7.3 طرائق تكامل الدوال المثلثية Trigonometric Techniques of Integration.

الحالة 1:

$$\int \sin^m x \cdot \cos^n x \, dx$$

حيث أن m أو n أحدهما عدد فردي

الدالة ذات القوة = u
الزوجية أو نسبة (البسط أو المقام)

مهم

ملا حظة هامة جداً.

يجب عدم فرض القوة مع الدالة

مثال:

$$\int \sin^3 x \cdot \cos x \, dx$$

$u = \sin x$

$u \neq \sin^2 x$

حل مثال 3.1 ص 499

$$\int \underbrace{\cos^4 x}_{u} \cdot \sin x \, dx$$

بدون القوة

$u = \cos^5 x$

$du = -\sin x \, dx$

$dx = \frac{du}{-\sin x}$

$$\Rightarrow \int u^4 \cdot \sin x \cdot \frac{du}{-\sin x}$$

$$\Rightarrow -\int u^4 \, du = -\frac{u^5}{5} + C$$

#

الجواب

$$= -\frac{\cos^5 x}{5} + C$$

حل سؤال 1 ص 507

2) $\int \cos x \cdot \sin^4 x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$dx = \frac{du}{\cos x}$

$\Rightarrow \int \cos x \cdot u^4 \frac{du}{\cos x}$

$\int u^4 du = \frac{\sin^5}{5} + C$ #

حل سؤال 2 ص 507

$\int \cos^3 x \cdot \underbrace{\sin x}_{u} \, dx$ ④

$u = \sin x$

$du = \cos x \, dx$

$dx = \frac{du}{\cos x}$

$\Rightarrow \int \cos^{2}\!\not{3} x \cdot u^4 \frac{du}{\cos x}$

$\cos^2 x = 1 - \sin^2 x = 1 - u^2$

$\int (1-u^2) u^4 \, du \Rightarrow \int u^4 - u^6 \, du$ #

$= \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

بعض متطابقات فيثاغورس

متطابقات أساسية

1) $\cos^2 x + \sin^2 x = 1$

2) $\sec^2 x = 1 + \tan^2 x$

3) $\csc^2 x = 1 + \cot^2 x$

متطابقات نصف الزاوية

4) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

5) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

3

حل سؤال خارجي 1

$$\int \cos^3 x \sqrt{\sin x} \cdot dx$$

$\sqrt{\sin x} \leftarrow u$ ذات قوة فردية.

$$\begin{cases} u = \sin x \\ du = \cos x\, dx \\ dx = \frac{du}{\cos x} \end{cases}$$

$$\Rightarrow \int \cos^2 x \sqrt{u} \cdot \frac{du}{\cos x} \Rightarrow \int \cos^2 x \cdot \sqrt{u}\, du$$

$\cos^2 x = 1 - u^2$

$$\Rightarrow \int (1-u^2)\sqrt{u}\, du \Rightarrow \int u^{\frac{1}{2}} + u^{\frac{5}{2}}\, du$$

$$= \frac{2u^{\frac{3}{2}}}{3} + \frac{2u^{\frac{7}{2}}}{7} + C = 2\left(\frac{\sin^{\frac{3}{2}} x}{3} + \frac{\sin^{\frac{7}{2}} x}{7}\right) + C \quad \#$$

س خارجي 2

$$\int \sin(2x) \cdot \cos^3(2x)\, dx$$

$u = \cos 2x$

$du = -2\sin 2x\, dx$

$dx = \frac{du}{-2\sin 2x}$

$$\Rightarrow \int \sin(2x) \cdot u^3 \cdot \frac{du}{-2\sin(2x)}$$

$$\Rightarrow -\frac{1}{2}\int u^3\, du = -\frac{1}{2}\left(\frac{u^4}{4}\right) + C$$

$$\Rightarrow -\frac{1}{8}u^4 + C = -\frac{(\cos 2x)^4}{8} + C$$

الجواب #

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الحالة 2 :

إذا كانت كلا m و n زوجيان

* في هذه الحالة نستخدم متطابقات الدوال المثلثية "نصف الكمية" :

