17. $f(x, y) = 2x - y$

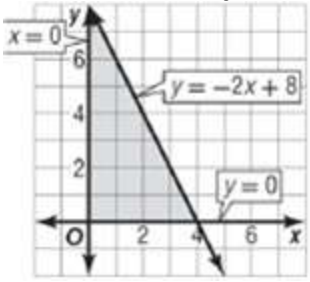
$x \geq 0$

$y \geq 0$

$y \geq -2x + 8$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has three vertices. Each vertex is located at an intersection of two of the boundaries. The vertices are (0, 0), (4, 0), and (0, 8).

Find the value of the objective function for each of the vertices.

$f(x, y) = 2x - y$

$f(0, 0) = 2(0) - 0$

$= 0$

$f(4, 0) = 2(4) - 0$

$= 8$

$f(0, 8) = 2(0) - 8$

$= -8$

max at (4, 0) = 8, min at (0, 8) = −8

18. $f(x, y) = -x + 2y$

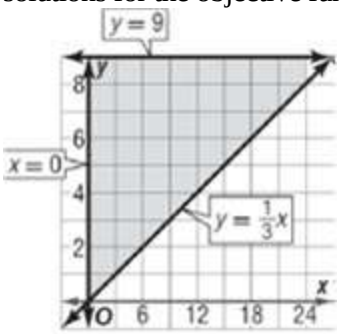
$x - 3y \le 0$

$x \ge 0$

$y \le 9$

*SOLUTION:*

Begin by graphing the given system of inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.




The polygonal region of feasible solutions has three vertices. Each vertex is located at an intersection of two of the boundaries. Two of the vertices are (0, 0) and (0, 9). Find the third vertex by solving the system of equations that form the intersection.

$y = 9$

$y = \frac{1}{3}x$

$9 = \frac{1}{3}x$

$27 = x$

The third vertex is (27, 9).

Find the value of the objective function for each of the vertices.

$f(x, y) = -x + 2y$

$f(0, 0) = -(0) + 2(0)$

$= 0$

$f(0, 9) = -(0) + 2(9)$

$= 18$

$f(27, 9) = -(27) + 2(9)$

$= -9$

max at (0, 9) = 18, min at (27, 9) = −9

19. **PRICING** The Harvest Nut Company sells create-your-own trail mixes where customers can choose whatever combinations they want. Colin's favorite mix contains peanuts, dried cranberries, and carob-coated pretzels. The prices for each are shown below. If Colin bought a 5-pound mixture for $16.80 that contained twice as many pounds of carob-coated pretzels as cranberries, how many pounds of peanuts, cranberries, and carob-coated pretzels did he buy?




*SOLUTION:*

Let $p$ = peanuts, $r$ = raisins, and $z$ = pretzels. Set up a system of equations. He bought a five-pound mixture, so $p + r + z = 5$. He spent $16.80, so $3.2p + 2.4r + 4z = 16.8$. There were twice as many pretzels as raisins, so $z = 2r$. Write the system of equations, aligning the variables.

$$\begin{aligned} p + r + z &= 5 \\ 3.2p + 2.4r + 4z &= 16.8 \\ 0p + 2r - z &= 0 \end{aligned}$$

Write the augmented matrix. Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 3.2 & 2.4 & 4 & 16.8 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -0.8 & 0.8 & 0.8 \\ 0 & 2 & -1 & 0 \end{array}\right] \leftarrow R_2 - 3.2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -1 & 0 \end{array}\right] \leftarrow R_2 + (-0.8)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \leftarrow R_3 - 2R_2$$

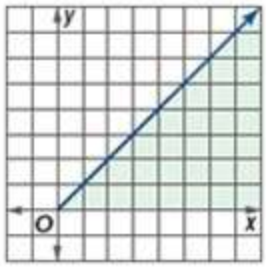
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \leftarrow R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \leftarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \leftarrow R_1 - R_2$$

peanuts, 2 lb; raisins, 1 lb; carob-coated pretzels, 2 lb

20. **MULTIPLE CHOICE** The graph displays the constraints for an objective function. Which of the following CANNOT be one of the constraints?




A $y \geq 0$
B $x \geq 0$
C $x - y \leq 0$
D $x - y \geq 0$

*SOLUTION:*

Identify which inequalities are constraints.

Choice A, $y \geq 0$, represents the bottom line of the shaded region. Choice B, $x \geq 0$, *can* be a constraint even though it has no direct effect on the boundaries of the shaded region. Choice D, $x - y \geq 0$, represents the diagonal boundary of the shaded region. Choice C, $x - y \leq 0$, cannot be a constraint because none of the possible values from the constraint fall in the region.

The correct choice is C.