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3. Electric Potential, Chapter 23, from University Physics with Modern Physics, 2e by Bauer and Westfall ©2014
4. Capacitors, Chapter 24, from University Physics with Modern Physics, 2e by Bauer and Westfall ©2014
5. Current and Resistance, Chapter 25, from University Physics with Modern Physics, 2e by Bauer and Westfall ©2014
6. Direct Current Circuits, Chapter 26, from University Physics with Modern Physics, 2e by Bauer and Westfall ©2014
7. Magnetism, Chapter 27, from University Physics with Modern Physics, 2e by Bauer and Westfall ©2014
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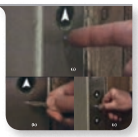

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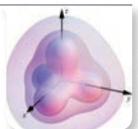

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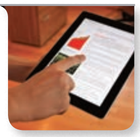

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PART 5 ELECTRICITY

1 Electrostatics

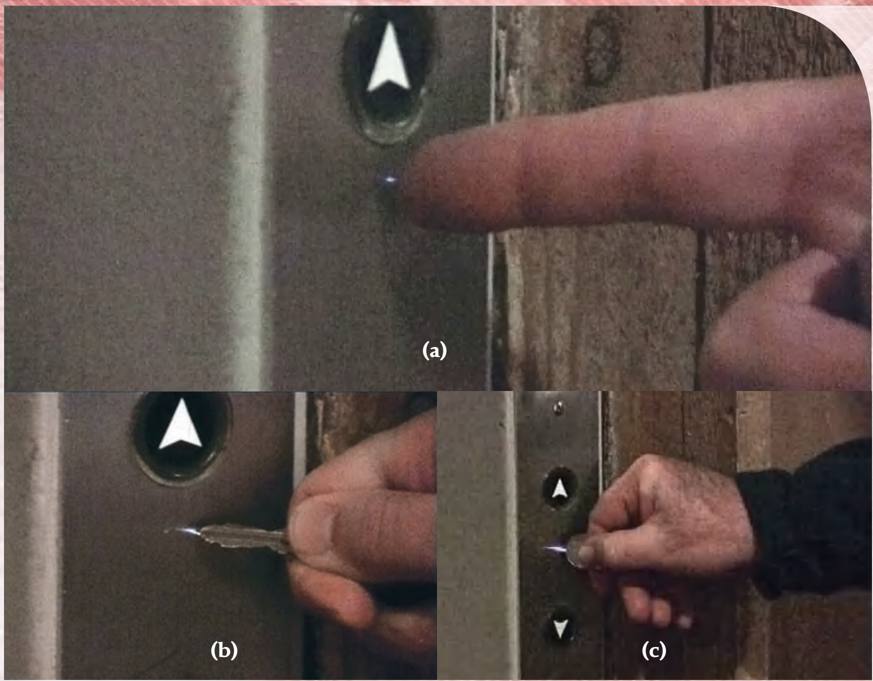

FIGURE 1.1 (a) A spark due to static electricity occurs between a person's finger and a metal surface near an elevator button. (b) and (c) Similar sparks are generated when the person holds a metal object like a car key or a coin, but are painless because the spark forms between the metal surface and the metal object.

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Many people think of static electricity as the annoying spark that occurs when they reach for a metal object like a doorknob on a dry day, after they have been walking on a carpet (Figure 1.1). In fact, many electronics manufacturers place small metal plates on equipment so that users can discharge any spark on the plate and not damage the more sensitive parts of the equipment. However, static electricity is more than just an occasional annoyance; it is the starting point for any study of electricity and magnetism, forces that have changed human society as radically as anything since the discovery of fire or the wheel.

In this chapter, we examine the properties of electric charge. A moving electric charge gives rise to a separate phenomenon, called *magnetism*, which is covered in later chapters. Here we look at charged objects that are not moving–hence the term *electrostatics*. All objects have charge, since charged particles make up atoms and molecules. We often don't notice the effects of electrical charge because most objects are electrically neutral. The forces that hold atoms together and that keep objects separate even when they're in contact, are all electric in nature.

WHAT WE WILL LEARN

- Electric charge gives rise to a force between charged particles or objects.
- Electricity and magnetism together make up the electromagnetic force, one of the four fundamental forces of nature.
- There are two kinds of electric charge, positive and negative. Like charges repel, and unlike charges attract.
- Electric charge is quantized, meaning that it occurs only in integral multiples of a smallest elementary quantity. Electric charge is also conserved.
- Most materials around us are electrically neutral.
- The electron is an elementary particle, and its charge is the smallest observable quantity of electric charge.
- Insulators conduct electricity poorly or not at all. Conductors conduct electricity well but not perfectly–some energy losses occur.
- Semiconductors can be made to change between a conducting state and a nonconducting state.
- Superconductors conduct electricity perfectly.
- Objects can be charged directly by contact or indirectly by induction.
- The force that two stationary electric charges exert on each other is proportional to the product of the charges and varies as the inverse square of the distance between the two charges.
- Electrostatic forces between particles can be added as vectors by the process of superposition.

1.1 Electromagnetism

FIGURE 1.2 Lightning strikes over a city.

Perhaps no mystery puzzled ancient civilizations more than electricity, which they observed primarily in the form of lightning strikes (Figure 1.2). The destructive force inherent in lightning, which could set objects on fire and kill people and animals, puzzled people because they did not understand what caused it or where the lightening came from.

The ancient Greeks knew that if you rubbed a piece of amber with a piece of cloth, you could attract small, light objects with the amber. We now know that rubbing amber with a cloth transfers negatively charged particles called *electrons* from the cloth to the amber. (The words *electron* and *electricity* derive from the Greek word for amber.) Lightning also consists of a flow of electrons. The early Greeks and others also knew about naturally occurring magnetic objects called *lodestones*, which were found in deposits of magnetite, a mineral consisting of iron oxide. These objects were used to construct compasses as early as 300 BC.

The relationship between electricity and magnetism was not understood until the middle of the 19th century. The following chapters will reveal how electricity and magnetism can be unified into a common framework called *electromagnetism*. However, unification of forces does not stop there. During the early part of the 20th century, two more fundamental forces were discovered: the weak force, which operates in beta decay (in which an electron and a neutrino are spontaneously emitted from certain types of nuclei), and the strong force, which acts inside the atomic nucleus. Currently, the electromagnetic and weak forces are viewed as two aspects of the electroweak force (Figure 1.3). For the phenomena discussed in this and the following chapters, this electroweak unification has no influence; it becomes important in the highest-energy particle collisions. Because the energy scale for the electroweak unification is so high, most textbooks continue to speak of four fundamental forces: gravitational, electromagnetic, weak, and strong.

Today, a large number of physicists believe that the electroweak force and the strong force can also be unified, that is, described in a common framework. Several theories propose ways to accomplish this, but so far experimental evidence is missing. Interestingly, the force that has been known longer than any of the other fundamental forces, gravity, seems to be hardest to shoehorn into a unified framework with the other fundamental forces. Quantum gravity, supersymmetry, and string theory are current foci of cutting-edge physics research in which theorists are attempting to construct this grand unification and discover the (hubristically

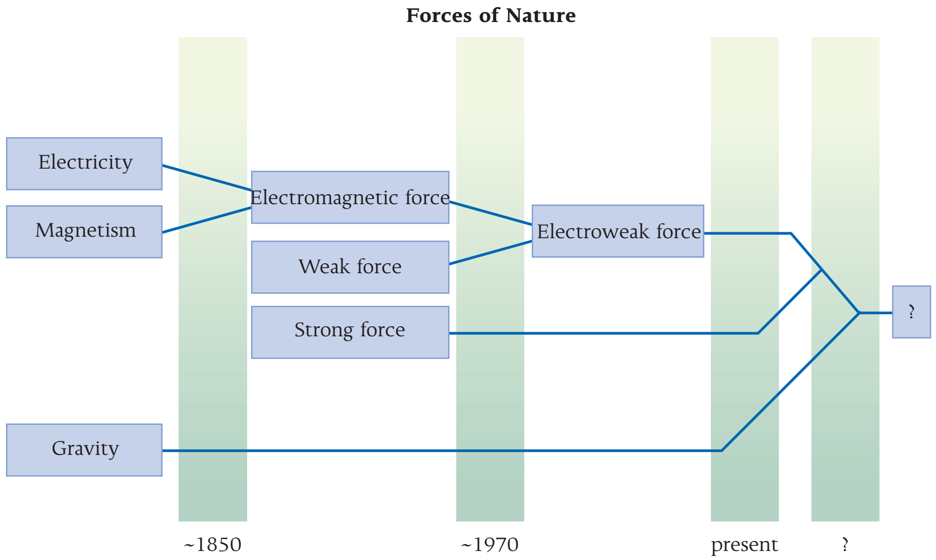


FIGURE 1.3 The history of the unification of fundamental forces.

named) Theory of Everything. They are mainly guided by symmetry principles and the conviction that nature must be elegant and simple.

In this chapter, we consider electric charge, how materials react to electric charge, static electricity, and the forces resulting from electric charges. **Electrostatics** covers situations where charges stay in place and do not move.

1.2 Electric Charge

Let's look a little deeper into the cause of the electric sparks that you occasionally receive on a dry winter day if you walk across a carpet and then touch a metal doorknob. (Electrostatic sparks have even ignited gas fumes while someone is filling the tank at a gas station. This is not an urban legend; a few of these cases have been caught on gas station surveillance cameras.) The process that causes this sparking is called **charging.** Charging consists of the transfer of negatively charged particles, called **electrons,** from the atoms and molecules of the material of the carpet to the soles of your shoes. This charge can move relatively easily through your body, including your hands. The built-up electric charge discharges through the metal of the doorknob, creating a spark.

The two types of electric charge found in nature are **positive charge** and **negative charge.** Normally, objects around us do not seem to be charged; instead, they are electrically neutral. Neutral objects contain roughly equal numbers of positive and negative charges that largely cancel each other. Only when positive and negative charges are not balanced do we observe the effects of electric charge.

If you rub a glass rod with a cloth, the glass rod becomes charged and the cloth acquires a charge of the opposite sign. If you rub a plastic rod with fur, the rod and fur also become oppositely charged. If you bring two charged glass rods together, they repel each other. Similarly, if you bring two charged plastic rods together, they also repel each other. However, a charged glass rod and a charged plastic rod will attract each other. This difference arises because the glass rod and the plastic rod have opposite charge. These observations led to the following law:

Law of Electric Charges

Like charges repel and opposite charges attract.

The unit of electric charge is the **coulomb** (C), named after the French physicist Charles-Augustine de Coulomb (1736–1806). The coulomb is defined in terms of the SI unit for current, the ampere (A), named after another French physicist, André-Marie Ampère (1775–1836). The ampere can not be derived in terms of the other SI units: meter, kilogram, and second. Instead, the ampere is another fundamental SI unit. For this reason, the SI system of units is sometimes called *MKSA (meter-kilogram-second-ampere) system*. The charge unit is defined as

$$1\ \mathrm{C} = 1\ \mathrm{A\ s} \tag{1.1}$$

The definition of the ampere must wait until we discuss current in later chapters. However, we can define the magnitude of the coulomb by simply specifying the charge of a single electron:

$$q_\mathrm{e} = -e \tag{1.2}$$

where q_e is the charge and e has the (currently best accepted and experimentally measured) value

$$e = 1.602176565(35) \times 10^{-19}\ \mathrm{C} \tag{1.3}$$

(Usually it is enough to carry only the first two to four significant digits of this mantissa. We will use a value of 1.602 in this chapter, but you should keep in mind that equation 1.3 gives the full accuracy to which this charge has been measured.)

The charge of the electron is an intrinsic property of the electron, just like its mass. The charge of the **proton,** another basic particle of atoms, is exactly the same magnitude as that of the electron, only the proton's charge is positive:

$$q_\mathrm{p} = +e \tag{1.4}$$

The choice of which charge is positive and which charge is negative is arbitrary. The conventional choice of $q_\mathrm{e} < 0$ and $q_\mathrm{p} > 0$ is due to the American statesman, scientist, and inventor Benjamin Franklin (1706–1790), who pioneered studies of electricity.

One coulomb is an extremely large unit of charge. We'll see later in this chapter just how big it is when we investigate the magnitude of the forces of charges on each other. Units of μC (microcoulombs, 10^{-6} C), nC (nanocoulombs, 10^{-9} C), and pC (picocoulombs, 10^{-12} C) are commonly used.

Benjamin Franklin also proposed that charge is conserved. Charge is not created or destroyed, simply moved from one object to another.

Concept Check 1.1

How many electrons does it take to make 1.00 C of charge?

a) 1.60×10^{19}
b) 6.60×10^{19}
c) 3.20×10^{16}
d) 6.24×10^{18}
e) 6.66×10^{17}

Law of Charge Conservation

The total electric charge of an isolated system is conserved.

This law is the fourth conservation law we have encountered so far, the first three being the conservation laws for total energy, momentum, and angular momentum. Conservation laws are a common thread that runs throughout all of physics and thus throughout this book as well.

It is important to note that there is a conservation law for charge, but *not* for mass. Mass and energy are not independent of each other. What is sometimes described in introductory chemistry as conservation of mass is not an exact conservation law, but only an approximation used to keep track of the number of atoms in chemical reactions. (It is a good approximation to a large number of significant figures but not an exact law, like charge conservation.) Conservation of charge applies to all systems, from the macroscopic system of plastic rod and fur down to systems of subatomic particles.

Elementary Charge

Electric charge occurs only in integral multiples of a minimum size. This is expressed by saying that charge is **quantized.** The smallest observable unit of electric charge is the charge of the electron, which is -1.602×10^{-19} C (as defined in equation 1.3).

The fact that electric charge is quantized was verified in an ingenious experiment carried out in 1910 by American physicist Robert A. Millikan (1868–1953) and known as the *Millikan oil drop experiment* (Figure 1.4). In this experiment, oil drops were sprayed into a chamber where electrons were knocked out of the drops by some form of radiation, usually X-rays. The resulting positively charged drops were allowed to fall between two electrically charged plates. Adjusting the charge of the plates caused the drops to stop falling and allowed their charge to be measured. What Millikan observed was that charge was quantized rather than continuous. (A quantitative analysis of this experiment will be presented in Chapter 3 on electric potential.) That is, this experiment and its subsequent refinements established that charge comes only in integer multiples of the charge of an electron. In everyday experiences with electricity, we do not notice that charge is quantized because most electrical phenomena involve huge numbers of electrons.

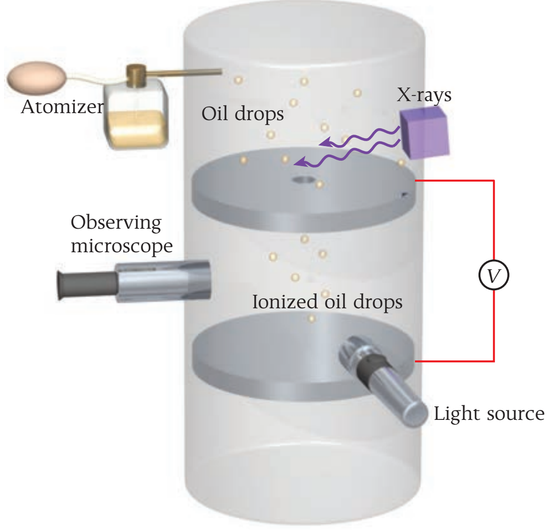


FIGURE 1.4 Schematic drawing of the Millikan oil drop experiment.

You studied earlier that matter is composed of atoms and that an atom consists of a nucleus containing charged protons and neutral neutrons. A schematic drawing of a carbon atom is shown in Figure 1.5. A carbon atom has six protons and (usually) six neutrons in its nucleus. This nucleus is surrounded by six electrons. Note that this drawing is not to scale. In the actual atom, the distance of the electrons from the nucleus is much larger (by a factor on the order of 10,000) than the size of the nucleus. In addition, the electrons are shown in circular orbits, which is also not quite correct. In Chapter 38, we'll see that the locations of electrons in the atom can be characterized only by probability distributions.

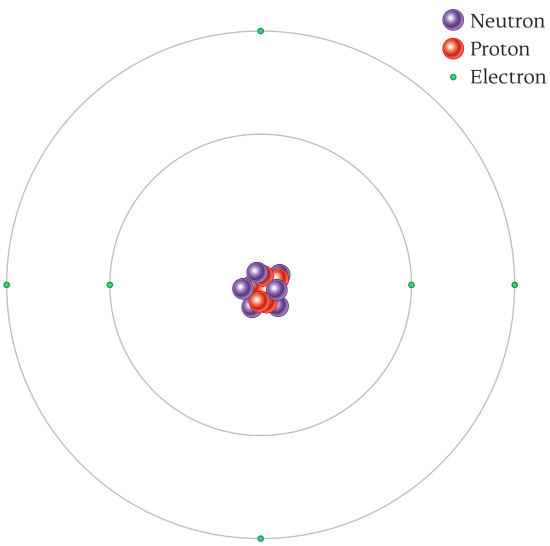


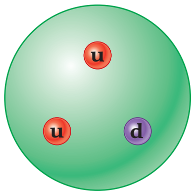
FIGURE 1.5 In a carbon atom, the nucleus contains six neutrons and six protons. The nucleus is surrounded by six electrons. Note that this drawing is schematic and not to scale.

As mentioned earlier, a proton has a positive charge with a magnitude that is *exactly* equal to the magnitude of the negative charge of an electron. In a neutral atom, the number of negatively charged electrons is equal to the number of positively charged protons. The mass of the electron is much smaller than the mass of the proton or the neutron. Therefore, most of the mass of an atom resides in the nucleus. Electrons can be removed from atoms relatively easily. For this reason, electrons are typically the carriers of electricity, rather than protons or atomic nuclei.

The electron is a fundamental particle and has no substructure: It is a point particle with zero radius (at least, according to current understanding). However, high-energy probes have been used to look inside the proton. A proton is composed of charged particles called *quarks*, held together by uncharged particles called *gluons*. Quarks have a charge of $\pm\frac{1}{3}$ or $\pm\frac{2}{3}$ times the charge of the electron. These fractionally charged particles cannot exist independently and have never been observed directly, despite numerous extensive searches. Just like the charge of an electron, the charges of quarks are intrinsic properties of these elementary particles.

A proton is composed of two *up quarks* (each with charge $+\frac{2}{3}e$) and one *down quark* (with charge $-\frac{1}{3}e$), giving the proton a charge of $q_p = (2)(+\frac{2}{3}e) + (1)(-\frac{1}{3}e) = +e$, as illustrated in Figure 1.6a. The electrically neutral neutron (hence the name!) is composed of an up quark and two down quarks, as shown in Figure 1.6b, so its charge is $q_n = (1)(+\frac{2}{3}e) + (2)(-\frac{1}{3}e) = 0$. There are also much more massive electron-like particles named *muon* and *tau*. But the basic fact remains that all of the matter in everyday experience is made up of electrons (with electrical charge $-e$), up and down quarks (with electrical charges $+\frac{2}{3}e$ and $-\frac{1}{3}e$, respectively), and gluons (zero charge).

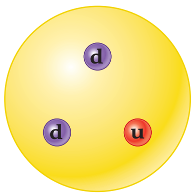
It is remarkable that the charges of the quarks inside a proton add up to *exactly* the same magnitude as the charge of the electron. This fact is still a puzzle, pointing to some deep symmetry in nature that is not yet understood.



Proton

$q_p = +\frac{2}{3}e + \frac{2}{3}e - \frac{1}{3}e = +e$

(a)



Neutron

$q_n = +\frac{2}{3}e - \frac{1}{3}e - \frac{1}{3}e = 0$

(b)

FIGURE 1.6 (a) A proton contains two up quarks (u) and one down quark (d). (b) A neutron contains one up quark (u) and two down quarks (d).

Self-Test Opportunity 1.1

Give the charge of the following elementary particles or atoms in terms of the elementary charge $e = 1.602 \times 10^{-19}$ C.

a) proton

b) neutron

c) helium atom (two protons, two neutron, and two electrons)

d) hydrogen atom (one proton and one electron)

e) up quark

f) down quark

g) electron

h) alpha particle (two protons and two neutrons)

Because all macroscopic objects are made of atoms, which in turn are made of electrons and atomic nuclei consisting of protons and neutrons, the charge, q, of any object can be expressed in terms of the sum of the number of protons, N_p, minus the sum of the number of electrons, N_e, that make up the object:

$$q = e(N_p - N_e) \qquad (1.5)$$

EXAMPLE 1.1 Net Charge

PROBLEM

If we wanted a block of iron of mass 3.25 kg to acquire a positive charge of 0.100 C, what fraction of the electrons would we have to remove?

SOLUTION

Iron has mass number 56. Therefore, the number of iron atoms in the 3.25 kg block is

$$N_{\text{atom}} = \frac{(3.25 \text{ kg})(6.022 \times 10^{23} \text{ atoms/mole})}{0.0560 \text{ kg/mole}} = 3.495 \times 10^{25} = 3.50 \times 10^{25} \text{ atoms}$$

Note that we have used Avogadro's number, 6.022×10^{23}, and the definition of the mole, which specifies that the mass of 1 mole of a substance in grams is just the mass number of the substance–in this case, 56.

Because the atomic number of iron is 26, which equals the number of protons or electrons in an iron atom, the total number of electrons in the 3.25 kg block is

$$N_e = 26N_{\text{atom}} = (26)(3.495 \times 10^{25}) = 9.09 \times 10^{26} \text{ electrons}$$

We use equation 1.5 to find the number of electrons, $N_{\Delta e}$, that we would have to remove. Because the number of electrons equals the number of protons in the original uncharged object, the difference in the number of protons and electrons is the number of removed electrons, $N_{\Delta e}$:

$$q = e \cdot N_{\Delta e} \Rightarrow N_{\Delta e} = \frac{q}{e} = \frac{0.100 \text{ C}}{1.602 \times 10^{-19} \text{ C}} = 6.24 \times 10^{17}.$$

Finally, we obtain the fraction of electrons we would have to remove:

$$\frac{N_{\Delta e}}{N_e} = \frac{6.24 \times 10^{17}}{9.09 \times 10^{26}} = 6.87 \times 10^{-10}.$$

We would have to remove fewer than one in a billion electrons from the iron block in order to put the sizable positive charge of 0.100 C on it.

1.3 Insulators, Conductors, Semiconductors, and Superconductors

Materials that conduct electricity well are called **conductors.** Materials that do not conduct electricity are called **insulators.** (Of course, there are good and poor conductors and good and poor insulators, depending on the properties of the specific materials.)

The electronic structure of a material refers to the way in which electrons are bound to nuclei, as we'll discuss in later chapters. For now, we are interested in the relative propensity of the atoms of a material to either give up or acquire electrons. For insulators, no free movement of electrons occurs because the material has no loosely bound electrons that can escape from its atoms and thereby move freely throughout the material. Even when external charge is placed on an insulator, this external charge cannot move appreciably. Typical insulators are glass, plastic, and cloth.

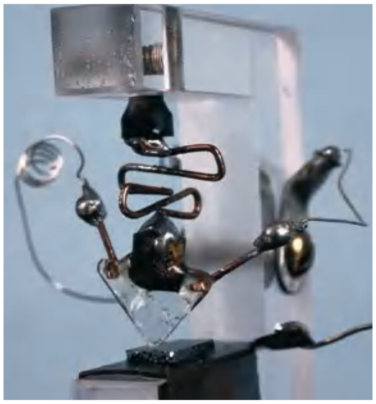
On the other hand, materials that are conductors have an electronic structure that allows the free movement of some electrons. The positive charges of the atoms of a conducting material do not move, since they reside in the heavy nuclei. Typical solid conductors are metals. Copper, for example, is a very good conductor and is therefore used in electrical wiring.

Fluids and organic tissue can also serve as conductors. Pure distilled water is not a very good conductor. However, dissolving common table salt (NaCl), for example, in water improves its conductivity tremendously, because the positively charged sodium ions (Na^+) and negatively charged chlorine ions (Cl^-) can move within the water to conduct electricity. In liquids, unlike solids, positive as well as negative charge carriers are mobile. Organic tissue is not a very good conductor, but it conducts electricity well enough to make large currents dangerous to us.

Semiconductors

A class of materials called **semiconductors** can change from being an insulator to being a conductor and back to an insulator again. Semiconductors were discovered only a little more than 50 years ago but are the backbone of the entire computer and consumer electronics industries. The first widespread use of semiconductors was in transistors (Figure 1.7a); modern computer chips (Figure 1.7b) perform the functions of millions of transistors. Computers and basically all modern consumer electronics products and devices (televisions, cameras, video game players, cell phones, etc.) would be impossible without semiconductors. Gordon Moore, cofounder of Intel, famously stated that due to advancing technology, the power of the average computer's CPU (central processing unit) doubles every 18 months, which is an empirical average over the last 5 decades. This doubling phenomenon is known as *Moore's Law*. Physicists have been and will undoubtedly continue to be the driving force behind this process of scientific discovery, invention, and improvement.

Semiconductors are of two kinds: intrinsic and extrinsic. Examples of *intrinsic semiconductors* are chemically pure crystals of gallium arsenide, germanium, or, especially, silicon. Engineers produce *extrinsic semiconductors* by *doping*, which is the addition of minute amounts (typically 1 part in 10^6) of other materials that can act as electron donors or electron receptors. Semiconductors doped with electron donors are called *n-type* (*n* stands for "negative charge"). If the doping substance acts as an electron receptor, the hole left behind by an electron that attaches to a receptor can also travel through the semiconductor and acts as an effective positive charge carrier. These semiconductors are consequently called *p-type* (*p* stand for "positive charge"). Thus, unlike normal solid conductors in which only negative charges move, semiconductors have movement of negative or positive charges (which are really electron holes, that is, missing electrons).



(a)

(b)

FIGURE 1.7 (a) Replica of the first transistor, invented in 1947 by John Bardeen, Walter H. Brattain, and William B. Shockley. (b) Modern computer chips made from silicon wafers contain many tens of millions of transistors.

Superconductors

Superconductors are materials that have zero resistance to the conduction of electricity, as opposed to normal conductors, which conduct electricity well but with some losses. Materials are superconducting only at very low temperatures. A typical superconductor is a niobium-titanium alloy that must be kept near the temperature of liquid helium (4.2 K) to retain its superconducting properties. During the last 20 years, new materials called *high-T_c superconductors* (T_c stands for "critical temperature," which is the maximum temperature that allows superconductivity) have been developed. These are superconducting at the temperature at which nitrogen can exist as a liquid (77.3 K). Materials that are superconductors at room temperature (300 K) have not yet been found, but they would be extremely useful. Research directed at developing such materials and theoretically explaining what physical phenomena cause high-T_c superconductivity is currently in progress.

1.4 Electrostatic Charging

Giving a static charge to an object is a process known as **electrostatic charging.** Electrostatic charging can be understood through a series of simple experiments. A power supply serves as a ready source of positive and negative charge. The battery in your car is a similar

FIGURE 1.8 A typical electroscope used in lecture demonstrations.

power supply; it uses chemical reactions to create a separation between positive and negative charge. Several insulating paddles can be charged with positive or negative charge from the power supply. In addition, a conducting connection is made to the Earth. The Earth is a nearly infinite reservoir of charge, capable of effectively neutralizing electrically charged objects in contact with it. This taking away of charge is called **grounding,** and an electrical connection to the Earth is called a **ground.**

An **electroscope** is a device that gives an observable response when it is charged. You can build a relatively simple electroscope by using two strips of very thin metal foil that are attached at one end and are allowed to hang straight down adjacent to each other from an isolating frame. Kitchen aluminum foil is not suitable, because it is too thick, but hobby shops sell thinner metal foils. For the isolating frame, you can use a Styrofoam coffee cup turned sideways, for example.

The lesson-demonstration-quality electroscope shown in Figure 1.8 has two conductors that in their neutral position are touching and oriented in a vertical direction. One of the conductors is hinged at its midpoint so that it will move away from the fixed conductor if a charge appears on the electroscope. These two conductors are in contact with a conducting ball on top of the electroscope, which allows charge to be applied or removed easily.

An uncharged electroscope is shown in Figure 1.9a. The power supply is used to give a negative charge to one of the insulating paddles. When the paddle is brought near the ball of the electroscope, as shown in Figure 1.9b, the electrons in the conducting ball of the electroscope are repelled, which produces a net negative charge on the conductors of the electroscope. This negative charge causes the movable conductor to rotate because the stationary conductor also has negative charge and repels it. Because the paddle did not touch the ball, the charge on the movable conductors is **induced.** If the charged paddle is then taken away, as illustrated in Figure 1.9c, the induced charge reduces to zero, and the movable conductor returns to its original position, because the total charge on the electroscope did not change in the process.

If the same process is carried out with a positively charged paddle, the electrons in the conductors are attracted to the paddle and flow into the conducting ball. This leaves a net positive charge on the conductors, causing the movable conducting arm to rotate again. Note that the net charge of the electroscope is zero in both cases and that the motion of the conductor indicates only that the paddle is charged. When the positively charged paddle is removed, the movable conductor again returns to its original position. It is important to note that we cannot determine the sign of this charge!

On the other hand, if a negatively charged insulating paddle *touches* the ball of the electroscope, as shown in Figure 1.10b, electrons will flow from the paddle to the conductor, producing a net negative charge. When the paddle is removed, the charge remains and the movable arm remains rotated, as shown in Figure 1.10c. Similarly, if a positively charged insulating paddle touches the ball of the uncharged electroscope, the electroscope transfers electrons to the positively charged paddle and becomes positively charged. Again, both a

Concept Check 1.2

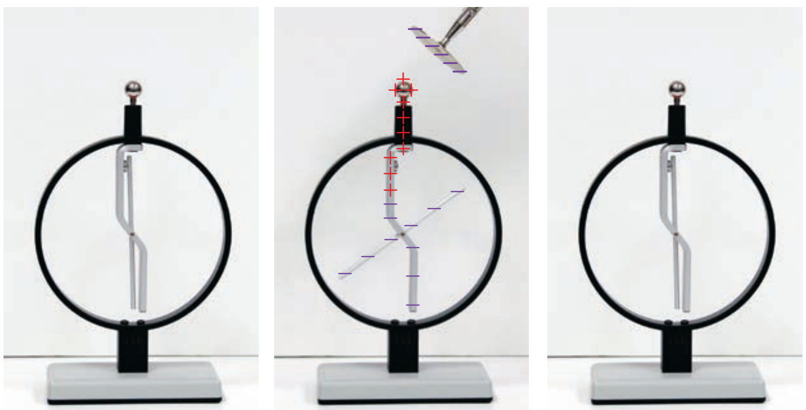
The hinged conductor moves away from the fixed conductor if a charge is applied to the electroscope, because

a) like charges repel each other.

b) like charges attract each other.

c) unlike charges attract each other.

d) unlike charges repel each other.



(a) (b) (c)

FIGURE 1.9 Inducing a charge: (a) An uncharged electroscope. (b) A negatively charged paddle is brought near the electroscope. (c) The negatively charged paddle is taken away.

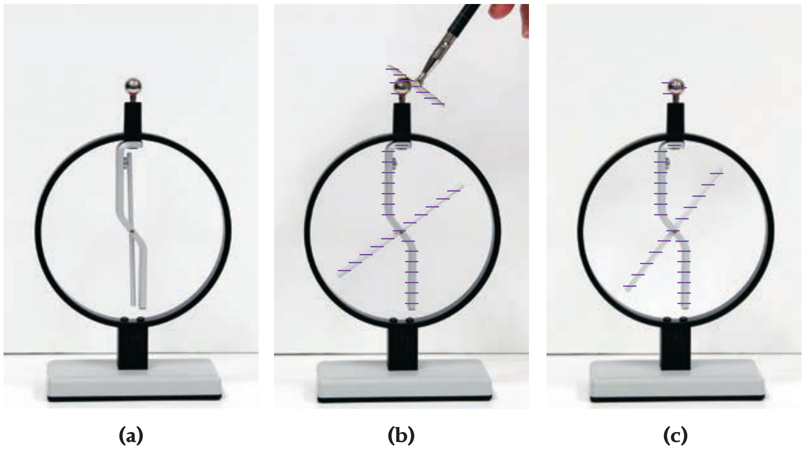


FIGURE 1.10 Charging by contact: (a) An uncharged electroscope. (b) A negatively charged paddle touches the electroscope. (c) The negatively charged paddle is removed.

positively charged paddle and a negatively charged paddle have the same effect on the electroscope, and we have no way of determining whether the paddles are positively charged or negatively charged. This process is called **charging by contact.**

The two different kinds of charge can be demonstrated by first touching a negatively charged paddle to the electroscope, producing a rotation of the movable arm, as shown in Figure 1.10. If a positively charged paddle is then brought into contact with the electroscope, the movable arm returns to the uncharged position. The charge is neutralized (assuming that both paddles originally had the same absolute value of charge). Thus, there are two kinds of charge. However, because charges are manifestations of mobile electrons, a negative charge is an excess of electrons and a positive charge is a deficit of electrons.

The electroscope can be given a charge without touching it with the charged paddle, as shown in Figure 1.11. The uncharged electroscope is shown in Figure 1.11a. A negatively charged paddle is brought close to the ball of the electroscope but not touching it, as shown in Figure 1.11b. In Figure 1.11c, the electroscope is connected to a ground. Then, while the charged paddle is still close to but not touching the ball of the electroscope, the ground connection is removed in Figure 1.11d. Next, when the paddle is moved away from the electroscope in Figure 1.11e, the electroscope is still positively charged (but with a smaller deflection than in Figure 1.11b). The same process also works with a positively charged paddle. This process is called **charging by induction** and yields an electroscope charge that has the opposite sign from the charge on the paddle.

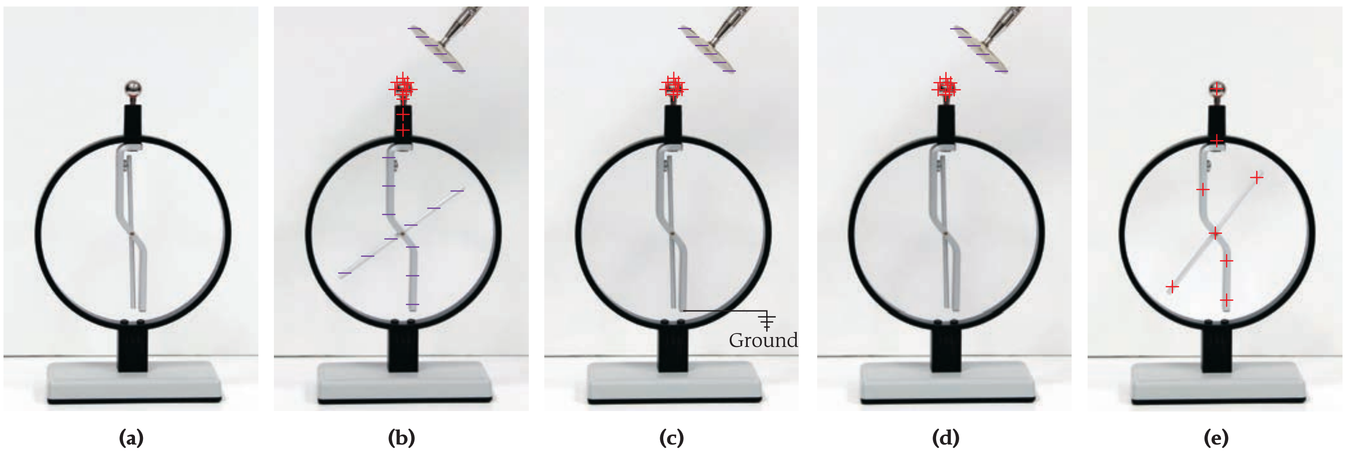


FIGURE 1.11 Charging by induction: (a) An uncharged electroscope. (b) A negatively charged paddle is brought close to the electroscope. (c) A ground is connected to the electroscope. (d) The connection to the ground is removed. (e) The negatively charged paddle is taken away, leaving the electroscope positively charged.

FIGURE 1.12 Triboelectric series for some common materials.

Triboelectric Charging

As mentioned earlier, rubbing two materials together will charge them. We have not addressed two basic questions about this effect: First, what really causes it? And second, which of two given materials gets charged positively and which negatively?

Amazingly, as with many aspects of friction, the microscopic causes of triboelectric charging are still not completely understood. The prevailing theory is that when the surfaces of the two materials involved in the charging process come into contact, adhesion takes place and chemical bonds are formed between atoms at the surfaces. As the surfaces separate, some of these newly formed bonds rupture and leave more of the electrons involved in the bonding on the material with the greater work function.

Recent research results obtained by examining surfaces with atomic force microscopes suggest, however, that triboelectric charging can also occur when two pieces of the *same* material are rubbed against each other. In addition, it has been found that the charging process transfers not only electrons but sometimes also small specks (a few nanometers across) of material.

Which material gets charged positively and which negatively? This question has been answered by a long series of experiments whose results are summarized in Figure 1.12 in the form of a list of some common materials that may be rubbed together. If you rub two materials from this list against each other, the one nearer the top will receive a net positive charge and the other a net negative charge.

Finally, as a rule of thumb, more intense rubbing creates greater charge transfer. This occurs because more intense contact increases friction, which in turn creates more microscopic points of charge transfer on the surfaces of the materials.

1.5 Electrostatic Force—Coulomb's Law

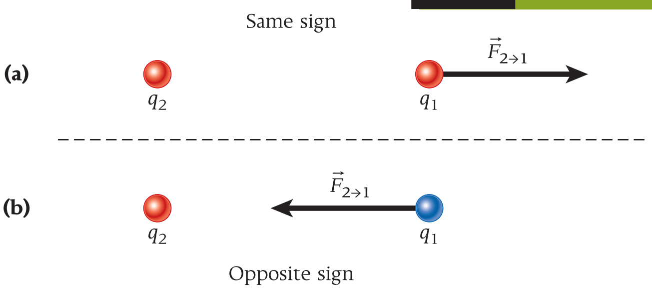


FIGURE 1.13 The force exerted by charge 2 on charge 1: (a) two charges with the same sign; (b) two charges with opposite signs.

The law of electric charges is evidence of a force between any two charges at rest. Experiments show that for the electrostatic force exerted by a charge q_2 on a charge q_1, $\vec{F}_{2\to1}$, the force on q_1 points toward q_2 if the charges have opposite signs and away from q_2 if the charges have like signs (Figure 1.13). This force on one charge due to another charge always acts along the line between the two charges. **Coulomb's Law** gives the magnitude of this force as

$$F = k\frac{|q_1 q_2|}{r^2}, \tag{1.6}$$

where q_1 and q_2 are electric charges, $r = |\vec{r}_1 - \vec{r}_2|$ is the distance between them, and

$$k = 8.99 \times 10^9 \ \frac{\text{N m}^2}{\text{C}^2} \tag{1.7}$$

is **Coulomb's constant.** You can see that 1 coulomb is a *very* large charge. If two charges of 1 C each were at a distance of 1 m apart, the magnitude of the force they would exert on each other would be 8.99 billion N. For comparison, this force equals the weight of 450 fully loaded space shuttles!

The relationship between Coulomb's constant and another constant, ε_0, called the **electric permittivity of free space,** is

$$k = \frac{1}{4\pi\varepsilon_0}. \tag{1.8}$$

Concept Check 1.3

You place two charges a distance *r* apart. Then you double each charge and double the distance between the charges. How does the force between the two charges change?

a) The new force is twice as large.

b) The new force is half as large.

c) The new force is four times larger.

d) The new force is four times smaller.

e) The new force is the same.

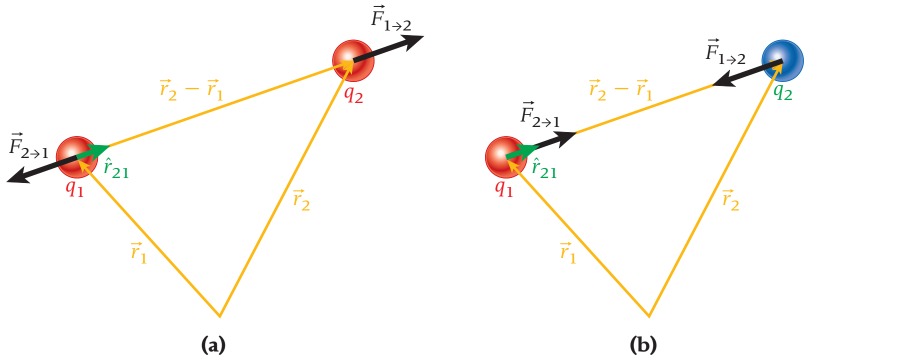


FIGURE 1.14 Vector representation of the electrostatic forces that two charges exert on each other: (a) two charges of like sign; (b) two charges of opposite sign.

Consequently, the value of ε_0 is

$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N\,m^2}. \tag{1.9}$$

An alternative way of writing equation 1.6 is then

$$F = \frac{1}{4\pi\varepsilon_0} \frac{|q_1 q_2|}{r^2}. \tag{1.10}$$

As you'll see in the next few chapters, some equations in electrostatics are more convenient to write with k, while others are more easily written in terms of $1/(4\pi\varepsilon_0)$.

Note that the charges in equations 1.6 and 1.10 can be positive or negative, so the product of the charges can also be positive or negative. Since opposite charges attract and like charges repel, a negative value for the product $q_1 q_2$ signifies attraction and a positive value means repulsion.

Finally, Coulomb's Law for the force due to charge 2 on charge 1 can be written in vector form:

$$\vec{F}_{2\to1} = -k \frac{q_1 q_2}{r^3} (\vec{r}_2 - \vec{r}_1) = -k \frac{q_1 q_2}{r^2} \hat{r}_{21}. \tag{1.11}$$

In this equation, $\hat{r}_{21}$ is a unit vector pointing from q_2 to q_1 (see Figure 1.14). The negative sign indicates that the force is repulsive if both charges are positive or both charges are negative. In that case, $\vec{F}_{2\to1}$ points away from charge 2, as depicted in Figure 1.14a. On the other hand, if one of the charges is positive and the other negative, then $\vec{F}_{2\to1}$ points toward charge 2, as shown in Figure 1.14b.

If charge 2 exerts the force $\vec{F}_{2\to1}$ on charge 1, then the force $\vec{F}_{1\to2}$ that charge 1 exerts on charge 2 is simply obtained from Newton's Third Law: $\vec{F}_{1\to2} = -\vec{F}_{2\to1}$.

Concept Check 1.4

What do the forces acting on the charge q_3 in Figure 1.15 indicate about the signs of the three charges?

a) All three charges must be positive.

b) All three charges must be negative.

c) Charge q_3 must be zero.

d) Charges q_1 and q_2 must have opposite signs.

e) Charges q_1 and q_2 must have the same sign, and q_3 must have the opposite sign.

Superposition Principle

So far in this chapter, we have been dealing with two charges. Now let's consider three point charges, q_1, q_2, and q_3, at positions x_1, x_2, and x_3, respectively, as shown in Figure 1.15. The force exerted by charge 1 on charge 3, $\vec{F}_{1\to3}$, is given by

$$\vec{F}_{1\to3} = -\frac{k q_1 q_3}{(x_3 - x_1)^2} \hat{x}.$$

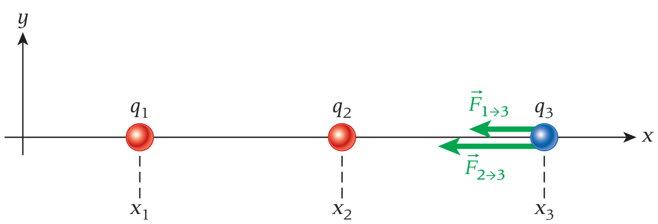


FIGURE 1.15 The forces exerted on charge 3 by charge 1 and charge 2.

Concept Check 1.5

Assuming that the lengths of the vectors in Figure 1.15 are proportional to the magnitudes of the forces they represent, what do they indicate about the magnitudes of the charges q_1 and q_2? (*Hint:* The distance between x_1 and x_2 is the same as the distance between x_2 and x_3.)

a) $|q_1| < |q_2|$

b) $|q_1| = |q_2|$

c) $|q_1| > |q_2|$

d) The answer cannot be determined from the information given in the figure.

The force exerted by charge 2 on charge 3 is

$$\vec{F}_{2\to3} = -\frac{kq_2q_3}{(x_3 - x_2)^2}\hat{x}.$$

The force that charge 1 exerts on charge 3 is not affected by the presence of charge 2. The force that charge 2 exerts on charge 3 is not affected by the presence of charge 1. In addition, the forces exerted by charge 1 and charge 2 on charge 3 add vectorially to produce a net force on charge 3:

$$\vec{F}_{\text{net}\to3} = \vec{F}_{1\to3} + \vec{F}_{2\to3}.$$

In general, we can express the electrostatic force, $\vec{F}(\vec{r})$, acting on a charge q at position $\vec{r}$ due to a collection of charges, q_i, at positions $\vec{r}_i$ as

$$\vec{F}(\vec{r}) = kq\sum_{i=1}^{n} q_i \frac{\vec{r}_i - \vec{r}}{|\vec{r}_i - \vec{r}|^3}. \qquad (1.12)$$

We obtain this result using superposition of the forces and equation 1.11 for each pairwise interaction.

EXAMPLE 1.2 Electrostatic Force inside the Atom

PROBLEM 1

What is the magnitude of the electrostatic force that the two protons inside the nucleus of a helium atom exert on each other?

SOLUTION 1

The two protons and two neutrons in the nucleus of the helium atom are held together by the strong force; the electrostatic force is pushing the protons apart. The charge of each proton is $q_p = +e$. A distance of approximately $r = 2 \cdot 10^{-15}$ m separates the two protons. Using Coulomb's Law, we can find the force:

$$F = k\frac{|q_pq_p|}{r^2} = \left(8.99\times10^9 \ \frac{\text{N m}^2}{\text{C}^2}\right)\frac{\left(+1.6\times10^{-19}\ \text{C}\right)\left(+1.6\times10^{-19}\ \text{C}\right)}{\left(2\times10^{-15}\ \text{m}\right)^2} = 58\ \text{N}.$$

Therefore, the two protons in the atomic nucleus of a helium atom are being pushed apart with a force of 58 N (approximately the weight of a small dog). Considering the size of the nucleus, this is an astonishingly large force. Why do atomic nuclei not simply explode? The answer is that an even stronger force, the aptly named strong force, keeps them together.

PROBLEM 2

What is the magnitude of the electrostatic force between a gold nucleus and an electron of the gold atom in an orbit with radius 4.88×10^{-12} m?

SOLUTION 2

The negatively charged electron and the positively charged gold nucleus attract each other with a force whose magnitude is

$$F = k\frac{|q_eq_N|}{r^2},$$

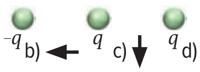
where the charge of the electron is $q_e = -e$ and the charge of the gold nucleus is $q_N = +79e$. The force between the electron and the nucleus is then

$$F = k\frac{|q_eq_N|}{r^2} = \left(8.99\times10^9 \ \frac{\text{N m}^2}{\text{C}^2}\right)\frac{\left(1.60\times10^{-19}\ \text{C}\right)\left[(79)\left(1.60\times10^{-19}\ \text{C}\right)\right]}{\left(4.88\times10^{-12}\ \text{m}\right)^2} = 7.63\times10^{-4}\ \text{N}.$$

Thus, the magnitude of the electrostatic force exerted on an electron in a gold atom by the nucleus is about 100,000 times less than that between protons inside a nucleus.

Concept Check 1.6

Three charges are arranged on a straight line as shown in the figure. What is the direction of the electrostatic force on the *middle* charge?



a) → b) ← c) ↓ d) ↑

e) There is no force on that charge.

Concept Check 1.7

Three charges are arranged on a straight line as shown in the figure. What is the direction of the electrostatic force on the *right* charge? (Note that the left charge is double what it was in Concept Check 1.6.)

a) → b) ← c) ↓ d) ↑

e) There is no force on that charge.

Note: The gold nucleus has a mass that is approximately 400,000 times that of the electron. But the force the gold nucleus exerts on the electron has exactly the same magnitude as the force that the electron exerts on the gold nucleus. You may say that this is obvious from Newton's Third Law, which is true. But it is worth emphasizing that this basic law holds for electrostatic forces as well.

EXAMPLE 1.3 Equilibrium Position

PROBLEM

Two charged particles are placed as shown in Figure 1.16: $q_1 = 0.15$ μC is located at the origin, and $q_2 = 0.35$ μC is located on the positive x-axis at $x_2 = 0.40$ m. Where should a third charged particle, q_3, be placed to be at an equilibrium point (such that the forces on it sum to zero)?

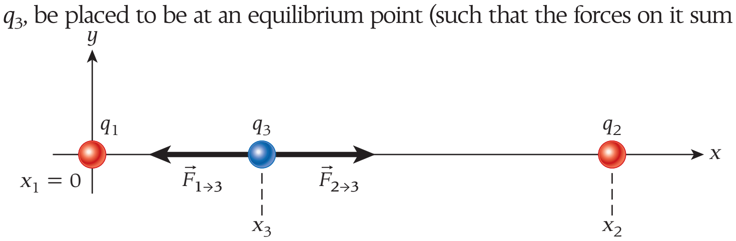


FIGURE 1.16 Placement of three charged particles. The third particle is shown as having a negative charge.

SOLUTION

Let's first determine where *not* to put the third charge. If the third charge is placed anywhere off the x-axis, there will always be a force component pointing toward or away from the x-axis. Thus, we can find an equilibrium point (a point where the forces sum to zero) only *on* the x-axis. The x-axis can be divided into three different segments: $x \le x_1 = 0$, $x_1 < x < x_2$, and $x_2 \le x$. For $x \le x_1 = 0$, the force vectors from both q_1 and q_2 acting on q_3 will point in the positive direction if the charge is negative and in the negative direction if the charge is positive. Because we are looking for a location where the two forces cancel, the segment $x \le x_1 = 0$ can be excluded. A similar argument excludes $x \ge x_2$.

In the remaining segment of the x-axis, $x_1 < x < x_2$, the forces from q_1 and q_2 on q_3 point in opposite directions. We look for the location, x_3, where the absolute magnitudes of both forces are equal and the forces thus sum to zero. We express the equality of the two forces as

$$\left|\vec{F}_{1\to3}\right| = \left|\vec{F}_{2\to3}\right|,$$

which we can rewrite as

$$k\frac{|q_1q_3|}{(x_3-x_1)^2} = k\frac{|q_3q_2|}{(x_2-x_3)^2}.$$

We now see that the magnitude and sign of the third charge do not matter because that charge cancels out, as does the constant k, giving us

$$\frac{q_1}{(x_3-x_1)^2} = \frac{q_2}{(x_2-x_3)^2}$$

or

$$q_1(x_2-x_3)^2 = q_2(x_3-x_1)^2. \qquad \text{(i)}$$

Taking the square root of both sides and solving for x_3, we find

$$\sqrt{q_1}(x_2-x_3) = \sqrt{q_2}(x_3-x_1),$$

or

$$x_3 = \frac{\sqrt{q_1}x_2 + \sqrt{q_2}x_1}{\sqrt{q_1}+\sqrt{q_2}}.$$

We can take the square root of both sides of equation (i) because $x_1 < x_3 < x_2$, and so both of the roots, $x_2 - x_3$ and $x_3 - x_1$, are assured to be positive.

Inserting the numbers given in the problem statement, we obtain

$$x_3 = \frac{\sqrt{q_1}x_2 + \sqrt{q_2}x_1}{\sqrt{q_1}+\sqrt{q_2}} = \frac{\sqrt{0.15\ \mu\text{C}}(0.4\ \text{m})}{\sqrt{0.15\ \mu\text{C}}+\sqrt{0.35\ \mu\text{C}}} = 0.16\ \text{m}.$$

This result makes sense because we expect the equilibrium point to reside closer to the smaller charge.

SOLVED PROBLEM 1.1 Charged Balls

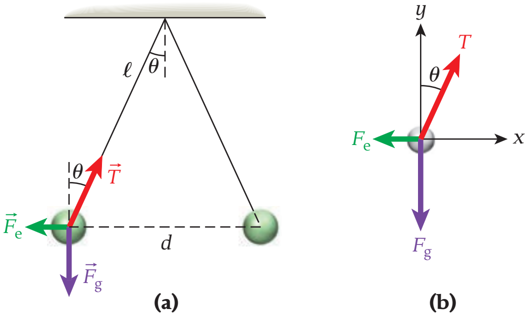


FIGURE 1.17 (a) Two charged balls hanging from the ceiling in their equilibrium position. (b) Free-body diagram for the left-hand charged ball.

PROBLEM

Two identical charged balls hang from the ceiling by insulated ropes of equal length, $\ell = 1.50$ m (Figure 1.17). A charge $q = 25.0$ μC is applied to each ball. Then the two balls hang at rest, and each supporting rope has an angle of 25.0° with respect to the vertical (Figure 1.17a). What is the mass of each ball?

SOLUTION

THINK Each charged ball has three forces acting on it: the force of gravity, the repulsive electrostatic force, and the tension in the supporting rope. We can resolve the components of the three forces and set them equal to zero, allowing us to solve for the mass of the charged balls.

SKETCH A free-body diagram for the left-hand ball is shown in Figure 1.17b.

RESEARCH The condition for static equilibrium says that the sum of the x-components of the three forces acting on the ball must equal zero and the sum of y-components of these forces must equal zero. The sum of the x-components of the forces is

$$T \sin\theta - F_e = 0, \tag{i}$$

where T is the magnitude of the string tension, θ is the angle of the string relative to the vertical, and F_e is the magnitude of the electrostatic force. The sum of the y-components of the forces is

$$T \cos\theta - F_g = 0. \tag{ii}$$

The force of gravity, F_g, is just the weight of the charged ball:

$$F_g = mg, \tag{iii}$$

where m is the mass of the charged ball. The electrostatic force the two balls exert on each other is given by

$$F_e = k\frac{q^2}{d^2}, \tag{iv}$$

where d is the distance between the two balls. We can express the distance between the two balls in terms of the length of the string, ℓ, by looking at Figure 1.17a. We see that

$$\sin\theta = \frac{d/2}{\ell}.$$

We can then express the electrostatic force in terms of the angle with respect to the vertical, θ, and the length of the string, ℓ:

$$F_e = k\frac{q^2}{(2\ell\sin\theta)^2} = k\frac{q^2}{4\ell^2\sin^2\theta}. \tag{v}$$

SIMPLIFY We divide equation (i) by equation (ii):

$$\frac{T\sin\theta}{T\cos\theta} = \frac{F_e}{F_g},$$

which, after the (unknown) string tension is canceled out, becomes

$$\tan\theta = \frac{F_e}{F_g}.$$

Substituting from equations (iii) and (v) for the force of gravity and the electrostatic force, we get

$$\tan\theta = \frac{k\dfrac{q^2}{4\ell^2\sin^2\theta}}{mg} = \frac{kq^2}{4mg\ell^2\sin^2\theta}.$$

Solving for the mass of the ball, we obtain

$$m=\frac{kq^2}{4g\ell^2\sin^2\theta\tan\theta}.$$

CALCULATE Putting in the numerical values gives

$$m=\frac{\left(8.99\times10^9\ \mathrm{N\ m^2/C^2}\right)(25.0\ \mu\mathrm{C})^2}{4\left(9.81\ \mathrm{m/s^2}\right)(1.50\ \mathrm{m})^2\left(\sin^2 25.0°\right)(\tan 25.0°)}=0.764116\ \mathrm{kg}.$$

ROUND We report our result to three significant figures:

$$m=0.764\ \mathrm{kg}.$$

DOUBLE-CHECK To double-check, we make the small-angle approximations that $\sin\theta\approx\tan\theta\approx\theta$ and $\cos\theta\approx1$. The tension in the string then approaches mg, and we can express the x-components of the forces as

$$T\sin\theta\approx mg\theta=F_e=k\frac{q^2}{d^2}\approx k\frac{q^2}{(2\ell\theta)^2}.$$

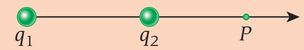
Solving for the mass of the charged ball, we get

$$m=\frac{kq^2}{4g\ell^2\theta^3}=\frac{\left(8.99\times10^9\ \mathrm{N\ m^2/C^2}\right)(25.0\ \mu\mathrm{C})^2}{4\left(9.81\ \mathrm{m/s^2}\right)(1.50\ \mathrm{m})^2(0.436\ \mathrm{rad})^3}=0.768\ \mathrm{kg},$$

which is close to our answer.

Self-Test Opportunity 1.2

A positive point charge $+q$ is placed at point P, to the right of two charges q_1 and q_2, as shown in the figure. The net electrostatic force on the positive charge $+q$ is found to be zero. Identify each of the following statements as true or false.



a) Charge q_2 must have the opposite sign from q_1 and be smaller in magnitude.

b) The magnitude of charge q_1 must be smaller than the magnitude of charge q_2.

c) Charges q_1 and q_2 must have the same sign.

d) If q_1 is negative, then q_2 must be positive.

e) Either q_1 or q_2 must be positive.

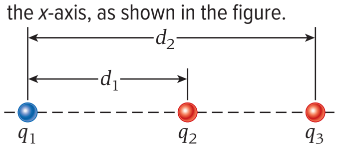
Electrostatic Precipitator

An application of electrostatic charging and electrostatic forces is the cleaning of emissions from coal-fired power plants. A device called an **electrostatic precipitator** (ESP) is used to remove ash and other particulates produced when coal is burned to generate electricity. The operation of this device is illustrated in Figure 1.18.

The ESP consists of wires and plates, with the wires held at a high negative voltage relative to the series of plates held at a positive voltage. (Here the term *voltage* is used colloquially; in Chapter 3, the concept will be defined in terms of electric potential difference.) In Figure 1.18, the exhaust gas from the coal-burning process enters the ESP from the left. Particulates passing near the wires pick up a negative charge. These particles are then attracted to one of the positive plates and stick there. The gas continues through the ESP, leaving the ash and other particulates behind. The accumulated material is then shaken off the plates to a hamper below. This waste can be used for many purposes, including construction materials and fertilizer. Figure 1.19 shows an example of a coal-fired power plant that incorporates an ESP.

Concept Check 1.8

Consider three charges placed along the x-axis, as shown in the figure.



The values of the charges are $q_1=-8.10\ \mu\mathrm{C}$, $q_2=2.16\ \mu\mathrm{C}$, and $q_3=2.16\ \mathrm{pC}$. The distance between q_1 and q_2 is $d_1=1.71$ m. The distance between q_1 and q_3 is $d_2=2.62$ m. What is the magnitude of the total electrostatic force exerted on q_3 by q_1 and q_2?

a) 2.77×10^{-8} N

b) 7.92×10^{-6} N

c) 1.44×10^{-5} N

d) 2.22×10^{-4} N

e) 6.71×10^{-2} N

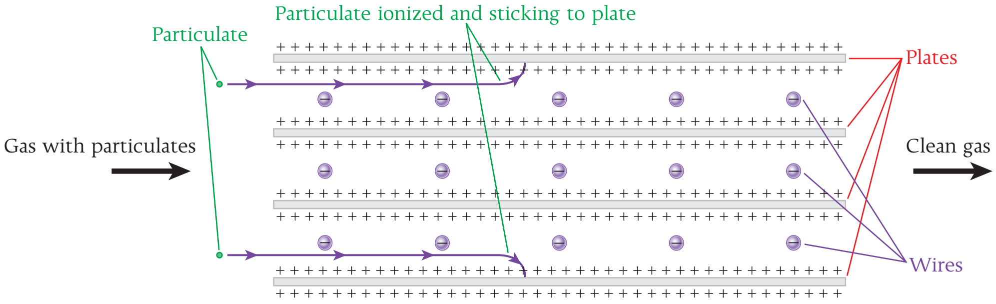


FIGURE 1.18 Operation of an electrostatic precipitator used to clean the exhaust gas of a coal-fired power plant. The view is from the top of the device.

FIGURE 1.19 A coal-fired power plant at Michigan State University that incorporates an electrostatic precipitator to remove particulates from its emissions.

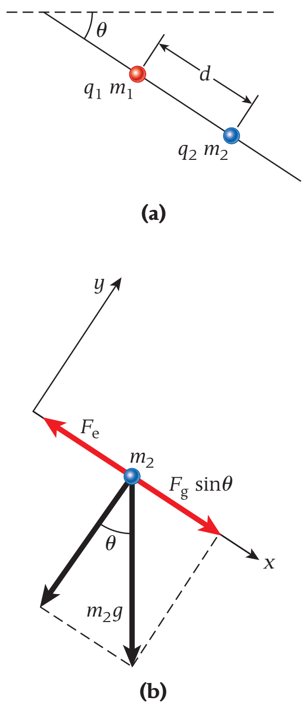


FIGURE 1.20 (a) Two charged beads on a wire. (b) Free-body diagram of the forces acting on the second bead.

SOLVED PROBLEM 1.2 Bead on a Wire

PROBLEM

A bead with charge $q_1 = +1.28\ \mu\text{C}$ is fixed in place on an insulating wire that makes an angle of $\theta = 42.3°$ with respect to the horizontal (Figure 1.20a). A second bead with charge $q_2 = -5.06\ \mu\text{C}$ slides without friction on the wire. At a distance $d = 0.380$ m between the beads, the net force on the second bead is zero. What is the mass, m_2, of the second bead?

SOLUTION

THINK The force of gravity pulling the bead of mass m_2 down the wire is offset by the attractive electrostatic force between the positive charge on the first bead and the negative charge on the second bead. The second bead can be thought of as sliding on an inclined plane.

SKETCH Figure 1.20b shows a free-body diagram of the forces acting on the second bead. We have defined a coordinate system in which the positive x-direction is down the wire. The force exerted on m_2 by the wire can be omitted because this force has only a y-component, and we can solve the problem by analyzing just the x-components of the forces.

RESEARCH The attractive electrostatic force between the two beads balances the component of the force of gravity that acts on the second bead down the wire. The electrostatic force acts in the negative x-direction and its magnitude is given by

$$F_e = k\frac{|q_1q_2|}{d^2}. \tag{i}$$

The x-component of the force of gravity acting on the second bead corresponds to the component of the weight of the second bead that is parallel to the wire. Figure 1.20b indicates that the component of the weight of the second bead down the wire is given by

$$F_g = m_2 g\sin\theta. \tag{ii}$$

SIMPLIFY For equilibrium, the electrostatic force and the gravitational force are equal: $F_e = F_g$. Substituting the expressions for these forces from equations (i) and (ii) yields

$$k\frac{|q_1q_2|}{d^2} = m_2 g\sin\theta.$$

Solving this equation for the mass of the second bead gives us

$$m_2 = \frac{k|q_1q_2|}{d^2 g\sin\theta}.$$

CALCULATE We put in the numerical values and get

$$m_2 = \frac{kq_1q_2}{d^2 g\sin\theta} = \frac{\left(8.99\times10^9\ \text{N m}^2/\text{C}^2\right)(1.28\ \mu\text{C})(5.06\ \mu\text{C})}{(0.380\ \text{m})^2\left(9.81\ \text{m/s}^2\right)(\sin 42.3°)} = 0.0610746\ \text{kg}.$$

ROUND We report our result to three significant figures:

$$m_2 = 0.0611\ \text{kg} = 61.1\ \text{g}.$$

DOUBLE-CHECK To double-check, let's calculate the mass of the second bead assuming that the wire is vertical, that is, $\theta = 90°$. We can then set the weight of the second bead equal to the electrostatic force between the two beads:

$$k\frac{|q_1q_2|}{d^2} = m_2 g.$$

Solving for the mass of the second bead, we obtain

$$m_2 = \frac{kq_1q_2}{d^2 g} = \frac{\left(8.99\times10^9\ \text{N m}^2/\text{C}^2\right)(1.28\ \mu\text{C})(5.06\ \mu\text{C})}{(0.380\ \text{m})^2\left(9.81\ \text{m/s}^2\right)} = 0.0411\ \text{kg}.$$

As the angle of the wire relative to the horizontal decreases, the calculated mass of the second bead will increase. Our result of 0.0611 kg is somewhat higher than the mass that can be supported with a vertical wire, so it seems reasonable.

Laser Printer

Another example of a device that applies electrostatic forces is the laser printer. The operation of such a printer is illustrated in Figure 1.21. The paper path follows the blue arrows. Paper is taken from the paper tray or fed manually through the alternate paper feed. The paper passes over a drum where the toner is placed on the surface of the paper and then passes through a fuser that melts the toner and permanently affixes it to the paper.

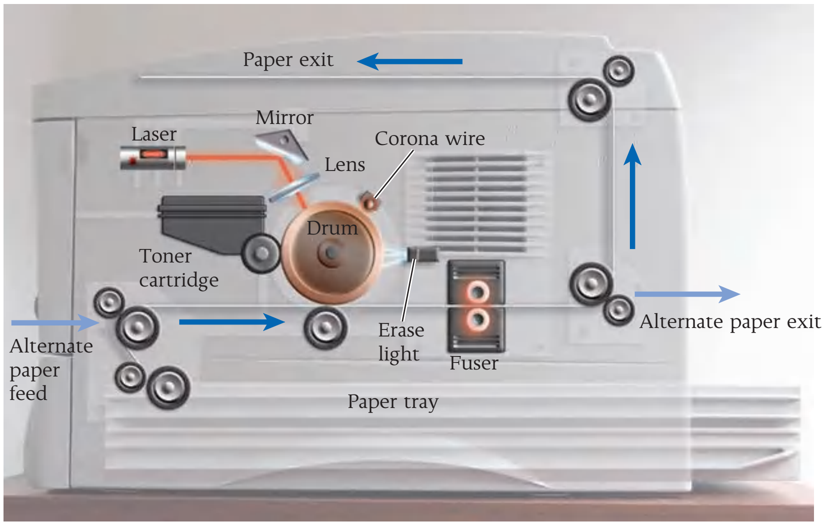


FIGURE 1.21 The operation of a typical laser printer.

The drum consists of a metal cylinder coated with a special photosensitive material. The photosensitive surface is an insulator that retains charge in the absence of light, but discharges quickly if light is incident on the surface. The drum rotates so that its surface speed is the same as the speed of the moving paper. The basic principle of the operation of the drum is illustrated in Figure 1.22.

The drum is negatively charged with electrons using a wire held at high voltage. Then laser light is directed at the surface of the drum. Wherever the laser light strikes the surface of the drum, the surface at that point is discharged. A laser is used because its beam is narrow and remains focused. A line of the image being printed is written one pixel (picture element or dot) at a time using a laser beam directed by a moving mirror and a lens. A typical laser printer can write 600–1200 pixels per inch. The surface of the drum then passes by a roller that picks up toner from the toner cartridge. Toner consists of small, black, insulating particles composed of a plastic-like material. The toner roller is charged to the same negative voltage as the drum. Therefore, wherever the surface of the drum has been discharged, electrostatic forces deposit toner on the surface of the drum. Any portion of the drum surface that has not been exposed to the laser will not pick up toner.

As the drum rotates, it next comes in contact with the paper. The toner is then transferred from the surface of the drum to the paper. As the drum rotates, any remaining toner is scraped off and the surface is neutralized with an erase light or a rotating erase drum in preparation for printing the next image. The paper then continues on to the fuser, which melts the toner, producing a permanent image on the paper.

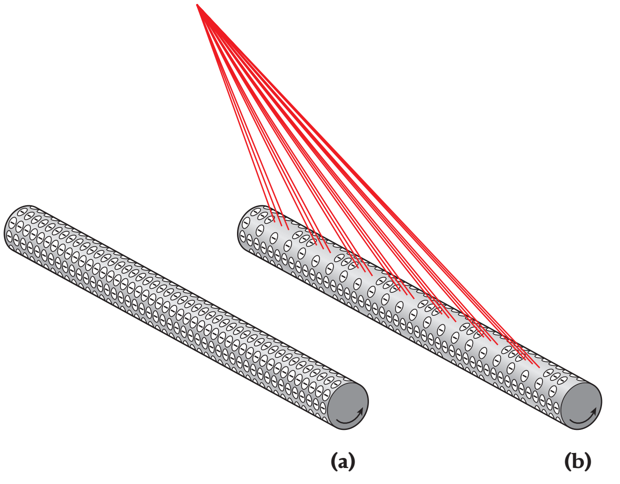


FIGURE 1.22 (a) The completely charged drum of a laser printer. This drum will produce a blank page. (b) A drum on which one line of information is being recorded by a laser. Wherever the laser strikes the charged drum, the negative charge is neutralized, and the discharged area will attract toner that will produce an image on the paper.

SOLVED PROBLEM 1.3 Four Charged Objects

Consider four charges placed at the corners of a square with side length 1.25 m, as shown in Figure 1.23a.

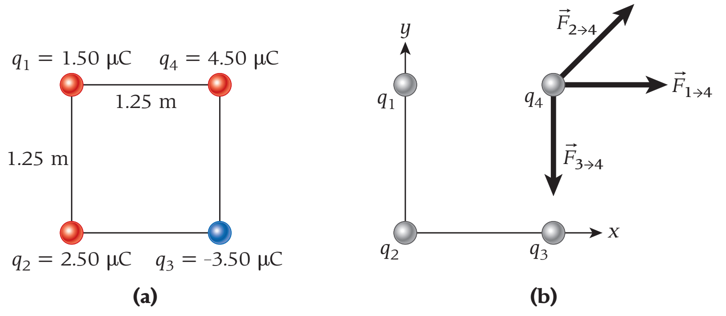


FIGURE 1.23 (a) Four charges placed at the corners of a square. (b) The forces exerted on q_4 by the other three charges.

– Continued

PROBLEM

What are the magnitude and the direction of the electrostatic force on q_4 resulting from the other three charges?

SOLUTION

THINK The electrostatic force on q_4 is the vector sum of the forces resulting from its interactions with the other three charges. Thus, it is important to avoid simply adding the individual force magnitudes algebraically. Instead we need to determine the individual force components in each spatial direction and add those to find the components of the net force vector. Then we need to calculate the length of that net force vector.

SKETCH Figure 1.23b shows the four charges in an xy-coordinate system with its origin at the location of q_2.

RESEARCH The net force on q_4 is the vector sum of the forces $\vec{F}_{1\to 4}$, $\vec{F}_{2\to 4}$, and $\vec{F}_{3\to 4}$. The x-component of the summed forces is

$$F_x = k\frac{|q_1q_4|}{d^2} + k\frac{|q_2q_4|}{\left(\sqrt{2}d\right)^2}\cos 45° = \frac{kq_4}{d^2}\left(q_1 + \frac{q_2}{2}\cos 45°\right), \qquad \text{(i)}$$

where d is the length of a side of the square and, as Figure 1.23b indicates, the x-component of $\vec{F}_{3\to 4}$ is zero. The y-component of the summed forces is

$$F_y = k\frac{|q_2q_4|}{\left(\sqrt{2}d\right)^2}\sin 45° - k\frac{|q_3q_4|}{d^2} = \frac{kq_4}{d^2}\left(\frac{q_2}{2}\sin 45° + q_3\right), \qquad \text{(ii)}$$

where, as Figure 1.23b indicates, the y-component of $\vec{F}_{1\to 4}$ is zero.

The magnitude of the net force is given by

$$F = \sqrt{F_x^2 + F_y^2}, \qquad \text{(iii)}$$

and the angle of the net force is given by

$$\tan\theta = \frac{F_y}{F_x}.$$

SIMPLIFY We substitute the expressions for F_x and F_y from equations (i) and (ii) into equation (iii):

$$F = \sqrt{\left[\frac{kq_4}{d^2}\left(q_1 + \frac{q_2}{2}\cos 45°\right)\right]^2 + \left[\frac{kq_4}{d^2}\left(\frac{q_2}{2}\sin 45° + q_3\right)\right]^2}.$$

We can rewrite this as

$$F = \frac{kq_4}{d^2}\sqrt{\left(q_1 + \frac{q_2}{2}\cos 45°\right)^2 + \left(\frac{q_2}{2}\sin 45° + q_3\right)^2}.$$

For the angle of the force, we get

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{\dfrac{kq_4}{d^2}\left(\dfrac{q_2}{2}\sin 45° + q_3\right)}{\dfrac{kq_4}{d^2}\left(q_1 + \dfrac{q_2}{2}\cos 45°\right)}\right) = \tan^{-1}\left(\frac{\left(\dfrac{q_2}{2}\sin 45° + q_3\right)}{\left(q_1 + \dfrac{q_2}{2}\cos 45°\right)}\right).$$

CALCULATE Putting in the numerical values, we get

$$\frac{q_2}{2}\sin 45° = \frac{q_2}{2}\cos 45° = \frac{2.50\ \mu\text{C}}{2\sqrt{2}} = 0.883883\ \mu\text{C}.$$

The magnitude of the force is then

$$F = \frac{\left(8.99\times 10^9\ \text{N m}^2/\text{C}^2\right)(4.50\ \mu\text{C})}{(1.25\ \text{m})^2}\sqrt{(1.50\ \mu\text{C} + 0.883883\ \mu\text{C})^2 + (0.883883\ \mu\text{C} - 3.50\ \mu\text{C})^2}$$
$$= 0.0916379\ \text{N}$$

For the direction of the force, we obtain

$$\theta = \tan^{-1}\left(\frac{\left(\frac{q_2}{2}\sin 45^\circ + q_3\right)}{\left(q_1 + \frac{q_2}{2}\cos 45^\circ\right)}\right) = \tan^{-1}\left(\frac{(0.883883\ \mu\text{C} - 3.50\ \mu\text{C})}{(1.50\ \mu\text{C} + 0.883883\ \mu\text{C})}\right) = -47.6593^\circ.$$

ROUND We report our results to three significant figures:

$$F = 0.0916\ \text{N}$$

and

$$\theta = -47.7^\circ.$$

DOUBLE-CHECK To double-check our result, we calculate the magnitude of the three forces acting on q_4. For $\vec{F}_{1\to4}$, we get

$$F_{1\to4} = k\frac{|q_1q_4|}{r_{14}^2} = \frac{\left(8.99\times10^9\ \text{N m}^2/\text{C}^2\right)(1.50\ \mu\text{C})(4.50\ \mu\text{C})}{(1.25\ \text{m})^2} = 0.0388\ \text{N}.$$

For $\vec{F}_{2\to4}$, we get

$$F_{2\to4} = k\frac{|q_2q_4|}{r_{24}^2} = \frac{\left(8.99\times10^9\ \text{N m}^2/\text{C}^2\right)(2.50\ \mu\text{C})(4.50\ \mu\text{C})}{\left[\sqrt{2}(1.25\ \text{m})\right]^2} = 0.0324\ \text{N}.$$

For $F_{3\to4}$ we get

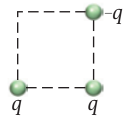
$$F_{3\to4} = k\frac{|q_3q_4|}{r_{34}^2} = \frac{\left(8.99\times10^9\ \text{N m}^2/\text{C}^2\right)(3.50\ \mu\text{C})(4.50\ \mu\text{C})}{(1.25\ \text{m})^2} = 0.0906\ \text{N}$$

All three of the magnitudes of the individual forces are of the same order as our result for the net force. This gives us confidence that our answer is not off by a large factor.

The direction we obtained also seems reasonable, because it orients the resulting force downward and to the right, as could be expected from looking at Figure 1.23b.

Concept Check 1.9

Three charges are arranged at the corners of a square as shown in the figure. What is the direction of the electrostatic force on the *lower-right* charge?

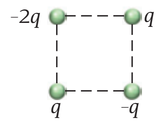


a) ↙ b) ↖ c) ↗ d) ↘

e) There is no force on that charge.

Concept Check 1.10

Four charges are arranged at the corners of a square as shown in the figure. What is the direction of the electrostatic force on the *lower-right* charge?



a) ↙ b) ↖ c) ↗ d) ↘

e) There is no force on that charge.

1.6 Coulomb's Law and Newton's Law of Gravitation

Coulomb's Law describing the electrostatic force between two electric charges, F_e, has a form similar to Newton's Law describing the gravitational force between two masses, F_g:

$$F_g = G\frac{m_1m_2}{r^2} \quad \text{and} \quad F_e = k\frac{|q_1q_2|}{r^2}$$

where m_1 and m_2 are the two masses, q_1 and q_2 are the two electric charges, and r is the distance of separation. Both forces vary with the inverse square of the distance. The electrostatic force can be attractive or repulsive because charges can have positive or negative signs. (See Figure 1.14a and b.) The gravitational force is always attractive because there is only one kind of mass. (For the gravitational force, only the case depicted in Figure 1.14b is possible.) The relative strengths of the forces are given by the proportionality constants k and G.

EXAMPLE 1.4 Forces between Electrons

Let's evaluate the relative strengths of the two interactions by calculating the ratio of the electrostatic force and the gravitational force that two electrons exert on each other. This ratio is given by

$$\frac{F_e}{F_g} = \frac{kq_e^2}{Gm_e^2}.$$

Because the dependence on distance is the same for both forces, there is no dependence on distance in the ratio of the two forces—it cancels out. The mass of an electron is $m_e = 9.109\times10^{-31}$ kg,

– *Continued*

Concept Check 1.11

The proton's mass is ~2000 times larger than the electron's mass. Therefore, the ratio F_e/F_g for two protons is ____________ the value calculated in Example 1.4 for two electrons.

a) ~4 million times smaller than

b) ~2000 times smaller than

c) the same as

d) ~2000 times larger than

e) ~4 million times larger than

and its charge is $q_e = -1.602 \times 10^{-19}$ C. Using the value of Coulomb's constant given in equation 1.7, $k = 8.99 \times 10^9$ N m^2/C^2, and the value of the universal gravitational constant, $G = 6.67 \times 10^{-11}$ N m^2/kg^2, we find numerically

$$\frac{F_e}{F_g} = \frac{(8.99 \times 10^9 \text{ N m}^2/\text{C}^2)(1.602 \times 10^{-19} \text{ C})^2}{(6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2)(9.109 \times 10^{-31} \text{ kg})^2} = 4.17 \times 10^{42}.$$

Therefore, the electrostatic force between electrons is stronger than the gravitational force between them by more than 42 orders of magnitude.

Despite the relative weakness of the gravitational force, it is the only force that matters on the astronomical scale. The reason for this dominance is that all stars, planets, and other objects of astronomical relevance carry no net charge. Therefore, there is no net electrostatic interaction between them, and gravity dominates.

Coulomb's Law of electrostatics applies to macroscopic systems down to the atom, though subtle effects in atomic and subatomic systems require use of a more sophisticated approach called *quantum electrodynamics*. Newton's law of gravitation fails in subatomic systems and also must be modified for some phenomena in astronomical systems, such as the precessional motion of Mercury around the Sun. These fine details of the gravitational interaction are governed by Einstein's theory of general relativity.

The similarities between the gravitational and electrostatic interactions will be covered further in the next two chapters, which address electric fields and electric potential.

WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- There are two kinds of electric charge, positive and negative. Like charges repel, and unlike charges attract.
- The quantum (elementary quantity) of electric charge is $e = 1.602 \times 10^{-19}$ C.
- The electron has charge $q_e = -e$, and the proton has charge $q_p = +e$. The neutron has zero charge.
- The net charge of an object is given by e times the number of protons, N_p, minus e times the number of electrons, N_e, that make up the object: $q = e \cdot (N_p - N_e)$.
- The total charge in an isolated system is always conserved.
- Objects can be charged directly by contact or indirectly by induction.
- Coulomb's Law describes the force that two stationary charges exert on each other: $F = k\dfrac{|q_1 q_2|}{r^2} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{|q_1 q_2|}{r^2}$.
- The constant in Coulomb's Law is
$$k = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}.$$
- The electric permittivity of free space is
$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N m}^2}.$$

ANSWERS TO SELF-TEST OPPORTUNITIES

21.1 a) $+1$ b) 0 c) 0 d) 0 e) $+\frac{2}{3}$ f) $-\frac{1}{3}$ g) -1 h) $+2$

22.2 a) true b) false c) false d) true e) true

PROBLEM-SOLVING GUIDELINES

1. For problems involving Coulomb's Law, drawing a free-body diagram showing the electrostatic force vectors acting on a charged particle is often helpful. Pay careful attention to signs; a negative force between two particles indicates attraction, and a positive force indicates repulsion. Be sure that the directions of forces in the diagram match the signs of forces in the calculations.

2. Use symmetry to simplify your work. However, be careful to take account of charge magnitudes and signs as well as distances. Two charges at equal distances from a third charge do not exert equal forces on that charge if they have different magnitudes or signs.

3. Units in electrostatics often have prefixes indicating powers of 10: Distances may be given in cm or mm; charges may be given in μC, nC, or pC; masses may be given in kg or g. Other units are also common. The best way to proceed is to convert all quantities to basic SI units, to be compatible with the value of k or $1/4\pi\epsilon_0$.

MULTIPLE-CHOICE QUESTIONS

1.1 When a metal plate is given a positive charge, which of the following is taking place?

a) Protons (positive charges) are transferred to the plate from another object.

b) Electrons (negative charges) are transferred from the plate to another object.

c) Electrons (negative charges) are transferred from the plate to another object, and protons (positive charges) are also transferred to the plate from another object.

d) It depends on whether the object conveying the charge is a conductor or an insulator.

1.2 The force between a charge of 25 μC and a charge of −10 μC is 8.0 N. What is the separation between the two charges?

a) 0.28 m
c) 0.45 m
b) 0.53 m
d) 0.15 m

1.3 A charge Q_1 is positioned on the x-axis at $x = a$. Where should a charge $Q_2 = -4Q_1$ be placed to produce a net electrostatic force of zero on a third charge, $Q_3 = Q_1$, located at the origin?

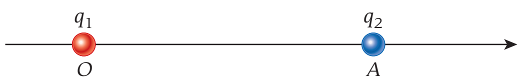
a) at the origin
c) at $x = -2a$
b) at $x = 2a$
d) at $x = -a$

1.4 Which one of these systems has the most negative charge?

a) 2 electrons
b) 3 electrons and 1 proton
c) 5 electrons and 5 protons
d) N electrons and $N - 3$ protons
e) 1 electron

1.5 Two point charges are fixed on the x-axis: $q_1 = 6.0$ μC is located at the origin, O, with $x_1 = 0.0$ cm, and $q_2 = -3.0$ μC is located at point A, with $x_2 = 8.0$ cm. Where should a third charge, q_3, be placed on the x-axis so that the total electrostatic force acting on it is zero?

a) 19 cm
c) 0.0 cm
e) −19 cm
b) 27 cm
d) 8.0 cm



1.6 Which of the following situations produces the largest net force on the charge Q?

a) Charge $Q = 1$ C is 1 m from a charge of −2 C.

b) Charge $Q = 1$ C is 0.5 m from a charge of −1 C.

c) Charge $Q = 1$ C is halfway between a charge of −1 C and a charge of 1 C that are 2 m apart.

d) Charge $Q = 1$ C is halfway between two charges of −2 C that are 2 m apart.

e) Charge $Q = 1$ C is a distance of 2 m from a charge of −4 C.

1.7 Two protons placed near one another with no other objects close by would

a) accelerate away from each other.
b) remain motionless.
c) accelerate toward each other.
d) be pulled together at constant speed.
e) move away from each other at constant speed.

1.8 Two lightweight metal spheres are suspended near each other from insulating threads. One sphere has a net charge; the other sphere has no net charge. The spheres will

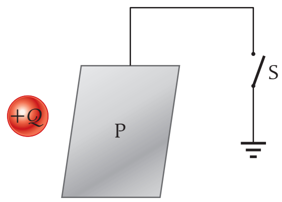
a) attract each other.

b) exert no net electrostatic force on each other.

c) repel each other.

d) do any of these things depending on the sign of the net charge on the one sphere.

1.9 A metal plate is connected by a conductor to a ground through a switch. The switch is initially closed. A charge $+Q$ is brought close to the plate without touching it, and then the switch is opened. After the switch is opened, the charge $+Q$ is removed. What is the charge on the plate then?



a) The plate is uncharged.

b) The plate is positively charged.

c) The plate is negatively charged.

d) The plate could be either positively or negatively charged, depending on the charge it had before $+Q$ was brought near.

1.10 You bring a negatively charged rubber rod close to a grounded conductor without touching it. Then you disconnect the ground. What is the sign of the charge on the conductor after you remove the charged rod?

a) negative
b) positive
c) no charge
d) cannot be determined from the information given

1.11 When a rubber rod is rubbed with rabbit fur, the rod becomes

a) negatively charged.

b) positively charged.

c) neutral.

d) either negatively charged or positively charged, depending on whether the fur is always moved in the same direction or is moved back and forth.

1.12 When a glass rod is rubbed with a polyester scarf, the rod becomes

a) negatively charged.

b) positively charged.

c) neutral.

d) either negatively charged or positively charged, depending on whether the scarf is always moved in the same direction or is moved back and forth.

1.13 Consider an electron with mass m and charge $-e$ moving in a circular orbit with radius r around a fixed proton with mass M and charge $+e$. The electron is held in orbit by the electrostatic force between itself and the proton. Which one of the following expressions for the speed of the electron is correct?

a) $v = \sqrt{\dfrac{ke^2}{mr}}$

b) $v = \sqrt{\dfrac{GM}{r}}$

c) $v = \sqrt{\dfrac{2ke^2}{mr^2}}$

d) $v = \sqrt{\dfrac{me^2}{kr}}$

e) $v = \sqrt{\dfrac{ke^2}{2Mr}}$

1.14 Consider an electron with mass m and charge $-e$ located a distance r from a fixed proton with mass M and charge $+e$. The electron is released from rest. Which one of the following expressions for the magnitude of the initial acceleration of the electron is correct?

a) $a = \dfrac{2ke^2}{mMr}$

b) $a = \sqrt{\dfrac{2e^2}{mkr}}$

c) $a = \frac{1}{2}me^2k^2$

d) $a = \dfrac{2ke^2}{mr}$

e) $a = \dfrac{ke^2}{mr^2}$

CONCEPTUAL QUESTIONS

1.15 If two charged particles (the charge on each is Q) are separated by a distance d, there is a force F between them. What is the force if the magnitude of each charge is doubled and the distance between them changes to $2d$?

1.16 Suppose the Sun and the Earth were each given an equal amount of charge of the same sign, just sufficient to cancel their gravitational attraction. How many times the charge on an electron would that charge be? Is this number a large fraction of the number of charges of either sign in the Earth?

1.17 It is apparent that the electrostatic force is *extremely* strong, compared to gravity. In fact, the electrostatic force is the basic force governing phenomena in daily life–the tension in a string, the normal forces between surfaces, friction, chemical reactions, and so forth–except weight. Why then did it take so long for scientists to understand this force? Newton came up with his gravitational law long before electricity was even crudely understood.

1.18 Occasionally, people who gain static charge by shuffling their feet on the carpet will have their hair stand on end. Why does this happen?

1.19 Two positive charges, each equal to Q, are placed a distance $2d$ apart. A third charge, $-0.2Q$, is placed exactly halfway between the two positive charges and is displaced a distance $x \ll d$ (that is, x is much smaller than d) perpendicular to the line connecting the positive charges. What is the force on this charge? For $x \ll d$, how can you approximate the motion of the negative charge?

1.20 Why does a garment taken out of a clothes dryer sometimes cling to your body when you wear it?

1.21 Two charged spheres are initially a distance d apart. The magnitude of the force on each sphere is F. They are moved closer to each other such that the magnitude of the force on each of them is $9F$. By what factor has the distance between the two spheres changed?

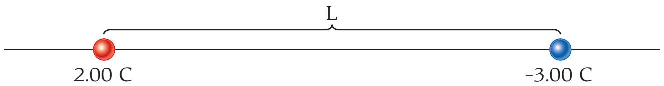
1.22 How is it possible for one electrically neutral atom to exert an electrostatic force on another electrically neutral atom?

1.23 The scientists who first contributed to the understanding of the electrostatic force in the 18th century were well aware of Newton's law of gravitation. How could they deduce that the force they were studying was *not* a variant or some manifestation of the gravitational force?

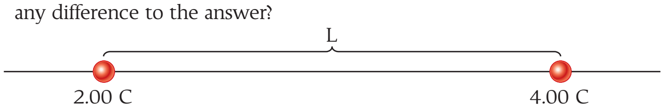
1.24 Two charged particles move solely under the influence of the electrostatic forces between them. What shapes can their trajectories have?

1.25 Rubbing a balloon causes it to become negatively charged. The balloon then tends to cling to the wall of a room. For this to happen, must the wall be positively charged?

1.26 Two electric charges are placed on a line, as shown in the figure. Is it possible to place a charged particle (that is free to move) anywhere on the line between the two charges and have it not move?



1.27 Two electric charges are placed on a line as shown in the figure. Where on the line can a third charge be placed so that the force on that charge is zero? Does the sign or the magnitude of the third charge make any difference to the answer?



1.28 When a positively charged rod is brought close to a neutral conductor without touching it, will the rod experience an attractive force, a repulsive force, or no force at all? Explain.

1.29 When you exit a car and the humidity is low, you often experience a shock from static electricity created by sliding across the seat. How can you discharge yourself without experiencing a painful shock? Why is it dangerous to get back into your car while fueling your car?

EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

Section 1.2

1.30 How many electrons are required to yield a total charge of 1.00 C?

1.31 The *faraday* is a unit of charge frequently encountered in electrochemical applications and named for the British physicist and chemist Michael Faraday. It consists of exactly 1 mole of elementary charges. Calculate the number of coulombs in 1.000 faraday.

1.32 Another unit of charge is the *electrostatic unit* (esu). It is defined as follows: Two point charges, each of 1 esu and separated by 1 cm, exert a force of exactly 1 dyne on each other: 1 dyne $= 1$ g cm/s^2 $= 1 \times 10^{-5}$ N.

a) Determine the relationship between the esu and the coulomb.

b) Determine the relationship between the esu and the elementary charge.

1.33 A current of 5.00 mA is enough to make your muscles twitch. Calculate how many electrons flow through your skin if you are exposed to such a current for 10.0 s.

•**1.34** How many electrons does 1.00 kg of water contain?

•**1.35** The Earth is constantly being bombarded by cosmic rays, which consist mostly of protons. These protons are incident on the Earth's atmosphere from all directions at a rate of 1245. protons per square meter per second. Assuming that the depth of Earth's atmosphere is 120.0 km, what is the total charge incident on the atmosphere in 5.000 min? Assume that the radius of the surface of the Earth is 6378. km.

•**1.36** Performing an experiment similar to Millikan's oil drop experiment, a student measures these charge magnitudes:

3.26×10^{-19} C $\quad$ 5.09×10^{-19} C $\quad$ 1.53×10^{-19} C

6.39×10^{-19} C $\quad$ 4.66×10^{-19} C

Find the charge on the electron using these measurements.

Section 1.3

•**1.37** A silicon sample is doped with phosphorus at 1 part per 1.00×10^6. Phosphorus acts as an electron donor, providing one free electron per atom. The density of silicon is 2.33 g/cm^3, and its atomic mass is 28.09 g/mol.

a) Calculate the number of free (conduction) electrons per unit volume of the doped silicon.

b) Compare the result from part (a) with the number of conduction electrons per unit volume of copper wire, assuming that each copper atom produces one free (conduction) electron. The density of copper is 8.96 g/cm^3, and its atomic mass is 63.54 g/mol.

Section 1.5

1.38 Two charged spheres are 8.00 cm apart. They are moved closer to each other by enough that the force on each of them increases four times. How far apart are they now?

1.39 Two identically charged particles separated by a distance of 1.00 m repel each other with a force of 1.00 N. What is the magnitude of the charges?

1.40 How far apart must two electrons be placed on the Earth's surface for there to be an electrostatic force between them equal to the weight of one of the electrons?

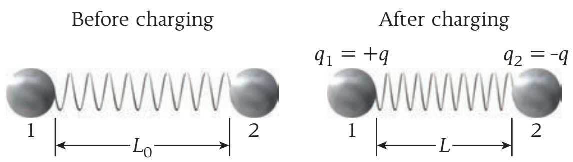
1.41 In solid sodium chloride (table salt), chloride ions have one more electron than they have protons, and sodium ions have one more proton than they have electrons. These ions are separated by about 0.28 nm. Calculate the electrostatic force between a sodium ion and a chloride ion.

1.42 In gaseous sodium chloride, chloride ions have one more electron than they have protons, and sodium ions have one more proton than they have electrons. These ions are separated by about 0.24 nm. Suppose a free electron is located 0.48 nm above the midpoint of the sodium chloride molecule. What are the magnitude and the direction of the electrostatic force the molecule exerts on it?

1.43 Calculate the magnitude of the electrostatic force the two up quarks inside a proton exert on each other if they are separated by a distance of 0.900 fm.

1.44 A −4.00 μC charge lies 20.0 cm to the right of a 2.00 μC charge on the x-axis. What is the force on the 2.00-μC charge?

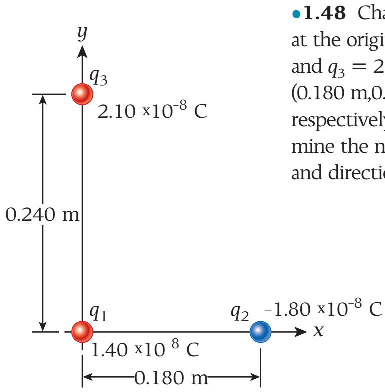
•**1.45** Two initially uncharged identical metal spheres, 1 and 2, are connected by an insulating spring (unstretched length $L_0 = 1.00$ m, spring constant $k = 25.0$ N/m), as shown in the figure. Charges $+q$ and $-q$ are then placed on the spheres, and the spring contracts to length $L = 0.635$ m. Recall that the force exerted by a spring is $F_s = k\Delta x$, where Δx is the change in the spring's length from its equilibrium length. Determine the charge q. If the spring is coated with metal to make it conducting, what is the new length of the spring?



•**1.46** A point charge $+3q$ is located at the origin, and a point charge $-q$ is located on the x-axis at $D = 0.500$ m. At what location on the x-axis will a third charge, q_0, experience no net force from the other two charges?

•**1.47** Identical point charges Q are placed at each of the four corners of a rectangle measuring 2.00 m by 3.00 m. If $Q = 32.0$ μC, what is the magnitude of the electrostatic force on any one of the charges?

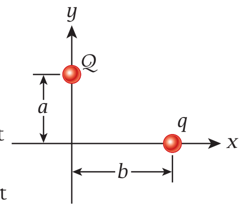
•**1.48** Charge $q_1 = 1.40 \times 10^{-8}$ C is placed at the origin. Charges $q_2 = -1.80 \times 10^{-8}$ C and $q_3 = 2.10 \times 10^{-8}$ C are placed at points (0.180 m,0.000 m) and (0.000 m,0.240 m), respectively, as shown in the figure. Determine the net electrostatic force (magnitude and direction) on charge q_3.