1 $\sin^2 x = \frac{1}{2}(1-\cos(2x))$

2 $\cos^2 x = \frac{1}{2}(1+\cos(2x))$

حل مثال 3.4 ص 501

$$\int \sin^2(2x)\,dx$$

$$\sin^2 x = \frac{1}{2}(1-\cos(2x))$$

$$\int \frac{1}{2}(1-\cos 4x)\,dx \Rightarrow = \frac{1}{2}\left(x - \frac{\sin 4x}{4}\right)+C \quad \#$$

بعض التكاملات المثلثية

$$\int \cos x\,dx = \sin x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

[5] $\int \cos^4 x \, dx$ حل مثال 501 ص 3.5

$$\Rightarrow \int (\cos^2 x)^2 \, dx \Rightarrow \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx$$

متطابقة

$$\Rightarrow \frac{1}{4}\int (1+\cos 2x)^2 \, dx \Rightarrow \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\, dx$$

فك التربيع .

متطابقة مرة اخرى ولكن هذه المرة الزاوية $2x$

$$\Rightarrow \frac{1}{4}\int \left(1+2\cos 2x+\left(\frac{1}{2}(1+\cos 4x)\right)\right) dx$$

$$\Rightarrow \frac{1}{4}\int \left(1+2\cos 2x+\frac{1}{2}+\frac{\cos 4x}{2}\right) dx$$

$$\Rightarrow \frac{1}{4}\left[\frac{3}{2}x+\frac{2\sin 2x}{2}+\frac{1}{2}\cdot\frac{\sin 4x}{4}\right]+C$$

$$\Rightarrow \frac{3}{8}x+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+C$$

الحل

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حل سؤال 17 ص 507

$$\int \underbrace{\sin^2 x}_{\text{متطابقة}} \cdot \underbrace{\cos^2 x}_{\text{متطابقة}}\, dx$$

$$\Rightarrow \int \left(\frac{1}{2}(1-\cos 2x)\right)\left(\frac{1}{2}(1+\cos 2x)\right) dx$$

ملاحظة:
$(a-b)(a+b) = a^2 - b^2$

$$\Rightarrow \int \frac{1}{4}(1-\cos^2 2x)\, dx$$

متطابقة ولكن الزاوية $2x$ فتصبح $4x$.

$$\Rightarrow \frac{1}{4}\int 1 - \frac{1}{2}(1+\cos 4x)\, dx \Rightarrow \frac{1}{4}\left(x - \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right)\right) + C$$

الجواب

$$\Rightarrow \frac{1}{4}\left[\frac{x}{2} + \frac{\sin 4x}{8}\right] + C$$

8- الصيغة الثانية

$$\int \cot^m x \csc^n x\, dx \text{ OR } \int \tan^m x \sec^n x\, dx$$

الحالة الأولى:- إذا كانت قوة الـ Sec عدد زوجي

$u = \tan x$

حل مثال خارجي [1]

$$\int \tan^3 x \sec^2 x\, dx$$

$u = \tan x$

$du = \sec^2 x\, dx$

$dx = \frac{du}{\sec^2 x}$

$$\int u^3 \sec^2 x \frac{du}{\sec^2 x}$$

$$\Rightarrow \int u^3\, du \Rightarrow \frac{u^4}{4} + C$$

الجواب

$$= \frac{\tan^4 x}{4} + C$$

7

حل سؤال خارجي 2 ص٨

$$\int \sqrt{\tan x}\ \sec^4 x\ dx$$

u

$u = \tan x$

$du = \sec^2 x\, dx$

$dx = \frac{du}{\sec^2 x}$

$$\Rightarrow \int \sqrt{u}\ \sec^{\frac{2}{4}} x\ \frac{du}{\sec^2 x}$$

$$\Rightarrow \int \sqrt{u}\ \sec^2 x\ du \Rightarrow \int \sqrt{u}\,(1+u^2)\,du$$

