5-6- التكامل باستخدام الكسور الجزئية

① $f(x) = \frac{2}{(x-5)(x-3)} = \frac{A}{x-5} + \frac{B}{x-3}$

$2 = A(x-3) + B(x-5)$

بوضع $x=5$

$2 = A(5-3) + B(0) \Rightarrow 2 = 2A \Rightarrow A = 1$

بوضع $x=3$

$2 = A(0) + B(3-5) \Rightarrow 2 = -2B \Rightarrow B = -1$

$\therefore f(x) = \frac{1}{x-5} + \frac{-1}{x-3}$

$\int f(x)\,dx = \int \frac{1}{x-5}\,dx + \int \frac{-1}{x-3}\,dx$

$= \ln|x-5| - \ln|x-3| + c$

② $f(x) = \frac{1}{x^2+2x} = \frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$1 = A(x+2) + Bx$

بوضع $x=0$

$1 = A(0+2) \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

بوضع $x=-2$

$1 = A(0) + B(-2) \Rightarrow 1 = -2B \Rightarrow B = \frac{1}{-2}$

$f(x) = \frac{\frac{1}{2}}{x} + \frac{\frac{-1}{2}}{x+2}$

$\int f(x)\,dx = \int \frac{\frac{1}{2}}{x}\,dx + \int \frac{\frac{-1}{2}}{x+2}\,dx = \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + c$

③ $$f(x)=\frac{-x+10}{x^2+x-12}=\frac{-x+10}{(x-3)(x+4)}$$

$$=\frac{A}{x-3}+\frac{B}{x+4}=\frac{A(x+4)+B(x-3)}{(x-3)(x+4)}$$

$$-x+10=A(x+4)+B(x-3)$$

بفرض $x=3$

$$-3+10=A(3+4)+B(0) \Rightarrow 7=7A \Rightarrow A=1$$

بفرض $x=-4$

$$-(-4)+10=A(-4+4)+B(-4-3)$$

$$14=A(0)-7B \Rightarrow B=-2$$

$$f(x)=\frac{1}{x-3}+\frac{-2}{x+4}$$

$$\int f(x)\,dx=\int\frac{1}{x-3}\,dx+\int\frac{-2}{x+4}\,dx$$

$$=\ln|x-3|-2\ln|x+4|+C$$

④ $$f(x)=\frac{12}{x^3+2x^2-3x}=\frac{12}{x(x+3)(x-1)}$$

$$=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-1}$$

$$12=A(x+3)(x-1)+B(x)(x-1)+C(x)(x+3)$$

بفرض $x=0$

$$12=A(0+3)(0-1)+B(0)+C(0) \Rightarrow 12=-3A \Rightarrow A=-4$$

بفرض $x=1$

$$12=A(0)+B(0)+C(1) \Rightarrow C=12$$

بفرض $x=-3$

$$12=A(0)+B(-3)(-3-1)+C(0) \Rightarrow 12=12B \Rightarrow B=1$$

$$\int f(x)\,dx=\int\frac{-4}{x}\,dx+\int\frac{1}{x+3}\,dx+\int\frac{12}{x-1}\,dx$$

$$=-4\ln|x|+\ln|x+3|+12\ln|x-1|+C$$

⑤ $\int \frac{x+17}{2x^2+5x-3}dx \qquad \frac{x+17}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$

$$x+17 = A(2x-1) + B(x+3)$$

بوضع $x = +\frac{1}{2}$

$$\frac{1}{2}+17 = A(0) + B\left(\frac{1}{2}+3\right) \Rightarrow 17.5 = 3.5B \Rightarrow B = 5$$