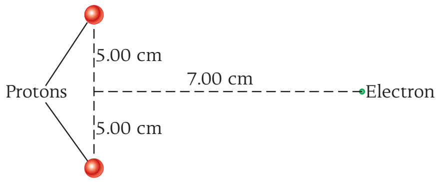
•**1.49** A positive charge Q is on the y-axis at a distance a from the origin, and another positive charge q is on the x-axis at a distance b from the origin.

a) For what value(s) of b is the x-component of the force on q a minimum?

b) For what value(s) of b is the x-component of the force on q a maximum?



•**1.50** Find the magnitude and direction of the electrostatic force acting on the electron in the figure.



•**1.51** In a region of two-dimensional space, there are three fixed charges: +1.00 mC at (0,0), −2.00 mC at (17.0 mm,−5.00 mm), and +3.00 mC at (−2.00 mm,11.0 mm). What is the net force on the −2.00 mC charge?

•**1.52** Two cylindrical glass beads each of mass $m = 10.0$ mg are set on their flat ends on a horizontal insulating surface separated by a distance $d = 2.00$ cm. The coefficient of static friction between the beads and the surface is $\mu_s = 0.200$. The beads are then given identical charges (magnitude and sign). What is the minimum charge needed to start the beads moving?

•**1.53** A small ball with a mass of 30.0 g and a charge of −0.200 μC is suspended from the ceiling by a string. The ball hangs at a distance of 5.00 cm above an insulating floor. If a second small ball with a mass of 50.0 g and a charge of 0.400 μC is rolled directly beneath the first ball, will the second ball leave the floor? What is the tension in the string when the second ball is directly beneath the first ball?

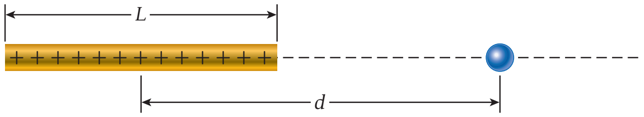
•**1.54** A +3.00 mC charge and a −4.00 mC charge are fixed in position and separated by 5.00 m.

a) Where can a +7.00 mC charge be placed so that the net force on it is zero?

b) Where can a −7.00 mC charge be placed so that the net force on it is zero?

•**1.55** Four point charges, q, are fixed to the four corners of a square that is 10.0 cm on a side. An electron is suspended above a point at which its weight is balanced by the electrostatic force due to the four electrons, at a distance of 15.0 nm above the center of the square. What is the magnitude of the fixed charges? Express the charge both in coulombs and as a multiple of the electron's charge.

••**1.56** The figure shows a uniformly charged thin rod of length L that has total charge Q. Find an expression for the magnitude of the electrostatic force acting on an electron positioned on the axis of the rod at a distance d from the midpoint of the rod.



••**1.57** A negative charge, $-q$, is fixed at the coordinate (0,0). It is exerting an attractive force on a positive charge, $+q$, that is initially at coordinate (x,0). As a result, the positive charge accelerates toward the negative charge. Use the binomial expansion $(1+x)^n \approx 1+nx$, for $x \ll 1$, to show that when the positive charge moves a distance $\delta \ll x$ closer to the negative charge, the force that the negative charge exerts on it increases by $\Delta F = 2kq^2\delta/x^3$.

••**1.58** Two negative charges ($-q$ and $-q$) of equal magnitude are fixed at coordinates ($-d$,0) and (d,0). A positive charge of the same magnitude, q, and with mass m is placed at coordinate (0,0), midway between the two negative charges. If the positive charge is moved a distance $\delta \ll d$ in the positive y-direction and then released, the resulting motion will be that of a harmonic oscillator–the positive charge will oscillate between coordinates $(0,\delta)$ and $(0,-\delta)$. Find the net force acting on the positive charge when it moves to $(0,\delta)$ and use the binomial expansion $(1+x)^n \approx 1+nx$, for $x \ll 1$, to find an expression for the frequency of the resulting oscillation. (*Hint*: Keep only terms that are linear in δ.)

Section 1.6

1.59 Suppose the Earth and the Moon carried positive charges of equal magnitude. How large would the charge need to be to produce an electrostatic repulsion equal to 1.00% of the gravitational attraction between the two bodies?

1.60 The similarity of form of Newton's law of gravitation and Coulomb's Law caused some to speculate that the force of gravity is related to the electrostatic force. Suppose that gravitation is entirely electrical in nature–that an excess charge Q on the Earth and an equal and opposite excess charge $-Q$ on the Moon are responsible for the gravitational force that causes the observed orbital motion of the Moon about the Earth. What is the required size of Q to reproduce the observed magnitude of the gravitational force?

•**1.61** In the Bohr model of the hydrogen atom, the electron moves around the one-proton nucleus on circular orbits of well-determined radii, given by $r_n = n^2 a_B$, where $n = 1, 2, 3, \ldots$ is an integer that defines the orbit and $a_B = 5.29 \times 10^{-11}$m is the radius of the first (minimum) orbit, called the *Bohr radius*. Calculate the force of electrostatic interaction between the electron and the proton in the hydrogen atom for the first four orbits. Compare the strength of this interaction to the gravitational interaction between the proton and the electron.

•**1.62** Some of the earliest atomic models held that the orbital velocity of an electron in an atom could be correlated with the radius of the atom. If the radius of the hydrogen atom is 5.29×10^{-11} m and the electrostatic force is responsible for the circular motion of the electron, what is the kinetic energy of this orbital electron?

1.63 For the atom described in Problem 1.62, what is the ratio of the gravitational force between electron and proton to the electrostatic force? How does this ratio change if the radius of the atom is doubled?

•**1.64** In general, astronomical objects are not exactly electrically neutral. Suppose the Earth and the Moon each carry a charge of -1.00×10^6 C (this is approximately correct; a more precise value is identified in Chapter 2).

a) Compare the resulting electrostatic repulsion with the gravitational attraction between the Moon and the Earth. Look up any necessary data.

b) What effects does this electrostatic force have on the size, shape, and stability of the Moon's orbit around the Earth?

Additional Exercises

1.65 Eight 1.00 μC charges are arrayed along the y-axis located every 2.00 cm starting at $y = 0$ and extending to $y = 14.0$ cm. Find the force on the charge at $y = 4.00$ cm.

1.66 In a simplified Bohr model of the hydrogen atom, an electron is assumed to be traveling in a circular orbit of radius of about 5.29×10^{-11} m around a proton. Calculate the speed of the electron in that orbit.

1.67 The nucleus of a carbon-14 atom (mass = 14 amu) has diameter of 3.01 fm. It has 6 protons and a charge of $+6e$.

a) What is the force on a proton located at 3.00 fm from the surface of this nucleus? Assume that the nucleus is a point charge.

b) What is the proton's acceleration?

1.68 Two charged objects experience a mutual repulsive force of 0.100 N.
If the charge of one of the objects is reduced by half and the distance separating the objects is doubled, what is the new force?

1.69 A particle (charge = +19.0 μC) is located on the x-axis at $x = -10.0$ cm, and a second particle (charge = −57.0 μC) is placed on the x-axis at $x = +20.0$ cm. What is the magnitude of the total electrostatic force on a third particle (charge = −3.80 μC) placed at the origin ($x = 0$)?

1.70 Three point charges are positioned on the x-axis: +64.0 μC at $x = 0.00$ cm, +80.0 μC at $x = 25.0$ cm, and −160.0 μC at $x = 50.0$ cm. What is the magnitude of the electrostatic force acting on the +64.0−μC charge?

1.71 From collisions with cosmic rays and from the solar wind, the Earth has a net electric charge of approximately -6.8×10^5 C. Find the charge that must be given to a 1.0 g object for it to be electrostatically levitated close to the Earth's surface.

•**1.72** A 10.0 g mass is suspended 5.00 cm above a nonconducting flat plate, directly above an embedded charge of q (in coulombs). If the mass has the same charge, q, how much must q be so that the mass levitates (just floats, neither rising nor falling)? If the charge q is produced by adding electrons to the mass, by how much will the mass be changed?

•**1.73** Four point charges are placed at the following xy-coordinates:

$Q_1 = -1.00$ mC, at (−3.00 cm,0.00 cm)

$Q_2 = -1.00$ mC, at (+3.00 cm,0.00 cm)

$Q_3 = +1.024$ mC, at (0.00 cm,0.00 cm)

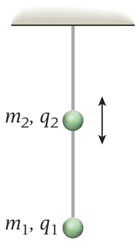
$Q_4 = +2.00$ mC, at (0.00 cm,−4.00 cm)

Calculate the net force on charge Q4 due to charges Q1, Q2, and Q3.

•**1.74** Three 5.00 g Styrofoam balls of radius 2.00 cm are coated with carbon black to make them conducting and then are tied to 1.00 m-long threads and suspended freely from a common point. Each ball is given the same charge, q. At equilibrium, the balls form an equilateral triangle with sides of length 25.0 cm in the horizontal plane. Determine q.

•**1.75** Two point charges lie on the x-axis. If one point charge is 6.00 μC and lies at the origin and the other is −2.00 μC and lies at 20.0 cm, at what position must a third charge be placed to be in equilibrium?

•**1.76** Two beads with charges $q_1 = q_2 = +2.67$ μC are on an insulating string that hangs straight down from the ceiling as shown in the figure. The lower bead is fixed in place on the end of the string and has a mass $m_1 = 0.280$ kg. The second bead slides without friction on the string. At a distance $d = 0.360$ m between the centers of



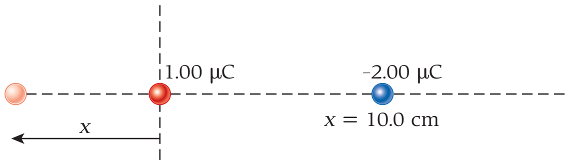
the beads, the force of the Earth's gravity on m_2 is balanced by the electrostatic force between the two beads. What is the mass, m_2, of the second bead? (*Hint:* You can neglect the gravitational interaction between the two beads.)

•**1.77** Find the net force on a +2.00-C charge at the origin of an xy-coordinate system if there is a +5.00 C charge at (3.00 m,0.00) and a −3.00 C charge at (0.00,4.00 m).

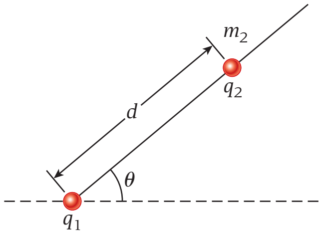
•**1.78** Two spheres, each of mass $M = 2.33$ g, are attached by pieces of string of length $L = 45.0$ cm to a common point. The strings initially hang straight down, with the spheres just touching one another. An equal amount of charge, q, is placed on each sphere. The resulting forces on the spheres cause each string to hang at an angle of $\theta = 10.0°$ from the vertical. Determine q, the amount of charge on each sphere.

•**1.79** A point charge $q_1 = 100.$ nC is at the origin of an xy-coordinate system, a point charge $q_2 = -80.0$ nC is on the x-axis at $x = 2.00$ m, and a point charge $q_3 = -60.0$ nC is on the y-axis at $y = -2.00$ m. Determine the net force (magnitude and direction) on q_1.

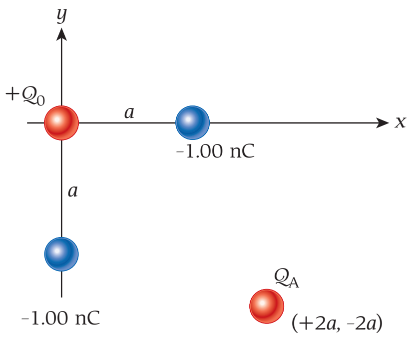
•**1.80** A positive charge $q_1 = 1.00$ μC is fixed at the origin, and a second charge $q_2 = -2.00$ μC is fixed at $x = 10.0$ cm. Where along the x-axis should a third charge be positioned so that it experiences no force?



•**1.81** A bead with charge $q_1 = 1.27$ μC is fixed in place at the end of a wire that makes an angle of $\theta = 51.3°$ with the horizontal. A second bead with mass $m_2 = 3.77$ g and a charge of 6.79 μC slides without friction on the wire. What is the distance d at which the force of the Earth's gravity on m_2 is balanced by the electrostatic force between the two beads? Neglect the gravitational interaction between the two beads.

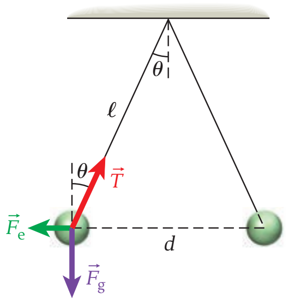


•**1.82** In the figure, the net electrostatic force on charge Q_A is zero. If $Q_A = +1.00$ nC, determine the magnitude of Q_0.



MULTI-VERSION EXERCISES

1.83 Two balls have the same mass, 0.9680 kg, and the same charge, 29.59 μC. They hang from the ceiling on strings of identical length, ℓ, as shown in the figure. If the angle of the strings with respect to the vertical is 29.79°, what is the length of the strings?



1.84 Two balls have the same mass and the same charge, 15.71 μC. They hang from the ceiling on strings of identical length, $\ell = 1.223$ m, as shown in the figure. The angle of the strings with respect to the vertical is 21.07°. What is the mass of each ball?

1.85 Two balls have the same mass, 0.9935 kg, and the same charge. They hang from the ceiling on strings of identical length, $\ell = 1.235$ m, as shown in the figure. The angle of the strings with respect to the vertical is 22.35°. What is the charge on each ball?

1.86 As shown in the figure, point charge q_1 is 3.979 μC and is located at $x_1 = -5.689$ m, and point charge q_2 is 8.669 μC and is located at $x_2 = 14.13$ m. What is the x-coordinate of the point at which the net force on a point charge of 5.000 μC will be zero?

1.87 As shown in the figure, point charge q_1 is 4.325 μC and is located at x_1, and point charge q_2 is 7.757 μC and is located at $x_2 = 14.33$ m. The x-coordinate of the point where the net force on a point charge of −3.000 μC is zero is 2.358 m. What is the value of x_1?

1.88 As shown in the figure, point charge q_1 is 4.671 μC and is located at $x_1 = -3.573$ m, and point charge q_2 is 6.845 μC and is located at x_2. The x-coordinate of the point where the net force on a point charge of −1.000 μC is zero is 4.625 m. What is the value of x_2?

Electric Fields and Gauss's Law

2

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FIGURE 2.1 A great white shark can detect tiny electric fields generated by its prey.

The great white shark is one of the most feared predators on Earth (Figure 2.1). It has several senses that have evolved for hunting prey; for example, it can smell tiny amounts of blood from as far away as 5 km (3 mi). Perhaps more amazing, it has developed special organs (called the *ampullae of Lorenzini*) that can detect the tiny electric fields generated by the movement of muscles in an organism, whether a fish, a seal, or a human. However, just what are electric fields? In addition, how are they related to electric charges?

The concept of vector fields is one of the most useful and productive ideas in all of physics. This chapter explains what an electric field is and how it is connected to electrostatic charges and forces and then examines how to determine the electric field due to some given distribution of charge. This study leads us to one of the most important laws of electricity–Gauss's Law–which provides a relationship between electric fields and electrostatic charge. However, Gauss's Law has practical application only when the charge distribution has enough geometric symmetry to simplify the calculation, and even then, some other concepts related to electric fields are necessary in order to apply the equations.

WHAT WE WILL LEARN

- An electric field represents the electric force at different points in space.
- Electric field lines represent the net force vectors exerted on a unit positive electric charge. They originate on positive charges and terminate on negative charges.
- The electric field of a point charge is radial, proportional to the charge, and inversely proportional to the square of the distance from the charge.
- An electric dipole consists of a positive charge and a negative charge of equal magnitude.
- The electric flux is the electric field component normal to an area times the area.
- Gauss's Law states that the electric flux through a closed surface is proportional to the net electric charge enclosed within the surface. This law provides simple ways to solve seemingly complicated electric field problems.
- The electric field inside a conductor is zero.
- The magnitude of the electric field due to a uniformly charged, infinitely long wire varies as the inverse of the perpendicular distance from the wire.
- The electric field due to an infinite sheet of charge does not depend on the distance from the sheet.
- The electric field outside a spherical distribution of charge is the same as the field of a point charge with the same total charge located at the sphere's center.

2.1 Definition of an Electric Field

You previously learned that the force between two or more point charges. When determining the net force exerted by other charges on a particular charge at some point in space, we obtain different directions for this force, depending on the sign of the charge that is the reference point. In addition, the net force is also proportional to the magnitude of the reference charge. The techniques used in Chapter 1 require us to redo the calculation for the net force each time we consider a different charge.

Dealing with this situation requires the concept of a **field,** which can be used to describe certain forces. An **electric field,** $E(r)$, is defined at any point in space, $\vec{r}$ as the net electric force on a charge, divided by that charge:

$$\vec{E}(\vec{r}) = \frac{\vec{F}(\vec{r})}{q} \tag{2.1}$$

The units of the electric field are newtons per coulomb (N/C). This simple definition eliminates the cumbersome dependence of the electric force on the particular charge being used to measure the force. We can quickly determine the net force on any charge by using $\vec{F}(\vec{r}) = q\vec{E}(\vec{r})$ which is a trivial rearrangement of equation 2.1.

The electric force on a charge at a point is parallel or antiparallel, depending on the sign of the charge in question) to the electric field at that point and proportional to the magnitude of the charge. The magnitude of the force is given by $F = |q|E$ The direction of the force on a positive charge is along $\vec{E}(\vec{r})$ the direction of the force on a negative charge is in the direction opposite to $\vec{E}(\vec{r})$

If several sources of electric fields are present at the same time, such as several point charges, the electric field at any given point is determined by the superposition of the electric fields from all sources. This superposition follows directly from the superposition of forces introduced in our study of mechanics and discussed earlier for electrostatic forces. The **superposition principle** for the total electric field, $\vec{E}_t$ at any point in space with coordinate $\vec{r}$ due to n electric field sources can be stated as

$$\vec{E}_t(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) + \cdots + \vec{E}_n(\vec{r}) \tag{2.2}$$

2.2 Field Lines

An electric field can (and in most applications does) change as a function of the spatial coordinate. The changing direction and strength of the electric field can be visualized by means of **electric field lines.** These graphically represent the net vector force exerted on a

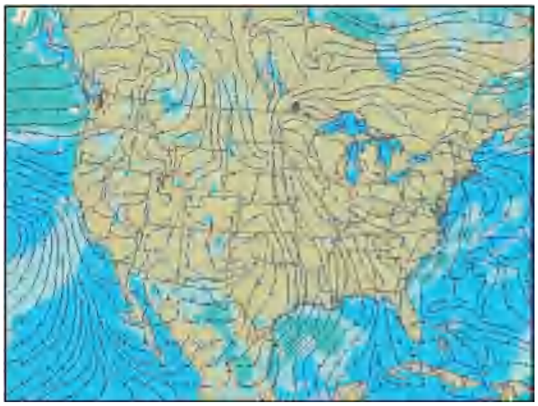


FIGURE 2.2 Streamlines of wind directions at the surface in the United States on March 23, 2008, from the National Weather Service.

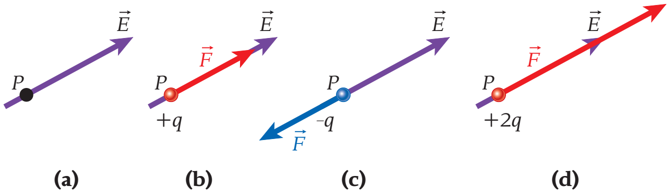


FIGURE 2.3 The force resulting from placing a charge in an electric field. (a) A point P on an electric field line. (b) A positive charge $+q$ placed at point P. (c) A negative charge $-q$ placed at point P. (d) A positive charge $+2q$ placed at point P.

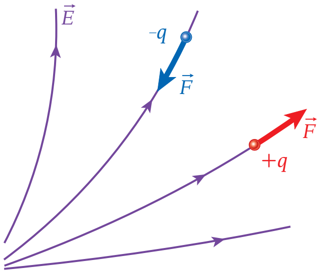


FIGURE 2.4 A nonuniform electric field. A positive charge $+q$ and a negative charge $-q$ placed in the field experience forces as shown. Each force is tangent to the electric field line.

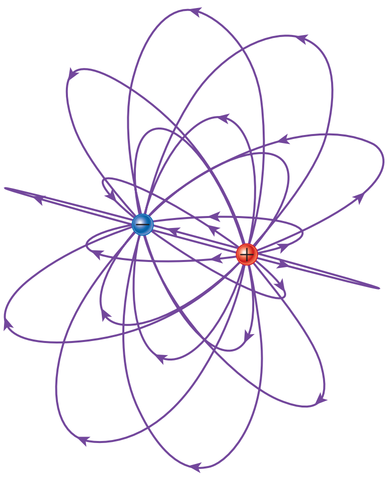


FIGURE 2.5 Three-dimensional representation of electric field lines from two point charges with opposite signs.

unit positive test charge. The representation applies separately for each point in space where the test charge might be placed. The direction of the field line at each point is the same as the direction of the force at that point, and the density of field lines is proportional to the magnitude of the force.

Electric field lines can be compared to the streamlines of wind directions, shown in Figure 2.2. These streamlines represent the force of the wind on objects at given locations, just as the electric field lines represent the electric force at specific points. A hot-air balloon can be used as a test particle for determining these wind streamlines. For example, a hot-air balloon launched in Dallas, Texas, would float from north to south in the situation depicted in Figure 2.2. Where the wind streamlines are close together, the speed of the wind is higher, so the balloon would move faster.

To draw an electric field line, we imagine placing a tiny positive charge at each point in the electric field. This charge is small enough that it does not affect the surrounding field. A small charge like this is sometimes called a **test charge.** We calculate the resultant force on the charge, and the direction of the force gives the direction of the field line. For example, Figure 2.3a shows a point in an electric field. In Figure 2.3b, a charge $+q$ is placed at point P, on an electric field line. The force on the charge is in the same direction as the electric field. In Figure 2.3c, a charge $-q$ is placed at point P, and the resulting force is in the direction opposite to the electric field. In Figure 2.3d, a charge $+2q$ is placed at point P, and the resulting force on the charge is in the direction of the electric field, with twice the magnitude of the force on the charge $+q$. We will follow the convention of depicting a positive charge as red and a negative charge as blue.

In a nonuniform electric field, the electric force at a given point is tangent to the electric field lines at that point, as illustrated in Figure 2.4. The force on a positive charge is in the direction of the electric field, and the force on a negative charge is in the direction opposite to the electric field.

Electric field lines point away from sources of positive charge and toward sources of negative charge. Each field line starts at a charge and ends at another charge. Electric field lines always originate on positive charges and terminate on negative charges.

Electric fields exist in three dimensions (Figure 2.5); however, this chapter usually presents two-dimensional depictions of electric fields for simplicity.

Point Charge

The electric field lines arising from an isolated point charge are shown in Figure 2.6. The field lines emanate in radial directions from the point charge. If the point charge is positive (Figure 2.6a), the field lines point outward, away from the charge; if the point charge is negative, the field lines point inward, toward the charge (Figure 2.6b). For an isolated positive point charge, the electric field lines originate at the charge and terminate on negative charges at infinity, and for a negative point charge, the electric field lines originate at positive charges at infinity and terminate at the charge. Note that the electric field lines are

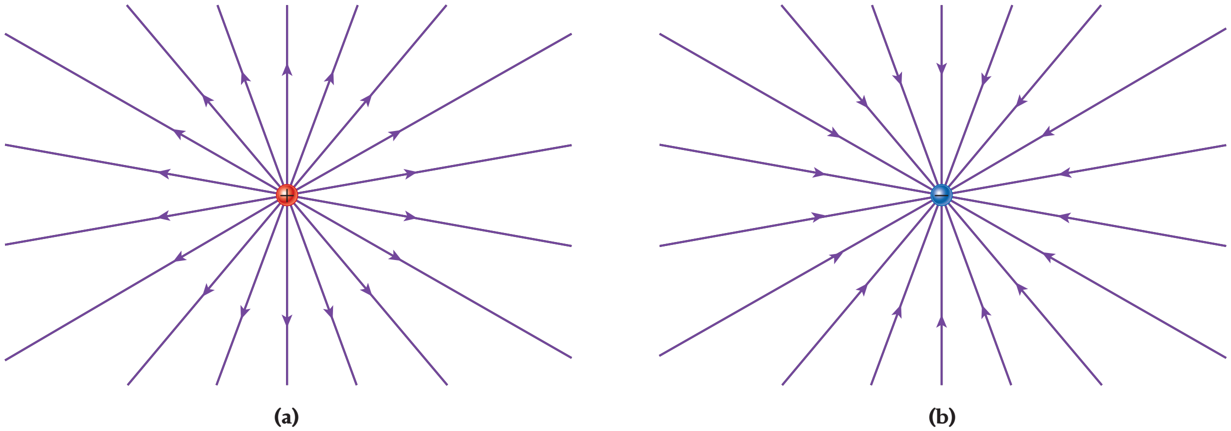


FIGURE 2.6 Electric field lines (a) from a single positive point charge and (b) to a single negative point charge.

closer together near the point charge and farther apart away from the point charge, indicating that the electric field becomes weaker with increasing distance from the charge. We'll examine the magnitude of the field quantitatively in Section 2.3.

Two Point Charges of Opposite Sign

We can use the superposition principle to determine the electric field from two point charges. Figure 2.7 shows the electric field lines for two oppositely charged point charges with the same magnitude. At each point in the plane, the electric field from the positive charge and the electric field from the negative charge add as vectors to give the magnitude and the direction of the resulting electric field. (Figure 2.5 shows the same field lines in three dimensions.)

As noted earlier, the electric field lines originate on the positive charge and terminate on the negative charge. At a point very close to either charge, the field lines are similar to those for a single point charge, since the effect of the more distant charge is small. Near the charges, the electric field lines are close together, indicating that the field is stronger in those regions. The fact that the field lines between the two charges connect indicates that an attractive force exists between the two charges.

Two Point Charges with the Same Sign

We can also apply the principle of superposition to two point charges with the same sign. Figure 2.8 shows the electric field lines for two point charges with the same sign and same magnitude. If both charges are positive (as in Figure 2.8), the electric field lines originate at the charges and terminate at infinity. If both charges are negative, the field lines originate at infinity and terminate at the charges. For two charges of the same sign, the field lines do not

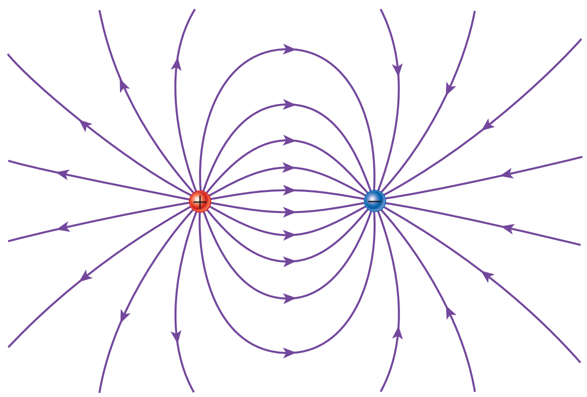


FIGURE 2.7 Electric field lines from two oppositely charged point charges. Each charge has the same magnitude.

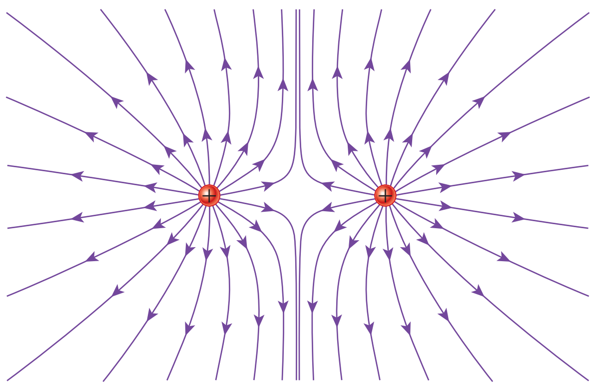
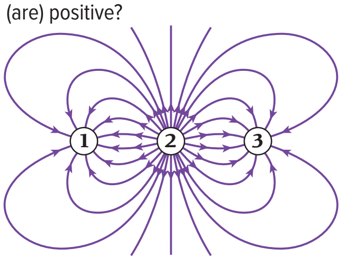


FIGURE 2.8 Electric field lines from two positive point charges with the same magnitude.

Concept Check 2.1

Which of the charges in the figure is (are) positive?



a) 1

b) 2

c) 3

d) 1 and 3

e) All three charges are positive.

connect the two charges. Rather, the field lines terminate on opposite charges at infinity or at the two charges themselves. The fact that the field lines never terminate on the other charge signifies that the charges repel each other.

General Observations

The three simplest possible cases that we just examined lead to two general rules that apply to all field lines of all charge configurations:

1. *Field lines originate at positive charges and terminate at negative charges.*
2. *Field lines never cross.* This result is a consequence of the fact that the lines represent the electric field, which in turn is proportional to the net force that acts on a charge placed at a particular point. Field lines that crossed would imply that the net force points in two different directions at the same point, which is impossible.

Concept Check 2.2

Assuming that there are no charges in the four regions shown in the figure, which of the patterns could represent an electric field?

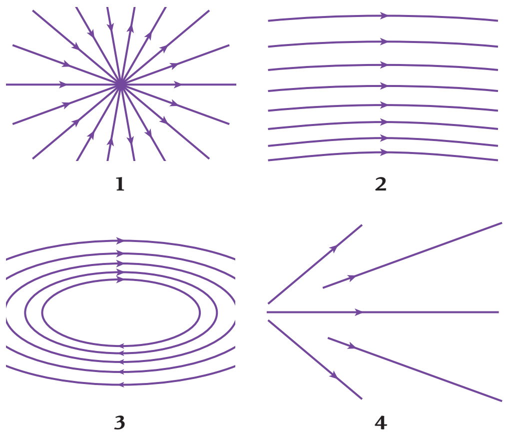
a) only 1

b) only 2

c) 2 and 3

d) 1 and 4

e) None of the patterns represent an electric field.



2.3 Electric Field due to Point Charges

The magnitude of the electric force on a point charge q_0 due to another point charge, q, is given by

$$F = \frac{1}{4\pi\varepsilon_0}\frac{|qq_0|}{r^2} \tag{2.3}$$

Taking q_0 to be a small test charge, we can express the magnitude of the electric field at the point where q_0 is and due to the point charge q as

$$E = \left|\frac{F}{q_0}\right| = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{r^2} \tag{2.4}$$

where r is the distance from the test charge to the point charge. The direction of this electric field is radial. The field points outward for a positive point charge and inward for a negative point charge.

An electric field is a vector quantity, and thus the components of the field must be added separately. Example 2.1 demonstrates the addition of electric fields created by three point charges.

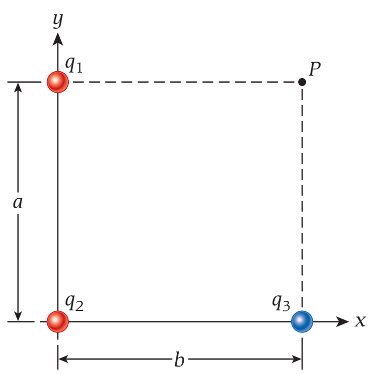


FIGURE 2.9 Locations of three point charges.

EXAMPLE 2.1 Three Charges

Figure 2.9 shows three fixed point charges: $q_1 = +1.50\ \mu C$, $q_2 = +2.50\ \mu C$, and $q_3 = -3.50\ \mu C$. Charge q_1 is located at $(0,a)$, q_2 is located at $(0,0)$, and q_3 is located at $(b,0)$, where $a = 8.00$ m and $b = 6.00$ m.

PROBLEM

What electric field, $\vec{E}$ do these three charges produce at the point $P = (b,a)$?

SOLUTION

We must sum the electric fields from the three charges using equation 2.2. We proceed by summing component by component, starting with the field due to q_1:

$$\vec{E}_1 = E_{1,x}\hat{x} + E_{1,y}\hat{y}$$

The field due to q_1 acts only in the x-direction at point (b,a), because q_1 has the same y-coordinate as P. Thus, $\vec{E}_1 = E_{1,x}\hat{x}$ We can determine $E_{1,x}$ using equation 2.4:

$$E_{1,x} = \frac{kq_1}{b^2}$$

Similarly, the field due to q_3 acts only in the y-direction at point (b,a). Thus, $\vec{E}_3 = E_{3,y}\hat{y}$ where

$$E_{3,y} = \frac{kq_3}{a^2}$$

As shown in Figure 2.10, the electric field due to q_2 at P is given by

$$\vec{E}_2 = E_{2,x}\hat{x} + E_{2,y}\hat{y}$$

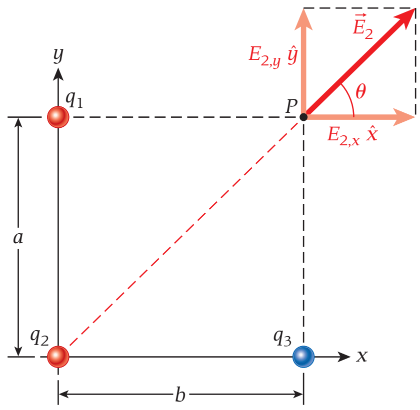


FIGURE 2.10 Electric field due to q_2 and its x- and y-components at point P.

Note that $\vec{E}_2$ the electric field due to q_2 at point P, points directly away from q_2, because $q_2 > 0$. (It would point directly toward q_2 if this charge were negative.) The magnitude of this electric field is given by

$$E_2 = \frac{k|q_2|}{a^2 + b^2}$$

The component $E_{2,x}$ is given by $E_2 \cos\theta$, where $\theta = \tan^{-1}(a/b)$, and the component $E_{2,y}$ is given by $E_2 \sin\theta$.

Adding the components, the total electric field at point P is

$$\begin{aligned}\vec{E} &= \left(E_{1,x} + E_{2,x}\right)\hat{x} + \left(E_{2,y} + E_{3,y}\right)\hat{y}\\ &= \underbrace{\left(\frac{kq_1}{b^2} + \frac{kq_2\cos\theta}{a^2+b^2}\right)}_{E_x}\hat{x} + \underbrace{\left(\frac{kq_2\sin\theta}{a^2+b^2} + \frac{kq_3}{a^2}\right)}_{E_y}\hat{y}\end{aligned}$$

With the given values for a and b, we find $\theta = \tan^{-1}(8/6) = 53.1°$, and $a^2 + b^2 = (8.00\text{ m})^2 + (6.00\text{ m})^2 = 100\text{ m}^2$. We can then calculate the x-component of the total electric field as

$$E_x = \left(8.99{\times}10^9\ \text{N m}^2/\text{C}^2\right)\left(\frac{1.50{\times}10^{-6}\ \text{C}}{(6.00\text{ m})^2} + \frac{\left(2.50{\times}10^{-6}\ \text{C}\right)\left(\cos 53.1°\right)}{100\text{ m}^2}\right) = 509\text{ N/C}$$

The y-component is

$$E_y = \left(8.99{\times}10^9\ \text{N m}^2/\text{C}^2\right)\left(\frac{\left(2.50{\times}10^{-6}\ \text{C}\right)\left(\sin 53.1°\right)}{100\text{ m}^2} + \frac{-3.50{\times}10^{-6}\ \text{C}}{(8.00\text{ m})^2}\right) = -312\text{ N/C}$$

The magnitude of the field is

$$E = \sqrt{E_x^2 + E_y^2} = \sqrt{(509\text{ N/C})^2 + (-312\text{ N/C})^2} = 597\text{ N/C}$$

The direction of the field at point P is

$$\varphi = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}\left(\frac{-312\text{ N/C}}{509\text{ N/C}}\right) = -31.5°$$

which means that the electric field points to the right and downward.

Note that even though the charges in this example are in microcoulombs and the distances are in meters, the electric fields are still large, showing that a microcoulomb is a large amount of charge.

2.4 Electric Field due to a Dipole

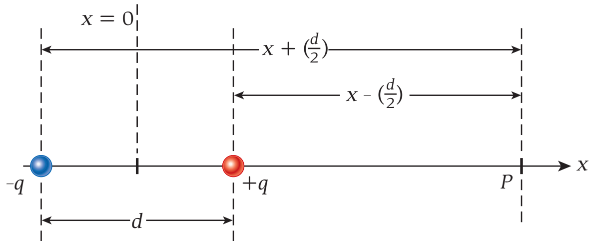


FIGURE 2.11 Calculation of the electric field from an electric dipole.

A system of two equal (in magnitude) but oppositely charged point particles is called an **electric dipole.** The electric field from an electric dipole is given by the vector sum of the electric fields from the two charges. Figure 2.7 shows the electric field lines in two dimensions for an electric dipole.

The superposition principle allows us to determine the electric field due to two point charges through vector addition of the electric fields of the two charges. Let's consider the special case of the electric field due to a dipole along the axis of the dipole, defined as the line connecting the charges. This main symmetry axis of the dipole is assumed to be oriented along the x-axis (Figure 2.11).

The electric field, $\vec{E}$ at point P on the dipole axis is the sum of the field due to $+q$, denoted as $\vec{E}_+$ and the field due to $-q$, denoted as $\vec{E}_-$

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

Using equation 2.4, we can express the magnitude of the dipole's electric field along the x-axis, for $x > d/2$, as

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_+^2} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{r_-^2}$$

where r_+ is the distance between P and $+q$ and r_- is the distance between P and $-q$. Absolute value bars are not needed in this equation, because the first term on the right-hand side is positive and is greater than the second (negative) term. The electric field at all points on the x-axis (except at $x = \pm d/2$, where the two charges are located) is given by

$$\vec{E} = E_x\hat{x} = \frac{1}{4\pi\varepsilon_0}\frac{q(x - d/2)}{r_+^3}\hat{x} + \frac{1}{4\pi\varepsilon_0}\frac{-q(x + d/2)}{r_-^3}\hat{x} \tag{2.5}$$

Now we examine the magnitude of $\vec{E}$ and restrict the value of x to $x > d/2$, where $E = E_x > 0$. Then we have

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(x - \frac{1}{2}d\right)^2} - \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(x + \frac{1}{2}d\right)^2}$$

With some rearrangement and keeping in mind that we want to obtain an expression that has the same form as the electric field from a point charge, we write the preceding equation as

$$E = \frac{q}{4\pi\varepsilon_0 x^2}\left[\left(1 - \frac{d}{2x}\right)^{-2} - \left(1 + \frac{d}{2x}\right)^{-2}\right]$$

To find an expression for the electric field at a large distance from the dipole, we can make the approximation $x \gg d$ and use the binomial expansion. (Since $x \gg d$, we can drop terms containing the square of d/x and higher powers.) We obtain

$$E \approx \frac{q}{4\pi\varepsilon_0 x^2}\left[\left(1 + \frac{d}{x} - \cdots\right) - \left(1 - \frac{d}{x} + \cdots\right)\right] = \frac{q}{4\pi\varepsilon_0 x^2}\left(\frac{2d}{x}\right)$$

which can be rewritten as

$$E \approx \frac{qd}{2\pi\varepsilon_0 x^3} \tag{2.6}$$

Equation 2.6 can be simplified by defining a vector quantity called the **electric dipole moment,** $\vec{p}$ The direction of this dipole moment is from the negative charge to the positive charge, which is opposite to the direction of the electric field lines. The magnitude, p, of the electric dipole moment is given by

$$p = qd \tag{2.7}$$

where q is the magnitude of either of the charges and d is the distance separating the two charges. With this definition, the expression for the magnitude of the electric field due to the dipole along the positive x-axis at a distance that is large compared with the separation between the two charges is

$$E = \frac{p}{2\pi\varepsilon_0 |x|^3} \tag{2.8}$$

Although not shown explicitly here, equation 2.8 is also valid for $x = \ll -d$. Also, an examination of equation 2.5 for $\vec{E}$ shows that $E_x > 0$ on either side of the dipole. In contrast to the field due to a point charge, which is inversely proportional to the square of the distance, the field due to a dipole is inversely proportional to the cube of the distance, according to equation 2.8.

EXAMPLE 2.2 Water Molecule

The water molecule, H_2O, is arguably the most important one for life. It has a nonzero dipole moment, which is the basic reason why many organic molecules are able to bind to water. This dipole moment also allows water to be an excellent solvent for many inorganic and organic compounds.

Each water molecule consists of two atoms of hydrogen and one atom of oxygen, as shown in Figure 2.12a. The charge distribution of each of the individual atoms is approximately spherical. The oxygen atom tends to pull the negatively charged electrons toward itself, giving the hydrogen atoms slight positive charges. The three atoms are arranged so that the lines connecting the centers of the hydrogen atoms with the center of the oxygen atom have an angle of 105° between them (see Figure 2.12a).

PROBLEM

Suppose we approximate a water molecule by two positive charges at the locations of the two hydrogen nuclei (protons) and two negative charges at the location of the oxygen nucleus, with all charges of equal magnitude. What is the resulting electric dipole moment of water?

SOLUTION

The center of charge of the two positive charges, analogous to the center of mass of two masses, is located exactly halfway between the centers of the hydrogen atoms, as shown in Figure 2.12b. With the hydrogen-oxygen distance of $\Delta r = 10^{-10}$ m, as indicated in Figure 2.12a, the distance between the positive and negative charge centers is

$$d = \Delta r \cos\left(\frac{\theta}{2}\right) = \left(10^{-10}\ \text{m}\right)\left(\cos 52.5°\right) = 0.6\times10^{-10}\,\text{m}$$

This distance times the transferred charge, $q = 2e$, is the magnitude of the dipole moment of water:

$$p = 2ed = \left(3.2\times10^{-19}\ \text{C}\right)\left(0.6\times10^{-10}\ \text{m}\right) = 2\times10^{-29}\ \text{C m}$$

This result of an extremely oversimplified calculation actually comes close, within a factor of 3, to the measured value of 6.2×10^{-30} C m. The fact that the real dipole moment of water is smaller than this calculated result is an indication that the two electrons of the hydrogen atoms are not pulled all the way to the oxygen but, on average, only one-third of the way.

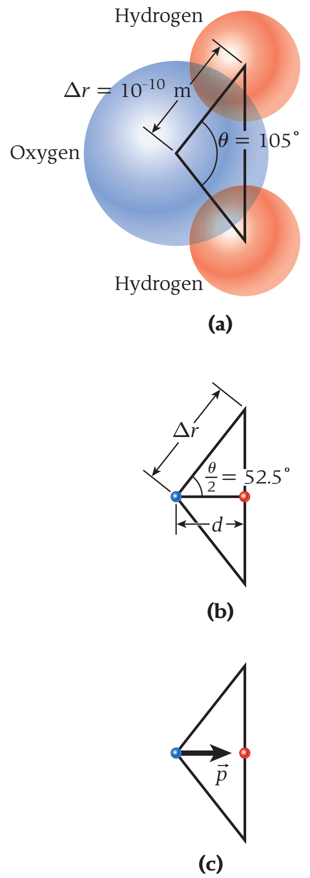


FIGURE 2.12 (a) Schematic drawing showing the geometry of a water molecule, H_2O, with atoms as spheres. (b) Diagram showing the effective positive (red dot on the right) and negative (blue dot on the left) charge centers. (c) Dipole moment assuming pointlike charges.

Concept Check 2.4

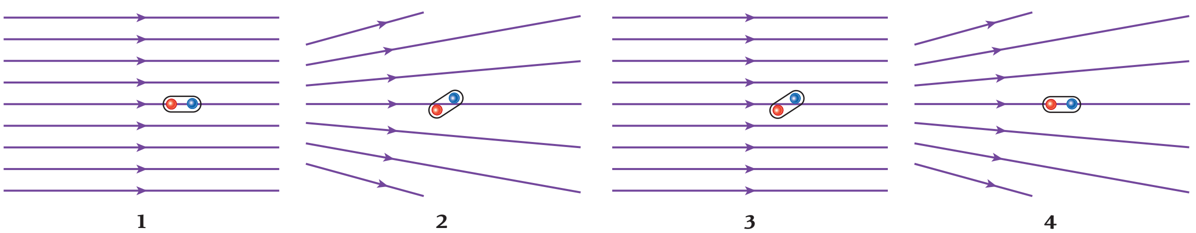
An electrically neutral dipole is placed in an external electric field as shown in the figure in Concept Check 2.3. In which situation(s) is the net *torque* on the dipole zero?

a) 1 and 3
b) 2 and 4
c) 1 and 4
d) 2 and 3
e) 1 only

Concept Check 2.3

An electrically neutral dipole is placed in an external electric field as shown in the figure. In which situation(s) is the net *force* on the dipole zero?

a) 1 and 3
b) 2 and 4
c) 1 and 4
d) 2 and 3
e) 1 only



2.5 General Charge Distributions

We have determined the electric fields of a single point charge and of two point charges (an electric dipole). What if we want to determine the electric field due to many charges? Each individual charge creates an electric field, as described by equation 2.4, and because of the superposition principle, all of these electric fields can be added to find the net field at any point in space. But we have already seen in Example 2.1 that the addition of electric field vectors can be cumbersome for a collection of only three point charges. If we had to apply this method to, say, trillions of point charges, the task would be unmanageable even if we could use a supercomputer. Since real-world applications usually involve a very large number of charges, it is clear that we need a way to simplify the calculations. This can be accomplished by using an integral, if the large number of charges are arranged in space in some regular distribution. Of particular interest are two-dimensional distributions, where charges are located on the surface of a metallic object, and one-dimensional distributions, where charges are arranged along a wire. As we will see, integration can be a surprisingly simple way to solve problems involving such charge distributions, which would be very hard to analyze by the method of direct summation.

To prepare for the integration procedure, we divide the charge into differential elements of charge, dq, and find the electric field resulting from each differential charge element as if it were a point charge. If the charge is distributed along a one-dimensional object (a line), the differential charge may be expressed in terms of a charge per unit length times a differential length, or λdx. If the charge is distributed over a surface (a two-dimensional object), dq is expressed in terms of a charge per unit area times a differential area, or σdA. And, finally, if the charge is distributed over a three-dimensional volume, then dq is written as the product of a charge per unit volume and a differential volume, or ρdV. That is,

$$\left.\begin{aligned} dq &= \lambda dx \\ dq &= \sigma dA \\ dq &= \rho dV \end{aligned}\right\} \text{ for a charge distribution } \begin{cases} \text{along a line;} \\ \text{over a surface;} \\ \text{throughout a volume.} \end{cases} \tag{2.9}$$

The magnitude of the electric field resulting from the charge distribution is then obtained from the differential charge:

$$dE = k\frac{dq}{r^2} \tag{2.10}$$

In the following example, we find the electric field due to a finite line of charge.

EXAMPLE 2.3 Finite Line of Charge

To find the electric field along a line bisecting a finite length of wire with linear charge density λ, we integrate the contributions to the electric field from all the charge in the wire. We assume that the wire lies along the x-axis (Figure 2.13).

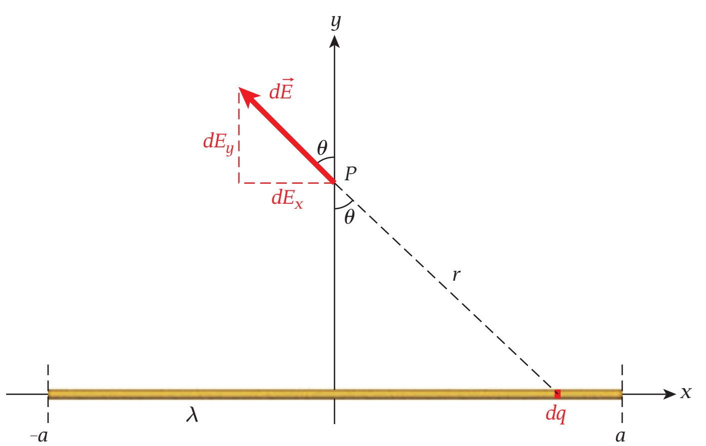


FIGURE 2.13 Calculating the electric field due to all the charge in a long wire by integrating the contributions to the electric field over the length of the wire.

We also assume that the wire is positioned with its midpoint at $x = 0$, one end at $x = a$, and the other end at $x = -a$. The symmetry of the situation then allows us to conclude that there cannot be any electric force parallel to the wire (in the x-direction) along the line bisecting the wire. Along this line, the electric field can be only in the y-direction. We can then calculate the electric field due to all the charge for $x \geq 0$ and multiply the result by 2 to get the electric field for the whole wire.

We consider a differential charge, dq, on the x-axis, as shown in Figure 2.13. The magnitude of the electric field, dE, at a point $(0,y)$ due to this charge is given by equation 2.10,

$$dE = k\frac{dq}{r^2}$$

where $r = \sqrt{x^2 + y^2}$ is the distance from dq to point P. The component of the electric field perpendicular to the wire (in the y-direction) is then given by

$$dE_y = k\frac{dq}{r^2}\cos\theta$$

where θ is the angle between the electric field produced by dq and the y-axis (see Figure 2.13). The angle θ is related to r and y because $\cos\theta = y/r$.

We can relate the differential charge to the differential distance along the x-axis through the linear charge density, λ: $dq = \lambda dx$. The electric field at a distance y from the long wire is then

$$E_y = 2\int_0^a dE_y = 2\int_0^a k\frac{dq}{r^2}\cos\theta = 2k\int_0^a \frac{\lambda dx}{r^2}\frac{y}{r} = 2k\lambda y\int_0^a \frac{dx}{\left(x^2+y^2\right)^{3/2}}$$

Evaluation of the integral on the right-hand side (with the aid of an integral table or a software package like Mathematica or Maple) gives us

$$\int_0^a \frac{dx}{\left(x^2+y^2\right)^{3/2}} = \left[\frac{1}{y^2}\frac{x}{\sqrt{x^2+y^2}}\right]_0^a = \frac{1}{y^2}\frac{a}{\sqrt{y^2+a^2}}$$

Thus, the electric field at a distance y along a line bisecting the wire is given by

$$E_y = 2k\lambda y\frac{1}{y^2}\frac{a}{\sqrt{y^2+a^2}} = \frac{2k\lambda}{y}\frac{a}{\sqrt{y^2+a^2}}$$

Finally, when $a \to \infty$, that is, the wire becomes infinitely long, $a/\sqrt{y^2+a^2} \to 1$ and we have for an infinitely long wire

$$E_y = \frac{2k\lambda}{y}$$

In other words, the electric field decreases in inverse proportion to the distance from the wire.

Now let's tackle a problem with a slightly more complicated geometry, finding the electric field due to a ring of charge along the axis of the ring.

SOLVED PROBLEM 2.1 Ring of Charge

PROBLEM

Consider a charged ring with radius $R = 0.250$ m (Figure 2.14). The ring has uniform linear charge density, and the total charge on the ring is $Q = +5.00\ \mu\text{C}$. What is the electric field at a distance $b = 0.500$ m along the axis of the ring?

SOLUTION

THINK The charge is evenly distributed around the ring. The electric field at position $x = b$ can be calculated by integrating the differential electric field due to a differential electric charge. By symmetry, the components of the electric field perpendicular to the axis of the ring integrate to zero, because the electric fields of charge elements on opposite sides of the axis cancel one another out. The resulting electric field is parallel to the axis of the circle.

– Continued

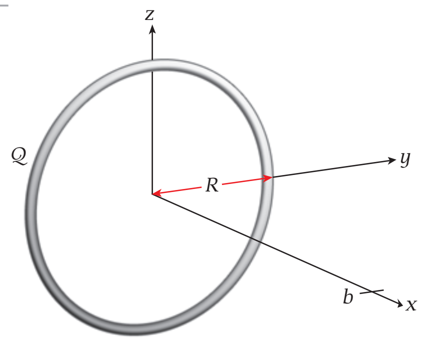


FIGURE 2.14 Charged ring with radius R and total charge Q.

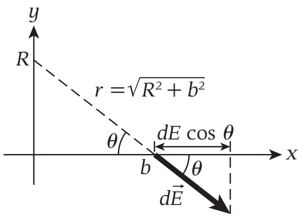


FIGURE 2.15 The geometry for the electric field along the axis of a ring of charge.

SKETCH Figure 2.15 shows the geometry for the electric field along the axis of the ring of charge.

RESEARCH The differential electric field, dE, at $x = b$ is due to a differential charge dq located at $y = R$ (see Figure 2.15). The distance from the point $(x = b, y = 0)$ to the point $(x = 0, y = R)$ is

$$r = \sqrt{R^2 + b^2}$$

Again, the magnitude of $d\vec{E}$ is given by equation 2.10:

$$dE = k\frac{dq}{r^2}$$

The magnitude of the component of $d\vec{E}$ parallel to the x-axis is given by

$$dE_x = dE\cos\theta = dE\frac{b}{r}$$

SIMPLIFY We can find the total electric field by integrating its x-components over all the charge on the ring:

$$E_x = \int_{\text{ring}} dE_x = \int_{\text{ring}} \frac{b}{r}k\frac{dq}{r^2}$$

We need to integrate around the circumference of the ring of charge. We can relate the differential charge to the differential arc length, ds, as follows:

$$dq = \frac{Q}{2\pi R}ds$$

We can then express the integral over the entire ring of charge as an integral around the arc length of a circle:

$$E_x = \int_0^{2\pi R} k\left(\frac{Q}{2\pi R}ds\right)\frac{b}{r^3} = \left(\frac{kQb}{2\pi R r^3}\right)\int_0^{2\pi R} ds = kQ\frac{b}{r^3} = \frac{kQb}{\left(R^2 + b^2\right)^{3/2}}$$

CALCULATE Putting in the numerical values, we get

$$E_x = \frac{kQb}{\left(R^2 + b^2\right)^{3/2}} = \frac{\left(8.99\times10^9\ \text{N m}^2/\text{C}^2\right)\left(5.00\times10^{-6}\ \text{C}\right)(0.500\ \text{m})}{\left[(0.250\ \text{m})^2 + (0.500\ \text{m})^2\right]^{3/2}} = 128{,}654\ \text{N/C}$$

ROUND We report our result to three significant figures:

$$E_x = 1.29\times10^5\ \text{N/C}$$

DOUBLE-CHECK We can check the validity of the formula we derived for the electric field by using a large distance from the ring of charge, such that $b \gg R$. In this case,

$$E_x = \frac{kQb}{\left(R^2 + b^2\right)^{3/2}} \overset{b \gg R}{\Rightarrow} E_x = \frac{kQb}{b^3} = k\frac{Q}{b^2}$$

which is the expression for the electric field due to a point charge Q at a distance b. We can also check the formula with $b = 0$:

$$E_x = \frac{kQb}{\left(R^2 + b^2\right)^{3/2}} \overset{b=0}{\Rightarrow} E_x = 0$$

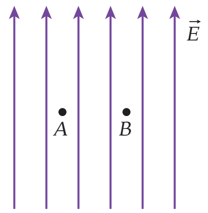
which is what we would expect at the center of a ring of charge. Thus, our result seems reasonable.

2.6 Force due to an Electric Field

The force $\vec{F}$ exerted by an electric field $\vec{E}$ on a point charge q is given by $\vec{F} = q\vec{E}$ a simple restatement of the definition of the electric field in equation 2.1. Thus, the force exerted by the electric field on a positive charge acts in the same direction as the electric field. The force vector is always tangent to the electric field lines and points in the direction of the electric field if $q > 0$.

Concept Check 2.6

A small positively charged object could be placed in a uniform electric field at position *A* or position *B* in the figure. How do the electric forces on the object at the two positions compare?



a) The magnitude of the electric force on the object is greater at position *A*.

b) The magnitude of the electric force on the object is greater at position *B*.

c) There is no electric force on the object at either position *A* or position *B*.

d) The electric force on the object at position *A* has the same magnitude as the force on the object at position *B* but is in the opposite direction.

e) The electric force on the object at position *A* is the same nonzero electric force as that on the object at position *B*.

The force at various locations on a positive charge due to the electric field in three dimensions is shown in Figure 2.16 for the case of two oppositely charged particles. (This is the same field as in Figure 2.5, but with some representative force vectors added.) You can see that the force on the positive charge is always tangent to the field lines and points in the same direction as the electric field. The force on the negative charge would point in the opposite direction.

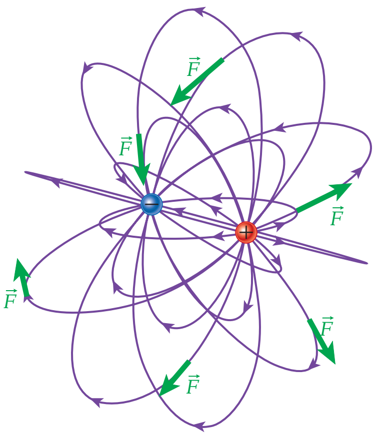


FIGURE 2.16 Direction of the force that an electric field produced by two opposite point charges exerts on a positive charge at various points in space.

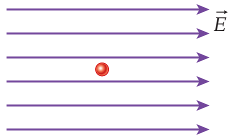
EXAMPLE 2.4 Time Projection Chamber

Nuclear physicists study new forms of matter by colliding gold nuclei at very high energies. In particle physics, new elementary particles are created and studied by colliding protons and antiprotons at the highest energies. These collisions create many particles that stream away from the interaction point at high speeds. A simple particle detector is not sufficient to identify these particles. A device that helps physicists study these collisions is a time projection chamber (TPC), found in most large particle detectors.

– Continued

Concept Check 2.5

A small positively charged object is placed at rest in a uniform electric field as shown in the figure. When the object is released, it will



a) not move.

b) begin to move with a constant speed.

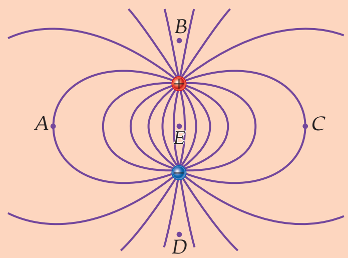
c) begin to move with a constant acceleration.

d) begin to move with an increasing acceleration.

e) move back and forth in simple harmonic motion.

Self-Test Opportunity 2.1

The figure shows a two-dimensional view of electric field lines due to two opposite charges. What is the direction of the electric field at the five points *A*, *B*, *C*, *D*, and *E*? At which of the five points is the magnitude of the electric field the largest?



One example of a TPC is the STAR TPC of the Relativistic Heavy Ion Collider at Brookhaven National Laboratory on Long Island, New York. The STAR TPC consists of a large cylinder filled with a gas (90% argon, 10% methane) that allows free electrons to move within it without being captured by the gas atoms or molecules.

Figure 2.17 shows the results of a collision of two gold nuclei that occurred in the STAR TPC. In such a collision, thousands of charged particles are created that pass through the gas inside the TPC. As these charged particles pass through the gas, they ionize the atoms of the gas, releasing free electrons. A constant electric field of magnitude 13,500 N/C is applied between the center of the TPC and the caps on the ends of the cylinder, and the field exerts an electric force on the freed electrons. Because the electrons have a negative charge, the electric field exerts a force in the direction opposite to the electric field. The electrons attempt to accelerate in the direction of the electric force, but they interact with the electrons of the molecules of the gas and begin to drift toward the caps with a constant speed of 5 cm/μs $= 5 \cdot 10^4$ m/s ≈ 100,000 mph.

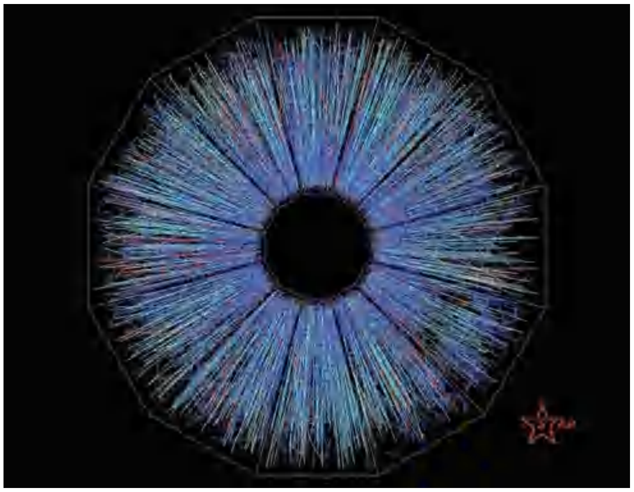


FIGURE 2.17 An event in the STAR TPC in which two gold nuclei have collided at very high energies at the point in the center of the image. Each colored line represents the track left behind by a subatomic particle produced in the collision.

Each end cap of the cylinder has 68,304 detectors that can measure the charge as a function of the drift time of the electrons from the point where they were freed. Each detector has a specific (x,y) position. From measurements of the arrival time of the charge and the known drift speed of the electrons, the z-components of their positions can be calculated. Thus, the STAR TPC can produce a complete three-dimensional representation of the ionization track of each charged particle. These tracks are shown in Figure 2.17, where the colors represent the amount of ionization produced by each track.

SOLVED PROBLEM 2.2 Electron Moving over a Charged Plate

PROBLEM

An electron with a kinetic energy of 2.00 keV ($1 \text{ eV} = 1.602 \times 10^{-19}$ J) is fired horizontally across a horizontally oriented charged conducting plate with a surface charge density of $+4.00 \times 10^{-6}$ C/m^2. Taking the positive direction to be upward (away from the plate), what is the vertical deflection of the electron after it has traveled a horizontal distance of 4.00 cm?

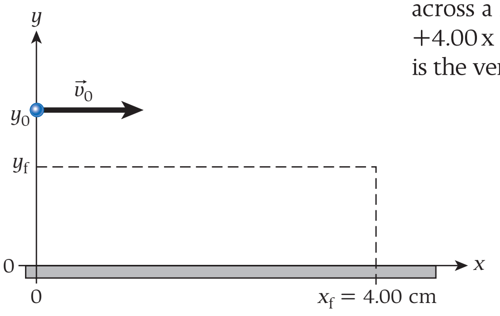


FIGURE 2.18 An electron moving to the right with initial velocity $\vec{v}_0$ over a charged conducting plate.

SOLUTION

THINK The initial velocity of the electron is horizontal. During its motion, the electron experiences a constant attractive force from the positively charged plate, which causes a constant acceleration downward. We can calculate the time it takes the electron to travel 4.00 cm in the horizontal direction and use this time to calculate the vertical deflection of the electron.

SKETCH Figure 2.18 shows the electron with initial velocity $\vec{v}_0$ in the horizontal direction. The initial position of the electron is taken to be at $x_0 = 0$ and $y = y_0$.

RESEARCH The time the electron takes to travel the given distance is

$$t = x_f/v_0 \tag{i}$$

where x_f is the final horizontal position and v_0 is the initial speed of the electron. While the electron is in motion, it experiences a force from the charged conducting plate. This force is directed downward (toward the plate) and has a magnitude given by

$$F = qE = e\frac{\sigma}{\varepsilon_0} \tag{ii}$$

where σ is the charge density on the conducting plate and e is the charge of an electron. This force causes a constant acceleration in the downward direction whose magnitude is given by $a = F/m$, where m is the mass of the electron. Using the expression for the force from equation (ii), we can express the magnitude of this acceleration as

$$a = \frac{F}{m} = \frac{e\sigma}{m\varepsilon_0} \tag{iii}$$

Note that this acceleration is constant. Thus, the vertical position of the electron as a function of time is given by

$$y_f = y_0 - \tfrac{1}{2}at^2 \Rightarrow y_f - y_0 = -\tfrac{1}{2}at^2 \tag{iv}$$

Finally, we can relate the electron's initial kinetic energy to its initial velocity through

$$K = \tfrac{1}{2}mv_0^2 \Rightarrow v_0^2 = \frac{2K}{m} \tag{v}$$

SIMPLIFY We substitute the expressions for the time and the acceleration from equations (i) and (iii) into equation (iv) and obtain

$$y_f - y_0 = -\tfrac{1}{2}at^2 = -\tfrac{1}{2}\left(\frac{e\sigma}{m\varepsilon_0}\right)\left(\frac{x_f}{v_0}\right)^2 = -\frac{e\sigma x_f^2}{2m\varepsilon_0 v_0^2} \tag{vi}$$

Now substituting the expression for the square of the initial speed from equation (v) into the right-hand side of equation (vi) gives us

$$y_f - y_0 = -\frac{e\sigma x_f^2}{2m\varepsilon_0\left(\frac{2K}{m}\right)} = -\frac{e\sigma x_f^2}{4\varepsilon_0 K} \tag{vii}$$

CALCULATE We first convert the kinetic energy of the electron from electron-volts to joules:

$$K = (2.00 \text{ keV})\frac{1.602\times10^{-19}\text{ J}}{1\text{ eV}} = 3.204\times10^{-16}\text{ J}$$

Putting the numerical values into equation (vii), we get

$$y_f - y_0 = -\frac{e\sigma x_f^2}{4\varepsilon_0 K} = -\frac{(1.602\times10^{-19}\text{ C})(4.00\times10^{-6}\text{ C/m}^2)(0.0400\text{ m})^2}{4(8.85\times10^{-12}\text{ C}^2/(\text{N m}^2))(3.204\times10^{-16}\text{ J})} = -0.0903955\text{ m}$$

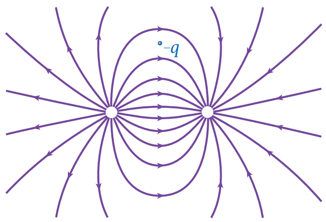
ROUND We report our result to three significant figures:

$$y_f - y_0 = -0.0904\text{ m} = -9.04\text{ cm}$$

DOUBLE-CHECK The vertical deflection that we calculated is about twice the distance that the electron travels in the x-direction, which seems reasonable, at least in the sense of being of the same order of magnitude. Also, equation (vii) for the deflection has several features that should be present. First, the trajectory is parabolic, which we expect for a constant force and thus constant acceleration. Second, for zero surface charge density, we obtain zero deflection. Third, for very high kinetic energy, there is negligible deflection, which is also intuitively what we expect.

Concept Check 2.7

A negative charge $-q$ is placed in a nonuniform electric field as shown in the figure. What is the direction of the electric force on this negative charge?



a) →

b) ↑

c) ←

d) ↓

e) The force is zero.

Dipole in an Electric Field

A point charge in an electric field experiences a force, given by equation 2.1. The electric force is always tangent to the electric field line passing through the point. The effect of an electric field on a dipole can be described in terms of the vector electric field, $\vec{E}$ and the

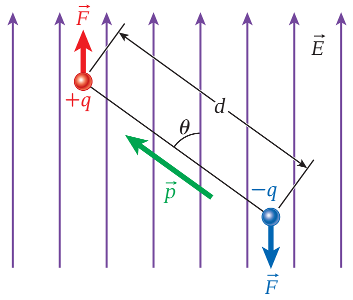


FIGURE 2.19 Electric dipole in an electric field.

Self-Test Opportunity 2.2

Use the center of mass of the dipole as the pivot point, and show that you again obtain the expression $\tau = qEd\sin\theta$ for the torque.

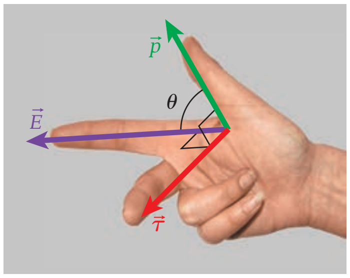


FIGURE 2.20 Right-hand rule for the vector product of the electric dipole moment and the electric field, producing the torque vector.

vector electric dipole moment, $\vec{p}$ without detailed knowledge of the charges making up the electric dipole.

To examine the behavior of an electric dipole, let's consider two charges, $+q$ and $-q$, separated by a distance d in a constant uniform electric field, $\vec{E}$ (Figure 2.19). (Note that we are now considering the forces acting on a dipole placed in an external field, as opposed to considering the field caused by the dipole, which we did in Section 2.4, and we also assume that the dipole field is small compared to $\vec{E}$ [so its effect on the uniform field can be ignored].) The electric field exerts an upward force on the positive charge and a downward force on the negative charge. Both forces have the magnitude qE. we saw that this situation gives rise to a torque, $\vec{\tau}$ given by $\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{r}$ is the moment arm and $\vec{F}$ is the force. The magnitude of the torque is $\tau = rF\sin\theta$.

As always, we can calculate the torque about any pivot point, so we can pick the location of the negative charge. Then, only the force on the positive charge contributes to the torque, and the length of the position vector is $r = d$, that is, the length of the dipole. Since, as already stated, $F = qE$, the expression for the torque on an electric dipole in an external electric field can be written as

$$\tau = qEd\sin\theta$$

Remembering that the electric dipole moment is defined as $p = qd$, we obtain the magnitude of the torque:

$$\tau = pE\sin\theta \tag{2.11}$$

Because the torque is a vector and must be perpendicular to both the electric dipole moment and the electric field, the relationship in equation 2.11 can be written as a vector product:

$$\vec{\tau} = \vec{p} \times \vec{E} \tag{2.12}$$

As with all vector products, the direction of the torque is given by a right-hand rule. As shown in Figure 2.20, the thumb indicates the direction of the first term of the vector product, in this case $\vec{p}$ and the index finger indicates the direction of the second term, $\vec{E}$ The result of the vector product, $\vec{\tau}$ is then directed along the middle finger and is perpendicular to each of the two terms.

SOLVED PROBLEM 2.3 Electric Dipole in an Electric Field

PROBLEM

An electric dipole with dipole moment of magnitude $p = 1.40 \times 10^{-12}$ C m is placed in a uniform electric field of magnitude $E = 498$ N/C (Figure 2.21a).

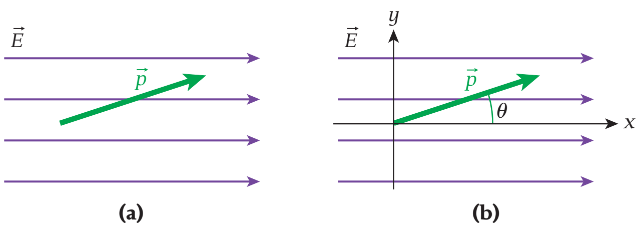


FIGURE 2.21 (a) An electric dipole in a uniform electric field. (b) The electric field oriented in the x-direction and the dipole moment in the xy-plane.

At some instant (in time) the angle between the electric dipole moment and the electric field is $\theta = 14.5°$. What are the Cartesian components of the torque on the dipole?

SOLUTION

THINK The torque on the dipole is equal to the vector product of the electric field and the electric dipole moment.

SKETCH We assume that the electric field lines point in the x-direction and the electric dipole moment is in the xy-plane (Figure 2.21b). The z-direction is perpendicular to the plane of the page.

RESEARCH The torque on the electric dipole due to the electric field is given by

$$\vec{\tau} = \vec{p} \times \vec{E}$$

Since the dipole is located in the xy-plane, the Cartesian components of the electric dipole moment are

$$\vec{p} = \left(p_x, p_y, 0\right)$$

Since the electric field is acting in the x-direction, its Cartesian components are

$$\vec{E} = \left(E_x, 0, 0\right) = \left(E, 0, 0\right)$$

SIMPLIFY From the definition of the vector product, we express the Cartesian components of the torque as

$$\vec{\tau} = \left(p_y E_z - p_z E_y\right)\hat{x} + \left(p_z E_x - p_x E_z\right)\hat{y} + \left(p_x E_y - p_y E_x\right)\hat{z}$$

In this particular case, with E_y, E_z, and p_z all equal to zero, we have

$$\vec{\tau} = -p_y E_x \hat{z}$$

The y-component of the dipole moment is $p_y = p\sin\theta$, and the x-component of the electric field is simply $E_x = E$. The magnitude of the torque is then

$$\tau = (p\sin\theta)E = pE\sin\theta$$

and the direction of the torque is in the negative z-direction.

CALCULATE We insert the given numerical data and get

$$\tau = pE\sin\theta = \left(1.40\times10^{-12}\ \text{C m}\right)\left(498\ \text{N/C}\right)\left(\sin 14.5°\right) = 1.74565\times10^{-10}\ \text{N m}$$

ROUND We report our result to three significant figures:

$$\tau = 1.75\times10^{-10}\ \text{N m}$$

DOUBLE-CHECK From equation 2.11, we know that the magnitude of the torque is

$$\tau = pE\sin\theta$$

which is the result we obtained using the explicit vector product. Applying the right-hand rule illustrated in Figure 2.20, we can determine the direction of the torque: With the right thumb representing the electric dipole moment and the right index finger representing the electric field, the right middle finger points into the page, which agrees with the result we found using the vector product. Thus, our result is correct.

Example 2.2 looked at the dipole moment of the water molecule. If water molecules are exposed to an external electric field, they experience a torque and thus begin to rotate. If the direction of the external electric field changes very rapidly, the water molecules perform rotational oscillations, which create heat. This is the principle of operation of a microwave oven. Microwave ovens use a frequency of 2.45 GHz for the oscillating electric field.

Electric fields also play a key role in human physiology, but these fields are time-varying and not static, like those studied in this chapter. The human brain also generates continuously changing electrical fields through the activity of the neurons. These fields can be measured by inserting electrodes through the skull and into the brain or by placing electrodes onto the surface of the exposed brain, usually during brain surgery. This technique is called electrocorticography (ECoG). An intense area of current research focuses on measuring and imaging brain electric fields noninvasively by attaching electrodes to the outside of the skull. However, since the skull itself dampens the electric fields, these techniques require great instrumental sensitivity and are still in their infancy. Perhaps the most exciting (or scary, depending on your point of view) research developments are in brain-computer interfaces. In this emerging field, electrical activity in the brain is used directly to control computers, and external stimuli are used to create electric fields inside the brain. Researchers in this area are motivated by the goal of helping people overcome physical disabilities, such as blindness or paralysis.

2.7 Electric Flux

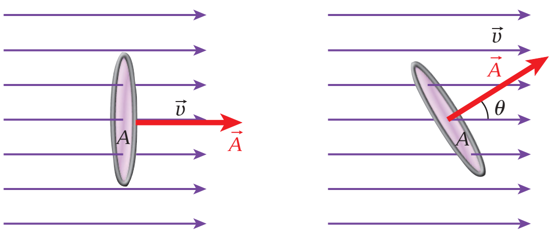


FIGURE 2.22 Water flowing with velocity of magnitude v through a ring of area A. (a) The area vector is parallel to the flow velocity. (b) The area vector is at an angle θ to the flow velocity.

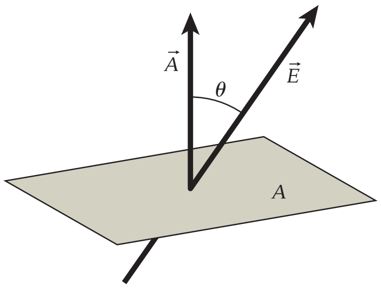


FIGURE 2.23 A uniform electric field $\vec{E}$ passing through an area $\vec{A}$

Electric field calculations, like those in Example 2.3, can require quite a bit of work. However, in many common situations, particularly those with some geometric symmetry, a powerful technique for determining electric fields without having to explicitly calculate integrals can be used. This technique is based on *Gauss's Law,* one of the fundamental relations concerning electric fields. It will allow us to solve seemingly very complicated problems involving electric fields in an amazingly straightforward and simple fashion. However, to use Gauss's Law requires understanding of a concept called *electric flux*.

Imagine holding a ring with inside area A in a stream of water flowing with velocity $\vec{v}$ as shown in Figure 22. The area vector, $\vec{A}$ of the ring is defined as a vector with magnitude A pointing in a direction perpendicular to the plane of the ring. In Figure 2.22a, the area vector of the ring is parallel to the flow velocity, and the flow velocity is perpendicular to the plane of the ring. The product Av gives the amount of water passing through the ring per unit time, where v is the magnitude of the flow velocity. If the plane of the ring is tilted with respect to the direction of the flowing water (Figure 2.22b), the amount of water flowing through the ring is given by $Av\cos\theta$, where θ is the angle between the area vector of the ring and the direction of the velocity of the flowing water. The amount of water flowing through the ring is called the *flux*, $\Phi = Av\cos\theta = \vec{A}\bullet\vec{v}$ Since flux is a measure of volume per unit time, its units are cubic meters per second (m^3/s).

An electric field is analogous to flowing water. Consider a uniform electric field of magnitude E passing through a given area A (Figure 2.23). Again, the area vector is $\vec{A}$, with a direction normal to the surface of the area and a magnitude A. The angle θ is the angle between the vector electric field and the area vector, as shown in Figure 2.23. The electric field passing through a given area A is called the **electric flux** and is given by

$$\Phi = EA\cos\theta \tag{2.13}$$

In simple terms, the electric flux is proportional to the number of electric field lines passing through the area. We'll assume that the electric field is given by $\vec{E}(\vec{r})$ and that the area is a closed surface, rather than the open surface of a simple ring in flowing water. In this closed-surface case, the total, or net, electric flux is given by an integral of the electric field over the closed surface:

$$\Phi = \oiint \vec{E}\bullet d\vec{A} \tag{2.14}$$

where $\vec{E}$ is the electric field at each differential area element $d\vec{A}$ of the closed surface. The direction of $d\vec{A}$ is outward from the closed surface. In equation 2.14, the loop on the integrals means that the integration is over a closed surface, and the two integral signs signify an integration over two variables. (*Note:* Some books use different notation for the integral over a closed surface, $\iint_S dA$ or just $\int_S dA$ but these refer to the same integration procedure as is represented in equation 2.14.) The differential area element $d\vec{A}$ must be described by two spatial variables, such as x and y in Cartesian coordinates or θ and ϕ in spherical coordinates.

Figure 2.24 shows a nonuniform electric field, $\vec{E}$ passing through a differential area element, $d\vec{A}$ A portion of the closed surface is also shown. The angle between the electric field and the differential area element is θ.

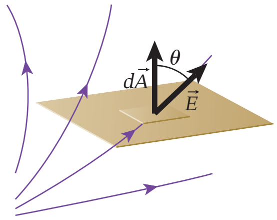


FIGURE 2.24 A nonuniform electric field, $\vec{E}$, passing through a differential area, $d\vec{A}$

EXAMPLE 2.5 Electric Flux through a Cube

Figure 2.25 shows a cube that has faces of area A in a uniform electric field, $\vec{E}$ that is perpendicular to the plane of one face of the cube.

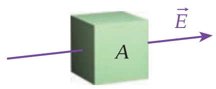


FIGURE 2.25 A cube with faces of area A in a uniform electric field, $\vec{E}$

PROBLEM

What is the net electric flux passing though the cube?

SOLUTION

The electric field in Figure 2.25 is perpendicular to the plane of one of the cube's six faces and therefore is also perpendicular to the opposite face. The area vectors of these two faces, $\vec{A}_1$ and $\vec{A}_2$ are shown in Figure 2.26a. The net electric flux passing through these two faces is

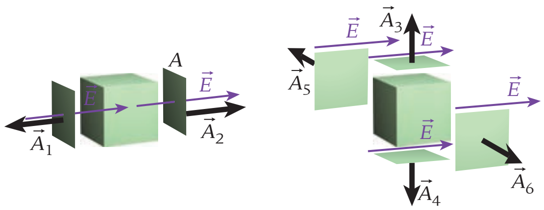
$$\Phi_{12} = \Phi_1 + \Phi_2 = \vec{E} \bullet \vec{A}_1 + \vec{E} \bullet \vec{A}_2 = -EA_1 + EA_2 = 0$$

The negative sign arises for the flux through face 1 because the electric field and the area vector, $\vec{A}_1$ are in opposite directions. The area vectors of the remaining four faces are all perpendicular to the electric field, as shown in Figure 2.26b. The net electric flux passing through these four faces is

$$\Phi_{3456} = \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6 = \vec{E} \bullet \vec{A}_3 + \vec{E} \bullet \vec{A}_4 + \vec{E} \bullet \vec{A}_5 + \vec{E} \bullet \vec{A}_6 = 0$$

All the scalar products are zero because the area vectors of these four faces are perpendicular to the electric field. Thus, the net electric flux passing through the cube is

$$\Phi = \Phi_{12} + \Phi_{3456} = 0$$



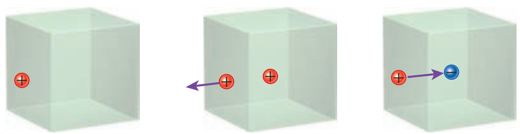
(a) **(b)**

FIGURE 2.26 (a) The two faces of the cube that are perpendicular to the electric field. The area vectors are parallel and antiparallel to the electric field. (b) The four faces of the cube that are parallel to the electric field. The area vectors are perpendicular to the electric field.

2.8 Gauss's Law

To begin our discussion of Gauss's Law, let's imagine a box in the shape of a cube (Figure 2.27a), which is constructed of a material that does not affect electric fields. A positive test charge brought close to any surface of the box will experience no force. Now suppose a positive charge is inside the box and the positive test charge is brought close to the surface of the box (Figure 2.27b). The positive test charge experiences an outward force due to the positive charge inside the box. If the test charge is close to any surface of the box, it experiences the outward force. If twice as much positive charge is inside the box, the positive test charge experiences twice the outward force when brought close to any surface of the box.

Now suppose there is a negative charge inside the box (Figure 2.27c). When the positive test charge is brought close to one surface of the box, the charge experiences an inward force. If the positive test charge is close to any surface of the box, it experiences an inward force. Doubling the negative charge in the box doubles the inward force on the test charge when it is close to any surface of the box.

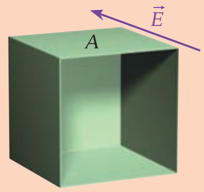


(a) **(b)** **(c)**

FIGURE 2.27 Three imaginary boxes constructed of material that does not affect electric fields. A positive test charge is brought up to the box from the left toward: (a) an empty box; (b) a box with a positive charge inside; (c) a box with a negative charge inside.

Self-Test Opportunity 2.3

The figure shows a cube with faces of area A and one face missing. This five-sided cubical object is in a uniform electric field, $\vec{E}$ perpendicular to one face. What is the net electric flux passing through the object?



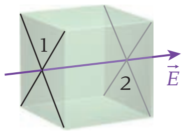
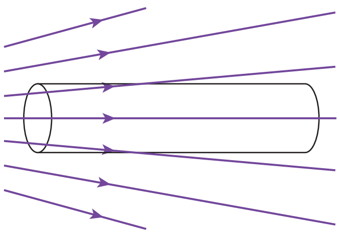


FIGURE 2.28 Imaginary empty box in a uniform electric field.

Concept Check 2.8

A cylinder made of an insulating material is placed in an electric field as shown in the figure. The net electric flux passing through the surface of the cylinder is



a) positive.

b) negative.

c) zero.

In analogy with flowing water, the electric field lines seem to be flowing out of the box containing positive charge and into the box containing negative charge.

Now let's imagine an empty box in a uniform electric field (Figure 2.28). If a positive test charge is brought close to side 1, it experiences an inward force. If the charge is close to side 2, it experiences an outward force. The electric field is parallel to the other four sides, so the positive test charge does not experience any inward or outward force when brought close to those sides. Thus, in analogy with flowing water, the net amount of electric field that seems to be flowing in and out of the box is zero.

Whenever a charge is inside the box, the electric field lines seem to be flowing in or out of the box. When there is no charge in the box, the net flow of electric field lines in or out of the box is zero. These observations and the definition of electric flux, which quantifies the concept of the flow of the electric field lines, lead to **Gauss's Law:**

$$\Phi = \frac{q}{\varepsilon_0} \tag{2.15}$$

Here q is the net charge inside a closed surface, called a **Gaussian surface.** The closed surface could be a box like that we have been discussing or any arbitrarily shaped closed surface. Usually, the shape of the Gaussian surface is chosen so as to reflect the symmetries of the problem situation.

Concept Check 2.9

The lines in the figure are electric field lines, and the circle is a Gaussian surface. For which case(s) is (are) the total electric flux nonzero?

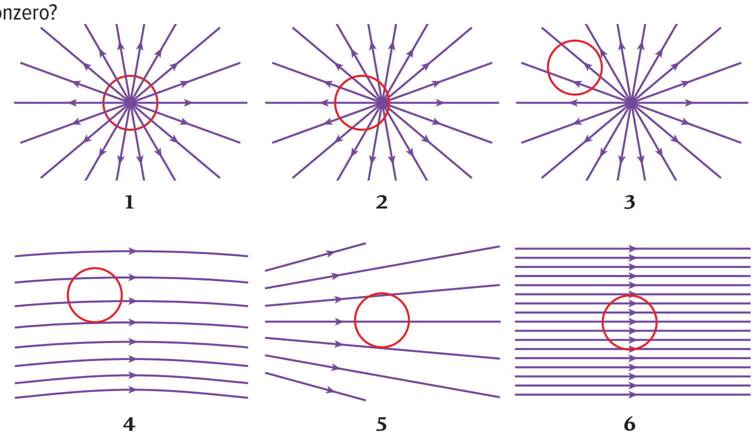
a) 1 only

b) 2 only

c) 4, 5, and 6

d) 6 only

e) 1 and 2



An alternative formulation of Gauss's Law incorporates the definition of the electric flux (equation 2.14):

$$\oiint \vec{E} \bullet d\vec{A} = \frac{q}{\varepsilon_0} \tag{2.16}$$

According to equation 2.16, Gauss's Law states that the surface integral of the electric field components perpendicular to the area times the area is proportional to the net charge within the closed surface. This expression may look daunting, but it simplifies considerably in many cases and allows us to perform very quickly calculations that would otherwise be quite complicated.

Gauss's Law and Coulomb's Law

We can derive Gauss's Law from Coulomb's Law. To do this, we start with a positive point charge, q. The electric field due to this charge is radial and pointing outward, as we saw in Section 2.3. According to Coulomb's Law (Section 1.5), the magnitude of the electric field from this charge is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

We now find the electric flux passing through a closed surface resulting from this point charge. For the Gaussian surface, we choose a spherical surface with radius r, with the charge at the center of the sphere, as shown in Figure 2.29. The electric field due to the positive point charge intersects each differential element of the surface of the Gaussian sphere perpendicularly. Therefore, at each point of this Gaussian surface, the electric field vector, $\vec{E}$ and the differential surface area vector, $d\vec{A}$ are parallel. The surface area vector will always point outward from the spherical Gaussian surface, but the electric field vector can point outward or inward depending on the sign of the charge. For a positive charge, the scalar product of the electric field and the surface area element is $\vec{E} \bullet d\vec{A} = E\,dA \cos 0° = E\,dA$ The electric flux in this case, according to equation 2.14, is

$$\Phi = \oiint \vec{E} \bullet d\vec{A} = \oiint E\,dA$$

Because the electric field has the same magnitude anywhere in space at a distance r from the point charge q, we can take E outside the integral:

$$\Phi = \oiint E\,dA = E \oiint dA$$

Now what we have left to evaluate is the integral of the differential area over a spherical surface, which is given by $\oiint dA = 4\pi r^2$ Therefore, we have found from Coulomb's Law for the case of a point charge

$$\Phi = (E)\left(\oiint dA\right) = \left(\frac{1}{4\pi\,\varepsilon_0}\frac{q}{r^2}\right)\left(4\pi r^2\right) = \frac{q}{\varepsilon_0}$$

which is the same as the expression for Gauss's Law in equation 2.15. We have shown that Gauss's Law can be derived from Coulomb's Law for a positive point charge, but it can also be shown that Gauss's Law holds for any distribution of charge inside a closed surface.

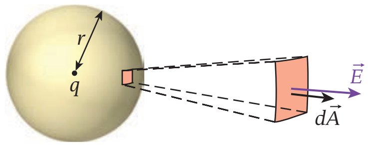


FIGURE 2.29 A spherical Gaussian surface with radius r surrounding a charge q. A closeup view of a differential surface element with area dA is shown.

Self-Test Opportunity 2.4

What changes in the preceding derivation of Gauss's Law if a negative point charge is used?

Shielding

Two important consequences of Gauss's Law are evident:

1. The electrostatic field inside any isolated conductor is always zero.
2. Cavities inside conductors are shielded from electric fields.

To examine these consequences, let's suppose a net electric field exists at some moment at some point inside an isolated conductor; see Figure 2.30a. But every conductor has free electrons inside it (blue circles in Figure 2.30b), which can move rapidly in response to any net external electric field, leaving behind positively charged ions (red circles in Figure 2.30b). The charges will move to the outer surface of the conductor, leaving no net accumulation of charge inside the volume of the conductor. These charges will in turn create an electric field inside the conductor (yellow arrows in Figure 2.30b), and they will move around until the electric field produced by them exactly cancels the external electric field. The net electric field thus becomes zero everywhere inside the conductor (Figure 2.30c).

If a cavity is scooped out of a conducting body, the net charge and thus the electric field inside this cavity is always zero, no matter how strongly the conductor is charged or how strong an external electric field acts on it. To prove this, we assume a closed Gaussian surface surrounds the cavity, completely inside the conductor. From the preceding discussion (see Figure 2.30), we know that at each point of this surface, the field is zero. Therefore, the net flux over this surface is also zero. By Gauss's Law, it then follows that this surface encloses zero net charge. If there were equal amounts of positive and negative charge on the cavity surface (and thus no net charge), this charge would not be stationary, as the positive and negative charges would be attracted to each other and would be free to move around the cavity surface to cancel each other. Therefore, any cavity inside a conductor is totally shielded from any external electric field. This effect is sometimes called **electrostatic shielding.**

A convincing demonstration of this shielding is provided by placing a plastic container filled with Styrofoam peanuts on top of a Van de Graaff generator, which serves as the source of strong electric field (Figure 2.31a). Charging the generator results in a large net charge accumulation on the dome, producing a strong electric field in the vicinity. Because

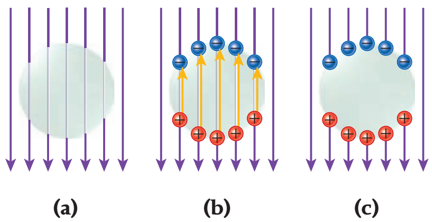
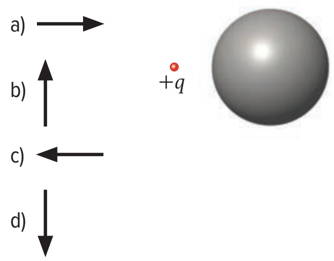


FIGURE 2.30 Shielding of an external electric field (purple vertical arrows) from the inside of a conductor.

Concept Check 2.10

A hollow, conducting sphere is initially given an evenly distributed negative charge. A positive charge $+q$ is brought near the sphere and placed at rest as shown in the figure. What is the direction of the electric field inside the hollow sphere?



e) The field is zero.

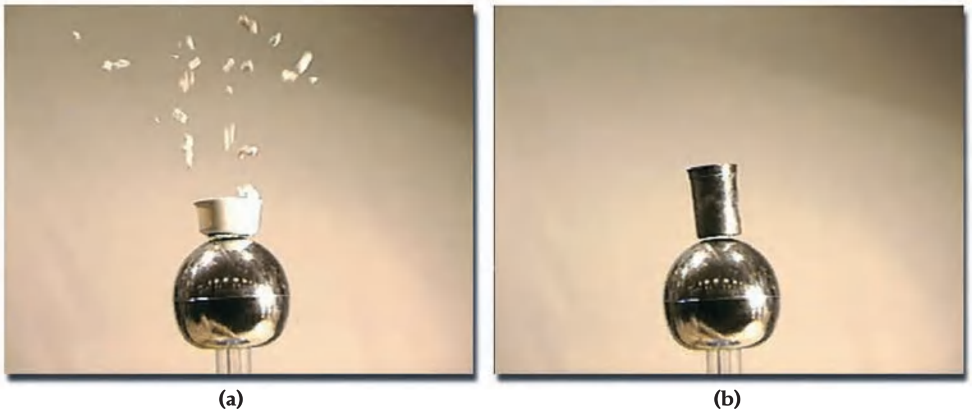


FIGURE 2.31 Styrofoam peanuts are put inside a container that is placed on top of a Van de Graaff generator, which is then charged. (a) The peanuts fly out of a nonconducting plastic container. (b) The peanuts remain within a metal can.

of this field, the charges in the Styrofoam peanuts separate slightly, and the peanuts acquire small dipole moments. If the field were uniform, there would be no force on these dipoles. However, the nonuniform electric field does exert a force, even though the peanuts are electrically neutral. The peanuts thus fly out of the container. If the same Styrofoam peanuts are placed inside an open metal can, they do not fly out when the generator is charged (Figure 2.31b). The electric field easily penetrates the walls of the plastic container and reaches the Styrofoam peanuts, whereas, in accord with Gauss's Law, the conducting metal can provide shielding inside and prevents the Styrofoam peanuts from acquiring dipole moments.

The conductor surrounding the cavity does not have to be a solid piece of metal; even a wire mesh is sufficient to provide shielding. This can be demonstrated most impressively by seating a person inside a cage and then hitting the cage with a lightning-like electrical discharge (Figure 2.32). The person inside the cage is unhurt, even if he or she touches the metal of the cage *from the inside.* (It is important to realize that severe injuries can result if any body parts stick out of the cage, for example, if hands are wrapped around the bars of the cage!) This cage is called a *Faraday cage,* after British physicist Michael Faraday (1791–1867), who invented it.

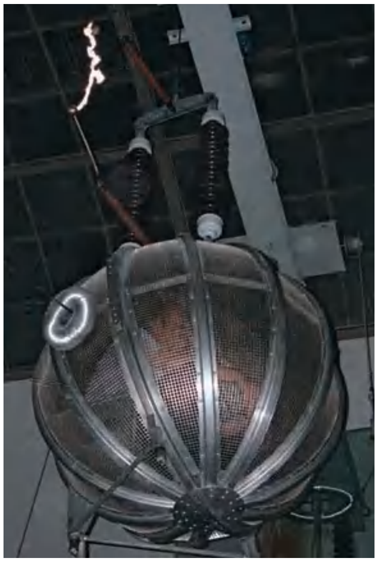


FIGURE 2.32 A person inside a Faraday cage is unharmed by a large voltage applied outside the cage, which produces a huge spark. This demonstration is performed several times daily at the Deutsches Museum in Munich, Germany.

A Faraday cage has important consequences, probably the most relevant of which is the fact that your car protects you from being hit by lightning while inside it–unless you drive a convertible. The sheet metal and steel frame that surround the passenger compartment provide the necessary shielding. (But as fiberglass, plastic, and carbon fiber begin to replace sheet metal in auto bodies, this shielding is not assured any more.)

Concept Check 2.11

A hollow, conducting sphere is initially uncharged. A positive charge, $+q_1$, is placed inside the sphere, as shown in the figure. Then, a second positive charge, $+q_2$, is placed near the sphere but outside it. Which of the following statements describes the net electric force on each charge?