$\sec^2 x = 1 + \tan^2 x \Rightarrow 1 + u^2$

#الجواب

$$\Rightarrow \frac{2}{7} u^{\frac{7}{2}} + \frac{2}{3} u^{\frac{3}{2}} + C \Rightarrow \frac{2}{7} \tan^{\frac{7}{2}} x + \frac{2}{3} \tan^{\frac{3}{2}} x + C$$

الحالة 2 ص٨- إذا كانت قوة $\tan x$ عدد فردي

$u = \sec x$

حل سؤال خارجي ص٨

$$\int \tan x\ \sec^5 x\ dx$$

u

$u = \sec x$

$du = \sec x \tan x\, dx$

$dx = \frac{du}{\sec x \tan x}$

u

$$\Rightarrow \int \tan x\ u^{\frac{4}{5}}\ \frac{du}{u \tan x} \Rightarrow \int u^4\, du$$

$$\Rightarrow \frac{\sec^5 x}{5} + C$$

#الجواب.

$$dx = \frac{du}{u \tan x}$$

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حل سؤال خارجي ٨ ج

$$\int \tan^3 x \sec x \, dx$$

u

$u = \sec x$

$du = \sec x \tan x \, dx$

$dx = \frac{du}{\sec x \tan x}$

$$\int \tan^2 x \cdot u \cdot \frac{du}{\tan x \sec x}$$

$$\Rightarrow \int \tan^2 x \, du \Rightarrow \int u^2 - 1 \, du$$

$\tan^2 x = \sec^2 x - 1 \Rightarrow u^2 - 1$

$$\Rightarrow \frac{u^3}{3} - u + C \Rightarrow \frac{1}{3}\sec^3 x - \sec x + C$$

#

حل سؤال خارجي ٨ ج

$$\int \cot x \cdot \csc^3 x \, dx$$

u

$u = \csc x$

$du = -\csc x \cdot \cot x \, dx$

$dx = \frac{du}{-\csc x \cdot \cot x}$

$$\Rightarrow \int \cot x \cdot u^2 \cdot \frac{du}{-\csc x \cdot \cot x}$$

$$\Rightarrow -\int u^2 \, du$$

$$\Rightarrow -\frac{u^3}{3} + C \Rightarrow -\frac{\csc^3 x}{3} + C$$

الجواب

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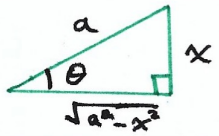
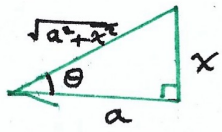
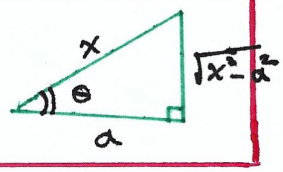
9

التكامل بالتعويضات المثلثية

الصيغة	التعويض	المثلث
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	

أكتب التعويض المناسب لإيجاد كل من التكاملات التالية:

1) $\int \frac{x^2}{\sqrt{x^2-9}}\,dx \Rightarrow$ $x = 3\sec\theta$

2) $\int \frac{x^2}{\sqrt{1-x^2}}\,dx \Rightarrow$ $x = \sin\theta$

3) $\int \frac{1}{\sqrt{3+x^2}}\,dx \Rightarrow$ $x = \sqrt{3}\tan\theta$

4) $\int \frac{x^2}{\sqrt{4-9x^2}}\,dx \Rightarrow \int \frac{x^2}{\sqrt{9(\frac{4}{9}-x^2)}}\,dx \Rightarrow a = \sqrt{\frac{4}{9}} = \frac{2}{3}$

$\Rightarrow x = \frac{2}{3}\tan\theta$

5) $\int \frac{x}{\sqrt{x^2+4x+5}}\,dx \Rightarrow$ (إكمال مربع) $\int \frac{x}{\sqrt{x^2+4x+4-4+5}}\,dx$

$\Rightarrow \int \frac{x}{\sqrt{(x^2+4x+4)+1}}\,dx \Rightarrow \int \frac{x}{\sqrt{(x+2)^2+1}}\,dx \Rightarrow x+2 = \tan\theta$