بوضع $x = -3$

$$-3+17 = A(2(-3)-1) + B(-3+3) \Rightarrow 14 = A(-7) + B(0) \Rightarrow A = -2$$

$$\therefore \int \frac{x+17}{2x^2+5x-3}dx = \int \frac{-2}{x+3}dx + \int \frac{5}{2x-1}dx$$

$$= -2\ln|x+3| + 5\ln|2x-1| + C$$

⑥ $\int \frac{-6x+25}{x^3-6x^2+9x}dx$

$$\frac{-6x+25}{x(x^2-6x+9)} = \frac{-6x+25}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$-6x+25 = A(x-3)^2 + B(x)(x-3) + C(x)$$

بوضع $x=0$

$$25 = A(-3)^2 + B(0) + C(0) \Rightarrow A = \frac{25}{9}$$

بوضع $x=3$

$$-18+25 = A(0) + B(0) + C(3) \Rightarrow C = \frac{7}{3}$$

بوضع $x=1$

$$-6+25 = \frac{25}{9}(1-3)^2 + B(1)(1-3) + \frac{7}{3}(1) \Rightarrow B = \frac{25}{9}$$

$$\int \frac{-6x+25}{x^3-6x+9x}dx = \int \frac{\frac{25}{9}}{x}dx + \int \frac{\frac{25}{9}}{x-3}dx + \int \frac{\frac{7}{3}}{(x-3)^2}dx$$

$$= \frac{25}{9}\ln|x| + \frac{25}{9}\ln|x-3| + \frac{7}{3}\,\frac{-1}{x-3} + C$$

⑦ $\int \frac{3x^2-4x+3}{x^3-3x^2}\,dx$

$$\frac{3x^2-4x+3}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

$$3x^2-4x+3 = A(x)(x-3) + B(x-3) + C(x^2)$$

$x=0$ نعوض

$$3 = A(0) + B(-3) + C(0)$$

$B = -1$

$x=3$ نعوض

$$18 = A(0) + B(0) + C(9)$$

$C = 2$

$x=1$ نعوض

$$2 = A(1)(-2) + (-1)(-2) + 2(1)$$

$$-2 = -2A \Rightarrow \quad A = 1$$

$$I = \int \frac{1}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{2}{x-3}\,dx$$

$$= \ln|x| + \frac{1}{x} + 2\ln|x-3| + c$$

⑧ $\int \frac{x^2+3x+2}{(x-3)^2}\,dx$

$(x-3)^2 = x^2-6x+9$

$$\begin{array}{r|l} & 1 \\ \hline x^2-6x+9 & x^2+3x+2 \\ & -\left(x^2-6x+9\right) \\ \hline & 0 \quad 9x-7 \end{array}$$

$I = \int 1\,dx + \int \frac{9x-7}{(x-3)^2}$

$\frac{9x-7}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$

$9x-7 = A(x-3) + B$

$x=3$ نعوض

$20 = A(0) + B \Rightarrow B = 20$

$x=1$ نعوض

$2 = A(-2) + 20 \Rightarrow A = 9$

$I = \int 1\,dx + \int \frac{9}{x-3}\,dx + \int \frac{20}{(x-3)^2}\,dx$

$= x + 9\ln|x-3| - \frac{20}{x-3} + c$

(9) $\int \frac{2x^2+x+3}{x^2-1}\,dx$

$$\begin{array}{r|l} & 2 \\ \hline x^2-1 & 2x^2+x+3 \\ & \ominus 2x^2 \quad \oplus -2 \\ \hline & 0 \quad x+5 \end{array}$$

$$I = \int 2\,dx + \int \frac{x+5}{x^2-1}$$

$$\frac{x+5}{x^2-1} = \frac{x+5}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$x+5 = A(x+1) + B(x-1)$$