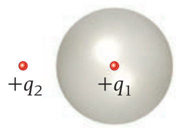
a) There is a net electric force on $+q_2$ but not on $+q_1$.

b) There is a net electric force on $+q_1$ but not on $+q_2$.

c) Both charges are acted on by a net electric force with the same magnitude and in the same direction.

d) Both charges are acted on by a net electric force with the same magnitude but in opposite directions.

e) There is no net electric force on either charge.



2.9 Special Symmetries

In this section we'll determine the electric field due to charged objects of different shapes. In Section 2.5, the charge distributions for different geometries were defined; see equation 2.9. Table 2.1 lists the symbols for these charge distributions and their units.

Table 2.1 Symbols for Charge Distributions

Symbol	Name	Unit
λ	Charge per length	C/m
σ	Charge per area	C/m^2
ρ	Charge per volume	C/m^3

Cylindrical Symmetry

Using Gauss's Law, we can calculate the magnitude of the electric field due to a long straight conducting wire with uniform charge per unit length $\lambda > 0$. We first imagine a Gaussian surface in the form of a right cylinder with radius r and length L surrounding the wire so that the wire is along the axis of the cylinder (Figure 2.33). We can apply Gauss's Law to this Gaussian surface. From symmetry, we know that the electric field produced by the wire must be radial and perpendicular to the wire. What invoking symmetry means deserves further explanation because such arguments are very common.

First, we imagine rotating the wire about an axis along its length. This rotation would include all charges on the wire and their electric fields. However, the wire would still look the same after a rotation through any angle. The electric field created by the charge on the wire would therefore also be the same. From this argument, we conclude that the electric field cannot depend on the rotation angle around the wire. This conclusion is general: If an object has *rotational symmetry*, its electric field cannot depend on the rotation angle.

Second, if the wire is very long, it will look the same no matter where along its length it is viewed. If the wire is unchanged, its electric field is also unchanged. This observation means that there is no dependence on the coordinate along the wire. This symmetry is called *translational symmetry*. Since there is no preferred direction in space along the wire, there can be no electric field component parallel to the wire.

Returning to the Gaussian surface, we can see that the contribution to the integral in Gauss's Law (equation 2.16) from the ends of the cylinder is zero because the electric field is parallel to these surfaces and is thus perpendicular to the normal vectors from the surface. The electric field is perpendicular to the wall of the cylinder everywhere, so we have

$$\oiint \vec{E} \bullet d\vec{A} = EA = E(2\pi rL) = \frac{q}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$$

where $2\pi rL$ is the area of the wall of the cylinder. Solving this equation, we find the magnitude of the electric field due to a uniformly charged long straight wire:

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{2k\lambda}{r} \qquad (2.17)$$

where r is the perpendicular distance to the wire. For $\lambda < 0$, equation 2.17 still applies, but the electric field points inward instead of outward. Note that this is the same result we obtained in Example 2.3 for the electric field due to a wire of infinite length–but attained here in a much simpler way!

You begin to see the great computational power contained in Gauss's Law, which can be used to calculate the electric field resulting from all kinds of charge distributions, both discrete and continuous. However, it is practical to use Gauss's Law only in situations where you can exploit some symmetry; otherwise, it is too difficult to calculate the flux.

It is instructive to compare the dependence of the electric field on the distance from a point charge and the distance from a long straight wire. For the point charge, the electric field falls off with the square of the distance, much faster than does the electric field due to the long wire, which decreases in inverse proportion to the distance.

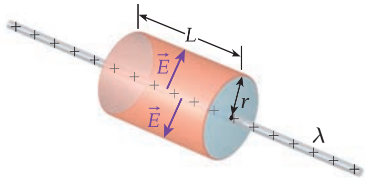


FIGURE 2.33 Long wire with charge per unit length λ surrounded by a Gaussian surface in the form of a right cylinder with radius r and length L. Representative electric field vectors are shown inside the cylinder.

Concept Check 2.12

A total of 1.45×10^6 excess electrons are placed on an initially electrically neutral wire of length 1.13 m. What is the magnitude of the electric field at a point at a perpendicular distance of 0.401 m away from the center of wire? (*Hint:* Assume that 1.13 m is close enough to "infinitely long.")

a) 9.21×10^{-3} N/C

b) 2.92×10^{-1} N/C

c) 6.77×10^{1} N/C

d) 8.12×10^{2} N/C

e) 3.31×10^{3} N/C

Self-Test Opportunity 2.5

By how much does the answer to Concept Check 2.12 change if the assumption that the wire can be treated as being infinitely long is not made? (*Hint:* See Example 2.3.)

Planar Symmetry

Assume a flat thin, infinite, nonconducting sheet of positive charge (Figure 2.34), with uniform charge per unit area $\sigma > 0$. Let's find the electric field a distance r from the surface of this infinite plane of charge.

To do this, we choose a Gaussian surface in the form of a closed right cylinder with cross-sectional area A and length $2r$, which cuts through the plane perpendicularly, as shown in Figure 2.34. Because the plane is infinite and the charge is positive, the electric field must be perpendicular to the ends of the cylinder and parallel to the cylinder wall. Using Gauss's Law, we obtain

$$\oiint \vec{E} \bullet d\vec{A} = (EA + EA) = \frac{q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

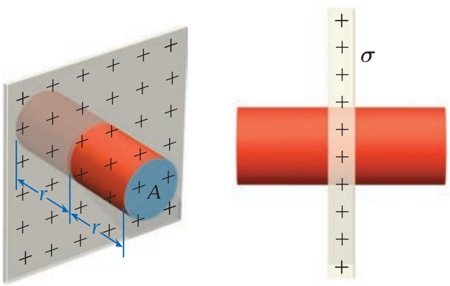


FIGURE 2.34 Infinite, flat, nonconducting sheet with charge density σ. Cutting through the plane perpendicularly is a Gaussian surface in the form of a right cylinder with cross-sectional area A parallel to the plane and height r above and below the plane.

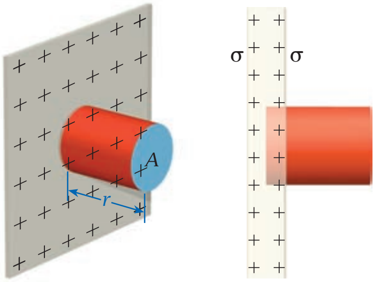


FIGURE 2.35 Infinite conducting plane with charge density σ on each surface and a Gaussian surface in the form of a right cylinder embedded in one side.

where σA is the charge enclosed in the cylinder. Thus, the magnitude of the electric field due to an infinite plane of charge is

$$E = \frac{\sigma}{2\varepsilon_0} \tag{2.18}$$

If $\sigma < 0$, then equation 2.18 still holds, but the electric field points toward the plane instead of away from it.

For an infinite conducting sheet with charge density $\sigma > 0$ on each surface, we can find the electric field by choosing a Gaussian surface in the form of a right cylinder. However, for this case, one end of the cylinder is embedded inside the conductor (Figure 2.35). The electric field inside the conductor is zero; therefore, there is no flux through the end of the cylinder enclosed in the conductor. The electric field outside the conductor must be perpendicular to the surface and therefore parallel to the wall of the cylinder and perpendicular to the end of the cylinder that is outside the conductor. Thus, the flux through the Gaussian surface is EA. The enclosed charge is given by σA, so Gauss's Law becomes

$$\oiint \vec{E} \bullet d\vec{A} = EA = \frac{\sigma A}{\varepsilon_0}$$

Thus, the magnitude of the electric field just outside the surface of a flat charged conductor is

$$E = \frac{\sigma}{\varepsilon_0} \tag{2.19}$$

Spherical Symmetry

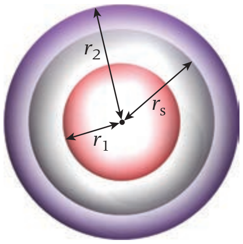


FIGURE 2.36 Spherical shell of charge with radius r_s along with a Gaussian surface with radius $r_2 > r_s$ and a second Gaussian surface with radius $r_1 < r_s$.

To find the electric field due to a spherically symmetrical distribution of charge, we consider a thin spherical shell with charge $q > 0$ and radius r_s (Figure 2.36).

Here we use a spherical Gaussian surface with $r_2 > r_s$ that is concentric with the charged sphere. Applying Gauss's Law, we get

$$\oiint \vec{E} \bullet d\vec{A} = E\left(4\pi r_2^2\right) = \frac{q}{\varepsilon_0}$$

We can solve for the magnitude of the electric field, E, which is

$$E = \frac{1}{4\pi\,\varepsilon_0} \frac{q}{r_2^2}$$

If $q < 0$, the field points radially inward instead of radially outward from the spherical surfaces. For another spherical Gaussian surface, with $r_1 < r_s$, that is also concentric with the charged spherical shell, we obtain

$$\oiint \vec{E} \bullet d\vec{A} = E\left(4\pi r_1^2\right) = 0$$

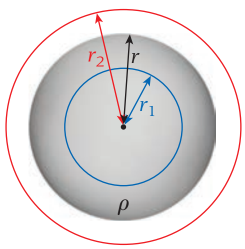


FIGURE 2.37 Spherical distribution of charge with uniform charge per unit volume ρ and radius r. Two spherical Gaussian surfaces are also shown, one with radius $r_1 < r$ and one with radius $r_2 > r$.

Thus, the electric field outside a spherical shell of charge behaves as if the charge were a point charge located at the center of the sphere, whereas the electric field is zero inside the spherical shell of charge.

Now let's find the electric field due to charge that is equally distributed throughout a spherical volume, with uniform charge density $\rho > 0$ (Figure 2.37). The radius of the sphere is r. We use a Gaussian surface in the form of a sphere with radius $r_1 < r$. From the symmetry of the charge distribution, we know that the electric field resulting from the charge is perpendicular to the Gaussian surface. Thus, we can write

$$\oiint \vec{E} \bullet d\vec{A} = E\left(4\pi r_1^2\right) = \frac{q}{\varepsilon_0} = \frac{\rho}{\varepsilon_0}\left(\tfrac{4}{3}\pi r_1^3\right)$$

where $4\pi r_1^2$ is the area of the spherical Gaussian surface and $\frac{4}{3}\pi r_1^3$ is the volume enclosed by the Gaussian surface. From the preceding equation, we obtain the electric field at a radius r_1 inside a uniform distribution of charge:

$$E = \frac{\rho r_1}{3\varepsilon_0} \tag{2.20}$$

The total charge on the sphere can be called q_t, and it equals the total volume of the spherical charge distribution times the charge density:

$$q_t = \rho \tfrac{4}{3}\pi r^3$$

The charge enclosed by the Gaussian surface then is

$$q = \frac{\text{volume inside } r_1}{\text{volume of charge distribution}} q_t = \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r^3} q_t = \frac{r_1^3}{r^3} q_t .$$

With this the expression for the enclosed charge, we can rewrite Gauss's Law for this case as

$$\oint\!\!\oint \vec{E} \bullet d\vec{A} = E\left(4\pi r_1^2\right) = \frac{q_t}{\varepsilon_0}\frac{r_1^3}{r^3}$$

which gives us

$$E = \frac{q_t r_1}{4\pi\varepsilon_0 r^3} = \frac{k q_t r_1}{r^3} \tag{2.21}$$

If we consider a Gaussian surface with a radius larger than the radius of the charge distribution, $r_2 > r$, we can apply Gauss's Law as follows:

$$\oint\!\!\oint \vec{E} \bullet d\vec{A} = E\left(4\pi r_2^2\right) = \frac{q_t}{\varepsilon_0}$$

or

$$E = \frac{q_t}{4\pi\varepsilon_0 r_2^2} = \frac{k q_t}{r_2^2} \tag{2.22}$$

Thus, the electric field outside a uniform spherical distribution of charge is the same as the field due to a point charge of the same magnitude located at the center of the sphere.

Self-Test Opportunity 2.6

Consider a sphere of radius *R* with charge *q* uniformly distributed throughout the volume of the sphere. What is the magnitude of the electric field at a point 2*R* away from the center of the sphere?

SOLVED PROBLEM 2.4 Nonuniform Spherical Charge Distribution

A spherically symmetrical but nonuniform charge distribution is given by

$$\rho(r) = \begin{cases} \rho_0\left(1 - \dfrac{r}{R}\right) & \text{for } r \le R \\ 0 & \text{for } r > R, \end{cases}$$

where $\rho_0 = 10.0\ \mu\text{C/m}^3$ and $R = 0.250$ m.

PROBLEM

What is the electric field produced by this charge distribution at $r_1 = 0.125$ m and at $r_2 = 0.500$ m?

SOLUTION

THINK We can use Gauss's Law to determine the electric field as a function of radius if we employ a spherical Gaussian surface. The radius $r_1 = 0.125$ m is located inside the charge distribution. The charge enclosed inside the spherical surface at $r = r_1$ is given by an integral of the charge density from $r = 0$ to $r = r_1$. Outside the spherical charge distribution, the electric field is the same as that of a point charge whose magnitude is equal to the total charge of the spherical distribution.

SKETCH The charge density, ρ, as a function of radius, r, is plotted in Figure 2.38.

RESEARCH Gauss's Law (equation 2.16) tells us that $\oint\!\!\oint \vec{E} \bullet d\vec{A} = q/\varepsilon_0$ Inside the nonuniform spherical charge distribution at a radius $r_1 < R$, Gauss's Law becomes

$$\varepsilon_0 E\left(4\pi r_1^2\right) = \int_0^{V_1} \rho(r)\, dV = \int_0^{r_1} \rho_0\left(1 - \frac{r}{R}\right)\left(4\pi r^2\right) dr \tag{i}$$

– Continued

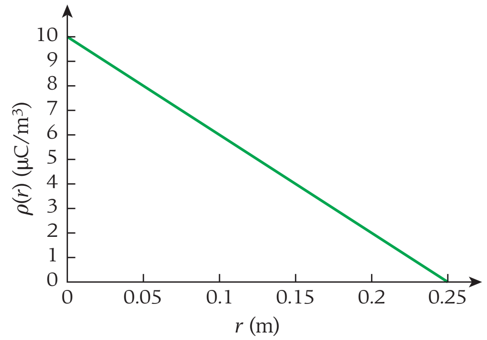


FIGURE 2.38 Charge density as a function of radius for a nonuniform spherical charge distribution.

Carrying out the integral on the right-hand side of equation (i), we obtain

$$\int_0^{r_1} \rho_0\left(1-\frac{r}{R}\right)\left(4\pi r^2\right)dr = 4\pi\rho_0\int_0^{r_1}\left(r^2-\frac{r^3}{R}\right)dr = 4\pi\rho_0\left(\frac{r_1^3}{3}-\frac{r_1^4}{4R}\right) \qquad \text{(ii)}$$

SIMPLIFY The electric field due to the charge inside $r_1 \le R$ is then given by

$$E = \frac{4\pi\rho_0\left(\dfrac{r_1^3}{3}-\dfrac{r_1^4}{4R}\right)}{\varepsilon_0\left(4\pi r_1^2\right)} = \frac{\rho_0}{\varepsilon_0}\left(\frac{r_1}{3}-\frac{r_1^2}{4R}\right) \qquad \text{(iii)}$$

In order to calculate the electric field due to the charge inside $r_2 > R$, we need the total charge contained in the spherical charge distribution. We can obtain the total charge using equation (ii) with $r_1 = R$:

$$q_t = 4\pi\rho_0\left(\frac{R^3}{3}-\frac{R^4}{4R}\right) = 4\pi\rho_0\left(\frac{R^3}{3}-\frac{R^3}{4}\right) = 4\pi\rho_0\frac{R^3}{12} = \frac{\pi\rho_0 R^3}{3}$$

The electric field outside the spherical charge distribution ($r_2 > R$) is then

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q_t}{r_2^2} = \frac{1}{4\pi\varepsilon_0}\frac{\frac{1}{3}\pi\rho_0 R^3}{r_2^2} = \frac{\rho_0 R^3}{12\varepsilon_0 r_2^2} \qquad \text{(iv)}$$

CALCULATE The electric field at $r_1 = 0.125$ m is

$$E = \frac{\rho_0}{\varepsilon_0}\left(\frac{r_1}{3}-\frac{r_1^2}{4R}\right) = \frac{10.0\ \mu\text{C/m}^3}{8.85\times10^{-12}\,\text{C}^2/\text{N m}^2}\left(\frac{0.125\text{ m}}{3}-\frac{(0.125\text{ m})^2}{4(0.250\text{ m})}\right) = 29{,}425.6\text{ N/C}$$

The electric field at $r_2 = 0.500$ m is

$$E = \frac{\rho_0 R^3}{12\varepsilon_0 r_2^2} = \frac{\left(10.0\ \mu\text{C/m}^3\right)(0.250\text{ m})^3}{12\left(8.85\times10^{-12}\,\text{C}^2/\text{N m}^2\right)(0.500\text{ m})^2} = 5885.12\text{ N/C}$$

ROUND We report our results to three significant figures. The electric field at $r_1 = 0.125$ m is

$$E = 2.94\times10^4\ \text{N/C}$$

The electric field at $r_2 = 0.500$ m is

$$E = 5.89\times10^3\ \text{N/C}$$

DOUBLE-CHECK The electric field at $r_1 = R$ can be calculated using equation (iii):

$$E = \frac{\rho_0}{\varepsilon_0}\left(\frac{R}{3}-\frac{R^2}{4R}\right) = \frac{\rho_0 R}{12\varepsilon_0} = \frac{\left(10.0\ \mu\text{C/m}^3\right)(0.250\text{ m})}{12\left(8.85\times10^{-12}\,\text{C}^2/\text{N m}^2\right)} = 2.35\times10^4\ \text{N/C}$$

We can also use equation (iv) to find the electric field outside the spherical charge distribution but very close to the surface, where $r_2 \approx R$:

$$E = \frac{\rho_0 R^3}{12\varepsilon_0 R^2} = \frac{\rho_0 R}{12\varepsilon_0}$$

which is the same result we obtained using our result for $r_1 \leq R$. The calculated electric field at the surface of the charge distribution is lower than that at $r_1 = 0.125$, which may seem counterintuitive. An idea of the dependence of the magnitude of E on r is provided by the plot in Figure 2.39, which was created using equations (iii) and (iv).

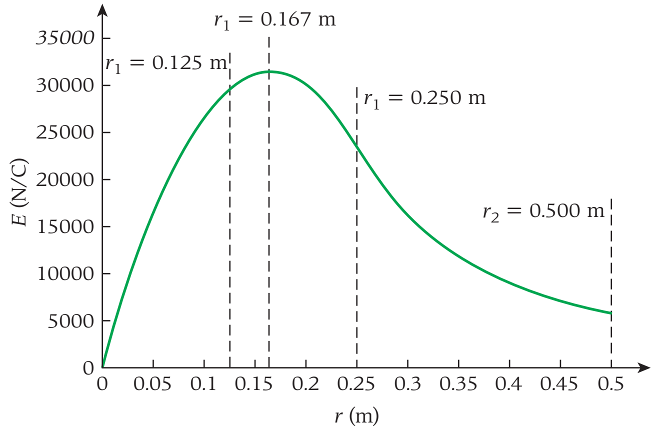


FIGURE 2.39 The electric field due to a nonuniform spherical distribution of charge as a function of the distance from the center of the sphere.

You can see that a maximum occurs in the electric field and that our result for $r_1 = 0.125$ m is less than this maximum value. We can calculate the radius at which the maximum occurs by differentiating equation (iii) with respect to r_1, setting the result equal to zero, and solving for r_1:

$$\frac{dE}{dr_1} = \frac{\rho_0}{\varepsilon_0}\left(\frac{1}{3} - \frac{r_1}{2R}\right) = 0 \Rightarrow$$

$$\frac{1}{3} = \frac{r_1}{2R} \Rightarrow r_1 = \frac{2}{3}R$$

Thus, we expect a maximum in the electric field at $r_1 = \frac{2}{3}R = 0.167$ m. The plot in Figure 2.39 does indeed show a maximum at that radius. It also shows the value of E at $r = 0.250$ to be smaller than that at $r = 0.125$ as we found in our calculation. Thus, our answers seem reasonable.

Concept Check 2.13

Suppose an uncharged solid steel ball, for example, one of the steel balls used in an old-fashioned pinball machine, is resting on a perfect insulator. Some small amount of negative charge (say, a few hundred electrons) is placed at the north pole of the ball. If you could check the distribution of the charge after a few seconds, what would you detect?

a) All of the added charge has vanished, and the ball is again electrically neutral.

b) All of the added charge has moved to the center of the ball.

c) All of the added charge is distributed uniformly over the surface of the ball.

d) The added charge is still located at or very near the north pole of the ball.

e) The added charge is performing a simple harmonic oscillation on a straight line between the south and north poles of the ball.

Concept Check 2.14

Suppose an uncharged hollow sphere made of a perfect insulator, for example a ping-pong ball, is resting on a perfect insulator. Some small amount of negative charge (say, a few hundred electrons) is placed at the north pole of the sphere. If you could check the distribution of the charge after a few seconds, what would you detect?

a) All of the added charge has vanished, and the sphere is again electrically neutral.

b) All of the added charge has moved to the center of the sphere.

c) All of the added charge is distributed uniformly over the surface of the sphere.

d) The added charge is still located at or very near the north pole of the sphere.

e) The added charge is performing a simple harmonic oscillation on a straight line between the south and north poles of the sphere.

Sharp Points and Lightning Rods

We have already seen that the electric field is perpendicular to the surface of a conductor. (To repeat, if there were a field component parallel to the surface of the conductor, then the charges inside the conductor would move until they reached equilibrium, which means no force or electric field component in the direction of motion, that is, along the surface of the conductor.) Figure 2.40a shows the distribution of charges on the surface of the end of a pointed conductor. Note that the charges are closer together at the sharp tip, where the curvature is largest. Near that sharp tip on the end of the conductor, the electric field looks much more like that due to a point charge, with the field lines spreading out radially (Figure 2.40b). Since the field lines are closer together

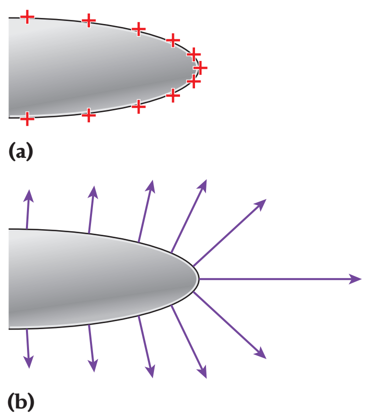


FIGURE 2.40 A sharp end of a conductor (with large curvature): (a) distribution of charges; (b) electric field at the surface of the conductor.

near a sharp point on a conductor, the field is stronger near the sharp tip than on the flat part of the conductor.

Benjamin Franklin proposed metal rods with sharp points as lightning rods. He reasoned that the sharp points would dissipate the electric charge built up in a storm, preventing the discharge of lightning. When Franklin installed such lightning rods, they were struck by lightning instead of the buildings to which they were attached. However, recent findings indicate that lightning rods used to protect structures from lightning should have blunt, rounded ends. When charged during thunderstorm conditions, a lightning rod with a sharp point creates a strong electric field that locally ionizes the air, producing a condition that actually causes lightning. Conversely, round-ended lightning rods are just as effective in protecting structures from lightning and do not increase lightning strikes. Any lightning rod should be carefully grounded to carry charge from a lightning strike away from the structure on which the lightning rod is mounted.

WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- The electric force, $\vec{F}(\vec{r})$ on a charge, q, due to an electric field, $\vec{E}(\vec{r})$ is given by $\vec{F}(\vec{r}) = q\vec{E}(\vec{r})$
- The electric field at any point is equal to the sum of the electric fields from all sources: $\vec{E}_t(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) + \cdots + \vec{E}_n(\vec{r})$
- The magnitude of the electric field due to a point charge q at a distance r is given by $E(r) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{|q|}{r^2} = \dfrac{k|q|}{r^2}$
  The electric field points radially away from a positive point charge and radially toward a negative charge.
- A system of two equal (in magnitude) but oppositely charged point particles is an electric dipole. The magnitude, p, of the electric dipole moment is given by $p = qd$, where q is the magnitude of either of the charges and d is the distance separating them. The electric dipole moment is a vector pointing from the negative toward the positive charge. On the dipole axis, the dipole produces an electric field of magnitude $E = \dfrac{p}{2\pi\varepsilon_0|x|^3}$ where $|x| \gg d$.
- Gauss's Law states that the electric flux over an entire closed surface is equal to the enclosed charge divided by ϵ_0: $\oiint \vec{E} \bullet d\vec{A} = \dfrac{q}{\varepsilon_0}$
- The differential electrical field is given by $dE = k\dfrac{dq}{r^2}$ and the differential charge is

$$\left.\begin{aligned} dq &= \lambda\,dx \\ dq &= \sigma\,dA \\ dq &= \rho\,dV \end{aligned}\right\}\text{ for a charge distribution }\left\{\begin{aligned} &\text{along a line;} \\ &\text{over a surface;} \\ &\text{throughout a volume.} \end{aligned}\right.$$

- The magnitude of the electric field at a distance r from a long straight wire with uniform linear charge density $\lambda > 0$ is given by $E = \dfrac{\lambda}{2\pi\varepsilon_0 r} = \dfrac{2k\lambda}{r}$
- The magnitude of the electric field produced by an infinite nonconducting plane that has uniform charge density $\sigma > 0$ is $E = \frac{1}{2}\sigma/\epsilon_0$.
- The magnitude of the electric field produced by an infinite conducting plane that has uniform charge density $\sigma > 0$ on each side is $E = \sigma/\epsilon_0$.
- The electric field inside a closed conductor is zero.
- The electric field outside a charged spherical conductor is the same as that due to a point charge of the same magnitude located at the center of the sphere.

ANSWERS TO SELF-TEST OPPORTUNITIES

2.1 The direction of the electric field is downward at points A, C, and E and upward at points B and D. (There is an electric field at point E, even though there is no line drawn there; the field lines are only sample representations of the electric field, which also exists between the field lines.) The field is largest in magnitude at point E, which can be inferred from the fact that it is located where the field lines have the highest density.

2.2 The two forces acting on the two charges in the electric field create a torque on the electric dipole around its center of mass, given by

$$\tau = (\text{force}_+)(\text{moment arm}_+)(\sin\theta) + (\text{force}_-)(\text{moment arm}_-)(\sin\theta).$$

The length of the moment arm in both cases is $\frac{1}{2}d$, and the magnitude of the force is $F = qE$ for both charges. Thus, the torque on the electric dipole is

$$\tau = qE\left(\frac{d}{2}\sin\theta\right) + qE\left(\frac{d}{2}\sin\theta\right) = qEd\sin\theta$$

2.3 The net electric flux passing though the object is EA. Remember, the object is not a closed surface; otherwise, the result would be zero.

2.4 The sign of the scalar product changes, because the electric field points radially inward: $\vec{E} \bullet d\vec{A} = EdA\cos 180° = -EdA$. But the magnitude of the electric field due to the negative charge is $E = \frac{1}{4\pi\varepsilon_0}\frac{-q}{r^2}$ The two minus signs cancel, giving the same results for Coulomb's and Gauss's Laws for a point charge, independent of the sign of the charge.

2.5 For a wire of infinite length, $E_y = \frac{2k\lambda}{y}$ for a wire of finite length, $E_y = \frac{2k\lambda}{y}\frac{a}{\sqrt{y^2+a^2}}$

With the values given in Concept Check 2.12, $\frac{a}{\sqrt{y^2+a^2}} = \frac{0.565}{\sqrt{0.401^2+0.565^2}} = 0.815$ Thus, the "infinitely long" approximation is off by ~18%.

2.6 The charged sphere acts like a point charge, so the electric field at $2R$ is

$$E = k\frac{q}{(2R)^2} = k\frac{q}{4R^2}$$

PROBLEM-SOLVING GUIDELINES

1. Be sure to distinguish between the point where an electric field is being generated and the point where the electric field is being determined.

2. Some of the same guidelines for dealing with electrostatic charges and forces also apply to electric fields: Use symmetry to simplify your calculations; remember that the field is composed of vectors and thus you have to use vector operations instead of simple addition, multiplication, and so on; convert units to meters and coulombs for consistency with the given values of constants.

3. Remember to use the correct form of the charge density for field calculations: λ for linear charge density, σ for surface charge density, and ρ for volume charge density.

4. The key to using Gauss's Law is to choose the right shape for the Gaussian surface to exploit the symmetry of the problem situation. Cubical, cylindrical, and spherical Gaussian surfaces are typically useful.

5. Often, you can break a Gaussian surface into surface elements that are either perpendicular to or parallel to the electric field lines. If the field lines are perpendicular to the surface, the electric flux is simply the field strength times the area, EA, or $-EA$ if the field points inward instead of outward. If the field lines are parallel to the surface, the flux through that surface is zero. The total flux is the sum of the flux through each surface element of the Gaussian surface. Remember that zero flux through a Gaussian surface does not necessarily mean that the electric field is zero.

MULTIPLE-CHOICE QUESTIONS

2.1 In order to use Gauss's Law to calculate the electric field created by a known distribution of charge, which of the following *must* be true?

a) The charge distribution must be in a nonconducting medium.

b) The charge distribution must be in a conducting medium.

c) The charge distribution must have spherical or cylindrical symmetry.

d) The charge distribution must be uniform.

e) The charge distribution must have a high degree of symmetry that allows assumptions about the symmetry of its electric field to be made.

2.2 An electric dipole consists of two equal and opposite charges situated a small distance from each other. When the dipole is placed in a uniform electric field, which of the following statements is (are) true?

a) The dipole will not experience any net force from the electric field; since the charges are equal and have opposite signs, the individual effects will cancel out.

b) There will be no net force and no net torque acting on the dipole.

c) There will be a net force but no net torque acting on the dipole.

d) There will be no net force, but there will (in general) be a net torque acting on dipole.

2.3 A point charge, $+Q$, is located on the x-axis at $x = a$, and a second point charge, $-Q$, is located on the x-axis at $x = -a$. A Gaussian surface with radius $r = 2a$ is centered at the origin. The flux through this Gaussian surface is

a) zero.

b) greater than zero.

c) less than zero.

d) none of the above.

2.4 A charge of $+2q$ is placed at the center of an uncharged conducting shell. What will be the charges on the inner and outer surfaces of the shell, respectively?

a) $-2q$, $+2q$

b) $-q$, $+q$

c) $-2q$, $-2q$

d) $-2q$, $+4q$

2.5 Two infinite nonconducting plates are parallel to each other, with a distance $d = 10.0$ cm between them, as shown in the figure. Each plate carries a uniform charge distribution of $\sigma = 4.5\ \mu C/m^2$. What is the electric field, $\vec{E}$ at point P (with $x_P = 20.0$ cm)?

a) 0 N/C

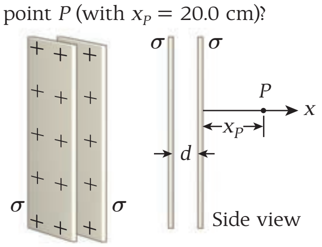
b) $2.54\hat{x}$ N/C

c) $(-5.08 \times 10^5)\hat{x}$ N/C

d) $(5.08 \times 10^5)\hat{x}$ N/C

e) $(-1.02 \times 10^6)\hat{x}$ N/C

f) $(1.02 \times 10^6)\hat{x}$ N/C



2.6 At which of the following locations is the electric field the strongest?

a) a point 1 m from a 1-C point charge

b) a point 1 m (perpendicular distance) from the center of a 1-m-long wire with 1 C of charge distributed on it

c) a point 1 m (perpendicular distance) from the center of a 1-m^2 sheet of charge with 1 C of charge distributed on it

d) a point 1 m from the surface of a charged spherical shell with a radius of 1 m

e) a point 1 m from the surface of a charged spherical shell with a radius of 0.5 m and a charge of 1 C

2.7 The electric flux through a spherical Gaussian surface of radius R centered on a charge Q is 1200 N/(Cm^2). What is the electric flux through a cubic Gaussian surface of side R centered on the same charge Q?

a) less than 1200 N/(C m^2)

b) more than 1200 N/(C m^2)

c) equal to 1200 N/(C m^2)

d) cannot be determined from the information given

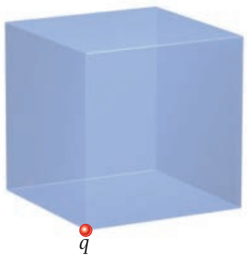
2.8 A single positive point charge, q, is at one corner of a cube with sides of length L, as shown in the figure. The net electric flux through the three adjacent sides is zero. The net electric flux through *each* of the other three sides is

a) $q/3\epsilon_0$.

b) $q/6\epsilon_0$.

c) $q/24\epsilon_0$.

d) $q/8\epsilon_0$.



2.9 Three −9mC point charges are located at (0,0), (3 m,3 m), and (3 m,−3 m). What is the magnitude of the electric field at (3 m,0)?

a) 0.9×10^7 N/C

b) 1.2×10^7 N/C

c) 1.8×10^7 N/C

d) 2.4×10^7 N/C

e) 3.6×10^7 N/C

f) 5.4×10^7 N/C

g) 10.8×10^7 N/C

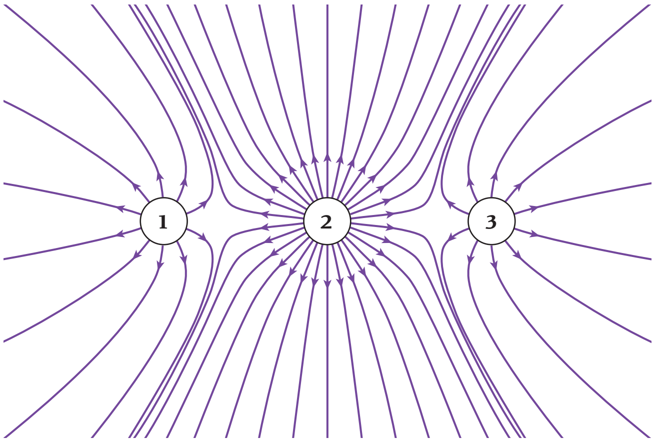
2.10 Which of the following statements is (are) true?

a) There will be *no* change in the charge on the inner surface of a hollow conducting sphere if additional charge is placed on the outer surface.

b) There will be some change in the charge on the inner surface of a hollow conducting sphere if additional charge is placed on the outer surface.

c) There will be *no* change in the charge on the inner surface of a hollow conducting sphere if additional charge is placed at the center of the sphere.

d) There will be some change in the charge on the inner surface of a hollow conducting sphere if additional charge is placed at the center of the sphere.



2.11 What are the signs of the charges in the configuration shown in the figure?

a) Charges 1, 2, and 3 are negative.

b) Charges 1, 2, and 3 are positive.

c) Charges 1 and 3 are positive, and 2 is negative.

d) Charges 1 and 3 are negative, and 2 is positive.

e) All that can be said is that the charges have the same sign.

2.12 Which of the following statements is (are) true?

a) Electric field lines point inward toward negative charges.

b) Electric field lines form circles around positive charges.

c) Electric field lines may cross.

d) Electric field lines point outward from positive charges.

e) A positive point charge released from rest will initially accelerate along a tangent to the electric field line at that point.

CONCEPTUAL QUESTIONS

2.13 Many people have been sitting in a car when it was struck by lightning. Why were they able to survive such an experience?

2.14 Why is it a bad idea to stand under a tree in a thunderstorm? What should one do instead to avoid getting struck by lightning?

2.15 Why do electric field lines never cross?

2.16 How is it possible that the flux through a closed surface does not depend on where inside the surface the charge is located (that is, the charge can be moved around inside the surface with no effect whatsoever on the flux)? If the charge is moved from just inside to just outside the surface, the flux changes discontinuously to zero, according to Gauss's Law. Does this really happen? Explain.

2.17 A solid conducting sphere of radius r_1 has a total charge of $+3Q$. It is placed inside (and concentric with) a conducting spherical shell of inner radius r_2 and outer radius r_3. Find the electric field in these regions: $r < r_1$, $r_1 < r < r_2$, $r_2 < r < r_3$, and $r > r_3$.

2.18 A thin rod has end points at $x = \pm 100$ cm. There is a total charge Q uniformly distributed along the rod.

a) What is the electric field very close to the midpoint of the rod?

b) What is the electric field a few centimeters (perpendicularly) from the midpoint of the rod?

c) What is the electric field very far (perpendicularly) from the midpoint of the rod?

2.19 A dipole is completely enclosed by a spherical surface. Describe how the total electric flux through this surface varies with the strength of the dipole.

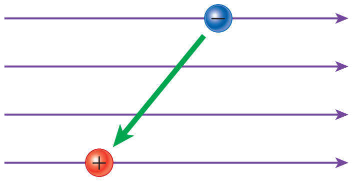
2.20 Repeat Example 2.3, assuming that the charge distribution is $-\lambda$ for $-a < x < 0$ and $+\lambda$ for $0 < x < a$.

2.21 A negative charge is placed on a solid prolate spheroidal conductor (shown in cross section in the figure). Sketch the distribution of the charge on the conductor and the electric field lines due to the charge.

2.2 *Saint Elmo's fire* is an eerie glow that appears at the tips of masts and yardarms of sailing ships in stormy weather and at the tips and edges of the wings of aircraft in flight. St. Elmo's fire is an electrical phenomenon. Explain it, concisely.

2.23 A charge placed on a conductor of any shape forms a layer on the outer surface of the conductor. Mutual repulsion of the individual charge elements creates an outward pressure on this layer, called *electrostatic stress*. Treating the infinitesimal charge elements like tiles of a mosaic, calculate the magnitude of this electrostatic stress in terms of the surface charge density, σ. Note that σ need not be uniform over the surface.

2.24 An electric dipole is placed in a uniform electric field as shown in the figure. What motion will the dipole have in the electric field? Which way will it move? Which way will it rotate?



EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

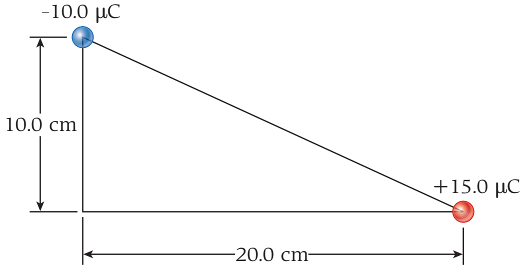
Section 2.3

2.25 A point charge, $q = 4.00 \times 10^{-9}$ C, is placed on the x-axis at the origin. What is the electric field produced at $x = 25.0$ cm?

2.26 A +1.60 nC point charge is placed at one corner of a square (1.00 m on a side), and a −2.40 nC charge is placed on the corner diagonally opposite. What is the magnitude of the electric field at either of the other two corners?

2.27 A +48.00 nC point charge is placed on the x-axis at $x = 4.000$ m, and a −24.00-nC point charge is placed on the y-axis at $y = -6.000$ m. What is the direction of the electric field at the origin?

•2.28 Two point charges are placed at two of the corners of a triangle as shown in the figure. Find the magnitude and the direction of the electric field at the third corner of the triangle.

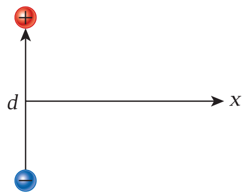


•2.29 A +5.00 C charge is located at the origin. A −3.00 C charge is placed at $x = 1.00$ m. At what finite distance(s) along the x-axis will the electric field be equal to zero?

•2.30 Three charges are on the y-axis. Two of the charges, each $-q$, are located $y = \pm d$, and the third charge, $+2q$, is located at $y = 0$. Derive an expression for the electric field at a point P on the x-axis.

Section 2.4

2.31 For the electric dipole shown in the figure, express the magnitude of the resulting electric field as a function of the perpendicular distance x from the center of the dipole axis. Comment on what the magnitude is when $x \gg d$.



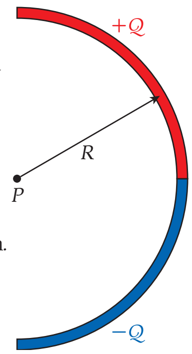
•2.32 Consider an electric dipole on the x-axis and centered at the origin. At a distance h along the positive x-axis, the magnitude of electric field due to the electric dipole is given by $k(2qd)/h^3$. Find a distance perpendicular to the x-axis and measured from the origin at which the magnitude of the electric field is the same.

Section 2.5

•2.33 A small metal ball with a mass of 4.00 g and a charge of 5.00 mC is located at a distance of 0.700 m above the ground in an electric field of 12.0 N/C directed to the east. The ball is then released from rest. What is the velocity of the ball after it has moved downward a vertical distance of 0.300 m?

•2.34 A charge per unit length $+\lambda$ is uniformly distributed along the positive y-axis from $y = 0$ to $y = +a$. A charge per unit length $-\lambda$ is uniformly distributed along the negative y-axis from $y = 0$ to $y = -a$. Write an expression for the electric field (magnitude and direction) at a point on the x-axis a distance x from the origin.

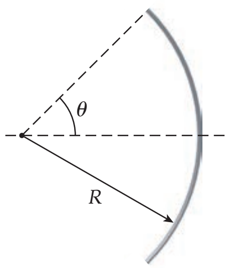
•2.35 A thin glass rod is bent into a semicircle of radius R. A charge $+Q$ is uniformly distributed along the upper half, and a charge $-Q$ is uniformly distributed along the lower half as shown in the figure. Find the magnitude and direction of the electric field $\vec{E}$ (in component form) at point P, the center of the semicircle.



•2.36 Two uniformly charged insulating rods are bent in a semicircular shape with radius $r = 10.0$ cm. If they are positioned so that they form a circle but do not touch and if they have opposite charges of +1.00 μC and −1.00 μC, find the magnitude and the direction of the electric field at the center of the composite circular charge configuration.

•2.37 A uniformly charged rod of length L with total charge Q lies along the y-axis, from $y = 0$ to $y = L$. Find an expression for the electric field at the point $(d,0)$ (that is, the point at $x = d$ on the x-axis).

••2.38 A charge Q is distributed evenly on a wire bent into an arc of radius R, as shown in the figure. What is the electric field at the center of the arc as a function of the angle θ? Sketch a graph of the electric field as a function of θ for $0 < \theta < 180°$.



••2.39 A thin, flat washer is a disk with an outer diameter of 10.0 cm and a hole of diameter 4.00 cm in the center. The washer has a uniform charge distribution and a total charge of 7.00 nC. What is the electric field on the axis of the washer at a distance of 30.0 cm from the center of the washer?

Section 2.6

2.40 Research suggests that the electric fields in some thunderstorm clouds can be on the order of 10.3 kN/C. Calculate the magnitude of the electric force acting on a particle with two excess electrons in the presence of a 10.0 kN/C field.

2.41 An electric dipole has opposite charges of 5.00×10^{-15} C separated by a distance of 0.400 mm. It is oriented at 60.0° with respect to a uniform electric field of magnitude 2.00×10^{3} N/C. Determine the magnitude of the torque exerted on the dipole by the electric field.

2.42 Electric dipole moments of molecules are often measured in debyes (D), where $1\text{ D} = 3.34 \times 10^{-30}$ C m. For instance, the dipole moment of hydrogen chloride gas molecules is 1.05 D. Calculate the maximum torque such a molecule can experience in the presence of an electric field of magnitude 160.0 N/C.

2.43 An electron is observed traveling at a speed of 27.5×10^6 m/s parallel to an electric field of magnitude 11,400 N/C. How far will the electron travel before coming to a stop?

2.44 Two charges, $+e$ and $-e$, are a distance of 0.680 nm apart in an electric field, E, that has a magnitude of 4.40 kN/C and is directed at an angle of 45.0° with respect to the dipole axis. Calculate the dipole moment and thus the torque on the dipole in the electric field.

•2.45 A body of mass M, carrying charge Q, falls from rest from a height h (above the ground) near the surface of the Earth, where the gravitational acceleration is g and there is an electric field with a constant component E in the vertical direction.

a) Find an expression for the speed, v, of the body when it reaches the ground, in terms of M, Q, h, g, and E.

b) The expression from part (a) is not meaningful for certain values of M, g, Q, and E. Explain what happens in such cases.

•2.46 A water molecule, which is electrically neutral but has a dipole moment of magnitude $p = 6.20 \times 10^{-30}$ C m, is 1.00 cm away from a point charge $q = +1.00$ μC. The dipole will align with the electric field due to the charge. It will also experience a net force, since the field is not uniform.

a) Calculate the magnitude of the net force. (*Hint:* You do not need to know the precise size of the molecule, only that it is much smaller than 1 cm.)

b) Is the molecule attracted to or repelled by the point charge? Explain.

•2.47 A total of 3.05×10^6 electrons are placed on an initially uncharged wire of length 1.33 m.

a) What is the magnitude of the electric field a perpendicular distance of 0.401 m away from the midpoint of the wire?

b) What is the magnitude of the acceleration of a proton placed at that point in space?

c) In which direction does the electric field force point in this case?

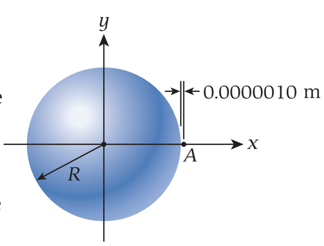
Sections 2.7 and 2.8

2.48 Four charges are placed in three-dimensional space. The charges have magnitudes $+3q$, $-q$, $+2q$, and $-7q$. If a Gaussian surface encloses all the charges, what will be the electric flux through that surface?

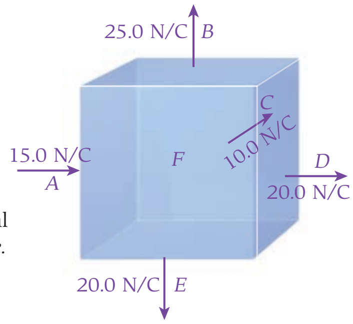
2.49 The six faces of a cubical box each measure 20.0 cm by 20.0 cm, and the faces are numbered such that faces 1 and 6 are opposite to each other, as are faces 2 and 5, and faces 3 and 4. The flux through each face is given in the table. Find the net charge inside the cube.

Face	Flux (N m^2/C)
1	−70.0
2	−300.0
3	−300.0
4	+300.0
5	−400.0
6	−500.0

2.50 A conducting solid sphere ($R = 0.15$ m, $q = 6.1 \times 10^{-6}$ C) is shown in the figure. Using Gauss's Law and two different Gaussian surfaces, determine the electric field (magnitude and direction) at point A, which is 0.0000010 m outside the conducting sphere. (*Hint:* One Gaussian surface is a sphere, and the other is a small right cylinder.)



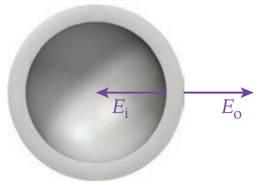
2.51 Electric fields of varying magnitudes are directed either inward or outward at right angles on the faces of a cube, as shown in the figure. What is the strength and direction of the field on the face F?



2.52 Consider a hollow spherical conductor with total charge $+5e$. The outer and inner radii are a and b, respectively. (a) Calculate the charge on the sphere's inner and outer surfaces if a charge of $-3e$ is placed at the center of the sphere. (b) What is the total net charge of the sphere?

•2.53 A spherical aluminized Mylar balloon carries a charge Q on its surface. You are measuring the electric field at a distance R from the balloon's center. The balloon is slowly inflated, and its radius approaches but never reaches R. What happens to the electric field you measure as the balloon increases in radius? Explain.

•2.54 A hollow conducting spherical shell has an inner radius of 8.00 cm and an outer radius of 10.0 cm. The electric field at the inner surface of the shell, E_i, has a magnitude of 80.0 N/C and points toward the center of the sphere, and the electric field at the outer surface, E_o, has a magnitude of 80.0 N/C and points away from the center of the sphere (see the figure). Determine the magnitude of the charge on the inner surface and on the outer surface of the spherical shell.



•2.55 A −6.00 nC point charge is located at the center of a conducting spherical shell. The shell has an inner radius of 2.00 m, an outer radius of 4.00 m, and a charge of +7.00 nC.

a) What is the electric field at $r = 1.00$ m?

b) What is the electric field at $r = 3.00$ m?

c) What is the electric field at $r = 5.00$ m?

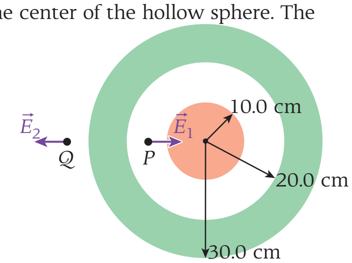
d) What is the surface charge distribution, σ, on the outside surface of the shell?

Section 2.9

2.56 A solid, nonconducting sphere of radius a has total charge Q and a uniform charge distribution. Using Gauss's Law, determine the electric field (as a vector) in the regions $r < a$ and $r > a$ in terms of Q.

2.57 There is an electric field of magnitude 150.0 N/C, directed downward, near the surface of the Earth. What is the net electric charge on the Earth? You can treat the Earth as a spherical conductor of radius 6371 km.

2.58 A hollow metal sphere has inner and outer radii of 20.0 cm and 30.0 cm, respectively. As shown in the figure, a solid metal sphere of radius 10.0 cm is located at the center of the hollow sphere. The electric field at a point P, a distance of 15.0 cm from the center, is found to be $E_1 = 1.00 \times 10^4$ N/C, directed radially inward. At point Q, a distance of 35.0 cm from the center, the electric field is found to be $E_2 = 1.00 \times 10^4$ N/C, directed radially outward. Determine the total charge on (a) the surface of the inner sphere, (b) the inner surface of the hollow sphere, and (c) the outer surface of the hollow sphere.



2.59 Two parallel, infinite, nonconducting plates are 10.0 cm apart and have charge distributions of $+1.00$ μC/m^2 and -1.00 μC/m^2. What is the force on an electron in the space between the plates? What is the force on an electron located outside the two plates near the surface of one of the two plates?

2.60 An infinitely long charged wire produces an electric field of magnitude 1.23×10^3 N/C at a distance of 50.0 cm perpendicular to the wire. The direction of the electric field is toward the wire.

a) What is the charge distribution?

b) How many electrons per unit length are on the wire?

•2.61 A solid sphere of radius R has a nonuniform charge distribution $\rho = Ar^2$, where A is a constant. Determine the total charge, Q, within the volume of the sphere.

•2.62 Two parallel, uniformly charged, infinitely long wires are 6.00 cm apart and carry opposite charges with a linear charge density of $\lambda = 1.00$ μC/m. What are the magnitude and the direction of the electric field at a point midway between the two wires and 40.0 cm above the plane containing them?

•2.63 A sphere centered at the origin has a volume charge distribution of 120. nC/cm^3 and a radius of 12.0 cm. The sphere is centered inside a conducting spherical shell with an inner radius of 30.0 cm and an outer radius of 50.0 cm. The charge on the spherical shell is -2.00 mC. What are the magnitude and the direction of the electric field at each of the following distances from the origin?

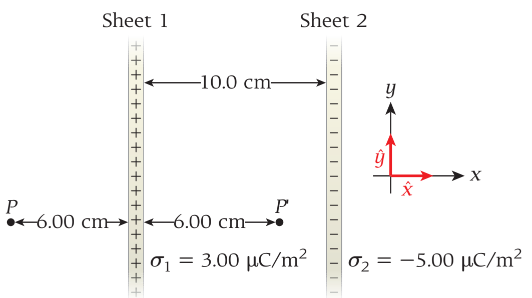
a) at $r = 10.0$ cm c) at $r = 40.0$ cm

b) at $r = 20.0$ cm d) at $r = 80.0$ cm

•2.64 A thin, hollow, metal cylinder of radius R has a surface charge distribution σ. A long, thin wire with a linear charge density $\lambda/2$ runs through the center of the cylinder. Find an expression for the electric field and determine the direction of the field at each of the following locations:

a) $r \le R$ b) $r \ge R$

•2.65 Two infinite sheets of charge are separated by 10.0 cm as shown in the figure. Sheet 1 has a surface charge distribution of $\sigma_1 = 3.00$ μC/m^2 and sheet 2 has a surface charge distribution of $\sigma_2 = -5.00$ μC/m^2. Find the total electric field (magnitude and direction) at each of the following locations:



a) at point P, 6.00 cm to the left of sheet 1

b) at point P', 6.00 cm to the right of sheet 1

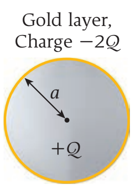
•2.66 A conducting solid sphere of radius 20.0 cm is located with its center at the origin of a three-dimensional coordinate system. A charge of 0.271 nC is placed on the sphere.

a) What is the magnitude of the electric field at point $(x,y,z) = (23.1\text{ cm}, 1.10\text{ cm}, 0.00\text{ cm})$?

b) What is the angle of this electric field with the x-axis at this point?

c) What is the magnitude of the electric field at point $(x,y,z) = (4.10\text{ cm}, 1.10\text{ cm}, 0.00\text{ cm})$?

••2.67 A solid nonconducting sphere of radius a has a total charge $+Q$ uniformly distributed throughout its volume. The surface of the sphere is coated with a very thin (negligible thickness) conducting layer of gold. A total charge of $-2Q$ is placed on this conducting layer. Use Gauss's Law to do the following.



a) Find the electric field $E(r)$ for $r < a$ (inside the sphere, up to and excluding the gold layer).

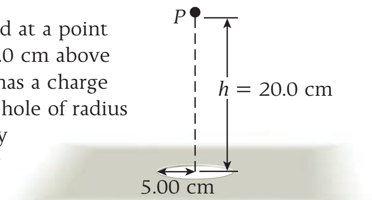
b) Find the electric field $E(r)$ for $r > a$ (outside the coated sphere, beyond the sphere and the gold layer).

c) Sketch the graph of $E(r)$ versus r. Comment on the continuity or discontinuity of the electric field, and relate this to the surface charge distribution on the gold layer.

••2.68 A solid nonconducting sphere has a volume charge distribution given by $\rho(r) = (\beta/r)\sin(\pi r/2R)$. Find the total charge contained in the spherical volume and the electric field in the regions $r < R$ and $r > R$. Show that the two expressions for the electric field equal each other at $r = R$.

••2.69 A very long cylindrical rod of nonconducting material with a 3.00-cm radius is given a uniformly distributed positive charge of 6.00 nC per centimeter of its length. Then a cylindrical cavity is drilled all the way through the rod, of radius 1 cm, with its axis located 1.50 cm from the axis of the rod. That is, if, at some cross section of the rod, x- and y-axes are placed so that the center of the rod is at $(x,y) = (0,0)$; then the center of the cylindrical cavity is at $(x,y) = (0,1.50)$. The creation of the cavity does not disturb the charge on the remainder of the rod that has not been drilled away; it just removes the charge from the region in the cavity. Find the electric field at the point $(x,y) = (2.00,1.00)$.

••2.70 What is the electric field at a point P, which is at a distance $h = 20.0$ cm above an infinite sheet of charge that has a charge distribution of 1.30 C/m^2 and a hole of radius 5.00 cm whose center is directly below P, as shown in the figure? Plot the electric field as a function of h in terms of $\sigma/(2e_0)$.



Additional Exercises

2.71 A cube has an edge length of 1.00 m. An electric field acting on the cube from outside has a constant magnitude of 150 N/C and its direction is also constant but unspecified (not necessarily along any edges of the cube). What is the total charge within the cube?

2.72 A carbon monoxide (CO) molecule has a dipole moment of approximately 8.0×10^{-30} C m. If the carbon and oxygen atoms are separated by 1.2×10^{-10} m, find the net charge on each atom and the maximum amount of torque the molecule would experience in an electric field of 500.0 N/C.

2.73 An infinitely long, solid cylinder of radius $R = 9.00$ cm, with a uniform charge per unit of volume of $\rho = 6.40 \times 10^{-8}$ C/m^3, is centered about the y-axis. Find the magnitude of the electric field at a radius $r = 4.00$ cm from the center of this cylinder.

2.74 Find the magnitudes and the directions of the electric fields needed to counteract the weight of (a) an electron and (b) a proton at the Earth's surface.

2.75 A solid metal sphere of radius 8.00 cm, with a total charge of 10.0 μC, is surrounded by a metallic shell with a radius of 15.0 cm carrying a -5.00 μC charge. The sphere and the shell are both inside a larger metallic shell of inner radius 20.0 cm and outer radius 24.0 cm. The sphere and the two shells are concentric.

a) What is the charge on the inner wall of the larger shell?

b) If the electric field outside the larger shell is zero, what is the charge on the outer wall of the shell?

2.76 Two infinite, uniformly charged, flat, nonconducting surfaces are mutually perpendicular. One of the surfaces has a charge distribution of $+30.0$ pC/m^2, and the other has a charge distribution of -40.0 pC/m^2. What is the magnitude of the electric field at any point not on either surface?

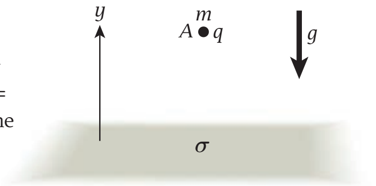
2.77 There is an electric field of magnitude 150. N/C, directed vertically downward, near the surface of the Earth. Find the acceleration (magnitude and direction) of an electron released near the Earth's surface.

2.78 Suppose you have a large spherical balloon and you are able to measure the component E_n of the electric field normal to its surface. If you sum $E_n\,dA$ over the whole surface area of the balloon and obtain a magnitude of 10.0 N m^2/C, what is the electric charge enclosed by the balloon?

2.79 A 30.0 cm-long uniformly charged rod is sealed in a container. The total electric flux leaving the container is 1.46×10^6 N m^2/C. Determine the linear charge distribution on the rod.

•**2.80** A long conducting wire with charge distribution λ and radius r produces an electric field of 2.73 N/C just outside its surface. What is the magnitude of the electric field just outside the surface of another wire with charge distribution 0.810λ and radius $6.50r$?

•**2.81** An object with mass $m = 1.00$ g and charge q is placed at point A, which is 0.0500 m above an infinitely large, uniformly charged, nonconducting sheet ($\sigma = -3.50 \times 10^{-5}$ C/m^2), as shown in the figure. Gravity is acting downward ($g = 9.81$ m/s^2). Determine the number, N, of electrons that must be added to or removed from the object for the object to remain motionless above the charged plane.



•**2.82** A proton enters the gap between a pair of metal plates (an electrostatic separator) that produce a uniform, vertical electric field between them. Ignore the effect of gravity on the proton.

a) Assuming that the length of the plates is 15.0 cm and that the proton approaches the plates with a speed of 15.0 km/s, what electric field strength should the plates be designed to provide so that the proton will be deflected vertically by 1.50×10^{-3} rad?

b) What speed will the proton have after exiting the electric field?

c) Suppose the proton is one in a beam of protons that has been contaminated with positively charged kaons, particles whose mass is 494 MeV/c^2 (8.81×10^{-28} kg), while the mass of the proton is 938 MeV/c^2 (1.67×10^{-27} kg). The kaons have a charge of $+1e$, just like the protons. If the electrostatic separator is designed to give the protons a deflection of 1.20×10^{-3} rad, what deflection will kaons with the same momentum as the protons experience?

•**2.83** Consider a uniform nonconducting sphere with a surface charge density $\rho = 3.57 \times 10^{-6}$ C/m^3 and a radius $R = 1.72$ m. What is the magnitude of the electric field 0.530 m from the center of the sphere?

••**2.84** A uniform sphere has a radius R and a total charge $+Q$, uniformly distributed throughout its volume. It is surrounded by a thick spherical shell carrying a total charge $-Q$, also uniformly distributed, and having an outer radius of $2R$. What is the electric field as a function of R?

••**2.85** If a charge is held in place above a large, flat, grounded, conducting slab, such as a floor, it will experience a downward force toward the floor. In fact, the electric field in the room above the floor will be exactly the same as that produced by the original charge plus a "mirror image" charge, equal in magnitude and opposite in sign, as far below the floor as the original charge is above it. Of course, there is no charge below the floor; the effect is produced by the surface charge distribution induced on the floor by the original charge.

a) Describe or sketch the electric field lines in the room above the floor.

b) If the original charge is 1.00 μC at a distance of 50.0 cm above the floor, calculate the downward force on this charge.

c) Find the electric field at (just above) the floor, as a function of the horizontal distance from the point on the floor directly under the original charge. Assume that the original charge is a point charge, $+q$, at a distance a above the floor. Ignore any effects of walls or ceiling.

d) Find the surface charge distribution $\sigma(\rho)$ induced on the floor.

e) Calculate the total surface charge induced on the floor.

MULTI-VERSION EXERCISES

2.86 A long, horizontal, conducting wire has the charge density $\lambda = 2.849 \times 10^{-12}$ C/m. A proton (mass $= 1.673 \times 10^{-27}$ kg) is placed 0.6815 m above the wire and released. What is the magnitude of the initial acceleration of the proton?

2.87 A long, horizontal, conducting wire has the charge density λ. A proton (mass $= 1.673 \times 10^{-27}$ kg) is placed 0.6897 m above the wire and released. The magnitude of the initial acceleration of the proton is 1.111×10^7 m/s^2. What is the charge density on the wire?

2.88 A long, horizontal, conducting wire has the charge density $\lambda = 6.055 \times 10^{-12}$ C/m. A proton (mass $= 1.673 \times 10^{-27}$ kg) is placed a distance d above the wire and released. The magnitude of the initial acceleration of the proton is 1.494×10^7 m/s^2. What is the distance d?

2.89 There is a uniform charge distribution of $\lambda = 5.635 \times 10^{-8}$ C/m along a thin wire of length $L = 2.13$ cm. The wire is then curved into a semicircle that is centered at the origin and has a radius of $R = L/\pi$. Find the magnitude of the electric field at the center of the semicircle.

2.90 There is a uniform charge distribution λ along a thin wire of length $L = 10.55$ cm. The wire is then curved into a semicircle that is centered at the origin and has a radius of $R = L/\pi$. The magnitude of the electric field at the center of the semicircle is 3.117×10^4 N/C. What is the value of λ?

2.91 There is a uniform charge distribution of $\lambda = 6.005 \times 10^{-8}$ C/m along a thin wire of length L. The wire is then curved into a semicircle that is centered at the origin and has a radius of $R = L/\pi$. The magnitude of the electric field at the center of the semicircle is 2.425×10^4 N/C. What is the value of L?

3 Electric Potential

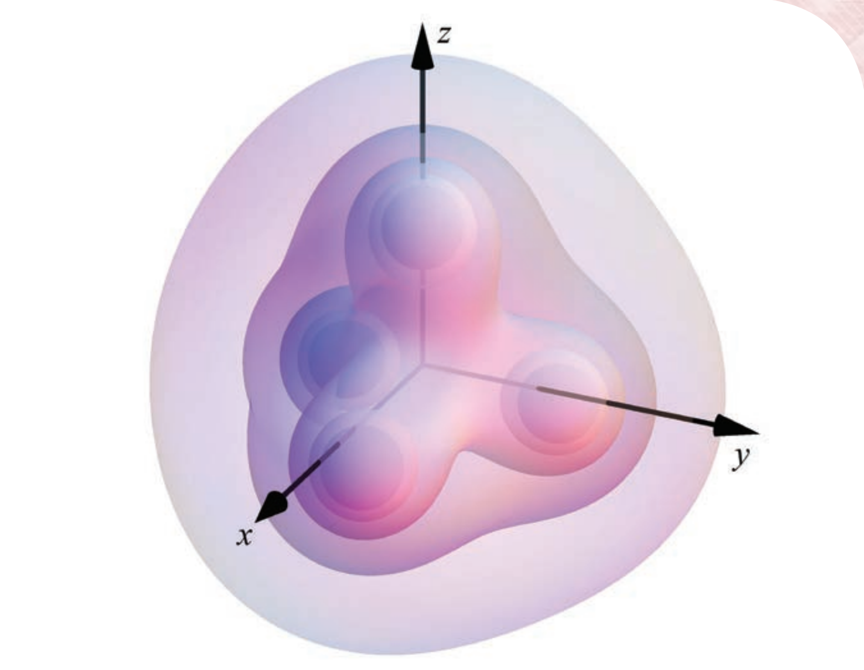


FIGURE 3.1 Five equipotential surfaces of a tetrahedral arrangement of four identical charges.

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Energy is vital to all kinds of systems, from our own bodies to the world economy. A large fraction of the devices we rely on–including pacemakers, televisions, computers, cell phones, hair dryers, and locomotives–use electric energy. It is therefore important to understand how electrically powered devices process and store this energy. We start by looking at electric energy and electric potential in order to understand the relationship between them and see how to calculate both. Figure 3.1 shows the result of calculating the electric potential due to four identical charges arranged at the corners of a tetrahedron. The electric potential is visualized as five nested surfaces representing points at which its magnitude is constant. The highest values of the potential occur closest to the charges, and the equipotential surfaces approximate spheres around each charge. The equipotential surfaces with the lowest values are farthest away from the charges and approximate spheres around the center of mass of the combined charges.

WHAT WE WILL LEARN

- Electric potential energy is analogous to gravitational potential energy.
- The change in electric potential energy is proportional to the work done by the electric field on a charge.
- The electric potential at a given point in space is a scalar.
- The electric potential, V, of a point charge, q, is inversely proportional to the distance from that point charge.
- The electric potential can be derived from the electric field by integrating the electric field over a displacement.
- The electric potential at a given point in space due to a distribution of point charges equals the algebraic sum of the electric potentials due to the individual charges.
- The electric field can be derived from the electric potential by differentiating the electric potential with respect to displacement.

3.1 Electric Potential Energy

Throughout this book, we have encountered various forms of energy and have seen how energy conservation affects different physical systems. We have also noted the importance of energy conversion in processes that are vital to daily life and the world economy. Now we turn our attention to electric energy, in particular, to the storage of electric potential energy in batteries. An electric field has many similarities to a gravitational field, including its mathematical formulation. We saw in Chapter 1 that the magnitude of the gravitational force is given by

$$F_g = G\frac{m_1 m_2}{r^2}$$

where G is the universal gravitational constant, m_1 and m_2 are two masses, and r is the distance between the two masses. In Chapter 21, we saw that the magnitude of the electrostatic force is

$$F_e = k\frac{|q_1 q_2|}{r^2} \tag{3.1}$$

where k is Coulomb's constant, q_1 and q_2 are two electric charges, and r is the distance between the two charges. Both gravitational and electrostatic forces depend only on the inverse square of the distance between the objects, and it can be shown that all such forces are conservative. Therefore, the electric potential energy, U, can be defined in analogy with the gravitational potential energy.

In Chapter 6, we saw that for any conservative force, the change in potential energy due to some spatial rearrangement of a system is equal to the negative of the work done by the conservative force during this spatial rearrangement. For a system of two or more particles, the work done by an electric force, W_e, when the system configuration changes from an initial state to a final state, is given in terms of the change in **electric potential energy,** ΔU:

$$\Delta U = U_f - U_i = -W_e \tag{3.2}$$

where U_i is the initial electric potential energy and U_f is the final electric potential energy. Note that it does not matter *how* the system gets from the initial to the final state. The work is always the same, independent of the path taken. Noted that this path-independence of the work done by a force is a general feature of conservative forces.

As is the case for gravitational potential energy (see Chapter 1), a reference point for the electric potential energy must always be specified. It simplifies the equations and calculations if the zero point of the electric potential energy is assumed to be the configuration in which an infinitely large distance separates all the charges, which is exactly the same convention used for the gravitational potential energy. This assumption allows equation 3.2 for the change in electric potential energy to be rewritten as $\Delta U = U_f - 0 = U$, or

$$U = -W_{e,\infty} \tag{3.3}$$

Even though the convention of zero potential energy at infinity is very useful and is universally accepted for a collection of point charges, in some physical situations there is a reason to select a reference potential energy at some point in space, which will not lead to a value of zero potential energy at infinite separation. Remember, all potential energies of conservative forces are fixed only within an arbitrary additive constant. So you need to pay attention to how this constant is chosen in a particular situation. One situation in which the potential energy at infinity is not set to zero is that involving a constant electric field.

Special Case: Charge in a Constant Electric Field

Let's consider a point charge, q, moving through a displacement, $\vec{d}$ in a constant electric field, $\vec{E}$ (Figure 3.2). The work done by a constant force $\vec{F}$ is $W = \vec{F} \bullet \vec{d}$ For this case, the constant force is created by a constant electric field, $\vec{F} = q\vec{E}$ Thus, the work done by the field on the charge is given by

$$W = q\vec{E} \bullet \vec{d} = qEd\cos\theta \tag{3.4}$$

where θ is the angle between the electric force and the displacement. When the displacement is parallel to the electric field ($\theta = 0°$), the work done by the field on the charge is $W = qEd$. When the displacement is antiparallel to the electric field ($\theta = 180°$), the work done by the field is $W = -qEd$. Because the change in electric potential energy is related to the work done on the charge by $\Delta U = -W$, if $q > 0$, the charge loses potential energy when the displacement is in the same direction as the electric field and gains potential energy when the displacement is in the direction opposite to the electric field.

Figure 3.3a shows a mass, m, near the surface of the Earth, where it can be considered to be in a constant gravitational field, which points downward. We know that when the mass moves toward the surface of the Earth a distance h, the change in the gravitational potential energy of the mass is

$$\Delta U = -W = -\vec{F}_g \bullet \vec{d} = -mgh$$

It is intuitive that the mass has less potential energy if it is closer to the surface of the Earth. Figure 3.3b shows a positive charge, q, in a constant electric field. If the charge moves a distance, d, in the same direction as the electric field, the change in the electric potential energy is

$$\Delta U = -W = -q\vec{E} \bullet \vec{d} = -qEd$$

Thus, the electric potential energy of a charge in an electric field is analogous to the gravitational potential energy of a mass in Earth's gravitational field near the surface of Earth. (But, of course, the important difference between the two interactions is that masses come in only one variety, and exert gravitational attraction for one another, whereas charges can attract or repel each other. Thus, ΔU can change sign, depending on the signs of the charges.)

Special Case: Dipole in a Constant Electric Field

Now let's consider an electric dipole with dipole moment $\vec{p}$ moving through a constant electric field (see Figure 3.4). In Chapter 2, we saw that an electric dipole consists of a positive charge and a negative charge, which have equal magnitudes, meaning that the dipole has zero net charge. Since, according to equation 3.4, the work done in moving an object through a constant electric field is proportional to the charge on that object, the net work done in moving an electric dipole through a constant electric field is zero.

This fact may make it seem impossible to store potential energy in a system consisting of a dipole in a constant field. However, this is not the case. In Chapter 2, we saw that a dipole in a constant electric field experiences a torque, $\vec{\tau} = \vec{p} \times \vec{E}$ and so it is clear that the orientation of the dipole relative to the electric field is key. Let's see how the orientation of the dipole can result in the storage of potential energy.

Previously, we saw that the work done by a torque is given by $W = \int \vec{\tau}(\theta')\, d\vec{\theta}'$ If we apply an external torque opposing the torque that the dipole experiences from the electric field, we can express the work done by this external torque as follows:

$$W = \int_{\theta_0}^{\theta} \vec{\tau}(\theta')\, d\vec{\theta}' = \int_{\theta_0}^{\theta} -pE\sin\theta'\, d\theta' = -pE\int_{\theta_0}^{\theta} \sin\theta'\, d\theta' = pE(\cos\theta - \cos\theta_0)$$

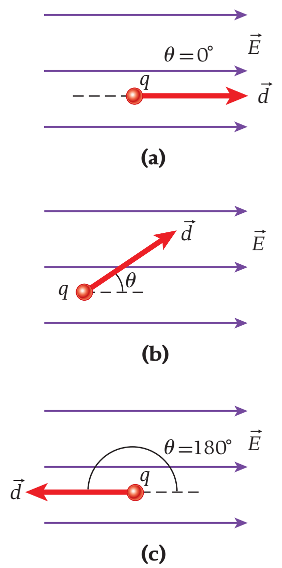


FIGURE 3.2 Work done by an electric field, $\vec{E}$ on a moving charge, q: (a) case where the displacement is in the same direction as the electric field, (b) general case, (c) case where the displacement is opposite to the direction of the electric field.

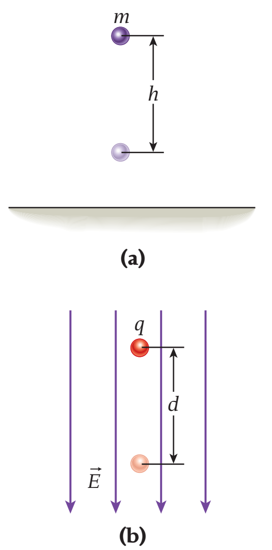


FIGURE 3.3 The analogy between gravitational potential energy and electric potential energy. (a) A mass falls in a gravitational field. (b) A positive charge moves in the same direction as an electric field.

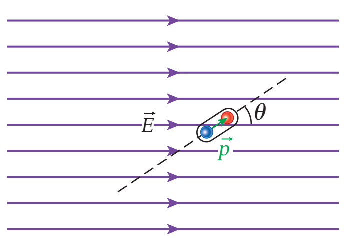


FIGURE 3.4 An electric dipole in a uniform electric field.

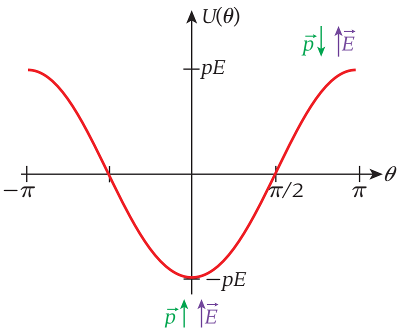


FIGURE 3.5 Potential energy as a function of the angle between an electric dipole and a constant external electric field.

With $W = -\Delta U = -(U - U_0) = U_0 - U$, we then have the potential energy of an electric dipole in a constant electric field:

$$U = -pE\cos\theta = -\vec{p}\bullet\vec{E} \tag{3.5}$$

where we have chosen the integration constant U_0 so that the potential energy is zero for $\theta = \pi/2$.

Equation 3.5 indicates that the potential energy has a minimum at $\theta = 0$, where the dipole moment is parallel to the electric field; see Figure 3.5. Remember that electric field lines point from positive to negative charges and that the dipole moment is defined to point from the negative to the positive charge. When the dipole moment and electric field vectors are parallel, the negative charge of the dipole is closest to the positive charge that is generating the external electric field, and it makes physical sense that that configuration has the lowest energy.

3.2 Definition of Electric Potential

The potential energy of a charged particle, q, in an electric field depends on the magnitude of the charge as well as that of the electric field. A quantity that is independent of the charge on the particle is the **electric potential,** V, defined in terms of the electric potential energy as

$$V = \frac{U}{q} \tag{3.6}$$

Because U is proportional to q, V is independent of q, which makes it a useful variable. The electric potential, V, characterizes an electrical property of a point in space even when no charge, q, is placed at that point. In contrast to the electric field, which is a vector, the electric potential is a scalar. It has a value everywhere in space, but has no direction. In Chapter 6, we saw that we can always add an arbitrary constant to the potential energy without changing any observable consequence and that only differences in potential energy are physically meaningful. Since the electric potential is proportional to the potential energy, the same is true for it. Stating a difference in electric potentials is unambiguous, but stating a value for the electric potential itself always implies a normalization condition, which is usually that the potential is zero at an infinite distance.

The difference in electric potential, ΔV, between an initial point and final point, $V_f - V_i$, can be expressed in terms of the electric potential energy at each point:

$$\Delta V = V_f - V_i = \frac{U_f}{q} - \frac{U_i}{q} = \frac{\Delta U}{q} \tag{3.7}$$

Combining equations 3.2 and 3.7 yields a relationship between the change in electric potential and the work done by an electric field on a charge:

$$\Delta V = -\frac{W_e}{q} \tag{3.8}$$

Taking the electric potential energy to be zero at infinity, as in equation 3.3, gives the electric potential at a point as

$$V = -\frac{W_{e,\infty}}{q} \tag{3.9}$$

where $W_{e,\infty}$ is the work done by the electric field on the charge when it is brought in to the point from infinity. An electric potential can have a positive, a negative, or a zero value, but it has no direction.

The SI units for electric potential are joules/coulomb (J/C). This combination has been named the **volt** (V) for Italian physicist Alessandro Volta (1745–1827) (note the use of the roman V for the unit, whereas the italicized V is used for the physical quantity of electric potential):

$$1\ \mathrm{V} \equiv \frac{1\ \mathrm{J}}{1\ \mathrm{C}}$$

With this definition of the volt, the units for the magnitude of the electric field are

$$[E] = \frac{[F]}{[q]} = \frac{1\ \mathrm{N}}{1\ \mathrm{C}} = \left(\frac{1\ \mathrm{N}}{1\ \mathrm{C}}\right)\frac{1\ \mathrm{V}}{\left(\frac{1\ \mathrm{J}}{1\ \mathrm{C}}\right)}\left(\frac{1\ \mathrm{J}}{(1\ \mathrm{N})(1\ \mathrm{m})}\right) = \frac{1\ \mathrm{V}}{1\ \mathrm{m}}$$

For the remainder of this book, the magnitude of an electric field will have units of V/m, which is the standard convention, instead of N/C. Note that an electric potential difference is often referred to as a "voltage," particularly in circuit analysis, because it is measured in volts.

EXAMPLE 3.1 Energy Gain of a Proton

A proton is placed between two parallel conducting plates in a vacuum (Figure 3.6). The difference in electric potential between the two plates is 450 V. The proton is released from rest close to the positive plate.

PROBLEM

What is the kinetic energy of the proton when it reaches the negative plate?

SOLUTION

The difference in electric potential, ΔV, between the two plates is 450 V. We can relate this potential difference across the two plates to the change in electric potential energy, ΔU, of the proton using equation 3.7:

$$\Delta V = \frac{\Delta U}{q}$$

Because of the conservation of total energy, all the electric potential energy lost by the proton in crossing between the two plates is turned into kinetic energy due to the motion of the proton. We apply the law of conservation of energy, $\Delta K + \Delta U = 0$, where ΔU is the change in the proton's electric potential energy:

$$\Delta K = -\Delta U = -q\Delta V$$

Because the proton started from rest, we can express its final kinetic energy as $K = -q\Delta V$. Therefore, the kinetic energy of the proton after crossing the gap between the two plates is

$$K = -\left(1.602\times10^{-19}\ \mathrm{C}\right)\left(-450\ \mathrm{V}\right) = 7.21\times10^{-17}\ \mathrm{J}$$

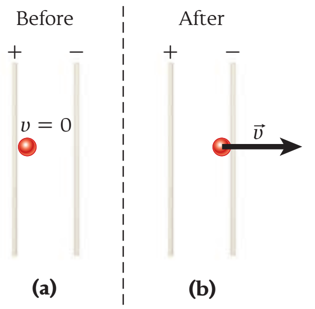


FIGURE 3.6 A proton between two charged parallel conducting plates in a vacuum. (a) The proton is released from rest. (b) The proton has moved from the positive plate to the negative plate, gaining kinetic energy.

Concept Check 3.1

An electron is positioned and then released on the x-axis, where the electric potential has the value −20 V. Which of the following statements describes the subsequent motion of the electron?

a) The electron will move to the left (negative x-direction) because it is negatively charged.

b) The electron will move to the right (positive x-direction) because it is negatively charged.

c) The electron will move to the left (negative x-direction) because the electric potential is negative.

d) The electron will move to the right (positive x-direction) because the electric potential is negative.

e) Not enough information is given to predict the motion of the electron.

Because the acceleration of charged particles across a potential difference is often used in the measurement of physical quantities, a common unit for the kinetic energy of a singly charged particle, such as a proton or an electron, is the **electron-volt** (eV): 1 eV represents the energy gained by a proton ($q = 1.602 \times 10^{-19}$ C) accelerated across a potential difference of 1 V. The conversion between electron-volts and joules is

$$1\ \mathrm{eV} = 1.602\times10^{-19}\ \mathrm{J}$$

The kinetic energy of the proton in Example 3.1 is then 450 eV, or 0.450 keV, which we could have obtained from the definition of the electron-volt without performing any calculations.

Batteries

A common means of creating electric potential is a battery. We'll see in Chapters 4 and 5 how a battery uses chemical reactions to provide a source of (nearly) constant potential difference between its two terminals. An assortment of batteries is shown in Figure 3.7.

At its simplest, a battery consists of two half-cells, filled with a conducting electrolyte (originally a liquid but now almost always a solid); see Figure 3.8. The electrolyte is separated into two equal portions by a barrier, which prevents the bulk of the electrolyte from passing through but allows charged ions to pass through. The negatively charged ions (anions) move toward the anode, and the positively charged ions move toward the cathode. This creates a potential difference between the two terminals of the battery. Thus, a battery is basically a device that converts chemical energy directly into electrical energy.

Research on battery technology is of current importance, because many mobile applications require a great deal of energy, from cell phones to laptop computers, from electrical cars to military gear. The weight of the batteries needs to be as small as possible, they need to be rapidly rechargeable for hundreds of cycles, they need to deliver as constant a potential difference as possible, and they need to be available at an affordable price. Thus, such research provides many scientific and engineering challenges.

One example of relatively recent battery technology is the lithium ion cell, which is often used in applications such as laptop computer batteries. A lithium ion battery has a much higher energy density (energy content per unit volume) than conventional batteries. A typical lithium ion cell, like the one in Figure 3.9, has a potential difference of 3.6 V. Lithium ion batteries have several other advantages over conventional batteries. They can be recharged hundreds of times. They have no "memory" effect and thus do not need to be conditioned to hold their charge. They hold their charge on the shelf. They also have some disadvantages. For example, if a lithium ion battery is completely discharged, it can no longer be recharged. The battery performs best if it is not charged to more than 80% of capacity and not discharged to less than 20% of its capacity. Heat degrades lithium ion batteries. If the batteries are discharged too quickly, the constituents can catch fire or explode. To deal with these problems, most commercial lithium ion battery packs have a small built-in electronic circuit that protects the battery pack. The circuit will not allow the battery to be overcharged or overly discharged; it will not allow charge to flow out of the battery so quickly that the battery will overheat. If the battery becomes too warm, the circuit disconnects the battery.

Currently, lithium ion batteries are being used in some electric-powered cars. The following example compares the energy carried by a battery-powered car and a gasoline-powered car.

(a)

(b)

FIGURE 3.7 (a) Some representative batteries (clockwise from upper left): rechargeable AA nickel metal hydride (NiMH) batteries in their charger, disposable 1.5-V AAA batteries, a 12-V lantern battery, a D-size battery, a lithium ion laptop battery, and a watch battery; (b) 330-V battery for a gas-electric hybrid SUV, filling the entire floor of the trunk.

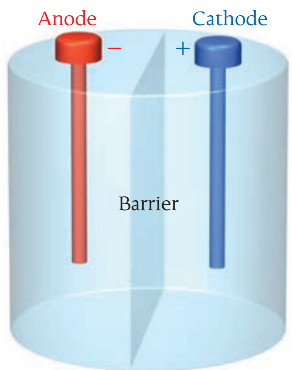


FIGURE 3.8 Schematic drawing of a battery.

FIGURE 3.9 Tesla machine shop in Frankfurt center. Also shown is a 2012 Ford Focus Electric car Lithium Ion battery pack.

EXAMPLE 3.2 Battery-Powered Cars

FIGURE 3.10 The Tesla electric-powered sports car.

Battery-powered cars produce no emissions and thus are an attractive alternative to gasoline-powered cars. Some of these cars, such as the Tesla sports car shown in Figure 3.10, are powered by batteries constructed of lithium ion cells.

The battery pack of the Tesla electric sports car (Figure 3.9) has the capacity to hold 53 kWh of energy. The battery pack is usually charged to 80% of its capacity and discharged to 20% of its capacity. A gasoline-powered car typically carries 50 L of gasoline, and gasoline has an energy content of 34.8 MJ/L.

PROBLEM

How does the available energy in a lithium ion battery pack of an electric-powered car compare with the energy carried by a gasoline-powered car?

SOLUTION

Because not all of the energy can be extracted from a lithium ion battery without damaging it, the total usable energy is

$$E_{\text{electric}} = (80\% - 20\%)(53\ \text{kWh})\left(\frac{1000\ \text{W}}{1\ \text{kW}}\right)\left(\frac{3600\ \text{s}}{1\ \text{h}}\right) = 1.14\times 10^{8}\ \text{J} = 114\ \text{MJ}$$

A typical gasoline-powered car can carry 50 L of gasoline, which has an energy content of

$$E_{\text{gasoline}} = (50\ \text{L})(34.8\ \text{MJ/L}) = 1740\ \text{MJ}$$

Thus, a typical gasoline-powered car carries 15 times as much energy as the Tesla electric-powered car. However, the efficiency of a gasoline-powered car is approximately 20%, while an electric-powered car can be close to 90% efficient. Thus, the usable energy of the electric-powered car is

$$E_{\text{electric, usable}} = 0.9\times(114\ \text{MJ}) = 103\ \text{MJ}$$

and the usable energy of the gasoline-powered car is

$$E_{\text{gasoline, usable}} = 0.2\times(1740\ \text{MJ}) = 348\ \text{M}$$

The final numbers for the usable energy should be rounded to 1 significant digit in both cases. But the basic point is clear: You can see that electric-powered cars, even with lithium ion batteries, can carry less energy than gasoline-powered cars.

Van de Graaff Generator

One means of creating large electric potentials is a **Van de Graaff generator,** a device invented by the American physicist Robert J. Van de Graaff (1901–1967). Large Van de Graaff generators can produce electric potentials of millions of volts. More modest Van de Graaff generators, such as the one shown in Figure 3.11, can produce several hundred thousand volts and are often used in physics classrooms.

A Van de Graaff generator uses a *corona discharge* to apply a positive charge to a nonconducting moving belt. Putting a high positive voltage on a conductor with a sharp point creates the corona discharge. The electric field on the sharp point is much stronger than on the flat surface of the conductor (see Chapter 2). The air around the sharp point is ionized. The ionized air molecules have a net positive charge, which causes the ions to be repelled away from the sharp point and deposited on the rubber belt. The moving belt, driven by an electric motor, carries the charge up into a hollow metal sphere, where the charge is taken from the belt by a pointed contact connected to the metal sphere. The charge that builds up on the metal sphere distributes itself uniformly around the outside of the sphere. On the Van de Graaff generator shown in Figure 3.11, a voltage limiter is used to keep the generator from producing sparks larger than desired.

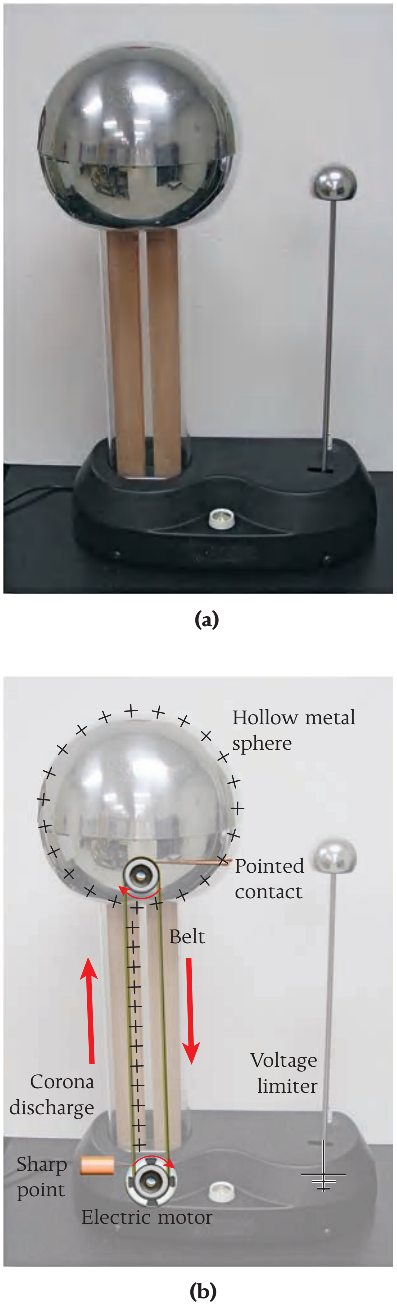


FIGURE 3.11 (a) A Van de Graaff generator used in physics classrooms. (b) The Van de Graaff generator can produce very high electric potentials by carrying charge from a corona discharge on a rubber belt up to a hollow metal sphere, where the charge is extracted from the belt by a sharp piece of metal attached to the inner surface of the sphere.

EXAMPLE 3.3 Tandem Van de Graaff Accelerator

A Van de Graaff accelerator is a particle accelerator that uses high electric potentials for studying nuclear physics processes of astrophysical relevance. A tandem Van de Graaff accelerator with a terminal potential difference of 10.0 MV (10.0 million volts), is diagrammed in Figure 3.12. This terminal potential difference is created in the center of the accelerator by a larger, more sophisticated version of the classroom Van de Graaff generator. Negative ions are created in the ion source by attaching an electron to the atoms to be accelerated. The negative ions then accelerate toward the positively charged terminal. Inside the terminal, the ions pass through a thin foil that strips off electrons, producing positively charged ions that then are accelerated away from the terminal and out of the tandem accelerator.

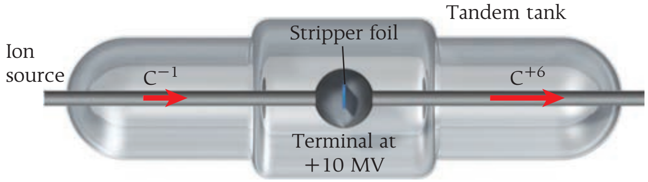


FIGURE 3.12 A tandem Van de Graaff accelerator.

PROBLEM 1

What is the highest kinetic energy that carbon nuclei can attain in this tandem accelerator?

SOLUTION 1

A tandem Van de Graaff accelerator has two stages of acceleration. In the first stage, each carbon ion has a net charge of $q_1 = -e$. After the stripper foil, the maximum charge any carbon ion can have is $q_2 = +6e$. The potential difference over which the ions are accelerated is $\Delta V = 10$ MV. The kinetic energy gained by each carbon ion is

$$\Delta K = |\Delta U| = |q_1 \Delta V| + |q_2 \Delta V| = K$$

or

$$K = e\Delta V + 6e\Delta V = 7e\Delta V$$

assuming that the initial speed of the ions is close to zero.

Putting in the numerical values, we get

$$K = 7\left(1.602\times10^{-19}\text{ C}\right)\left(10\times10^{6}\text{ V}\right) = 1.12\times10^{-11}\text{ J}$$

Nuclear physicists often use electron-volts instead of joules to express the kinetic energy of accelerated nuclei:

$$K = 7e\Delta V = 7e\left(10\times10^{6}\text{ V}\right) = 7\times10^{7}\text{ eV} = 70\text{ MeV}$$

PROBLEM 2

What is the highest speed that carbon nuclei can attain in this tandem accelerator?

SOLUTION 2

To determine the speed, we use the relationship between kinetic energy and speed:

$$K = \tfrac{1}{2}mv^2$$

where $m = 1.99 \times 10^{-26}$ kg is the mass of the carbon nucleus. Solving this equation for the speed, we get

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2\left(1.12\times10^{-11}\text{ J}\right)}{1.99\times10^{-26}\text{ kg}}} = 3.36\times10^{7}\text{ m/s}$$

which is 11% of the speed of light.

Concept Check 3.2

A cathode ray tube uses a potential difference of 5.0 kV to accelerate electrons and produce an electron beam that makes images on a phosphor screen. What is the speed of these electrons as a percentage of the speed of light?

a) 0.025%
b) 0.22%
c) 1.3%
d) 4.5%
e) 14%

SOLVED PROBLEM 3.1 Beam of Oxygen Ions

PROBLEM

Fully stripped (all electrons removed) oxygen (^{16}O) ions are accelerated from rest in a particle accelerator using a total potential difference of 10.0 MV $= 1.00 \times 10^7$ V. The ^{16}O nucleus has 8 protons and 8 neutrons. The accelerator produces a beam of 3.13×10^{12} ions per second.

This ion beam is completely stopped in a beam dump. What is the total power the beam dump has to absorb?

SOLUTION

THINK Power is energy per unit time. We can calculate the energy of each ion and then the total energy in the beam per unit time to obtain the power dissipated in the beam dump.

SKETCH Figure 3.13 illustrates a beam of fully stripped oxygen ions being stopped in a beam dump.

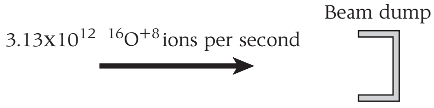


FIGURE 3.13 A beam of fully stripped oxygen ions stops in a beam dump.

RESEARCH The electric potential energy gained by each ion during the acceleration process is

$$U_{\text{ion}} = q\Delta V = ZeV$$

where $Z = 8$ is the atomic number of oxygen, $e = 1.602 \times 10^{-19}$ C is the charge of a proton, and $V = 1.00 \times 10^7$ V is the electric potential across which the ions are accelerated.

SIMPLIFY The power of the beam, which is dissipated in the beam dump, is then

$$P = NU_{\text{ion}} = NZeV$$

where $N = 3.13 \times 10^{12}$ ions/s is the number of ions per second stopped in the beam dump.

CALCULATE Putting in the numerical values, we get

$$P = NZeV = \left(3.13\times10^{12}\ \text{s}^{-1}\right)(8)\left(1.602\times10^{-19}\ \text{C}\right)\left(1.00\times10^{7}\ \text{V}\right)$$
$$= 40.1141\ \text{W}$$

ROUND We report our result to three significant figures:

$$P = 40.1\ \text{W}$$

DOUBLE-CHECK We can relate the change in kinetic energy for each ion to the change in electric potential energy of each ion:

$$\Delta K = \Delta U = \tfrac{1}{2}mv^2 = U_{\text{ion}} = ZeV$$

The mass of an oxygen nucleus is 2.66×10^{-26} kg. The velocity of each ion is then

$$v = \sqrt{\frac{2ZeV}{m}} = \sqrt{\frac{2(8)\left(1.602\times10^{-19}\ \text{C}\right)\left(1.00\times10^{7}\,\text{V}\right)}{2.66\times10^{-26}\ \text{kg}}} = 3.10\times10^{7}\ \text{m/s}$$

which is about 10% of the speed of light, which seems reasonable for the velocity of the ions. Thus, our result seems reasonable.

3.3 Equipotential Surfaces and Lines

Imagine you had to map out a ski resort with three peaks, like the one shown in Figure 3.14a. In Figure 3.14b, lines of equal elevation have been superimposed on the peaks. You could walk along each of these lines, without ever going uphill or downhill, and would be guaranteed to reach the point from which you started. These lines are lines of constant gravitational potential energy, because the gravitational potential energy is a function of the elevation only, and the elevation remains constant on each of the lines. Figure 3.14c shows a top view of the contour lines of equal elevation, which mark the equipotential lines for the gravitational potential energy. If you have understood this figure, the following discussion of electric potential lines and surfaces should be easy to follow.