$x = \tan\theta - 2$

Mode 5,3

$\dot{x}$

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حل مثال خارجي 8:

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$a = 1$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\, d\theta$$

$\sqrt{1-\sin^2\theta} = \cos^2\theta$

$$\Rightarrow \int \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta\, d\theta \Rightarrow \int \sin^2\theta\, d\theta$$

متطابقة نصف الزاوية

$$\Rightarrow \int \frac{1}{2}(1-\cos 2\theta)\, d\theta \Rightarrow \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

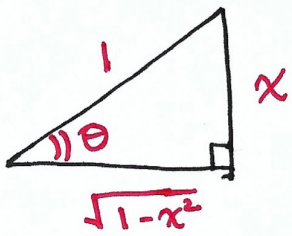
$\sin 2\theta = 2\sin\theta\cos\theta$

$$\Rightarrow \frac{1}{2}\left(\theta - \frac{2\sin\theta\cos\theta}{2}\right) + C$$

يجب التخلص من θ من خلال المثلث.

$x = \sin\theta \Rightarrow \boxed{\theta = \sin^{-1} x}$ من الفرض نجد θ

ومن المثلث نجد الـ $\sin\theta$ والـ $\cos\theta$



$$= \frac{1}{2}\left(\sin^{-1} x - x\sqrt{1-x^2}\right) + C$$

$x = \sin\theta = \frac{opp}{hypo}$

$\frac{\sqrt{1-x^2}}{1} = \cos\theta = \frac{adj}{hypo}$

الحل

II

حل سؤال خارجي 8)

$$\int \frac{1}{\sqrt{16+x^2}}\,dx$$

$a = 4$

$x = 4\tan\theta$

$dx = 4\sec^2\theta\,d\theta$

$$\Rightarrow \int \frac{1}{\sqrt{16+(4\tan\theta)^2}}\,4\sec^2\theta\,d\theta$$

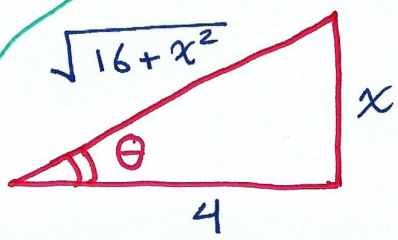
$$\Rightarrow \int \frac{1}{\sqrt{16+16\tan^2\theta}}\,4\sec^2\theta\,d\theta \Rightarrow \int \frac{1}{4\sqrt{1+\tan^2\theta}}\cdot\frac{4\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$\Rightarrow \int \frac{1}{\sqrt{\sec^2\theta}}\sec^2\theta\,d\theta \Rightarrow \int \frac{\sec^2\theta}{\sec\theta}\,d\theta$$

حفظ $\Rightarrow \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$

من المثلث

$$\sec\theta = \frac{\text{وتر}}{\text{مجاور}} = \frac{\sqrt{16+x^2}}{4}$$



من الفرض

$x = 4\tan\theta$

$$\tan\theta = \frac{x}{4} = \frac{\text{مقابل}}{\text{مجاور}}$$

$$\Rightarrow \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$$

الجواب

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حل سؤال خارجي ٥٠

$$\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$$

$a=1$

$x = \sin\theta$

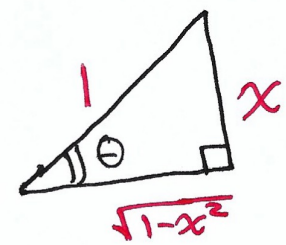
$dx = \cos\theta\, d\theta$

$$\Rightarrow \int \frac{1}{\sin^2\theta\sqrt{1-\sin^2\theta}}\cdot\cos\theta\, d\theta \Rightarrow \int \frac{\cancel{\cos\theta}}{\sin^2\theta\sqrt{\cancel{\cos^2\theta}}}\,d\theta$$

متطابقة

$$\Rightarrow \int \frac{1}{\sin^2\theta}\,d\theta \Rightarrow \int \csc^2\theta\, d\theta = -\cot\theta + C$$