بوضع $x=1$

$$6 = A(2) + B(0) \Rightarrow A = 3$$

بوضع $x=-1$

$$4 = A(0) + B(-2) \Rightarrow B = -2$$

$$I = \int 2\,dx + \int \frac{3}{x-1}\,dx + \int \frac{-2}{x+1}\,dx$$

$$= 2x + 3\ln|x-1| - 2\ln|x+1| + c$$

(10) $\displaystyle\int \frac{x^3-2}{x^2+x}\,dx$

$$
\begin{array}{r|l}
 & x \quad -1 \\
\hline
x^2+x & x^3 \qquad\qquad -2 \\
 & -\;(x^3 \ominus x^2) \\
\cline{2-2}
 & 0 \;\; -x^2 \qquad -2 \\
 & -\;(\oplus x^2 \oplus x) \\
\cline{2-2}
 & \quad 0 \quad x \; -2
\end{array}
$$

$$\frac{x^3-2}{x^2+x} = x-1 + \frac{x-2}{x^2+x}$$

$$\frac{x-2}{x^2+x} = \frac{x-2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$x-2 = A(x+1) + B(x)$$

بفرض $x=0$

$$-2 = A(1) + B(0) \Rightarrow A = -2$$

بفرض $x=-1$

$$-3 = A(0) + B(-1) \Rightarrow B = 3$$

$$\int \frac{x^3-2}{x^2+x}\,dx = \int x-1\,dx + \int \frac{-2}{x}\,dx + \int \frac{3}{x+1}\,dx$$

$$= \frac{x^2}{2} - x - 2\ln|x| + 3\ln|x+1| + c$$

(11) $\displaystyle\int \frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1}\,dx$

$$\begin{array}{r|l} & x^2 \\ \hline x^2-2x+1 & x^4-2x^3+x^2+2x-1 \\ & \ominus x^4 \oplus{-2x^3} \ominus{+x^2} \\ \hline & 0 \quad 0 \quad 0 \quad 2x-1 \end{array}$$

$$\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} = x^2 + \frac{2x-1}{x^2-2x+1}$$

$$\frac{2x-1}{x^2-2x+1} = \frac{2x-1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$2x-1 = A(x-1) + B$$

بوضع $x=1$

$$1 = A(0) + B \Rightarrow B = 1$$

بوضع $x=0$

$$-1 = A(-1) + 1 \Rightarrow A = 2$$

$$I = \int x^2\,dx + \int \frac{2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \frac{x^3}{3} + 2\ln|x-1| - \frac{1}{x-1} + C$$

(12) $$\int \frac{2x^4-5x^3-7x^2+32x-28}{x^3-2x^2-4x+8}$$

$$\begin{array}{r|l}
 & 2x \quad -1 \\
\hline
x^3-2x^2-4x+8 & 2x^4-5x^3-7x^2+32x-28 \\
 & \ominus 2x^4 \overset{\oplus}{-} 4x^3 \overset{\oplus}{-} 8x^2 \overset{\ominus}{+} 16x \\
\hline
 & 0 \quad -x^3+x^2+16x-28 \\
 & \overset{\oplus}{-}x^3 \overset{\ominus}{+} 2x^2 \overset{\ominus}{+} 4x \overset{\oplus}{-} 8 \\
\hline
 & 0 \quad -x^2+12x-20
\end{array}$$

$$f(x) = 2x-1 + \frac{-x^2+12x-20}{x^3-2x^2-4x+8}$$

$$= 2x-1 + \frac{-x^2+12x-20}{(x+2)(x-2)^2} = 2x-1$$

$$\frac{-x^2+12x-20}{(x+2)(x-2)^2} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$-x^2+12x-20 = A(x-2)^2 + B(x+2)(x-2) + C(x+2)$$

$0 = A(0) + B(0) + C(4) \Rightarrow C = 0 \quad \Leftarrow$ بجعل $x=2$

$-48 = A(16) + B(0) + C(0) \Rightarrow A = -3$ بجعل $x=-2$

بجعل $x=0$

$-20 = (-3)(4) + B(-4) + 0 \Rightarrow B = 2$

$$I = \int 2x-1\,dx - 3\int \frac{1}{x+2}\,dx + 2\int \frac{1}{x-2}\,dx + 0$$

$$= x^2 - x - 3\ln|x+2| + 2\ln|x-2| + C$$