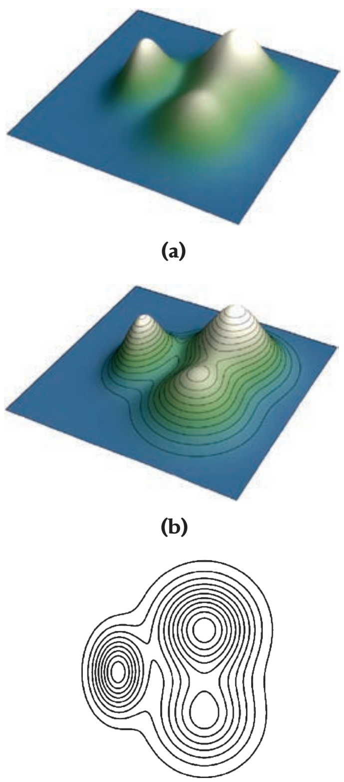


FIGURE 3.14 (a) Ski resort with three peaks; (b) the same peaks with lines of equal elevation superimposed; (c) the contour lines of equal elevation in a two-dimensional plot.

When an electric field is present, the electric potential has a value everywhere in space. Points that have the same electric potential form an **equipotential surface.** Charged particles can move along an equipotential surface without having any work done on them by the electric field. According to principles of electrostatics, the surface of a conductor must be an equipotential surface; otherwise, the free electrons on the conductor surface would accelerate. The discussion in Chapter 2 established that the electric field is

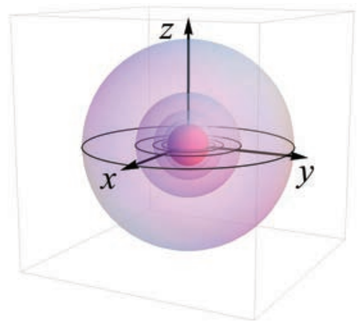


FIGURE 3.15 Concentric equipotential surfaces with values of 5 V, 4 V, 3 V, 2 V, and 1 V around a spherical conductor with a potential of 5 V centered at the origin of an *xyz*-coordinate system. The circles represent the intersections of the equipotential spheres with the *xy*-plane and are equipotential lines.

zero everywhere inside the body of a conductor. This means that the entire volume of the conductor must be at the same potential; that is, the entire conductor is an equipotential.

Equipotential surfaces exist in three dimensions (Figure 3.15); however, symmetries in the electric potential allow us to represent equipotential surfaces in two dimensions, as **equipotential lines** in the plane in which the charges reside. Before determining the shape and location of these equipotential surfaces, let's first look at some qualitative features of some of the simplest cases (for which the electric fields were determined in Chapter 2).

In drawing equipotential lines, we note that charges can move perpendicular to any electric field line without having any work done on them by the electric field, because according to equation 3.4, the scalar product of the electric field and the displacement is then zero. If the work done by the electric field is zero, the potential remains the same, by equation 3.8. Thus, *equipotential lines and planes are always perpendicular to the direction of the electric field.* (In Figure 3.14b, the elevation map of the ski resort, the equivalent of electric field lines would be the lines of steepest descent, which are, of course, always perpendicular to the lines of equal elevation.)

Before examining the particular equipotential surfaces resulting from different electric field configurations, let's note the two most important general observations of this section, which hold for all of the following cases:

1. The surface of any conductor forms an equipotential surface.
2. Equipotential surfaces are always perpendicular to the electric field lines at any point in space.

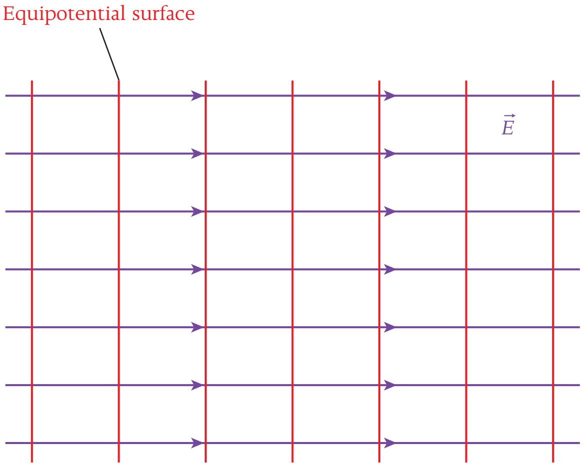


FIGURE 3.16 Equipotential surfaces (red lines) from a constant electric field. The purple lines with the arrowheads represent the electric field.

Constant Electric Field

A constant electric field has straight, equally spaced, and parallel field lines. Thus, such a field produces equipotential surfaces in the form of parallel planes, because of the condition that the equipotential surfaces or equipotential lines have to be perpendicular to the field lines. These planes are represented in two dimensions as equally spaced equipotential lines (Figure 3.16).

Single Point Charge

Figure 3.17 shows the electric field and corresponding equipotential lines due to a single point charge. The electric field lines extend radially from a positive point charge, as shown in Figure 3.17a. In this case, the field lines point away from the positive charge and terminate at infinity. For a negative charge, as shown in Figure 3.17b, the field lines originate at infinity and terminate at the negative charge. The equipotential lines are spheres centered on the point charge. (In the two-dimensional views shown in the figure, the circles represent the lines where the

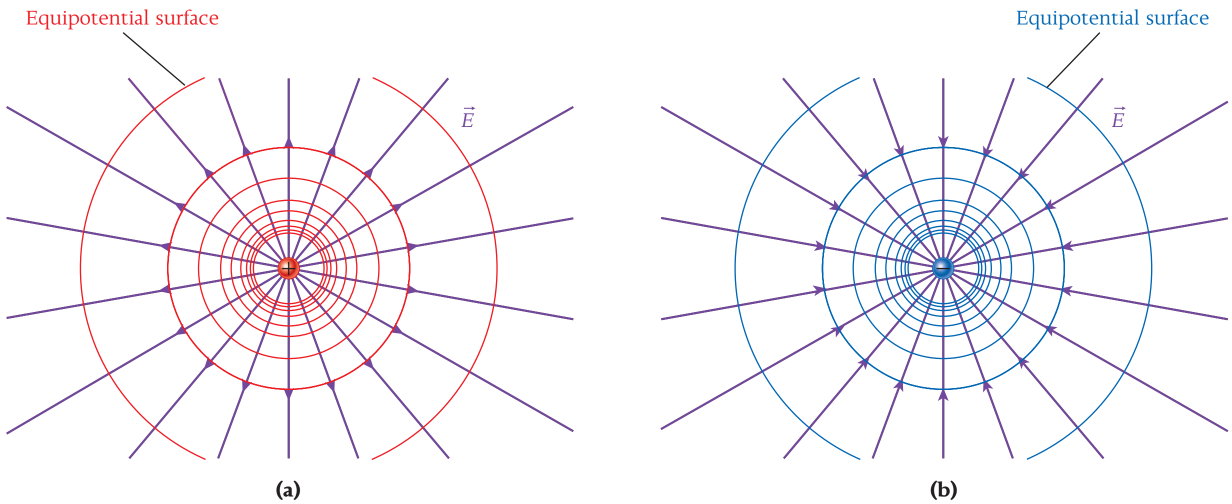


FIGURE 3.17 Equipotential surfaces and electric field lines from (a) a single positive point charge and (b) a single negative point charge.

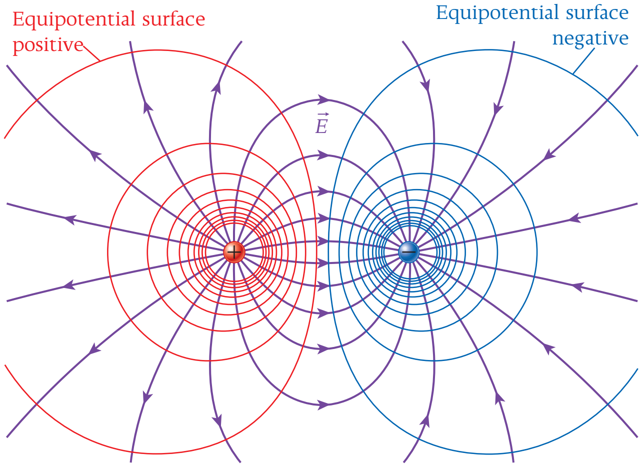


FIGURE 3.18 Equipotential surfaces created by point charges of the same magnitude but opposite sign. The red lines represent positive potential, and the blue lines represent negative potential. The purple lines with the arrowheads represent the electric field.

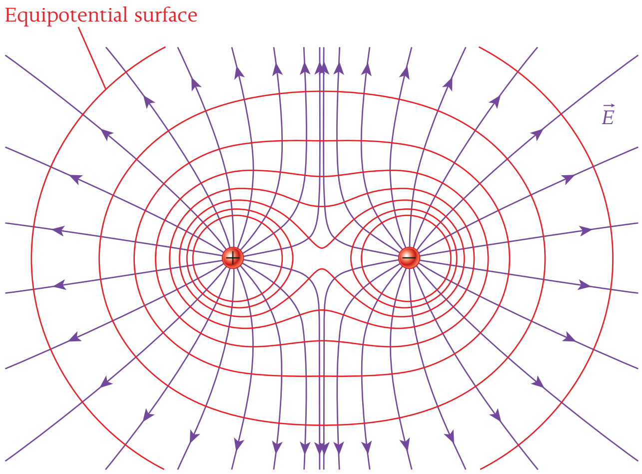


FIGURE 3.19 Equipotential surfaces (red lines) from two identical positive point charges. The purple lines with the arrowheads represent the electric field.

plane of the page cuts through equipotential spheres.) The values of the potential difference between neighboring equipotential lines are equal, producing equipotential lines that are close together near the charge and more widely spaced away from the charge. Note again that the equipotential lines are always perpendicular to the electric field lines. Equipotential surfaces do not have arrows like the field lines, because the potential is a scalar.

Two Oppositely Charged Point Charges

Figure 3.18 shows the electric field lines from two oppositely charged point charges, along with equipotential surfaces depicted as equipotential lines. An electrostatic force would attract these two point charges toward each other, but this discussion assumes that the charges are fixed in space and cannot move. The electric field lines originate at the positive charge and terminate on the negative charge. Again, the equipotential lines are always perpendicular to the electric field lines. The red lines in this figure represent positive equipotential surfaces, and the blue lines represent negative equipotential surfaces. Positive charges produce positive potential, and negative charges produce negative potential (relative to the value of the potential at infinity). Close to each charge, the resultant electric field lines and the resultant equipotential lines resemble those for a single point charge. Away from the vicinity of each charge, the electric field and the electric potential are the sums of the fields and potentials due to the two charges. The electric fields add as vectors, while the electric potentials add as scalars. Thus, the electric field is defined at all points in space in terms of a magnitude and a direction, while the electric potential is defined solely by its value at a given point in space and has no direction associated with it.

Self-Test Opportunity 3.1

Suppose the charges in Figure 3.18 were located at $(x,y) = (-10\text{ cm},0)$ and $(x,y) = (+10\text{ cm},0)$. What would the electric potential be along the y-axis ($x = 0$)?

Two Identical Point Charges

Figure 3.19 shows electric field lines and equipotential surfaces resulting from two identical positive point charges. These two charges experience a repulsive electrostatic force. Because both charges are positive, the equipotential surfaces represent positive potentials. Again, the electric field and electric potential result from the sums of the fields and potentials, respectively, due to the two charges.

Self-Test Opportunity 3.2

Suppose the charges in Figure 3.19 were located at $(x,y) = (-10\text{ cm},0)$ and $(x,y) = (+10\text{ cm},0)$. Would $(x,y) = (0,0)$ correspond to a maximum, a minimum, or a saddle point in the electric potential?

3.4 Electric Potential of Various Charge Distributions

The electric potential is defined as the work required to place a unit charge at a point, and work is a force acting over a distance. Also, the electric field can be defined as the force acting on a unit charge at a point. Therefore, it seems that the potential at a point should be related to the field strength at that point. In fact, electric potential and electric field are directly related; we can determine either one given an expression for the other.

To determine the electric potential from the electric field, we start with the definition of the work done on a particle with charge q by a force, $\vec{F}$ over a displacement, $d\vec{s}$

$$dW = \vec{F} \bullet d\vec{s}$$

In this case, the force is given by $\vec{F} = q\vec{E}$ so

$$dW = q\vec{E} \bullet d\vec{s} \tag{3.10}$$

Integration of equation 3.10 as the particle moves in the electric field from some initial point to some final point gives

$$W = W_e = \int_i^f q\vec{E} \bullet d\vec{s} = q\int_i^f \vec{E} \bullet d\vec{s}$$

Using equation 3.8 to relate the work done to the change in electric potential, we get

$$\Delta V = V_f - V_i = -\frac{W_e}{q} = -\int_i^f \vec{E} \bullet d\vec{s}$$

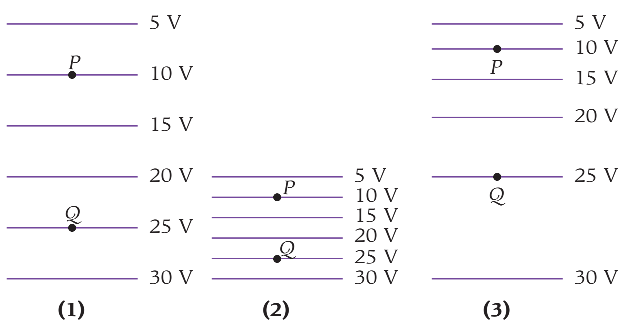
As mentioned earlier, the usual convention is to set the electric potential to zero at infinity. With this convention, we can express the potential at some poin $\vec{r}$ in space as

$$V(\vec{r}) - V(\infty) \equiv V(\vec{r}) = -\int_\infty^{\vec{r}} \vec{E} \bullet d\vec{s} \tag{3.11}$$

Concept Check 3.3

In the figure, the lines represent equipotential lines. A charged object is moved from point *P* to point *Q*. How does the amount of work done on the object compare for these three cases?

a) All three cases involve the same work.
b) The most work is done in case 1.
c) The most work is done in case 2.
d) The most work is done in case 3.
e) Cases 1 and 3 involve the same amount of work, which is more than is involved in case 2.



Point Charge

Let's use equation 3.11 to determine the electric potential due to a point charge, q. The electric field due to a point charge, q (for now, taken as positive), at a distance r from the charge is given by

$$E = \frac{kq}{r^2}$$

The direction of the electric field is radial from the point charge. Assume that the integration is carried out along a radial line from infinity to a point at a distance R from the point charge, such that $\vec{E} \bullet d\vec{s} = E dr$ Then we can use equation 3.11 to obtain

$$V(R) = -\int_\infty^R \vec{E} \bullet d\vec{s} = -\int_\infty^R \frac{kq}{r^2}\,dr = \left[\frac{kq}{r}\right]_\infty^R = \frac{kq}{R}$$

Thus, the electric potential due to a point charge at a distance r from the charge is given by

$$V = \frac{kq}{r} \tag{3.12}$$

Self-Test Opportunity 3.3

Obtaining equation 3.12 for the electric potential from a point charge involved integrating along a radial line from infinity to a point at a distance *R* from the point charge. How would the result change if the integration were carried out over a different path?

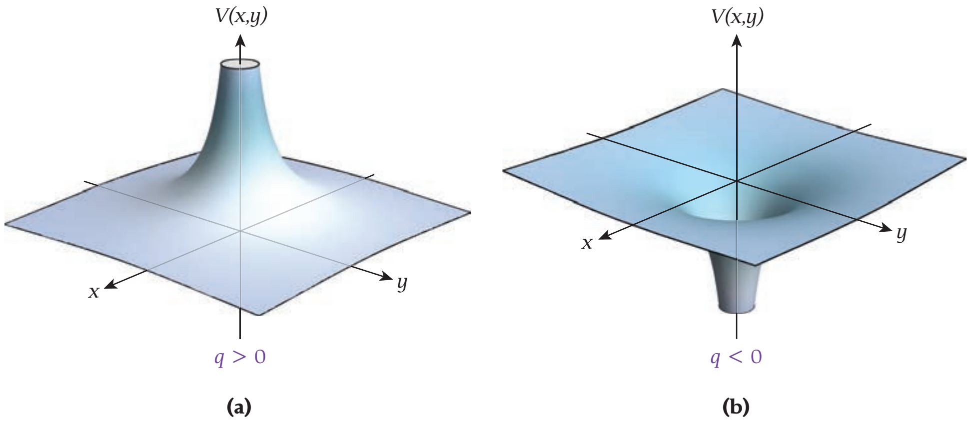


FIGURE 3.20 Electric potential due to (a) a positive point charge and (b) a negative point charge.

Concept Check 3.4

What is the electric potential 45.5 cm away from a point charge of 12.5 pC?

a) 0.247 V
b) 1.45 V
c) 4.22 V
d) 10.2 V
e) 25.7 V

Equation 3.12 also holds when $q < 0$. A positive charge produces a positive potential, and a negative charge produces a negative potential, as shown in Figure 3.20.

In Figure 3.20, the electric potential is calculated for all points in the xy-plane. The vertical axis represents the value of the potential at each point on the plane, $V(x,y)$, found using $r = \sqrt{x^2 + y^2}$ The potential is not calculated close to $r = 0$ because it becomes infinite there. You can see from Figure 3.20 how the circular equipotential lines shown in Figure 3.17 originate.

SOLVED PROBLEM 3.2 Fixed and Moving Positive Charges

PROBLEM

A positive charge of 4.50 μC is fixed in place. A particle of mass 6.00 g and charge +3.00 μC is fired with an initial speed of 66.0 m/s directly toward the fixed charge from a distance of 4.20 cm away. How close does the moving charge get to the fixed charge before it comes to rest and starts moving away from the fixed charge?

SOLUTION

THINK The moving charge will gain electric potential energy as it nears the fixed charge. The negative of the change in potential energy of the moving charge is equal to the change in kinetic energy of the moving charge because $\Delta K + \Delta U = 0$.

SKETCH We set the location of the fixed charge at $x = 0$, as shown in Figure 3.21. The moving charge starts at $x = d_i$, moves with initial speed $v = v_0$, and comes to rest at $x = d_f$.

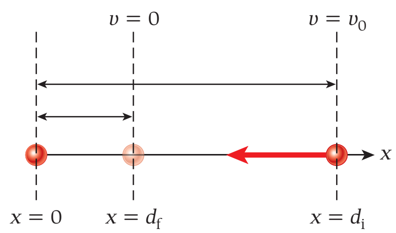


FIGURE 3.21 Two positive charges. One charge is fixed in place at $x = 0$, and the second charge begins moving with velocity $\vec{v}_0$ at $x = d_i$ and has zero velocity at $x = d_f$.

RESEARCH The moving charge gains electric potential energy as it approaches the fixed charge and loses kinetic energy until it stops. At that point, all the original kinetic energy of the moving charge has been converted to electric potential energy. Using energy conservation, we can write this relationship as

$$\Delta K + \Delta U = 0 \Rightarrow \Delta K = -\Delta U \Rightarrow$$

$$0 - \tfrac{1}{2}mv_0^2 = -q_{\text{moving}}\Delta V \Rightarrow$$

$$\tfrac{1}{2}mv_0^2 = q_{\text{moving}}\Delta V \qquad \text{(i)}$$

The electric potential experienced by the moving charge is due to the fixed charge, so we can write the change in potential as

$$\Delta V = V_f - V_i = k\frac{q_{\text{fixed}}}{d_f} - k\frac{q_{\text{fixed}}}{d_i} = kq_{\text{fixed}}\left(\frac{1}{d_f} - \frac{1}{d_i}\right) \qquad \text{(ii)}$$

– *Continued*

SIMPLIFY Substituting the expression for the potential difference from equation (ii) into equation (i), we find

$$\tfrac{1}{2}mv_0^2 = q_{\text{moving}}\Delta V = kq_{\text{moving}}q_{\text{fixed}}\left(\frac{1}{d_\text{f}} - \frac{1}{d_\text{i}}\right) \Rightarrow$$

$$\frac{1}{d_\text{f}} - \frac{1}{d_\text{i}} = \frac{mv_0^2}{2kq_{\text{moving}}q_{\text{fixed}}} \Rightarrow$$

$$\frac{1}{d_\text{f}} = \frac{1}{d_\text{i}} + \frac{mv_0^2}{2kq_{\text{moving}}q_{\text{fixed}}}$$

CALCULATE Putting in the numerical values, we get

$$\frac{1}{d_\text{f}} = \frac{1}{0.0420 \text{ m}} + \frac{(0.00600 \text{ kg})(66.0 \text{ m/s})^2}{2\left(8.99\times10^9 \text{ N m}^2/\text{C}^2\right)\left(3.00\times10^{-6}\text{C}\right)\left(4.50\times10^{-6}\text{C}\right)} = 131.485$$

or

$$d_\text{f} = 0.00760545 \text{ m}$$

ROUND We report our result to three significant figures:

$$d_\text{f} = 0.00761 \text{ m} = 0.761 \text{ cm}$$

DOUBLE-CHECK The final distance of 0.761 cm is less than the initial distance of 4.20 cm. At the final distance, the electric potential energy of the moving charge is

$$U = q_{\text{moving}}V = q_{\text{moving}}\left(k\frac{q_{\text{fixed}}}{d_\text{f}}\right) = k\frac{q_{\text{moving}}q_{\text{fixed}}}{d_\text{f}}$$

$$= \left(8.99\times10^9 \text{ N m}^2/\text{C}^2\right)\frac{\left(3.00\times10^{-6} \text{ C}\right)\left(4.50\times10^{-6} \text{ C}\right)}{0.00761 \text{ m}} = 16.0 \text{ J}$$

The electric potential energy at the initial distance is

$$U = q_{\text{moving}}V = q_{\text{moving}}\left(k\frac{q_{\text{fixed}}}{d_\text{i}}\right) = k\frac{q_{\text{moving}}q_{\text{fixed}}}{d_\text{i}}$$

$$= \left(8.99\times10^9 \text{ N m}^2/\text{C}^2\right)\frac{\left(3.00\times10^{-6} \text{ C}\right)\left(4.50\times10^{-6} \text{ C}\right)}{(0.0420 \text{ m})} = 2.9 \text{ J}$$

The initial kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \frac{(0.00600 \text{ kg})(66.0 \text{ m/s})^2}{2} = 13.1 \text{ J}$$

We can see that the equation based on energy conservation, from which the solution process started, is satisfied:

$$\tfrac{1}{2}mv^2 = \Delta U$$

$$13.1 \text{ J} = 16.0 \text{ J} - 2.9 \text{ J} = 13.1 \text{ J}$$

This gives us confidence that our result for the final distance is correct.

System of Point Charges

Assuming again that the electric potential is zero at an infinite distance from the origin, we calculate the electric potential due to a system of n point charges by adding the potentials due to all the charges:

$$V = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} \frac{kq_i}{r_i} \tag{3.13}$$

Equation 3.13 can be proved by inserting the expression for the total electric field from n charges $\left(\vec{E}_\text{t} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_n\right)$ into equation 3.11 and integrating term by term. The summation in equation 3.13 produces a potential at any point in space that has a value but no direction. Thus, calculating the potential due to a group of point charges is usually much simpler than calculating the electric field, which involves the addition of vectors.

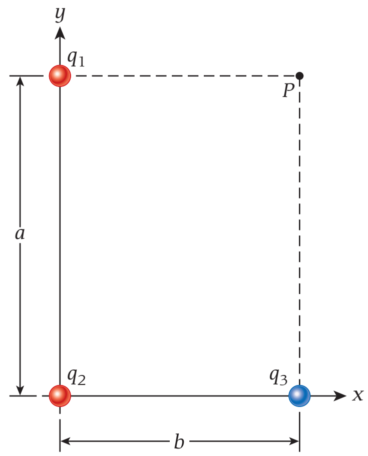


FIGURE 3.22 Electric potential at a point due to three point charges.

EXAMPLE 3.4 Superposition of Electric Potentials

Let's calculate the electric potential at a given point due to a system of point charges. Figure 3.22 shows three point charges: $q_1 = +1.50\ \mu\text{C}$, $q_2 = +2.50\ \mu\text{C}$, and $q_3 = -3.50\ \mu\text{C}$. Charge q_1 is located at $(0,a)$, q_2 is located at $(0,0)$, and q_3 is located at $(b,0)$, where $a = 8.00$ m and $b = 6.00$ m.

The electric potential at point P is the sum of the potentials due to the three charges:

$$V = \sum_{i=1}^{3} \frac{kq_i}{r_i} = k\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3}\right) = k\left(\frac{q_1}{b} + \frac{q_2}{\sqrt{a^2 + b^2}} + \frac{q_3}{a}\right)$$

$$= \left(8.99\times10^9\ \text{N m}^2/\text{C}^2\right)\left(\frac{1.50\times10^{-6}\ \text{C}}{6.00\ \text{m}} + \frac{2.50\times10^{-6}\ \text{C}}{\sqrt{(8.00\ \text{m})^2 + (6.00\ \text{m})^2}} + \frac{-3.50\times10^{-6}\ \text{C}}{8.00\ \text{m}}\right)$$

$$= 562\ \text{V}$$

Note that the potential due to q_3 is negative at point P, but the sum of the potentials is positive.

This example is similar to Example 2.1, in which we calculated the electric field at point P due to three charges. Note that this calculation of the electric potential due to three charges is much simpler than that calculation.

Concept Check 3.5

Two protons are located in space in the three ways shown in the figure. Rank the three cases from highest to lowest net electric potential, V, produced at point P.

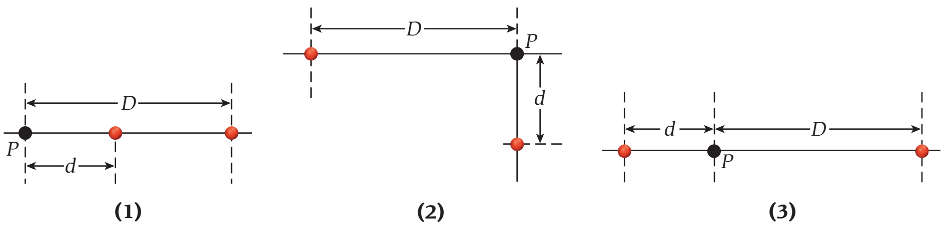
a) $2 > 3 > 1$

b) All three potentials are the same.

c) $3 > 2 > 1$

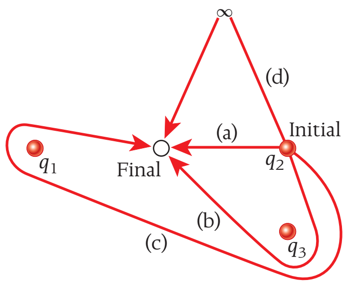
d) The potentials are equal for cases 1 and 3, with the potential for case 2 lower.

e) $1 > 2 > 3$



Concept Check 3.6

Three identical positive point charges are located at fixed points in space. Then charge q_2 is moved from its initial location to a final location as shown in the figure. Four different paths, marked (a) through (d), are shown. Path (a) follows the shortest line; path (b) takes q_2 around q_3; path (c) takes q_2 around q_3 and q_1; path (d) takes q_2 out to infinity and then to the final location. Which path requires the least work?



a) path (a)

b) path (b)

c) path (c)

d) path (d)

e) The work is the same for all the paths.

SOLVED PROBLEM 3.3 Minimum Potential

PROBLEM

A charge of $q_1 = 0.829$ nC is placed at $r_1 = 0$ on the x-axis. Another charge of $q_2 = 0.275$ nC is placed at $r_2 = 11.9$ cm on the x-axis. At which point along the x-axis between the two charges does the electric potential resulting from both of them have a minimum?

SOLUTION

THINK We can express the electric potential due to the two charges as the sum of the electric potentials from the individual charges. To obtain the minimum potential, we take the derivative of the potential and set it equal to zero. We can then solve for the distance where the derivative is zero.

SKETCH Figure 3.23 shows the locations of the two charges.

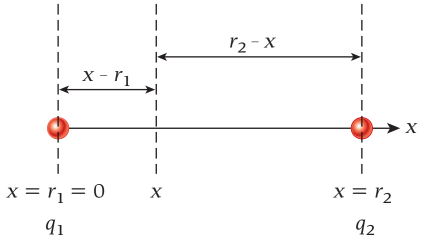


FIGURE 3.23 Two charges placed along the x-axis.

– *Continued*

RESEARCH We can express the electric potential produced along the x-axis by the two charges as

$$V = V_1 + V_2 = k\frac{q_1}{x - r_1} + k\frac{q_2}{r_2 - x} = k\frac{q_1}{x} + k\frac{q_2}{r_2 - x}$$

Note that the quantities x and $r_2 - x$ are always positive for $0 < x < r_2$. To find the minimum, we take the derivative of the electric potential:

$$\frac{dV}{dx} = -k\frac{q_1}{x^2} - k\frac{q_2}{(r_2 - x)^2}(-1) = k\frac{q_2}{(r_2 - x)^2} - k\frac{q_1}{x^2}$$

SIMPLIFY Setting the derivative of the electric potential equal to zero and rearranging, we obtain

$$k\frac{q_2}{(r_2 - x)^2} = k\frac{q_1}{x^2}$$

Dividing out k and rearranging, we get

$$\frac{x^2}{(r_2 - x)^2} = \frac{q_1}{q_2}$$

Now we can take the square root and rearrange:

$$x = \pm(r_2 - x)\sqrt{\frac{q_1}{q_2}}$$

Because $x > 0$ and $(r_2 - x) > 0$, the sign must be positive. Solving for x, we get

$$x = \frac{r_2\sqrt{\dfrac{q_1}{q_2}}}{1 + \sqrt{\dfrac{q_1}{q_2}}} = \frac{r_2}{\sqrt{\dfrac{q_2}{q_1}} + 1}$$

CALCULATE Putting in the numerical values results in

$$x = \frac{0.119 \text{ m}}{1 + \sqrt{\dfrac{0.275 \text{ nC}}{0.829 \text{ nC}}}} = 0.0755097 \text{ m}$$

ROUND We report our result to three significant figures:

$$x = 0.0755 \text{ m} = 7.55 \text{ cm}$$

DOUBLE-CHECK We can double-check our result by plotting (for example, with a graphing calculator) the electric potential resulting from the two charges and graphically determining the minimum (Figure 3.24).

The minimum of the electric potential is located at $x = 7.55$ cm, which confirms our calculated result.

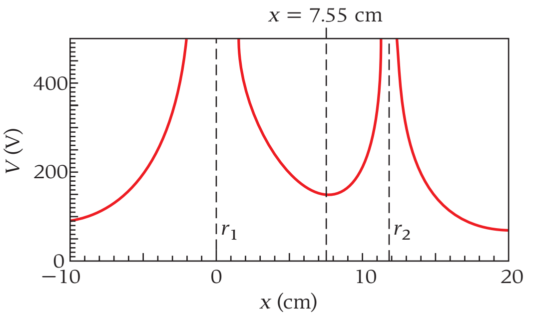


FIGURE 3.24 Graph of the electric potential resulting from two charges.

Continuous Charge Distribution

We can also determine the electric potential due to a continuous distribution of charge. To do this, we divide the charge into differential elements of charge, dq, and find the electric potential resulting from that differential charge as if it were a point charge. This is the way charge distributions were treated in determining electric fields in Chapter 2. The differential charge, dq, can be expressed in terms of a charge per unit length times a differential length, λdx; in terms of a charge per unit area times a differential area, σdA; or in terms of a charge per unit volume times a differential volume, ρdV. The electric potential resulting from the charge distribution is obtained by integrating over the contributions from the differential charges. Let's consider an example involving the electric potential due to a one-dimensional charge distribution.

EXAMPLE 3.5 Finite Line of Charge

What is the electric potential at a distance d along the perpendicular bisector of a thin wire with length $2a$ and linear charge distribution λ (Figure 3.25)?

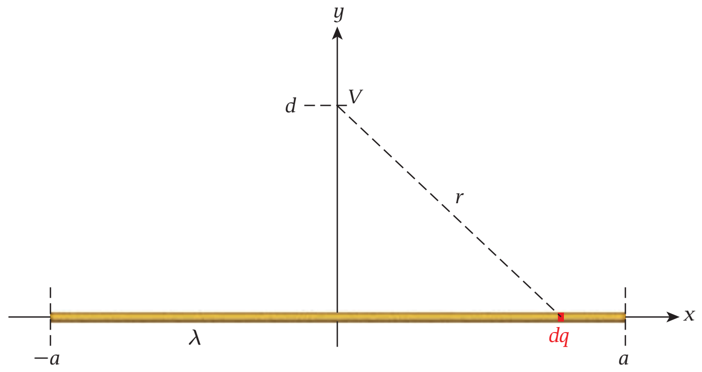


FIGURE 3.25 Calculating the electric potential due to a line of charge.

The differential electric potential, dV, at a distance d along the perpendicular bisector of the wire due to a differential charge, dq, is given by

$$dV = k\frac{dq}{r}$$

The electric potential due to the whole wire is found by integrating dV along the length of the wire:

$$V = \int_{-a}^{a} dV = \int_{-a}^{a} k\frac{dq}{r} \tag{i}$$

With $dq = \lambda dx$ and $r = \sqrt{x^2 + d^2}$ we can rewrite equation (i) as

$$V = \int_{-a}^{a} k\frac{\lambda dx}{\sqrt{x^2 + d^2}} = k\lambda \int_{-a}^{a} \frac{dx}{\sqrt{x^2 + d^2}}$$

Finding this integral in a table or evaluating it with software gives

$$\int_{-a}^{a} \frac{dx}{\sqrt{x^2 + d^2}} = \left[\ln\left(x + \sqrt{x^2 + d^2}\right)\right]_{-a}^{a} = \ln\left(\frac{\sqrt{a^2 + d^2} + a}{\sqrt{a^2 + d^2} - a}\right)$$

Thus, the electric potential at a distance d along the perpendicular bisector of a finite line of charge is given by

$$V = k\lambda \ln\left(\frac{\sqrt{a^2 + d^2} + a}{\sqrt{a^2 + d^2} - a}\right)$$

SOLVED PROBLEM 3.4 Charged Disk

PROBLEM

A charge of 3.50 nC is uniformly applied to a disk of radius 1.00 cm. What is the electric potential at a distance of 4.50 mm from the disk along its symmetry axis, assuming, as usual, that the potential is zero at an infinite distance?

SOLUTION

THINK A point charge creates the electric potential $V(r) = kq/r$, but since the charge in this case is distributed over an area, we cannot use this relationship but have to perform an

– Continued

Self-Test Opportunity 3.4

Sketch the graph of the electric potential due to a hollow conducting charged sphere as a function of the radial coordinate r from zero to three times the sphere's radius, R.

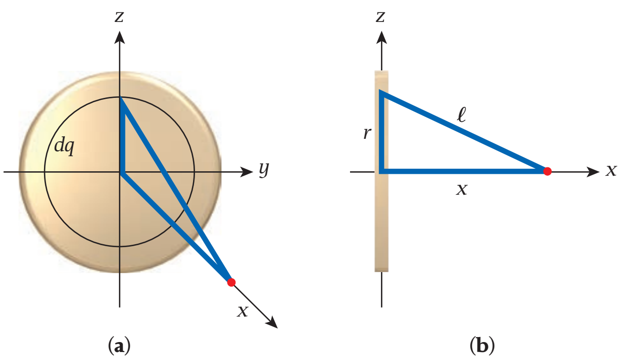


FIGURE 3.26 Electric potential on the symmetry axis of a disk: (a) front view, (b) side view.

integration. The general procedure for this integration is always the same: We divide the total charge into small increments, dq, compute the electric potential for each, and then integrate over all the charge increments. In this case, our task is to find the potential on a point along the symmetry axis of the disk, and so we should make use of this symmetry in our integration procedure.

SKETCH A sketch of the problem situation is given in Figure 3.26.

RESEARCH The surface charge density on the disk is $\sigma = q/A$, where A is the area of the disk, $A = \pi R^2$. Further, the charge is distributed symmetrically around the symmetry axis, the x-axis in Figure 3.26. This motivates the use of a thin ring of width dr for our differential charge unit: $dq = \sigma dA$, with $dA = 2\pi r dr$. You can see in Figure 3.26 that every point on the ring is an equal distance ℓ from the point (marked by the red dot) at which we want to evaluate the potential. The contribution of dq to the electric potential is then $dV = kdq/\ell$, and the total potential is $V = \int dV$. The last thing we need to do is to relate the distance ℓ to the distance x between the point and the center of the disk. From part (b) of Figure 3.26, you can see that this relationship is given by $\ell = \sqrt{r^2 + x^2}$

SIMPLIFY Putting the pieces together, we find that the potential on the symmetry axis of the disk as a function of the distance to the center is given by

$$V(x) = \int dV = \int \frac{k}{\ell} dq = \int \frac{k\sigma}{\ell} dA = \int \frac{k\sigma}{\ell} 2\pi r \, dr$$

After inserting the expressions we found for the charge density, σ, and the distance, ℓ, we are ready to integrate over r from zero to the radius of the disk, R:

$$V(x) = \frac{2kq}{R^2} \int_0^R \frac{r}{\sqrt{x^2 + r^2}} dr = \frac{2kq}{R^2} \sqrt{x^2 + r^2} \Big|_0^R = \frac{2kq}{R^2}\left(\sqrt{x^2 + R^2} - x\right)$$

CALCULATE Inserting the given numerical values, we find

$$V(4.5 \text{ mm}) = \frac{2(8.98755 \times 10^9 \text{ N m}^2/\text{C}^2)(3.5 \times 10^{-9} \text{ C})}{(0.01 \text{ m})^2}\left(\sqrt{(4.5 \times 10^{-3} \text{ m})^2 + (0.01 \text{ m})^2} - 4.5 \times 10^{-3} \text{ m}\right)$$
$$= 4067.85 \text{ N m/C}$$

ROUND We round our final result to three significant figures: $V(4.5 \text{ mm}) = 4.07$ kV. (We have used 1 N m = 1 J and 1 J/1 C = 1 V to get the proper unit, the volt, for the potential.)

DOUBLE-CHECK We have already done the simple check of noting that the units of our answer are correct. As another check, we can look at the limiting case where the radius of the disk shrinks to zero, that is, where it becomes a point charge. The potential at a distance of 4.50 mm from a 3.50-nC point charge is

$$V_{\text{point}}(4.5 \text{ mm}) = \frac{(8.988 \times 10^9 \text{ N m}^2/\text{C}^2)(3.5 \times 10^{-9} \text{ C})}{0.0045 \text{ m}} = 6.99 \text{ kV}$$

This result is comforting because it is of the same order of magnitude as our calculated answer, just slightly larger, as we would expect. Figure 26.27 compares the graph of the electric potential due to the charged disk (blue curve) with the graph of the potential due to a point charge (red curve). As expected, the distribution of the charge does not matter at large distances, and our calculated potential approaches that for a point charge. For distances smaller than the radius of the disk, however, the difference becomes very pronounced. In particular, note that for $x \rightarrow 0$ the potential due to a uniformly charged disk does not diverge but has a maximum of $2kq/R$.

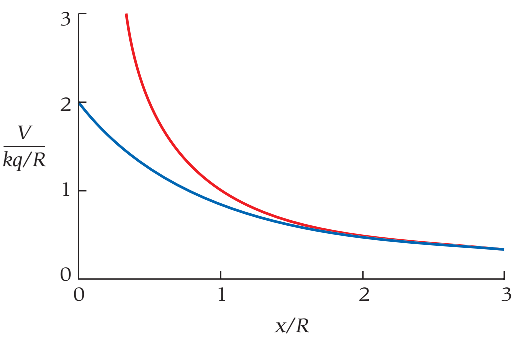


FIGURE 3.27 Comparison of the electric potential due to the uniformly charged disk with radius R (blue curve) and that due to a point charge (red curve).

3.5 Finding the Electric Field from the Electric Potential

As we mentioned earlier, we can determine the electric field starting with the electric potential. This calculation uses equations 3.8 and 3.10:

$$-q dV = q\vec{E} \bullet d\vec{s}$$

where $d\vec{s}$ is a vector from an initial point to a final point located a small (infinitesimal) distance away. The component of the electric field, E_s, along the direction of $d\vec{s}$ is given by the partial derivative

$$E_s = -\frac{\partial V}{\partial s} \qquad (3.14)$$

Thus, we can find any component of the electric field by taking the partial derivative of the potential along the direction of that component. We can then write the components of the electric field in terms of partial derivatives of the potential:

$$E_x = -\frac{\partial V}{\partial x}; \quad E_y = -\frac{\partial V}{\partial y}; \quad E_z = -\frac{\partial V}{\partial z} \qquad (3.15)$$

The equivalent vector calculus formulation is $\vec{E} = -\vec{\nabla}V \equiv -(\partial V/\partial x, \partial V/\partial y, \partial V/\partial z)$, where the operator $\vec{\nabla}$ is called the **gradient.** Thus, the electric field can be determined either graphically, by measuring the negative of the change of the potential per unit distance perpendicular to an equipotential line, or analytically, by using equation 3.15.

To visually reinforce the concepts of electric fields and potentials, the following example shows how a graphical technique can be used to find the field given the potential.

Concept Check 3.7

Suppose an electric potential is described by $V(x, y, z) = -(5x^2 + y + z)$ in volts. Which of the following expressions describes the associated electric field, in units of volts per meter?

a) $\vec{E} = 5\hat{x} + 2\hat{y} + 2\hat{z}$

b) $\vec{E} = 10x\hat{x}$

c) $\vec{E} = 5x\hat{x} + 2\hat{y}$

d) $\vec{E} = 10x\hat{x} + \hat{y} + \hat{z}$

e) $\vec{E} = 0$

Concept Check 3.8

In the figure, the lines represent equipotential lines. How does the magnitude of the electric field, E, at point P compare for the three cases?

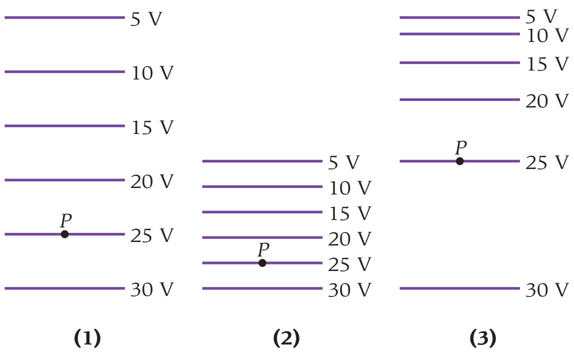
a) $E_1 = E_2 = E_3$

b) $E_1 > E_2 > E_3$

c) $E_1 < E_2 < E_3$

d) $E_3 > E_1 > E_2$

e) $E_3 < E_1 < E_2$



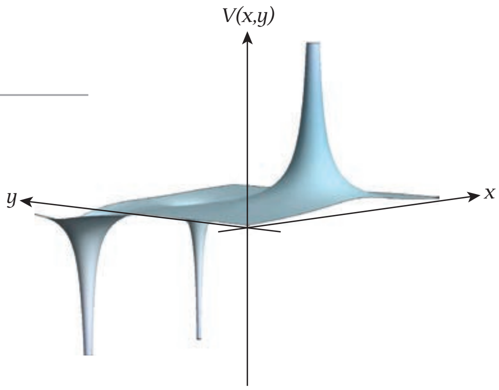


FIGURE 3.28 Electric potential due to three charges.

EXAMPLE 3.6 Graphical Extraction of the Electric Field

Let's consider a system of three point charges with values $q_1 = -6.00\ \mu C$, $q_2 = -3.00\ \mu C$, and $q_3 = +9.00\ \mu C$, located at positions $(x_1,y_1) = (1.5\text{ cm}, 9.0\text{ cm})$, $(x_2,y_2) = (6.0\text{ cm}, 8.0\text{ cm})$, and $(x_3,y_3) = (5.3\text{ cm}, 2.0\text{ cm})$. Figure 3.28 shows the electric potential, $V(x,y)$, resulting from these three charges, with equipotential lines calculated at potential values from -5000 V to 5000 V in 1000-V increments shown in Figure 3.29.

We can calculate the magnitude of the electric field at point P using equation 3.14 and graphical techniques. To perform this task, we use the green line

– Continued

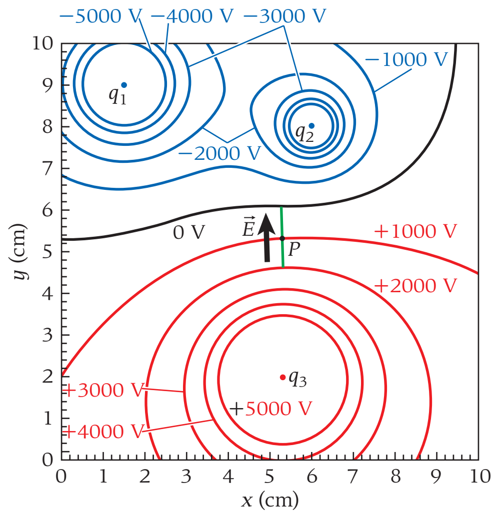


FIGURE 3.29 Equipotential lines for the electric potential due to three point charges.

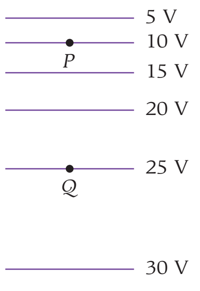
in Figure 3.29, which is drawn through point P perpendicular to the equipotential line because the electric field is always perpendicular to the equipotential lines, reaching from the equipotential line of 0 V to the line of 2000 V. As you can see from Figure 3.29, the length of the green line is 1.5 cm. Therefore, the magnitude of the electric field can be approximated as

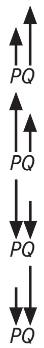
$$|E_s| = \left|-\frac{\Delta V}{\Delta s}\right| = \left|\frac{(+2000\ \text{V}) - (0\ \text{V})}{1.5\ \text{cm}}\right| = 1.3\times 10^5\ \text{V/m}$$

where Δs is the length of the line through point P. The negative sign in equation 3.14 indicates that the direction of the electric field between neighboring equipotential lines points from the 2000 V equipotential line to the zero potential line.

Concept Check 3.9

In the figure, the lines represent equipotential lines. A positive charge is placed at point P, and then another positive charge is placed at point Q. Which set of vectors best represents the relative magnitudes and directions of the electric field forces exerted on the positive charges at P and Q?





a) *PQ*

b) *PQ*

c) *PQ*

d) *PQ*

00

e) *PQ*

In Chapter 2, we derived an expression for the electric field along the perpendicular bisector of a finite line of charge:

$$E_y = \frac{2k\lambda}{y}\frac{a}{\sqrt{y^2 + a^2}}$$

In Example 3.5, we found an expression for the electric potential along the perpendicular bisector of a finite line of charge; here we replace the coordinate d used in that example with the distance in the y-direction:

$$V = k\lambda \ln\left(\frac{\sqrt{y^2 + a^2} + a}{\sqrt{y^2 + a^2} - a}\right) \tag{3.16}$$

We can find the y-component of the electric field from the potential using equation 3.15:

$$\begin{aligned}
E_y &= -\frac{\partial V}{\partial y} \\
&= -\frac{\partial\left(k\lambda \ln\left(\dfrac{\sqrt{y^2+a^2}+a}{\sqrt{y^2+a^2}-a}\right)\right)}{\partial y} \\
&= -k\lambda\left(\frac{\partial\left(\ln\left(\sqrt{y^2+a^2}+a\right)\right)}{\partial y} - \frac{\partial\left(\ln\left(\sqrt{y^2+a^2}-a\right)\right)}{\partial y}\right)
\end{aligned}$$

Taking the partial derivative (remember that we can treat the partial derivative like a regular derivative), we obtain for the first term

$$\frac{\partial}{\partial y}\left(\ln\left(\sqrt{y^2+a^2}+a\right)\right)=\underbrace{\left(\frac{1}{\sqrt{y^2+a^2}+a}\right)}_{\text{derivative of ln}}\underbrace{\left(\frac{1}{2}\frac{1}{\sqrt{y^2+a^2}}\right)}_{\text{derivative of }\sqrt{y^2+a^2}}\underbrace{(2y)}_{\text{derivative of }y^2}=\frac{y}{y^2+a^2+a\sqrt{y^2+a^2}}$$

where the fact that the derivative of the natural log function is $d(\ln x)/dx = 1/x$ and the chain rule of differentiation have been used. (The outer and inner derivatives are indicated under the terms that they generate.) A similar expression can be found for the second term. Using the values of the derivatives, we find the component of the electric field:

$$E_y=-k\lambda\left(\frac{y}{y^2+a^2+a\sqrt{y^2+a^2}}-\frac{y}{y^2+a^2-a\sqrt{y^2+a^2}}\right)=\frac{2k\lambda}{y}\frac{a}{\sqrt{y^2+a^2}}$$

This result is the same as that for the electric field in the y-direction derived in Chapter 2 by integrating over a finite line of charge.

3.6 Electric Potential Energy of a System of Point Charges

Section 3.1 discussed the electric potential energy of a point charge in a given external electric field, and Section 3.4 described how to calculate the electric potential due to a system of point charges. This section combines these two pieces of information to find the electric potential energy of a system of point charges. Consider a system of charges that are infinitely far apart. To bring these charges into proximity with each other, work must be done on the charges, which changes the electric potential energy of the system. The electric potential energy of a system of point charges is defined as the work required to bring the charges together from being infinitely far apart.

As an example, let's find the electric potential energy of a system of two point charges (Figure 3.30). Assume that the two charges start at an infinite separation. We then bring point charge q_1 into the system. Because the system without charges has no electric field and no corresponding electric force, this action does not require that any work be done on the charge. Keeping this charge stationary, we bring the second point charge, q_2, from infinity to a distance r from q_1. Using equation 3.6, we can write the electric potential energy of the system as

$$U = q_2 V \tag{3.17}$$

where

$$V = \frac{kq_1}{r} \tag{3.18}$$

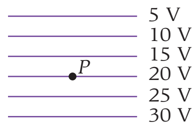
Thus, the electric potential energy of this system of two point charges is

$$U = \frac{kq_1q_2}{r} \tag{3.19}$$

From the work-energy theorem, the work, W, that must be done on the particles to bring them together and keep them stationary is equal to U. If the two charges have the same sign, $W = U > 0$, positive work must be done to bring them together from infinity and keep them motionless. If the two charges have opposite signs, negative work must be done to bring them together from infinity and hold them motionless. To determine U for more than two point charges, we assemble them from infinity one charge at a time, in any order.

Concept Check 3.10

In the figure, the lines represent equipotential lines. What is the direction of the electric field at point P?



a) up

b) down

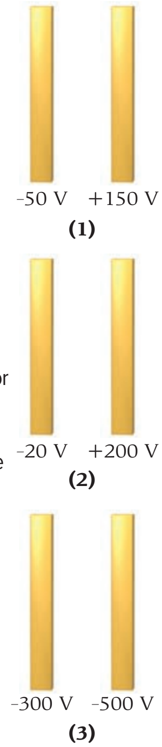
c) left

d) right

e) The electric field at P is zero.

Concept Check 3.11

Three pairs of parallel plates have the same plate separation and potentials on each plate as indicated in the drawing. The electric field, E, is uniform between each pair of plates and perpendicular to them. Rank the magnitude of E between the plates, from highest to lowest.



a) $1 > 2 > 3$

b) $3 > 2 > 1$

c) The magnitudes for 3 and 2 are equal and greater than the magnitude for 1.

d) The three magnitudes are equal.

e) The magnitude for 2 is greater than that for 1 and 3, which are the same.

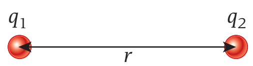


FIGURE 3.30 Two point charges separated by a distance r.

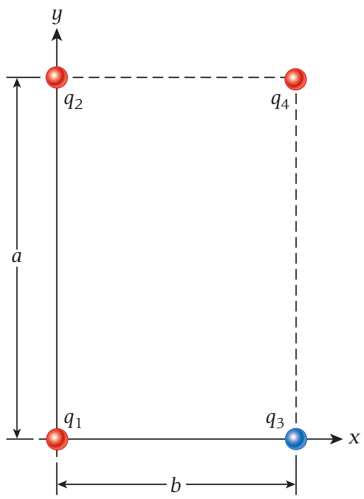


FIGURE 3.31 Calculating the potential energy of a system of four point charges.

EXAMPLE 3.7 Four Point Charges

Let's calculate $\mu$the electric potential energy of a system of four point charges, shown in Figure 3.31. The four point charges have the values $q_1 = +1.0\ \mu\text{C}$, $q_2 = +2.0\ \mu\text{C}$, $q_3 = -3.0\ \mu\text{C}$, and $q_4 = +4.0\ \mu\text{C}$. The charges are placed with $a = 6.0$ m and $b = 4.0$ m.

PROBLEM

What is the electric potential energy of this system of four point charges?

SOLUTION

We begin the calculation with the four charges infinitely far apart and assume that the electric potential energy is zero in that configuration. We bring in q_1 and position that charge at (0,0). This action does not change the electric potential energy of the system. Now we bring in q_2 and place that charge at (0,a). The electric potential energy of the system is now

$$U = \frac{kq_1q_2}{a}$$

Bringing q_3 in from an infinite distance and placing it at (b,0) changes the potential energy of the system through the interaction of q_3 with q_1 and the interaction of q_3 with q_2. The new potential energy is

$$U = \frac{kq_1q_2}{a} + \frac{kq_1q_3}{b} + \frac{kq_2q_3}{\sqrt{a^2 + b^2}}$$

Finally, bringing in q_4 and placing it at (b,a) changes the potential energy of the system through interactions with q_1, q_2, and q_3, bringing the total electric potential energy of the system to

$$U = \frac{kq_1q_2}{a} + \frac{kq_1q_3}{b} + \frac{kq_2q_3}{\sqrt{a^2 + b^2}} + \frac{kq_1q_4}{\sqrt{a^2 + b^2}} + \frac{kq_2q_4}{b} + \frac{kq_3q_4}{a}$$

Note that the order in which the charges are brought from infinity will not change this result. (You can try a different order to verify this statement.) Putting in the numerical values, we obtain

$$U = \left(3.0\times10^{-3}\ \text{J}\right) + \left(-6.7\times10^{-3}\ \text{J}\right) + \left(-7.5\times10^{-3}\ \text{J}\right) +$$
$$\left(5.0\times10^{-3}\ \text{J}\right) + \left(1.8\times10^{-2}\ \text{J}\right) + \left(-1.8\times10^{-2}\ \text{J}\right) = -\,6.2\times10^{-3}\ \text{J}$$

From the calculation in Example 3.7, we extrapolate the result to obtain a formula for the electric potential energy of a collection of point charges:

$$U = k \sum_{ij(\text{pairings})} \frac{q_iq_j}{r_{ij}} \tag{3.20}$$

where i and j label each pair of charges, the summation is over each pair ij (for all $i \neq j$), and r_{ij} is the distance between the charges in each pair. An alternative way to write this double sum is

$$U = \tfrac{1}{2}k\sum_{j=1}^{n}\ \sum_{i=1,i\ j}^{n} \frac{q_iq_j}{\left|\vec{r}_i - \vec{r}_j\right|}$$

which is more explicit than the equivalent formulation of equation 3.20.

WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- The change in the electric potential energy, ΔU, of a point charge moving in an electric field is equal to the negative of the work done on the point charge by the electric field W_e: $\Delta U = U_\text{f} - U_\text{i} = -W_\text{e}$.
- The change in electric potential energy, ΔU, is equal to the charge, q, times the change in electric potential, ΔV: $\Delta U = q\Delta V$.
- Equipotential surfaces and equipotential lines represent locations in space that have the same electric potential. Equipotential surfaces are always perpendicular to the electric field lines.
- A surface of a conductor is an equipotential surface.

- The change in electric potential can be determined from the electric field by integrating over the field: $\Delta V = -\int_i^f \vec{E} \bullet d\vec{s}$ Setting the potential equal to zero at infinity gives $V = \int_i^\infty \vec{E} \bullet d\vec{s}$
- The electric potential due to a point charge, q, at a distance r from the charge is given by $V = \frac{kq}{r}$
- The electric potential due to a system of n point charges can be expressed as an algebraic sum of the individual potentials: $V = \sum_{i=1}^{n} V_i$
- The electric field can be determined from gradients of the electric potential in each component direction: $E_x = -\frac{\partial V}{\partial x}$, $E_y = -\frac{\partial V}{\partial y}$, $E_z = -\frac{\partial V}{\partial z}$
- The electric potential energy of a system of two point charges is given by $U = \frac{kq_1q_2}{r}$

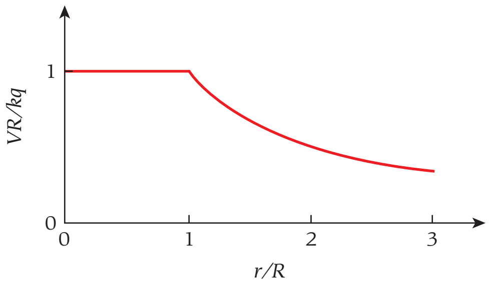
ANSWERS TO SELF-TEST OPPORTUNITIES

3.1 The electric potential along the y-axis is zero.

3.2 $(x,y) = (0,0)$ corresponds to a saddle point.

3.3 Nothing would change. The electrostatic force is conservative, and for a conservative force, the work is path-independent.

3.4



PROBLEM-SOLVING GUIDELINES

1. A common source of error in calculations is confusing the electric field, $\vec{E}$ the electric potential energy, U, and the electric potential, V. Remember that an electric field is a vector quantity produced by a charge distribution; electric potential energy is a property of the charge distribution; and electric potential is a property of the field. Be sure you know what it is that you're calculating.

2. Be sure to identify the point with respect to which you are calculating the potential energy or potential. Like calculations involving electric fields, calculations involving potentials can use a linear charge distribution (λ), a planar charge distribution (σ), or a volume charge distribution (ρ).

3. Since potential is a scalar, the total potential due to a system of point charges is calculated by simply adding the individual potentials due to all the charges. For a continuous charge distribution, you need to calculate the potential by integrating over the differential charge. Assume that the potential produced by the differential charge is the same as the potential from a point charge!

MULTIPLE-CHOICE QUESTIONS

3.1 A positive charge is released and moves along an electric field line. This charge moves to a position of

a) lower potential and lower potential energy.
b) lower potential and higher potential energy.
c) higher potential and lower potential energy.
d) higher potential and higher potential energy.

3.2 A proton is placed midway between points A and B. The potential at point A is -20 V, and the potential at point B $+20$ V. The potential at the midpoint is 0 V. The proton will

a) remain at rest.
b) move toward point B with constant velocity.
c) accelerate toward point A.
d) accelerate toward point B.
e) move toward point A with constant velocity.

3.3 What would be the consequence of setting the potential at +100 V at infinity, rather than taking it to be zero there?

a) Nothing; the field and the potential would have the same values at every finite point.

b) The electric potential would become infinite at every finite point, and the electric field could not be defined.

c) The electric potential everywhere would be 100 V higher, and the electric field would be the same.

d) It would depend on the situation. For example, the potential due to a positive point charge would drop off more slowly with distance, so the magnitude of the electric field would be less.

3.4 In which situation is the electric potential the highest?

a) at a point 1 m from a point charge of 1 C

b) at a point 1 m from the center of a uniformly charged spherical shell with a radius of 0.5 m and a total charge of 1 C

c) at a point 1 m from the center of a uniformly charged rod with a length of 1 m and a total charge of 1 C

d) at a point 2 m from a point charge of 2 C

e) at a point 0.5 m from a point charge of 0.5 C

3.5 The amount of work done to move a positive point charge q on an equipotential surface of 1000 V relative to that done to move the charge on an equipotential surface of 10 V is

a) the same.

b) less.

c) more.

d) dependent on the distance the charge moves.

3.6 A solid conducting sphere of radius R is centered about the origin of an xyz-coordinate system. A total charge Q is distributed uniformly on the surface of the sphere. Assuming, as usual, that the electric potential is zero at an infinite distance, what is the electric potential at the center of the conducting sphere?

a) zero

b) $Q/\varepsilon_0 R$

c) $Q/2\pi\varepsilon_0 R$

d) $Q/4\pi\varepsilon_0 R$

3.7 Which of the following angles between an electric dipole moment and an applied electric field will result in the most stable state?

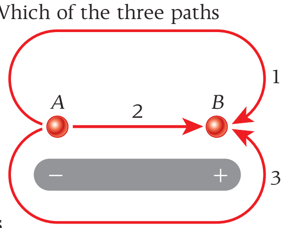
a) 0 rad

b) $\pi/2$ rad

c) π rad

d) The electric dipole moment is not stable under any condition in an applied electric field.

3.8 A positive point charge is to be moved from point A to point B in the vicinity of an electric dipole. Which of the three paths shown in the figure will result in the most work being done by the dipole's electric field on the point charge?



a) path 1

b) path 2

c) path 3

d) The work is the same on all three paths.

3.9 Each of the following pairs of charges are separated by a distance d. Which pair has the highest potential energy?

a) +5 C and +3 C

b) +5 C and −3 C

c) −5 C and +3 C

d) All pairs have the same potential energy.

3.10 A negatively charged particle revolves in a clockwise direction around a positively charged sphere. The work done on the negatively charged particle by the electric field of the sphere is

a) positive.

b) negative.

c) zero.

3.11 A hollow conducting sphere of radius R is centered about the origin of an xyz-coordinate system. A total charge Q is distributed uniformly over the surface of the sphere. Assuming, as usual, that the electric potential is zero at an infinite distance, what is the electric potential at the center of the sphere?

a) zero

b) $2kQ/R$

c) kQ/R

d) $kQ/2R$

e) $kQ/4R$

3.12 A solid conducting sphere of radius R has a charge Q evenly distributed over its surface, producing an electric potential V_0 at the surface. How much charge must be added to the sphere to increase the potential at the surface to $2V_0$?

a) $Q/2$

b) Q

c) $2Q$

d) Q^2

e) $2Q^2$

3.13 Which one of the following statements is not true?

a) Equipotential lines are parallel to the electric field lines.

b) Equipotential lines for a point charge are circular.

c) Equipotential surfaces exist for any charge distribution.

d) When a charge moves on an equipotential surface, the work done on the charge is zero.

3.14 If a proton and an alpha particle (composed of two protons and two neutrons) are each accelerated from rest through the same potential difference, how do their resulting speeds compare?

a) The proton has twice the speed of the alpha particle.

b) The proton has the same speed as the alpha particle.

c) The proton has half the speed of the alpha particle.

d) The speed of the proton is $\sqrt{2}$ times the speed of the alpha particle.

e) The speed of the alpha particle is $\sqrt{2}$ times the speed of the proton.

CONCEPTUAL QUESTIONS

3.15 High-voltage power lines are used to transport electricity cross country. These wires are favored resting places for birds. Why don't the birds die when they touch the wires?

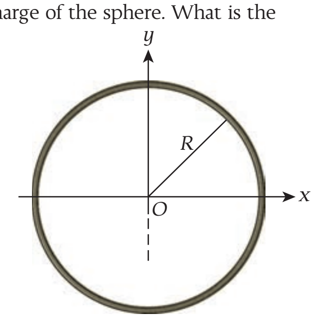
3.16 You have heard that it is dangerous to stand under trees in electrical storms. Why?

3.17 Can two equipotential lines cross? Why or why not?

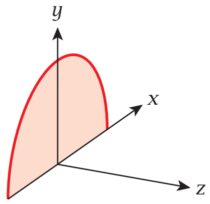
3.18 Why is it important, when soldering connectors onto a piece of electronic circuitry, to leave no pointy protrusions from the solder joints?

3.19 Using Gauss's Law and the relation between electric potential and electric field, show that the potential outside a uniformly charged sphere is identical to the potential of a point charge placed at the center of the sphere and equal to the total charge of the sphere. What is the potential at the surface of the sphere? How does the potential change if the charge distribution is not uniform but has spherical (radial) symmetry?

3.20 A metal ring has a total charge q and a radius R, as shown in the figure. Without performing any calculations, predict the value of the electric potential and the electric field at the center of the circle.



3.21 Find an integral expression for the electric potential at a point on the z-axis a distance H from a half-disk of radius R (see the figure). The half-disk has uniformly distributed charge over its surface, with charge distribution σ.



3.22 An electron moves away from a proton. Describe how the potential it encounters changes. Describe how its potential energy is changing.

3.23 The electric potential energy of a continuous charge distribution can be found in a way similar to that used for systems of point charges in Section 3.6, by breaking the distribution up into suitable pieces. Find the electric potential energy of an *arbitrary* spherically symmetrical charge distribution, $\rho(r)$. Do ***not*** assume that $\rho(r)$ represents a point charge, that it is constant, that it is piecewise-constant, or that it does or does not end at any finite radius, r. Your expression must cover all possibilities. Your expression may include an integral or integrals that cannot be evaluated without knowing the specific form of $\rho(r)$. (*Hint:* A spherical pearl is built up of thin layers of nacre added one by one.)

EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

Section 3.1

3.24 In molecules of gaseous sodium chloride, the chloride ion has one more electron than proton, and the sodium ion has one more proton than electron. These ions are separated by about 0.236 nm. How much work would be required to increase the distance between these ions to 1.00 cm?

•3.25 A metal ball with a mass of 3.00×10^{-6} kg and a charge of +5.00 mC has a kinetic energy of 6.00×10^{8} J. It is traveling directly at an infinite plane of charge with a charge distribution of +4.00 C/m^2. If it is currently 1.00 m away from the plane of charge, how close will it come to the plane before stopping?

Section 3.2

3.26 An electron is accelerated from rest through a potential difference of 370. V. What is its final speed?

3.27 How much work would be done by an electric field in moving a proton from a point at a potential of +180. V to a point at a potential of −60.0 V?

3.28 What potential difference is needed to give an alpha particle (composed of 2 protons and 2 neutrons) 200. keV of kinetic energy?

3.29 A proton, initially at rest, is accelerated through a potential difference of 500. V. What is its final velocity?

3.30 A 10.0V battery is connected to two parallel metal plates placed in a vacuum. An electron is accelerated from rest from the negative plate toward the positive plate.

a) What kinetic energy does the electron have just as it reaches the positive plate?

b) What is the speed of the electron just as it reaches the positive plate?

•3.31 A proton gun fires a proton from midway between two plates, A and B, which are separated by a distance of 10.0 cm; the proton initially moves at a speed of 150.0 km/s toward plate B. Plate A is kept at zero potential, and plate B at a potential of 400.0 V.

a) Will the proton reach plate B?

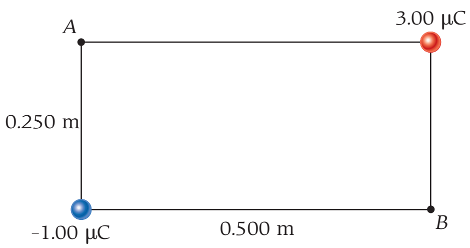
b) If not, where will it turn around?

c) With what speed will it hit plate A?

•3.32 Fully stripped (all electrons removed) sulfur (^{32}S) ions are accelerated from rest in an accelerator that uses a total voltage of 1.00×10^{9} V. ^{32}S has 16 protons and 16 neutrons. The accelerator produces a beam consisting of 6.61×10^{12} ions per second. This beam of ions is completely stopped in a beam dump. What is the total power the beam dump has to absorb?

Section 3.4

3.33 Two point charges are located at two corners of a rectangle, as shown in the figure.



a) What is the electric potential at point A?

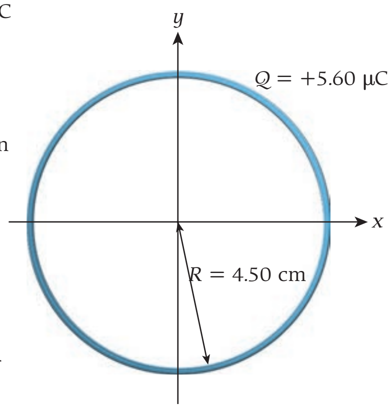
b) What is the potential difference between points A and B?

3.34 Four identical point charges (+1.61 nC) are placed at the corners of a rectangle, which measures 3.00 m by 5.00 m. If the electric potential is taken to be zero at infinity, what is the potential at the geometric center of this rectangle?

3.35 If a Van de Graaff generator has an electric potential of 1.00×10^{5} V and a diameter of 20.0 cm, find how many more protons than electrons are on its surface.

3.36 One issue encountered during the exploration of Mars has been the accumulation of static charge on land-roving vehicles, resulting in a potential of 100. V or more. Calculate how much charge must be placed on the surface of a sphere of radius 1.00 m for the electric potential just above the surface to be 100. V. Assume that the charge is uniformly distributed.

3.37 A charge $Q = +5.60\ \mu$C is uniformly distributed on a thin cylindrical plastic shell. The radius, R, of the shell is 4.50 cm. Calculate the electric potential at the origin of the xy-coordinate system shown in the figure. Assume that the electric potential is zero at points infinitely far away from the origin.



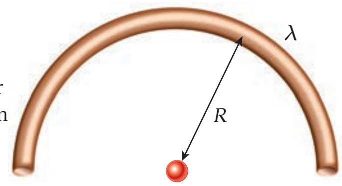
3.38 A hollow spherical conductor with a 5.00-cm radius has a surface charge of 8.00 nC.

a) What is the potential 8.00 cm from the center of the sphere?

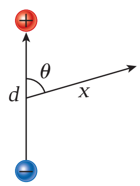
b) What is the potential 3.00 cm from the center of the sphere?

c) What is the potential at the center of the sphere?

3.39 Find the potential at the center of curvature of the (thin) wire shown in the figure. It has a (uniformly distributed) charge per unit length of $\lambda = 3.00 \times 10^{-8}$ C/m and a radius of curvature of $R = 8.00$ cm.



•3.40 Consider a dipole with charge q and separation d. What is the potential a distance x from the center of this dipole at an angle θ with respect to the dipole axis, as shown in the figure?



•3.41 A spherical water drop 50.0 μm in diameter has a uniformly distributed charge of +20.0 pC. Find (a) the potential at its surface and (b) the potential at its center.

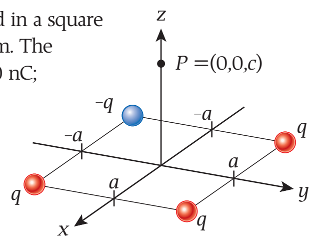
•3.42 Consider an electron in the ground state of the hydrogen atom, separated from the proton by a distance of 0.0529 nm.

a) Viewing the electron as a satellite orbiting the proton in the electric potential, calculate the speed of the electron in its orbit.

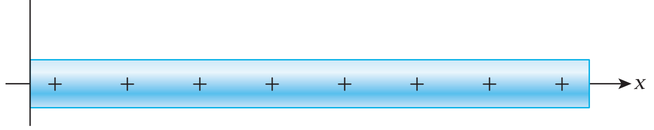
b) Calculate an effective escape speed for the electron.

c) Calculate the energy of an electron having this speed, and from it determine the energy that must be given to the electron to ionize the hydrogen atom.

•3.43 Four point charges are arranged in a square with side length $2a$, where $a = 2.70$ cm. The charges have the same magnitude, 1.50 nC; three of them are positive and one is negative as shown in the figure. What is the value of the electric potential generated by these four point charges at point $P = (0,0,c)$, where $c = 4.10$ cm?



•3.44 The plastic rod of length L shown in the figure has the nonuniform linear charge distribution $\lambda = cx$, where c is a positive constant. Find an expression for the electric potential at point P on the y-axis, a distance y from one end of the rod.

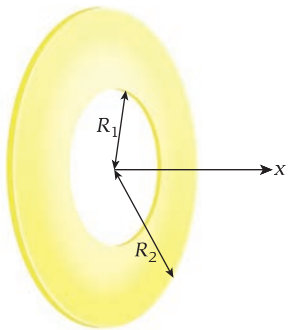


••3.45 An electric field varies in space according to this equation: $\vec{E} = E_0 x e^{-x}\hat{x}$

a) For what value of x does the electric field have its largest value, x_{max}?

b) What is the potential difference between the points at $x = 0$ and $x = x_{max}$?

••3.46 Derive an expression for electric potential along the axis (the x-axis) of a disk with a hole in the center, as shown in the figure, where R_1 and R_2 are the inner and outer radii of the disk. What would the potential be if $R_1 = 0$?



Section 3.5

3.47 An electric field is established in a nonuniform rod. A voltmeter is used to measure the potential difference between the left end of the rod and a point a distance x from the left end. The process is repeated, and it is found that the data are described by the relationship $\Delta V = 270.x^2$, where ΔV has the units V/m^2. What is the x-component of the electric field at a point 13.0 cm from the left end?

3.48 Two parallel plates are held at potentials of +200.0 V and −100.0 V. The plates are separated by 1.00 cm.

a) Find the electric field between the plates.

b) An electron is initially placed halfway between the plates. Find its kinetic energy when it hits the positive plate.

3.49 A 2.50-mg dust particle with a charge of 1.00 μC falls at a point $x = 2.00$ m in a region where the electric potential varies according to $V(x) = (2.00\ \text{V/m}^2)x^2 - (3.00\ \text{V/m}^3)x^3$. With what acceleration will the particle start moving after it touches down?

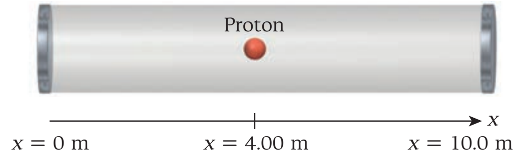
3.50 The electric potential in a volume of space is given by $V(x,y,z) = x^2 + xy^2 + yz$. Determine the electric field in this region at the coordinate (3,4,5).

•3.51 The electric potential inside a 10.0-m-long linear particle accelerator is given by $V = (3000 - 5x^2/\text{m}^2)$ V, where x is the distance from the left plate along the accelerator tube, as shown in the figure.

a) Determine an expression for the electric field along the accelerator tube.

b) A proton is released (from rest) at $x = 4.00$ m. Calculate the acceleration of the proton just after it is released.

c) What is the impact speed of the proton when (and if) it collides with the plate?



•3.52 An infinite plane of charge has a uniform charge distribution of +4.00 nC/m^2 and is located in the yz-plane at $x = 0$. A +11.0-nC fixed point charge is located at $x = +2.00$ m.

a) Find the electric potential $V(x)$ on the x-axis from $0 < x < +2.00$ m.

b) At what position(s) on the x-axis between $x = 0$ and $x = +2.00$ m is the electric potential a minimum?

c) Where on the x-axis between $x = 0$ m and $x = +2.00$ m could a positive point charge be placed and not move?

•3.53 Use $V = \dfrac{kq}{r}$ $E_x = -\dfrac{\partial V}{\partial x}$ $E_y = -\dfrac{\partial V}{\partial y}$ and $E_z = -\dfrac{\partial V}{\partial z}$ to derive the expression for the electric field of a point charge, q.

•3.54 Show that an electron in a one-dimensional electrical potential, $V(x) = Ax^2$, where the constant A is a positive real number, will execute simple harmonic motion about the origin. What is the period of that motion?

••3.55 The electric field, $\vec{E}(\vec{r})$ and the electric potential, $V(\vec{r})$ are calculated from the charge distribution, $\rho(\vec{r})$ by integrating Coulomb's Law and then the electric field. In the other direction, the field and the charge distribution are determined from the potential by suitably differentiating. Suppose the electric potential in a large region of space is given by $V(r) = V_0 \exp(-r^2/a^2)$, where V_0 and a are constants and $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin.

a) Find the electric field $\vec{E}(\vec{r})$ in this region.

b) Determine the charge density, $\rho(\vec{r})$ in this region, which gives rise to the potential and field.

c) Find the total charge in this region.

d) Roughly sketch the charge distribution that could give rise to such an electric field.

••3.56 The electron beam emitted by an electron gun is controlled (steered) with two sets of parallel conducting plates: a horizontal set to control the vertical motion of the beam, and a vertical set to control the horizontal motion of the beam. The beam is emitted with an initial velocity of 2.00×10^7 m/s. The width of the plates is $d = 5.00$ cm, the separation between the plates is $D = 4.00$ cm, and the distance between the edge of the plates and a target screen is $L = 40.0$ cm. In the absence of any applied voltage, the electron beam hits the origin of the xy-coordinate system on the observation screen. What voltages need to be applied to the two sets of plates for the electron beam to hit a target placed on the observation screen at coordinates $(x,y) = (0\ \text{cm}, 8.00\ \text{cm})$?

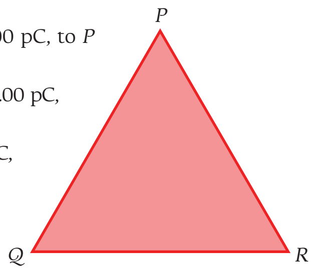
Section 3.6

3.57 Nuclear fusion reactions require that positively charged nuclei be brought into close proximity, against the electrostatic repulsion. As a simple example, suppose a proton is fired at a second, stationary proton from a large distance away. What kinetic energy must be given to the moving proton to get it to come within 1.00×10^{-15} m of the target? Assume that there is a head-on collision and that the target is fixed in place.

3.58 Fission of a uranium nucleus (containing 92 protons) produces a barium nucleus (56 protons) and a krypton nucleus (36 protons). The fragments fly apart as a result of electrostatic repulsion; they ultimately emerge with a total of 200. MeV of kinetic energy. Use this information to estimate the size of the uranium nucleus; that is, treat the barium and krypton nuclei as point charges and calculate the separation between them at the start of the process.

3.59 A deuterium ion and a tritium ion each have charge $+e$. What work has to be done on the deuterium ion in order to bring it within 1.00×10^{-14} m of the tritium ion? This is the distance within which the two ions can fuse, as a result of strong nuclear interactions that overcome electrostatic repulsion, to produce a helium-5 nucleus. Express the work in electron-volts.

•**3.60** Three charges, q_1, q_2, and q_3, are located at the corners of an equilateral triangle with side length of 1.20 m. Find the work done in each of the following cases:



a) to bring the first particle, $q_1 = 1.00$ pC, to P from infinity

b) to bring the second particle, $q_2 = 2.00$ pC, to Q from infinity

c) to bring the last particle, $q_3 = 3.00$ pC, to R from infinity

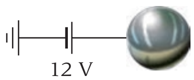
d) Find the total potential energy stored in the final configuration of q_1, q_2, and q_3.

•**3.61** Two metal balls of mass $m_1 = 5.00$ g (diameter = 5.00 mm) and $m_2 = 8.00$ g (diameter = 8.00 mm) have positive charges of $q_1 = 5.00$ nC and $q_2 = 8.00$ nC, respectively. A force holds them in place so that their centers are separated by 8.00 mm. What will their velocities be after the force has been removed and they are separated by a large distance?

Additional Exercises

3.62 Two protons at rest and separated by 1.00 mm are released simultaneously. What is the speed of either at the instant when the two are 10.0 mm apart?

3.63 A 12V battery is connected between a hollow metal sphere with a radius of 1 m and a ground, as shown in the figure. What are the electric field and the electric potential inside the hollow metal sphere?



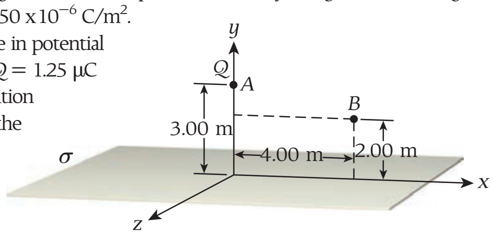
3.64 A solid metal ball with a radius of 3.00 m has a charge of 4.00 mC. If the electric potential is zero far away from the ball, what is the electric potential at each of the following positions?

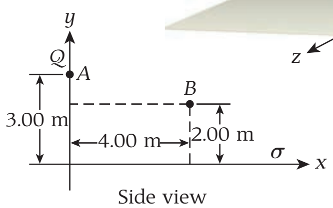
a) at $r = 0$ m, the center of the ball

b) at $r = 3.00$ m, on the surface of the ball

c) at $r = 5.00$ m

3.65 An insulating sheet in the xz-plane is uniformly charged with a charge distribution $\sigma = 3.50 \times 10^{-6}$ C/m^2. What is the change in potential when a charge of $Q = 1.25$ μC is moved from position A to position B in the figure?

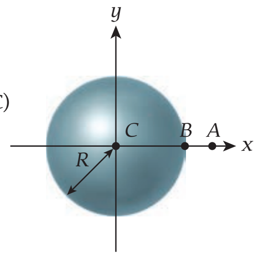




Side view

3.66 Suppose that an electron inside a cathode ray tube starts from rest and is accelerated by the tube's voltage of 21.9 kV. What is the speed (in km/s) with which the electron (mass = 9.11×10^{-31} kg) hits the screen of the tube?

3.67 A conducting solid sphere (radius of $R = 18.0$ cm, charge of $q = 6.10 \times 10^{-6}$ C) is shown in the figure. Calculate the electric potential at a point 24.0 cm from the center (point A), a point on the surface (point B), and at the center of the sphere (point C). Assume that the electric potential is zero at points infinitely far away from the origin of the coordinate system.



3.68 A classroom Van de Graaff generator accumulates a charge of 1.00×10^{-6} C on its spherical conductor, which has a radius of 10.0 cm and stands on an insulating column. Neglecting the effects of the generator base or any other objects or fields, find the potential at the surface of the sphere. Assume that the potential is zero at infinity.

3.69 A Van de Graaff generator has a spherical conductor with a radius of 25.0 cm. It can produce a maximum electric field of $2.00 \cdot 10^6$ V/m. What are the maximum voltage and charge that it can hold?

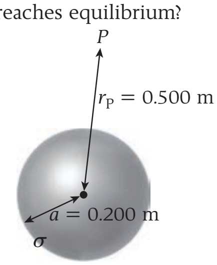
3.70 A proton with a speed of 1.23×10^4 m/s is moving from infinity directly toward a second proton. Assuming that the second proton is fixed in place, find the position where the moving proton stops momentarily before turning around.

3.71 Two metal spheres of radii $r_1 = 10.0$ cm and $r_2 = 20.0$ cm, respectively, have been positively charged so that both have a total charge of 100. μC.

a) What is the ratio of their surface charge distributions?

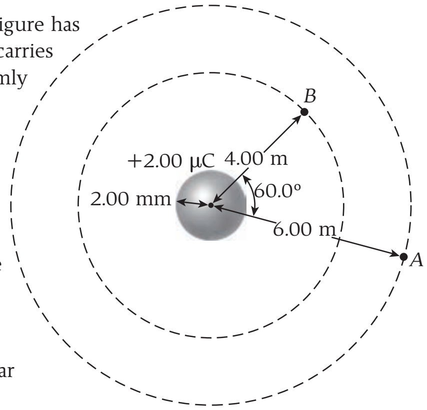
b) If the two spheres are connected by a copper wire, how much charge flows through the wire before the system reaches equilibrium?

3.72 The solid metal sphere of radius $a = 0.200$ m shown in the figure has a surface charge distribution of σ. The potential difference between the surface of the sphere and a point P at a distance $r_P = 0.500$ m from the center of the sphere is $\Delta V = V_{\text{surface}} - V_P = +4\pi$ V = +12.566 V. Determine the value of σ.



3.73 A particle with a charge of +5.00 μC is released from rest at a point on the x-axis, where $x = 0.100$ m. It begins to move as a result of the presence of a +9.00μC charge that remains fixed at the origin. What is the kinetic energy of the particle at the instant it passes the point $x = 0.200$ m?

3.74 The sphere in the figure has a radius of 2.00 mm and carries a +2.00μC charge uniformly distributed throughout its volume. What is the potential difference, $V_B - V_A$, if the angle between the two radii to points A and B is 60.0°? Is the potential difference dependent on the angle? Would the answer be the same if the charge distribution had an angular dependence, $\rho = \rho(\theta)$?



•**3.75** Two metallic spheres have radii of 10.0 cm and 5.00 cm, respectively. The magnitude of the electric field on the surface of each sphere is 3600. V/m. The two spheres are then connected by a long, thin metal wire. Determine the magnitude of the electric field on the surface of each sphere when they are connected.

•**3.76** A ring with charge Q and radius R is in the yz-plane and centered on the origin. What is the electric potential a distance x above the center of the ring? Derive the electric field from this relationship.

•**3.77** A charge of 0.681 nC is placed at $x = 0$. Another charge of 0.167 nC is placed at $x_1 = 10.9$ cm on the x-axis.

a) What is the combined electric potential of these two charges at $x = 20.1$ cm, also on the x-axis?

b) At which point(s) on the x-axis does this potential have a minimum?

•**3.78** A point charge of $+2.00\ \mu C$ is located at (2.50 m,3.20 m). A second point charge of $-3.10\ \mu C$ is located at (-2.10 m,1.00 m).

a) What is the electric potential at the origin?

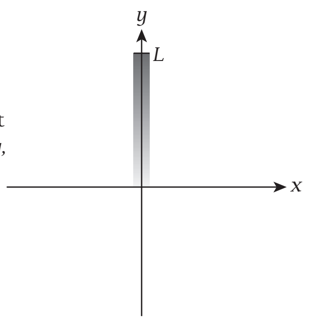
b) Along a line passing through both point charges, at what point(s) is (are) the electric potential(s) equal to zero?

•**3.79** A total charge of $Q = 4.20 \text{x} 10^{-6}$ C is placed on a conducting sphere (sphere 1) of radius $R = 0.400$ m.

a) What is the electric potential, V_1, at the surface of sphere 1 assuming that the potential infinitely far away from it is zero? (*Hint:* What is the change in potential if a charge is brought from infinitely far away, where $V(\infty) = 0$, to the surface of the sphere?)

b) A second conducting sphere (sphere 2) of radius $r = 0.100$ m with an initial net charge of zero ($q = 0$) is connected to sphere 1 using a long thin metal wire. How much charge flows from sphere 1 to sphere 2 to bring them into equilibrium? What are the electric fields at the surfaces of the two spheres at equilibrium?

•**3.80** A thin line of charge is aligned along the positive y-axis from $0 \le y \le L$, with $L = 4.0$ cm. The charge is not uniformly distributed but has a charge per unit length of $\lambda = Ay$, with $A = 8.00 \text{x} 10^{-7}\ \text{C/m}^2$. Assuming that the electric potential is zero at infinite distance, find the electric potential at a point on the x-axis as a function of x. Give the value of the electric potential at $x = 3.00$ cm.



•**3.81** Two fixed point charges are on the x-axis. A charge of -3.00 mC is located at $x = +2.00$ m and a charge of $+5.00$ mC is located at $x = -4.00$ m.

a) Find the electric potential, $V(x)$, for an arbitrary point on the x-axis.

b) At what position(s) on the x-axis is $V(x) = 0$?

c) Find $E(x)$ for an arbitrary point on the x-axis.

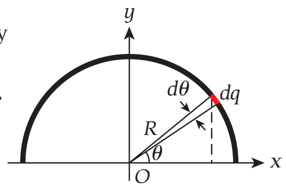
•**3.82** One of the greatest physics experiments in history measured the charge-to-mass ratio of an electron, q/m. If a uniform potential difference is created between two plates, atomized particles–each with an integral amount of charge–can be suspended in space. The assumption is that the particles of unknown mass, M, contain a net number, n, of electrons of mass m and charge q. For a plate separation of d, what is the potential difference necessary to suspend a particle of mass M containing n net electrons? What is the acceleration of the particle if the voltage is cut in half? What is the acceleration of the particle if the voltage is doubled?

•**3.83** A uniform linear charge distribution of total positive charge Q has the shape of a half-circle of radius R, as shown in the figure.

a) Without performing any calculations, predict the electric potential produced by this linear charge distribution at point O.

b) Confirm, through direct calculations, your prediction of part (a).

c) Make a similar prediction for the electric field.



••**3.84** A point charge Q is placed a distance R from the center of a conducting sphere of radius a, with $R > a$ (the point charge is outside the sphere). The sphere is grounded, that is, connected to a distant, unlimited source and/or sink of charge at zero potential. (Neither the distant ground nor the connection directly affects the electric field in the vicinity of the charge and sphere.) As a result, the sphere acquires a charge opposite in sign to Q, and the point charge experiences an attractive force toward the sphere.

a) Remarkably, the electric field *outside the sphere* is the same as would be produced by the point charge Q plus an imaginary *mirror-image* point charge q, with magnitude and location that make the set of points corresponding to the surface of the sphere an equipotential of potential zero. That is, the imaginary point charge produces the same field contribution outside the sphere as the actual surface charge on the sphere. Calculate the value and location of q. (*Hint:* By symmetry, q must lie somewhere on the axis that passes through the center of the sphere and the location of Q.)

b) Calculate the force exerted on point charge Q and directed toward the sphere, in terms of the original quantities Q, R, and a.

c) Determine the actual nonuniform surface charge distribution on the conducting sphere.

MULTI-VERSION EXERCISES

3.85 A solid conducting sphere of radius $R_1 = 1.206$ m has a charge of $Q = 1.953\ \mu C$ evenly distributed over its surface. A second solid conducting sphere of radius $R_2 = 0.6115$ m is initially uncharged and at a distance of 10.00 m from the first sphere. The two spheres are momentarily connected with a wire, which is then removed. What is the charge on the second sphere?

3.86 A solid conducting sphere of radius $R_1 = 1.435$ m has a charge of Q evenly distributed over its surface. A second solid conducting sphere of radius $R_2 = 0.6177$ m is initially uncharged and at a distance of 10.00 m from the first sphere. The two spheres are momentarily connected with a wire, which is then removed. The resulting charge on the second sphere is $0.9356\ \mu C$. What was the original charge, Q, on the first sphere?

3.87 A solid conducting sphere of radius R_1 has a charge of $Q = 4.263\ \mu C$ evenly distributed over its surface. A second solid conducting sphere of radius $R_2 = 0.6239$ m is initially uncharged and at a distance of 10.00 m from the first sphere. The two spheres are momentarily connected with a wire, which is then removed. The resulting charge on the second sphere is $1.162\ \mu C$. What is the radius of the first sphere?

3.88 A solid conducting sphere of radius $R = 1.895$ m is charged, and the magnitude of the electric field at the surface of the sphere is $3.165 \text{x} 10^5$ V/m. What is the electric potential 29.81 cm from the surface of the sphere?

3.89 A solid conducting sphere of radius R is charged, and the magnitude of the electric field at the surface of the sphere is $3.269 \text{x} 10^5$ V/m. The electric potential 32.37 cm from the surface of the sphere is $2.843 \text{x} 10^5$ V. What is the radius, R, of the sphere?

3.90 A solid conducting sphere of radius $R = 1.351$ m is charged, and the magnitude of the electric potential 34.95 cm from the surface of the sphere is $3.618 \text{x} 10^5$ V. What is the magnitude of the electric field at the surface of the sphere?

4 Capacitors

FIGURE 4.1 Interacting with the touch screen of an iPad.

Touch screens, such as the one shown in Figure 4.1, have become very common, found on everything from computer screens to cell phones to voting machines. They work in several ways, one of which involves using a property of conductors called *capacitance*, which we'll study in this chapter. Capacitance appears whenever two conductors–any two conductors–are separated by a small distance. The contact of a finger with a touch screen causes a change in capacitance that can be detected.

Capacitors have the very useful capability of storing electric charge and then releasing it very quickly. Thus, they are useful in camera flash attachments, cardiac defibrillators, and even experimental fusion reactors–anything that needs a large electric charge delivered quickly. Most circuits of any kind contain at least one capacitor. However, capacitance has a downside. It can also appear where it's not wanted, for example, between neighboring conductors in a tiny electronics circuit, where it can create "cross-talk"–unwanted interference between circuit components.

Because capacitors are one of the basic elements of electric circuits, this chapter examines how they function in simple circuits. The next two chapters will cover additional basic circuit elements and their uses.

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WHAT WE WILL LEARN

- Capacitance is the ability to store electric charge.
- Capacitors usually consist of two separated conductors or conducting plates.
- A capacitor can store charge on one plate, and there is typically an equal and opposite charge on the other plate.
- The capacitance of a capacitor is the charge stored on the plates divided by the resulting electric potential difference.
- A capacitor can store electric potential energy.
- A common type of capacitor is the parallel plate capacitor, consisting of two flat parallel conducting plates.
- The capacitance of a given capacitor depends on its geometry.
- In a circuit, capacitors wired in parallel or in series can be replaced by an equivalent capacitance.
- The capacitance of a given capacitor is increased when a dielectric material is placed between the plates.
- A dielectric material reduces the electric field between the plates of a capacitor as a result of the alignment of molecular dipole moments in the dielectric material.

4.1 Capacitance

FIGURE 4.2 Some representative types of capacitors.

Figure 4.2, shows that capacitors come in a variety of sizes and shapes. In general, a **capacitor** consists of two separated conductors, which are usually called *plates* even if they are not simple planes. If we take apart one of these capacitors, we might find two sheets of metal foil separated by an insulating layer of Mylar, as shown in Figure 4.3. The sandwiched layers of metal foil and Mylar can be rolled up with another insulating layer into a compact form that does not resemble two parallel conductors, as shown in Figure 4.4. This technique produces capacitors with some of the physical formats shown in Figure 4.2. The insulating layer between the two metal foils plays a crucial role in the characteristics of the capacitor.

To study the properties of capacitors, we'll assume a convenient geometry and then generalize the results. Figure 4.5 shows a **parallel plate capacitor,** which consists of two parallel conducting plates, each with area A, separated by a distance, d, and assumed to be in a vacuum. The capacitor is charged by placing a charge of $+q$ on one plate and a charge of $-q$ on the other plate. (It is not necessary to put exactly opposite charges onto the two plates of the capacitor to charge it; any difference in charge will do. But, for practical purposes, the overall device should remain neutral, and this requires charges of equal magnitude and opposite sign on the two plates.) Because the plates are conductors, they are equipotential surfaces; thus, the electrons on the plates will distribute themselves uniformly over the surfaces.

FIGURE 4.3 Two sheets of metal foil separated by an insulating layer.

Let's apply the results obtained in Chapter 3 to determine the electric potential and electric field for the parallel plate capacitor. (In principle, we could do this by calculating the electric potential and electric field for continuous charge distributions. However, for this physical configuration we would need to use a computer to provide the solution.) Let's place the origin of the coordinate system in the middle between the two plates, with the x-axis aligned with the two plates. Figure 4.6 shows a three-dimensional plot of the electric potential, $V(x,y)$, in the xy-plane, similar to the plots in Chapter 3.

FIGURE 4.4 The metal foil and Mylar sandwich shown in Figure 4.3 can be rolled up with an insulating layer to produce a capacitor with a compact geometry.