$\csc^2\theta$



$$\frac{\sqrt{1-x^2}}{x} = \cot\theta = \frac{adj}{opp} \Leftarrow$$

$$\Rightarrow -\frac{\sqrt{1-x^2}}{x} + C$$

الجواب #

للتذكير ٥٠

$\sin\theta = \frac{opp}{hypo}$ مقلوب $\csc\theta = \frac{hypo}{opp}$

$\cos\theta = \frac{adj}{hypo}$ $\sec\theta = \frac{hypo}{adj}$

$\tan\theta = \frac{opp}{adj}$ $\cot\theta = \frac{adj}{opp}$

حل سؤال خارجي 08

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$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx$$

$a = 2$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$$\Rightarrow \int \frac{\sqrt{4-4\sin^2\theta}}{4\sin^2\theta}\cdot 2\cos\theta\,d\theta$$

مشتركة

$$\Rightarrow \int \frac{\sqrt{4(1-\sin^2\theta)}\cdot 2\cos\theta}{4\sin^2\theta}\,d\theta \Rightarrow \frac{4}{4}\int \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}\cdot\cos\theta\,d\theta$$

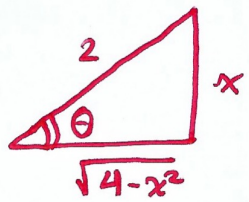
$$\Rightarrow \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta \Rightarrow \int \cot^2\theta\,d\theta \Rightarrow \int(\csc^2\theta - 1)\,d\theta$$

$$\Rightarrow = -\cot\theta - \theta + C$$

$$\Rightarrow -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

الجواب #.

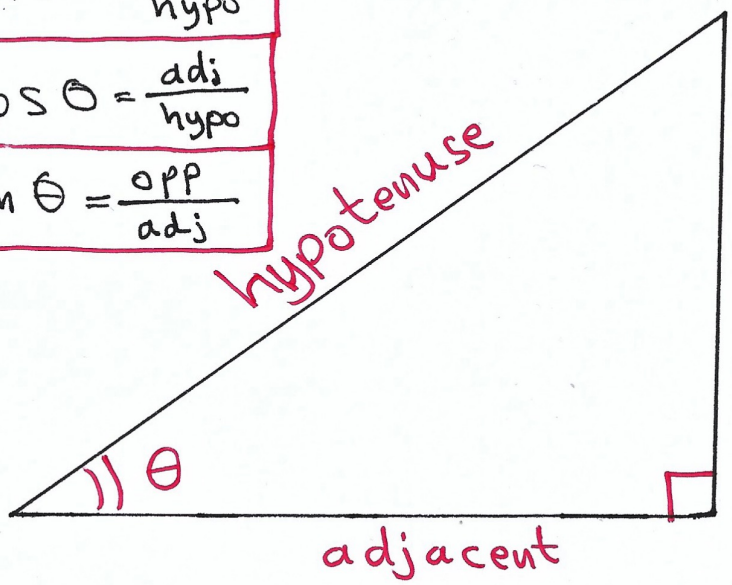
$\frac{\sqrt{4-x^2}}{x} = \cot\theta = \frac{adj}{opp}$



$\theta = \sin^{-1}\left(\frac{x}{2}\right)$ من الفرض

* $\sin\theta = \frac{opp}{hypo}$
* $\cos\theta = \frac{adj}{hypo}$
* $\tan\theta = \frac{opp}{adj}$

* $\csc\theta = \frac{hypo}{opp}$
* $\sec\theta = \frac{hypo}{adj}$
* $\cot\theta = \frac{adj}{opp}$



14

حل سؤال خارجي ق

$$\int \frac{1}{x\sqrt{1+x^2}}\,dx$$

$a = 1$

$x = \tan\theta$

$dx = \sec^2\theta$

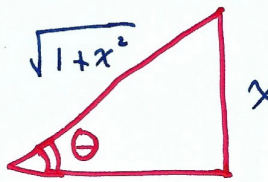
$$\Rightarrow \int \frac{1}{\tan\theta\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta$$