The potential in Figure 4.6 has a very steep (and approximately linear) drop between the two plates and a more gradual drop outside the plates. This means that the electric field can be expected to be strongest between the plates and weaker outside. Figure 4.7a presents a contour plot of the electric potential shown in Figure 4.6 for the two parallel plates. Negative potential values are shaded in green, and positive values in pink. The equipotential lines, which are the lines where the three-dimensional equipotential surfaces intersect the xy-plane, displayed in Figure 4.6 are also shown in this plot, as are representations of the two plates. Note that the equipotential lines between the two plates are all parallel to each other and equally spaced.

In Figure 4.7b, the electric field lines have been added to the contour plot. The electric field is determined using $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ introduced in Chapter 3. Far away from the two plates, the electric field looks very similar to that generated by a dipole composed of two point charges. It is easy to see that the electric field lines are perpendicular to the potential contour lines (which represent the equipotential surfaces!) everywhere in space.

But the electric field lines in Figure 4.7b do not convey adequate information about the magnitude of the electric field. Another representation of the electric field, in Figure 4.7c, displays the electric field vectors at regularly spaced grid points in the *xy*-plane. (The contour shading of the potential has been removed to reduce visual clutter.) In this plot, the field strength at each point of the grid is proportional to the size of the arrow at that point. You can clearly see that the electric field between the two plates is perpendicular to the plates and much larger in magnitude than the field outside the plates. The field in the space outside the plates is called the *fringe field*. If the plates are moved closer together, the electric field between the plates remains the same, while the fringe field is reduced.

The potential difference, ΔV, between the two parallel plates of the capacitor is proportional to the amount of charge on the plates. The proportionality constant is the **capacitance,** C, of the device, defined as

$$C = \left| \frac{q}{\Delta V} \right| . \tag{4.1}$$

The capacitance of a device depends on the area of the plates and the distance between them but not on the charge or the potential difference. (This will be shown for this and other geometries in the following sections.) By definition, the capacitance is a positive number. It tells how much charge is required to produce a given potential difference between the plates. The larger the capacitance, the more charge is required to produce a given potential difference. (Note that it is a common practice to use V, not ΔV, to represent potential difference. Be sure you understand when V is being used for potential and when it is being used for potential difference.)

Equation 4.1, the definition of capacitance, can be rewritten in this commonly used form:

$$q = C\Delta V .$$

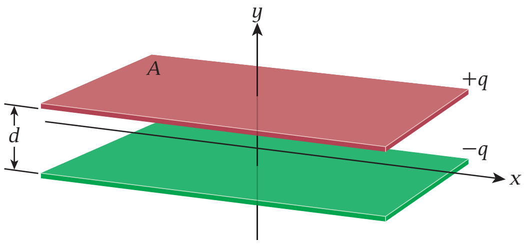


FIGURE 4.5 Parallel plate capacitor consisting of two conducting plates, each having area *A*, separated by a distance *d*.

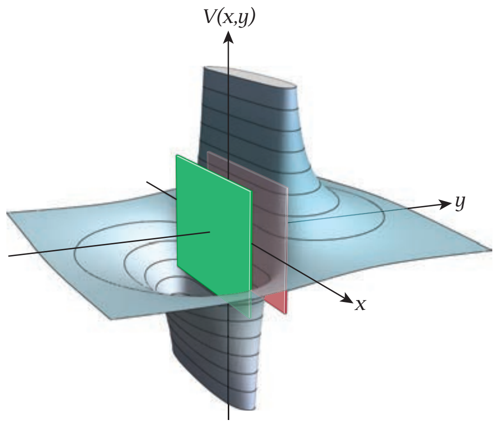


FIGURE 4.6 Electric potential in the *xy*-plane for the two oppositely charged parallel plates (superimposed) of Figure 4.5.

Equation 4.1 indicates that the units of capacitance are the units of charge divided by the units of potential, or coulombs per volt. A new unit was assigned to capacitance, named after British physicist Michael Faraday (1791–1867). This unit is called the **farad** (F):

$$1 \text{ F} = \frac{1 \text{ C}}{1 \text{ V}} . \tag{4.2}$$

One farad represents a very large capacitance. Typically, capacitors have a capacitance in the range from 1 μF $= 1 \times 10^{-6}$ F to 1 pF $= 1 \times 10^{-12}$ F.

With the definition of the farad, we can write the electric permittivity of free space, ε_0 (introduced in Chapter 1), as 8.85×10^{-12} F/m.

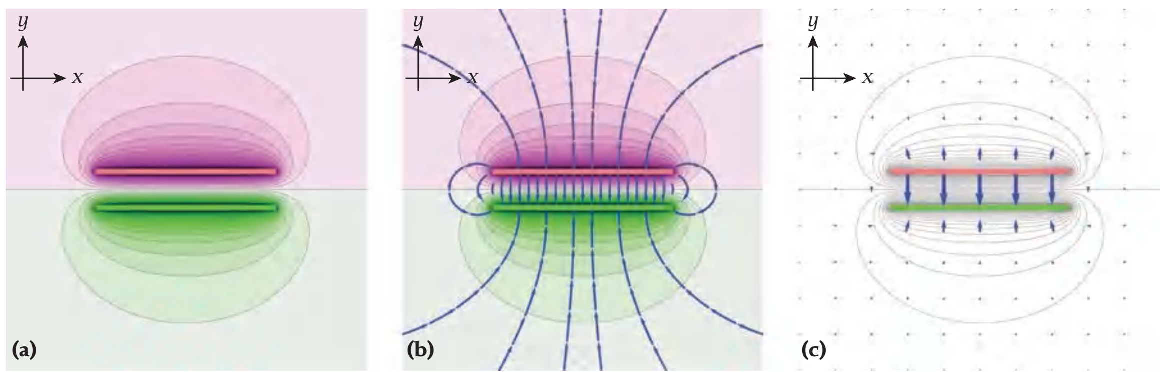


FIGURE 4.7 (a) Two-dimensional contour plot of the same potential as in Figure 4.6. (b) Contour plot with electric field lines superimposed. (c) Electric field strength at regularly spaced points in the *xy*-plane represented by the sizes of the arrows.

4.2 Circuits

The next few chapters will introduce more and more complex and interesting circuits. So let's look at what a circuit is, in general.

An **electric circuit** consists of simple wires or other conducting paths that connect circuit elements. These circuit elements can be capacitors, which we'll examine in depth in this chapter.

Circuits usually need some kind of power, which can be provided either by a battery or by an AC (alternating current) power source. The concept of a battery, a device that maintains a potential difference across its terminals through chemical reactions, was introduced in Chapter 3; for the purpose of a circuit, it can be viewed simply as an external source of electrostatic potential difference, something that delivers a fixed potential difference (which is commonly called *voltage*). An AC power source can produce the same result with a specially designed circuit that maintains a fixed potential difference. Figure 4.8 shows the symbols for circuit elements, which are used throughout this and the following chapters.

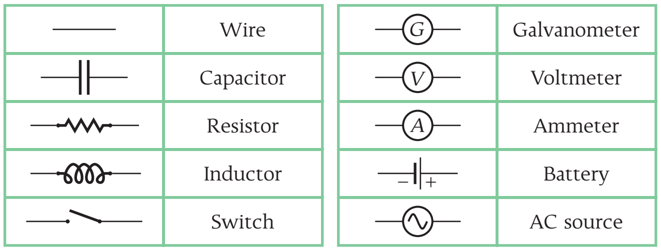
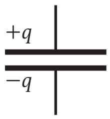


FIGURE 4.8 Commonly used symbols for circuit elements.

Concept Check 4.1

The figure shows a charged capacitor. What is the net charge on the capacitor?



a) $(+q) + (-q) = 0$

b) $|+q| + |-q| = 0$

c) $|+q| + |-q| = 2q$

d) $(+q) + (-q) = 2q$

e) q

Charging and Discharging a Capacitor

A capacitor is charged by connecting it to a battery or to a constant-voltage power supply to create a circuit. Charge flows to the capacitor from the battery or power supply until the potential difference across the capacitor is the same as the supplied voltage. If the capacitor is disconnected, it retains its charge and potential difference. A real capacitor is subject to charge leaking away over time. However, in this chapter, we'll assume that an isolated capacitor retains its charge and potential difference indefinitely.

Figure 4.9 illustrates this charging process with a circuit diagram. In this diagram, the lines represent conducting wires. The battery (power supply) is represented by the symbol, which is labeled with plus and minus signs indicating the potential assignments of the terminals and with the potential difference, V. The capacitor is represented by the symbol, which is labeled C. This circuit also contains a switch. When the switch is between positions a and b, the battery is not connected and the circuit is open. When the switch is at position a, the circuit is closed; the battery is connected across the capacitor, and the capacitor charges. When the switch is at position b, the circuit is closed in a different manner. The battery is removed from the circuit, the two plates of the capacitor are connected to each other, and charge can flow from one plate to the other through the wire, which now forms a physical connection between the plates. When the charge has dissipated on the two plates, the potential difference between the plates drops to zero, and the capacitor is said to be discharged.

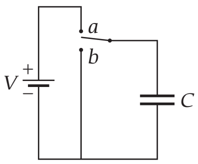


FIGURE 4.9 Simple circuit used for charging and discharging a capacitor.

4.3 Parallel Plate Capacitor and Other Types of Capacitors

Section 4.1 discussed the general features of the electric potential and the electric field of two parallel plates of opposite charge. This section examines how to determine the electric field strength between the plates and the potential difference between the two plates. Let's consider an ideal parallel plate capacitor in the form of a pair of parallel conducting plates in a vacuum with charge $+q$ on one plate and charge $-q$ on the other plate (Figure 4.10). (This ideal parallel plate capacitor has very large plates, that are very close together, much closer than shown in Figure 4.10. This configuration allows us to neglect the fringe field, the small electric field outside the space between the plates, shown in Figure 4.7c.) When the plates are charged, the upper plate has charge $+q$ and the lower plate has charge $-q$. The electric field between the two plates points from the positively charged plate downward toward the negatively charged plate. The field near the ends of the plates, the fringe field (compare Figure 4.7), can be neglected; that is, we can assume that the electric field is constant, with magnitude E, everywhere between the plates and zero elsewhere. The electric field is always perpendicular to the surface of the two parallel plates.

The electric field can be found using Gauss's Law:

$$\oiint \vec{E} \bullet d\vec{A} = \frac{q}{\varepsilon_0} \tag{4.3}$$

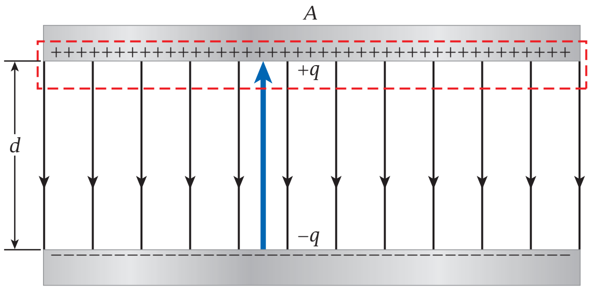


FIGURE 4.10 Side view of a parallel plate capacitor consisting of two plates with the same surface area, A, and separated by a small distance, d. The red dashed line is a Gaussian surface. The black arrows pointing downward represent the electric field. The blue arrow indicates an integration path.

How do we evaluate the integral over the Gaussian surface (whose cross section is outlined by a red dashed line in Figure 4.10)? We add the contributions from the top, the bottom, and the sides. The sides of the Gaussian surface are very small, so we can ignore the contributions from the fringe field. The top surface passes through the conductor, where the electric field is zero (remember shielding; see Chapter 2). This leaves only the bottom part of the Gaussian surface. The electric field vectors point straight down and are perpendicular to the conductor surfaces. The vector normal to the surface, $d\vec{A}$ points in the same direction and is thus parallel to $\vec{E}$ Therefore, the scalar product is $\vec{E} \bullet d\vec{A} = E\,dA\cos 0° = E\,dA$. For the integral over the Gaussian surface, we then have

$$\oiint \vec{E} \bullet d\vec{A} = \iint_{\text{bottom}} E\,dA = E \iint_{\text{bottom}} dA = EA,$$

where A is the area of the plate. In other words, for the parallel plate capacitor, Gauss's Law yields

$$EA = \frac{q}{\varepsilon_0}, \tag{4.4}$$

where A is the surface area of the positively charged plate and q is the magnitude of the charge on the positively charged plate. The charge on each plate resides entirely on the inside surface because of the presence of opposite charge on the other plate.

The electric potential difference across the two plates in terms of the electric field is

$$\Delta V = -\int_{\text{i}}^{\text{f}} \vec{E} \bullet d\vec{s}\,. \tag{4.5}$$

The path of integration is chosen to be from the negatively charged plate to the positively charged plate, along the blue arrow in Figure 4.10. Since the electric field is antiparallel to this integration path (see Figure 4.10), the scalar product is $\vec{E} \bullet d\vec{s} = Eds\cos 180° = -Eds$. Thus, the integral in equation 4.5 reduces to

$$\Delta V = Ed = \frac{qd}{\varepsilon_0 A},$$

where we used equation 4.4 to relate the electric field to the charge. Combining this expression for the potential difference and the definition of capacitance (equation 4.1) gives an expression for the capacitance of a parallel plate capacitor:

$$C = \left|\frac{q}{\Delta V}\right| = \frac{\varepsilon_0 A}{d}. \tag{4.6}$$

Note that the capacitance of a parallel plate capacitor depends only on the area of the plates and the distance between the plates. In other words, only the geometry of a capacitor affects its capacitance. The amount of charge on the capacitor or the potential difference between its plates does not affect its capacitance.

Concept Check 4.2

Suppose you charge a parallel plate capacitor using a battery and then remove the battery, isolating the capacitor and leaving it charged. You then move the plates of the capacitor farther apart. The potential difference between the plates will

a) increase.

b) decrease.

c) stay the same.

d) not be determinable.

Self-Test Opportunity 4.1

You charge a parallel plate capacitor using a battery. You then remove the battery and isolate the capacitor. If you decrease the distance between the plates of the capacitor, what will happen to the electric field between the plates?

Concept Check 4.3

Suppose you have a parallel plate capacitor with area *A* and plate separation *d*, but space constraints on a circuit board force you to reduce the area of the capacitor by a factor of 2. What do you have to do to compensate and retain the same value of the capacitance?

a) reduce *d* by a factor of 2

b) increase *d* by a factor of 2

c) reduce *d* by a factor of 4

d) increase *d* by a factor of 4

EXAMPLE 4.1 Area of a Parallel Plate Capacitor

A parallel plate capacitor has plates that are separated by 1.00 mm (Figure 4.11).

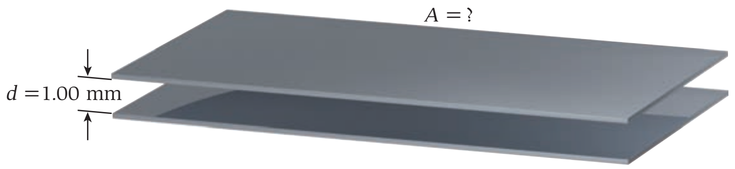


FIGURE 4.11 A parallel plate capacitor with plates separated by 1.00 mm.

PROBLEM

What is the area required to give this capacitor a capacitance of 1.00 F?

SOLUTION

The capacitance is given by

$$C = \frac{\varepsilon_0 A}{d}. \tag{i}$$

Solving equation (i) for the area and putting in $d = 1.00 \times 10^{-3}$ m and $C = 1.00$ F, we get

$$A = \frac{dC}{\varepsilon_0} = \frac{\left(1.00 \times 10^{-3}\ \text{m}\right)\left(1.00\ \text{F}\right)}{\left(8.85 \times 10^{-12}\ \text{F/m}\right)} = 1.13 \times 10^{8}\ \text{m}^2.$$

If these plates were square, each one would be 10.6 km by 10.6 km! This result emphasizes that a farad is an extremely large amount of capacitance.

Cylindrical Capacitor

Consider a capacitor constructed of two collinear conducting cylinders with vacuum between them (Figure 4.12). The inner cylinder has radius r_1, and the outer cylinder has radius r_2. The inner cylinder has charge $-q$, and the outer cylinder has charge $+q$. The electric field between the two cylinders is then directed radially inward and perpendicular to the surfaces of both cylinders. As for a parallel plate capacitor, we assume that the cylinders are long and that there is essentially no fringe field near their ends.

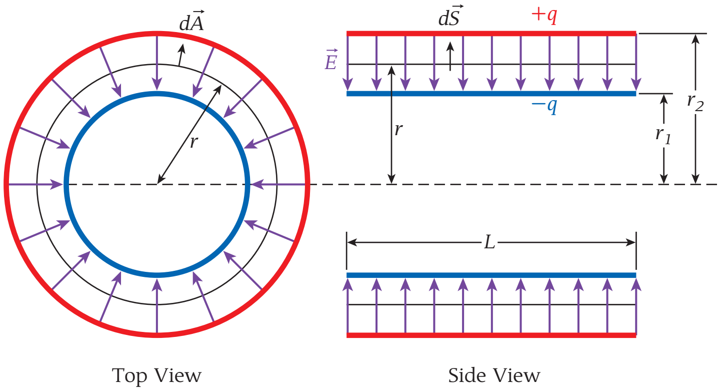


FIGURE 4.12 Cylindrical capacitor consisting of two long collinear conducting cylinders. The black circle represents a Gaussian surface. The purple arrows represent the electric field.

We can apply Gauss's Law to find the electric field between the two cylinders, using a Gaussian surface in the form of a cylinder with radius r and length L that is collinear with the two cylinders of the capacitor, as shown in Figure 4.12. The enclosed charge is then $-q$, because only the negatively charged surface of the capacitor is inside the Gaussian surface. The normal vector to the Gaussian surface, $d\vec{A}$ points radially outward and is thus antiparallel to the electric field. This means that $\vec{E} \bullet d\vec{A} = E\,dA \cos 180° = -E\,dA$. Applying Gauss's Law and using the fact that the surface of the cylinder has area $A = 2\pi rL$ results in

$$\oiint \vec{E} \bullet d\vec{A} = -E \oiint dA = -E 2\pi rL = \frac{-q}{\varepsilon_0}. \tag{4.7}$$

Equation 4.7 can be rearranged to give an expression for the magnitude of the electric field:

$$E = \frac{q}{\varepsilon_0 2\pi rL}, \quad \text{for } r_1 < r < r_2.$$

The potential difference between the two cylindrical capacitor plates is obtained by integrating over the electric field, $\Delta V = -\int_i^f \vec{E} \bullet d\vec{s}$. For the integration path in the radial direction from the negatively charged cylinder at r_1 to the positively charged cylinder at r_2, the electric field is antiparallel to the path. Thus, $\vec{E} \bullet d\vec{s}$ in equation 4.5 becomes $-E\,dr$. Therefore,

$$\Delta V = -\int_i^f \vec{E} \circ d\vec{s} = \int_{r_1}^{r_2} E dr = \int_{r_1}^{r_2} \frac{q}{\varepsilon_0 2\pi r L} dr = \frac{q}{\varepsilon_0 2\pi L} \ln\left(\frac{r_2}{r_1}\right).$$

This expression for the potential difference and equation 4.1 yield an expression for the capacitance:

$$C = \left|\frac{q}{\Delta V}\right| = \frac{q}{\dfrac{q}{\varepsilon_0 2\pi L}\ln(r_2/r_1)} = \frac{2\pi\varepsilon_0 L}{\ln(r_2/r_1)}. \qquad (4.8)$$

Just as for a parallel plate capacitor, the capacitance of a cylindrical capacitor depends only on its geometry.

Spherical Capacitor

Now let's consider a spherical capacitor formed by two concentric conducting spheres with radii r_1 and r_2 and vacuum between them (Figure 4.13). The inner sphere has charge $+q$, and the outer sphere has charge $-q$. The electric field is perpendicular to the surfaces of both spheres and points radially from the inner, positively charged sphere to the outer, negatively charged sphere, as shown by the purple arrows in Figure 4.13. (Previously, for the parallel plate and cylindrical capacitors, the integration was from the negative to the positive charge. Here we'll see what happens when the direction is reversed.) To determine the magnitude of the electric field, we employ Gauss's Law, using a Gaussian surface consisting of a sphere concentric with the two spherical conductors and having a radius r such that $r_1 < r < r_2$. The electric field is also perpendicular to the Gaussian surface everywhere, so we have

$$\oiint \vec{E} \bullet d\vec{A} = EA = E\left(4\pi r^2\right) = \frac{q}{\varepsilon_0}. \qquad (4.9)$$

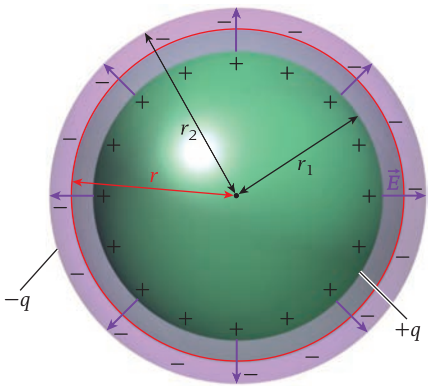


FIGURE 4.13 Spherical capacitor consisting of two concentric conducting spheres. The Gaussian surface is represented by the red circle of radius *r*.

Solving equation 4.9 for E gives

$$E = \frac{q}{4\pi\varepsilon_0 r^2}, \quad \text{for } r_1 < r < r_2.$$

For the potential difference, we proceed in a fashion similar to that used for the cylindrical capacitor and obtain

$$\Delta V = -\int_i^f \vec{E} \circ d\vec{s} = -\int_{r_1}^{r_2} E dr = -\int_{r_1}^{r_2} \frac{q}{4\pi\varepsilon_0 r^2} dr = -\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

In this case, $\Delta V < 0$. Why? Because we integrated from the positive charge to the negative charge! The positive charge is at a higher potential than the negative one, resulting in a negative potential difference. Equation 4.1 gives the capacitance of a spherical capacitor as the absolute value of the charge divided by the absolute value of the potential difference:

$$C = \left|\frac{q}{\Delta V}\right| = \frac{q}{\dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)} = \frac{4\pi\varepsilon_0}{\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)}.$$

This can be rewritten in a more convenient form:

$$C = 4\pi\varepsilon_0 \frac{r_1 r_2}{r_2 - r_1}. \qquad (4.10)$$

Note that again the capacitance depends only on the geometry of the device.

Concept Check 4.4

If the inner and outer radii of a spherical capacitor are increased by a factor of 2, what happens to the capacitance?

a) It is reduced by a factor of 4.

b) It is reduced by a factor of 2.

c) It stays the same.

d) It is increased by a factor of 2.

e) It is increased by a factor of 4.

We can obtain the capacitance of a single conducting sphere from equation 4.10 by assuming that the outer spherical conductor is infinitely far away. With $r_2 = \infty$ and $r_1 = R$, the capacitance of an isolated spherical conductor is given by

$$C = 4\pi\varepsilon_0 R. \tag{4.11}$$

4.4 Capacitors in Circuits

As stated earlier, a circuit is a set of electrical devices connected by conducting wires. Capacitors can be wired in circuits in different ways, but the two most fundamental ones are parallel connection and series connection.

Capacitors in Parallel

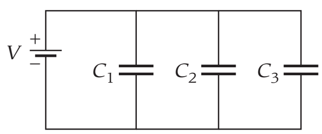


FIGURE 4.14 Simple circuit with a battery and three capacitors in parallel.

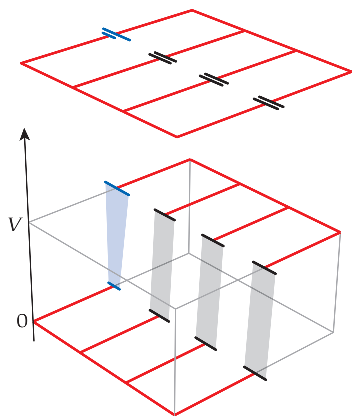


FIGURE 4.15 The potential in different parts of the circuit of Figure 4.14.

Figure 4.14 shows a circuit with three capacitors in **parallel connection.** Each of the three capacitors has one plate wired directly to the positive terminal of a battery with potential difference V and one plate wired directly to the negative terminal of that battery. The same circuit appears in the upper part of Figure 4.15, and the lower part of this figure shows the value of the potential at each part of the circuit in a three-dimensional plot. This illustrates that all capacitor plates connected to the positive terminal of the battery are at the same potential. The other plates of the capacitors are all at the potential of the negative terminal of the battery (set to zero). (The negative and positive terminals of the battery are joined with a light blue sheet to show that these two terminals are part of the same device and to provide a better visual representation of the potential difference between the two terminals. The plates of each capacitor are joined by a light gray band.)

The key insight provided by Figure 4.15 is that the potential difference across each of the three capacitors is the same, ΔV. Thus, for the three capacitors in this circuit, we have

$$q_1 = C_1 \Delta V$$
$$q_2 = C_2 \Delta V$$
$$q_3 = C_3 \Delta V\ .$$

In general, the charge on each capacitor can have a different value. The three capacitors can be viewed as one equivalent capacitor that holds a total charge q, given by

$$q = q_1 + q_2 + q_3 = C_1\Delta V + C_2\Delta V + C_3\Delta V = (C_1 + C_2 + C_3)\Delta V$$

Thus, the equivalent capacitance for this capacitor is

$$C_{\text{eq}} = C_1 + C_2 + C_3\,.$$

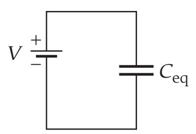


FIGURE 4.16 The three capacitors in Figure 4.14 can be replaced with an equivalent capacitance.

This result can be extended to any number, n, of capacitors connected in parallel:

$$C_{\text{eq}} = \sum_{i=1}^{n} C_i\,. \tag{4.12}$$

In other words, the equivalent capacitance of a system of capacitors in parallel is just the sum of the capacitances. Thus, several capacitors in parallel in a circuit can be replaced with an equivalent capacitance given by equation 4.12, as shown in Figure 4.16.

Capacitors in Series

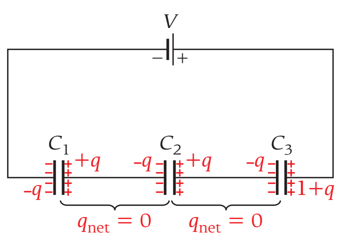


FIGURE 4.17 Simple circuit with three capacitors in series.

Figure 4.17 shows a circuit with three capacitors in **series connection.** In this configuration, the battery produces an equal charge of $+q$ on the right plate of each capacitor and an equal charge of $-q$ on the left plate of each capacitor. This fact can be made clear by starting when the capacitors are uncharged. The battery is then connected to the series arrangement of the three capacitors. The positive plate of C_3 is connected to the positive terminal of the battery and begins to collect positive charge supplied by the battery. This positive charge induces a negative charge of equal magnitude onto the other plate of C_3.

The negatively charged plate of C_3 is connected to the right plate of C_2, which then becomes positively charged because no net charge can accumulate on the isolated section consisting of the left plate of C_3 and the right plate of C_2. The positively charged plate of C_2 induces a negative charge of equal magnitude onto the other plate of C_2. In turn, the negatively charged plate of C_2 leaves a positive charge on the plate of C_1 connected to it, which induces a negative charge onto the left plate of C_1. The negatively charged plate of C_1 is connected to the negative terminal of the battery. Thus, charge flows from the battery, charging the positive plate of C_3 to a charge of value $+q$, and inducing a corresponding charge of $-q$ on the negatively charged plate of C_1. Therefore, each capacitor does indeed end up with the same charge.

When the three capacitors in the circuit in Figure 4.17 are charged, the sum of the potential drops across all three must equal the potential difference supplied by the battery. This is illustrated in Figure 4.18, a three-dimensional representation of the potential in the circuit with the three capacitors in series, similar to that in Figure 4.15. (Note that the potential drops at the three capacitors in series are not equal; this is true in general for a series connection.)

As you can see from Figure 4.18, the potential drops across the three capacitors must add up to the total potential difference, ΔV, supplied by the battery. Because each capacitor has the same charge, we have

$$\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3 = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right).$$

The equivalent capacitance can be written as

$$\Delta V = \frac{q}{C_{\text{eq}}},$$

where

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}. \tag{4.13}$$

Thus, the three capacitors in series in the circuit shown in Figure 4.17 can be replaced with an equivalent capacitance given by equation 4.13, yielding the same circuit diagram as that in Figure 4.16.

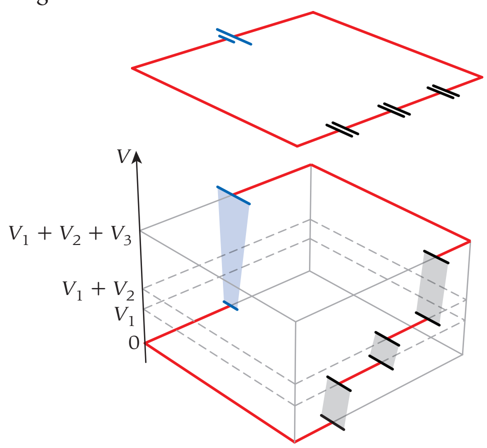


FIGURE 4.18 The potential in a circuit with three capacitors in series.

Concept Check 4.5

For a circuit with three capacitors in series, the equivalent capacitance must always be

a) equal to the largest of the three individual capacitances.

b) equal to the smallest of the three individual capacitances.

c) larger than the largest of the three individual capacitances.

d) smaller than the smallest of the three individual capacitances.

Concept Check 4.6

The potential drop for a circuit with three capacitors of different individual capacitances in series connection is

a) the same across each capacitor and has the same value as the potential difference supplied by the battery.

b) the same across each capacitor and has $\frac{1}{3}$ of the value of the potential difference supplied by the battery.

c) largest across the capacitor with the smallest capacitance.

d) largest across the capacitor with the largest capacitance.

For a system of n capacitors, equation 4.13 generalizes to

$$\frac{1}{C_{\text{eq}}} = \sum_{i=1}^{n} \frac{1}{C_i}. \tag{4.14}$$

Thus, the capacitance of a system of capacitors in series is always less than the smallest capacitance in the system.

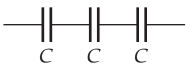
Finding equivalent capacitances for capacitors in series and in parallel allows problems involving complicated circuits to be solved, as the following example illustrates.

Self-Test Opportunity 4.2

What is the equivalent capacitance for four 10.0-μF capacitors connected in series? What is the equivalent capacitance for four 10.0-μF capacitors connected in parallel?

Concept Check 4.7

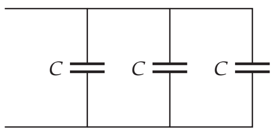
Three capacitors, each with capacitance C, are connected as shown in the figure. What is the equivalent capacitance for this arrangement of capacitors?



a) $C/3$
b) $3C$
c) $C/9$
d) $9C$
e) none of the above

Concept Check 4.8

Three capacitors, each with capacitance C, are connected as shown in the figure. What is the equivalent capacitance for this arrangement of capacitors?



a) $C/3$
b) $3C$
c) $C/9$
d) $9C$
e) none of the above

EXAMPLE 4.2 System of Capacitors

PROBLEM

Consider the circuit shown in Figure 4.19a, a complicated-looking arrangement of five capacitors with a battery. What is the combined capacitance of this set of five capacitors? If each capacitor has a capacitance of 5.0 nF, what is the equivalent capacitance of the arrangement? If the potential difference of the battery is 12 V, what is the charge on each capacitor?

SOLUTION

This problem may look complicated at first, but it can be simplified into sequential steps, using the rules for equivalent capacitances of capacitors in series and in parallel. We begin with the innermost circuit structures and work outward.

STEP 1

Looking at capacitors 1 and 2 in Figure 4.19a, we see right away that they are in parallel. Because capacitor 3 is some distance away, it is less obvious that it is also in parallel with 1 and 2. However, the upper plates of all three of these capacitors are connected by wires and are thus at the same potential. The same goes for their lower plates, so all three are indeed in parallel. According to equation 4.12, the equivalent capacity for these three capacitors is

$$C_{123} = \sum_{i=1}^{3} C_i = C_1 + C_2 + C_3$$

We can replace three capacitors with one with the equivalent capacity. This replacement is shown in Figure 4.19b.

STEP 2

In Figure 4.19b, C_{123} and C_4 are in series. Thus, their equivalent capacitance is, according to equation 4.14,

$$\frac{1}{C_{1234}} = \frac{1}{C_{123}} + \frac{1}{C_4} \Rightarrow C_{1234} = \frac{C_{123}C_4}{C_{123} + C_4}$$

This replacement is shown in Figure 4.19c.

STEP 3

Finally, C_{1234} and C_5 are in parallel in Figure 4.19c. Therefore, we can repeat the calculation for two capacitors in parallel and find the equivalent capacitance of all five capacitors:

$$C_{12345} = C_{1234} + C_5 = \frac{C_{123}C_4}{C_{123} + C_4} + C_5 = \frac{(C_1 + C_2 + C_3)C_4}{C_1 + C_2 + C_3 + C_4} + C_5$$

This result gives us the simple circuit shown in Figure 4.19d.

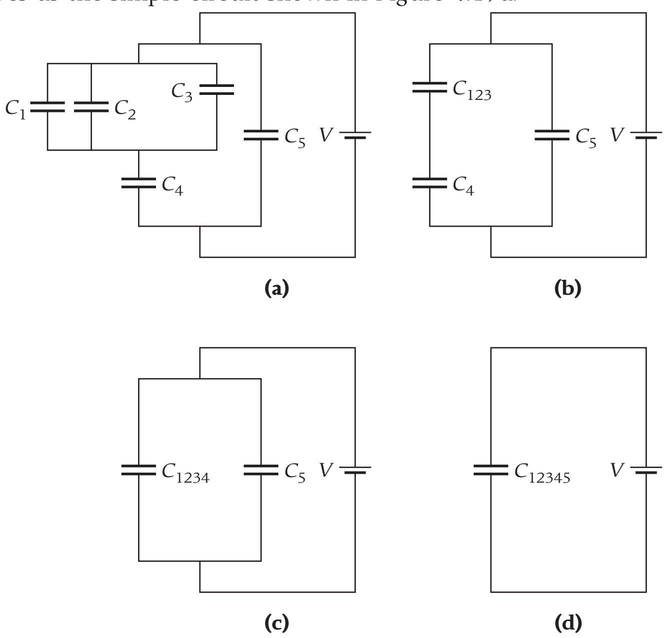


FIGURE 4.19 System of capacitors: (a) original circuit configuration; (b) reducing parallel capacitors to their equivalent; (c) reducing series capacitors to their equivalent; (d) equivalent capacitance for the entire set of capacitors.

STEP 4: INSERT THE NUMBERS FOR THE CAPACITORS

We can now find the equivalent capacitance if all the capacitors have identical 5.0-nF capacitances:

$$\left(\frac{(5.0+5.0+5.0)5.0}{5.0+5.0+5.0+5.0}+5.0\right)\text{ nF}=8.8\text{ nF}$$

As you can see, more than half of the total capacitance of this arrangement is provided by capacitor 5 alone. This result shows that you need to be extremely careful about how you arrange capacitors in circuits.

STEP 5: CALCULATE THE CHARGES ON THE CAPACITORS

C_{1234} and C_5 are in parallel. Thus, they have the same potential difference across them, 12 V. The charge on C_5 is then

$$q_5=C_5\Delta V=(5.0\text{ nF})(12\text{ V})=60.\text{ nC}$$

C_{1234} is composed of C_{123} and C_4 in series. Thus, C_{123} and C_4 must have the same charge q_4, so

$$\Delta V=\Delta V_{123}+\Delta V_4=\frac{q_4}{C_{123}}+\frac{q_4}{C_4}=q_4\left(\frac{1}{C_{123}}+\frac{1}{C_4}\right)$$

The charge on C_4 is then

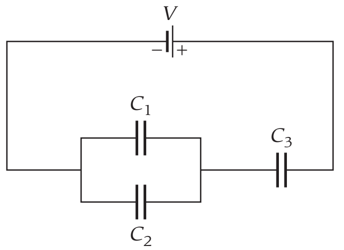
$$q_4=\Delta V\frac{C_{123}C_4}{C_{123}+C_4}=\Delta V\frac{(C_1+C_2+C_3)C_4}{C_1+C_2+C_3+C_4}=(12\text{ V})\frac{(15\text{ nF})(5.0\text{ nF})}{20.0\text{ nF}}=45\text{ nC}$$

C_{123} is equivalent to three capacitors in parallel, and it also has the same charge as C_4, or 45 nC. The three capacitors C_1, C_2, and C_3 have the same capacitance, the same potential difference across them as they are in parallel, and the sum of the charge on these three capacitors must equal 45 nC. Therefore, we can calculate the charge on C_1, C_2, and C_3:

$$q_1=q_2=q_3=\frac{45\text{ nC}}{3}=15\text{ nC}$$

Concept Check 4.9

Three capacitors are connected to a battery as shown in the figure. If $C_1=C_2=C_3=10.0\ \mu\text{F}$ and $V=10.0$ V, what is the charge on capacitor C_3?



a) 66.7 μC
b) 100. μC
c) 150. μC
d) 300. μC
e) 457. μC

4.5 Energy Stored in Capacitors

Capacitors are extremely useful for storing electric potential energy. They are much more useful than batteries if the potential energy has to be converted into other energy forms very quickly. One application of capacitors for the storage and rapid release of electric potential energy is described in Example 4.4, about the high-powered laser at the National Ignition Facility. Let's examine how much energy can be stored in a capacitor.

A battery must do work to charge a capacitor. This work can be conceptualized in terms of changing the electric potential energy of the capacitor. In order to accomplish the charging process, charge has to be moved against the electric field between the two capacitor plates. As noted earlier in this chapter, the larger the charge of the capacitor is, the larger the potential difference between the plates. This means that the more charge already on the capacitor, the harder it becomes to add a differential amount of charge to the capacitor. The differential work, dW, done by a battery with potential difference ΔV to put a differential charge, dq, on a capacitor with capacitance C is

$$dW=\Delta V'dq'=\frac{q'}{C}dq'$$

where $\Delta V'$ and q' are the instantaneous (increasing) potential difference and charge, respectively, on the capacitor during the charging process. The total work, W_t, required to bring the capacitor to its full charge, q, is given by

$$W_t=\int dW=\int_0^q\frac{q'}{C}dq'=\frac{1}{2}\frac{q^2}{C}$$

This work is stored as electric potential energy:

$$U=\frac{1}{2}\frac{q^2}{C}=\frac{1}{2}C(\Delta V)^2=\frac{1}{2}q\Delta V \qquad (4.15)$$

All three of the formulations for the stored electric potential energy in equation 4.15 are equally valid. Each can be transformed to one of the others by using $q = C\Delta V$ and eliminating one of the three quantities in favor of the other two.

The **electric energy density,** u, is defined as the electric potential energy per unit volume:

$$u = \frac{U}{\text{volume}}.$$

(*Note:* V is not used to represent the volume here, because in this context it is reserved for the potential.)

For the special case of a parallel plate capacitor that has no fringe field, it is easy to calculate the volume enclosed between two plates of area A separated by a perpendicular distance d. It is the area of each plate times the distance between the plates, or Ad. Using equation 4.15 for the electric potential energy, we obtain

$$u = \frac{U}{Ad} = \frac{\frac{1}{2}C(\Delta V)^2}{Ad} = \frac{C(\Delta V)^2}{2Ad}$$

Using equation 4.6 for the capacitance of a parallel plate capacitor with vacuum between the plates, we get

$$u = \frac{(\varepsilon_0 A/d)(\Delta V)^2}{2Ad} = \frac{1}{2}\varepsilon_0\left(\frac{\Delta V}{d}\right)^2$$

Recognizing that $\Delta V/d$ is the magnitude of the electric field, E, we obtain an expression for the electric energy density for a parallel plate capacitor:

$$u = \frac{1}{2}\varepsilon_0 E^2 \tag{4.16}$$

This result, although derived for a parallel plate capacitor, is in fact much more general. The electric potential energy stored in any electric field per unit volume occupied by that field can be described using equation 4.16.

Concept Check 4.10

How much energy is stored in the 180-μF capacitor of a camera flash unit charged to 300.0 V?

a) 1.22 J

b) 8.10 J

c) 45.0 J

d) 115 J

e) 300 J

EXAMPLE 4.3 Thundercloud

Suppose a thundercloud with a width of 2.0 km and a length of 3.0 km hovers at an altitude of 0.50 km over a flat area. The cloud carries a charge of 160 C, and the ground has no charge.

PROBLEM 1

What is the potential difference between the cloud and the ground?

SOLUTION 1

We can approximate the cloud-ground system as a parallel plate capacitor. Its capacitance is, according to equation 4.6,

$$C = \frac{\varepsilon_0 A}{d} = \frac{(8.85\times 10^{-12}\ \text{F/m})(2.0\ \text{km})(3.0\ \text{km})}{0.50\ \text{km}} = 0.11\ \mu\text{F}$$

Because we know the charge carried by the cloud, 160 C, it is tempting to insert this value into the relationship among charge, capacitance, and potential difference (equation 4.1) to find the desired answer. However, a parallel plate capacitor with a charge of $+q$ on one plate and $-q$ on the other has a charge difference of $2q$ between the plates. For the cloud-ground system, $2q = 160$ C, or $q = 80.$ C. Alternatively, we can think of the cloud as a charged insulator and use the result from Section 22.9, that the field due to a plane sheet of charge is $E = \sigma/2\varepsilon_0$, to justify the factor of $\frac{1}{2}$. Now we can use equation 4.1 and obtain

$$\Delta V = \frac{q}{C} = \frac{80.\ \text{C}}{0.11\ \mu\text{F}} = 7.3\times 10^8\ \text{V}$$

The potential difference is more than 700 million volts!

PROBLEM 2

Lightning strikes require electric field strengths of approximately 2.5 MV/m. Are the conditions described in the problem statement sufficient for a lightning strike?

SOLUTION 2

We use the potential difference between cloud and ground and the given distance between them to calculate the magnitude of the electric field:

$$E = \frac{\Delta V}{d} = \frac{7.3 \times 10^8 \text{ V}}{0.50 \text{ km}} = 1.5 \text{ MV/m}$$

From this result, we can conclude that no lightning will develop in these conditions. However, if the cloud drifted over a radio tower, the electric field strength would likely increase and lead to a lightning discharge.

PROBLEM 3

What is the total electric potential energy contained in the field between this thundercloud and the ground?

SOLUTION 3

From equation 4.15, the total electric potential energy stored in the cloud-ground system is

$$U = \tfrac{1}{2} q \Delta V = 0.5(80.\text{ C})(7.3 \times 10^8 \text{ V}) = 2.9 \times 10^{10} \text{ J}$$

For comparison, this energy is sufficient to run a typical 1500-W hair dryer for more than 5000 hours.

SOLVED PROBLEM 4.1 Energy Stored in Capacitors

PROBLEM

Suppose many capacitors, each with $C = 90.0\ \mu\text{F}$, are connected in parallel across a battery with a potential difference of $\Delta V = 160.0$ V. How many capacitors are needed to store 95.6 J of energy?

SOLUTION

THINK The equivalent capacitance of many capacitors connected in parallel is given by the sum of the capacitances of all the capacitors. We can calculate the energy stored from the equivalent capacitance of the capacitors in parallel and the potential difference of the battery.

SKETCH Figure 4.20 shows a circuit with n capacitors connected in parallel across a battery.

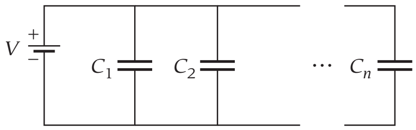


FIGURE 4.20 A circuit with n capacitors connected in parallel across a battery.

RESEARCH The equivalent capacitance, C_{eq}, of n capacitors, each with capacitance C, connected in parallel is

$$C_{eq} = C_1 + C_2 + \cdots + C_n = nC$$

The energy stored in the capacitors is then given by

$$U = \tfrac{1}{2} C_{eq} (\Delta V)^2 = \tfrac{1}{2} nC (\Delta V)^2 \qquad \text{(i)}$$

SIMPLIFY Solving equation (i) for the required number of capacitors gives us

$$n = \frac{2U}{C(\Delta V)^2}$$

CALCULATE Putting in the numerical values, we get

$$n = \frac{2(95.6 \text{ J})}{(90.0 \times 10^{-6} \text{ C})(160.0 \text{ V})^2} = 82.986$$

ROUND We report our result as an integer number of capacitors:

$$n = 83 \text{ capacitors.}$$

DOUBLE-CHECK The capacitance of 83 capacitors with $C = 90.0\ \mu\text{F}$ is

$$C_{eq} = 83(90.0\ \mu\text{F}) = 0.00747 \text{ F}$$

– *Continued*

Charging this capacitor with a 160-V battery produces a stored energy of

$$U = \tfrac{1}{2} C_{eq} (\Delta V)^2 = \tfrac{1}{2}(0.00747 \text{ F})(160.0 \text{ V})^2 = 95.6 \text{ J}$$

Thus, our answer for the number of capacitors is consistent.

FIGURE 4.21 Forty-eight power-conditioning modules in one of four bays at the National Ignition Facility at Lawrence Livermore National Laboratory.

EXAMPLE 4.4 The National Ignition Facility

The National Ignition Facility (NIF) is a high-powered laser designed to produce fusion reactions similar to those that occur in the Sun. The laser uses a short, high-energy pulse of light to heat and compress a small pellet containing isotopes of hydrogen. The laser is powered by 192 power-conditioning modules (Figure 4.21), each containing twenty 300. μF capacitors connected in parallel and charged to 24.0 kV. The capacitors are charged over a period of 90.0 s. The laser is then fired by discharging all the energy stored in the capacitors in 400. μs.

PROBLEM 1
How much energy is stored in the capacitors of NIF?

SOLUTION 1
The capacitors are connected in parallel. Thus, the equivalent capacitance of each power-conditioning module is

$$C_{eq} = 20(300.\ \mu\text{F}) = 6.00 \text{ mF}$$

The energy stored in each power-conditioning module is

$$U = \tfrac{1}{2} C_{eq} (\Delta V)^2 = \tfrac{1}{2}\left(6.00 \times 10^{-3} \text{ F}\right)\left(24.0 \times 10^3 \text{ V}\right)^2 = 1.73 \text{ MJ}$$

Thus, the total energy stored in all the capacitors of NIF is

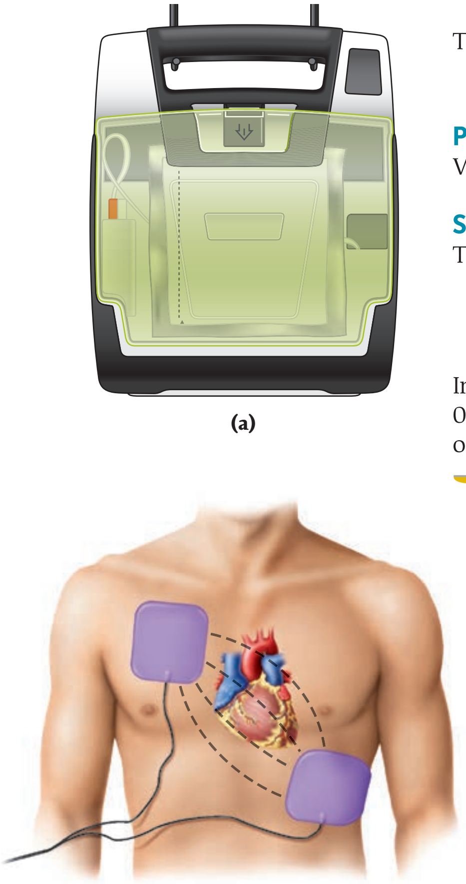
$$U_{total} = 192(1.73 \text{ MJ}) = 332 \text{ MJ}.$$

PROBLEM 2
What is the average power released by the power-conditioning modules during the laser pulse?

SOLUTION 2
The power is the energy per unit time, which is given by

$$P = \frac{\Delta U}{\Delta t} = \frac{332 \text{ MJ}}{400.\ \mu\text{s}} = \frac{332 \times 10^6 \text{ J}}{400. \times 10^{-6} \text{ s}} = 8.30 \times 10^{11} \text{ W} = 0.830 \text{ TW}$$

In comparison, the average electrical power generated in the United States in 2010 was 0.47 TW. Of course, the 0.830 TW of power delivered to the NIF laser is maintained for only a small fraction of a second.



(a)

(b)

FIGURE 4.22 (a) An automatic external defibrillator (AED) in its holder on a wall. (b) Schematic diagram showing where to place the hands-free electrodes.

Defibrillator

An important application of capacitors is the portable automatic external defibrillator (AED), a device designed to shock the heart of a person who is in ventricular fibrillation. A typical AED is shown in Figure 4.22a.

Being in ventricular fibrillation means that the heart is not beating in a regular pattern. Instead, the signals that control the beating of the heart are erratic, preventing the heart from performing its function of maintaining regular blood circulation throughout the body. This condition must be treated within a few minutes to avoid permanent damage or death. Having many AED devices located in accessible public places allows quick treatment of this condition.

An AED provides a pulse of electrical current intended to stimulate the heart to beat regularly. Typically, an AED is designed to analyze a person's heartbeat automatically, determine if the person is in ventricular fibrillation, and administer the electrical pulse if required. The operator of the AED must attach the electrodes of the AED to the chest of the person experiencing the problem and push the start button. The AED will do nothing if the person is not in ventricular defibrillation. If the AED determines that the person is in ventricular fibrillation, the AED will instruct the operator to press the button to initiate the electrical pulse. Note that an AED is not designed to restart a heart that is not beating. Rather it is designed to restore a regular heartbeat when the heart is beating erratically.

Typically, an AED delivers 150 J of electric energy to the patient, administered through a pair of electrodes that are attached to the chest area (see Figure 4.22b). This energy is stored by charging a capacitor through a special circuit from a low-voltage battery. This capacitor typically has a capacitance of 100. μF and is charged in 10. s. The power used during charging is

$$P = \frac{E}{t} = \frac{150\ \text{J}}{10.\ \text{s}} = 15\ \text{W}$$

which is within the capability of a simple battery. The energy of the capacitor is then discharged in 10. ms. The instantaneous power during the discharge is

$$P = \frac{E}{t} = \frac{150\ \text{J}}{10.\ \text{ms}} = 15\ \text{kW}$$

which is beyond the capability of a small, portable battery but well within the capabilities of a well-designed capacitor.

The energy stored in the capacitor is $U = \frac{1}{2}C(\Delta V)^2$. When the capacitor is charged, its potential difference is

$$\Delta V = \sqrt{\frac{2U}{C}} = \sqrt{\frac{2(150\ \text{J})}{100.\times 10^{-6}\ \text{F}}} = 1.7\ \text{kV}$$

When the AED delivers an electrical current, the capacitor is charged from a battery contained in the AED. The capacitor is then discharged through the person to stimulate the heart to beat in a regular manner. Most AEDs can deliver the electrical current many times without recharging the battery.

4.6 Capacitors with Dielectrics

The capacitors we have been discussing have air or vacuum between the plates. However, capacitors for just about any commercial application have an insulating material, called a **dielectric,** between the two plates. This dielectric serves several purposes: First, it maintains the separation between the plates. Second, the dielectric insulates the two plates from each other electrically. Third, the dielectric allows the capacitor to maintain a higher potential difference than it could with only air between the plates. Lastly, a dielectric increases the capacitance of the capacitor. We'll see that this ability to increase the capacitance is due to the molecular structure of the dielectric.

Filling the space between the plates of a capacitor completely with a dielectric increases its capacitance by a numerical factor called the **dielectric constant,** κ. We'll assume that the dielectric fills the entire volume between the capacitor plates, unless explicitly stated otherwise. Solved Problem 4.2 considers an example where the filling is only partial.

The capacitance, C, of a capacitor containing a dielectric with dielectric constant κ between its plates is given by

$$C = \kappa C_{\text{air}}, \tag{4.17}$$

where C_{air} is the capacitance of the capacitor without the dielectric.

Placing a dielectric between the plates of a capacitor has the effect of lowering the electric field between the plates (see Section 4.7) and allowing more charge to be stored in the capacitor. For example, the electric field between the plates of a parallel plate capacitor, given by equation 4.4, is modified for a parallel plate capacitor with a dielectric to

$$E = \frac{E_{\text{air}}}{\kappa} = \frac{q}{\kappa\varepsilon_0 A} = \frac{q}{\varepsilon A}. \tag{4.18}$$

The constant ε_0 is the electric permittivity of free space, previously encountered in Coulomb's Law. The right-hand side of equation 4.18 was obtained by replacing the factor $\kappa\varepsilon_0$ with ε,

Self-Test Opportunity 4.3

One way to increase the capacitance of a parallel plate capacitor, other than by adding a dielectric between the plates, is to decrease the distance between the plates. What is the minimum distance between the plates of a parallel plate capacitor if the space is filled with air and if the maximum potential difference between the plates is to be 100.0 V? (*Hint:* Table 4.1 may be useful.)

Concept Check 4.11

Suppose you charge a parallel plate capacitor with a dielectric between the plates using a battery and then remove the battery, isolating the capacitor and leaving it charged. You then remove the dielectric from between the plates. The potential difference between the plates will

a) increase.

b) decrease.

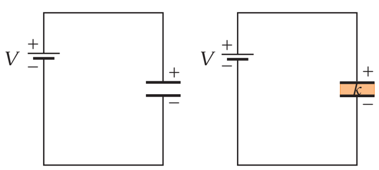
c) stay the same.

d) not be determinable.

Table 4.1 Dielectric Constants and Dielectric Strengths for Some Representative Materials

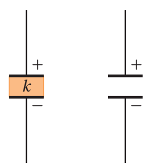
Material	Dielectric Constant, κ	Dielectric Strength (kV/mm)
Vacuum	1	
Air (1 atm)	1.00059	2.5
Liquid nitrogen	1.454	
Teflon	2.1	60
Polyethylene	2.25	50
Benzene	2.28	
Polystyrene	2.6	24
Lexan	2.96	16
Mica	3–6	150–220
Paper	3	16
Mylar	3.1	280
Plexiglas	3.4	30
Polyvinyl chloride (PVC)	3.4	29
Glass	5	14
Neoprene	16	12
Germanium	16	
Glycerin	42.5	
Water	80.4	65
Strontium titanate	310	8

Note that these values are approximate and are for room temperature.



(a) (b)

FIGURE 4.23 Parallel plate capacitor connected to a battery: (a) with no dielectric; (b) with a dielectric inserted between the plates.



(a) (b)

FIGURE 4.24 Isolated capacitor: (a) with dielectric and (b) with dielectric removed.

the **electric permittivity** of the dielectric. In other words, the electric permittivity of a dielectric is the product of the electric permittivity of free space (the vacuum) and the dielectric constant of the dielectric:

$$\varepsilon = \kappa\varepsilon_0 \tag{4.19}$$

Note that this replacement of ε_0 by ε is all that is needed to generalize expressions for the capacitance, such as equations 4.6, 4.8, and 4.10, from the values applicable for a capacitor with vacuum between its plates to those appropriate when the capacitor is completely filled with a dielectric. We can now see how the capacitance is increased by adding a dielectric between the plates. The potential difference across a parallel plate capacitor is

$$\Delta V = Ed = \frac{qd}{\kappa\varepsilon_0 A}$$

Therefore, we can write the capacitance as

$$C = \frac{q}{\Delta V} = \frac{\kappa\varepsilon_0 A}{d} = \kappa C_{\text{air}}.$$

The **dielectric strength** of a material is a measure of its ability to withstand potential difference. If the electric field strength in the dielectric exceeds the dielectric strength, the dielectric breaks down and begins to conduct charge between the plates via a spark, which usually destroys the capacitor. Thus, a useful capacitor must contain a dielectric that not only provides a given capacitance but also enables the device to hold the required potential difference without breaking down. Capacitors are normally specified by the value of their capacitance and by the maximum potential difference that they are designed to handle.

The dielectric constant of vacuum is defined to be 1, and the dielectric constant of air is close to 1.0. The dielectric constants and dielectric strengths of air and of other common materials used as dielectrics are listed in Table 4.1.

EXAMPLE 4.5 Parallel Plate Capacitor with a Dielectric

PROBLEM 1
Consider a parallel plate capacitor without a dielectric and with capacitance $C = 2.00\ \mu\text{F}$ connected to a battery with potential difference $\Delta V = 12.0$ V (Figure 4.23a). What is the charge stored in the capacitor?

SOLUTION 1
Using the definition of capacitance (equation 4.1), we have

$$q = C\Delta V = \left(2.00 \times 10^{-6}\ \text{F}\right)(12.0\ \text{V}) = 2.40 \times 10^{-5}\ \text{C}$$

PROBLEM 2
In Figure 4.23b, a dielectric with $\kappa = 2.50$ has been inserted between the plates of the capacitor, completely filling the space between them. Now what is the charge on the capacitor?

SOLUTION 2
The capacitance of the capacitor is increased by the dielectric:

$$C = \kappa C_{\text{air}}.$$

The charge is

$$q = \kappa C_{\text{air}}\Delta V = (2.50)\left(2.00 \times 10^{-6}\ \text{F}\right)(12.0\ \text{V}) = 6.00 \times 10^{-5}\ \text{C}.$$

The charge on the capacitor increases when the capacitance increases because the battery maintains a constant potential difference across the capacitor. The battery provides the additional charge until the capacitor is fully charged.

PROBLEM 3
Now suppose the capacitor is disconnected from the battery (Figure 4.24a). The capacitor, which is now isolated, maintains its charge of $q = 6.00 \times 10^{-5}$ C and its potential difference of

$\Delta V = 12.0$ V. What happens to the charge and potential difference if the dielectric is removed, keeping the capacitor isolated (Figure 4.24b)?

SOLUTION 3

The charge on the isolated capacitor cannot change when the dielectric is removed because there is nowhere for the charge to flow. Thus, the potential difference on the capacitor is

$$\Delta V = \frac{q}{C} = \frac{6.00 \times 10^{-5}\ \text{C}}{2.00 \times 10^{-6}\ \text{F}} = 30.0\ \text{V}$$

The potential difference increases because removing the dielectric increases the electric field and the resulting potential difference between the plates.

PROBLEM 4

Does removing the dielectric change the energy stored in the capacitor?

SOLUTION 4

The energy stored in a capacitor is given by equation 4.15. Before the dielectric was removed, the energy in the capacitor was

$$U = \tfrac{1}{2}C(\Delta V)^2 = \tfrac{1}{2}\kappa C_{\text{air}}(\Delta V)^2 = \tfrac{1}{2}(2.50)\left(2.00 \times 10^{-6}\ \text{F}\right)(12.0\ \text{V})^2 = 3.60 \times 10^{-4}\ \text{J}.$$

After the dielectric is removed, the energy is

$$U = \tfrac{1}{2}C_{\text{air}}(\Delta V)^2 = \tfrac{1}{2}\left(2.00 \times 10^{-6}\ \text{F}\right)(30.0\ \text{V})^2 = 9.00 \times 10^{-4}\ \text{J}.$$

The increase in energy from 3.60×10^{-4} J to 9.00×10^{-4} J when the dielectric is removed is due to the work done on the dielectric in pulling it out of the electric field between the plates.

Concept Check 4.12

What would happen if the dielectric in the capacitor of Example 4.5 were pulled halfway out and then released?

a) The dielectric would be pulled back into the capacitor.

b) The dielectric would heat up rapidly.

c) The dielectric would be pushed out of the capacitor.

d) The capacitor plates would heat up rapidly.

e) The dielectric would remain in the halfway position, and no heating would be observed.

SOLVED PROBLEM 4.2 Capacitor Partially Filled with a Dielectric

PROBLEM

A parallel plate capacitor is constructed of two square conducting plates with side length $L = 10.0$ cm (Figure 4.25a). The distance between the plates is $d = 0.250$ cm. A dielectric with dielectric constant $\kappa = 15.0$ and thickness 0.250 cm is inserted between the plates. The dielectric is $L = 10.0$ cm wide and $L/2 = 5.00$ cm long, as shown in Figure 4.25a. What is the capacitance of this capacitor?

SOLUTION

THINK We have a parallel plate capacitor that is partially filled with a dielectric. We can treat this capacitor as two capacitors in parallel. One capacitor is a parallel plate capacitor with plate area $A = L(L/2)$ and air between the plates; the second capacitor is a parallel plate capacitor with plate area $A = L(L/2)$ and a dielectric between the plates.

SKETCH Figure 4.25b shows a representation of the partially filled capacitor as two capacitors in parallel: one filled with a dielectric and the other filled with air.

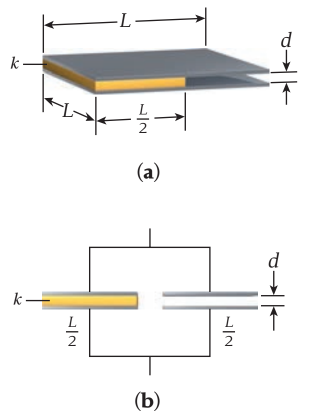


FIGURE 4.25 (a) A parallel plate capacitor with square plates of side length L separated by distance d with a dielectric that is L wide and $L/2$ long and has a dielectric constant κ inserted between the plates.
(b) The partially filled capacitor represented as two capacitors in parallel.

RESEARCH The capacitance, C_1, of a parallel plate capacitor is given by equation 4.6:

$$C_1 = \frac{\varepsilon_0 A}{d}$$

where A is the area of the plates and d is the separation between them. If a dielectric is placed between the plates, the capacitance becomes

$$C_2 = \kappa\frac{\varepsilon_0 A}{d}$$

where κ is the dielectric constant. For two capacitors, C_1 and C_2, in parallel, the effective capacitance, C_{12}, is given by

$$C_{12} = C_1 + C_2$$

– Continued

SIMPLIFY Substituting the expressions for the two individual capacitances into the sum, we get

$$C_{12} = \frac{\varepsilon_0 A}{d} + \kappa \frac{\varepsilon_0 A}{d} = (\kappa + 1)\frac{\varepsilon_0 A}{d} \qquad \text{(i)}$$

The area of the plates for each capacitor is

$$A = (L)(L\ /\ 2) = L^2/2$$

Inserting the expression for the area into equation (i) gives the capacitance of the partially filled capacitor:

$$C_{12} = (\kappa + 1)\frac{\varepsilon_0 \left(L^2/2\right)}{d} = \frac{(\kappa + 1)\varepsilon_0 L^2}{2d}$$

CALCULATE Putting in the numerical values, we get

$$C_{12} = \frac{(15.0 + 1)\left(8.85 \times 10^{-12}\ \text{F/m}\right)(0.100\ \text{m})^2}{2(0.00250\ \text{m})} = 2.832 \times 10^{-10}\ \text{F}$$

ROUND We report our result to three significant figures:

$$C_{12} = 2.83 \times 10^{-10}\ \text{F} = 283\ \text{pF}$$

DOUBLE-CHECK To double-check our answer, we calculate the capacitance of the capacitor without any dielectric:

$$C = \frac{\varepsilon_0 A}{d} = \frac{\left(8.85 \times 10^{-12}\ \text{F/m}\right)(0.100\ \text{m})^2}{0.0025\ \text{m}} = 3.54 \times 10^{-11}\ \text{F} = 35.4\ \text{pF}$$

We then calculate the capacitance of the capacitor if completely filled with dielectric:

$$C = \kappa \frac{\varepsilon_0 A}{d} = (15.0)(35.4\ \text{pF}) = 5.31 \times 10^{-10}\ \text{F} = 531\ \text{pF}$$

Our result for the partially filled capacitor is half of the sum of these two results, so it seems reasonable.

EXAMPLE 4.6 Capacitance of a Coaxial Cable

Coaxial cables are used to transport signals, for example, TV signals, between devices with minimum interference from the surroundings. A 20.0 m-long coaxial cable is composed of a conductor and a coaxial conducting shield around the conductor. The space between the conductor and the shield is filled with polystyrene. The radius of the conductor is 0.250 mm, and the radius of the shield is 2.00 mm (Figure 4.26).

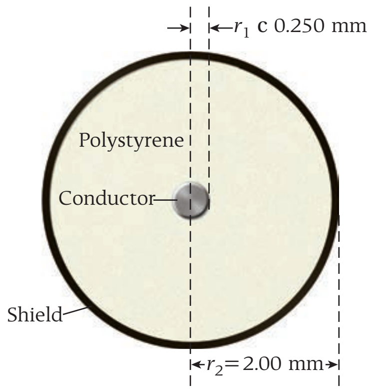


FIGURE 4.26 Cross section of a coaxial cable.

PROBLEM

What is the capacitance of the coaxial cable?

SOLUTION

We can think of the conductor of the coaxial cable as a cylinder because all the charge on the conductor resides on its surface. From Table 4.1, the dielectric constant for polystyrene is 2.6. We can treat the coaxial cable as a cylindrical capacitor with $r_1 = 0.250$ mm and $r_2 = 2.00$ mm, filled with a dielectric with $\kappa = 2.6$. Then, we can use equation 4.8 to find the capacitance of the coaxial cable:

$$C = \kappa \frac{2\pi\varepsilon_0 L}{\ln(r_2/r_1)} = \frac{2.6(2\pi)\left(8.85 \times 10^{-12}\ \text{F/m}\right)(20.0\ \text{m})}{\ln\left[\left(2.00 \times 10^{-3}\text{m}\right)/\left(2.50 \times 10^{-4}\ \text{m}\right)\right]} = 1.4 \times 10^{-9}\ \text{F} = 1.4\ \text{nF}$$

One interesting application of capacitance and the dielectric constant is in measuring liquid nitrogen levels in cryostats (containers insulated to maintain cold temperatures). It is often difficult to conduct a visual exam to determine how much liquid nitrogen is left in a cryostat. However, if one determines the capacitance, C, of the empty cryostat, then, when completely filled with liquid nitrogen, the cryostat should have a capacitance of $\kappa C = 1.454C$, because liquid nitrogen has a dielectric constant of 1.454. The capacitance varies smoothly as a function of the fullness between the maximum value $\kappa C = 1.454C$ for the completely full cryostat and the value C for the empty cryostat, giving an easy way to determine how full the cryostat is.

SOLVED PROBLEM 4.3 Charge on a Cylindrical Capacitor

PROBLEM

Consider a cylindrical capacitor with inner radius $r_1 = 10.0$ cm, outer radius $r_2 = 12.0$ cm, and length $L = 50.0$ cm (Figure 4.27a). A dielectric with dielectric constant $\kappa = 12.5$ fills the volume between the two cylinders (Figure 4.27b). The capacitor is connected to a 100.0-V battery and charged completely. What is the charge on the capacitor?

SOLUTION

THINK We have a cylindrical capacitor filled with a dielectric. When the capacitor is connected to the battery, charge will accumulate on the capacitor until the capacitor is fully charged. We can calculate the amount of charge on the capacitor.

SKETCH A circuit diagram with the cylindrical capacitor connected to a battery is shown in Figure 4.28.

RESEARCH The capacitance, C, of a cylindrical capacitor is given by equation 4.8:

$$C = \frac{2\pi\varepsilon_0 L}{\ln(r_2/r_1)}$$

where r_1 is the inner radius of the capacitor, r_2 is the outer radius of the capacitor, and L is the length of the capacitor. With a dielectric between the plates, the capacitance becomes

$$C = \kappa\frac{2\pi\varepsilon_0 L}{\ln(r_2/r_1)} \qquad \text{(i)}$$

where κ is the dielectric constant. For a capacitor with capacitance C charged to a potential difference ΔV, the charge q is given by equation 4.1:

$$q = C\Delta V \qquad \text{(ii)}$$

SIMPLIFY Combining equations (i) and (ii) gives

$$q = C\Delta V = \left(\kappa\frac{2\pi\varepsilon_0 L}{\ln(r_2/r_1)}\right)\Delta V = \frac{2\kappa\pi\varepsilon_0 L\Delta V}{\ln(r_2/r_1)}$$

CALCULATE Putting in the numerical values, we get

$$q = \frac{2\kappa\pi\varepsilon_0 L\Delta V}{\ln(r_2/r_1)} = \frac{2(12.5)\pi\left(8.85\times10^{-12}\text{ F/m}\right)(0.500\text{ m})(100.0\text{ V})}{\ln[(0.120\text{ m})/(0.100\text{ m})]} = 19.0618\times10^{-8}\text{ C}$$

ROUND We report our result to three significant figures:

$$q = 19.1\times10^{-8}\text{ C} = 191\text{ nC}$$

DOUBLE-CHECK Our answer is a very small fraction of a coulomb of charge, so it seems reasonable.

4.7 Microscopic Perspective on Dielectrics

Let's consider what happens at the atomic and molecular level when a dielectric is placed in an electric field. There are two types of dielectric materials: polar dielectrics and nonpolar dielectrics.

A **polar dielectric** is a material composed of molecules that have a permanent electric dipole moment due to their structure. A common example of such a molecule is

Concept Check 4.13

State whether each of the following statements about an *isolated* parallel plate capacitor is true or false.

a) When the distance between the plates of the capacitor is doubled, the energy stored in the capacitor doubles.
b) Increasing the distance between the plates increases the electric field between the plates.
c) When the distance between the plates is halved, the charge on the plates stays the same.
d) Inserting a dielectric between the plates increases the charge on the plates.
e) Inserting a dielectric between the plates decreases the energy stored in the capacitor.

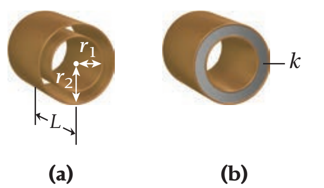


FIGURE 4.27 (a) A cylindrical capacitor with inner radius r_1, outer radius r_2, and length L. (b) A dielectric with dielectric constant κ is inserted between the cylinders.

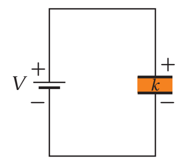


FIGURE 4.28 A cylindrical capacitor connected to a battery.

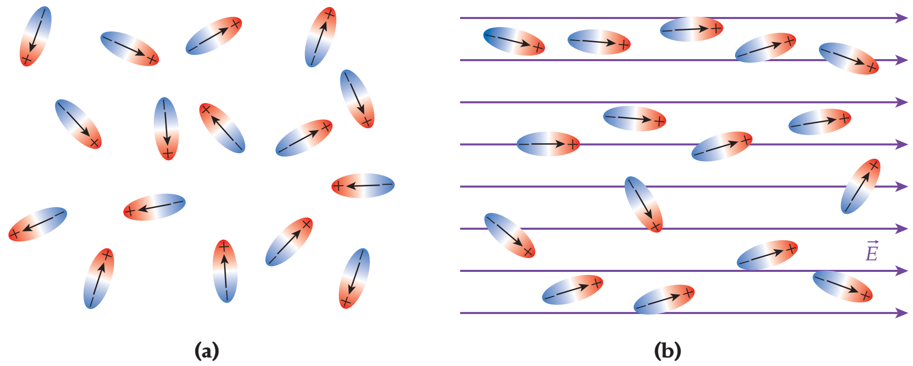


FIGURE 4.29 Polar molecules: (a) randomly distributed and (b) oriented by an external electric field.

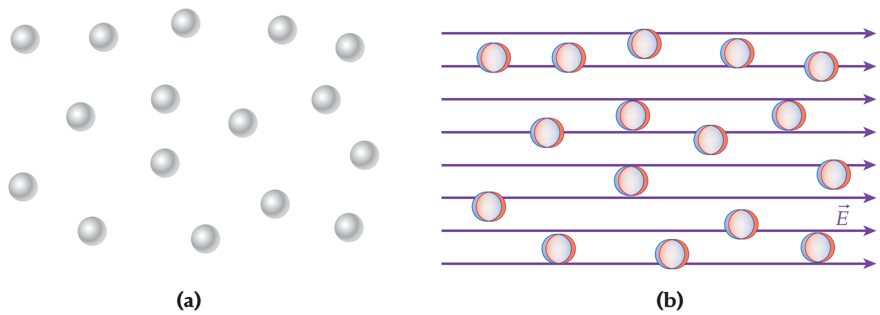


FIGURE 4.30 Nonpolar molecules: (a) with no electric dipole moment and (b) with an electric dipole moment induced by an external electric field.

water. Normally, the directions of electric dipoles are randomly distributed (Figure 4.29a). However, when an electric field is applied to these polar molecules, they tend to align with the field (Figure 4.29b).

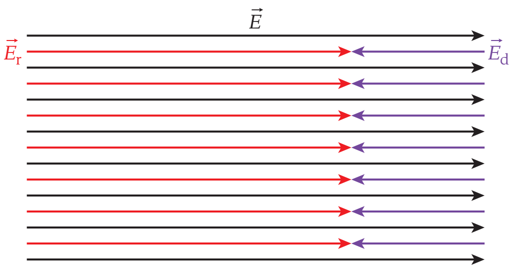


FIGURE 4.31 Partial cancellation of the applied electric field across a parallel plate capacitor by the electric dipoles of a dielectric.

A **nonpolar dielectric** is a material composed of atoms or molecules that have no inherent electric dipole moment (Figure 4.30a). These atoms or molecules can be induced to have a dipole moment under the influence of an external electric field (Figure 4.30b). The opposite directions of the electric forces acting on the negative and positive charges in the atom or molecule displace these two charge distributions and produce an induced electric dipole moment.

In both polar and nonpolar dielectrics, the fields resulting from the aligned electric dipole moments tend to partially cancel the original external electric field (Figure 4.31). For an electric field, $\vec{E}$ applied across a capacitor with a dielectric between the plates, the resulting electric field, $\vec{E}_r$ inside the capacitor is just the original field plus the electric field induced in the dielectric material, $\vec{E}_d$

$$\vec{E}_r = \vec{E} + \vec{E}_d$$

or

$$E_r = E - E_d$$

Note that the resulting electric field points in the same direction as the original field but is smaller in magnitude. The dielectric constant is given by $\kappa = E/E_r$.

Electrolytic Capacitors

Instead of consisting of two conducting plates with the gap filled with a dielectric, a capacitor can have one of the plates replaced by an ion-conducting liquid, or electrolyte, that is, a liquid that contains ions that can move freely through it. Most commonly, these *electrolytic capacitors* are constructed of two pieces of aluminum foil, of which one is coated with an insulating oxide layer. The two foil pieces are separated by a paper spacer, which is saturated with the electrolyte. The oxide layer typically has a dielectric constant of ~10 and a dielectric strength of 20–30 kV/mm. This layer can therefore be very thin, and this type of electrolytic capacitor has a relatively high charge capacity.

The main drawback of an electrolytic capacitor is that it is polarized and one electrode must always be maintained at a positive potential relative to the other. A reverse potential difference as low as 1–2 V will destroy the oxide layer and lead to short-circuiting and destruction of the capacitor.

Supercapacitors

As we've seen in this chapter, 1 F is a huge amount of capacitance. Even the National Ignition Facility (NIF), which needs the highest possible energy storage, uses only 300-μF capacitors. However, it is possible to create *supercapacitors* (also called *ultracapacitors*) with vastly greater capacitance. This is accomplished by using a material with a very large surface area between the capacitor plates. Activated charcoal is one possibility, as it has a very large surface area because of its foamlike structure at a nanoscale level. Two layers of activated charcoal are given charges of opposite polarity and are separated by an insulating material (represented by the red line in Figure 4.32b). This allows each side of the supercapacitor to store oppositely charged free ions from the electrolyte. The separation between the electrolyte ions and the charges on the activated charcoal is typically on the order of nanometers (nm), that is, millions of times smaller than in conventional capacitors. The activated charcoal provides a surface area many orders of magnitude larger than in conventional capacitors. Since, as noted in Section 4.3, the capacitance is proportional to the surface area and inversely proportional to the plate separation, this technology has resulted in commercially available capacitors with capacitances on the order of kilofarads (kF), that is, millions of times larger than those used in the NIF.

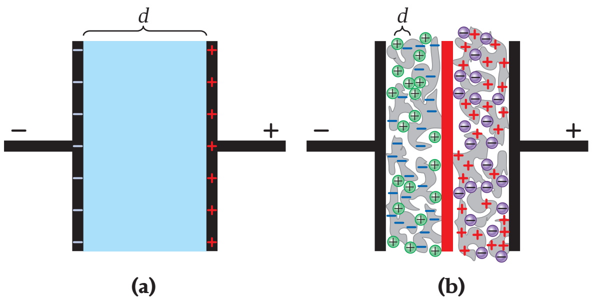


FIGURE 4.32 Comparison of (a) a conventional parallel plate capacitor and (b) a supercapacitor filled with activated charcoal.

Why is the NIF not using supercapacitors? The answer is that these supercapacitors can only function with potential differences of up to 2–3 V. The highest-capacity commercially available supercapacitors have capacitance values of up to 5 kF. Using $U = \frac{1}{2}C(\Delta V)^2$ and $\Delta V = 2$ V shows that a supercapacitor can hold 10 kJ. The 300-μF capacitors used at NIF, when charged to 24 kV, can hold 86.4 kJ. In addition, they can also be discharged much more rapidly, which is crucial for fulfilling the high power requirement of the NIF's laser.

However, supercapacitors can reach energy storage capabilities that rival those of conventional batteries. In addition, supercapacitors can be charged and discharged millions of times, compared to perhaps thousands of times for rechargeable batteries. This, along with their very short charging time, makes them potentially suitable for many applications. For example, there is intense research into using these supercapacitors for electric vehicles. A bus based on this energy storage technology, named *capabus*, is currently in use in Shanghai, China (see Figure 4.33).

A promising line of research on improving the potential difference that supercapacitors can employ is investigating the use of carbon nanotubes and graphene instead of activated charcoal. The first laboratory prototypes are very promising, and commercial products based on this approach could be in use within a few years. Research has also succeeded in improving the energy storage capabilities of supercapacitors while lowering the prices. A 5kF supercapacitor that cost about $5,000 in 2000 can now be purchased for 1% of that price.

FIGURE 4.33 Supercapacitor-powered bus recharging at a bus stop in Shanghai, China.

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- The capacitance of a capacitor–its ability to store charge–is defined in terms of the charge, q, that can be stored on the capacitor and the potential difference, V, across the plates: $q = C\Delta V$.
- The farad is the unit of capacitance: $1\text{ F} = \frac{1\text{ C}}{1\text{ V}}$
- The capacitance of a parallel plate capacitor with plates of area A with a vacuum (or air) between the plates separated by distance d is given by $C = \frac{\varepsilon_0 A}{d}$
- The capacitance of a cylindrical capacitor of length L consisting of two collinear cylinders with a vacuum (or air) between the cylinders with inner radius r_1 and outer radius r_2 is given by $C = \frac{2\pi\varepsilon_0 L}{\ln(r_2/r_1)}$
- The capacitance of a spherical capacitor consisting of two concentric spheres with vacuum (or air) between the spheres with inner radius r_1 and outer radius r_2 is given by $C = 4\pi\varepsilon_0 \frac{r_1 r_2}{r_2 - r_1}$
- The electric energy density, u, between the plates of a parallel plate capacitor with vacuum (or air) between the plates is given by $u = \frac{1}{2}\varepsilon_0 E^2$
- A system of n capacitors connected in parallel in a circuit can be replaced by an equivalent capacitance given by the sum of the capacitances of the capacitors: $C_{eq} = \sum_{i=1}^{n} C_i$
- A system of n capacitors connected in series in a circuit can be replaced by an equivalent capacitance given by the reciprocal of the sum of the reciprocal capacitances of the capacitors: $\frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i}$
- When the space between the plates of a capacitor is filled with a dielectric whose dielectric constant is κ, the capacitance increases relative to the capacitance in air: $C = \kappa C_{air}$.
- The electric potential energy stored in a capacitor is $U = \frac{1}{2}q^2/C = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}q\Delta V$

ANSWERS TO SELF-TEST OPPORTUNITIES

4.1 The electric field remains constant.

4.2 Series: $\frac{1}{C_{eq}} = \frac{4}{C} \Rightarrow C_{eq} = \frac{1}{4}C = 2.50\,\mu\text{F}$

Parallel: $C_{eq} = 4C = 40.0\ \mu\text{F}$

4.3 $100\text{ V} = d(2500\text{ V/mm}) \Rightarrow d = 0.04\text{ mm}$.

PROBLEM-SOLVING GUIDELINES

1. Remember that saying that a capacitor has charge q means that one plate has charge $+q$ and the other plate has charge $-q$. Be sure you understand how a charge applied to a capacitor is distributed between the two conducting plates; review Example 4.3 if you're uncertain about this.

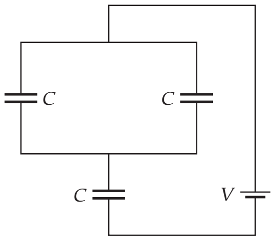
2. It is always a good idea to draw a circuit diagram when solving a problem involving a circuit, if one is not supplied. Identifying series and parallel connections can take some practice, but is usually an important first step in reducing a complicated-looking circuit to an equivalent circuit that is straightforward to deal with. Remember that capacitors connected in series all have the same charge, and capacitors connected in parallel all have the same potential difference.

3. You can remember most of the important results for a capacitor with a dielectric if you remember that a dielectric increases the capacitance. (This is what makes a dielectric useful.) If your calculations show a reduced capacitance with a dielectric, recheck your work.

4. You can calculate the energy stored in a capacitor if you know two out of these three quantities: the charge on a plate, the capacitance of the capacitor, and the potential difference between the plates. Make sure you take advantage of equation 4.15 in the appropriate form.

MULTIPLE-CHOICE QUESTIONS

4.1 In the circuit shown in the figure, the capacitance for each capacitor is C. The equivalent capacitance for these three capacitors is



a) $\frac{1}{3}C$.
b) $\frac{2}{3}C$.
c) $\frac{2}{5}C$.
d) $\frac{3}{5}C$.
e) C.
f) $\frac{5}{3}C$.

4.2 A parallel plate capacitor of capacitance C has plates of area A with distance d between them. When the capacitor is connected to a battery supplying potential difference V, it has a charge of magnitude Q on its plates. While the capacitor is connected to the battery, the distance between the plates is decreased by a factor of 3. The magnitude of the charge on the plates and the capacitance will then be

a) $\frac{1}{3}Q$ and $\frac{1}{3}C$.
b) $\frac{1}{3}Q$ and $3C$.
c) $3Q$ and $3C$.
d) $3Q$ and $\frac{1}{3}C$.

4.3 The distance between the plates of a parallel plate capacitor is reduced by half and the area of the plates is doubled. What happens to the capacitance?

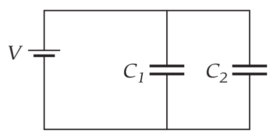
a) It remains unchanged.
b) It doubles.
c) It quadruples.
d) It is reduced by half.

4.4 Which of the following capacitors has the largest charge?

a) a parallel plate capacitor with an area of 10 cm^2 and a plate separation of 2 mm connected to a 10-V battery
b) a parallel plate capacitor with an area of 5 cm^2 and a plate separation of 1 mm connected to a 10-V battery
c) a parallel plate capacitor with an area of 10 cm^2 and a plate separation of 4 mm connected to a 5-V battery
d) a parallel plate capacitor with an area of 20 cm^2 and a plate separation of 2 mm connected to a 20-V battery
e) All of the capacitors have the same charge.

4.5 Two identical parallel plate capacitors are connected in a circuit as shown in the figure. Initially the space between the plates of each capacitor is filled with air. Which of the following changes will double the total amount of charge stored on both capacitors with the same applied potential difference?

a) Fill the space between the plates of C_1 with glass (dielectric constant of 4) and leave C_2 as is.
b) Fill the space between the plates of C_1 with Teflon (dielectric constant of 2) and leave C_2 as is.



c) Fill the space between the plates of both C_1 and C_2 with Teflon (dielectric constant of 2).
d) Fill the space between the plates of both C_1 and C_2 with glass (dielectric constant of 4).

4.6 The space between the plates of an isolated parallel plate capacitor is filled with a slab of dielectric material. The magnitude of the charge Q on each plate is kept constant. If the dielectric material is removed the energy stored in the capacitor

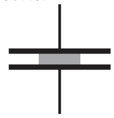
a) increases.
b) stays the same.
c) decreases.
d) may increase or decrease.

4.7 Which of the following is (are) proportional to the capacitance of a parallel plate capacitor?

a) the charge stored on each conducting plate
b) the potential difference between the two plates
c) the separation distance between the two plates
d) the area of each plate
e) all of the above
f) none of the above

4.8 A dielectric with the dielectric constant $\kappa = 4$ is inserted into a parallel plate capacitor, filling $\frac{1}{3}$ of the volume, as shown in the figure. If the capacitance of the capacitor without the dielectric is C, what is the capacitance of the capacitor with the dielectric?

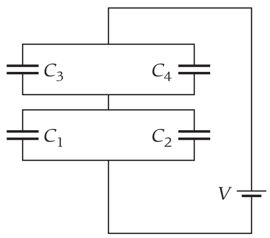
a) $0.75C$
b) C
c) $2C$
d) $4C$
e) $6C$



4.9 A parallel plate capacitor is connected to a battery for charging. After some time, while the battery is still connected to the capacitor, the distance between the capacitor plates is doubled. Which of the following is (are) true?

a) The electric field between the plates is halved.
b) The potential difference of the battery is halved.
c) The capacitance doubles.
d) The potential difference across the plates does not change.
e) The charge on the plates does not change.

4.10 Referring to the figure, decide which of the following equations is (are) true. Assume that all of the capacitors have different capacitances. The potential difference across capacitor C_1 is V_1. The potential difference across capacitor C_2 is V_2. The potential difference across capacitor C_3 is V_3. The potential difference across capacitor C_4 is V_4. The charge stored in capacitor C_1 is q_1. The charge stored in capacitor C_2 is q_2. The charge stored in capacitor C_3 is q_3. The charge stored in capacitor C_4 is q_4.



a) $q_1 = q_3$
b) $V_1 + V_2 = V$
c) $q_1 + q_2 = q_3 + q_4$
d) $V_1 + V_2 = V_3 + V_4$
e) $V_1 + V_3 = V$

4.11 You have N identical capacitors, each with capacitance C, connected in series. The equivalent capacitance of this system of capacitors is

a) NC.
b) C/N.
c) N^2C.
d) C/N^2.
e) C.

4.12 You have N identical capacitors, each with capacitance C, connected in parallel. The equivalent capacitance of this system of capacitors is

a) NC.
b) C/N.
c) N^2C.
d) C/N^2.
e) C.

4.13 When a dielectric is placed between the plates of a charged, isolated capacitor, the electric field inside the capacitor

a) increases.
b) decreases.
c) stays the same.
d) increases if the charge on the plates is positive.
e) decreases if the charge on the plates is positive.

4.14 A parallel plate capacitor with a dielectric filling the volume between its plates is charged. The charge is

a) stored on the plates.
b) stored on the dielectric.
c) stored both on the plates and in the dielectric.

CONCEPTUAL QUESTIONS

4.15 Must a capacitor's plates be made of conducting material? What would happen if two insulating plates were used instead of conducting plates?

4.16 Does it take more work to separate the plates of a charged parallel plate capacitor while it remains connected to the charging battery or after it has been disconnected from the charging battery?

4.17 When working on a piece of equipment, electricians and electronics technicians sometimes attach a grounding wire to the equipment even after turning the device off and unplugging it. Why would they do this?

4.18 Table 4.1 does not list a value of the dielectric constant for any good conductor. What value would you assign to it?

4.19 A parallel plate capacitor is charged with a battery and then disconnected from the battery, leaving a certain amount of energy stored in the capacitor. The separation between the plates is then increased. What happens to the energy stored in the capacitor? Discuss your answer in terms of energy conservation.

4.20 You have an electric device containing a 10.0 μF capacitor, but an application requires an 18.0 μF capacitor. What modification can you make to your device to increase its capacitance to 18.0 μF?

4.21 Two capacitors with capacitances C_1 and C_2 are connected in series. Show that, no matter what the values of C_1 and C_2 are, the equivalent capacitance is always less than the smaller of the two capacitances.

4.22 Two capacitors, with capacitances C_1 and C_2, are connected in series. A potential difference, V_0, is applied across the combination of capacitors. Find the potential differences V_1 and V_2 across the individual capacitors, in terms of V_0, C_1, and C_2.

4.23 An isolated solid spherical conductor of radius 5.00 cm is surrounded by dry air. It is given a charge and acquires potential V, with the potential at infinity assumed to be zero.

a) Calculate the maximum magnitude V can have.

b) Explain clearly and concisely *why* there is a maximum.

4.24 A parallel plate capacitor with capacitance C is connected to a power supply that maintains a constant potential difference, V. A slab of dielectric, with dielectric constant κ, is then inserted into and completely fills the previously empty space between the plates.

a) What was the energy stored on the capacitor before the insertion of the dielectric?

b) What was the energy stored after the insertion of the dielectric?

c) Was the dielectric pulled into the space between the plates, or did it have to be pushed in? Explain.

4.25 A parallel plate capacitor with square plates of edge length L separated by a distance d is given a charge Q, then disconnected from its power source. A close-fitting square slab of dielectric, with dielectric constant κ, is then inserted into the previously empty space between the plates. Calculate the force with which the slab is pulled into the capacitor during the insertion process.

4.26 A cylindrical capacitor has an outer radius R and a separation d between the cylinders. Determine what the capacitance approaches in the limit where $d \ll R$. (*Hint:* Express the capacitance in terms of the ratio d/R and then examine what happens as that ratio becomes very small compared to 1.) Explain why the limit on the capacitance makes sense.

4.27 A parallel plate capacitor is constructed from two plates of different areas. If this capacitor is initially uncharged and then connected to a battery, how will the amount of charge on the big plate compare to the amount of charge on the small plate?

4.28 A parallel plate capacitor is connected to a battery. As the plates are moved farther apart, what happens to each of the following?

a) the potential difference across the plates

b) the charge on the plates

c) the electric field between the plates

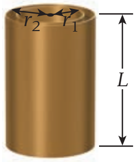
EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

Section 4.3

4.29 Supercapacitors, with capacitances of 1.00 F or more, are made with plates that have a spongelike structure with a very large surface area. Determine the surface area of a supercapacitor that has a capacitance of 1.00 F and an effective separation between the plates of $d = 1.00$ mm.

4.30 A potential difference of 100. V is applied across the two collinear conducting cylinders shown in the figure. The radius of the outer cylinder is 15.0 cm, the radius of the inner cylinder is 10.0 cm, and the length of the two cylinders is 40.0 cm. How much charge is applied to each of the cylinders? What is the magnitude of the electric field between the two cylinders?



4.31 What is the radius of an isolated spherical conductor that has a capacitance of 1.00 F?

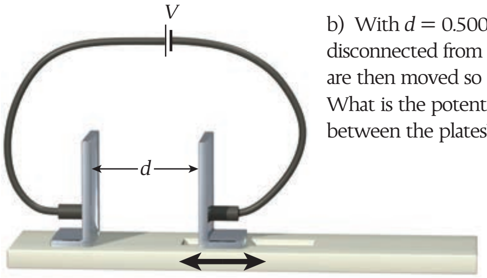
4.32 A spherical capacitor is made from two thin concentric conducting shells. The inner shell has radius r_1, and the outer shell has radius r_2. What is the fractional difference in the capacitances of this spherical capacitor and a parallel plate capacitor made from plates that have the same area as the inner sphere and the same separation $d = r_2 - r_1$ between them?

4.33 Calculate the capacitance of the Earth. Treat the Earth as an isolated spherical conductor of radius 6371 km.

4.34 Two concentric metal spheres are found to have a potential difference of 900. V when a charge of 6.726×10^{-8} C is applied to them. The radius of the outer sphere is 0.210 m. What is the radius of the inner sphere?

•4.35 A capacitor consists of two parallel plates, but one of them can move relative to the other as shown in the figure. Air fills the space between the plates, and the capacitance is 32.0 pF when the separation between plates is $d = 0.500$ cm.

a) A battery supplying a potential difference $V = 9.00$ V is connected to the plates. What is the charge distribution, σ, on the left plate? What are the capacitance, C', and the charge distribution, σ', when d is changed to 0.250 cm?



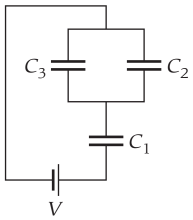
b) With $d = 0.500$ cm, the battery is disconnected from the plates. The plates are then moved so that $d = 0.250$ cm. What is the potential difference V', between the plates?

Section 4.4

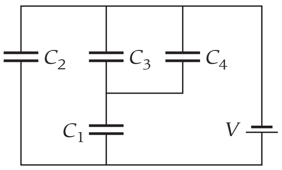
4.36 Determine all the values of equivalent capacitance you can create using any combination of three identical capacitors with capacitance C.

4.37 A large parallel plate capacitor with plates that are square with side length 1.00 cm and are separated by a distance of 1.00 mm is dropped and damaged. Half of the areas of the two plates are pushed closer together to a distance of 0.500 mm. What is the capacitance of the damaged capacitor?

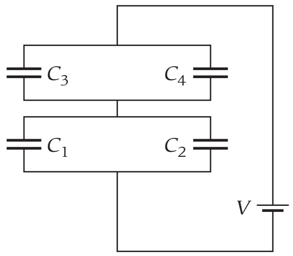
4.38 Three capacitors with capacitances $C_1 = 3.10$ nF, $C_2 = 1.30$ nF, and $C_3 = 3.70$ nF are wired to a battery with $V = 14.9$ V, as shown in the figure. What is the potential drop across capacitor C_2?



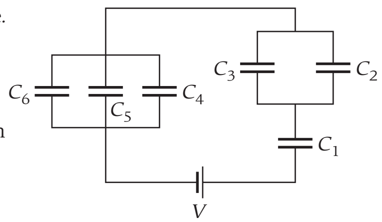
4.39 Four capacitors with capacitances $C_1 = 3.50$ nF, $C_2 = 2.10$ nF, $C_3 = 1.30$ nF, and $C_4 = 4.90$ nF are wired to a battery with $V = 10.3$ V, as shown in the figure. What is the equivalent capacitance of this set of capacitors?



4.40 The capacitors in the circuit shown in the figure have capacitances $C_1 = 18.0$ µF, $C_2 = 11.3$ µF, $C_3 = 33.0$ µF, and $C_4 = 44.0$ µF. The potential difference is $V = 10.0$ V. What is the total charge the power source must supply to charge this arrangement of capacitors?



4.41 Six capacitors are connected as shown in the figure.



a) If $C_3 = 2.300$ nF, what does C_2 have to be to yield an equivalent capacitance of 5.000 nF for the combination of the two capacitors?

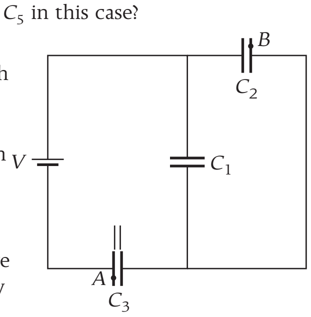
b) For the same values of C_2 and C_3 as in part (a), what is the value of C_1 that will give an equivalent capacitance of 1.914 nF for the combination of the three capacitors?

c) For the same values of C_1, C_2, and C_3 as in part (b), what is the equivalent capacitance of the whole set of capacitors if the values of the other capacitances are $C_4 = 1.300$ nF, $C_5 = 1.700$ nF, and $C_6 = 4.700$ nF?

d) If a battery with a potential difference of 11.70 V is connected to the capacitors as shown in the figure, what is the total charge on the six capacitors?

e) What is the potential drop across C_5 in this case?

•**4.42** A potential difference of $V = 80.0$ V is applied across a circuit with capacitances $C_1 = 15.0$ nF, $C_2 = 7.00$ nF, and $C_3 = 20.0$ nF, as shown in the figure. What is the magnitude and sign of q_{3l}, the charge on the left plate of C_3 (marked by point A)? What is the electric potential, V_3, across C_3? What is the magnitude and sign of the charge q_{2r}, on the right plate of C_2 (marked by point B)?



•**4.43** Fifty parallel plate capacitors are connected in series. The distance between the plates is d for the first capacitor, $2d$ for the second capacitor, $3d$ for the third capacitor, and so on. The area of the plates is the same for all the capacitors. Express the equivalent capacitance of the whole set in terms of C_1 (the capacitance of the first capacitor).

•**4.44** A 5.00 nF capacitor charged to 60.0 V and a 7.00-nF capacitor charged to 40.0 V are connected negative plate to negative plate. What is the final charge on the 7.00-nF capacitor?

Section 4.5

4.45 When a capacitor has a charge of magnitude 60.0 µC on each plate, the potential difference across the plates is 12.0 V. How much energy is stored in this capacitor when the potential difference across its plates is 120. V?

4.46 The capacitor in an automatic external defibrillator is charged to 7.50 kV and stores 2400. J of energy. What is its capacitance?

4.47 The Earth has an electric field of 150. V/m near its surface. Find the electrical energy contained in each cubic meter of air near the surface.

•**4.48** The potential difference across two capacitors in series is 120. V. The capacitances are $C_1 = 1.00 \times 10^3$ µF and $C_2 = 1.50 \times 10^3$ µF.

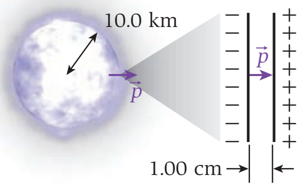
a) What is the total capacitance of this pair of capacitors?

b) What is the charge on each capacitor?

c) What is the potential difference across each capacitor?

d) What is the total energy stored by the capacitors?

•**4.49** Neutron stars are thought to have electric dipole $(\vec{p})$ layers at their surfaces. If a neutron star with a 10.0 km radius has a dipole layer 1.00 cm thick with charge distributions of $+1.00$ µC/cm^2 and -1.00 µC/cm^2 on the surface, as indicated in the figure, what is the capacitance of this star? What is the electric potential energy stored in the neutron star's dipole layer?



•**4.50** A 4.00×10^3 nF parallel plate capacitor is connected to a 12.0V battery and charged.

a) What is the charge Q on the positive plate of the capacitor?

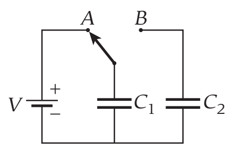
b) What is the electric potential energy stored in the capacitor?

The 4.00×10^3 nF capacitor is then disconnected from the 12.0 V battery and used to charge three uncharged capacitors, a 100. nF capacitor, a 200.-nF capacitor, and a 300. nF capacitor, connected in series.

c) After charging, what is the potential difference across each of the four capacitors?

d) How much of the electrical energy stored in the 4.00×10^3 nF capacitor was transferred to the other three capacitors?

•**4.51** The figure shows a circuit with $V = 12.0$ V, $C_1 = 500.$ pF, and $C_2 = 500.$ pF. The switch is closed, to A, and the capacitor C_1 is fully charged. Find (a) the energy delivered by the battery and (b) the energy stored in C_1. Then the switch is thrown to B and the circuit is allowed to reach equilibrium. Find (c) the total energy stored at C_1 and C_2. (d) Explain the energy loss, if there is any.



••**4.52** The Earth is held together by its own gravity. But it is also a charge-bearing conductor.

a) The Earth can be regarded as a conducting sphere of radius 6371 km, with electric field $\vec{E} = (-150.\ \text{V/m})\hat{r}$ at its surface, where $\hat{r}$ is a unit vector directed radially outward. Calculate the total electric potential energy associated with the Earth's electric charge and field.

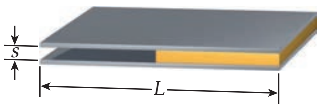
b) The Earth has gravitational potential energy, akin to the electric potential energy. Calculate this energy, treating the Earth as a uniform solid sphere. (*Hint:* $dU = -(Gm/r)dm$.)

c) Use the results of parts (a) and (b) to address this question: To what extent do electrostatic forces affect the structure of the Earth?

Section 4.6

4.53 Two parallel plate capacitors have identical plate areas and identical plate separations. The maximum energy each can store is determined by the maximum potential difference that can be applied before dielectric breakdown occurs. One capacitor has air between its plates, and the other has Mylar. Find the ratio of the maximum energy the Mylar capacitor can store to the maximum energy the air capacitor can store.

4.54 A capacitor has parallel plates, with half of the space between the plates filled with a dielectric material of constant κ and the other half filled with air as shown in the figure. Assume that the plates are square, with sides of length L, and that the separation between the plates is s. Determine the capacitance as a function of L.



4.55 Calculate the maximum surface charge distribution that can be maintained on any surface surrounded by dry air.

4.56 Thermocoax is a type of coaxial cable used for high-frequency filtering in cryogenic quantum computing experiments. Its stainless steel shield has an inner diameter of 0.350 mm, and its Nichrome conductor has a diameter of 0.170 mm. Nichrome is used because its resistance doesn't change much in going from room temperature to near absolute zero. The insulating dielectric is magnesium oxide (MgO), which has a dielectric constant of 9.70. Calculate the capacitance per meter of Thermocoax.

4.57 A parallel plate capacitor has square plates of side $L = 10.0$ cm and a distance $d = 1.00$ cm between the plates. Of the space between the plates, $\frac{1}{5}$ is filled with a dielectric with dielectric constant $\kappa_1 = 20.0$. The remaining $\frac{4}{5}$ of the space is filled with a different dielectric, with $\kappa_2 = 5.00$. Find the capacitance of the capacitor.

•**4.58** A 4.0 nF parallel plate capacitor with a sheet of Mylar ($\kappa = 3.1$) filling the space between the plates is charged to a potential difference of 120 V and is then disconnected.

a) How much work is required to completely remove the sheet of Mylar from the space between the two plates?

b) What is the potential difference between the plates of the capacitor once the Mylar is completely removed?

•**4.59** The volume between the two cylinders of a cylindrical capacitor is half filled with a dielectric whose dielectric constant is κ and is connected to a battery with a potential difference ΔV. What is the charge placed on the capacitor? What is the ratio of this charge to the charge placed on an identical capacitor with no dielectric connected in the same way across the same potential drop?

•**4.60** A dielectric slab with thickness d and dielectric constant $\kappa = 2.31$ is inserted in a parallel place capacitor that has been charged by a 110.-V battery and has area $A = 100.$ cm^2 and separation distance $d = 2.50$ cm.

a) Find the capacitance, C, the potential difference, V, the electric field, E, the total charge stored on the capacitor Q, and electric potential energy stored in the capacitor, U, before the dielectric material is inserted.

b) Find C, V, E, Q, and U when the dielectric slab has been inserted and the battery is still connected.

c) Find C, V, E, Q, and U when the dielectric slab is in place and the battery is disconnected.

•**4.61** A parallel plate capacitor has a capacitance of 120. pF and a plate area of 100. cm^2. The space between the plates is filled with mica whose dielectric constant is 5.40. The plates of the capacitor are kept at 50.0 V.

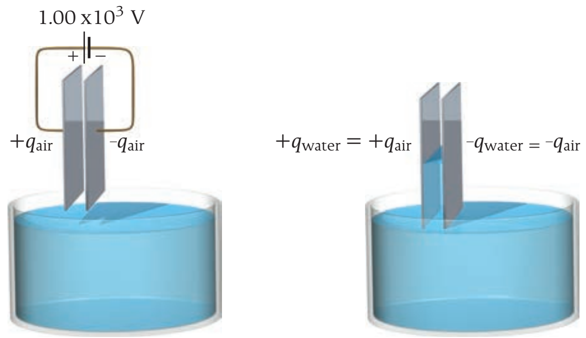
a) What is the strength of the electric field in the mica?

b) What is the amount of free charge on the plates?

c) What is the amount of charge induced on the mica?

••**4.62** Design a parallel plate capacitor with a capacitance of 47.0 pF and a capacity of 7.50 nC. You have available conducting plates, which can be cut to any size, and Plexiglas sheets, which can be cut to any size and machined to any thickness. Plexiglas has a dielectric constant of 3.40 and a dielectric strength of 3.00×10^7 V/m. You must make your capacitor as compact as possible. Specify all relevant dimensions. Ignore any fringe field at the edges of the capacitor plates.

••**4.63** A parallel plate capacitor consisting of a pair of rectangular plates, each measuring 1.00 cm by 10.0 cm, with a separation between the plates of 0.100 mm, is charged by a power supply at a potential difference of 1.00×10^3 V. The power supply is then removed, and without being discharged, the capacitor is placed in a vertical position over a container holding de-ionized water, with the short sides of the plates in contact with the water, as shown in the figure. Using energy considerations, show that the water will rise between the plates. Neglecting other effects, determine the system of equations that can be used to calculate the height to which the water rises between the plates. You do not have to solve the system.

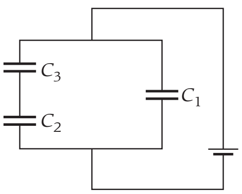


Additional Exercises

4.64 Two circular metal plates of radius 0.610 m and thickness 7.10 mm are used in a parallel plate capacitor. A gap of 2.10 mm is left between the plates, and half of the space (a semicircle) between them is filled with a dielectric for which $\kappa = 11.1$ and the other half is filled with air. What is the capacitance of this capacitor?

4.65 Considering the dielectric strength of air, what is the maximum amount of charge that can be stored on the plates of a capacitor that are a distance of 15 mm apart and have an area of 25 cm^2?

4.66 The figure shows three capacitors in a circuit: $C_1 = 2.00$ nF and $C_2 = C_3 = 4.00$ nF. Find the charge on each capacitor when the potential difference applied is $V = 1.50$ V.



4.67 A capacitor with a vacuum between its plates is connected to a battery and then the gap is filled with Mylar. By what percentage is its energy-storing capacity increased?

4.68 A parallel plate capacitor with a plate area of 12.0 cm^2 and air in the space between the plates, which are separated by 1.50 mm, is connected to a 9.00 V battery. If the plates are pulled back so that the separation increases to 2.75 mm, how much work is done?

4.69 Suppose you want to make a 1.00 F capacitor using two square sheets of aluminum foil. If the sheets of foil are separated by a single piece of paper (thickness of about 0.100 mm and $\kappa = 5.00$), find the size of the sheets of foil (the length of each edge).

4.70 A 4.00 pF parallel plate capacitor has a potential difference of 10.0 V across it. The plates are 3.00 mm apart, and the space between them contains air.

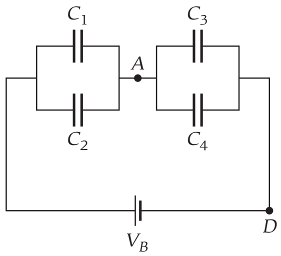
a) What is the charge on the capacitor?

b) How much energy is stored in the capacitor?

c) What is the area of the plates?

d) What would the capacitance of this capacitor be if the space between the plates were filled with polystyrene?

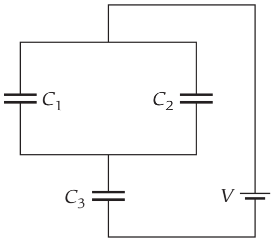
4.71 A four-capacitor circuit is charged by a battery, as shown in the figure. The capacitances are $C_1 = 1.00$ mF, $C_2 = 2.00$ mF, $C_3 = 3.00$ mF, and $C_4 = 4.00$ mF, and the battery potential is $V_B = 1.00$ V. When the circuit is at equilibrium, point D has potential $V_D = 0.00$ V. What is the potential, V_A, at point A?



4.72 How much energy can be stored in a capacitor with two parallel plates, each with an area of 64.0 cm^2 and separated by a gap of 1.30 mm, filled with porcelain whose dielectric constant is 7.00, and holding equal and opposite charges of magnitude 420. μC?

4.73 A quantum mechanical device known as the *Josephson junction* consists of two overlapping layers of superconducting metal (for example, aluminum at 1.00 K) separated by 20.0 nm of aluminum oxide, which has a dielectric constant of 9.10. If this device has an area of 100. μm^2 and a parallel plate configuration, estimate its capacitance.

4.74 Three capacitors with capacitances $C_1 = 6.00$ μF, $C_2 = 3.00$ μF, and $C_3 = 5.00$ μF are connected in a circuit as shown in the figure, with an applied potential of V. After the charges on the capacitors have reached their equilibrium values, the charge Q_2 on the second capacitor is found to be 40.0 μC.

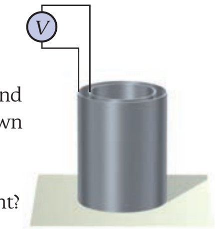


a) What is the charge, Q_1, on capacitor C_1?

b) What is the charge, Q_3, on capacitor C_3?

c) How much voltage, V, was applied across the capacitors?

4.75 For a science project, a fourth-grader cuts the tops and bottoms off two soup cans of equal height, 7.24 cm, and with radii of 3.02 cm and 4.16 cm, puts the smaller one inside the larger, and hot-glues them both on a sheet of plastic, as shown in the figure. Then she fills the gap between the cans with a special "soup" (dielectric constant of 63.0). What is the capacitance of this arrangement?



4.76 The Earth can be thought of as a spherical capacitor. If the net charge on the Earth is -7.80×10^5 C , find (a) the capacitance of the Earth and (b) the electric potential energy stored on the Earth's surface.

•**4.77** A parallel plate capacitor with air in the gap between the plates is connected to a 6.00 V battery. After charging, the energy stored in the capacitor is 72.0 nJ. Without disconnecting the capacitor from the battery, a dielectric is inserted into the gap and an additional 317 nJ of energy flows from the battery to the capacitor.

a) What is the dielectric constant of the dielectric?

b) If each of the plates has an area of 50.0 cm^2, what is the charge on the positive plate of the capacitor after the dielectric has been inserted?

c) What is the magnitude of the electric field between the plates before the dielectric is inserted?

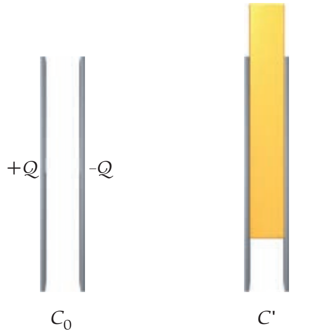
d) What is the magnitude of the electric field between the plates after the dielectric is inserted?

•**4.78** An 8.00 μF capacitor is fully charged by a 240. V battery, which is then disconnected. Next, the capacitor is connected to an initially uncharged capacitor of capacitance C, and the potential difference across it is found to be 80.0 V. What is C? How much energy ends up being stored in the second capacitor?

•**4.79** A parallel plate capacitor consists of square plates of edge length 2.00 cm separated by a distance of 1.00 mm. The capacitor is charged with a 15.0 V battery, and the battery is then removed. A 1.00 mm-thick sheet of nylon (dielectric constant of 3.50) is slid between the plates. What is the average force (magnitude and direction) on the nylon sheet as it is inserted into the capacitor?

•**4.80** A proton traveling along the x-axis at a speed of 1.00×10^6 m/s enters the gap between the plates of a 2.00 cm-wide parallel plate capacitor. The surface charge distributions on the plates are given by $\sigma = \pm 1.00 \times 10^{-6}$ C/m^2. How far has the proton been deflected sideways (Δy) when it reaches the far edge of the capacitor? Assume that the electric field is uniform inside the capacitor and zero outside.

•**4.81** A parallel plate capacitor has square plates of side $L = 10.0$ cm separated by a distance $d = 2.50$ mm, as shown in the figure. The capacitor is charged by a battery with potential difference $V_0 = 75.0$ V; the battery is then disconnected.



Charged capacitor

Charged capacitor with Plexiglas slab inserted

a) Determine the capacitance, C_0, and the electric potential energy, U_0, stored in the capacitor at this point.

b) A slab made of Plexiglas ($\kappa = 3.40$) is then inserted so that it fills $\frac{2}{3}$ of the volume between the plates, as shown in the figure. Determine the new capacitance, C', the new potential difference between the plates, V', and the new electric potential energy, U', stored in the capacitor.

c) Neglecting gravity, did the inserter of the dielectric slab have to do work or not?

•**4.82** A typical AAA battery has stored energy of about 3400 J. (Battery capacity is typically listed as 625 mA h, meaning that much charge can be delivered at approximately 1.5 V.) Suppose you want to build a parallel plate capacitor to store this amount of energy, using a plate separation of 1.0 mm and with air filling the space between the plates.

a) Assuming that the potential difference across the capacitor is 1.50 V, what must the area of each plate be?

b) Assuming that the potential difference across the capacitor is the maximum that can be applied without dielectric breakdown occurring, what must the area of each plate be?

c) Is either capacitor a practical replacement for the AAA battery?

•**4.83** Two parallel plate capacitors, C_1 and C_2, are connected in series to a 96.0V battery. Both capacitors have plates with an area of 1.00 cm^2 and a separation of 0.100 mm; C_1 has air between its plates, and C_2 has that space filled with porcelain (dielectric constant of 7.00 and dielectric strength of 5.70 kV/mm).

a) After charging, what are the charges on each capacitor?

b) What is the total energy stored in the two capacitors?

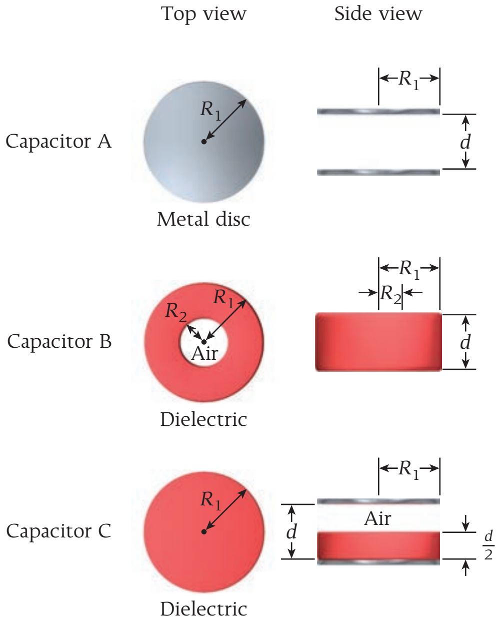
c) What is the electric field between the plates of C_2?

•**4.84** The plates of parallel plate capacitor A consist of two metal discs of identical radius, $R_1 = 4.00$ cm, separated by a distance $d = 2.00$ mm, as shown in the figure.

a) Calculate the capacitance of this parallel plate capacitor with the space between the plates filled with air.

b) A dielectric in the shape of a thick-walled cylinder of outer radius $R_1 = 4.00$ cm, inner radius $R_2 = 2.00$ cm, thickness $d = 2.00$ mm, and dielectric constant $\kappa = 2.00$ is placed between the plates, coaxial with them, as shown in the figure. Calculate the capacitance of capacitor B, with this dielectric.

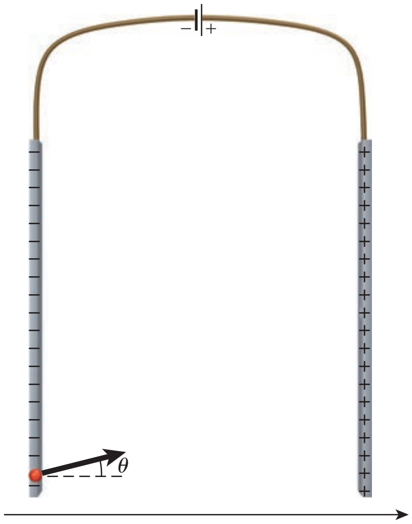
c) The dielectric cylinder is removed, and instead a solid disc of radius R_1 made of the same dielectric is placed between the plates to form capacitor C, as shown in the figure. What is the new capacitance?



•4.85 A 1.00-μF capacitor charged to 50.0 V and a 2.00 μF capacitor charged to 20.0 V are connected, with the positive plate of each connected to the negative plate of the other. What is the final charge on the 1.00-μF capacitor?

•4.86 The capacitance of a spherical capacitor consisting of two concentric conducting spheres with radii r_1 and r_2 ($r_2 > r_1$) is given by $C = 4\pi\epsilon_0 r_1 r_2/(r_2 - r_1)$. Suppose that the space between the spheres, from r_1 up to a radius R ($r_1 < R < r_2$) is filled with a dielectric for which $\varepsilon = 10\varepsilon_0$. Find an expression for the capacitance, and check the limits when $R = r_1$ and $R = r_2$.

•4.87 In the figure, a parallel plate capacitor is connected to a 300.-V battery. With the capacitor connected, a proton is fired with a speed of 2.00×10^5 m/s from (through) the negative plate of the capacitor at an angle θ with the normal to the plate.



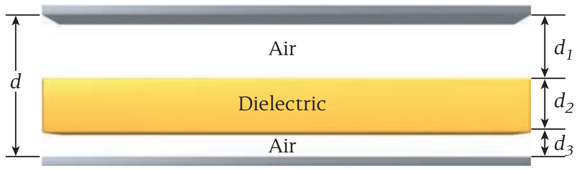
a) Show that the proton cannot reach the positive plate of the capacitor, regardless of what the angle θ is.

b) Sketch the trajectory of the proton between the plates.

c) Assuming that $V = 0$ at the negative plate, calculate the potential at the point between the plates where the proton reverses its motion in the x-direction.

d) Assuming that the plates are long enough for the proton to stay between them throughout its motion, calculate the speed (magnitude only) of the proton as it collides with the negative plate.

••4.88 For the parallel plate capacitor with dielectric shown in the figure, prove that for a given thickness of the dielectric slab, the capacitance does not depend on the position of the slab relative to the two conducting plates (that is, it does not depend on the values of d_1 and d_3).



MULTI-VERSION EXERCISES

4.89 The battery of an electric car stores 53.63 MJ of energy. How many supercapacitors, each with capacitance $C = 3.361$ kF at a potential difference of 2.121 V, are required to supply this amount of energy?

4.90 The battery of an electric car stores 60.51 MJ of energy. If 6990 supercapacitors, each with capacitance $C = 3.423$ kF, are required to supply this amount of energy, what is the potential difference across each supercapacitor?

4.91 The battery of an electric car stores 67.39 MJ of energy. If 6845 supercapacitors, each with capacitance C and charged to a potential difference of 2.377 V, can supply this amount of energy, what is the value of C for each supercapacitor?

4.92 A parallel plate capacitor with vacuum between the plates has a capacitance of 3.547 μF. A dielectric material with $\kappa = 4.617$ is placed between the plates, completely filling the volume between them. The capacitor is then connected to a battery that maintains a potential difference of 10.03 V across the plates. How much work is required to pull the dielectric material out of the capacitor?

4.93 A parallel plate capacitor with vacuum between the plates has a capacitance of 3.607 μF. A dielectric material is placed between the plates, completely filling the volume between them. The capacitor is then connected to a battery that maintains a potential difference of 11.33 V across the plates. The dielectric material is pulled out of the capacitor, which requires 4.804×10^{-4} J of work. What is the dielectric constant of the material?

4.94 A parallel plate capacitor with vacuum between the plates has a capacitance of 3.669 μF. A dielectric material with $\kappa = 3.533$ is placed between the plates, completely filling the volume between them. The capacitor is then connected to a battery that maintains a potential difference V across the plates. The dielectric material is pulled out of the capacitor, which requires 7.389×10^{-4} J of work. What is the potential difference, V?

5 Current and Resistance

FIGURE 5.1 A current flowing through a wire makes this light bulb shine.

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Electric lighting is so commonplace that you don't even think about it. You walk into a dark room and simply flick on a switch, bringing the room to nearly daytime brightness (Figure 5.1). However, what happens when the switch is flicked depends ultimately on principles of physics and engineering devices that took decades to develop and refine.

This chapter is the first to focus on electric charges in motion. It presents some of the fundamental concepts that we'll use in Chapter 6 to analyze basic electric circuits, which are integral to all electronics applications. These chapters concentrate on the electrical effects of moving charges, but you should be aware that moving charges also give rise to other effects, which we'll start to examine in Chapter 7 on magnetism.

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WHAT WE WILL LEARN

- Electric current at a point in a circuit is the rate at which net charge moves past that point.
- Direct current is current flowing in a direction that does not change with time. The direction of current is defined as the direction in which positive charge would be moving.
- The current density passing a given point in a conductor is the current per cross-sectional area.
- The conductivity of a material characterizes the ability of that material to conduct current. Its inverse is resistivity.
- The resistance of a device depends on its geometry and on the material of which it is made.
- The resistivity of a conductor increases approximately linearly with temperature.
- Electromotive force (usually referred to as emf) is a potential difference in an electric circuit.
- Ohm's Law states that the potential drop across a device is equal to the current flowing through the device times the resistance of the device.
- A simple circuit consists of a source of emf and resistors connected in series or in parallel.
- In a circuit diagram, an equivalent resistance can replace resistors connected in series or in parallel.
- The power in a circuit is the product of the current and the voltage drop.
- A diode conducts current in one direction but not in the opposite direction.

5.1 Electric Current

Up to this point, our study of electricity has focused on electrostatics, which deals with the properties of stationary electric charges and fields. If electrostatics were all there was to electricity, it would not be nearly as important to modern society as it is. The world-changing impact of electricity is due to the properties of charges in motion, or electric current. All electrical devices rely on some kind of current for their operation.

Let's start by looking at a few very simple experiments. When you were a kid, you very likely had some toys that ran on batteries, and probably some of them contained small light bulbs. Consider a very simple circuit consisting only of a battery, a switch, and a light bulb (see Figure 5.2). If the switch is open, as in Figure 5.2a, the light bulb does not shine. If the switch is closed, as in Figure 5.2b, the light bulb turns on. We all know why this happens—because a current flows through the closed circuit. In this chapter, we'll examine what it means for the current to flow, what the physical basis of current is, and how it is related to the potential difference provided by the battery. We'll see that the light bulb acts as a resistor in the circuit, and we'll examine how resistors behave.

Let's begin by considering the simple experiment performed in Figure 5.2c, where the battery's orientation is the reverse of what it is in Figure 5.2b. The light bulb shines just

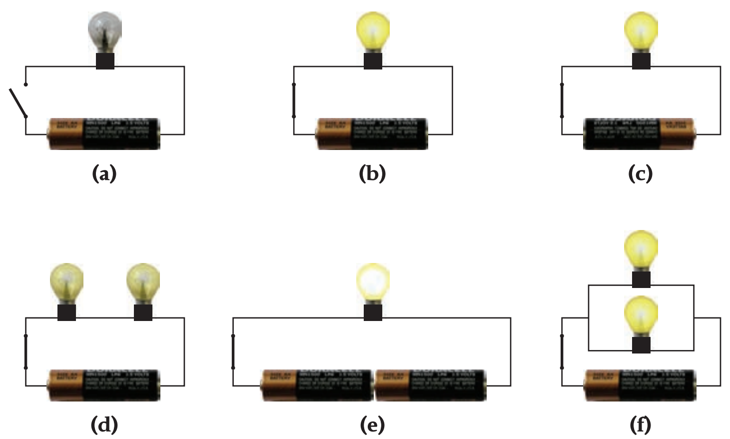


FIGURE 5.2 Experiments with batteries and light bulbs.

the same, despite the fact that the sign of the potential difference provided by the battery is reversed. (The positive terminal of the battery is on the end with the copper color.) In Figure 5.2d, two light bulbs are in the circuit, one behind the other. (Section 5.5 will focus on this arrangement of resistors, which is a connection *in series*.) Each of the two light bulbs shines with significantly less intensity than the single bulb in Figure 5.2c, so the current in the circuit may be smaller than before. On the other hand, two batteries in series, as in Figure 5.2e, double the potential difference in the circuit, and the bulb shines significantly brighter. Finally, if separate wires are used to connect the two light bulbs to a single battery, as shown in Figure 5.2f, the light bulbs shine with about the same intensity as in Figure 5.2b or Figure 5.2c. This way of wiring resistors in a circuit is called a *parallel connection* and will be explored in Section 5.6.

Quantitatively, the **electric current,** i, is the net charge passing a given point in a given time, divided by that time. The random motion of electrons in a conductor is not a current, in spite of the fact that large amounts of charge are moving past a given point, because no *net* charge flows. If net charge dq passes a point during time dt, the current at that point is, by definition,

$$i = \frac{dq}{dt}. \tag{5.1}$$

The net amount of charge passing a given point in time t is the integral of the current with respect to time:

$$q = \int dq = \int_0^t i\,dt'. \tag{5.2}$$

Total charge is conserved, implying that charge flowing in a conductor is never lost. Therefore, the same amount of charge flows into one end of a conductor as emerges from the other end.

The unit of current, coulombs per second, was given the name **ampere** (abbreviated A, or sometimes amp), after the French physicist André Ampère (1775–1836):

$$1\text{ A} = \frac{1\text{ C}}{1\text{ s}}.$$

Some typical currents are 1 A for a light bulb, 200 A for the starter in your car, $1\text{ mA} = 1 \cdot 10^{-3}\text{A}$ to power your MP3 player, 1 nA for the currents in your brain's neurons and synaptic connections, and $10{,}000\text{ A} = 10^4\text{ A}$ in a lightning strike (for a short time). The smallest currents that can be measured are those from individual electrons tunneling in scanning tunneling microscopes and are on the order of 10 pA. The largest current in the Solar System is the solar wind, which is in the GA range. Other examples of the broad range of currents are shown in Figure 5.3.

Hohlraums

Hohlraums are devices for inertial confinement fusion.

You should remember this handy safety rule related to the orders of magnitudes of currents: 1-10-100. That is, 1 mA of current flowing through a human body can be felt (as

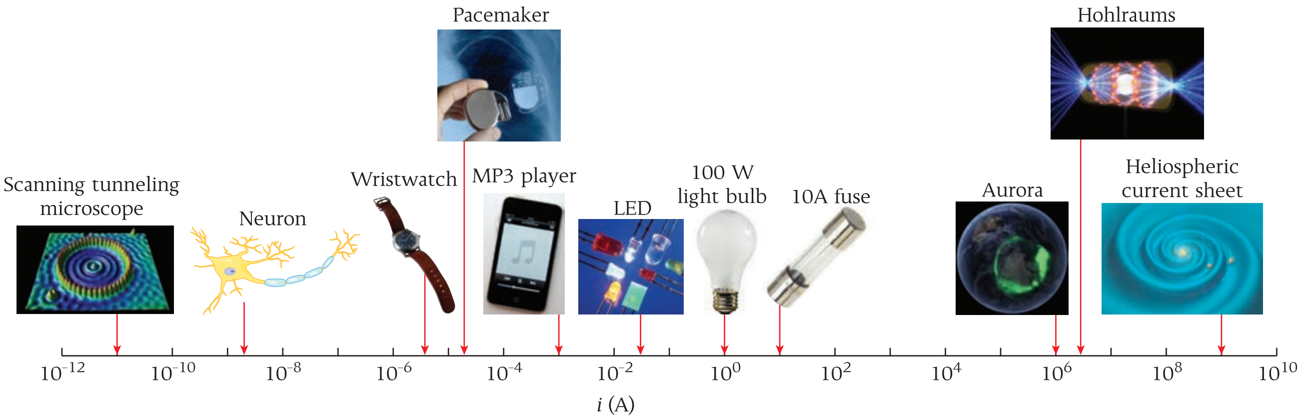


FIGURE 5.3 Examples of electrical currents ranging from 1 pA to 10 GA.

a tingle, usually), 10 mA of current makes muscles contract to the point that the person cannot let go of the wire carrying the current, and 100 mA is sufficient to stop the heart.

A current that flows in only one direction, which does not change with time, is called **direct current.** In this chapter, the direction of the current flowing in a conductor is indicated by an arrow.

Physically, the charge carriers in a conductor are electrons, which are negatively charged. However, by convention, positive current is defined as flowing from the positive to the negative terminal. The reason for this counterintuitive definition of current direction is that the definition originated in the second half of the 19th century, when it was not known that electrons are the charge carriers responsible for current. And so the current direction was simply defined as the direction in which the positive charges would flow.

Self-Test Opportunity 5.1

A typical rechargeable AA battery is rated at 700 mAh. How long can this battery provide a current of 100 μA ?

EXAMPLE 5.1 Iontophoresis

There are three ways to administer anti-inflammatory medication. The painless way is the oral one–simply swallowing the drug. However, this method typically leads to a small amount of the drug in the affected tissue, on the order of 1 μg. The second way is to have the drug injected locally with a needle. This hurts but can deposit on the order of 10 mg of the drug in the affected tissue–four orders of magnitude more than with the oral method. However, since the 1990s, a third method has been available, which is also painless and can deposit on the order of 100 μg of a drug in the area where it is needed. This method, called *iontophoresis*, uses (very weak) electrical currents that are sent through the patient's tissue (Figure 5.4). The iontophoresis device consists of a battery and two electrodes (plus other electronic circuitry that allows the nurse to control the strength of the current applied). The anti-inflammatory drug, usually dexamethasone, is applied to the underside of the negatively charged electrode. A current flows through the patient's skin and deposits the drug in the tissue to a depth of up to 1.7 cm.

PROBLEM

A nurse wants to administer 80 μg of dexamethasone to the heel of an injured soccer player. If she uses an iontophoresis device that applies a current of 0.14 mA, as shown in Figure 5.4, how long does the administration of the dose take? Assume that the instrument has an application rate of 650 μg/C and that the current flows at a constant rate.

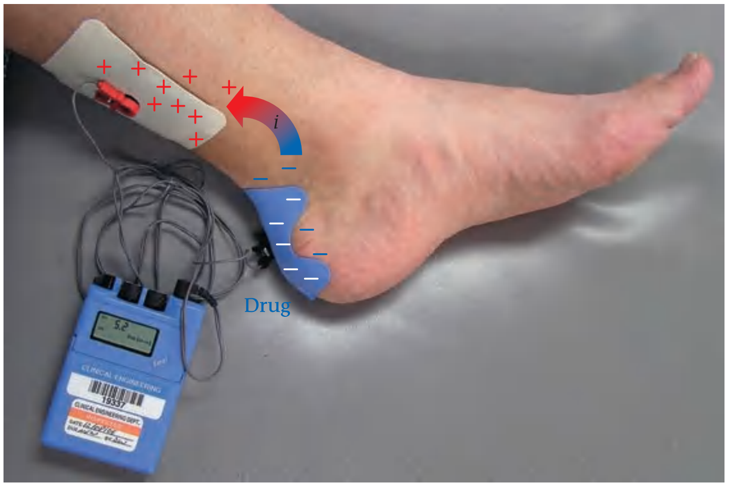


FIGURE 5.4 Iontophoresis is the application of medication under the skin with the aid of electrical current.

SOLUTION

If the drug application rate is 650 μg/C, to apply 80 μg requires a total charge of

$$q = \frac{80\ \mu\text{g}}{650\ \mu\text{g/C}} = 0.123\ \text{C}.$$

The current flows at a constant rate, so the integral in equation 5.2 is simply

$$q = \int_0^t i\,dt' = it.$$

Solving for t and inserting the numbers, we find

$$q = it \Rightarrow t = \frac{q}{i} = \frac{0.123\ \text{C}}{0.14 \times 10^{-3}\ \text{A}} = 880\ \text{s}.$$

The iontophoresis treatment of the athlete will take approximately 15 min.

5.2 Current Density

Consider a current flowing in a conductor. For a perpendicular plane through the conductor, the current per unit area flowing through the conductor at that point (the cross-sectional area A in Figure 5.5) is the **current density,** $\vec{J}$. The direction of $\vec{J}$ is defined as the direction of the velocity of the positive charges (or opposite to the direction of negative charges) crossing the plane. The current flowing through the plane is

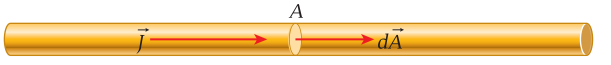


FIGURE 5.5 Segment of a conductor (wire), with a perpendicular plane intersecting it and forming a cross-sectional area A.

$$i = \int \vec{J} \bullet d\vec{A}, \qquad (5.3)$$

where $d\vec{A}$ is the differential area element of the perpendicular plane, as indicated in Figure 5.5. If the current is uniform and perpendicular to the plane, then $i = JA$, and the magnitude of the current density can be expressed as

$$J = \frac{i}{A}. \qquad (5.4)$$

In a conductor that is not carrying current, the conduction electrons move randomly. When current flows through the conductor, electrons still move randomly but also have an additional **drift velocity,** $\vec{v}_{\text{d}}$, in the direction opposite to that of the electric field driving the current. The magnitude of the velocity of random motion is on the order of 10^6 m/s, while the magnitude of the drift velocity is on the order of 10^{-4} m/s or even less. With such a slow drift velocity, you might wonder why a light comes on almost immediately after you turn on a switch. The answer is that the switch establishes an electric field almost immediately throughout the circuit (with a speed on the order of $3 \cdot 10^8$ m/s), causing the free electrons in the entire circuit (including in the light bulb) to move almost instantly.

The current density is related to the drift velocity of the moving electrons. Consider a conductor with cross-sectional area A and electric field $\vec{E}$ applied to it. Suppose the conductor has n conduction electrons per unit volume, and assume that all the electrons have the same drift velocity and that the current density is uniform. The negatively charged electrons will drift in a direction opposite to that of the electric field. In a time interval dt, each electron moves a net distance $v_{\text{d}}dt$. The volume of electrons passing a cross section of the conductor in time dt is then $Av_{\text{d}}dt$, and the number of electrons in this volume is $nAv_{\text{d}}dt$. Each electron has charge $-e$, so the charge dq that flows through the area in time dt is

$$dq = -nev_{\text{d}}A\,dt. \qquad (5.5)$$

Therefore, the current is

$$i = \frac{dq}{dt} = -nev_{\text{d}}A. \qquad (5.6)$$

The resulting current density is

$$J = \frac{i}{A} = -nev_{\text{d}}. \qquad (5.7)$$

Equation 5.7 was derived in one spatial dimension, as is appropriate for a wire. However, it can be readily generalized to arbitrary directions in three-dimensional space:

$$\vec{J} = -(ne)\vec{v}_{\rm d}.$$

You can see that the drift velocity vector is antiparallel to the current density vector, as stated before.

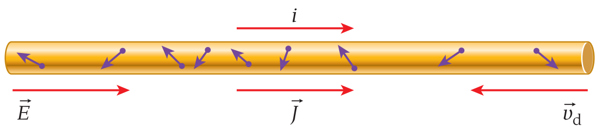


FIGURE 5.6 Electrons moving in a wire from right to left, causing a current in the direction from left to right.

Figure 5.6 shows a schematic drawing of a wire carrying a current. The physical current carriers are negatively charged electrons. In Figure 5.6, these electrons are moving to the left with drift velocity $\vec{v}_{\rm d}$. However, the electric field, the current density, and the current are all directed to the right because of the convention that these quantities refer to positive charges. You may find this convention somewhat confusing, but you'll need to keep it in mind.

SOLVED PROBLEM 5.1 Drift Velocity of Electrons in a Copper Wire

PROBLEM

You are playing "Galactic Destroyer" on your video game console. Your game controller operates at 12 V and is connected to the main box with an 18 gauge copper wire of length 1.5 m. As you fly your spaceship into battle, you hold the joystick in the forward position for 5.3 s, sending a current of 0.78 mA to the console. How far have the electrons in the wire moved during those few seconds, while on the screen your spaceship crossed half of a star system?

SOLUTION

THINK To find out how far electrons in a wire move during a given time interval, we need to calculate their drift velocity. To determine the drift velocity for electrons in a copper wire carrying a current, we need to find the density of charge-carrying electrons in copper. Then, we can apply the definition of the charge density to calculate the drift velocity.

SKETCH A copper wire with cross-sectional area A carrying a current, i, is shown in Figure 5.7, which also shows that, by convention, the electrons drift in the direction opposite to the direction of the current.

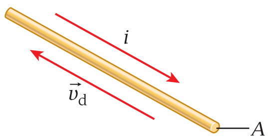


FIGURE 5.7 A copper wire with cross-sectional area A carrying a current, i.

RESEARCH We obtain the distance x traveled by the electrons during time t from

$$x = v_{\rm d}t,$$

where $v_{\rm d}$ is the magnitude of the drift velocity of the electrons. The drift velocity is related to the current density via equation 5.7:

$$\frac{i}{A} = -nev_{\rm d} \qquad \text{(i)}$$

where i is the current, A is the cross-sectional area (0.823 mm^2 for an 18 gauge wire), n is the density of electrons, and $-e$ is the charge of an electron. The density of electrons is defined as

$$n = \frac{\text{number of conduction electrons}}{\text{volume}}.$$

We can calculate the density of electrons by assuming there is one conduction electron per copper atom. The density of copper is

$$\rho_{\rm Cu} = 8.96 \text{ g/cm}^3 = 8960 \text{ kg/m}^3.$$

One mole of copper has a mass of 63.5 g and contains 6.02×10^{23} atoms. Thus, the density of electrons is

$$n = \left(\frac{1 \text{ electron}}{1 \text{ atom}}\right)\left(\frac{6.02 \times 10^{23} \text{ atoms}}{63.5 \text{ g}}\right)\left(\frac{8.96 \text{ g}}{1 \text{ cm}^3}\right)\left(\frac{10^6 \text{ cm}^3}{1 \text{ m}^3}\right) = 8.49 \times 10^{28} \ \frac{\text{electrons}}{\text{m}^3}.$$

SIMPLIFY We solve equation (i) for the magnitude of the drift velocity:

$$v_{\rm d} = \frac{i}{neA}.$$

Thus, the distance traveled by the electrons is

$$x = v_d t = \frac{i}{neA} t.$$

CALCULATE Putting in the numerical values, we get

$$x = v_d t = \frac{it}{neA} = \frac{(0.78 \times 10^{-3}\ \text{A})(5.3\ \text{s})}{(8.49 \times 10^{28}\ \text{m}^{-3})(1.602 \times 10^{-19}\ \text{C})(0.823\ \text{mm}^2)}$$
$$= (6.96826 \times 10^{-8}\ \text{m/s})(5.3\ \text{s})$$
$$= 3.69318 \times 10^{-7}\ \text{m}.$$

ROUND We report our results to two significant figures:

$$v_d = 7.0 \times 10^{-8}\ \text{m/s},$$

and

$$x = 3.7 \times 10^{-7}\ \text{m} = 0.37\ \mu\ \text{m}.$$

DOUBLE-CHECK Our result for the magnitude of the drift velocity turns out to be a stunningly small number. Earlier, it was stated that typical drift velocities are on the order of 10^{-4} m/s or smaller. Since the current is proportional to the drift velocity, a relatively small current implies a relatively small drift velocity. An 18 gauge wire can carry a current of several amperes, so the current specified in the problem statement is less than 1% of the maximum current. Therefore, the fact that our calculated drift velocity is less than 1% of 10^{-4} m/s, a typical drift velocity for high currents, is reasonable.

The distance we calculated for the movement of the electrons is less than 0.001 of the thickness of a fingernail, a very small distance compared to the length of the wire. This result provides a valuable reminder that the electromagnetic field moves with nearly the speed of light (in vacuum) inside a conductor and causes all conduction electrons to drift basically at the same time. Therefore, the signal from your game controller arrives almost instantaneously at the console, despite the incredibly slow pace of the individual electrons.

5.3 Resistivity and Resistance

Some materials conduct electricity better than others. Applying a given potential difference across a good conductor results in a relatively large current; applying the same potential difference across an insulator produces little current. The **resistivity,** ρ, is a measure of how strongly a material opposes the flow of electric current. The **resistance,** R, is a material's opposition to the flow of electric current.

If a known electric potential difference, ΔV, is applied across a conductor (some physical device or material that conducts current) and the resulting current, i, is measured, the resistance of that conductor is given by

$$R = \frac{\Delta V}{i}. \tag{5.8}$$

The units of resistance are volts per ampere, a combination that was given the name **ohm** and the symbol Ω (the capital Greek letter omega), in honor of the German physicist Georg Simon Ohm (1789–1854):

$$1\ \Omega = \frac{1\ \text{V}}{1\ \text{A}}.$$

Rearrangement of equation 5.8 results in

$$i = \frac{\Delta V}{R}, \tag{5.9}$$

which states that for a given potential difference, ΔV, the current, i, is inversely proportional to the resistance, R. This equation is commonly referred to as **Ohm's Law.** A rearrangement of equation 5.9, $\Delta V = iR$, is sometimes also referred to as Ohm's Law.

Sometimes devices are described in terms of the **conductance,** G, defined as

$$G = \frac{i}{\Delta V} = \frac{1}{R}.$$

Conductance has the SI derived unit of siemens (S), in honor of German inventor and industrialist Ernst Werner von Siemens (1816–1892):

$$1\ \mathrm{S} = \frac{1\ \mathrm{A}}{1\ \mathrm{V}} = \frac{1}{1\ \Omega}.$$

In some conductors, the resistivity depends on the direction in which the current is flowing. This chapter assumes that the resistivity of a material is uniform for all directions of the current.

The resistance of a device depends on the material of which the device is made as well as its geometry. As stated earlier, the resistivity of a material characterizes how much it opposes the flow of current. The resistivity is defined in terms of the magnitude of the applied electric field, E, and the magnitude of the resulting current density, J:

$$\rho = \frac{E}{J}. \tag{5.10}$$

The units of resistivity are

$$[\rho] = \frac{[E]}{[J]} = \frac{\mathrm{V/m}}{\mathrm{A/m^2}} = \frac{\mathrm{V\ m}}{\mathrm{A}} = \Omega\ \mathrm{m}.$$

Table 5.1 lists the resistivities of some representative conductors at 20 °C. As you can see, typical values for the resistivity of metal conductors used in wires are on the order of $10^{-8}\ \Omega$ m. For example, copper has a resistivity of about $2 \cdot 10^{-8}\ \Omega$ m. Several metal alloys listed in Table 5.1 have useful properties. For example, wire made from Nichrome (80% nickel and 20% chromium) is often used as a heating element in devices such as toasters. The next time you are toasting an English muffin, look inside the toaster. The glowing elements are probably Nichrome wires. The resistivity of Nichrome ($108 \cdot 10^{-8}\ \Omega$ m) is about 50 times that of copper. Thus, when current is run through the Nichrome wires of the toaster, the wires dissipate power and heat up until they glow with a dull red color, while the copper wires of the electrical cord that connects the toaster to the wall remain cool.

Table 5.1 The Resistivity and the Temperature Coefficient of the Resistivity for Some Representative Conductors

Material	Resistivity, ρ, at 20 °C ($10^{-8}\ \Omega$ m)	Temperature Coefficient, α ($10^{-3}\ \mathrm{K}^{-1}$)
Silver	1.62	3.8
Copper	1.72	3.9
Gold	2.44	3.4
Aluminum	2.82	3.9
Brass	3.9	2
Tungsten	5.51	4.5
Nickel	7	5.9
Iron	9.7	5
Steel	11	5
Tantalum	13	3.1
Lead	22	4.3
Constantan	49	0.01
Stainless steel	70	1
Mercury	95.8	0.89
Nichrome	108	0.4

The values for steel and stainless steel depend strongly on the type of steel.

Sometimes materials are specified in terms of their **conductivity,** σ, rather than their resistivity, ρ; conductivity is defined as

$$\sigma = \frac{1}{\rho}.$$

The units of conductivity are $(\Omega\ \mathrm{m})^{-1}$.

The resistance of a conductor can be found from its resistivity and its geometry. For a homogeneous conductor of length L and constant cross-sectional area A, the equation $\Delta V = -\int \vec{E} \bullet d\vec{s}$ can be used to relate the electric field, E, and the electric potential difference, ΔV, across the conductor:

$$E = \frac{\Delta V}{L}.$$

Note that in contrast to electrostatics, where the surface of any conductor is an equipotential surface and has no electric field inside and no current flowing through it, the conductor in this situation has $\Delta V \neq 0$ and $\vec{E} \neq 0$, causing a current to flow. The magnitude of the current density is the current divided by the cross-sectional area:

$$J = \frac{i}{A}.$$

From the definition of resistivity (equation 5.10) and using $J = i/A$ and Ohm's Law (equation 5.8), we obtain

$$\rho = \frac{E}{J} = \frac{\Delta V/L}{i/A} = \frac{\Delta V}{i}\frac{A}{L} = \frac{iR}{i}\frac{A}{L} = R\frac{A}{L}.$$

Rearranging terms yields an expression for the resistance of a conductor in terms of the resistivity of its constituent material, the length, and the cross-sectional area:

$$R = \rho \frac{L}{A}. \tag{5.11}$$

Note that in the limit $A \to \infty$, equation 5.11 implies that $R \to 0$. Therefore, the equation is only an approximation, because the conduction electrons will still encounter some resistance in traveling through some length L of a material of density ρ. However, for all wires in actual use, equation 5.11 is an excellent approximation.

Size Convention for Wires

The American Wire Gauge (AWG) size convention for wires specifies diameters and thus cross-sectional areas on a logarithmic scale. The AWG size convention is shown in Table 5.2. The wire gauge is related to the diameter: The higher the gauge number, the thinner the wire. For large-diameter wires, gauge numbers consist of one or more zeros, as shown in Table 5.2. A 00 gauge wire is equivalent to a –1 gauge, a 000 gauge wire is equivalent to a –2 gauge, and so on. By definition, a 36 gauge wire has a diameter of exactly 0.005 in, and a 0000 gauge wire has a diameter of exactly 0.46 in. (These sizes appear in red in Table 5.2.) There are 39 gauge values from 0000-gauge to 36 gauge, and the gauge number is a logarithmic representation of the wire diameter. Therefore, the formula to convert from the AWG gauge to the wire diameter, in inches, is $d = (0.005)92^{(36-n)/39}$, where n is the gauge number. Typical residential wiring uses 12 gauge to 10 gauge wires. An important rule of thumb is that a reduction by 3 gauges doubles the cross-sectional area of the wire. Examining equation 5.11, you can see that to cut the resistance of a given length of wire in half, you have to reduce the gauge number by 3.

Table 5.2 Wire Diameters and Cross-Sectional Areas as Defined by the American Wire Gauge Convention

AWG	*d* (in)	*d* (mm)	*A* (mm²)
000000	0.5800	14.733	170.49
00000	0.5165	13.120	135.20
0000	**0.46**	11.684	107.22
000	0.4096	10.405	85.029
00	0.3648	9.2658	67.431
0	0.3249	8.2515	53.475
1	0.2893	7.3481	42.408
. . .			
8	0.1285	3.2636	8.3656
9	0.1144	2.9064	6.6342
10	0.1019	2.5882	5.2612
11	0.0907	2.3048	4.1723
12	0.0808	2.0525	3.3088
13	0.0720	1.8278	2.6240
14	0.0641	1.6277	2.0809
15	0.0571	1.4495	1.6502
16	0.0508	1.2908	1.3087
17	0.0453	1.1495	1.0378
18	0.0403	1.0237	0.8230
. . .			
35	0.0056	0.1426	0.0160
36	**0.005**	0.1270	0.0127
37	0.0045	0.1131	0.0100
. . .			

EXAMPLE 5.2 Resistance of a Copper Wire

Standard wires that electricians put into residential housing have fairly low resistance.

PROBLEM

What is the resistance of the 100.0 m standard 12 gauge copper wire that is typically used in wiring household electrical outlets?

SOLUTION

A 12 gauge copper wire has a diameter of 2.053 mm (see Table 5.2). Its cross-sectional area is then

$$A = 3.31 \text{ mm}^2.$$

Using the value for the resistivity of copper from Table 5.1 and equation 5.11, we find

$$R = \rho \frac{L}{A} = (1.72 \times 10^{-8}\ \Omega\ \text{m}) \frac{100.0 \text{ m}}{3.31 \times 10^{-6}\ \text{m}^2} = 0.520\ \Omega.$$

Concept Check 5.1

If the diameter of the wire in Example 5.2 is doubled, its resistance will

a) increase by a factor of 4.

b) increase by a factor of 2.

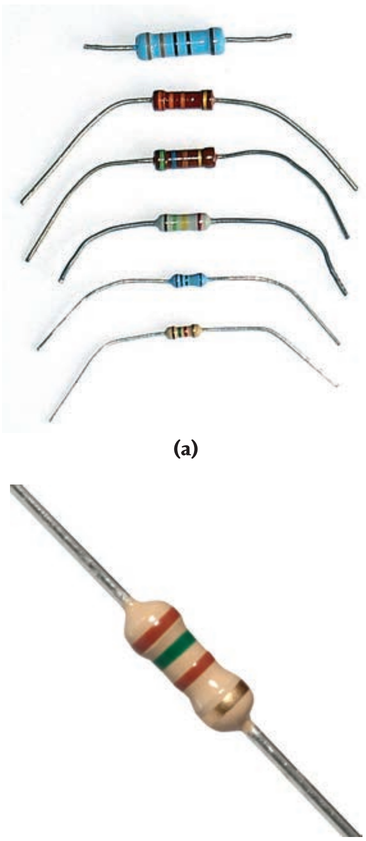
c) stay the same.

d) decrease by a factor of 2.

e) decrease by a factor of 4.

Resistor Codes

In many applications, circuit design calls for a range of resistances in various parts of a circuit. Commercially available resistors, such as those shown in Figure 5.8a, have a wide range of resistances. Resistors are commonly made of carbon enclosed in a plastic cover that looks like a medicine capsule, with wires sticking out at the ends for electrical connection. The value of the resistance is indicated by three or four color bands on the plastic covering. The first two bands indicate numbers for the mantissa, the third represents a power of 10, and the fourth indicates a tolerance for the range of values. For the mantissa and power of 10, the numbers associated with the colors are black = 0, brown = 1, red = 2, orange = 3, yellow = 4, green = 5, blue = 6, purple = 7, gray = 8, and white = 9. For the tolerance, brown means 1%, red means 2%, gold means 5%, silver means 10%, and no band at all means 20%. For example, the single resistor shown in Figure 5.8b has the colors (left to right) brown, green, brown, and gold. From the code, the resistance of this resistor is 15×10^1 $\Omega = 150\ \Omega$, with a tolerance of 5%.



(a)

(b)

FIGURE 5.8 (a) Selection of resistors with various resistances. (b) Color-coding of a 150 Ω resistor.

Temperature Dependence and Superconductivity

The values of resistivity and resistance vary with temperature. For metals, this dependence on temperature is linear over a broad range of temperatures. An empirical relationship for the temperature dependence of the resistivity of a metal is

$$\rho - \rho_0 = \rho_0 \alpha (T - T_0), \tag{5.12}$$

where ρ is the resistivity at temperature T, ρ_0 is the resistivity at temperature T_0, and α is the **temperature coefficient of electric resistivity** for the particular conductor.

In everyday applications, the temperature dependence of the resistance is often important. Equation 5.11 states that the resistance of a device depends on its length and cross-sectional area. Thus, the temperature dependence of the resistance of a conductor can be approximated as

$$R - R_0 = R_0 \alpha (T - T_0). \tag{5.13}$$

Note that equations 5.12 and 5.13 deal with temperature differences, so the temperatures can be expressed in degrees Celsius or kelvins (but not in degrees Fahrenheit!). Also note that T_0 does not necessarily have to be room temperature.

Values of α for representative conductors are listed in Table 5.1. Note that common metal conductors such as copper have a temperature coefficient of electric resistivity on the order of $4 \times 10^{-3}\ \mathrm{K}^{-1}$. However, one metal alloy, constantan (60% copper and 40% nickel), has the special characteristic that its temperature coefficient of electric resistivity is very small: $\alpha = 1 \times 10^{-5}\ \mathrm{K}^{-1}$. The name of this alloy comes from shortening the phrase "constant resistance." The small temperature coefficient of constantan combined with its relatively high resistivity of $4.9 \times 10^{-7}\ \Omega$ m makes it useful for precision resistors whose resistances have little dependence on temperature. Note also that Nichrome has a relative small temperature coefficient, $4 \times 10^{-4}\ \mathrm{K}^{-1}$, which makes it suitable for the construction of heating elements, as noted earlier.

According to equation 5.12, most materials have a resistivity that varies linearly with temperature under ordinary circumstances. However, some materials do not follow this rule at low temperatures. At very low temperatures, the resistivity of some materials goes to exactly zero. These materials are called **superconductors.** Superconductors have applications in the construction of magnets for devices such as magnetic resonance imagers (MRI). Magnets constructed with superconductors use less power and can produce stronger magnetic fields than magnets constructed with conventional resistive conductors. A more extensive discussion of superconductivity will be presented in Chapter 8.

The resistance of some semiconducting materials actually decreases as the temperature increases, which implies a negative temperature coefficient of electric resistivity. These materials are often employed in high-resolution detectors for optical measurements or in particle detectors. Such devices must be kept cold to keep their resistance high, which is accomplished with refrigerators or liquid nitrogen.

A *thermistor* is a semiconductor whose resistance depends strongly on temperature. Thermistors are used to measure temperature. The temperature dependence of the resistance of a typical thermistor is shown in Figure 5.9a. Here you can see that the resistance of a thermistor falls with increasing temperature. This drop is in contrast to the increase in resistance of a copper wire over the same temperature range, shown in Figure 5.9b.

Concept Check 5.2

If the temperature of a copper wire with a resistance of 100 Ω is increased by 25 K, the resistance will

a) increase by approximately 10 Ω.

b) increase by approximately 4 m Ω.

c) decrease by approximately 4 m Ω.

d) decrease by approximately 10 Ω.

e) stay the same.

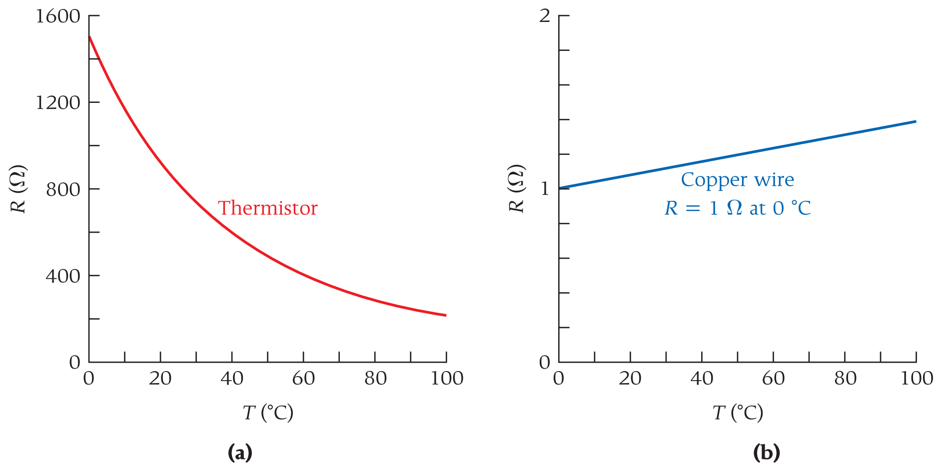


FIGURE 5.9 (a) The temperature dependence of the resistance of a thermistor. (b) The temperature dependence of the resistance of a copper wire that has a resistance of 1 Ω at $T = 0$ °C.

Microscopic Basis of Conduction in Solids

Conduction of current through solids results from the motion of electrons. In a metal conductor such as copper, the atoms of the metal form a regular array called a *crystal lattice*. The outermost electrons of each atom are essentially free to move randomly in this lattice. When an electric field is applied, the electrons drift in the direction opposite to that of the electric field. Resistance to drift occurs when electrons interact with the metal atoms in the lattice. When the temperature of the metal is increased, the motion of the atoms in the lattice increases. This, in turn, increases the probability that electrons will interact with the atoms, effectively increasing the resistance of the metal. A detailed discussion of the interaction between the lattice atoms and the conduction electrons is beyond the scope of this book. The approximate relations given by equations 5.12 and 5.13, which result from this interaction, are sufficient for our purposes.

The atoms of a semiconductor are also arranged in a crystal lattice. However, the outermost electrons of the atoms of the semiconductor are not free to move about within the lattice. To move about, the electrons must be given enough energy to attain an energy state where they can move freely. Thus, a typical semiconductor has a higher resistance than a metal conductor because it has many fewer conduction electrons. In addition, when a semiconductor is heated, many more electrons gain enough energy to move freely; thus, the resistance of the semiconductor decreases as its temperature increases.

5.4 Electromotive Force and Ohm's Law

For current to flow through a resistor, a potential difference must be established across the resistor. This potential difference, supplied by a battery or other device, is termed an **electromotive force,** abbreviated **emf.** (Electromotive force is not a force at all, but rather a potential difference. The term is still in widespread use but mainly in the form of its abbreviation, pronounced "ee-em-eff.") A device that maintains a potential difference is called an *emf device* and does work on the charge carriers. The potential difference created by the emf device is represented as V_{emf}. This text assumes that emf devices have terminals to which a circuit can be connected. The emf device is assumed to maintain a constant potential difference, V_{emf}, between these terminals.

Solar cells convert light energy from the Sun to electric energy. If you examine a battery, you will find its potential difference (sometimes colloquially called "voltage") written on it. This "voltage" is the potential difference (emf) that the battery can provide to a circuit.

(Note that a battery is a source of constant emf, it does *not* supply constant current to a circuit.) Rechargeable batteries also display a rating in mAh (milliampere-hour), which provides information on the total charge the battery can deliver when fully charged. The mAh is another unit of charge:

$$1 \text{ mAh} = (10^{-3} \text{ A})(3600 \text{ s}) = 3.6 \text{ As} = 3.6 \text{ C}.$$

Electrical components in a circuit can be sources of emf, capacitors, resistors, or other electrical devices. These components are connected with conducting wires. At least one component must be a source of emf because the potential difference created by the emf device is what drives the current through the circuit. You can think of an emf device as the pump in a water pipeline; without the pump, the water sits in the pipe and doesn't move. Once the pump is turned on, the water moves through the pipe in a continuous flow.

An electric circuit starts and ends at an emf device. Since the emf device maintains a constant potential difference, V_{emf}, between its terminals, positive current leaves the device at the higher potential of its positive terminal and enters its negative terminal at a lower potential. This lower potential is conventionally set to zero.

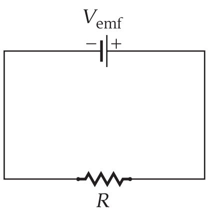


FIGURE 5.10 Simple circuit containing a source of emf and a resistor.

Consider a simple circuit of the form shown in Figure 5.10, where a source of emf provides a potential difference, V_{emf}, across a resistor with resistance R. Note an important convention for circuit diagrams: A resistor is always symbolized by a zigzag line, and it is assumed that all of the resistance, R, is concentrated there. The wires connecting the different circuit elements are represented by straight lines; it is implied that they do not have a resistance. Physical wires do, of course, have some resistance, but it is assumed to be negligible for the purpose of the diagram.

For a circuit like the one shown in Figure 5.10, the emf device provides the potential difference that creates the current flowing through the resistor. Therefore, in this case, Ohm's Law (equation 5.9) can be written in terms of the external emf as

$$V_{\text{emf}} = iR. \tag{5.14}$$

Note that, unlike Newton's law of gravitation or the law of conservation of energy, Ohm's Law is not a law of nature. It is not even obeyed by all resistors. For many resistors, called *ohmic resistors*, the current is directly proportional to the potential difference across the resistor over a wide range of temperatures and a wide range of applied potential differences. For other resistors, called *non-ohmic resistors*, current and potential difference are not directly proportional at all. Non-ohmic resistors include many kinds of transistors, which means that many modern electronic devices do not obey Ohm's Law. We'll take a closer look at one of these devices, the diode, in Section 5.8. Nevertheless, a large class of materials and devices (such as conventional wires, for example) do obey Ohm's Law, and thus it is worth devoting attention to its consequences. The remainder of this chapter (with the exception of Section 5.8) treats resistors as ohmic devices; that is, devices that obey Ohm's Law.

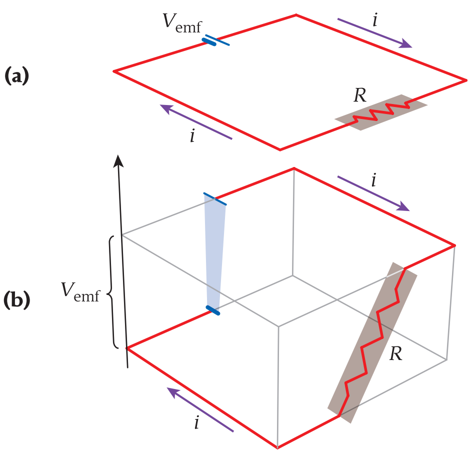


FIGURE 5.11 (a) Conventional representation of a simple circuit with a resistor and a source of emf. (b) Three-dimensional representation of the same circuit, displaying the potential at each point in the circuit. The current in the circuit is shown for both views.

The current, *i*, that flows through the resistor in Figure 5.10 also flows through the source of emf and the wires connecting the components. Because the wires are assumed to have zero resistance (as noted above), the change in potential of the current must occur in the resistor, according to Ohm's Law. This change is referred to as the **potential drop** across the resistor. Thus, the circuit shown in Figure 5.10 can be represented in a different way, making it clearer where the potential drop happens and showing which parts of the circuit are at which potential. Figure 5.11a shows the circuit in Figure 5.10. Figure 5.11b shows the same circuit but with the vertical dimension representing the value of the electric potential at different points around the circuit. The potential difference is supplied by the source of emf, and the entire potential drop occurs across the single resistor. (Remember, the convention is that the lines connecting the circuit elements in a circuit diagram represent wires with no resistance. Therefore, these connecting wires are represented in Figure 5.11b by horizontal lines, signifying that the entire wire is at exactly the same potential.) Ohm's Law applies for the potential drop across the resistor, and the current in the circuit can be calculated using equation 5.9.

Figure 5.11 illustrates an important point about circuits. Sources of emf add potential difference to a circuit, and potential drops through resistors reduce potential in the circuit. However, the total potential difference on any closed path around the complete circuit must be zero. This is a straightforward consequence of the law of conservation of energy. An analogy with gravity may help: You can gain and lose potential energy by moving up and down in a gravitational field (for example, by climbing up and down hills), but if you arrive back at the same point from which you started, the net energy gained or lost is exactly zero. The same holds for current flowing in a circuit: It does not matter how many potential drops or sources of emf are encountered on any closed loop; a given point always has the same value of the electric potential. The current can flow through the loop in either direction with the same result.

Self-Test Opportunity 5.2

A resistor with $R = 10.0\ \Omega$ is connected across a source of emf with potential difference $V_{emf} = 1.50$ V. What is the current flowing though the circuit?

Resistance of the Human Body

This short introduction of resistance and Ohm's Law leads to a point about electrical safety. It was mentioned earlier that currents above 100 mA can be deadly if they flow through human heart muscle. Ohm's Law makes it clear that the resistance of the human body determines whether a given potential difference–say, from a car battery–can be dangerous. Since we usually handle tools with our hands, the most relevant measure for the human body's resistance, R_{body}, is the resistance along a path from the fingertips of one hand to the fingertips of the other hand. (Note that the heart is pretty much in the middle of this path!) For most people this resistance is in the range $500\ \text{k}\Omega < R_{body} < 2\ \text{M}\Omega$. Most of this resistance comes from the skin, in particular, the layers of dead skin on the outside. However, if the skin is wet, its conductivity is drastically increased, and consequently, the body's resistance is drastically lowered. For a given potential difference, Ohm's Law implies that the current then drastically increases. Handling electrical devices in wet environments or touching them with your tongue is thus a very bad idea.

Wires in a circuit can have sharp points where they are cut. If these points penetrate the skin at the fingertips, the resistance of the skin is eliminated, and the fingertip-to-fingertip resistance is very drastically lowered. If a wire penetrates a blood vessel, the human body's resistance decreases even further, because blood has a high salinity and is thus a good conductor. In this case, even relatively small potential differences from batteries can have a deadly effect.

5.5 Resistors in Series

A circuit can contain more than one resistor and/or more than one source of emf. The analysis of circuits with multiple resistors requires different techniques. Let's first examine resistors connected in series.

Two resistors, R_1 and R_2, are connected in series with one source of emf with potential difference V_{emf} in the circuit shown in Figure 5.12. The potential drop across resistor R_1 is denoted by ΔV_1, and the potential drop across resistor R_2 by ΔV_2. The two potential drops must sum to the potential difference supplied by the source of emf:

$$V_{emf} = \Delta V_1 + \Delta V_2.$$

The crucial insight is that the same current must flow through all the elements of the circuit. How do we know this? Remember, at the beginning of this chapter, current was defined as the rate of change of the charge in time: $i = dq/dt$. Current has to be the same everywhere along a wire, and also in a resistor, because charge is conserved everywhere. No charge is lost or gained along the wire, and so the current is the same everywhere around the loop in Figure 5.12.

To clear up a common misconception, note that *there is no such thing as current getting "used up" in a resistor*. No matter how many resistors are connected in series, the current that flows into the first one is the same current that flows out of the last one. An analogy to water flowing in a pipe may help: No matter how long the pipe is and how many bends it may have, all water that flows into one end has to come out the other end.

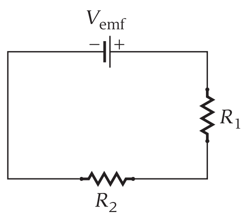


FIGURE 5.12 Circuit with two resistors in series with one source of emf.

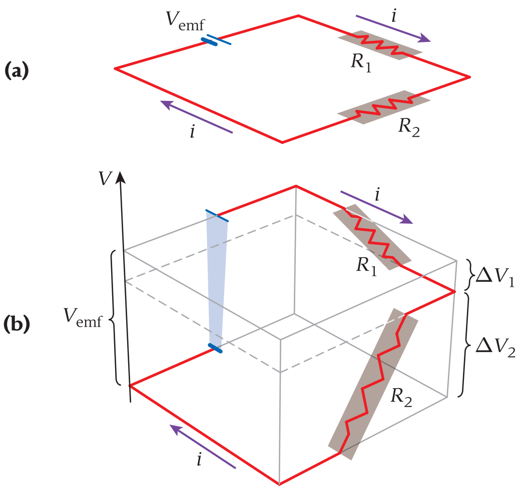


FIGURE 5.13 (a) Conventional representation of a simple circuit with two resistors in series and a source of emf. (b) Three-dimensional representation of the same circuit, displaying the potential at each point in the circuit. The current in the circuit is shown for both views.

Thus, the current flowing through each resistor in Figure 5.12 is the same. For each resistor, we can apply Ohm's Law and get

$$V_{emf} = iR_1 + iR_2.$$

An equivalent resistance, R_{eq}, can replace the two individual resistances:

$$V_{emf} = iR_1 + iR_2 = iR_{eq},$$

where

$$R_{eq} = R_1 + R_2.$$

Thus, two resistors in series can be replaced with an equivalent resistance equal to the sum of the two resistances. Figure 5.13 illustrates the potential drops in the series circuit of Figure 5.12, using a three-dimensional view.

The expression for the equivalent resistance of two resistors in series can be generalized to a circuit with n resistors in series:

$$R_{eq} = \sum_{i=1}^{n} R_i \quad \text{(for resistors in series).} \tag{5.15}$$

That is, if resistors are connected in a single path so that the same current flows through all of them, their total resistance is just the sum of their individual resistances.

Concept Check 5.3

What are the relative magnitudes of the two resistances in Figure 5.13?

a) $R_1 < R_2$

b) $R_1 = R_2$

c) $R_1 > R_2$

d) Not enough information is given in the figure to compare the resistances.

EXAMPLE 5.3 Internal Resistance of a Battery

When a battery is not connected in a circuit, the potential difference across its terminals is V_t. When the battery is connected in series with a resistor with resistance R, current i flows through the circuit. When the current is flowing, the potential difference, V_{emf}, across the terminals of the battery is less than V_t. This drop occurs because the battery has an internal resistance, R_i, which can be thought of as being in series with the external resistor (Figure 5.14). That is,

$$V_t = iR_{eq} = i(R + R_i).$$

The battery is depicted by the gray cylinder in Figure 5.14. The terminals of the battery are represented by points A and B.

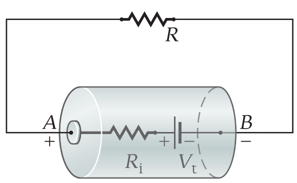


FIGURE 5.14 Battery (gray cylinder) with internal resistance R_i connected to an external resistor, R.

PROBLEM

Consider a battery that has $V_t = 12.0$ V when it is not connected to a circuit. When a 10.0-Ω resistor is connected with the battery, the potential difference across the battery's terminals drops to 10.9 V. What is the internal resistance of the battery?

SOLUTION

The current flowing through the external resistor is given by

$$i = \frac{\Delta V}{R} = \frac{10.9 \text{ V}}{10.0\ \Omega} = 1.09 \text{ A}.$$

The current flowing in the complete circuit, including the battery, must be the same as the current flowing in the external resistor. Thus, we have

$$V_t = iR_{eq} = i(R + R_i)$$

$$(R + R_i) = \frac{V_t}{i}$$

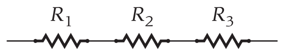
$$R_i = \frac{V_t}{i} - R = \frac{12.0 \text{ V}}{1.09 \text{ A}} - 10.0\ \Omega = 1.00\ \Omega.$$

The internal resistance of the battery is 1.00 Ω. Batteries with internal resistance are said to be *nonideal*. Unless otherwise specified, batteries in circuits will be assumed to have zero internal resistance. Such batteries are said to be *ideal*. An ideal battery maintains a constant potential difference between its terminals, independent of the current flowing.

Whether a battery can still provide energy cannot be determined by simply measuring the potential difference across the terminals. Instead, you must place a resistance on the battery and then measure the potential difference. If the battery is no longer functional, it may still provide its rated potential difference when not connected, but its potential difference may drop to zero when connected to an external resistance. Some brands of batteries have built-in devices to measure the functioning potential difference simply by pressing on a particular spot on the battery and observing an indicator.

Concept Check 5.4

Three identical resistors, R_1, R_2, and R_3, are wired together as shown in the figure. An electric current is flowing through the three resistors. The current through R_2



a) is the same as the current through R_1 and R_3.

b) is a third of the current through R_1 and R_3.

c) is twice the sum of the current through R_1 and R_3.

d) is three times the current through R_1 and R_3.

e) cannot be determined.

Resistor with a Nonconstant Cross Section

Up to now the discussion has assumed that a resistor has the same cross-sectional area, A, and the same resistivity, ρ, everywhere along its length (this was the implicit assumption in the derivation leading to equation 5.11). This is, of course, not always the case. How do we handle the analysis of a resistor whose cross-sectional area is a function of the position x along the resistor, $A(x)$, and/or whose resistivity can change as a function of position, $\rho(x)$? We simply divide the resistor into many very short pieces of length Δx and sum over all of them, since equation 5.15 says that the total resistance is the sum of all of the resistances of the individual short pieces; then we take the limit $\Delta x \rightarrow 0$. If this sounds like an integration to you, you are right. The general formula for computing the resistance of a resistor of length L with a nonuniform cross-sectional area, $A(x)$, is

$$R = \int_0^L \frac{\rho(x)}{A(x)} dx. \tag{5.16}$$

A concrete example will help clarify this equation.

SOLVED PROBLEM 5.2 Brain Probe

Reasearchers use a technique known as electrocorticography (ECoG) to measure the electric field generated by neurons in the brain. Some of these measurements can only be done by inserting very thin wires into the brain in order to probe directly into neurons. These wires are insulated, with only a very short tip exposed, which is pulled into a very fine conical tip. ECoG is being used to treat an epilepsy patient in Figure 5.15.

FIGURE 5.15 Electrocorticography performed with electrode grids on the cerebral cortex of an epilepsy patient.

PROBLEM

If the wire used for ECoG is made of tungsten and has a diameter of 0.74 mm and the tip has a length of 2.0 mm and is sharpened to a diameter of 2.4 μm at the end, what is the resistance of the tip? (The resistivity of tungsten is listed in Table 5.1 as $5.51 \times 10^{-8}\ \Omega$ m.)

SOLUTION

THINK First, why might one want to know the resistance? To measure electrical fields or potential differences in neurons, probes with a large resistance, say, on the order of kilo-ohms, cannot be used because the fields or differences will not be detectable. However, since resistance is inversely proportional to the cross-sectional area, a very small area means a relatively large resistance. The problem statement says that the probe has a very fine tip, much pointier than any sewing needle. Hence the need to find out the resistance of the probe before inserting it into the brain!

Clearly, we are dealing with a case of nonconstant cross-sectional area, and so we will need to perform the integration of equation 5.16. However, since the tip is entirely made of tungsten, the resistivity is constant throughout its volume, which will simplify the task.

SKETCH Figure 5.16a shows a three-dimensional view of the tip, and Figure 5.16b presents a cut through its symmetry plane and the integration path.

RESEARCH The research part is quite simple for this problem, because we already know which equation we need to use. However, equation 5.16 needs to be altered to reflect the fact that the resistivity is constant throughout the tip:

$$R = \rho \int_0^L \frac{1}{A(x)} dx, \tag{i}$$

– Continued

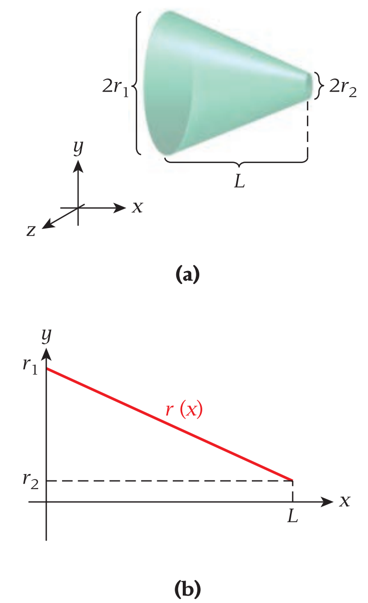


FIGURE 5.16 (a) Shape of the tip of the probe. (b) Coordinate system for the integration.

where $A(x)$ is the area of a circle, $A(x) = \pi[r(x)]^2$. The radius of the circle falls linearly from r_1 to r_2 (see Figure 5.16b):

$$r(x) = r_1 + \frac{(r_2 - r_1)x}{L}. \tag{ii}$$

SIMPLIFY We substitute the expression for the radius from equation (ii) into the formula for the area and then substitute the resulting expression for $A(x)$ into equation (i). We arrive at

$$R = \rho \int_0^L \frac{1}{\pi\left(r_1 + (r_2 - r_1)x/L\right)^2}\,dx.$$

This integral may look daunting at first sight, but except for x all the other quantities are constants. We consult an integration table or software and find

$$R = -\frac{\rho L}{\pi(r_2 - r_1)\left(r_1 + (r_2 - r_1)x/L\right)}\Bigg|_0^L = \frac{\rho L}{\pi r_1 r_2}.$$

CALCULATE Putting in the numerical values, we get

$$R = \frac{\left(5.51\times10^{-8}\,\Omega\;\;\text{m}\right)\left(2.0\times10^{-3}\,\text{m}\right)}{\pi(0.37\times10^{-3}\,\text{m})(1.2\times10^{-6}\,\text{m})} = 7.90039\times10^{-2}\,\Omega\;.$$

ROUND We report our result to the two significant figures to which the geometric properties of the tip were given:

$$R = 7.9\times10^{-2}\,\Omega\;.$$

DOUBLE-CHECK The value of 79 mΩ seems a very small resistance to a current that has to pass through a tip that is sharpened to a diameter of 2.4 μm. On the other hand, the tip has a very small length, which argues for a small resistance. An additional confidence builder is the fact that the units worked out properly.

However, there are also a few tests we can perform to convince ourselves that at least the asymptotic limits of the solution to $R = \rho L/(\pi r_1 r_2)$ are reasonable. First, as the length approaches zero, so does the resistance, as expected. Second, as the radius of either end of the tip approaches zero, the formula predicts an infinite resistance, which is also expected.

5.6 Resistors in Parallel

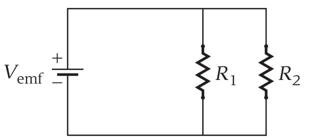


FIGURE 5.17 Circuit with two resistors connected in parallel and a single source of emf.

Instead of being connected in series, which causes all the current to pass through both resistors, two resistors can be connected in parallel, which divides the current between them, as shown in Figure 5.17. Again, to better illustrate the potential drops, Figure 5.18 shows the same circuit in a three-dimensional view.

In this case, the potential drop across each resistor is equal to the potential difference provided by the source of emf. Using Ohm's Law (equation 5.14) for the current i_1 in R_1 and the current i_2 in R_2, we have

$$i_1 = \frac{V_{\text{emf}}}{R_1}$$

and

$$i_2 = \frac{V_{\text{emf}}}{R_2}.$$

The total current from the source of emf, i, must be

$$i = i_1 + i_2.$$

Inserting the expressions for i_1 and i_2, we obtain

$$i = i_1 + i_2 = \frac{V_{\text{emf}}}{R_1} + \frac{V_{\text{emf}}}{R_2} = V_{\text{emf}}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

Ohm's Law (equation 5.14) can be rewritten as

$$i = V_{\text{emf}}\left(\frac{1}{R_{\text{eq}}}\right).$$

Thus, two resistors connected in parallel can be replaced with an equivalent resistance given by

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2}.$$

In general, the equivalent resistance for n resistors connected in parallel is given by

$$\frac{1}{R_{\text{eq}}} = \sum_{i=1}^{n} \frac{1}{R_i} \quad \text{(resistors in parallel).} \tag{5.17}$$

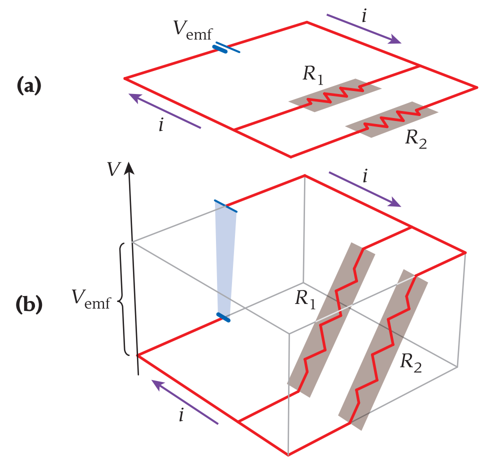
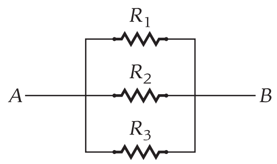


FIGURE 5.18 (a) Conventional representation of a simple circuit with two resistors in parallel and a source of emf. (b) Three-dimensional representation of the same circuit, displaying the potential at each point in the circuit.

Concept Check 5.5

Three identical resistors, R_1, R_2, and R_3, are wired together as shown in the figure. An electric current is flowing from point A to point B. The current flowing through R_2



a) is the same as the current through R_1 and R_3.

b) is a third of the current through R_1 and R_3.

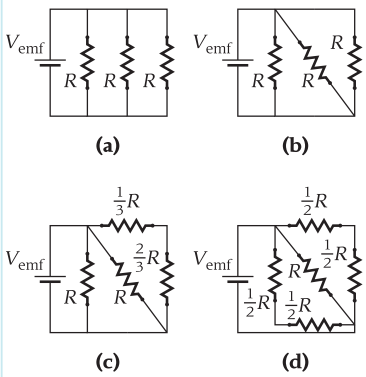
c) is twice the sum of the current through R_1 and R_3.

d) is three times the current through R_1 and R_3.

e) cannot be determined.

Concept Check 5.6

Which combination of resistors has the highest equivalent resistance?



a) combination (a)

b) combination (b)

c) combination (c)

d) combination (d)

e) The equivalent resistance is the same for all four.

EXAMPLE 5.4 Equivalent Resistance in a Circuit with Six Resistors

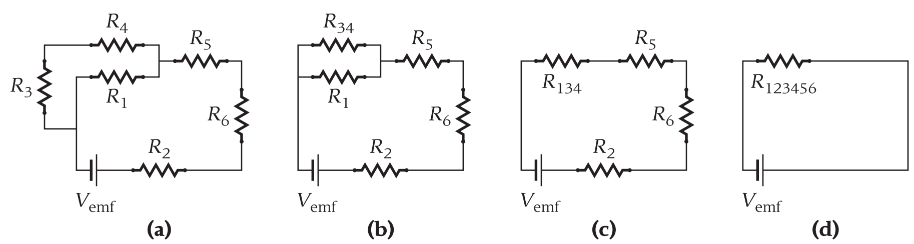


FIGURE 5.19 (a) Circuit with six resistors. (b)–(d) Steps in combining these resistors to determine the equivalent resistance.

PROBLEM

Figure 5.19a shows a circuit with six resistors, R_1 through R_6. What is the current flowing through resistors R_2 and R_3 in terms of V_{emf} and R_1 through R_6?

SOLUTION

We begin by identifying parts of the circuit that are clearly wired in parallel or in series. The current flowing through R_2 is the current flowing from the source of emf. We note that R_3 and R_4 are in series. Thus, we can write

$$R_{34} = R_3 + R_4. \tag{i}$$

– *Continued*

This substitution is made in Figure 5.19b. This figure shows us that R_{34} and R_1 are in parallel. We can then write

$$\frac{1}{R_{134}} = \frac{1}{R_1} + \frac{1}{R_{34}},$$

or

$$R_{134} = \frac{R_1 R_{34}}{R_1 + R_{34}}. \qquad \text{(ii)}$$

This substitution is depicted in Figure 5.19c. From this figure, we can see that R_2, R_5, R_6, and R_{134} are in series. Thus, we can write

$$R_{123456} = R_2 + R_5 + R_6 + R_{134}. \qquad \text{(iii)}$$

This substitution is shown in Figure 5.19d. We substitute for R_{34} and R_{134} from equations (i) and (ii) into equation (iii):

$$R_{123456} = R_2 + R_5 + R_6 + \frac{R_1 R_{34}}{R_1 + R_{34}} = R_2 + R_5 + R_6 + \frac{R_1(R_3 + R_4)}{R_1 + R_3 + R_4}.$$

Thus, i_2, the current flowing through R_2, is given by

$$i_2 = \frac{V_{\text{emf}}}{R_{123456}}.$$

Now we turn to the determination of the current flowing through R_3. Current i_2 is also flowing through the equivalent resistance R_{134} that contains R_3 (see Figure 5.19c). Thus, we can write

$$V_{134} = i_2 R_{134},$$

where V_{134} is the potential drop across the equivalent resistance R_{134}. The resistor R_1 and the equivalent resistance R_{34} are in parallel. Thus, V_{34}, the potential drop across R_{34}, is the same as the potential drop across R_{134} which is V_{134}. The resistors R_3 and R_4 are in series, and thus, i_3, the current flowing through R_3, is the same as i_{34}, the current flowing through R_{34}. We can thus write

$$V_{34} = V_{134} = i_{34} R_{34} = i_3 R_{34}.$$

Now we can express i_3 in terms of V and R_1 through R_6:

$$i_3 = \frac{V_{134}}{R_{34}} = \frac{i_2 R_{134}}{R_{34}} = \frac{\left(\dfrac{V_{\text{emf}}}{R_{123456}}\right) R_{134}}{R_{34}} = \frac{V_{\text{emf}} R_{134}}{R_{34} R_{123456}} = \frac{V_{\text{emf}}\left(\dfrac{R_1 R_{34}}{R_1 + R_{34}}\right)}{R_{34} R_{123456}} = \frac{V_{\text{emf}} R_1}{R_{123456}(R_1 + R_{34})}$$

or

$$i_3 = \frac{V_{\text{emf}} R_1}{\left(R_2 + R_5 + R_6 + \dfrac{R_1(R_3 + R_4)}{R_1 + R_3 + R_4}\right)(R_1 + R_3 + R_4)} = \frac{V_{\text{emf}} R_1}{(R_2 + R_5 + R_6)(R_1 + R_3 + R_4) + R_1(R_3 + R_4)}.$$

SOLVED PROBLEM 5.3 Potential Drop across a Resistor in a Circuit

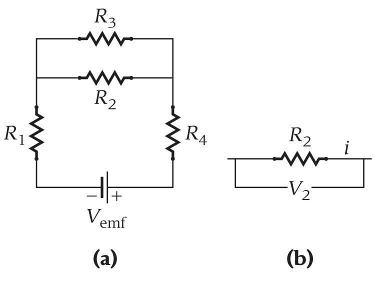


FIGURE 5.20 (a) A circuit with four resistors and a battery. (b) Potential drop across resistor R_2.

PROBLEM

The circuit shown in Figure 5.20a has four resistors and a battery with $V_{\text{emf}} = 149$ V. The values of the four resistors are $R_1 = 17.0\ \Omega$, $R_2 = 51.0\ \Omega$, $R_3 = 114.0\ \Omega$, and $R_4 = 55.0\ \Omega$. What is the magnitude of the potential drop across R_2?

SOLUTION

THINK The resistors R_2 and R_3 are in parallel and can be replaced with an equivalent resistance, R_{23}. The resistors R_1 and R_4 are in series with R_{23}. The current flowing through R_1, R_4, and R_{23} is the same because they are in series. We can obtain the current in the circuit by calculating the equivalent resistance for R_1, R_4, and R_{23} and using Ohm's Law. The potential drop across R_{23} is equal to the current flowing in the circuit times R_{23}. The potential drop across R_2 is the same as the potential drop across R_{23} because R_2 and R_3 are in parallel.

SKETCH The potential drop across resistor R_2 is illustrated in Figure 5.20b.

RESEARCH The equivalent resistance for R_2 and R_3 can be calculated using equation 5.17:

$$\frac{1}{R_{23}} = \frac{1}{R_2} + \frac{1}{R_3}. \qquad \text{(i)}$$

The equivalent resistance of the three resistors in series can be found using equation 5.15:

$$R_{\text{eq}} = \sum_{i=1}^{n} R_i = R_1 + R_{23} + R_4.$$

Finally, we obtain the current in the circuit using Ohm's Law:

$$V_{\text{emf}} = iR_{\text{eq}} = i(R_1 + R_{23} + R_4).$$

SIMPLIFY The potential drop, V_2, across R_2 is equal to the potential drop, V_{23}, across the equivalent resistance R_{23}:

$$V_2 = V_{23} = iR_{23} = \frac{V_{\text{emf}}}{R_1 + R_{23} + R_4} R_{23} = \frac{R_{23}V_{\text{emf}}}{R_1 + R_{23} + R_4}. \qquad \text{(ii)}$$

We can solve equation (i) for R_{23} to obtain

$$R_{23} = \frac{R_2 R_3}{R_2 + R_3}.$$

We can then use equation (ii) to determine the potential drop V_2 as

$$V_2 = \frac{\left(\dfrac{R_2R_3}{R_2 + R_3}\right)V_{\text{emf}}}{R_1 + \left(\dfrac{R_2R_3}{R_2 + R_3}\right) + R_4} = \frac{R_2R_3V_{\text{emf}}}{R_1(R_2 + R_3) + R_2R_3 + R_4(R_2 + R_3)},$$

which we can rewrite as

$$V_2 = \frac{R_2R_3V_{\text{emf}}}{(R_1 + R_4)(R_2 + R_3) + R_2R_3}.$$

CALCULATE Putting in the numerical values, we get

$$\begin{aligned} V_2 &= \frac{R_2R_3V_{\text{emf}}}{(R_1 + R_4)(R_2 + R_3) + R_2R_3} \\ &= \frac{(51.0\ \Omega)(114.0\ \Omega)(149\ \text{V})}{(17.0\ \Omega + 55.0\ \Omega)(51.0\ \Omega + 114.0\ \Omega) + (51.0\ \Omega)(114.0\ \Omega)} \\ &= 48.9593\ \text{V}. \end{aligned}$$

ROUND We report our result to three significant figures:

$$V = 49.0\ \text{V}.$$

DOUBLE-CHECK You may be tempted to avoid completing the analytic solution as we've done here. Instead, you may want to insert numbers earlier, for example, into the expression for R_{23}. So, to double-check our result, let's calculate the current in the circuit explicitly and then calculate the potential drop across R_{23} using that current. The equivalent resistance for R_2 and R_3 in parallel is

$$R_{23} = \frac{R_2R_3}{R_2 + R_3} = \frac{(51.0\ \Omega)(114.0\ \Omega)}{51.0\ \Omega + 114.0\ \Omega} = 35.2\ \Omega.$$

The current in the circuit is then

$$i = \frac{V_{\text{emf}}}{R_1 + R_{23} + R_4} = \frac{149.0\ \text{V}}{17.0\ \Omega + 35.2\ \Omega + 55.0\ \Omega} = 1.39\ \text{A}.$$

The potential drop across R_2 is then

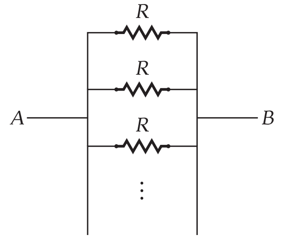
$$V_2 = iR_{23} = (1.39\ \text{A})(35.2\ \Omega) = 48.9\ \text{V},$$

which agrees with our result within rounding error. It is reassuring that both methods lead to the same answer.

We can also check that the potential drops across R_1, R_{23}, and R_4 sum to V_{emf}, as they should since R_1, R_{23}, and R_4 are in series. The potential drop across R_1 is $V_1 = iR_1 = (1.39\ \text{A})(17.0\ \Omega) = 23.6\ \text{V}$. The potential drop across R_4 is $V_4 = iR_4 = (1.39\ \text{A})(55.0\ \Omega) = 76.5\ \text{V}$. So the total potential drop is $V_{\text{total}} = V_1 + V_{23} + V_4 = (23.6\ \text{V}) + (48.9\ \text{V}) + (76.5\ \text{V}) = 149\ \text{V}$, which is equal to V_{emf}. Thus, our answer is consistent.