$\sec^2\theta =$ متطابقة

$$\Rightarrow \int \frac{\sec^2\theta}{\tan\theta \cdot \sec\theta}\,d\theta \Rightarrow \int \frac{\sec\theta}{\tan\theta}\,d\theta \Rightarrow \int \csc\theta\,d\theta$$

حفظ

$$\Rightarrow \ln|\csc\theta - \cot\theta| + C$$



$\csc\theta = \frac{hypo}{opp}$ 1 $\cot\theta = \frac{adj}{opp}$

\# الجواب

$$\ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + C$$

حل سؤال 43 ص 507

$$\int \frac{x}{\sqrt{x^2+2x+10}}\,dx$$

اكمال مربع

$x^2 + 2x + 10$

$x^2 + 2x + 1 + 10 - 1$

Mode 5, 3

$= (x+1)^2 + 9$

$$\int \frac{x}{\sqrt{(x+1)^2+9}}\,dx$$

نفرض أن

$a^2 = 9 \Rightarrow a = 3$

$x + 1 = 3\tan\theta$

$x = 3\tan\theta - 1$

$dx = 3\sec^2\theta\,d\theta$

$$\Rightarrow \int \frac{3\tan\theta - 1}{\sqrt{(3\tan\theta\cdot\,)^2+9}} \cdot 3\sec^2\theta\,d\theta$$

$$\Rightarrow \int \frac{3\tan\theta - 1}{\sqrt{9\tan^2\theta+9}} \cdot 3\sec^2\theta\,d\theta \Rightarrow \int \frac{\tan\theta - 1}{3\sqrt{\tan^2\theta+1}} \cdot 3\sec^2\theta\,d\theta$$

3

$\sec^2\theta$

التكملة في الصفحة التالية

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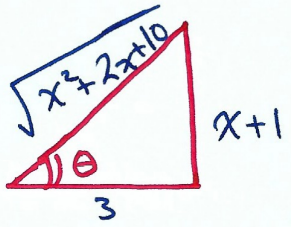
$$\int \frac{(3\tan\theta - 1)}{\sec\theta} \cdot \sec^2\theta \cdot d\theta$$

$$\Rightarrow \int 3\tan\theta\sec\theta - \sec\theta \, d\theta$$



$$\Rightarrow 3\sec\theta - \ln|\sec\theta + \tan\theta| + C$$

$$\Rightarrow 3\left(\frac{\sqrt{x^2+2x+10}}{3}\right) - \ln\left|\frac{\sqrt{x^2+2x+10}}{3} - \frac{x+1}{3}\right| + C$$

$\frac{\text{hypo}}{\text{adj}} = \sec\theta = \frac{\sqrt{x^2+2x+10}}{3}$

$\frac{\text{opp}}{\text{adj}} = \tan\theta = \frac{x+1}{3}$

حل سؤال خارجي

$$\int \frac{\sqrt{x^2-1}}{x} \, dx$$

$a = 1$

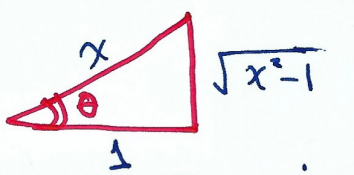
$x = \sec\theta$

$dx = \sec\theta\tan\theta \, d\theta$

$$\Rightarrow \int \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} \cdot \sec\theta\tan\theta \, d\theta$$

$$\Rightarrow \int \sqrt{\tan^2} \cdot \tan\theta \, d\theta \Rightarrow \int \tan^2\theta \, d\theta \Rightarrow \int (\sec^2\theta - 1) \, d\theta$$

$$\Rightarrow \tan\theta - \theta + C$$



$\theta = \sec^{-1}x$ من الفرض

$\frac{\text{opp}}{\text{adj}} = \tan\theta = \frac{\sqrt{x^2-1}}{1}$

$$\Rightarrow = \sqrt{x^2-1} - \sec^{-1}x + C$$

الجواب