Concept Check 5.7

As more identical resistors, R, are added to the circuit shown in the figure, the resistance between points A and B will



a) increase.

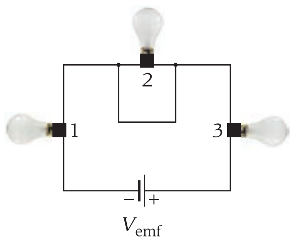
b) stay the same.

c) decrease.

d) change in an unpredictable manner.

Concept Check 5.8

Three light bulbs are connected in series with a battery that delivers a constant potential difference, V_{emf}. When a wire is connected across light bulb 2 as shown in the figure, light bulbs 1 and 3



a) burn just as brightly as they did before the wire was connected.

b) burn more brightly than they did before the wire was connected.

c) burn less brightly than they did before the wire was connected.

d) go out.

5.7 Energy and Power in Electric Circuits

Consider a simple circuit in which a source of emf with potential difference ΔV causes a current, i, to flow. The work required from the emf device to move a differential amount of charge, dq, from the negative terminal to the positive terminal (within the emf device) is equal to the increase in electric potential energy of that charge, dU:

$$dU = dq\,\Delta V.$$

Remembering that current is defined as $i = dq/dt$, we can rewrite the differential electric potential energy as

$$dU = i\,dt\,\Delta V.$$

Using the definition of power, $P = dU/dt$, and substituting into it the expression for the differential potential energy, we obtain

$$P = \frac{dU}{dt} = \frac{i\,dt\,\Delta V}{dt} = i\Delta V.$$

Thus, the product of the current times the potential difference gives the power supplied by the source of emf. By conservation of energy, this power is equal to the power dissipated in a circuit containing one resistor. In a more complicated circuit, each resistor will dissipate power at the rate given by this equation, where i and ΔV refer to the current through and potential difference across that resistor. Ohm's Law (equation 5.9) leads to different formulations of the power:

$$P = i\Delta V = i^2R = \frac{(\Delta V)^2}{R}. \tag{5.18}$$

The unit of power is the watt (W). Electrical devices, such as light bulbs, are rated in terms of how much power they consume. Your electric bill depends on how much electrical energy your appliances consume, and this energy is measured in kilowatt-hours (kWh).

Qualitatively, much or most of the energy dissipated in resistors is converted into heat. This phenomenon is employed in incandescent lighting, where heating a metal filament to a very high temperature causes it to emit light. The heat dissipated in electrical circuits is a huge problem for large-scale computer systems and server farms for the biggest Internet databases. These computer systems use thousands of processors for computing applications that can be parallelized. All of these processors emit heat, and very expensive cooling has to be provided to offset it. It turns out that the cost of cooling is one of the most stringent boundary conditions limiting the maximum size of these supercomputers.

Some of the power dissipated in circuits can be converted into mechanical energy by motors. The functioning of electric motors requires an understanding of magnetism and will be covered later.

EXAMPLE 5.5 Temperature Dependence of a Light Bulb's Resistance

A 100-W light bulb is connected in series to a source of emf with $V_{emf} = 100$ V. When the light bulb is lit, the temperature of its tungsten filament is 2520 °C.

PROBLEM

What is the resistance of the light bulb's tungsten filament at room temperature (20 °C)?

SOLUTION

The resistance of the filament when the light bulb is lit can be obtained using equation 5.18:

$$P = \frac{V_{emf}^2}{R}.$$

We rearrange this equation and substitute the numerical values to get the resistance of the filament:

$$R = \frac{V_{\text{emf}}^2}{P} = \frac{(100\ \text{V})^2}{100\ \text{W}} = 100\ \Omega.$$

The temperature dependence of the filament's resistance is given by equation 5.13:

$$R - R_0 = R_0\alpha(T - T_0).$$

We solve for the resistance at room temperature, R_0:

$$R = R_0 + R_0\alpha(T - T_0) = R_0\left[1 + \alpha(T - T_0)\right]$$

$$R_0 = \frac{R}{1 + \alpha(T - T_0)}.$$

Using the temperature coefficient of resistivity for tungsten from Table 5.1, we get

$$R_0 = \frac{R}{1 + \alpha(T - T_0)} = \frac{100\ \Omega}{1 + \left(4.5 \cdot 10^{-3}\ {}^\circ\text{C}^{-1}\right)(2520\ {}^\circ\text{C} - 20\ {}^\circ\text{C})} = 8.2\ \Omega.$$

High-Voltage Direct Current Power Transmission

The transmission of electrical power from power-generating stations to users of electricity is of great practical interest. Often, electrical power-generating stations are located in remote areas, and thus the power must be transmitted long distances. This is particularly true for clean power sources, such as hydroelectric dams and large solar farms in deserts.

The power, P, transmitted to users is the product of the current, i, and the potential difference, ΔV, in the power line: $P = i\Delta V$. Thus, the current required for a given power is $i = P/\Delta V$, and a higher potential difference means a lower current in the power line. Equation 5.18 indicates that the power dissipated in an electrical power transmission line, P_{loss}, is given by $P_{\text{loss}} = i^2R$. The resistance, R, of the power line is fixed; thus, decreasing the power lost during transmission means reducing the current carried in the transmission line. This reduction is accomplished by transmitting the power using a very high potential difference and a very low current. Looking at equation 5.18, you might argue that we could also write $P_{\text{loss}} = (\Delta V)^2/R$ and that a high potential difference means a large power loss instead of a small power loss. However, ΔV in this equation is the potential drop across the power line, not the potential difference at which the electrical power is being transmitted. The potential drop across the power line is $V_{\text{drop}} = iR$, which is much lower than the high potential difference used to transmit the electrical power. The expressions for the transmitted power and the dissipated power can be combined to give $P_{\text{loss}} = (P/\Delta V)^2R = P^2R/(\Delta V)^2$, which means that for a given amount of power, the dissipated power decreases as the square of the potential difference used to transmit the power.

However, alternating currents have the inherent disadvantage of high power losses. High-voltage direct current (HVDC) transmission lines do not have this problem and suffer only power losses due to the resistance of the power line. However, HVDC transmission lines have the extra requirement that alternating current must be converted to direct current for transmission and the direct current must be converted back to alternating current at the destination.

Part of the power produced by the hydroelectric plant shown in Figure 5.21 is transmitted via the world's largest HVDC transmission line a distance of about 800 km from the Itaipú Dam to São Paulo, Brazil, one of the ten largest metropolitan areas in the world. The transmission line carries 6300 MW of electrical power using direct current with a potential difference of ± 600 kV. The station at the Itaipú Dam that converts the alternating current to direct current is shown in Figure 5.21. The station in São Paulo that converts the transmitted direct current back to alternating current is shown in Figure 5.21.

FIGURE 5.21 The station that converts alternating current to direct current at the Itaipú Dam on the Paraná River in Brazil and Paraguay.

Self-Test Opportunity 5.3

Consider a battery with internal resistance R_i. What external resistance, R, will undergo the maximum heating when connected to this battery?

Future applications of HVDC power transmission include the transmission of power from solar power stations located in remote areas in the southwest United States to densely populated areas, such as large cities in California and Texas. Some of the most promising research on power transmission is focusing on the use of superconducting wires for this purpose. As mentioned earlier, once a material becomes superconducting, its resistance approaches zero. Therefore, it can transmit electrical power without significant losses. The most promising materials for superconducting power transmission wires are *high-temperature superconductors*, discovered only in 1986. These wires have to be cooled with liquid nitrogen, but we are approaching the point where their widespread use in long-distance power transmission can become a reality.

SOLVED PROBLEM 5.4 Size of Wire for a Power Line

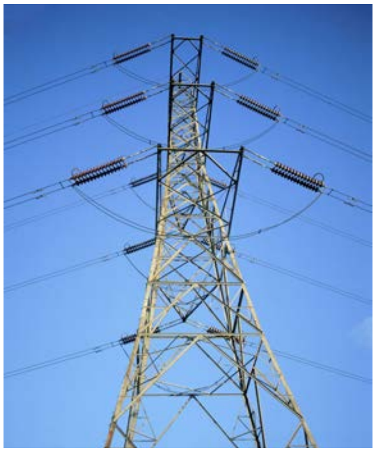


FIGURE 5.22 HVDC power transmission line.

PROBLEM

Imagine you are designing the HVDC power line from the Itaipú Dam on the Paraná River in Brazil and Paraguay to the city of São Paulo in Brazil. The power line is 800 km long and transmits 6300 MW of power at a potential difference of 1.20 MV. (Figure 5.22 shows an HVDC line.)

The electric company requires that no more than 25% of the power be lost in transmission. If the line consists of one wire made out of copper and having a circular cross section, what is the minimum diameter of the wire?

SOLUTION

THINK Knowing the power transmitted and the potential difference with which it is transmitted, we can calculate the current carried in the line. We can then express the power lost in terms of the resistance of the transmission line. With the current and the resistance of the wire, we can write an expression for the power lost during transmission. The resistance of the wire is a function of the diameter of the wire, the length of the wire, and the resistivity of copper. We can then solve for the diameter of the wire that will keep the power loss within the specified limit.

SKETCH A sketch of a copper wire of length L and diameter d is shown in Figure 5.23.

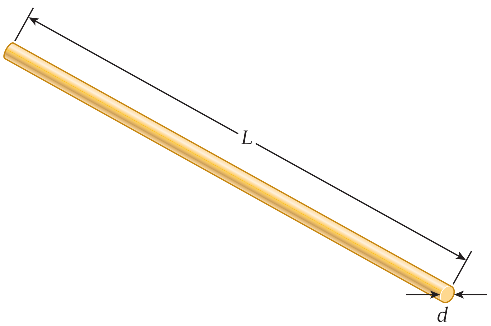


FIGURE 5.23 An HVDC transmission line consisting of a copper conductor (not to scale).

RESEARCH The power, P, carried in the line is related to the current, i, and the potential difference, ΔV: $P = i\Delta V$. The power lost in transmission, P_{lost}, can be related (see equation 5.18) to the current in the wire and the resistance, R, of the wire:

$$P_{\text{lost}} = i^2 R. \tag{i}$$

The resistance of the wire is given by equation 5.11:

$$R = \rho_{\text{Cu}} \frac{L}{A}, \tag{ii}$$

where ρ_{Cu} is the resistivity of copper, L is the length of the wire, and A is the cross-sectional area of the wire.

The cross-sectional area of the wire is the area of a circle:

$$A = \pi \left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4},$$

where d is the diameter of the wire. Thus, with the area of a circle substituted for A, equation (ii) becomes

$$R = \rho_{\text{Cu}} \frac{L}{\pi d^2/4}. \tag{iii}$$

SIMPLIFY We can solve $P = i\Delta V$ for the current in the wire:

$$i = \frac{P}{\Delta V}.$$

Substituting this expression for the current and that for the resistance from equation (iii) into equation (i) for the lost power gives

$$P_{\text{lost}} = \left(\frac{P}{\Delta V}\right)^2 \left(\rho_{\text{Cu}} \frac{L}{\pi d^2/4}\right) = \frac{4P^2 \rho_{\text{Cu}} L}{\pi (\Delta V)^2 d^2}.$$

The fraction of lost power relative to total power, *f*, is

$$\frac{P_{\text{lost}}}{P} = \frac{\left(\dfrac{4P^2 \rho_{\text{Cu}} L}{\pi (\Delta V)^2 d^2}\right)}{P} = \frac{4P \rho_{\text{Cu}} L}{\pi (\Delta V)^2 d^2} = f.$$

Solving this equation for the diameter of the wire gives

$$d = \sqrt{\frac{4P\rho_{\text{Cu}} L}{f\pi (\Delta V)^2}}.$$

CALCULATE Putting in the numerical values gives us

$$d = \sqrt{\frac{4\left(6300 \times 10^6\ \text{W}\right)\left(1.72 \times 10^{-8}\ \Omega\ \text{m}\right)\left(800 \times 10^3\ \text{m}\right)}{(0.25)\pi \left(1.20 \times 10^6\ \text{V}\right)^2}} = 0.0175099\ \text{m}.$$

ROUND Rounding to three significant figures gives us the minimum diameter of the copper wire:

$$d = 1.75\ \text{cm}.$$

DOUBLE-CHECK To double-check our result, let's calculate the resistance of this transmission line. Using our calculated value for the diameter, we can find the cross-sectional area and then, using equation 5.11, obtain

$$R = \rho_{\text{Cu}} \frac{L}{\pi d^2/4} = \frac{4\rho_{\text{Cu}} L}{\pi d^2} = \frac{4\left(1.72 \times 10^{-8}\,\Omega\ \text{m}\right)\left(800 \times 10^3\ \text{m}\right)}{\pi \left(1.75 \times 10^{-2}\ \text{m}\right)^2} = 57.2\ \Omega.$$

The current transmitted is

$$i = \frac{P}{V} = \frac{6300 \times 10^6\ \text{W}}{1.20 \times 10^6\ \text{V}} = 5250\ \text{A}.$$

The power lost is then

$$P = i^2 R = (5250\ \text{A})^2 (57.2\ \Omega) = 1580\ \text{MW},$$

which is close (within rounding error) to 25% of the total power of 6300 MW. Thus, our result seems reasonable.

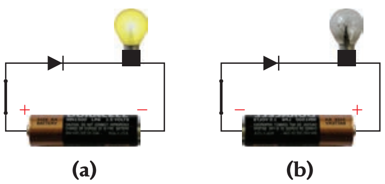


FIGURE 5.24 (a) The circuit of Figure 5.2c, but with a diode included. (b) Reversing the potential difference from the battery causes the current to stop flowing and the light bulb to stop shining.

5.8 Diodes: One-Way Streets in Circuits

Section 5.4 stated that many resistors obey Ohm's Law. It was noted, however, that there are also non-ohmic resistors that do not obey Ohm's Law. A very common and extremely useful example is a diode. A *diode* is an electronic device that is designed to conduct current in one direction and not in the other direction. Remember that Figure 5.2c showed that a light bulb was still shining with the same intensity when the battery it was connected to was reversed. If a diode (represented by the symbol ▶|) is added to the same circuit, the diode prevents the current from flowing when the potential difference delivered by the battery is reversed; see Figure 5.24. The diode acts like a one-way street for the current.

Figure 5.25 shows current versus potential difference for a 3-Ω ohmic resistor and a silicon diode. The resistor obeys Ohm's Law, with the current flowing in the opposite direction when the potential difference is negative. The plot of current versus potential difference for the resistor is a straight line with a slope of $\frac{1}{3}$ Ω. The silicon diode is wired so that it will not conduct

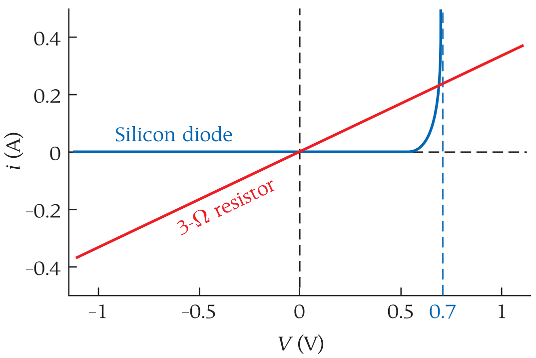


FIGURE 5.25 Current as a function of potential difference for a resistor (red) and a diode (blue).

any current when there is a negative potential difference. This silicon diode, like most, will conduct current if the potential difference is above 0.7 V. For potential differences above this threshold, the diode is essentially a conductor; below this threshold, the diode will not conduct current. The turn-on of the diode above the threshold potential difference increases exponentially; it can be close to instantaneous, as is visible in Figure 5.25.

The fundamental physics principles that underlie the functioning of diodes require an understanding of quantum mechanics.

One particularly useful kind of diode is the light-emitting diode (LED), which not only regulates current in a circuit but also emits light of a single wavelength in a very controlled way. LEDs that emit light of many different wavelengths have been manufactured, and they emit light much more efficiently than conventional incandescent bulbs do. Light intensity is measured in lumens (lm). Light sources can be compared in terms of how many lumens they produce per watt of electrical power. During the last decade, intensive research into LED technology has resulted in huge increases in LED output efficiency, which has reached values of 130 to 170 lm/W. This compares very favorably with conventional incandescent lights (which are in the range from 5 to 20 lm/W), halogen lights (20 to 30 lm/W), and even fluorescent high-efficiency lights (30 to 95 lm/W). Prices for LEDs (in particular, "white" LEDs) are still comparatively high but are expected to decrease significantly. The United States uses over 100 billion kWh of electrical energy for lighting alone each year, which is approximately 10% of the total U.S. energy consumption. Universal use of LED lighting could save 70% to 90% of those 100 billion kWh, approximately the annual energy output of 10 nuclear power plants (~1 GW power each).

LEDs are also used in large display screens, where high light output is desirable. Perhaps the most impressive of these was showcased during the opening ceremony of the 2008 Beijing Olympics (Figure 5.26). It used 44,000 individual LEDs and measured an astounding 147 m by 22 m.

FIGURE 5.26 Giant LED screen used during the opening ceremony of the 2008 Olympic Games in Beijing.

Self-Test Opportunity 5.4

Suppose the battery in Figure 5.24 has a potential difference of 1.5 V across its terminals and the diode is a silicon diode like the one in Figure 5.25. What are the potential drops across the diode and the light bulb in parts (a) and (b) of Figure 5.24?

WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- Current, i, is defined as the rate at which charge, q, flows past a particular point: $i = \dfrac{dq}{dt}$.
- The magnitude of the average current density, J, at a given cross-sectional area, A, in a conductor is given by $J = \dfrac{i}{A}$.
- The magnitude of the current density, J, is related to the magnitude of the drift velocity, v_d, of the current-carrying charges, $-e$, by $J = \dfrac{i}{A} = -nev_d$, where n is the number of charge carriers per unit volume.
- The resistivity, ρ, of a material is defined in terms of the magnitudes of the electric field applied across the material, E, and the resulting current density, J: $\rho = \dfrac{E}{J}$.
- The resistance, R, of a specific device having resistivity ρ, length L, and constant cross-sectional area A, is $R = \rho\dfrac{L}{A}$.
- The temperature dependence of the resistivity of a material is given by $\rho - \rho_0 = \rho_0\alpha(T - T_0)$, where ρ is the final resistivity, ρ_0 is the initial resistivity, α is the temperature coefficient of electric resistivity, T is the final temperature, and T_0 is the initial temperature.
- The electromotive force, or emf, is a potential difference created by a device that drives current through a circuit.
- Ohm's Law states that when a potential difference, ΔV, appears across a resistor, R, the current, i, flowing through the resistor is $i = \dfrac{\Delta V}{R}$.
- Resistors connected in series can be replaced with an equivalent resistance, R_{eq}, given by the sum of the resistances of the resistors: $R_{eq} = \sum_{i=1}^{n} R_i$.
- Resistors connected in parallel can be replaced with an equivalent resistance, R_{eq}, given by $\dfrac{1}{R_{eq}} = \sum_{i=1}^{n} \dfrac{1}{R_i}$.
- The power, P, dissipated by a resistor, R, through which a current, i, flows is given by $P = i\Delta V = i^2R = \dfrac{(\Delta V)^2}{R}$, where ΔV is the potential drop across the resistor.

ANSWERS TO SELF-TEST OPPORTUNITIES

5.1 $\dfrac{700 \text{ mAh}}{0.1 \text{ mA}} = 7000 \text{ h} \approx 292 \text{ days}.$

5.2 $\Delta V = iR \Rightarrow i = \dfrac{\Delta V}{R} = \dfrac{1.50 \text{ V}}{10.0\ \Omega} = 0.150 \text{ A}.$

5.3 The maximum heating of the external resistance occurs when the external resistance is equal to the internal resistance.

$$V_\text{t} = V_\text{emf} + iR_\text{i} = i(R + R_\text{i})$$

$$P_\text{heat} = i^2 R = \frac{V_\text{t}^2 R}{(R + R_\text{i})^2}$$

$$\frac{dP_\text{heat}}{dR} = -\frac{2V_\text{t}^2 R}{(R + R_\text{i})^3} + \frac{V_\text{t}^2}{(R + R_\text{i})^2} = 0 \text{ at extremum.}$$

From this, it follows that $R = R_\text{i}$.

You can check that this extremum is a maximum by taking the second derivative at $R = R_\text{i}$:

$$\left.\frac{d^2 P_\text{heat}}{dR^2}\right|_{R=R_\text{i}} = V_\text{t}^2\left(\frac{6}{16R^3} - \frac{5}{8R^3}\right) = -\frac{V_\text{t}^2}{R^3}\left(\frac{1}{2}\right) < 0.$$

5.4 The sum of the two potential drops, across the diode and across the light bulb, has to equal 1.5 V, the potential difference supplied by the battery. In part (b), the diode prevents any current from flowing; therefore, the potential drop across the diode is 1.5 V and that across the light bulb is zero. In part (a), the potential drop across the diode is 0.7 V (see Figure 5.25), and therefore that across the light bulb is 1.5 V − 0.7 V = 0.8 V.

PROBLEM-SOLVING GUIDELINES

1. If a circuit diagram is not given as part of the problem statement, draw one yourself and label all the given values and unknown components. Indicate the direction of the current, starting from the emf source. (Don't worry about getting the direction of the current wrong in your diagram; if you guess wrong, your final answer for the current will be a negative number.)

2. Sources of emf supply potential to a circuit, and resistors reduce potential in the circuit. However, be careful to check the direction of the potential of the emf source relative to that of the current; a current flowing in the direction opposite to the potential of an emf device picks up a negative potential difference.

3. The sum of the potential drops across resistors in a circuit equals the net amount of emf supplied to the circuit. (This is a consequence of the law of conservation of energy.)

4. In any given wire segment, the current is the same everywhere. (This is a consequence of the law of conservation of charge.)

MULTIPLE-CHOICE QUESTIONS

5.1 If the current through a resistor is increased by a factor of 2, how does this affect the power that is dissipated?

a) It decreases by a factor of 4.
b) It increases by a factor of 2.
c) It decreases by a factor of 8.
d) It increases by a factor of 4.

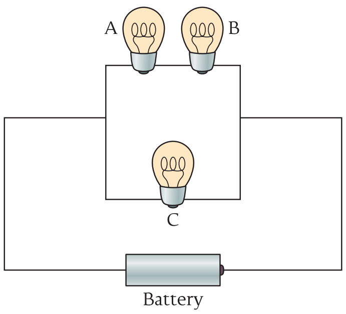
5.2 You make a parallel connection between two resistors, resistor A having a very large resistance and resistor B having a very small resistance. The equivalent resistance for this combination will be

a) slightly greater than the resistance of resistor A.
b) slightly less than the resistance of resistor A.
c) slightly greater than the resistance of resistor B.
d) slightly less than the resistance of resistor B.

5.3 Two cylindrical wires, 1 and 2, made of the same material, have the same resistance. If the length of wire 2 is twice that of wire 1, what is the ratio of their cross-sectional areas, A_1 and A_2?

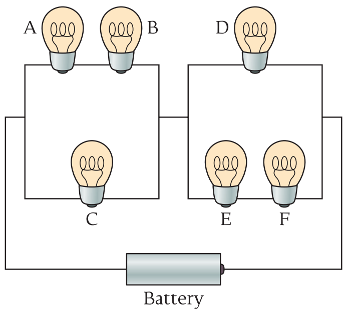
a) $A_1/A_2 = 2$
b) $A_1/A_2 = 4$
c) $A_1/A_2 = 0.5$
d) $A_1/A_2 = 0.25$

5.4 All three light bulbs in the circuit shown in the figure are identical. Which of the three shines the brightest?



a) A
b) B
c) C
d) A and B
e) All three are equally bright.

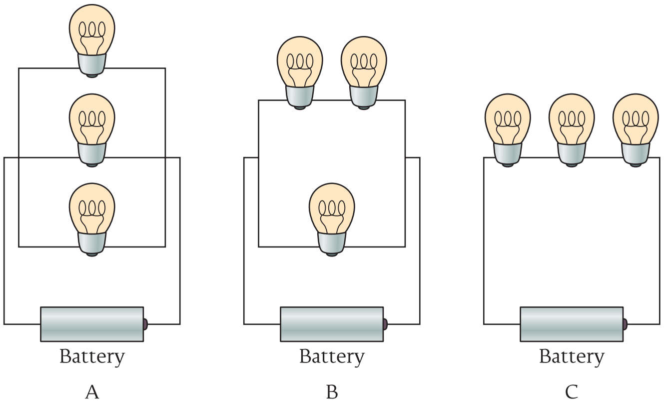
5.5 All of the six light bulbs in the circuit shown in the figure are identical. Which ordering correctly expresses the relative brightness of the bulbs? (*Hint:* The more current flowing through a light bulb, the brighter it is!)



a) A = B > C = D > E = F
b) A = B = E = F > C = D
c) C = D > A = B = E = F
d) A = B = C = D = E = F

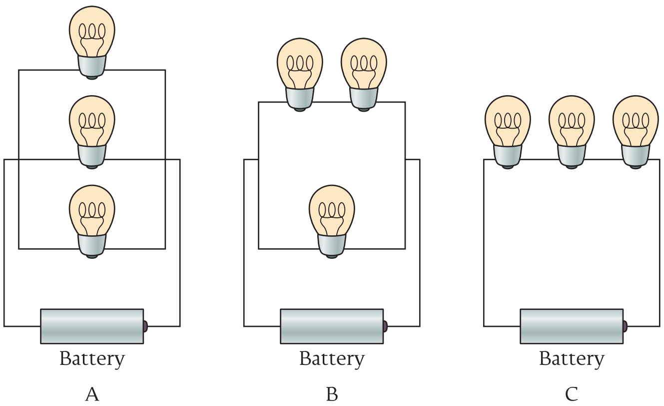
5.6 Which of the arrangements of three identical light bulbs shown in the figure draws the most current from the battery?

a) A
b) B
c) C
d) All three draw equal current.
e) A and C are tied for drawing the most current.

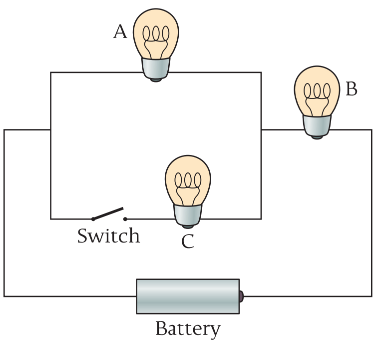


5.7 Which of the arrangements of three identical light bulbs shown in the figure has the highest resistance?

a) A
b) B
c) C
d) All three have equal resistance.
e) A and C are tied for having the highest resistance.



5.8 Three identical light bulbs are connected as shown in the figure. Initially the switch is closed. When the switch is opened (as shown in the figure), bulb C goes off. What happens to bulbs A and B?



a) Bulb A gets brighter, and bulb B gets dimmer.
b) Both bulbs A and B get brighter.
c) Both bulbs A and B get dimmer.
d) Bulb A gets dimmer, and bulb B gets brighter.

5.9 Which of the following wires has the largest current flowing through it?

a) a 1 m long copper wire of diameter 1 mm connected to a 10-V battery
b) a 0.5-m long copper wire of diameter 0.5 mm connected to a 5-V battery
c) a 2-m long copper wire of diameter 2 mm connected to a 20-V battery
d) a 1 m long copper wire of diameter 0.5 mm connected to a 5-V battery
e) All of the wires have the same current flowing through them.

5.10 Ohm's Law states that the potential difference across a device is equal to

a) the current flowing through the device times the resistance of the device.
b) the current flowing through the device divided by the resistance of the device.
c) the resistance of the device divided by the current flowing through the device.
d) the current flowing through the device times the cross-sectional area of the device.
e) the current flowing through the device times the length of the device.

5.11 A constant electric field is maintained inside a semiconductor. As the temperature is lowered, the magnitude of the current density inside the semiconductor

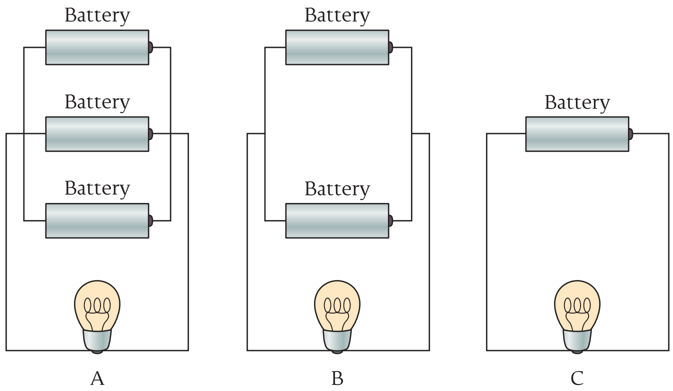
a) increases.
b) stays the same.
c) decreases.
d) may increase or decrease.

5.12 Which of the following is an incorrect statement?

a) The currents through electronic devices connected in series are equal.
b) The potential drops across electronic devices connected in parallel are equal.
c) More current flows across the smaller resistance when two resistors are connected in parallel.
d) More current flows across the smaller resistance when two resistors are connected in series.

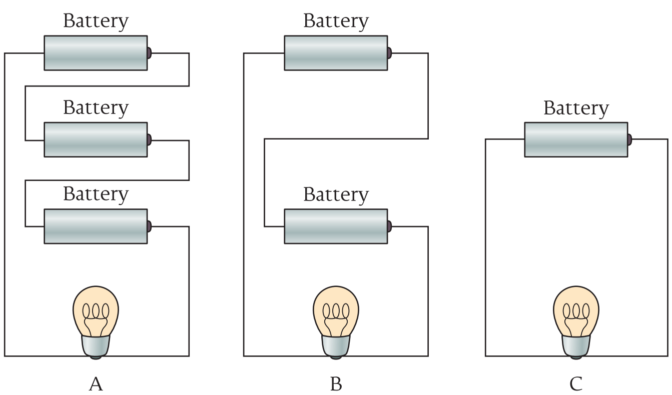
5.13 Identical batteries are connected in three different arrangements to the same light bulb as shown in the figure. Assume that the batteries have no internal resistance. In which arrangement will the light bulb shine the brightest?

a) A
b) B
c) C
d) The bulb will have the same brightness in all three arrangements.
e) The bulb will not light in any of the arrangements.



5.14 Identical batteries are connected in three different arrangements to the same light bulb as shown in the figure. Assume that the batteries have no internal resistance. In which arrangement will the light bulb shine the brightest?

a) A
b) B
c) C
d) The bulb will have the same brightness in all three arrangements.
e) The bulb will not light in any of the arrangements.



CONCEPTUAL QUESTIONS

5.15 What would happen to the drift velocity of electrons in a wire if the resistance due to collisions between the electrons and the atoms in the crystal lattice of the metal disappeared?

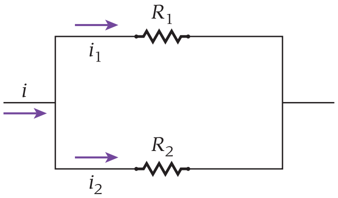
5.16 Why do light bulbs typically burn out just as they are turned on rather than while they are lit?

5.17 Two identical light bulbs are connected to a battery. Will the light bulbs be brighter if they are connected in series or in parallel?

5.18 Two resistors with resistances R_1 and R_2 are connected in parallel. Demonstrate that, no matter what the actual values of R_1 and R_2 are, the equivalent resistance is always less than the smaller of the two resistances.

5.19 Show that for resistors connected in series, it is always the highest resistance that dissipates the most power, while for resistors connected in parallel, it is always the lowest resistance that dissipates the most power.

5.20 For the connections shown in the figure, determine the current i_1 in terms of the total current, i, and R_1 and R_2.



5.21 An infinite number of resistors are connected in parallel. If $R_1 = 10\ \Omega$, $R_2 = 10^2\ \Omega$, $R_3 = 10^3\ \Omega$, and so on, show that $R_{eq} = 9\ \Omega$.

5.22 You are given two identical batteries and two pieces of wire. The red wire has a higher resistance than the black wire. You place the red wire across the terminals of one battery and the black wire across the terminals of the other battery. Which wire gets hotter?

5.23 Should light bulbs (ordinary incandescent bulbs with tungsten filaments) be considered ohmic resistors? Why or why not? How would this be determined experimentally?

5.24 A charged-particle beam is used to inject a charge, Q_0, into a small, irregularly shaped region (not a cavity, just some region within the solid block) in the interior of a block of ohmic material with conductivity σ and permittivity ε at time $t = 0$. Eventually, all the injected charge will move to the outer surface of the block, but how quickly?

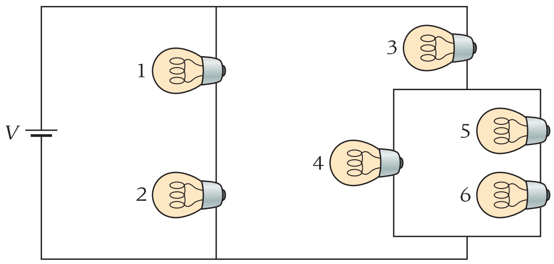
a) Derive a differential equation for the charge, $Q(t)$, in the injection region as a function of time.

b) Solve the equation from part (a) to find $Q(t)$ for all $t \geq 0$.

c) For copper, a good conductor, and for quartz (crystalline SiO_2), an insulator, calculate the time for the charge in the injection region to decrease by half. Look up the necessary values. Assume that the effective "dielectric constant" of copper is 1.00000.

5.25 Show that the drift speed of free electrons in a wire does not depend on the cross-sectional area of the wire.

5.26 Rank the brightness of the six identical light bulbs in the circuit in the figure. Each light bulb may be treated as an identical resistor with resistance R.



5.27 Two conductors of the same length and radius are connected to the same emf device. If the resistance of one is twice that of the other, to which conductor is more power delivered?

EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

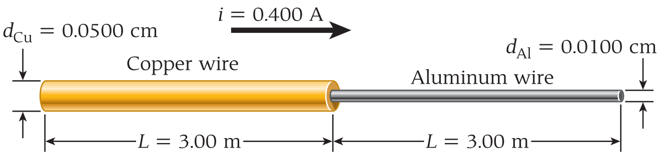
Sections 5.1 and 5.2

5.28 How many protons are in the beam traveling close to the speed of light in the Tevatron at Fermilab, which is carrying 11 mA of current around the 6.3 km circumference of the main Tevatron ring?

5.29 What is the current density in an aluminum wire having a radius of 1.00 mm and carrying a current of 1.00 mA? What is the drift speed of the electrons carrying this current? The density of aluminum is 2.70×10^3 kg/m^3, and 1 mole of aluminum has a mass of 26.98 g. There is one conduction electron per atom in aluminum.

•5.30 A copper wire has a diameter $d_{Cu} = 0.0500$ cm, is 3.00 m long, and has a charge-carrier density of 8.50×10^{28} electrons/m^3. As shown in the figure, the copper wire is attached to an equal length of aluminum wire with a diameter $d_{Al} = 0.0100$ cm and a charge-carrier density of 6.02×10^{28} electrons/m^3. A current of 0.400 A flows through the copper wire.

a) What is the ratio of the current densities in the two wires, J_{Cu}/J_{Al}?



b) What is the ratio of the drift velocities in the two wires, v_{d-Cu}/v_{d-Al}?

•5.31 A current of 0.123 mA flows in a silver wire whose cross-sectional area is 0.923 mm^2.

a) Find the density of electrons in the wire, assuming that there is one conduction electron per silver atom.

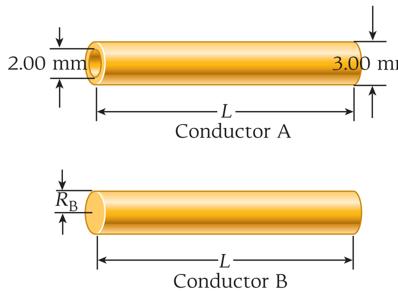
b) Find the current density in the wire assuming that the current is uniform.

c) Find the electrons' drift speed.

Section 5.3

5.32 What is the resistance of a copper wire of length $l = 10.9$ m and diameter $d = 1.30$ mm? The resistivity of copper is $1.72 \times 10^{-8}\ \Omega$ m.

5.33 Two conductors are made of the same material and have the same length L. Conductor A is a hollow tube with inside diameter 2.00 mm and outside diameter 3.00 mm; conductor B is a solid wire with radius R_B. What value of R_B is required for the two conductors to have the same resistance measured between their ends?



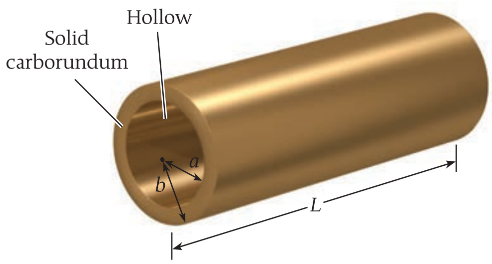
5.34 A copper coil has a resistance of 0.100 Ω at room temperature (20.0 °C). What is its resistance when it is cooled to $-100.$ °C?

5.35 What gauge of aluminum wire will have the same resistance per unit length as 12 gauge copper wire?

5.36 A rectangular wafer of pure silicon, with resistivity $\rho = 2300\ \Omega$ m, measures 2.00 cm by 3.00 cm by 0.0100 cm. Find the maximum resistance of this rectangular wafer between any two faces.

•5.37 A copper wire that is 1.00 m long and has a radius of 0.500 mm is stretched to a length of 2.00 m. What is the fractional change in resistance, $\Delta R/R$, as the wire is stretched? What is $\Delta R/R$ for a wire of the same initial dimensions made out of aluminum?

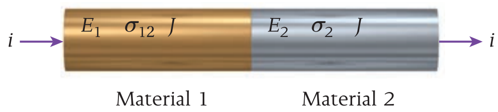
•5.38 The most common material used for sandpaper, silicon carbide, is also widely used in electrical applications. One common device is a tubular resistor made of a special grade of silicon carbide called *carborundum*. A particular carborundum resistor (see the figure) consists of a thick-walled cylindrical shell (a pipe) of inner radius $a = 1.50$ cm, outer radius $b = 2.50$ cm, and length $L = 60.0$ cm. The resistance of this carborundum resistor at 20.0 °C is 1.00 Ω.



a) Calculate the resistivity of carborundum at room temperature. Compare this to the resistivities of the most commonly used conductors (copper, aluminum, and silver).

b) Carborundum has a high temperature coefficient of resistivity: $\alpha = 2.14 \cdot 10^{-3}\ \text{K}^{-1}$. If, in a particular application, the carborundum resistor heats up to 300. °C, what is the percentage change in its resistance between room temperature (20.0 °C) and this operating temperature?

••5.39 As illustrated in the figure, a current, i, flows through the junction of two materials with the same cross-sectional area and with conductivities σ_1 and σ_2. Show that the total amount of charge at the junction is $\varepsilon_0 i(1/\sigma_2 - 1/\sigma_1)$.



Section 5.4

5.40 A potential difference of 12.0 V is applied across a wire of cross-sectional area 4.50 mm^2 and length 1000. km. The current passing through the wire is 3.20×10^{-3} A.

a) What is the resistance of the wire?

b) What type of wire is this?

5.41 One brand of 12.0 V automotive battery used to be advertised as providing "600 cold-cranking amps." Assuming that this is the current the battery supplies if its terminals are shorted, that is, connected to negligible resistance, determine the internal resistance of the battery. (*IMPORTANT*: **Do not** attempt such a connection as it could be lethal!)

5.42 A copper wire has radius $r = 0.0250$ cm, is 3.00 m long, has resistivity $\rho = 1.72 \cdot 10^{-8}\ \Omega$ m, and carries a current of 0.400 A. The wire has a charge-carrier density of $8.50 \cdot 10^{28}$ electrons/m^3.

a) What is the resistance, R, of the wire?

b) What is the electric potential difference, ΔV, across the wire?

c) What is the electric field, E, in the wire?

•5.43 A 34 gauge copper wire ($A = 0.0201$ mm^2), with a constant potential difference of 0.100 V applied across its 1.00 m length at room temperature (20.0 °C), is cooled to liquid nitrogen temperature (77 K = -196 °C).

a) Determine the percentage change in the wire's resistance during the drop in temperature.

b) Determine the percentage change in current flowing in the wire.

c) Compare the drift speeds of the electrons at the two temperatures.

Section 5.5

5.44 A resistor of unknown resistance and a 35.0-Ω resistor are connected across a 120. V emf device in such a way that an 11.0 A current flows. What is the value of the unknown resistance?

5.45 A battery has a potential difference of 14.50 V when it is not connected in a circuit. When a 17.91 Ω resistor is connected across the battery, the potential difference of the battery drops to 12.68 V. What is the internal resistance of the battery?

5.46 When a battery is connected to a 100. Ω resistor, the current is 4.00 A. When the same battery is connected to a 400. Ω resistor, the current is 1.01 A. Find the emf supplied by the battery and the internal resistance of the battery.

•5.47 A light bulb is connected to a source of emf. There is a 6.20 V drop across the light bulb and a current of 4.10 A flowing through the light bulb.

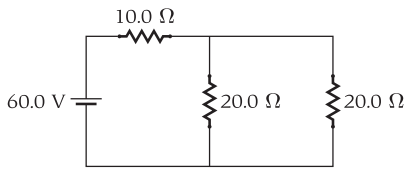
a) What is the resistance of the light bulb?

b) A second light bulb, identical to the first, is connected in series with the first bulb. The potential drop across the bulbs is now 6.29 V, and the current through the bulbs is 2.90 A. Calculate the resistance of each light bulb.

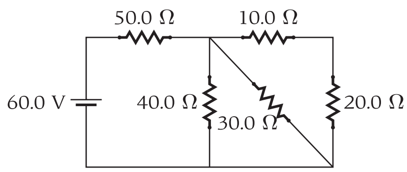
c) Why are your answers to parts (a) and (b) not the same?

Section 5.6

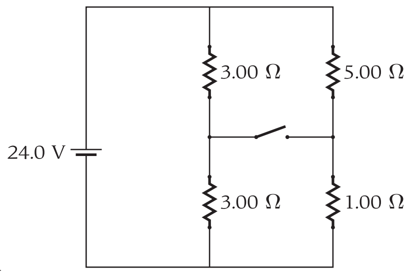
5.48 What is the current in the 10.0 Ω resistor in the circuit in the figure?



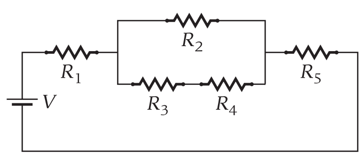
5.49 What is the equivalent resistance of the five resistors in the circuit in the figure?



•5.50 What is the current in the circuit shown in the figure when the switch is (a) open and (b) closed?



5.51 For the circuit shown in the figure, $R_1 = 6.00\ \Omega$, $R_2 = 6.00\ \Omega$, $R_3 = 2.00\ \Omega$, $R_4 = 4.00\ \Omega$, $R_5 = 3.00\ \Omega$, and the potential difference is 12.0 V.

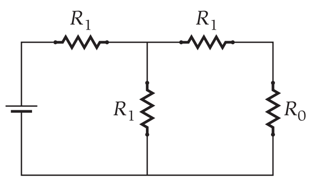


a) What is the equivalent resistance for the circuit?

b) What is the current through R_5?

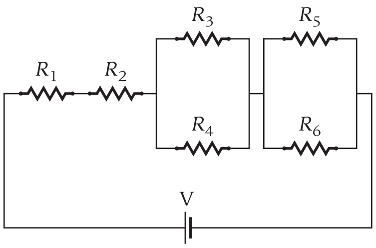
c) What is the potential drop across R_3?

5.52 Four resistors are connected in a circuit as shown in the figure. What value of R_1, expressed as a multiple of R_0, will make the equivalent resistance for the circuit equal to R_0?



•5.53 As shown in the figure, a circuit consists of an emf source with $V = 20.0$ V and six resistors. Resistors $R_1 = 5.00\ \Omega$ and $R_2 = 10.00\ \Omega$ are connected in series. Resistors

$R_3 = 5.00\ \Omega$ and $R_4 = 5.00\ \Omega$ are connected in parallel and are in series with R_1 and R_2. Resistors $R_5 = 2.00\ \Omega$ and $R_6 = 2.00\ \Omega$ are connected in parallel and are also in series with R_1 and R_2.



a) What is the potential drop across each resistor?

b) How much current flows through each resistor?

•**5.54** When a 40.0 V emf device is placed across two resistors in series, a current of 10.0 A flows through each of the resistors. When the same emf device is placed across the same two resistors in parallel, the current through the emf device is 50.0 A. What is the magnitude of the larger of the two resistances?

Section 5.7

5.55 A voltage spike causes the line voltage in a home to jump rapidly from 110. V to 150. V. What is the percentage increase in the power output of a 100.-W tungsten-filament incandescent light bulb during this spike, assuming that the bulb's resistance remains constant?

5.56 A thundercloud similar to the one described in Example 24.3 produces a lightning bolt that strikes a radio tower. If the lightning bolt transfers 5.00 C of charge in about 0.100 ms and the potential remains constant at 70.0 MV, find (a) the average current, (b) the average power, (c) the total energy, and (d) the effective resistance of the air during the lightning strike.

5.57 A hair dryer consumes 1600. W of power and operates at 110. V. (Assume that the current is DC. In fact, these are root-mean-square values of AC quantities, but the calculation is not affected.

a) Will the hair dryer trip a circuit breaker designed to interrupt the circuit if the current exceeds 15.0 A?

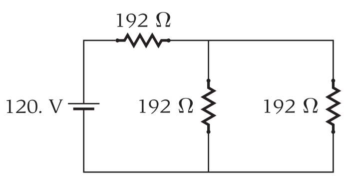
b) What is the resistance of the hair dryer when it is operating?

5.58 How much money will a homeowner owe an electric company if he turns on a 100.00 W incandescent light bulb and leaves it on for an entire year? (Assume that the cost of electricity is \$0.12000/kWh and that the light bulb lasts that long.) The same amount of light can be provided by a 26.000 W compact fluorescent light bulb. What would it cost the homeowner to leave one of those on for a year?

5.59 Three resistors are connected across a battery as shown in the figure.

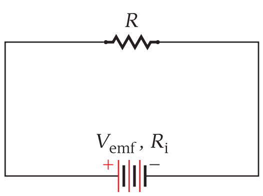
a) How much power is dissipated across the three resistors?

b) Determine the potential drop across each resistor.



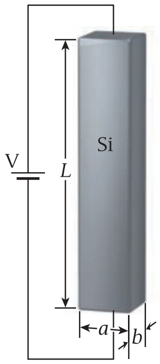
5.60 Suppose an AAA battery is able to supply 625 mAh before its potential drops below 1.50 V. How long will it be able to supply power to a 5.00 W bulb before the potential drops below 1.50 V?

•**5.61** Show that the power supplied to the circuit in the figure by the battery with internal resistance R_i is maximum when the resistance of the resistor in the circuit, R, is equal to R_i. Determine the power supplied to R. For practice, calculate the power dissipated by a 12.0 V battery with an internal resistance of 2.00 Ω when $R = 1.00\ \Omega$, $R = 2.00\ \Omega$, and $R = 3.00\ \Omega$.



•**5.62** A water heater consisting of a metal coil that is connected across the terminals of a 15.0 V power supply is able to heat 250 mL of water from room temperature to boiling point in 45.0 s. What is the resistance of the coil?

•**5.63** A potential difference of $V = 0.500$ V is applied across a block of silicon with resistivity $8.70 \times 10^{-4}\ \Omega$ m. As indicated in the figure, the dimensions of the silicon block are width $a = 2.00$ mm and length $L = 15.0$ cm. The resistance of the silicon block is 50.0 Ω, and the density of charge carriers is $1.23 \times 10^{23}\ m^{-3}$. Assume that the current density in the block is uniform and that current flows in silicon according to Ohm's Law. The total length of 0.500 mm-diameter copper wire in the circuit is 75.0 cm, and the resistivity of copper is $1.69 \times 10^{-8} \Omega$ m.



a) What is the resistance, R_w, of the copper wire?

b) What are the direction and the magnitude of the electric current, i, in the block?

c) What is the thickness, b, of the block?

d) On average, how long does it take an electron to pass from one end of the block to the other?

e) How much power, P, is dissipated by the block?

f) As what form of energy does this dissipated power appear?

Additional Exercises

5.64 In an emergency, you need to run a radio that uses 30.0 W of power when attached to a 10.0-V power supply. The only power supply you have access to provides 25.0 kV, but you do have a large number of 25.0-Ω resistors. If you want the power to the radio to be as close as possible to 30.0 W, how many resistors should you use, and how should they be connected (in series or in parallel)?

5.65 A certain brand of hot dog cooker applies a potential difference of 120. V to opposite ends of the hot dog and cooks it by means of the heat produced. If 48.0 kJ is needed to cook each hot dog, what current is needed to cook three hot dogs simultaneously in 2.00 min? Assume a parallel connection.

5.66 A circuit consists of a copper wire of length 10.0 m and radius 1.00 mm connected to a 10.0-V battery. An aluminum wire of length 5.00 m is connected to the same battery and dissipates the same amount of power. What is the radius of the aluminum wire?

5.67 The resistivity of a conductor is $\rho = 1.00 \times 10^{-5}\ \Omega$ m. If a cylindrical wire is made of this conductor, with a cross-sectional area of $1.00 \times 10^{-6}\ m^2$, what should the length of the wire be for its resistance to be 10.0 Ω?

5.68 Two cylindrical wires of identical length are made of copper and aluminum. If they carry the same current and have the same potential difference across their length, what is the ratio of their radii?

5.69 Two resistors with resistances 200. Ω and 400. Ω are connected (a) in series and (b) in parallel with an ideal 9.00-V battery. Compare the power delivered to the 200.-Ω resistor.

5.70 What are (a) the conductance and (b) the radius of a 3.50-m-long iron heating element for a 110.-V, 1500.-W heater?

5.71 A 100.-W, 240.-V European light bulb is used in an American household, where the electricity is delivered at 120. V. What power will it consume?

5.72 A modern house is wired for 115 V, and the current is limited by circuit breakers to a maximum of 200. A. (For the purpose of this problem, treat these as DC quantities.)

a) Calculate the minimum total resistance the circuitry in the house can have at any time.

b) Calculate the maximum electrical power the house can consume.

•**5.73** A 12.0 V battery with an internal resistance $R_i = 4.00\ \Omega$ is attached across an external resistor of resistance R. Find the maximum power that can be delivered to the resistor.

•**5.74** A multiclad wire consists of a zinc core of radius 1.00 mm surrounded by a copper sheath of thickness 1.00 mm. The resistivity of zinc is $\rho = 5.964 \times 10^{-8}\ \Omega$ m. What is the resistance of a 10.0 m long strand of this wire?

•**5.75** The Stanford Linear Accelerator accelerated a beam consisting of 2.0×10^{14} electrons per second through a potential difference of 2.0×10^{10} V.

a) Calculate the current in the beam.

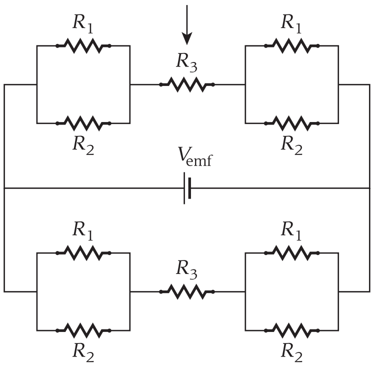
b) Calculate the power of the beam.

c) Calculate the effective ohmic resistance of the accelerator.

•**5.76** In the circuit shown in the figure, $R_1 = 3.00\ \Omega$, $R_2 = 6.00\ \Omega$, $R_3 = 20.0\ \Omega$, and $V_{emf} = 12.0$ V.

a) Determine a value for the equivalent resistance.

b) Calculate the magnitude of the current flowing through R_3 on the top branch of the circuit (marked with a vertical arrow).

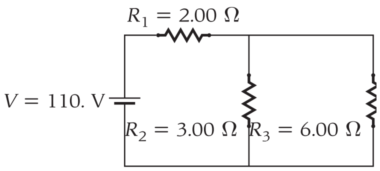


•**5.77** Three resistors are connected to a power supply with $V = 110.$ V as shown in the figure.

a) Find the potential drop across R_3.

b) Find the current in R_1.

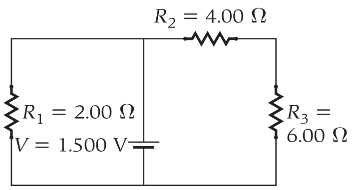
c) Find the rate at which thermal energy is dissipated from R_2.



•**5.78** A battery with $V = 1.500$ V is connected to three resistors as shown in the figure.

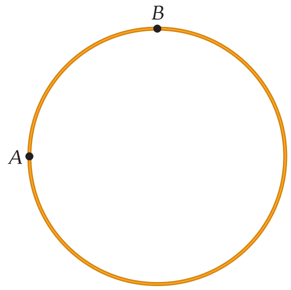
a) Find the potential drop across each resistor.

b) Find the current in each resistor.



•**5.79** A 2.50 m long copper cable is connected across the terminals of a 12.0 V car battery. Assuming that it is completely insulated from its environment, how long after the connection is made will the copper start to melt? (Useful information: copper has a mass density of 8960 kg/m^3, a melting point of 1359 K, and a specific heat of 386 J/kg/K.)

•**5.80** A piece of copper wire is used to form a circular loop of radius 10.0 cm. The wire has a cross-sectional area of 10.0 mm^2. Points A and B are 90.0° apart, as shown in the figure. Find the resistance between points A and B.



•**5.81** Two conducting wires have identical lengths $L_1 = L_2 = L = 10.0$ km and identical circular cross sections of radius $r_1 = r_2 = r = 1.00$ mm. One wire is made of steel (with resistivity $\rho_{steel} = 40.0 \times 10^{-8}\ \Omega$ m); the other is made of copper (with resistivity $\rho_{copper} = 1.68 \cdot 10^{-8}\ \Omega$ m).

a) Calculate the ratio of the power dissipated by the two wires, P_{copper}/P_{steel}, when they are connected in parallel and a potential difference of $V = 100.$ V is applied to them.

b) Based on this result, how do you explain the fact that conductors for power transmission are made of copper and not steel?

•**5.82** Before bendable tungsten filaments were developed, Thomas Edison used carbon filaments in his light bulbs. Though carbon has a very high melting temperature (3599 °C), its sublimation rate is high at high temperatures. So carbon-filament bulbs were kept at lower temperatures, thereby rendering them dimmer than later tungsten-based bulbs.

A typical carbon-filament bulb requires an average power of 40. W, when 110 volts is applied across it, and has a filament temperature of 1800 °C. Carbon, unlike copper, has a negative temperature coefficient of resistivity: $\alpha = -0.00050\ °C^{-1}$. Calculate the resistance at room temperature (20. °C) of this carbon filament.

••**5.83** A material is said to be *ohmic* if an electric field, $\vec{E}$, in the material gives rise to current density $\vec{J} = \sigma\vec{E}$, where the conductivity, σ, is a constant independent of $\vec{E}$ or $\vec{J}$. (This is the precise form of Ohm's Law.) Suppose in some material an electric field, $\vec{E}$, produces current density, $\vec{J}$, not necessarily related by Ohm's Law; that is, the material may or may not be ohmic.

a) Calculate the rate of energy dissipation (sometimes called *ohmic heating* or *joule heating*) per unit volume in this material, in terms of $\vec{E}$ and $\vec{J}$.

b) Express the result of part (a) in terms of $\vec{E}$ alone and $\vec{J}$ alone, for $\vec{E}$ and $\vec{J}$ related via Ohm's Law, that is, in an ohmic material with conductivity σ or resistivity ρ.

MULTI-VERSION EXERCISES

5.84 A high-voltage direct current (HVDC) transmission line carries electrical power a distance of 643.1 km. The line transmits 7935 MW of power at a potential difference of 1.177 MV. If the HVDC line consists of one copper wire of diameter 2.353 cm, what fraction of the power is lost in transmission?

5.85 A high-voltage direct current (HVDC) transmission line carries 5319 MW of electrical power a distance of 411.7 km. The HVDC line consists of one copper wire of diameter 2.125 cm. The fraction of the power lost in transmission is $7.538 \cdot 10^{-2}$. What is the potential difference in the line?

5.86 A high-voltage direct current (HVDC) transmission line carries 5703 MW of electrical power at a potential difference of 1.197 MV. The HVDC line consists of one copper wire of diameter 1.895 cm. The fraction of the power lost in transmission is 1.166×10^{-1}. How long is the line?

5.87 The reserve capacity (RC) of a car battery is defined as the number of minutes the battery can provide 25.0 A of current at a potential difference of 10.5 V. Thus, the RC indicates how long the battery can power a car whose charging system has failed. How much energy is stored in a car battery with an RC of 110.0?

5.88 The reserve capacity (RC) of a car battery is defined as the number of minutes the battery can provide 25.0 A of current at a potential difference of 10.5 V. Thus, the RC indicates how long the battery can power a car whose charging system has failed. If a car battery stores $1.843 \cdot 10^6$ J of energy, what is its RC?

5.89 A flashlight bulb with a tungsten filament draws 374.3 mA of current when a potential difference of 3.907 V is applied. When the bulb is at room temperature (20.00 °C) and is not lit, its resistance is 1.347 Ω. What is the temperature of the tungsten filament when the bulb is lit?

5.90 A flashlight bulb with a tungsten filament draws 420.1 mA when a potential difference of 3.949 V is applied. The temperature of the tungsten filament when the bulb is lit is 1291 °C. What is the resistance of the bulb when it is at room temperature (20.00 °C) and is not lit?

5.91 When a flashlight bulb with a tungsten filament is lit, the applied potential difference is 3.991 V and the temperature of the tungsten filament is 1.110×10^3 °C. The resistance of the bulb when it is at room temperature (20.00 °C) and is not lit is 1.451 Ω. What current does the bulb draw when it is lit?

6 Direct Current Circuits

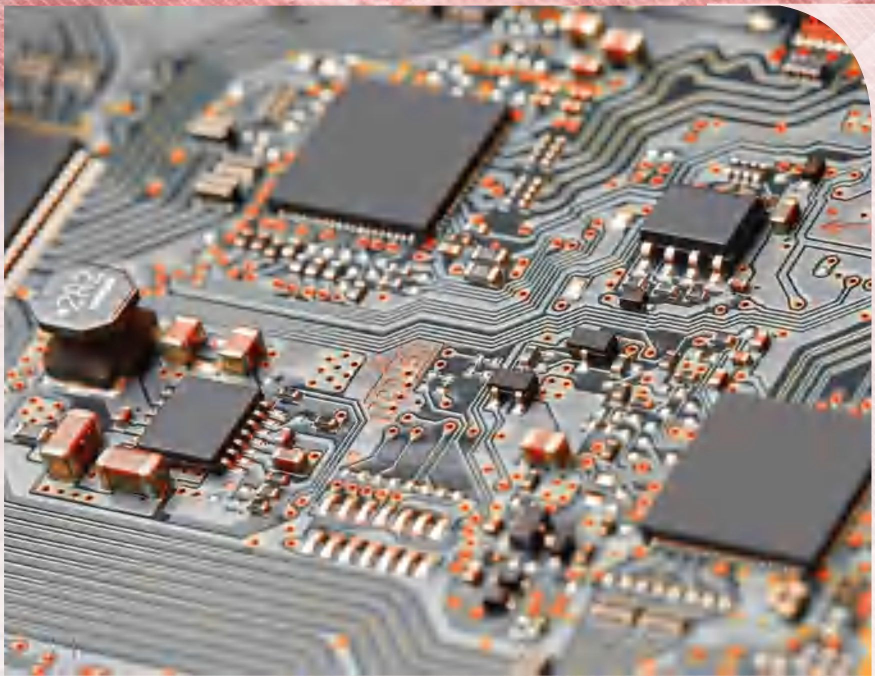


FIGURE 6.1 A circuit board can have hundreds of circuit components connected by metallic conducting paths.

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The electric circuit, such as the one shown in Figure 6.1, undoubtedly changed the world. Modern electronics continues to change human society, at a faster and faster pace. It took 38 years for radio to reach 50 million users in the United States. However, it took only 13 years for television to reach that number of users, 10 years for cable TV, 5 years for the Internet, and 3 years for cell phones.

This chapter examines the techniques used to analyze circuits that cannot be broken down into simple series and parallel connections. Modern electronics design depends on millions of different circuits, each with its own purpose and configuration. However, regardless of how complicated a circuit becomes, the basic rules for analyzing it are the ones presented in this chapter.

Some of the circuits analyzed in this chapter contain not only resistors and emf devices but also capacitors. In these circuits, the current is not steady, but changes with time. Time-varying currents will be covered more thoroughly in later chapters, which introduce additional circuit components.

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WHAT WE WILL LEARN

- Some circuits cannot be reduced to a single loop; complex circuits can be analyzed using Kirchhoff's rules.
- Kirchhoff's Junction Rule states that the algebraic sum of the currents at any junction in a circuit must be zero.
- Kirchhoff's Loop Rule states that the algebraic sum of the potential changes around any closed loop in a circuit must be zero.
- Single-loop circuits can be analyzed using Kirchhoff's Loop Rule.
- Multiloop circuits must be analyzed using both Kirchhoff's Junction Rule and Kirchhoff's Loop Rule.
- The current in a circuit that contains a resistor and a capacitor varies exponentially with time, with a characteristic time constant given by the product of the resistance and the capacitance.

6.1 Kirchhoff's Rules

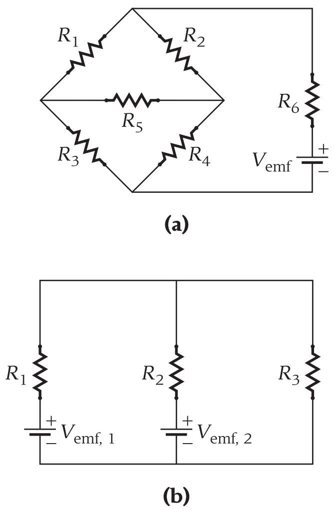


FIGURE 6.2 Two examples of circuits that cannot be reduced to simple combinations of parallel and series resistors.

In Chapter 5, we considered several kinds of direct current (DC) circuits, each containing one emf device along with resistors connected in series or in parallel. Some seemingly complicated circuits contain multiple resistors in series or in parallel that can be replaced with an equivalent resistance. However, we did not consider circuits containing multiple sources of emf. In addition, there are single-loop and multiloop circuits with emf devices and resistors that cannot be reduced to simple circuits containing parallel or series connections. Figure 6.2 shows two examples of such circuits. This chapter explains how to analyze these kinds of circuits using **Kirchhoff's rules.**

Kirchhoff's Junction Rule

A **junction** is a place in a circuit where three or more wires are connected to each other. Each connection between two junctions in a circuit is called a **branch.** A branch can contain any number of different circuit elements and the wires between them. Each branch can have a current flowing, and this current is the same everywhere in the branch. This fact leads to **Kirchhoff's Junction Rule:**

> The sum of the currents entering a junction must equal the sum of the currents leaving the junction.

With a positive sign assigned (arbitrarily) to currents entering the junction and a negative sign to those exiting the junction, Kirchhoff's Junction Rule is expressed mathematically as

$$\text{Junction:} \quad \sum_{k=1}^{n} i_k = 0. \qquad (6.1)$$

How do you know which currents enter a junction and which exit the junction when you make a drawing like the one shown in Figure 6.3? You don't; you simply assign a direction for each current along a given wire. If an assigned direction turns out to be wrong, you will obtain a negative number for that particular current in your final solution.

Kirchhoff's Junction Rule is a direct consequence of the conservation of electric charge. Junctions do not have the capability of storing charge. Thus, charge conservation requires that all charges streaming into a junction also leave the junction, which is exactly what Kirchhoff's Junction Rule states.

According to Kirchhoff's Junction Rule, at each junction in a multiloop circuit, the current flowing into the junction must equal the current flowing out of the junction. For example, Figure 6.3 shows a single junction, a, with a current, i_1, entering the junction and two currents, i_2 and i_3, leaving the junction. According to Kirchhoff's Junction Rule, in this case,

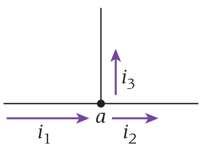


FIGURE 6.3 A single junction from a multiloop circuit.

$$\sum_{k=1}^{3} i_k = i_1 - i_2 - i_3 = 0 \Rightarrow i_1 = i_2 + i_3.$$

Kirchhoff's Loop Rule

A **loop** in a circuit is any set of connected wires and circuit elements forming a closed path. If you follow a loop, eventually you will get to the same point from which you started. For example, in the circuit diagram shown in Figure 6.2b, three possible loops can be identified. These three loops are shown in different colors (red, green, and blue) in Figure 6.4. The blue loop includes resistors 1 and 2, emf sources 1 and 2, and their connecting wires. The red loop includes resistors 2 and 3, emf source 2, and their connecting wires. Finally, the green loop includes resistors 1 and 3, emf source 1, and their connecting wires. Note that any given wire or circuit element can be and usually is part of more than one loop.

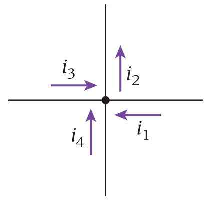
You can move through any loop in a circuit in either a clockwise or a counterclockwise direction. Figure 6.4 shows a clockwise path through each of the loops, as indicated by the arrows. But the direction of the path taken around the loop is irrelevant as long as your choice is followed consistently all the way around the loop.

Summing the potential differences from all circuit elements encountered along any given loop yields the total potential difference of the complete path along the loop. **Kirchhoff's Loop Rule** then states:

> The potential difference around a complete circuit loop must sum to zero.

Concept Check 6.1

For the junction shown in the figure, which equation correctly expresses the sum of the currents?



a) $i_1 + i_2 + i_3 + i_4 = 0$

b) $i_1 - i_2 + i_3 + i_4 = 0$

c) $-i_1 + i_2 + i_3 - i_4 = 0$

d) $i_1 - i_2 - i_3 - i_4 = 0$

e) $i_1 + i_2 - i_3 - i_4 = 0$

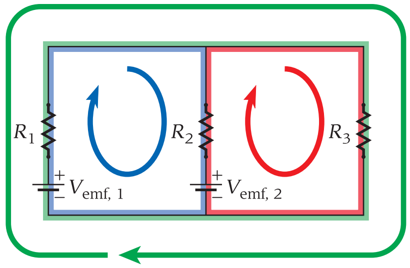


FIGURE 6.4 The three possible loops (indicated in red, green, and blue) for the circuit diagram shown in Figure 6.2b.

Kirchhoff's Loop Rule is a direct consequence of the fact that electric potential is single-valued. This means that the electric potential energy of a conduction electron at a point in the circuit has one specific value. Suppose this rule were not valid. Then we could analyze the potential changes of a conduction electron in going around a loop and find that the electron had a different potential energy when it returned to its starting point. The potential energy of this electron would change at a point in the circuit, in obvious contradiction of energy conservation. In other words, Kirchhoff's Loop Rule is simply a consequence of the law of conservation of energy.

Application of Kirchhoff's Loop Rule requires conventions for determining the potential drop across each element of the circuit. This depends on the assumed direction of the current and the direction of the analysis. For emf sources, the rules are straightforward, since minus and plus signs (as well as short and long lines) indicate which side of the emf source is at the higher potential. The potential drop for an emf source is in the direction from minus to plus or from short line to long line. As noted earlier, the assignment of the current directions and the choice of a clockwise or counterclockwise path around a loop are arbitrary. Any direction will give the same information, as long as it is applied consistently around a loop. The conventions used to analyze circuit elements in a loop are summarized in Table 6.1 and Figure 6.5, where the magnitude of the current through the circuit element is i. (The labels in the rightmost column of Table 6.1 correspond to the parts of Figure 6.5.)

Table 6.1 Conventions Used to Determine the Sign of Potential Changes Around a Single-Loop Circuit Containing Several Resistors and Sources of emf

Element	Direction of Analysis	Potential Change	
R	Same as current	$-iR$	(a)
R	Opposite to current	$+iR$	(b)
V_{emf}	Same as emf	$+V_{\text{emf}}$	(c)
V_{emf}	Opposite to emf	$-V_{\text{emf}}$	(d)

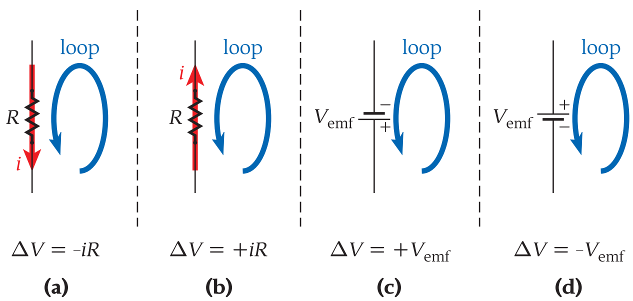


FIGURE 6.5 Sign convention for potential changes in analyzing loops.

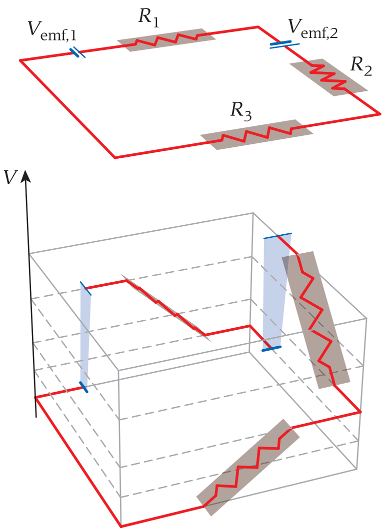


FIGURE 6.6 Loop with multiple sources of emf and multiple resistors.

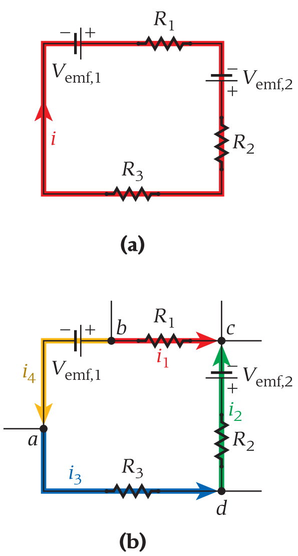


FIGURE 6.7 The same circuit loop as in Figure 6.6: (a) as an isolated single loop; (b) as a loop connected to other circuit branches.

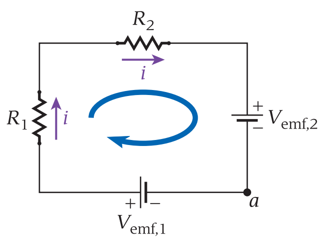


FIGURE 6.8 A single-loop circuit containing two resistors and two sources of emf in series.

If we move around a loop in a circuit in the same direction as the current, the potential changes across resistors will be negative. If we move around the loop in the opposite direction from the current, the potential changes across the resistors will be positive. If we move around a loop in such a way that we pass through an emf source from the negative to the positive terminal, this component contributes a positive potential difference. If we pass through an emf source from the positive to the negative terminal, that component contributes a negative potential difference.

With these conventions, Kirchhoff's Loop Rule is written in mathematical form as

$$\text{Closed loop:}\quad \sum_{j=1}^{m} V_{\text{emf},j} - \sum_{k=1}^{n} i_k R_k = 0. \tag{6.2}$$

The most important point visualized by Figure 6.6 is that one complete turn around the loop always ends up at the same value of the potential as at the starting point. This is exactly what Kirchhoff's Loop Rule (equation 6.2) claims.

A final point about loops is illustrated by Figure 6.7. Figure 6.7a reproduces the circuit shown in Figure 6.6 as a single isolated loop. Since this loop does not have junctions and thus has only one branch (which is the entire loop), the same current i flows everywhere in the loop. In Figure 6.7b, this loop is connected at four junctions (labeled a, b, c, and d) to other parts of a more extended circuit. Now this loop has four branches, each of which can have a different current flowing through it, as illustrated by the different-colored arrows in the figure. The point is this: In both parts of the figure, Kirchhoff's Loop Rule holds for the loop shown. The relative values of the electric potential between any two circuit elements in Figure 6.7b are the same as those shown in Figure 6.6, independent of the currents that are forced through the different branches of the loop by the rest of the circuit.

6.2 Single-Loop Circuits

Let's begin analyzing general circuits by considering a circuit containing two sources of emf, $V_{\text{emf},1}$ and $V_{\text{emf},2}$, and two resistors, R_1 and R_2, connected in series in a single loop, as shown in Figure 6.8. Note that $V_{\text{emf},1}$ and $V_{\text{emf},2}$ have opposite polarity. In this single-loop circuit, there are no junctions, and so the entire circuit consists of a single branch. The current is the same everywhere in the loop. To illustrate the potential changes across the components of this circuit, Figure 6.9 shows a three-dimensional view.

Although we could arbitrarily pick any point in the circuit of Figure 6.8 and assign it the value 0 V (or any other value of the potential, because we can always add a global additive constant to all potential values without changing the physical outcome), we start at point a with $V = 0$ V and proceed around the circuit in a clockwise direction (indicated by blue elliptical arrow in the figure). Because the components of the circuit are in series, the current, i, is the same in each component, and we assume that the current is flowing in the clockwise direction (purple arrows in the figure). The first circuit component along the clockwise path from point a is the source of emf, $V_{\text{emf},1}$, which produces a positive potential gain of $V_{\text{emf},1}$. Next is resistor R_1, which produces a potential drop given by $\Delta V_1 = iR_1$. Continuing around the loop, the next component is resistor R_2, which produces a potential drop given by $\Delta V_2 = iR_2$. Next, we encounter a second source of emf, $V_{\text{emf},2}$. This source of emf is wired into the circuit with its polarity opposite that of $V_{\text{emf},1}$. Thus, this component produces a potential *drop* with magnitude $V_{\text{emf},2}$, rather than a potential *gain*. We have now completed the loop and are back at $V = 0$ V. Using equation 6.2, we sum the potential changes of this loop as follows:

$$V_{\text{emf},1} - \Delta V_1 - \Delta V_2 - V_{\text{emf},2} = V_{\text{emf},1} - iR_1 - iR_2 - V_{\text{emf},2} = 0.$$

To show that the direction in which we move through a loop, clockwise or counterclockwise, is arbitrary, let's analyze the same circuit in the counterclockwise direction, starting at point a (see Figure 6.10). The first circuit element is $V_{\text{emf},2}$, which produces a positive potential gain. The next element is R_2. Because we have assumed that the current is in the clockwise direction and we are analyzing the loop in the counterclockwise direction, the potential change for R_2 is $+iR_2$, according to the conventions listed in Table 6.1. Proceeding to the next element in the loop, R_1, we use a similar argument to designate the potential change for this resistor as $+iR_1$. The final element in the circuit is $V_{\text{emf},1}$, which is aligned in a direction opposite to that of our analysis, so the potential change across this element is $-V_{\text{emf},1}$. Kirchhoff's Loop Rule then gives us

$$+V_{\text{emf},2} + iR_2 + iR_1 - V_{\text{emf},1} = 0.$$

You can see that the clockwise and counterclockwise loop directions give the same information, which means that the direction in which we choose to analyze the circuit does not matter.

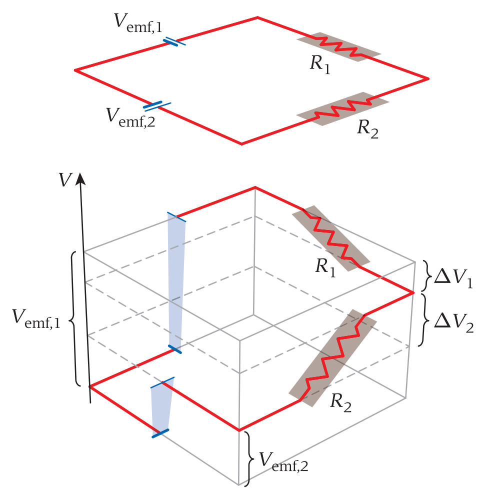


FIGURE 6.9 Three-dimensional representation of the single-loop circuit in Figure 6.8, containing two resistors and two sources of emf in series.

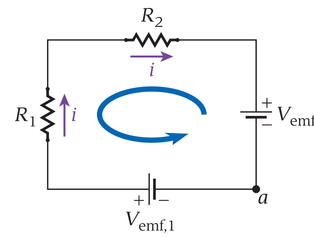


FIGURE 6.10 The same loop as in Figure 6.8, but analyzed in the counterclockwise direction.

SOLVED PROBLEM 6.1 Charging a Battery

A 12.0 V battery with internal resistance $R_i = 0.200\ \Omega$ is being charged by a battery charger that is capable of delivering a current of magnitude $i = 6.00$ A.

PROBLEM

What is the minimum emf the battery charger must supply to be able to charge the battery?

SOLUTION

THINK The battery charger, which is an external source of emf, must have enough potential difference to overcome the potential difference of the battery and the potential drop across the battery's internal resistance. The battery charger must be hooked up so that its positive terminal is connected to the positive terminal of the battery to be charged. We can think of the battery's internal resistance as a resistor in a single-loop circuit that also contains two sources of emf with opposite polarities.

SKETCH Figure 6.11 shows a diagram of the circuit, consisting of a battery with potential difference V_t and internal resistance R_i connected to an external source of emf, V_e. The yellow shaded area represents the battery's physical dimensions. Note that the positive terminal of the battery charger is connected to the positive terminal of the battery.

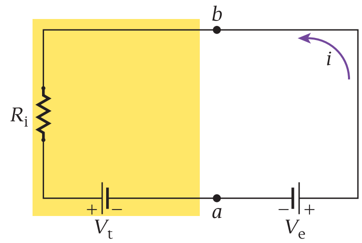


FIGURE 6.11 Circuit consisting of a battery with internal resistance connected to an external source of emf.

RESEARCH We can apply Kirchhoff's Loop Rule to this circuit. We assume a current flowing counterclockwise around the circuit, as shown in Figure 6.11. The potential changes around the circuit must sum to zero. We sum the potential changes starting at point b and moving in a counterclockwise direction:

$$-iR_i - V_t + V_e = 0.$$

– *Continued*

SIMPLIFY We can solve this equation for the required potential difference of the charger:

$$V_e = iR_i + V_t,$$

where i is the current that the charger supplies.

CALCULATE Putting in the numerical values gives us

$$V_e = iR_i + V_t = (6.00\text{ A})(0.200\ \Omega) + 12.0\text{ V} = 13.20\text{ V}.$$

ROUND We report our result to three significant figures:

$$V_e = 13.2\text{ V}.$$

DOUBLE-CHECK Our result indicates that the battery charger has to have a higher potential difference than the specified potential difference of the battery, which is reasonable. A typical charger for a 12 V battery has a potential difference of around 14 V.

6.3 Multiloop Circuits

Analyzing multiloop circuits requires both Kirchhoff's Loop Rule and Kirchhoff's Junction Rule. The procedure for analyzing a multiloop circuit consists of identifying complete loops and junction points in the circuit and applying Kirchhoff's rules to these parts of the circuit separately. Analyzing the single loops in a multiloop circuit with Kirchhoff's Loop Rule and the junctions with Kirchhoff's Junction Rule results in a system of coupled equations in several unknown variables. These equations can be solved for the quantities of interest using various techniques, including direct substitution. Example 6.1 illustrates the analysis of a multiloop circuit.

EXAMPLE 6.1 Multiloop Circuit

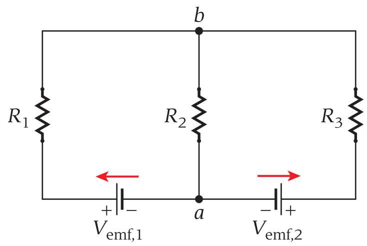


FIGURE 6.12 Multiloop circuit with three resistors and two sources of emf.

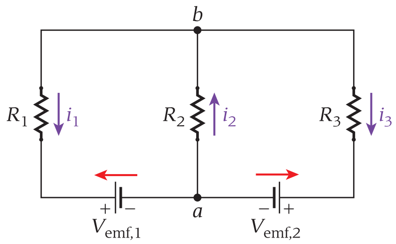


FIGURE 6.13 Multiloop circuit with the assumed direction of the current through the resistors indicated.

Consider the circuit shown in Figure 6.12. This circuit has three resistors, R_1, R_2, and R_3, and two sources of emf, $V_{emf,1}$ and $V_{emf,2}$. The red arrows show the direction of potential drop across the emf sources. This circuit cannot be resolved into simple series or parallel connections. To analyze this circuit, we need to assign directions to the currents flowing through the resistors. We can choose these directions arbitrarily (knowing that if we choose the wrong direction, the resulting current value will be negative). Figure 6.13 shows the circuit with the assigned directions for the currents shown by the purple arrows.

Let's consider junction b first. The current entering the junction must equal the current leaving it, so we can write

$$i_2 = i_1 + i_3. \tag{i}$$

Looking at junction a, we again equate the incoming current and the outgoing current to get

$$i_1 + i_3 = i_2,$$

which provides the same information obtained for junction b. Note that this is a typical result: If a circuit has n junctions, it is possible to obtain at most $n - 1$ independent equations from application of Kirchhoff's Junction Rule. (In this case, $n = 2$, so we can get only one independent equation.)

At this point we cannot determine the currents in the circuit because we have three unknown values and only one equation. Therefore, we need two more independent equations. To get these equations, we apply Kirchhoff's Loop Rule. We can identify three loops in the circuit shown in Figure 6.13:

1. the left half of the circuit, including the elements R_1, R_2, and $V_{emf,1}$;
2. the right half of the circuit, including the elements R_2, R_3, and $V_{emf,2}$; and
3. the outer loop, including the elements R_1, R_3, $V_{emf,1}$, and $V_{emf,2}$.

Applying Kirchhoff's Loop Rule to the left half of the circuit, using the assumed directions for the currents and analyzing the loop in a counterclockwise direction starting at junction b, we obtain

$$-i_1R_1 - V_{emf,1} - i_2R_2 = 0,$$

or

$$i_1R_1 + V_{\text{emf},1} + i_2R_2 = 0. \qquad \text{(ii)}$$

Applying the Loop Rule to the right half of the circuit, again starting at junction *b* and analyzing the loop in a clockwise direction, we get

$$-i_3R_3 - V_{\text{emf},2} - i_2R_2 = 0,$$

or

$$i_3R_3 + V_{\text{emf},2} + i_2R_2 = 0. \qquad \text{(iii)}$$

Applying the Loop Rule to the outer loop, starting at junction *b* and working in a clockwise direction, gives us

$$-i_3R_3 - V_{\text{emf},2} + V_{\text{emf},1} + i_1R_1 = 0.$$

This equation provides no new information because we can also obtain it by subtracting equation (iii) from equation (ii). For all three loops, we obtain equivalent information if we analyze them in either a counterclockwise direction or a clockwise direction or if we start at any other point and move around the loops from there.

With three equations, (i), (ii), and (iii), and three unknowns, i_1, i_2, and i_3, we can solve for the unknown currents in several ways. For example, we could put the three equations in matrix format and then solve them using Kramer's method on a calculator. This is a recommended method for complicated circuits with many equations and many unknowns. However, for this example, we can proceed by substituting from equation (i) into the other two, thus eliminating i_2. We then solve one of the two resulting equations for i_1 and substitute from that into the other to obtain an expression for i_3. Substituting back then gives solutions for i_2 and i_1:

$$i_1 = -\frac{(R_2 + R_3)V_{\text{emf},1} - R_2V_{\text{emf},2}}{R_1R_2 + R_1R_3 + R_2R_3}$$

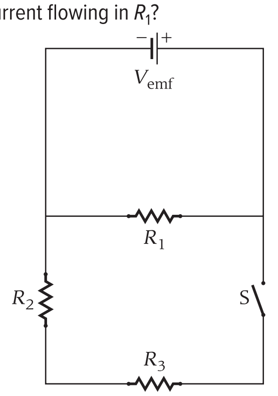
$$i_2 = -\frac{R_3V_{\text{emf},1} + R_1V_{\text{emf},2}}{R_1R_2 + R_1R_3 + R_2R_3}$$

$$i_3 = -\frac{-R_2V_{\text{emf},1} + (R_1 + R_2)V_{\text{emf},2}}{R_1R_2 + R_1R_3 + R_2R_3}$$

Note: You do not need to remember this particular solution or the linear algebra used to get there. However, the general method of applying Kirchhoff's rules for loops and junctions, and assigning currents in arbitrary directions is the central idea of circuit analysis.

Concept Check 6.2

In the circuit in the figure, there are three identical resistors. The switch, S, is initially open. When the switch is closed, what happens to the current flowing in R_1?



a) The current in R_1 decreases.

b) The current in R_1 increases.

c) The current in R_1 stays the same.

SOLVED PROBLEM 6.2 The Wheatstone Bridge

The Wheatstone bridge is a particular circuit used to measure unknown resistances. The circuit diagram of a Wheatstone bridge is shown in Figure 6.14. This circuit consists of three known resistances, R_1, R_3, and a variable resistor, R_v, as well as an unknown resistance, R_u. A source of emf, *V*, is connected across junctions *a* and *c*. A sensitive ammeter (a device used to measure current, discussed in Section 6.4) is connected between junctions *b* and *d*. The Wheatstone bridge is used to determine R_u by varying R_v until the ammeter between *b* and *d* shows no current flowing. When the ammeter reads zero, the bridge is said to be balanced.

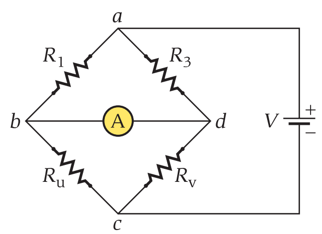


FIGURE 6.14 Circuit diagram of a Wheatstone bridge.

PROBLEM

Determine the unknown resistance, R_u, in the Wheatstone bridge shown in Figure 6.14. The known resistances are $R_1 = 100.0\ \Omega$ and $R_3 = 110.0\ \Omega$, and $R_v = 15.63\ \Omega$ when the current through the ammeter is zero, indicating that the bridge is balanced.

SOLUTION

THINK The circuit has four resistors and an ammeter, and each component can have a current flowing through it. However, in this case, with $R_v = 15.63\ \Omega$, there is no current flowing through the ammeter. Setting this current to zero leaves four unknown currents through the four resistors, and we therefore need four equations. We can use Kirchhoff's rules to analyze two loops, *adb* and *cbd*, and two junctions, *b* and *d*.

SKETCH Figure 6.15 shows the Wheatstone bridge with the assumed directions for currents i_1, i_3, i_u, i_v, and i_A.

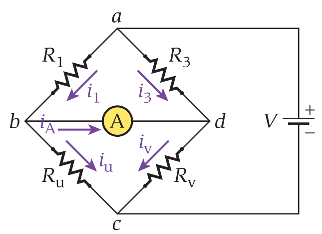


FIGURE 6.15 The Wheatstone bridge with the assumed current directions indicated.

– Continued

RESEARCH We first apply Kirchhoff's Loop Rule to loop *adb*, starting at *a* and going clockwise, to obtain

$$-i_3R_3 + i_AR_A + i_1R_1 = 0, \tag{i}$$

where R_A is the resistance of the ammeter. We apply Kirchhoff's Loop Rule again to loop *cbd*, starting at *c* and going clockwise, to get

$$+i_uR_u - i_AR_A - i_vR_v = 0. \tag{ii}$$

Now we can use Kirchhoff's Junction Rule at junction *b* to obtain

$$i_1 = i_A + i_u. \tag{iii}$$

Another application of Kirchhoff's Junction Rule, at junction *d*, gives

$$i_3 + i_A = i_v. \tag{iv}$$

SIMPLIFY When the current through the ammeter is zero ($i_A = 0$), we can rewrite equations (i) through (iv) as follows:

$$i_1R_1 = i_3R_3 \tag{v}$$

$$i_uR_u = i_vR_v \tag{vi}$$

$$i_1 = i_u \tag{vii}$$

and

$$i_3 = i_v. \tag{viii}$$

Dividing equation (vi) by equation (v) gives us

$$\frac{i_uR_u}{i_1R_1} = \frac{i_vR_v}{i_3R_3},$$

which we can rewrite using equations (vii) and (viii):

$$R_u = \frac{R_1}{R_3}R_v.$$

CALCULATE Putting in the numerical values, we get

$$R_u = \frac{R_1}{R_3}R_v = \frac{100.0\ \Omega}{110.0\ \Omega}15.63\ \Omega = 14.20901\ \Omega.$$

ROUND We report our result to four significant figures:

$$R_u = 14.21\ \Omega.$$

DOUBLE-CHECK Our result for the resistance of the unknown resistor is similar to the value for the variable resistor. Thus, our answer seems reasonable, because the other two resistors in the circuit also have resistances that are approximately equal.

Self-Test Opportunity 6.1

Show that when the current through the ammeter is zero, the equivalent resistance for the resistors in the Wheatstone bridge in Figure 6.14 is given by

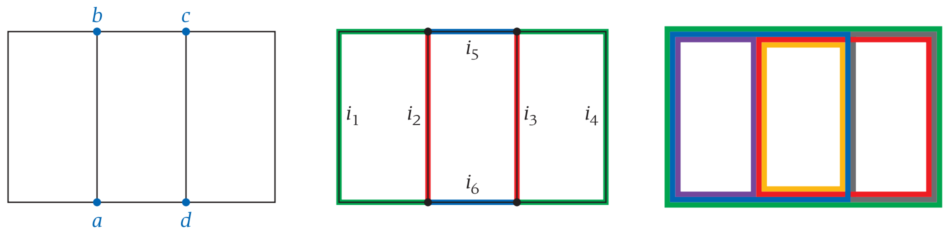
$$R_{eq} = \frac{(R_1 + R_u)(R_3 + R_v)}{R_1 + R_u + R_3 + R_v}.$$

General Observations on Circuit Networks

An important observation about solving circuit problems is that, in general, the complete analysis of a circuit requires knowing the current flowing in each branch of the circuit. We use Kirchhoff's Junction Rule and Loop Rule to establish equations relating the currents, and we need as many linearly independent equations as there are branches to guarantee that we can obtain a solution to the system.

Let's consider the abstract example shown in Figure 6.16, where all circuit components except the wires have been omitted. This circuit has four junctions, shown in blue in Figure 6.16a. Six branches connect these junctions, shown in Figure 6.16b. We therefore need six linearly independent equations relating the currents in these branches. It was noted earlier that not all equations obtained by applying Kirchhoff's rules to a circuit are linearly independent. This fact is worth repeating: If a circuit has n junctions, it is possible to obtain at most $n - 1$ independent equations from application of the Junction Rule. (For the circuit in Figure 6.16, $n = 4$, so we can get only three independent equations.)

The Junction Rule alone is not enough for the complete analysis of any circuit. It is generally best to write as many equations as possible for junctions and then augment them with equations obtained from loops. Figure 6.16c shows that there are six possible loops in



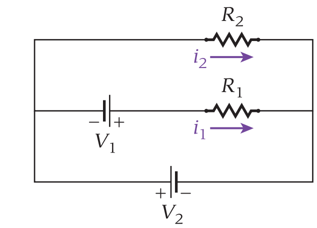
(a) (b) (c)

FIGURE 6.16 Circuit network consisting of (a) four junctions, (b) six branches, (c) six possible loops.

this network, which are marked in different colors. Clearly, there are more loops than we need to analyze to obtain three equations. This is again a general observation: The system of equations that can be set up by considering all possible loops is overdetermined. Thus, you'll always have the freedom to select particular loops to augment the equations obtained from analyzing the junctions. As a general rule of thumb, it is best to choose loops with fewer circuit elements, which often makes the subsequent linear algebra considerably simpler. In particular, if you are asked to find the current in a particular branch of a network, choosing the appropriate loop may allow you to avoid setting up a lengthy set of equations and let you solve the problem with just one equation. So it pays to devote some attention to the selection of loops! Do not write down more equations than you need to solve for the unknowns in any particular problem. This will only complicate the algebra. However, once you have the solution, you can use one or more of the unused loops to check your values.

Concept Check 6.3

In the multiloop circuit shown in the figure, $V_1 = 6.00$ V, $V_2 = 12.0$ V, $R_1 = 10.0\ \Omega$, and $R_2 = 12.0$ V. What is the magnitude of current i_2?



a) 0.500 A
b) 0.750 A
c) 1.00 A
d) 1.25 A
e) 1.50 A

6.4 Ammeters and Voltmeters

A device used to measure current is called an **ammeter.** A device used to measure potential difference is called a **voltmeter.** To measure the current, an ammeter must be wired in a circuit in *series*. Figure 6.17 shows an ammeter connected in a circuit in a way that allows it to measure the current *i*. To measure the potential difference, a voltmeter must be wired in *parallel* with the component across which the potential difference is to be measured. Figure 6.17 shows a voltmeter placed in the circuit to measure the potential drop across resistor R_1.

It is important to realize that these instruments must be able to make measurements while disturbing the circuit as little as possible. Thus, ammeters are designed to have as low a resistance as possible, usually on the order of 1 Ω, so they do not have an appreciable effect on the currents they measure. Voltmeters are designed to have as high a resistance as possible, usually on the order of 10 MΩ (10^7 Ω), so they have a negligible effect on the potential differences they are measuring.

In practice, measurements of current and potential difference are made with a digital multimeter that can switch between functioning as an ammeter and functioning as a voltmeter. It displays the results with a numerical digital display, which includes the sign of the potential difference or current. Most digital multimeters can also measure the resistance of a circuit component; that is, they can function as an **ohmmeter.** The digital multimeter performs this task by applying a known potential difference and measuring the resulting current. This test is useful for determining circuit continuity and the status of fuses, as well as measuring the resistance of resistors.

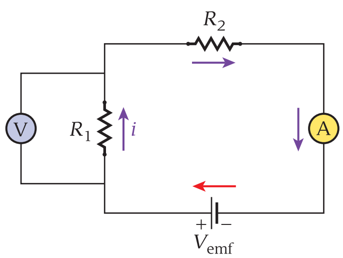


FIGURE 6.17 Placement of an ammeter and a voltmeter in a simple circuit.

EXAMPLE 6.2 Voltmeter in a Simple Circuit

Consider a simple circuit consisting of a source of emf with voltage $V_{emf} = 150.$ V and a resistor with resistance $R = 100.$ kΩ (Figure 6.18). A voltmeter with resistance $R_V = 10.0$ MΩ is connected across the resistor.

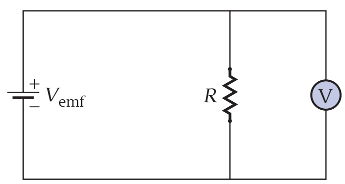


FIGURE 6.18 A simple circuit with a voltmeter connected in parallel across a resistor.

PROBLEM 1

What is the current in the circuit before the voltmeter is connected?

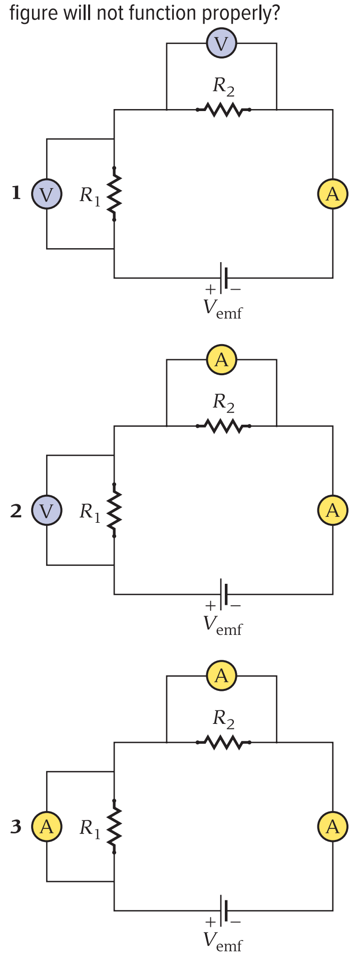
– Continued

Self-Test Opportunity 6.2

When the starter of a car is engaged while the headlights are on, the headlights dim. Explain.

Concept Check 6.4

Which of the circuits shown in the figure will not function properly?



a) 1

b) 2

c) 3

d) 1 and 2

e) 2 and 3

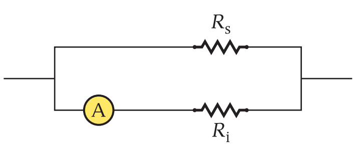


FIGURE 6.19 An ammeter with a shunt resistor connected across it in parallel.

SOLUTION 1

Ohm's Law says that $V = iR$, so we can find the current in the circuit:

$$i = \frac{V_{emf}}{R} = \frac{150.\ \text{V}}{100.\times 10^3\ \Omega} = 1.50\times 10^{-3}\ \text{A} = 1.50\ \text{mA}.$$

PROBLEM 2

What is the current in the circuit when the voltmeter is connected across the resistor?

SOLUTION 2

The equivalent resistance of the resistor and the voltmeter connected in parallel is given by

$$\frac{1}{R_{eq}} = \frac{1}{R} + \frac{1}{R_V}.$$

Solving for the equivalent resistance and putting in the numerical values, we get

$$R_{eq} = \frac{RR_V}{R + R_V} = \frac{(100.\times 10^3\ \Omega)(10.0\times 10^6\ \Omega)}{100.\ \times 10^3\ \Omega + 10.0\times 10^6\ \Omega} = 9.90\times 10^4\ \Omega = 99.0\ \text{k}\Omega.$$

The current is then

$$i = \frac{V_{emf}}{R_{eq}} = \frac{150.\ \text{V}}{9.90\times 10^4\ \Omega} = 1.52\times 10^{-3}\ \text{A} = 1.52\ \text{mA}.$$

The current in the circuit increases by 0.02 mA when the voltmeter is connected because the parallel combination of the resistor and the voltmeter has a lower resistance than that of the resistor alone. However, the effect is small, even with this relatively large resistance ($R = 100.\ \text{k}\Omega$).

SOLVED PROBLEM 6.3 Increasing the Range of an Ammeter

PROBLEM

An ammeter can be used to measure different ranges of current by adding a current divider in the form of a shunt resistor connected in parallel with the ammeter. A *shunt resistor* is simply a resistor with a very small resistance. Its name arises from the fact that when it is connected in parallel with the ammeter, whose resistance is larger, most of the current is shunted through it, bypassing the meter. The sensitivity of the ammeter is therefore decreased allowing it to measure larger currents. Suppose an ammeter produces a full-scale reading when a current of $i_{int} = 5.10$ mA passes through it. The ammeter has an internal resistance of $R_i = 16.8\ \Omega$. To use this ammeter to measure a maximum current of $i_{max} = 20.2$ A, what should be the resistance of the shunt resistor, R_s, connected in parallel with the ammeter?

SOLUTION

THINK The shunt resistor connected in parallel with the ammeter needs to have a substantially lower resistance than the internal resistance of the ammeter. Most of the current will then flow through the shunt resistor rather than the ammeter.

SKETCH Figure 6.19 shows a shunt resistor, R_s, connected in parallel with an ammeter.

RESEARCH The two resistors are connected in parallel, so the potential difference across each resistor is the same. The potential difference that gives a full-scale reading on the ammeter is

$$\Delta V_{fs} = i_{int} R_i. \tag{i}$$

From Chapter 5, we know that the equivalent resistance of the two resistors in parallel is given by

$$\frac{1}{R_{eq}} = \frac{1}{R_i} + \frac{1}{R_s}. \tag{ii}$$

The voltage drop across the equivalent resistance must equal the voltage drop across the ammeter that gives a full-scale reading when current i_{max} is flowing through the circuit. Therefore, we can write

$$\Delta V_{fs} = i_{max} R_{eq}. \tag{iii}$$

SIMPLIFY Combining equations (i) and (iii) for the potential difference gives

$$\Delta V_{fs} = i_{int} R_i = i_{max} R_{eq}. \tag{iv}$$

We can rearrange equation (iv) and substitute for R_{eq} in equation (ii):

$$\frac{i_{max}}{i_{int}R_i} = \frac{1}{R_{eq}} = \frac{1}{R_i} + \frac{1}{R_s}. \qquad \text{(v)}$$

Solving equation (v) for the shunt resistance gives us

$$\frac{1}{R_s} = \frac{i_{max}}{i_{int}R_i} - \frac{1}{R_i} = \frac{1}{R_i}\left(\frac{i_{max}}{i_{int}} - 1\right) = \frac{1}{R_i}\left(\frac{i_{max} - i_{int}}{i_{int}}\right),$$

or

$$R_s = R_i \frac{i_{int}}{i_{max} - i_{int}}.$$

CALCULATE Putting in the numerical values, we get

$$R_s = R_i \frac{i_{int}}{i_{max} - i_{int}} = (16.8\ \Omega)\frac{5.10\times 10^{-3}\ \text{A}}{20.2\ \text{A} - 5.10\times 10^{-3}\ \text{A}}$$
$$= 0.00424266\ \Omega.$$

ROUND We report our result to three significant figures:

$$R_s = 0.00424\ \Omega.$$

DOUBLE-CHECK The equivalent resistance of the ammeter and the shunt resistor connected in parallel is given by equation (ii). Solving that equation for the equivalent resistance and putting in the numbers gives

$$R_{eq} = \frac{R_i R_s}{R_i + R_s} = \frac{(16.8\ \Omega)(0.00424\ \Omega)}{16.8\ \Omega + 0.00424\ \Omega} = 0.00424\ \Omega.$$

Thus, the equivalent resistance of the ammeter and the shunt resistor connected in parallel is approximately equal to the resistance of the shunt resistor. This low equivalent resistance is necessary for a current-measuring instrument, which must be placed in series in a circuit. If the current-measuring device has a high resistance, its presence will disturb the measurement of the current.

6.5 RC Circuits

So far in this chapter, we have dealt with circuits containing sources of emf and resistors. The currents in these circuits do not vary in time. Now we consider circuits that contain capacitors, as well as sources of emf and resistors. Called **RC circuits,** these circuits have currents that *do* vary with time. The simplest circuit operations that involve time-dependent currents are the charging and discharging of a capacitor. Understanding these time-dependent processes involves the solution of some simple differential equations.

Charging a Capacitor

Consider a circuit with a source of emf, V_{emf}, a resistor, R, and a capacitor, C (Figure 6.20). Initially, the switch is open and the capacitor is uncharged, as shown in Figure 6.20a. When the switch is closed (Figure 6.20b), current begins to flow in the circuit, building up opposite charges on the plates of the capacitor and thus creating a potential difference, ΔV, across the capacitor. Current flows because of the source of emf, which maintains a constant voltage. When the capacitor is fully charged (Figure 6.20c), no more current flows in the circuit. The potential difference across the plates is then equal to the voltage provided by the source of emf, and the magnitude of the total charge, q_{tot}, on each plate of the capacitor is $q_{tot} = CV_{emf}$.

While the capacitor is charging, we can analyze the current, i, flowing in the circuit (assumed to flow from the negative to the positive terminal inside the voltage source) by applying Kirchhoff's Loop Rule to the loop in Figure 6.20b in the counterclockwise direction:

$$V_{emf} - V_R - V_C = V_{emf} - i(t)R - q(t)/C = 0,$$

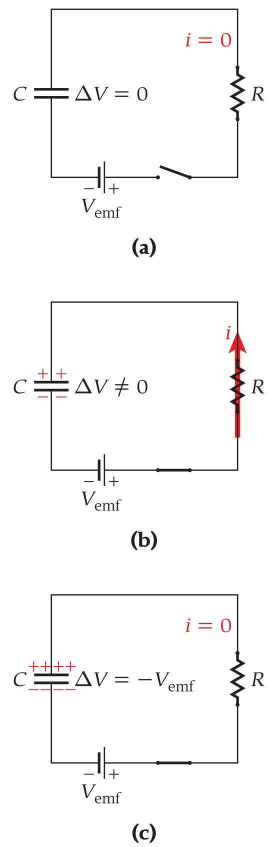


FIGURE 6.20 A basic RC circuit, containing a source of emf, a resistor, and a capacitor: (a) with the switch open; (b) a short time after the switch is closed; (c) a long time after the switch is closed.

where V_C is the potential drop across the capacitor and $q(t)$ is the charge on the capacitor at a given time t. The change of the charge on the capacitor plates due to the current is $i(t) = dq(t)/dt$, and we can rewrite the preceding equation as

$$R\frac{dq(t)}{dt} + \frac{q(t)}{C} = V_{\text{emf}},$$

or

$$\frac{dq(t)}{dt} + \frac{q(t)}{RC} = \frac{V_{\text{emf}}}{R}. \tag{6.3}$$

This differential equation relates the charge to its time derivative. It seems appropriate to try an exponential form for the solution of equation 6.3 because an exponential function is the only type that has the property of having a derivative that is identical to itself. Because equation 6.3 also has a constant term, the trial solution needs to have a constant term. We therefore try a solution with a constant and an exponential and for which $q(0) = 0$:

$$q(t) = q_{\max}\left(1 - e^{-t/\tau}\right),$$

where the constants $q_{\max}$ and τ are to be determined. Substituting this trial solution back into equation 6.3, we obtain

$$q_{\max}\frac{1}{\tau}e^{-t/\tau} + \frac{1}{RC}q_{\max}\left(1 - e^{-t/\tau}\right) = \frac{V_{\text{emf}}}{R}.$$

Now we collect the time-dependent terms on the left-hand side and the time-independent terms on the right-hand side:

$$q_{\max}e^{-t/\tau}\left(\frac{1}{\tau} - \frac{1}{RC}\right) = \frac{V_{\text{emf}}}{R} - \frac{1}{RC}q_{\max}.$$

This equation can only be true for all times if both sides are equal to zero. From the left-hand side, we then find

$$\tau = RC. \tag{6.4}$$

Thus, the constant τ (called the **time constant**) is simply the product of the capacitance and the resistance. From the right-hand side, we find an expression for the constant $q_{\max}$:

$$q_{\max} = CV_{\text{emf}}.$$

Thus, the differential equation for charging the capacitor (equation 6.3) has the solution

$$q(t) = CV_{\text{emf}}\left(1 - e^{-t/RC}\right). \tag{6.5}$$

Note that at $t = 0$, $q = 0$, which is the initial condition before the circuit components were connected. At $t = \infty$, $q = q_{\max} = CV_{\text{emf}}$, which is the steady-state condition in which the capacitor is fully charged. The time dependence of the charge on the capacitor is shown in Figure 6.21a for three different values of the time constant τ.

The current flowing in the circuit is obtained by differentiating equation 6.5 with respect to time:

$$i = \frac{dq}{dt} = \left(\frac{V_{\text{emf}}}{R}\right)e^{-t/RC}. \tag{6.6}$$

From equation 6.6, at $t = 0$, the current in the circuit is V_{emf}/R, and at $t = \infty$, the current is zero, as shown in Figure 6.21b.

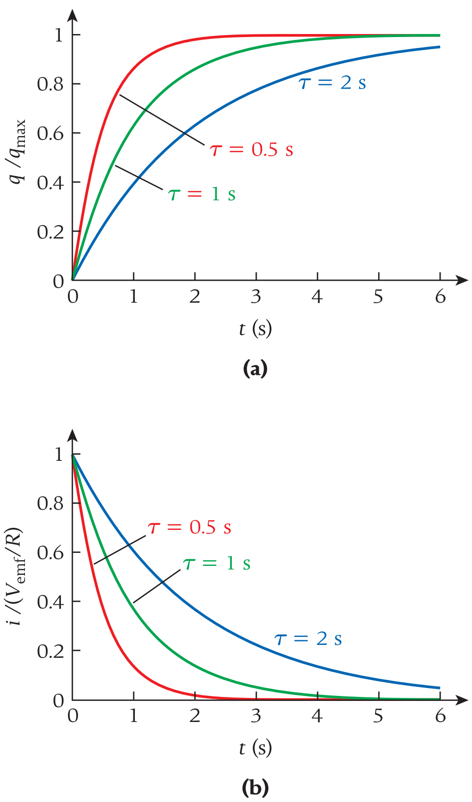


FIGURE 6.21 Charging a capacitor: (a) charge on the capacitor as a function of time; (b) current flowing through the resistor as a function of time.

Discharging a Capacitor

Now let's consider a circuit containing only a resistor, R, and a fully charged capacitor, C, obtained by moving the switch in Figure 6.22 from position 1 to position 2. The charge on the capacitor before the switch is moved is q_{max}. In this case, current will flow in the circuit until the capacitor is completely discharged. While the capacitor is discharging, we can apply Kirchhoff's Loop Rule around the loop in the clockwise direction and obtain

$$-i(t)R - V_C = -\,i(t)R - \frac{q(t)}{C} = 0.$$

We can rewrite this equation using the definition of current:

$$\frac{R\,dq(t)}{dt} + \frac{q(t)}{C} = 0. \tag{6.7}$$

The solution to equation 6.7 is obtained using the same method as for equation 6.3, except that equation 6.7 has no constant term and $q(0) > 0$. Thus, we try a solution of the form $q(t) = q_{max}e^{-t/\tau}$, which leads to

$$q(t) = q_{max}e^{-t/RC}. \tag{6.8}$$

At $t = 0$, the charge on the capacitor is q_{max}. At $t = \infty$, the charge on the capacitor is zero.

We can obtain the current by differentiating equation 6.8 as a function of time:

$$i(t) = \frac{dq}{dt} = -\left(\frac{q_{max}}{RC}\right)e^{-t/RC}. \tag{6.9}$$

At $t = 0$, the current in the circuit is $-q_{max}/RC$. At $t = \infty$, the current in the circuit is zero. Plotting the time dependence of the charge on the capacitor and the current flowing through the resistor for the discharging process would result in exponentially decreasing curves like those in Figure 6.21b.

The equations describing the time dependence of the charging and the discharging of a capacitor all involve the exponential factor $e^{-t/RC}$. Again, the product of the resistance and the capacitance is defined as the time constant of an RC circuit: $\tau = RC$. According to equation 6.5, after an amount of time equal to the time constant, the capacitor will have been charged to 63% of its maximum value. Thus, an RC circuit can be characterized by specifying the time constant. A large time constant means that it takes a long time to charge the capacitor; a small time constant means that it takes a short time to charge the capacitor.

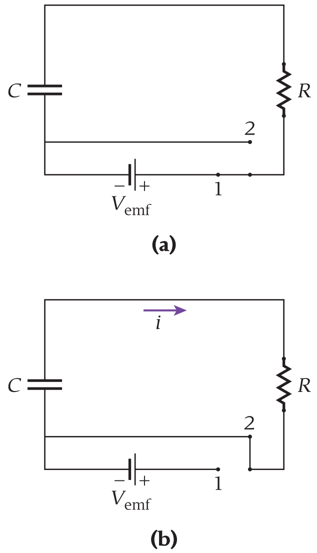


FIGURE 6.22 RC circuit containing an emf source, a resistor, a capacitor, and a switch. The capacitor is (a) charged with the switch in position 1 and (b) discharged with the switch in position 2.

Concept Check 6.5

To discharge a capacitor in an RC circuit very quickly, what should the values of the resistance and the capacitance be?

a) Both should be as large as possible.

b) Resistance should be as large as possible, and capacitance as small as possible.

c) Resistance should be as small as possible, and capacitance as large as possible.

d) Both should be as small as possible.

EXAMPLE 6.3 Time Required to Charge a Capacitor

Consider a circuit consisting of a 12.0 V battery, a 50.0 Ω resistor, and a 100.0 μF capacitor wired in series. The capacitor is initially completely discharged.

PROBLEM

How long after the circuit is closed will it take to charge the capacitor to 90% of its maximum charge?

SOLUTION

The charge on the capacitor as a function of time is given by

$$q(t) = q_{max}\left(1 - e^{-t/RC}\right),$$

where q_{max} is the maximum charge on the capacitor. We want to know the time until $q(t)/q_{max} = 0.90$, which can be obtained from

$$\left(1 - e^{-t/RC}\right) = \frac{q(t)}{q_{max}} = 0.90,$$

or

$$0.10 = e^{-t/RC}. \tag{i}$$

– Continued

Self-Test Opportunity 6.3

A 1.00-mF capacitor is fully charged, and a 100.0-Ω resistor is connected across the capacitor. How long will it take to remove 99.0% of the charge stored in the capacitor?

Taking the natural log of both sides of equation (i), we get

$$\ln 0.10 = -\frac{t}{RC},$$

or

$$t = -RC\ln 0.10 = -(50.0\ \Omega)\left(100\times 10^{-6}\ \text{F}\right)(-2.30) = 0.0115\ \text{s} = 11.5\ \text{ms}.$$

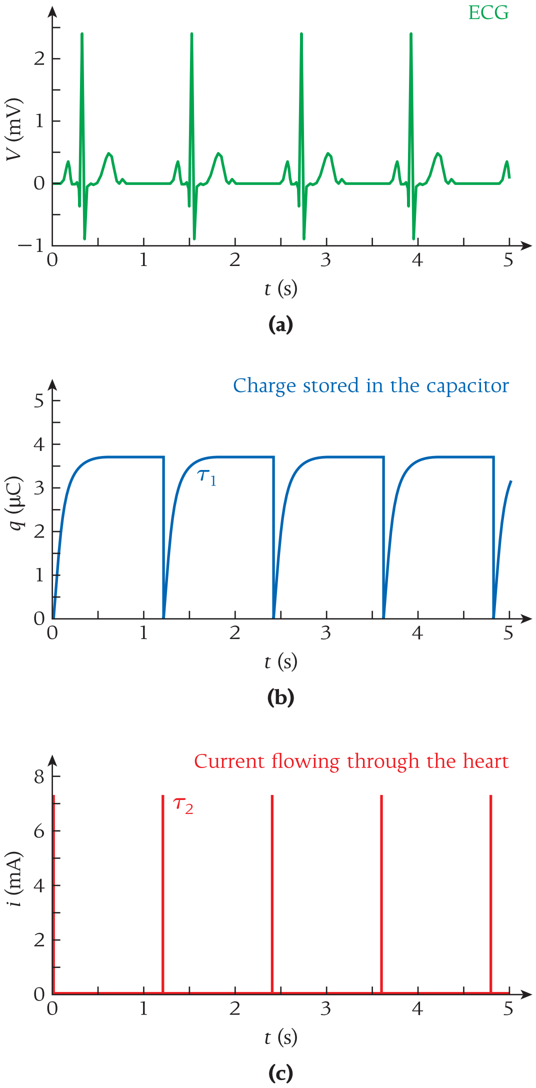


FIGURE 6.23 (a) An electrocardiogram (ECG) showing four regular heartbeats. (b) The charge stored in the pacemaker's capacitor as a function of time. (c) The current flowing through the heart due to the discharging of the pacemaker's capacitor.

Pacemaker

A normal human heart beats at regular intervals, sending blood through the body. The heart's own electrical signals regulate its beating. These electrical signals can be measured through the skin using an electrocardiograph. This device produces a graph of potential difference versus time, which is called an *electrocardiogram*, or *ECG* (sometimes *EKG*, from the German word *Electrokardiograph*). Figure 6.23a shows how an ECG would represent four regular heartbeats occurring at a rate of 72 beats per minute. Doctors and medical personnel can use an ECG to diagnose the health of the heart.

Sometimes the heart does not beat regularly and needs help to maintain its proper rhythm. This help can be provided by a pacemaker, an electrical circuit that sends electrical pulses to the heart at regular intervals, replacing the heart's usual electrical signals and stimulating the heart to beat at prescribed intervals. A pacemaker is implanted in the patient and connected directly to the heart, as illustrated in Figure 6.24.

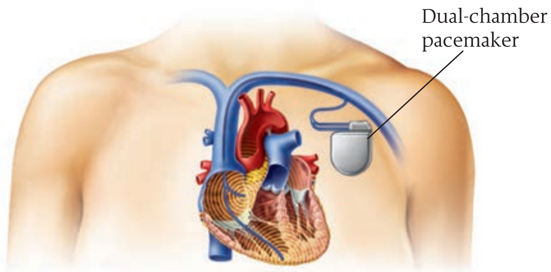
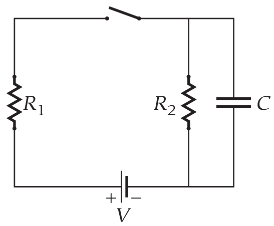


FIGURE 6.24 A modern pacemaker implanted in a patient. The pacemaker sends electrical pulses to two chambers of the heart to help keep it beating regularly.

Concept Check 6.6

In the circuit shown in the figure, the capacitor, C, is initially uncharged. Immediately after the switch is closed,



a) the current flowing through R_1 is zero.

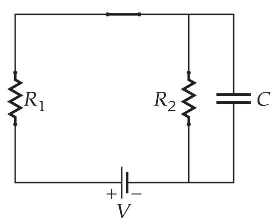
b) the current flowing through R_1 is larger than that through R_2.

c) the current flowing through R_2 is larger than that through R_1.

d) the current flowing through R_1 is the same as that through R_2.

Concept Check 6.7

In the circuit shown in the figure, the switch is closed. After a long time,



a) the current through R_1 is zero.

b) the current flowing through R_1 is larger than that through R_2.

c) the current flowing through R_2 is larger than that through R_1.

d) the current flowing through R_1 is the same as that through R_2.

EXAMPLE 6.4 Circuit Elements of a Pacemaker

Let's analyze the circuit shown in Figure 6.25, which simulates the function of a pacemaker. This pacemaker circuit operates by charging a capacitor, C, for some time using a battery with voltage V_{emf} and a resistor R_1, as illustrated in Figure 6.25a, where the switch is open. Closing the switch, as in Figure 6.25b, shorts the capacitor across the heart, and the capacitor discharges through the heart in a short time to stimulate the heart to beat. Thus, this circuit operates as a pacemaker by keeping the switch open for the time between heartbeats, closing the switch for a short time to stimulate a heartbeat, and then opening the switch again.

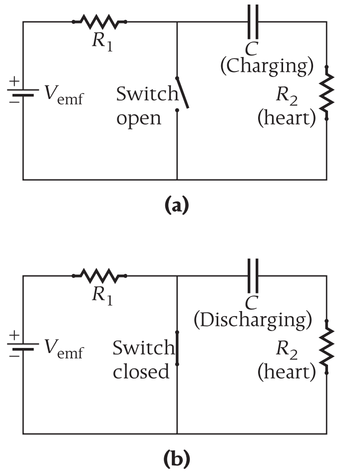


FIGURE 6.25 (a) A simplified pacemaker circuit in charging mode. (b) The pacemaker circuit in discharging mode.

PROBLEM

What values of the capacitance, C, and the resistance, R_1, should be used in a pacemaker?

SOLUTION

We assume that the heart acts as a resistor with a value $R_2 = 500\ \Omega$ and that the source of emf is a lithium ion battery. Such a battery has a very high energy density and a voltage of 3.7 V. A normal heart rate is in the range from 60 to 100 beats per minute. However, the pacemaker might need to stimulate the heart to beat faster, so it should be capable of running at 180 beats per minute, which means that the capacitor may be charged up to 180 times a minute. Thus, the minimum time between discharges, t_{min}, is

$$t_{min} = \frac{1}{180\ \text{beats/min}} = \left(\frac{1\ \text{min}}{180}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = 0.333\ \text{s}.$$

Equation 6.5 gives the charge, q, as a function of time, t, for the maximum charge, q_{max}, for a given time constant, $\tau_1 = R_1C$:

$$q = q_{max}\left(1 - e^{-t/\tau_1}\right).$$

Rearranging this equation gives us

$$\frac{q}{q_{max}} = f = 1 - e^{-t/\tau_1},$$

where f is the fraction of the charge capacity of the capacitor. Let's assume the capacitor must be charged to 95% of its maximum charge in time t_{min}. Solving for the time constant gives us

$$\tau_1 = R_1C = -\frac{t_{min}}{\ln(1-f)} = -\frac{0.333\ \text{s}}{\ln(1-0.95)} = 0.111\ \text{s}.$$

Thus, the time constant for the charging should be $\tau_1 = R_1C = 100$ ms. The time constant for discharging, τ_2, needs to be small to produce short pulses of a high current to stimulate the heart. Letting $\tau_2 = 0.500$ ms, which is on the order of the narrow electrical pulse in the ECG, we have

$$\tau_2 = R_2C = 0.500\ \text{ms}.$$

We can solve for the required capacitance and substitute the value 500 Ω for R_2:

$$C = \frac{0.000500\ \text{s}}{500\ \Omega} = 1.00\ \mu\text{F}.$$

The resistance required for the charging circuit can now be related to the time constant for the charging and the value of the capacitance we just calculated:

$$\tau_1 = R_1C = 0.100\ \text{s} = R_1(1.00\ \mu\text{F}),$$

which gives us

$$R_1 = \frac{0.100\ \text{s}}{1.00 \times 10^{-6}\ \text{F}} = 100\ \text{k}\Omega.$$

Figure 6.23b shows the charge in the capacitor of the simulated pacemaker as a function of time for a heart rate of 72 heartbeats per minute. You can see that the capacitor charges to nearly full capacity before it is discharged. Figure 6.23c shows the current that flows through the heart when the capacitor discharges. The current pulse is narrow, lasting less than a millisecond. This pulse stimulates the heart to beat, as illustrated by the ECG in Figure 6.23a. The rate at which the heart beats is controlled by the rate at which the pacemaker's switch is closed and opened, which is controlled by a microprocessor.

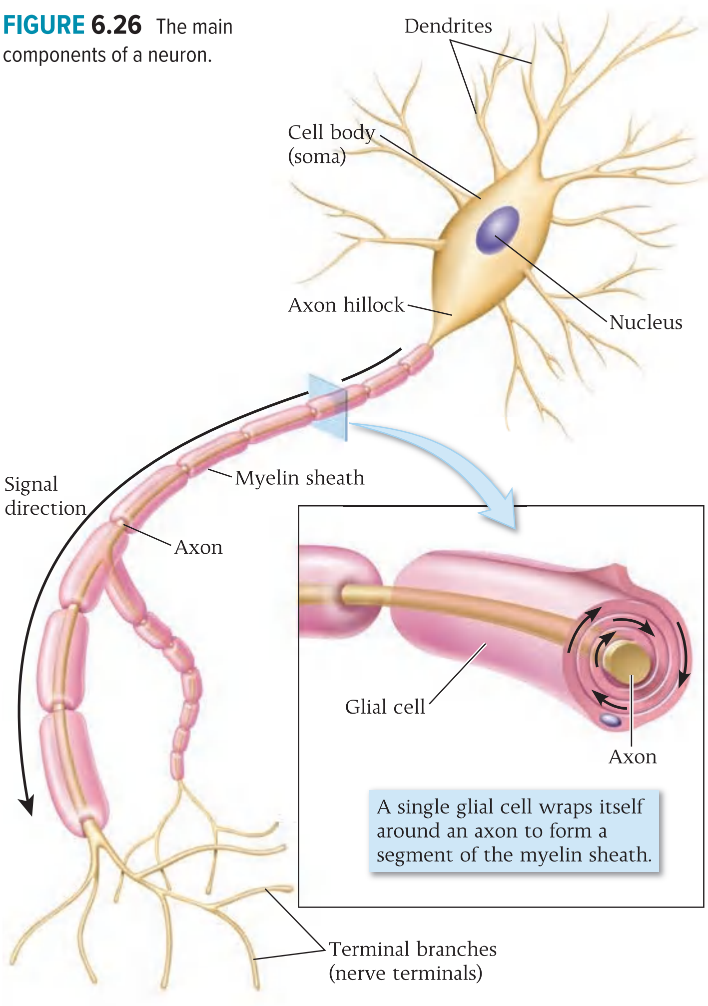


FIGURE 6.26 The main components of a neuron.

Neuron

The type of cell responsible for transmitting and processing signals in the nervous systems and brains of humans and other animals is a neuron (Figure 6.26). The neuron conducts the necessary currents by electrochemical means, via the movement of ions (mainly Na^+, K^+, and Cl^-). Neurons receive signals from other neurons through dendrites and send signals to other neurons through an axon. The axon can be quite long (for example, in the spinal cord) and is covered with an insulating myelin sheath. The signals are received from and sent to other neurons or cells in the sense organs and other tissues. All of this, and much more, can be learned in an introductory biology or physiology course. Here we'll take a look at a neuron as a basic circuit that processes signals.

An input signal has to be strong enough to get a neuron to fire, that is, to send an output signal down the axon. It is a crude but reasonable approximation to represent the main cell body of a neuron, the soma, as a basic RC circuit that processes these signals. A diagram of this RC circuit is shown in Figure 6.27. A capacitor and a resistor are connected in parallel to an input and an output potential. The typical potential values for neurons are on the order of ±50 mV relative to the background of the surrounding tissue. If $\Delta V = V_{in} - V_{out} \neq 0$, a current flows through the circuit. Part of this current flows through the resistor, but part of the current simultaneously charges the capacitor until it reaches the potential difference between input and output potentials. The potential difference between the capacitor plates rises exponentially, according to $V_C(t) = (V_{in} - V_{out})(1 - e^{-t/RC})$, just as for the process of charging a capacitor in an RC circuit. (The time constant, $\tau = RC$, is on the order of 10 ms, assuming a capacitance of 1 nF and a resistance of 10 MΩ.) If the external source of potential difference is then removed from the circuit, the capacitor discharges with the same time constant, and the potential across the capacitor decays exponentially, according to $V_C(t) = V_0 e^{-t/RC}$. This simple time dependence captures the basic response of a neuron. Figure 6.28 shows the potential difference across the capacitor of this model neuron, while it is charged for 30 ms and then discharged.

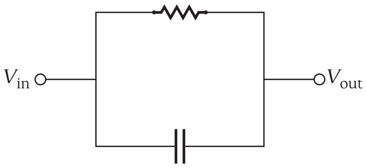


FIGURE 6.27 Simplified model of a neuron as an RC circuit.

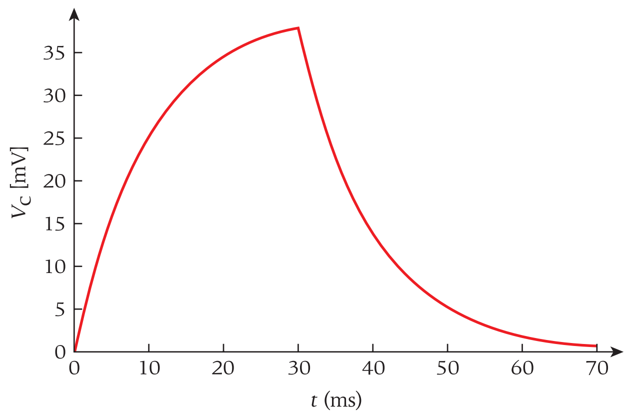


FIGURE 6.28 Potential difference on the capacitor in a model neuron.

SOLVED PROBLEM 6.4 Rate of Energy Storage in a Capacitor

A resistor with $R = 2.50\ \text{M}\Omega$ and a capacitor with $C = 1.25\ \mu\text{F}$ are connected in series with a battery for which $V_{\text{emf}} = 12.0$ V. At $t = 2.50$ s after the circuit is closed, what is the rate at which energy is being stored in the capacitor?

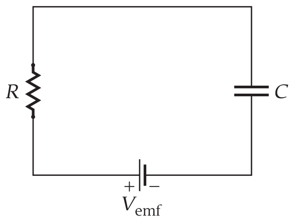


FIGURE 6.29 Series circuit containing a battery, a resistor, and a capacitor.

THINK When the circuit is closed, the capacitor begins to charge. The rate at which energy is stored in the capacitor is given by the time derivative of the amount of energy stored in the capacitor, which is a function of the charge on the capacitor.

SKETCH Figure 6.29 shows a diagram of the series circuit containing a battery, a resistor, and a capacitor.

RESEARCH The charge on the capacitor as a function of time is given by equation 6.5:

$$q(t) = CV_{\text{emf}}\left(1 - e^{-t/RC}\right).$$

The energy stored in a capacitor that has charge q is given by

$$U = \tfrac{1}{2}\frac{q^2}{C}. \tag{i}$$

The time derivative of the energy stored in the capacitor is then

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{q^2(t)}{C}\right) = \frac{q(t)}{C}\frac{dq(t)}{dt}. \tag{ii}$$

The time derivative of the charge is the current, i. Thus, we can replace dq/dt with the expression given by equation 6.6:

$$i(t) = \frac{dq(t)}{dt} = \left(\frac{V_{\text{emf}}}{R}\right)e^{-t/RC}. \tag{iii}$$

SIMPLIFY We can express the rate of change of the energy stored in the capacitor by combining equations (i) through (iii):

$$\frac{dU}{dt} = \frac{q(t)}{C}i(t) = \frac{CV_{\text{emf}}\left(1 - e^{-t/RC}\right)}{C}\left(\frac{V_{\text{emf}}}{R}\right)e^{-t/RC} = \frac{V_{\text{emf}}^2}{R}e^{-t/RC}\left(1 - e^{-t/RC}\right).$$

CALCULATE We first calculate the value of the time constant, $\tau = RC$:

$$RC = \left(2.50\times10^{6}\ \Omega\right)\left(1.25\times10^{-6}\ \text{F}\right) = 3.125\ \text{s}.$$

We can then calculate the rate of change of the energy stored in the capacitor:

$$\frac{dU}{dt} = \frac{(12.0\ \text{V})^2}{2.50\times10^{6}\,\Omega}e^{-(2.50\ \text{s})/(3.125\ \text{s})}\left(1 - e^{-(2.50\ \text{s})/(3.125\ \text{s})}\right) = 1.42521\times10^{-5}\ \text{W}.$$

ROUND We report our result to three significant figures:

$$\frac{dU}{dt} = 1.43\times10^{-5}\ \text{W}.$$

DOUBLE-CHECK The current at $t = 2.50$ s is

$$i(2.50\ \text{s}) = \left(\frac{12.0\ \text{V}}{2.50\ \text{M}\Omega}\right)e^{-(2.50\ \text{s})/(3.125\ \text{s})} = 2.16\times10^{-6}\ \text{A}.$$

The rate of energy dissipation in the resistor at this time is

$$P = \frac{dU}{dt} = i^2R = \left(2.16\times10^{-6}\ \text{A}\right)^2\left(2.50\times10^{6}\ \Omega\right) = 1.16\times10^{-5}\ \text{W}.$$

The rate at which the battery delivers energy to the circuit at this time is given by

$$P = \frac{dU}{dt} = iV_{\text{emf}} = \left(2.16\times10^{-6}\ \text{A}\right)(12.0\ \text{V}) = 2.59\times10^{-5}\ \text{W}.$$

Energy conservation dictates that at any time the energy supplied by the battery is either dissipated as heat in the resistor or stored in the capacitor. In this case, the power supplied by the battery, 2.59×10^{-5} W, is equal to the power dissipated as heat in the resistor, 1.16×10^{-5} W, plus the rate at which energy is stored in the capacitor, 1.43×10^{-5} W. Thus, our answer is consistent.

WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- Kirchhoff's rules for analyzing circuits are as follows:
  - Kirchhoff's Junction Rule: The sum of the currents entering a junction must equal the sum of the currents leaving a junction.
  - Kirchhoff's Loop Rule: The potential difference around a complete loop must sum to zero.
- In applying Kirchhoff's Loop Rule, the sign of the potential change for each circuit element is determined by the direction of the current and the direction of analysis. The conventions are
  - Sources of emf in the same direction as the direction of analysis provide potential gains, while sources opposite to the analysis direction give potential drops.
  - For resistors, the magnitude of the potential change is $|iR|$, where i is the assumed current and R is the resistance. The sign of the potential change depends on the (known or assumed) direction of the current as well as the direction of analysis. If these directions are the same, the resistor produces a potential drop. If the directions are opposite, the resistor produces a potential gain.
- An RC circuit contains a resistor of resistance R and a capacitor of capacitance C. The time constant, τ, is given by $\tau = RC$.
- In an RC circuit, the charge, q, as a function of time for a charging capacitor with capacitance C is given by $q(t) = CV_{emf}(1 - e^{-t/RC})$, where V_{emf} is the voltage supplied by the source of emf and R is the resistance of the resistor.
- In an RC circuit, the charge, q, as a function of time for a discharging capacitor with capacitance C is given by $q(t) = q_{max}e^{-t/RC}$, where q_{max} is the size of the charge on the capacitor plates at $t = 0$ and R is the resistance of the resistor.

ANSWERS TO SELF-TEST OPPORTUNITIES

6.1 The resistors R_1 and R_u are in series and have an equivalent resistance of $R_{1u} = R_1 + R_u$. The resistors R_3 and R_v are in series and have an equivalent resistance of $R_{3v} = R_3 + R_v$. The equivalent resistances R_{1u} and R_{3v} are in parallel. Thus, we can write

$$\frac{1}{R_{eq}} = \frac{1}{R_{1u}} + \frac{1}{R_{3v}},$$

or

$$R_{eq} = \frac{R_{1u}R_{3v}}{R_{1u} + R_{3v}} = \frac{(R_1 + R_u)(R_3 + R_v)}{R_1 + R_u + R_3 + R_v}.$$

6.2 When the headlights are on, the battery is supplying a modest amount of current to the lights, and the potential drop across the internal resistance of the battery is small. The starter motor is wired in parallel with the lights. When the starter motor engages, it draws a large current, producing a noticeable drop in potential across the internal resistance of the battery and causing less current to flow to the headlights.

6.3

$$q = q_{max}e^{-t/RC}$$

$$\frac{q}{q_{max}} = 0.01 = e^{-t/RC} \Rightarrow \ln 0.01 = -\frac{t}{RC}$$

$$t = -RC \ln 0.01 = -(100\ \Omega)(1.00 \times 10^{-3}\ \text{F})(\ln 0.01) = 0.461\ \text{s}.$$

PROBLEM-SOLVING GUIDELINES

1. It is always helpful to label everything in a circuit diagram, including all the given information and all the unknowns, as well as pertinent currents, branches, and junctions. Redraw the diagram at a larger scale if you need more space for clarity.

2. Remember that the directions you choose for currents and for the path around a circuit loop are arbitrary. If your choice turns out to be incorrect, a negative value for the current will result.

3. Review the signs for potential changes given in Table 6.1. Moving through a circuit loop in the same direction as the assumed current means that an emf device produces a positive potential change in the direction from negative to positive within the device and that the potential change across a resistor is negative. Sign errors are common, and it pays to stick to the conventions to avoid such errors.

4. Sources of emf or resistors may be parts of two separate loops. Count each circuit component as a part of each loop it is in, according to the sign conventions you've adopted for that loop. A resistor may yield a potential drop in one loop and a potential gain in the other loop.

5. It is always possible to use Kirchhoff's rules to write more equations than you need to solve for unknown currents in a circuit's branches. Write as many equations as possible for junctions, and then augment them with equations representing loops. But not all loops are created equal; you need to select them carefully. As a rule of thumb, choose loops with fewer circuit elements.

MULTIPLE-CHOICE QUESTIONS

6.1 A resistor and a capacitor are connected in series. If a second identical capacitor is connected in series in the same circuit, the time constant for the circuit will

a) decrease. b) increase. c) stay the same.

6.2 A resistor and a capacitor are connected in series. If a second identical resistor is connected in series in the same circuit, the time constant for the circuit will

a) decrease. b) increase. c) stay the same.

6.3 A circuit consists of a source of emf, a resistor, and a capacitor, all connected in series. The capacitor is fully charged. How much current is flowing through it?

a) $i = V/R$ b) zero c) neither (a) nor (b)

6.4 Which of the following will reduce the time constant in an RC circuit?

a) increasing the dielectric constant of the capacitor
b) adding an additional 20 m of wire between the capacitor and the resistor
c) increasing the voltage of the battery
d) adding an additional resistor in parallel with the first resistor
e) none of the above

6.5 Kirchhoff's Junction Rule states that

a) the algebraic sum of the currents at any junction in a circuit must be zero.
b) the algebraic sum of the potential changes around any closed loop in a circuit must be zero.
c) the current in a circuit with a resistor and a capacitor varies exponentially with time.
d) the current at a junction is given by the product of the resistance and the capacitance.
e) the time for the current development at a junction is given by the product of the resistance and the capacitance.

6.6 How long will it take, as a multiple of the time constant, τ, for the capacitor in an RC circuit to be 98% charged?

a) 9τ c) 90τ e) 0.98τ
b) 0.9τ d) 4τ

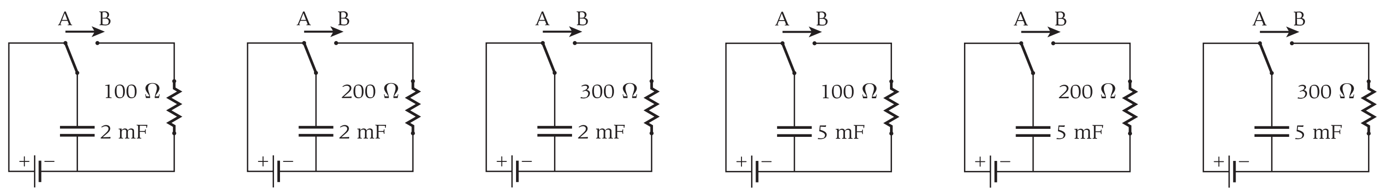
6.7 A capacitor C is initially uncharged. At time $t = 0$, the capacitor is attached through a resistor R to a battery. The energy stored in the capacitor increases, eventually reaching a value U as $t \to \infty$. After a time equal to the time constant $\tau = RC$, the energy stored in the capacitor is given by

a) U/e. c) $U(1 - 1/e)^2$.
b) U/e^2. d) $U(1 - 1/e)$.

6.8 Which of the following has the same unit as the electromotive force (emf)?

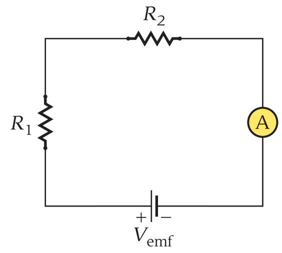
a) current
b) electric potential
c) electric field
d) electric power
e) none of the above

6.9 The capacitor in each circuit in the figure is first charged by a 10-V battery with no internal resistance. Then, the switch is flipped from position A to position B, and the capacitor is discharged through various resistors. For which circuit is the total energy dissipated by the resistor the largest?



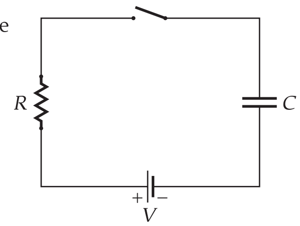
(a) (b) (c) (d) (e) (f)

6.10 Two resistors, $R_1 = 3.00\ \Omega$ and $R_2 = 5.00\ \Omega$, are connected in series with a battery and an ammeter, as shown in the figure. The battery supplies $V_{emf} = 8.00$ V, and the ammeter has the resistance $R_A = 1.00\ \Omega$. What is the current measured by the ammeter?



a) 0.500 A
b) 0.750 A
c) 0.889 A
d) 1.00 A
e) 1.50 A

6.11 An uncharged capacitor ($C = 14.9\ \mu$F), a resistor ($R = 24.3$ kΩ), and a battery ($V = 25.7$ V) are connected in series, as shown in the figure. What is the charge on the capacitor at $t = 0.3621$ s after the switch is closed?



a) 5.48×10^{-5} C
b) 7.94×10^{-5} C
c) $1.15 \times \cdot 10^{-5}$ C
d) 1.66×10^{-4} C
e) 2.42×10^{-4} C

6.12 Kirchhoff's Loop Rule states that

a) the algebraic sum of the currents around a complete circuit loop must be zero.
b) the resistances around a complete circuit loop must sum to zero.
c) the sources of emf around a complete circuit loop must sum to zero.
d) the sum of the potential differences around a complete circuit loop must be greater than zero.
e) the potential differences around a complete circuit loop must sum to zero.

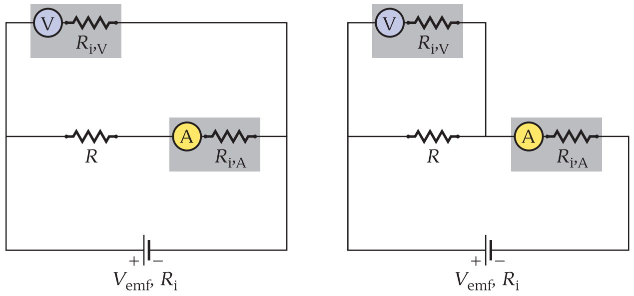
6.13 Which of the following statements are true?

1. An ideal ammeter should have infinite resistance.
2. An ideal ammeter should have zero resistance.
3. An ideal voltmeter should have infinite resistance.
4. An ideal voltmeter should have zero resistance.

a) 1 and 3
b) 2 and 4
c) 2 and 3
d) 1 and 4

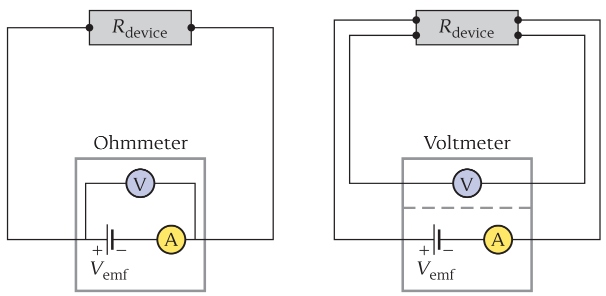
CONCEPTUAL QUESTIONS

6.14 You want to measure simultaneously the potential difference across and the current through a resistor, R. As the circuit diagrams show, there are two ways to connect the two instruments–ammeter and voltmeter–in the circuit. Comment on the result of the measurement using each configuration.



6.15 If the capacitor in an RC circuit is replaced with two identical capacitors connected in series, what happens to the time constant for the circuit?

6.16 You want to accurately measure the resistance, R_{device}, of a new device. The figure shows two ways to accomplish this task. On the left, an ohmmeter produces a current through the device and measures that current, i, and the potential difference, ΔV, across the device. This potential difference includes the potential drops across the wires leading to and from the device and across the contacts that connect the wires to the device. These extra resistances cannot always be neglected, especially if the device has low resistance. This technique is called *two-probe measurement* since two probe wires are connected to the device. The resulting current, i, is measured with an ammeter. The total resistance is then determined by dividing ΔV by i. For this configuration, what is the resistance that the ohmmeter measures? In the alternative configuration, shown on the right, a similar current source is used to produce and measure the current through the device, but the potential difference, ΔV, is measured directly across the device with a nearly ideal voltmeter with extremely large internal resistance. This technique is called a *four-probe measurement* since four probe wires are connected to the device. What resistance is being measured in this four-probe configuration? Is it different from that being assessed by the two-probe measurement? Why or why not? (*Hint:* Four-probe measurements are used extensively by scientists and engineers and are especially useful for accurate measurements of the resistance of materials or devices with low resistance.)



6.17 Explain why the time constant for an RC circuit increases with R and with C. (The answer "That's what the formula says" is not sufficient.)

6.18 A battery, a resistor, and a capacitor are connected in series in an RC circuit. What happens to the current through a resistor after a long time? Explain using Kirchhoff's rules.

6.19 How can you light a 1.0 W, 1.5 V bulb with your 12.0 V car battery?

6.20 A multiloop circuit contains a number of resistors and batteries. If the emf values of all the batteries are doubled, what happens to the currents in all the components of the circuit?

6.21 A multiloop circuit of resistors, capacitors, and batteries is switched on at $t = 0$, at which time all the capacitors are uncharged. The initial distribution of currents and potential differences in the circuit can be analyzed by treating the capacitors as if they were connecting wires or closed switches. The final distribution of currents and potential differences, which occurs after a long time has passed, can be analyzed by treating the capacitors as open segments or open switches. Explain why these tricks work.

6.22 Voltmeters are always connected in parallel with a circuit component, and ammeters are always connected in series. Explain why.

6.23 You wish to measure both the current through and the potential difference across some component of a circuit. It is not possible to do this simultaneously and accurately with ordinary voltmeters and ammeters. Explain why not.

6.24 Two light bulbs for use at 110 V are rated at 60 W and 100 W, respectively. Which has the filament with lower resistance?

6.25 Two capacitors in series are charged through a resistor. Identical capacitors are instead connected in parallel and charged through the same resistor. How do the times required to fully charge the two sets of capacitors compare?

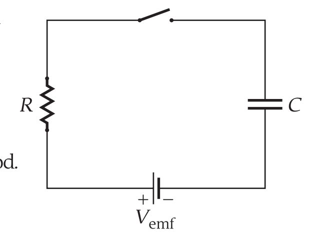
6.26 The figure shows a circuit consisting of a battery connected to a resistor and a capacitor, which is fully discharged initially, in series with a switch.

a) What is the current in the circuit at any time t?

b) Calculate the total energy provided by the battery from $t = 0$ to $t = \infty$.

c) Calculate the total energy dissipated from the resistor over the same time period.

d) Is energy conserved in this circuit?



EXERCISES

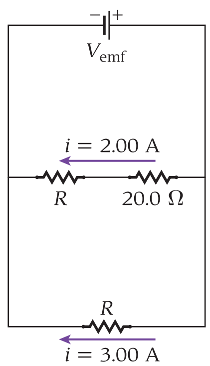
A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

Sections 6.1 through 6.3

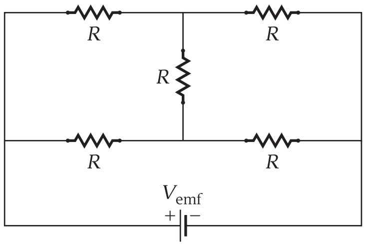
6.27 Two resistors, R_1 and R_2, are connected in series across a potential difference, ΔV_0. Express the potential drop across each resistor individually, in terms of these quantities. What is the significance of this arrangement?

6.28 A battery has $V_{emf} = 12.0$ V and internal resistance $r = 1.00\ \Omega$. What resistance, R, can be put across the battery to extract 10.0 W of power from it?

6.29 Three resistors are connected across a battery as shown in the figure. What values of R and V_{emf} will produce the indicated currents?



•**6.30** Find the equivalent resistance for the circuit in the figure.

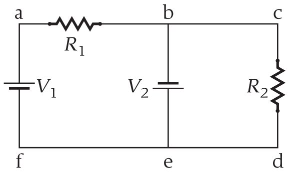


•**6.31** The dead battery of your car provides a potential difference of 9.950 V and has an internal resistance of 1.100 Ω. You charge it by connecting it with jumper cables to the live battery of another car. The live battery provides a potential difference of 12.00 V and has an internal resistance of 0.01000 Ω, and the starter resistance is 0.07000 Ω.

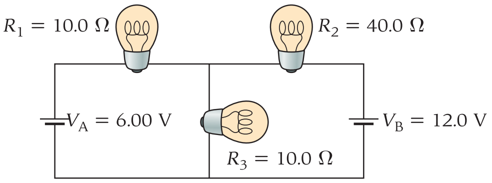
a) Draw the circuit diagram for the connected batteries.

b) Determine the current in the live battery, in the dead battery, and in the starter immediately after you closed the circuit.

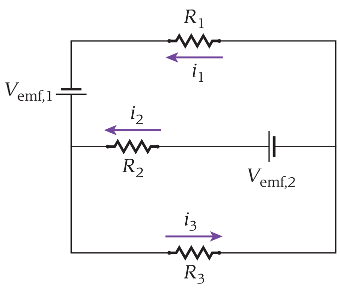
•**6.32** In the circuit shown in the figure, $V_1 = 1.50$ V, $V_2 = 2.50$ V, $R_1 = 4.00\ \Omega$, and $R_2 = 5.00\ \Omega$. What is the magnitude of the current, i_1, flowing through resistor R_1?

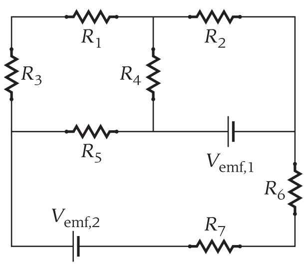


•**6.33** The circuit shown in the figure consists of two batteries supplying voltages V_A and V_B and three light bulbs with resistances R_1, R_2, and R_3. Calculate the magnitudes of the currents i_1, i_2, and i_3 flowing through the bulbs. Indicate the correct directions of current flow on the diagram. Calculate the power, P_A and P_B, supplied by battery A and by battery B.



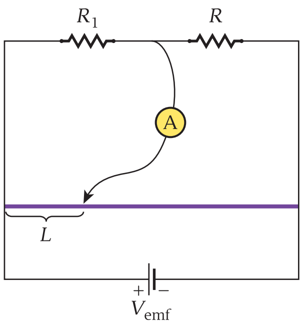
•**6.34** In the circuit shown in the figure, $R_1 = 5.00\ \Omega$, $R_2 = 10.0\ \Omega$, $R_3 = 15.0\ \Omega$, $V_{emf,1} = 10.0$ V, and $V_{emf,2} = 15.0$ V. Using Kirchhoff's Loop and Junction Rules, determine the currents i_1, i_2, and i_3 flowing through R_1, R_2, and R_3, respectively, in the direction indicated in the figure.



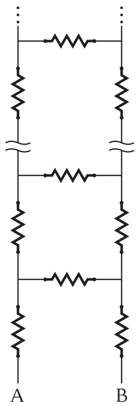


•**6.35** For the circuit shown in the figure, find the magnitude and the direction of the current through each resistor and the power supplied by each battery, using the following values: $R_1 = 4.00\ \Omega$, $R_2 = 6.00\ \Omega$, $R_3 = 8.00\ \Omega$, $R_4 = 6.00\ \Omega$, $R_5 = 5.00\ \Omega$, $R_6 = 10.0\ \Omega$, $R_7 = 3.00\ \Omega$, $V_{emf,1} = 6.00$ V, and $V_{emf,2} = 12.0$ V.

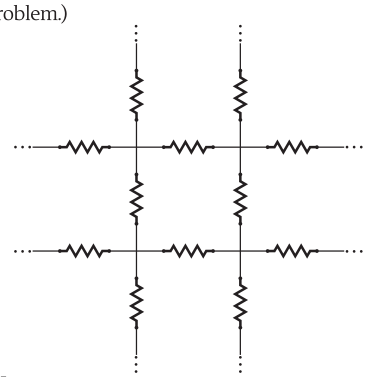
•**6.36** A Wheatstone bridge is constructed using a 1.00 m long Nichrome wire (the purple line in the figure) with a conducting contact that can slide along the wire. A resistor, $R_1 = 100.\ \Omega$, is placed on one side of the bridge, and another resistor, R, of unknown resistance, is placed on the other side. The contact is moved along the Nichrome wire, and it is found that the ammeter reading is zero for $L = 25.0$ cm. Knowing that the wire has a uniform cross section throughout its length, determine the unknown resistance.



••**6.37** A "resistive ladder" is constructed with identical resistors, R, making up its legs and rungs, as shown in the figure. The ladder has "infinite" height; that is, it extends very far in one direction. Find the equivalent resistance of the ladder, measured between its "feet" (points A and B), in terms of R.

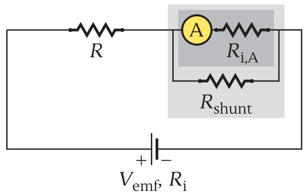


••**6.38** Consider an "infinite," that is, very large, two-dimensional square grid of identical resistors, R, as shown in the figure. Find the equivalent resistance of the grid, as measured across any individual resistor. (*Hint:* Symmetry and superposition are very helpful in solving this problem.)

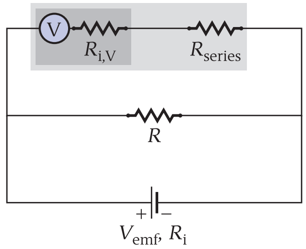


Section 6.4

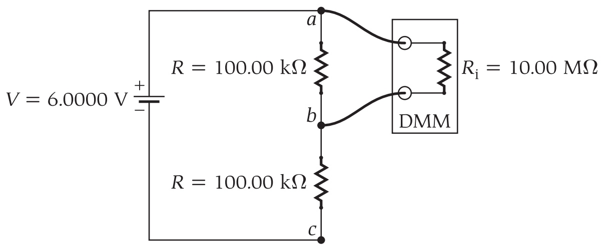
6.39 To extend the useful range of an ammeter, a shunt resistor, R_{shunt}, is placed in parallel with the ammeter as shown in the figure. If the internal resistance of the ammeter is $R_{i,A}$, determine the resistance that the shunt resistor has to have to extend the useful range of the ammeter by a factor N. Then, calculate the resistance the shunt resistor has to have to allow an ammeter with an internal resistance of 1.00 Ω and a maximum range of 1.00 A to measure currents up to 100. A. What fraction of the total 100.-A current flows through the ammeter, and what fraction flows through the shunt resistor?



6.40 To extend the useful range of a voltmeter, an additional resistor, R_{series}, is placed in series with the voltmeter as shown in the figure. If the internal resistance of the voltmeter is $R_{i,V}$, determine the resistance that the added series resistor has to have to extend the useful range of the voltmeter by a factor N. Then, calculate the resistance the series resistor has to have to allow a voltmeter with an internal resistance of 1.00 MΩ (10^6 Ω) and a maximum range of 1.00 V to measure potential differences up to 100. V. What fraction of the total 100.-V potential drop occurs across the voltmeter, and what fraction of that drop occurs across the added series resistor?



6.41 As shown in the figure, a 6.0000-V battery is used to produce a current through two identical resistors, R, each having a resistance of



100.00 kΩ. A digital multimeter (DMM) is used to measure the potential difference across the first resistor. DMMs typically have an internal resistance of 10.00 MΩ. Determine the potential differences V_{ab} (the potential difference between points *a* and *b*, which is the difference the DMM measures) and V_{bc} (the potential difference between points *b* and *c*, which is the difference across the second resistor). Nominally, $V_{ab} = V_{bc}$, but this may not be the case here. How can this measurement error be reduced?

6.42 You want to make an ohmmeter to measure the resistance of unknown resistors. You have a battery with voltage $V_{emf} = 9.00$ V, a variable resistor, *R*, and an ammeter that measures current on a linear scale from 0 to 10.0 mA.

a) What resistance should the variable resistor have so that the ammeter gives its full-scale (maximum) reading when the ohmmeter is shorted?

b) Using the resistance from part (a), what is the unknown resistance if the ammeter reads $\frac{1}{4}$ of its full scale?

•6.43 A circuit consists of two 1.00 kΩ resistors in series with an ideal 12.0 V battery.

a) Calculate the current flowing through each resistor.

b) A student trying to measure the current flowing through one of the resistors inadvertently connects an ammeter in parallel with that resistor rather than in series with it. How much current will flow through the ammeter, assuming that it has an internal resistance of 1.00 Ω?

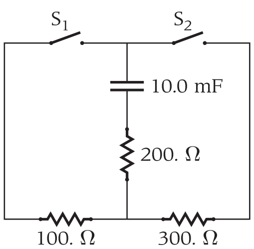
•6.44 A circuit consists of two 100. kΩ resistors in series with an ideal 12.0-V battery.

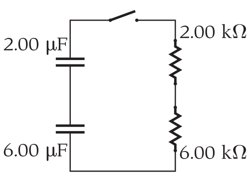
a) Calculate the potential drop across one of the resistors.

b) A voltmeter with internal resistance 10.0 MΩ is connected in parallel with one of the two resistors in order to measure the potential drop across the resistor. By what percentage will the voltmeter reading deviate from the value you determined in part (a)? (*Hint:* The difference is rather small so it is helpful to solve algebraically first to avoid a rounding error.)

Section 6.5

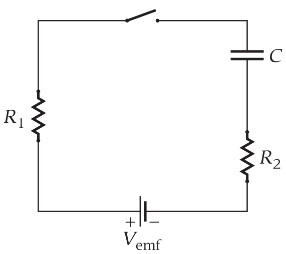
6.45 Initially, switches S_1 and S_2 in the circuit shown in the figure are open and the capacitor has a charge of 100. mC. About how long will it take after switch S_1 is closed for the charge on the capacitor to drop to 5.00 mC?





6.46 What is the time constant for the discharging of the capacitors in the circuit shown in the figure? If the 2.00 μF capacitor initially has a potential difference of 10.0 V across its plates, how much charge is left on it after the switch has been closed for a time equal to half of the time constant?

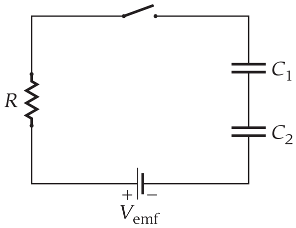
6.47 The circuit shown in the figure has a switch, S, two resistors, R_1 = 1.00 Ω and R_2 = 2.00 Ω, a 12.0-V battery, and a capacitor with C = 20.0 μF. After the switch is closed, what will the maximum charge on the capacitor be? How long after the switch has been closed will the capacitor have 50.0% of this maximum charge?



6.48 In the movie *Back to the Future*, time travel is made possible by a flux capacitor, which generates 1.21 GW of power. Assuming that a 1.00 F capacitor is charged to its maximum capacity with a 12.0 V car battery and is discharged through a resistor, what resistance is necessary to produce a peak power output of 1.21 GW in the resistor? How long would it take for a 12.0 V car battery to charge the capacitor to 90.0% of its maximum capacity through this resistor?

6.49 During a physics demonstration, a fully charged 90.0-μF capacitor is discharged through a 60.0 Ω resistor. How long will it take for the capacitor to lose 80.0% of its initial energy?

•6.50 Two parallel plate capacitors, C_1 and C_2, are connected in series with a 60.0-V battery and a 300. kΩ resistor, as shown in the figure. Both capacitors have plates with an area of 2.00 cm^2 and a separation of 0.100 mm. Capacitor C_1 has air between its plates, and capacitor C_2 has the gap filled with a certain porcelain (dielectric constant of 7.00 and dielectric strength of 5.70 kV/mm). The switch is closed, and a long time passes.



a) What is the charge on capacitor C_1?

b) What is the charge on capacitor C_2?

c) What is the total energy stored in the two capacitors?

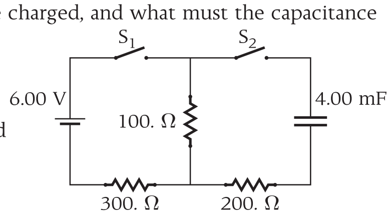
d) What is the electric field inside capacitor C_2?

•6.51 A parallel plate capacitor with $C = 0.0500$ μF has a separation between its plates of $d = 50.0$ μm. The dielectric that fills the space between the plates has dielectric constant $\kappa = 2.50$ and resistivity $r = 4.00 \times 10^{12}$ Ω m. What is the time constant for this capacitor? (*Hint:* First calculate the area of the plates for the given *C* and κ, and then determine the resistance of the dielectric between the plates.)

•6.52 A 12.0-V battery is attached to a 2.00 mF capacitor and a 100. Ω resistor. Once the capacitor is fully charged, what is the energy stored in it? What is the energy dissipated as heat by the resistor as the capacitor is charging?

•6.53 A capacitor bank is designed to discharge 5.00 J of energy through a 10.0 kΩ resistor array in under 2.00 ms. To what potential difference must the bank be charged, and what must the capacitance of the bank be?

•6.54 The circuit in the figure has a capacitor connected to a battery, two switches, and three resistors. Initially, the capacitor is uncharged and both of the switches are open.



a) Switch S_1 is closed. What is the current flowing out of the battery immediately after switch S_1 is closed?

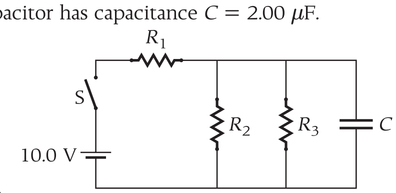
b) After about 10.0 min, switch S_2 is closed. What is the current flowing out of the battery immediately after switch S_2 is closed?

c) What is the current flowing out of the battery about 10.0 min after switch S_2 has been closed?

d) After another 10.0 min, switch S_1 is opened. How long will it take until the current in the 200.-Ω resistor is below 1.00 mA?

•6.55 In the circuit shown in the figure, R_1 = 10.0 Ω, R_2 = 4.00 Ω, and R_3 = 10.0 Ω, and the capacitor has capacitance C = 2.00 μF.

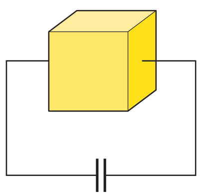
a) Determine the potential difference, ΔV_C, across the capacitor after switch S has been closed for a long time.



b) Determine the energy stored in the capacitor when switch S has been closed for a long time.

c) After switch S is opened, how much energy is dissipated through R_3?

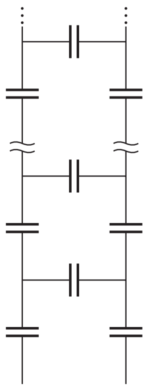
••6.56 A cube of gold that is 2.50 mm on a side is connected across the terminals of a 15.0 μF capacitor that initially has a potential difference of 100.0 V between its plates.



a) What time is required to fully discharge the capacitor?

b) When the capacitor is fully discharged, what is the temperature of the gold cube?

••**6.57** A "capacitive ladder" is constructed with identical capacitors, C, making up its legs and rungs, as shown in the figure. The ladder has "infinite" height; that is, it extends very far in one direction. Calculate the equivalent capacitance of the ladder, measured between its "feet" (points A and B), in terms of C.



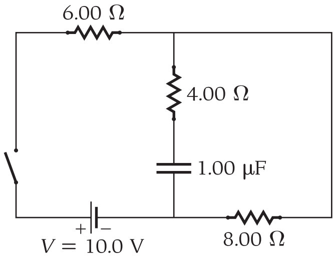
Additional Exercises

6.58 In the circuit in the figure, the capacitors are completely uncharged. The switch is then closed for a long time.

a) Calculate the current through the 4.00 Ω resistor.

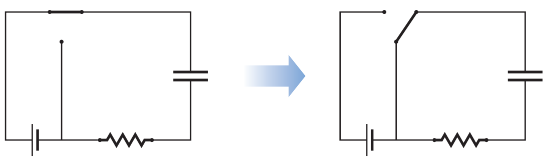
b) Find the potential difference across the 4.00 Ω, 6.00 Ω, and 8.00 Ω resistors.

c) Find the potential difference across the 1.00 μF capacitor.

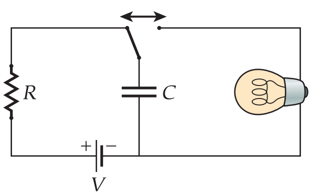


6.59 The ammeter your physics instructor uses for in-class demonstrations has internal resistance $R_i = 75.0\ \Omega$ and measures a maximum current of 1.50 mA. The same ammeter can be used to measure currents of much greater magnitudes by wiring a shunt resistor of relatively small resistance, R_{shunt}, in parallel with the ammeter. (a) Sketch the circuit diagram, and explain why the shunt resistor connected in parallel with the ammeter allows it to measure larger currents. (b) Calculate the resistance the shunt resistor has to have to allow the ammeter to measure a maximum current of 15.0 A.

6.60 Many electronics devices can be dangerous even after they are shut off. Consider an RC circuit with a 150. μF capacitor and a 1.00 MΩ resistor connected to a 200. V power source for a long time and then disconnected and shorted, as shown in the figure. How long will it be until the potential difference across the capacitor drops to below 50.0 V?

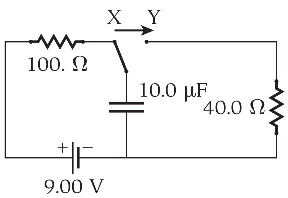


6.61 Design a circuit like that shown in the figure to operate a strobe light. The capacitor discharges power through the light bulb filament (resistance of 2.50 kΩ) in 0.200 ms and charges through a resistor R, with a repeat cycle of 1.00 kHz. What capacitor and resistor should be used?



6.62 An ammeter with an internal resistance of 53.0 Ω measures a current of 5.25 mA in a circuit containing a battery and a total resistance of 1130 Ω. The insertion of the ammeter alters the resistance of the circuit, and thus the measurement does not give the actual value of the current in the circuit without the ammeter. Determine the actual value of the current.

•**6.63** In the circuit shown in the figure, a 10.0 μF capacitor is charged by a 9.00 V battery with the two-way switch kept in position X for a long time. Then the switch is suddenly flicked to position Y. What current flows through the 40.0 Ω resistor



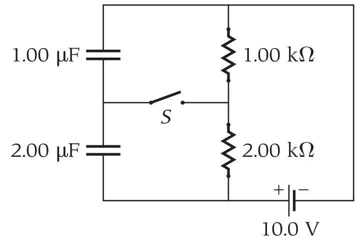
a) immediately after the switch moves to position Y?

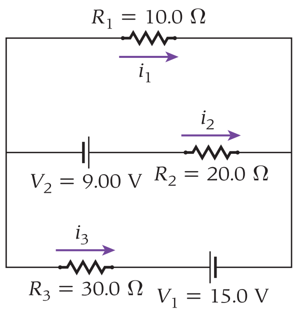
b) 1.00 ms after the switch moves to position Y?

•**6.64** How long will it take for the current in a circuit to drop from its initial value to 1.50 mA if the circuit contains two 3.80 μF capacitors that are initially uncharged, two 2.20 kΩ resistors, and a 12.0-V battery all connected in series?

6.65 An RC circuit has a time constant of 3.10 s. At $t = 0$, the process of charging the capacitor begins. At what time will the energy stored in the capacitor reach half of its maximum value?

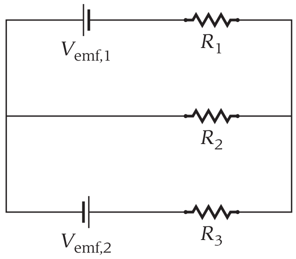
•**6.66** For the circuit shown in the figure, determine the charge on each capacitor when (a) switch S has been closed for a long time and (b) switch S has been open for a long time.

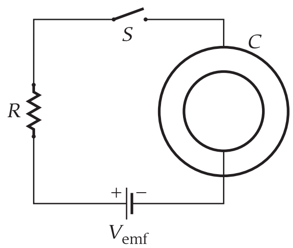




•**6.67** Three resistors, $R_1 = 10.0\ \Omega$, $R_2 = 20.0\ \Omega$, and $R_3 = 30.0\ \Omega$, are connected in a multiloop circuit, as shown in the figure. Determine the amount of power dissipated in the three resistors.

•**6.68** The figure shows a circuit containing two batteries and three resistors. The batteries provide $V_{\text{emf},1} = 12.0$ V and $V_{\text{emf},2} = 16.0$ V and have no internal resistance. The resistors have resistances of $R_1 = 30.0\ \Omega$, $R_2 = 40.0\ \Omega$, and $R_3 = 20.0\ \Omega$. Find the magnitude of the potential drop across R_2.



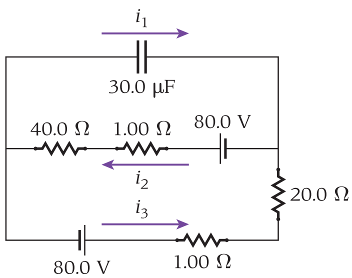


6.69 The figure shows a spherical capacitor. The inner sphere has radius $a = 1.00$ cm, and the outer sphere has radius $b = 1.10$ cm. The battery supplies $V_{\text{emf}} = 10.0$ V, and the resistor has a value of $R = 10.0$ MΩ.

a) Determine the time constant of the RC circuit.

b) Determine how much charge has accumulated on the capacitor after switch S has been closed for 0.1 ms.

•**6.70** Write the set of equations that determines the three currents in the circuit shown in the figure. (Assume that the capacitor is initially uncharged.)



•**6.71** Consider a series RC circuit with $R = 10.0\ \Omega$, $C = 10.0\ \mu$F and $V = 10.0$ V.

a) How much time, expressed as a multiple of the time constant, does it take for the capacitor to be charged to half of its maximum value?

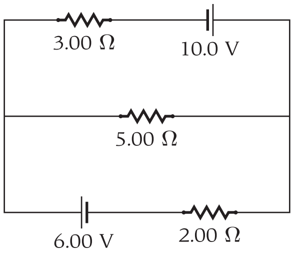
b) At this instant, what is the ratio of the energy stored in the capacitor to its maximum possible value?

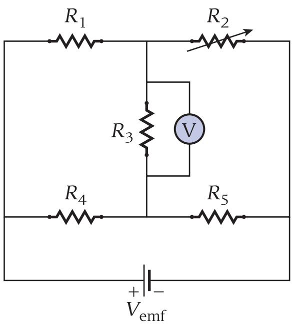
c) Now suppose the capacitor is fully charged. At time $t = 0$, the original circuit is opened and the capacitor is allowed to discharge across another resistor, $R' = 1.00\ \Omega$, that is connected across the capacitor. What is the time constant for the discharging of the capacitor?

d) How many seconds does it take for the capacitor to discharge half of its maximum stored charge, Q?

•6.72 a) What is the current in the $5.00\ \Omega$ resistor in the circuit shown in the figure?

b) What is the power dissipated in the $5.00\ \Omega$ resistor?



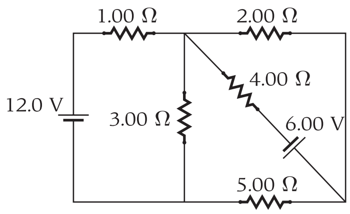


•6.73 In the Wheatstone bridge shown in the figure, the known resistances are $R_1 = 8.00\ \Omega$, $R_4 = 2.00\ \Omega$, and $R_5 = 6.00\ \Omega$, and the battery supplies $V_{\text{emf}} = 15.0$ V. The variable resistance R_2 is adjusted until the potential difference across R_3 is zero ($V = 0$). Find i_2 (the current through resistor R_2) at this time.

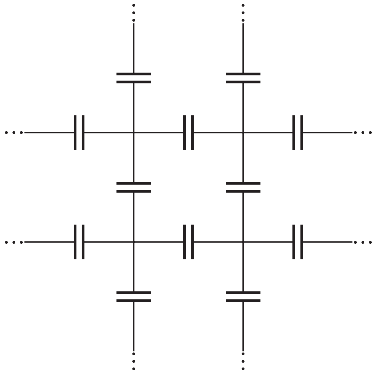
••6.74 Consider the circuit with five resistors and two batteries (with no internal resistance) shown in the figure.

a) Write a set of equations that will allow you to solve for the current in each of the resistors.

b) Solve the equations from part (a) for the current in the 4.00-Ω resistor.



••6.75 Consider an "infinite," that is, very large, two-dimensional square grid of identical capacitors, C, shown in the figure. Find the effective capacitance of the grid, as measured across any individual capacitor.



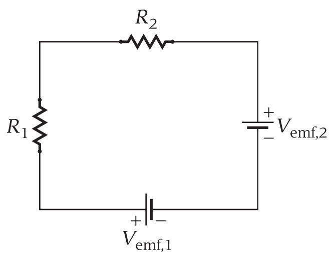
MULTI-VERSION EXERCISES

6.76 A 11.45 V battery with internal resistance $R_i = 0.1373\ \Omega$ is to be charged by a battery charger that is capable of delivering a current $i = 9.759$ A. What is the minimum emf the battery charger must be able to supply in order to charge the battery?

6.77 A battery with internal resistance $R_i = 0.1415\ \Omega$ is being charged by a battery charger that delivers a current $i = 5.399$ A. The battery charger supplies an emf of 14.51 V. What is the potential difference across the terminals of the battery?

6.78 A 16.05 V battery with internal resistance R_i is being charged by a battery charger that is capable of delivering a current $i = 6.041$ A. The battery charger supplies an emf of 16.93 V. What is the internal resistance, R_i, of the battery?

6.79 The single-loop circuit shown in the figure has $V_{\text{emf},1} = 21.01$ V, $V_{\text{emf},2} = 10.75$ V, $R_1 = 23.37\ \Omega$, and $R_2 = 11.61\ \Omega$. What is the current flowing in the circuit?



6.80 The single-loop circuit shown in the figure has $V_{\text{emf},1} = 16.37$ V, $V_{\text{emf},2} = 10.81$ V, and $R_1 = 24.65\ \Omega$. The current flowing in the circuit is 0.1600 A. What is the resistance R_2?

6.81 The single-loop circuit shown in the figure has $V_{\text{emf},1} = 17.75$ V, $R_1 = 25.95\ \Omega$, and $R_2 = 13.59\ \Omega$. The current flowing in the circuit is 0.1740 A. What is $V_{\text{emf},2}$?

6.82 A 15.19 mF capacitor is fully charged using a battery that supplies $V_{\text{emf}} = 131.1$ V. The battery is disconnected, and a 616.5 Ω resistor is connected across the capacitor. What current will be flowing through the resistor after 3.871 s?

6.83 A capacitor is fully charged using a battery that supplies $V_{\text{emf}} = 133.1$ V. The battery is disconnected, and a 655.1 Ω resistor is connected across the capacitor. The current flowing through the resistor after 1.743 s is 0.1745 A. What is the capacitance of the capacitor?

6.84 A 19.79 mF capacitor is fully charged using a battery. The battery is disconnected, and a 693.5 Ω resistor is connected across the capacitor. The current flowing through the resistor after 6.615 s is 0.1203 A. What is the emf supplied by the battery?

7 Magnetism

FIGURE 7.1 The Shanghai Maglev Train in the Pudong Airport station. The inset is a display inside the train showing the maximum speed of 430 km/h (267 mph) attained during the 7-minute 20-second trip from the airport to downtown Shanghai.

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While much of our knowledge of magnetism has been known for two centuries and some of it since ancient times, new and exciting technical applications of this phenomenon continue to be developed today. Many of the devices we use every day make use of magnetism. Cars, computers, power generators, almost anything that uses an electric motor, and almost any information storage technology are just a few examples. One particularly impressive example is Shanghai's Maglev Train, shown in Figure 7.1, which floats on artificially generated magnetic fields and is thus able to reach very high speeds. In order to appreciate these innovations, however, we need to start with the basics of magnetism.

This chapter is the first to consider magnetism, describing magnetic fields and magnetic forces and their effects on charged particles and currents. We'll continue to study magnetism in the next few chapters, describing the causes of magnetic fields and their connection to electric fields. You will see that electricity and magnetism are really parts of the same universal force, called the *electromagnetic force*; their connection is one of the most spectacular successes in physical theory.

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WHAT WE WILL LEARN

- Permanent magnets exist in nature. A magnet always has a north pole and a south pole. A single magnetic north pole or south pole cannot be isolated–magnetic poles always come in pairs.
- Opposite magnetic poles attract, and like poles repel.
- Breaking a bar magnet in half results in two new magnets, each with a north and a south pole.
- A magnetic field exerts a force on a moving charged particle.
- The force exerted on a charged particle moving in a magnetic field is perpendicular to both the magnetic field and the velocity of the particle.
- The torque on a current-carrying loop can be expressed in terms of the vector product of the magnetic dipole moment of the loop and the magnetic field.
- The Hall effect can be used to measure magnetic fields.

7.1 Permanent Magnets

In the region of Magnesia (in central Greece), the ancient Greeks found several types of naturally occurring minerals that attract and repel each other and attract certain kinds of metal, such as iron. They also, if floating freely, line up with the North and South Poles of the Earth. These minerals are various forms of iron oxide and are called **permanent magnets.** Other examples of permanent magnets include refrigerator magnets and magnetic door latches, which are made of compounds of iron, nickel, or cobalt. If you touch an iron bar to a piece of the mineral lodestone (magnetic magnetite), the iron bar will be magnetized. If you float this iron bar in water, it will align with the Earth's magnetic poles. The end of the magnet that points north is called the **north magnetic pole,** and the other end is called the **south magnetic pole.**

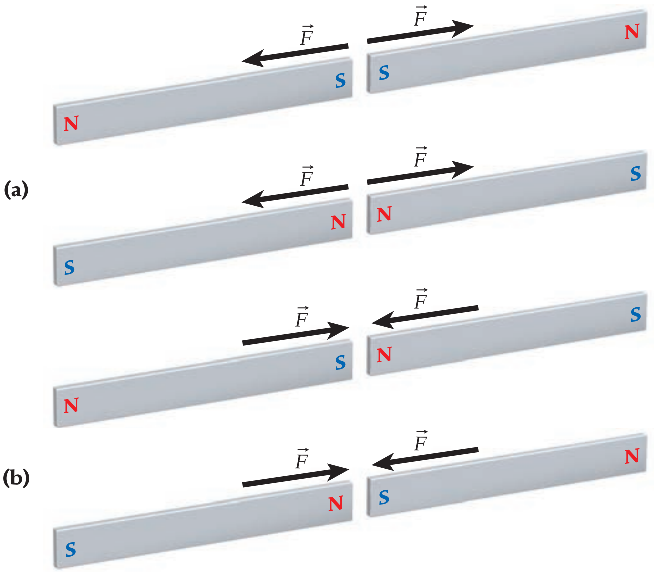


FIGURE 7.2 (a) Like magnetic poles repel; (b) unlike magnetic poles attract.

If two permanent magnets are brought close together with the two north poles or two south poles almost touching, the magnets repel each other (Figure 7.2a). If a north pole and a south pole are brought close together, the magnets attract each other (Figure 7.2b). What is called the North Pole of Earth is actually a magnetic south pole, which is why it attracts the north pole of permanent magnets.

Breaking a permanent magnet in half does not yield one north pole and one south pole. Instead, two new magnets, each with its own north and south pole result (Figure 7.3). Unlike electric charge, which exists as separate positive (proton) and negative (electron) charges, no separate magnetic monopoles (isolated north and south poles) exist. Scientists have carried out extensive searches for magnetic monopoles, and none has been found. The discussion of the source of magnetism in this chapter will help you understand why there are no magnetic monopoles.

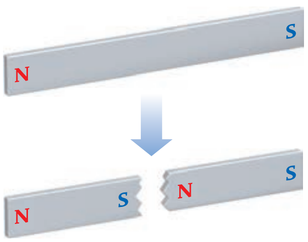
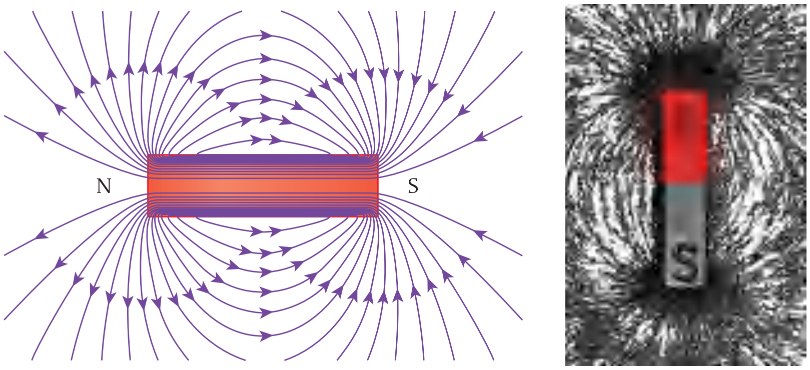


FIGURE 7.3 Breaking a bar magnet in half yields two magnets, each with its own north and south pole.

Magnetic Field Lines

Permanent magnets interact with each other at some distance, without touching. In analogy with the gravitational field and the electric field, the concept of a **magnetic field** is used to describe the magnetic force. The vector $\vec{B}(\vec{r})$ denotes the magnetic field vector at any given point in space.

Like an electric field, a magnetic field is represented using field lines. The magnetic field vector is always tangent to the **magnetic field lines.** The magnetic field lines from a permanent bar magnet are shown in Figure 7.4a. As with electric field lines, closer spacing between lines indicates higher field strength. In an electric field, the electric force on a positive test charge points in the same direction as the electric field vector.



(a) (b)

FIGURE 7.4 (a) Computer-generated magnetic field lines from a permanent bar magnet. (b) Iron filings align themselves with the magnetic field lines and make them visible.

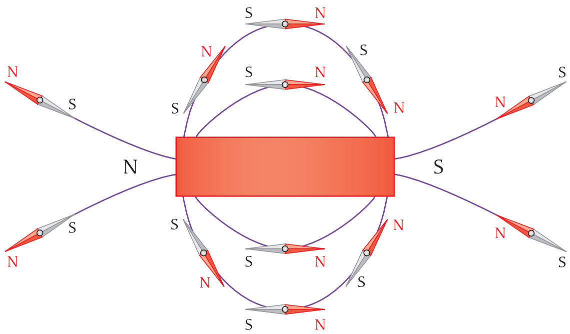


FIGURE 7.5 Using a compass needle to determine the direction of the magnetic field from a bar magnet.

However, because no magnetic monopole exists, the magnetic force cannot be described in an analogous way.

The direction of the magnetic field is established in terms of the direction in which a compass needle points. A compass needle, with a north pole and a south pole, will orient itself so that its north pole points in the direction of the magnetic field. Thus, the direction of the field can be determined at any point by noting the direction in which a compass needle placed at that point points, as illustrated in Figure 7.5 for a bar magnet.

Externally, magnetic field lines appear to originate on north poles and terminate on south poles, but these field lines are actually closed loops that penetrate the magnet itself. This formation of loops is an important difference between electric and magnetic field lines (for static fields—this statement does not apply to time-dependent fields, as we'll see in subsequent chapters). Recall that electric field lines start at positive charges and end on negative charges. However, because no magnetic monopoles exist, magnetic field lines cannot start or stop at particular points. Instead, they form closed loops that do not start or stop anywhere. We'll see later that this difference is important in describing the interaction of electric and magnetic fields. If you see a field pattern and don't know at first if it is an electric field or a magnetic field, check for closed loops. If you find some, it is a magnetic field; if the field lines do not form loops, it is an electric field.

Earth's Magnetic Field

Earth itself is a magnet, with a magnetic field similar to the magnetic field of a bar magnet (Figure 7.4). This magnetic field is important because it protects us from high-energy radiation from space called *cosmic rays*. These cosmic rays consist mostly of charged particles that are deflected away from Earth's surface by its magnetic field. The poles of Earth's magnetic field do not coincide with the geographic poles, defined as the points where Earth's rotation axis intersects its surface.

Figure 7.6 shows a cross section of Earth's magnetic field lines. The field lines are close together, forming a surface that wraps around Earth like a doughnut. Earth's magnetic field is distorted by the solar wind, a flow of ionized particles, mainly protons, emitted by the Sun and moving outward from the Sun at approximately 400 km/s. Two bands of charged particles captured from the solar wind circle the Earth. These are called the **Van Allen radiation belts** (Figure 7.6), after James A. Van Allen (1914–2006),

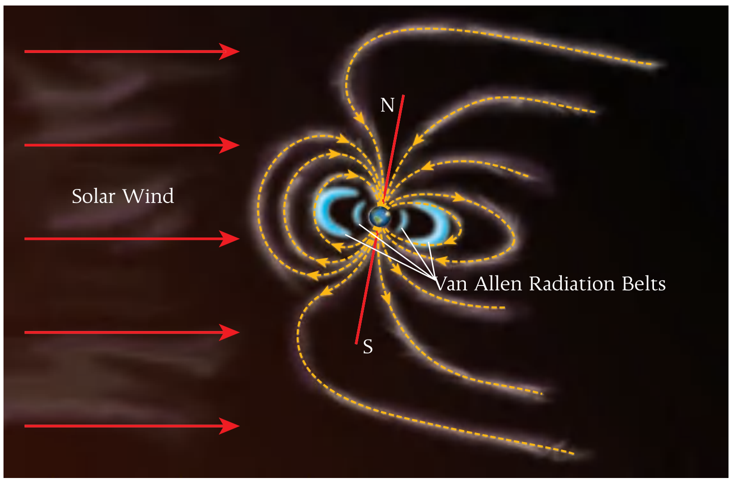


FIGURE 7.6 Cross section through Earth's magnetic field. The dashed lines represent the magnetic field lines. The axis defined by the north and south magnetic poles (red line) currently forms an angle of approximately 11° with the rotation axis.

FIGURE 7.7 Aurora borealis over Finland, photographed from the International Space Station.

FIGURE 7.8 Aurora on Saturn, photographed by the Hubble Space Telescope.

who discovered them in the early days of space flight by putting radiation counters on satellites. The Van Allen radiation belts come closest to Earth near the north and south magnetic poles, where the charged particles trapped within the belts often collide with atoms in the atmosphere, exciting them. These excited atoms emit light of different colors as they collide and lose energy; the results are the fabulous **aurora borealis** (northern lights) in the northern latitudes (see Figure 7.7) and the **aurora australis** (southern lights) in the southern latitudes. Aurorae are not unique to Earth; they have been seen on other planets with strong magnetic fields, such as Jupiter and Saturn (shown in Figure 7.8).

Earth's magnetic poles move at a current rate of up to 40 km in a single year. Right now, the magnetic north pole is located approximately 2800 km away from the geographic South Pole, at the edge of Antarctica, and is moving toward Australia. The magnetic south pole is located in the Canadian Arctic and, if its present rate of motion continues, will reach Siberia in 2050. Earth's magnetic field has decreased steadily at a rate of about 7% per century since it was first measured accurately around 1840. At that rate, Earth's magnetic field will disappear in a few thousand years. However, some geological evidence indicates that the magnetic field of Earth has reversed itself approximately 170 times in the past 100 million years. The last reversal occurred about 770,000 years ago. Thus, rather than disappear, Earth's magnetic field may reverse its direction. What is the cause of Earth's magnetic field? Surprisingly, the answer to this question is not known exactly and is under intense current research. Most likely, it is caused by strong electrical currents inside the Earth, caused by the spinning liquid iron-nickel core. This spinning is often referred to as the *dynamo effect*. (We'll see how currents create magnetic fields in Chapter 8.)

Because the geographic North Pole and the magnetic north pole are not in the same location, a compass needle generally does not point exactly to the geographic North Pole. This difference is called the **magnetic declination.** The magnetic declination is taken to be positive when magnetic north is east of true north and negative when magnetic north is west of true north. The magnetic north pole currently lies on a line that passes through southeastern Missouri, western Illinois, eastern Iowa, and western Wisconsin. Along this line, the magnetic declination is zero. West of this line, the magnetic declination is positive and reaches 18° in Seattle. East of this line, the declination is negative, up to −18° in Maine. A map showing the magnetic declinations in the United States as of 2004 is presented in Figure 7.9.

Because the positions of Earth's magnetic poles move with time, the magnetic declinations for all locations on Earth's surface also change with time. For example, Figure 7.10 shows the estimated magnetic declination for Lansing, Michigan, for the period 1900−2004. A similar graph can be drawn for any location on Earth.

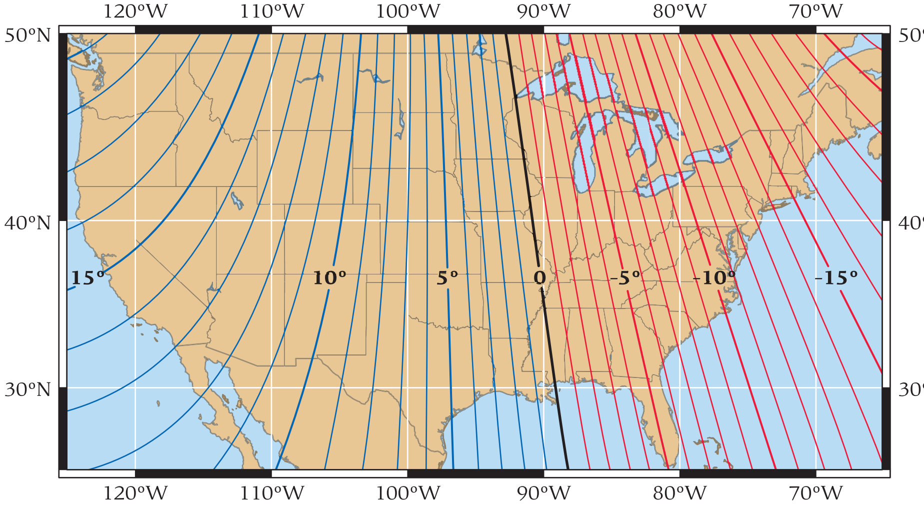


FIGURE 7.9 Magnetic declinations in the United States in 2004. Red lines represent negative magnetic declinations, and blue lines signify positive magnetic declinations. Lines of magnetic declination are separated by 1 degree.

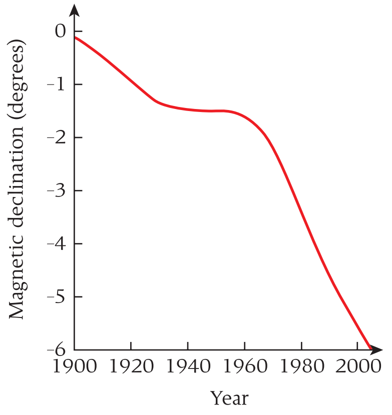


FIGURE 7.10 The magnetic declination at Lansing, Michigan, from 1900 to 2004.

Superposition of Magnetic Fields

If several sources of magnetic field, such as several permanent magnets, are close together, the magnetic field at any given point in space is given by the superposition of the magnetic fields from all the sources. The superposition principle for the total magnetic field, $\vec{B}_{\text{total}}(\vec{r})$, due to n magnetic field sources can be stated as

$$\vec{B}_{\text{total}}(\vec{r}) = \vec{B}_1(\vec{r}) + \vec{B}_2(\vec{r}) + \cdots + \vec{B}_n(\vec{r}). \tag{7.1}$$

7.2 Magnetic Force

The qualitative discussion in the preceding section pointed out that a magnetic field has a direction, along the magnetic field lines. The magnitude of a magnetic field is determined by examining its effect on a moving charged particle. We'll start with a constant magnetic field and study its effect on a single charge. As a reminder, the electric field exerts a force on a charge given by $\vec{F}_E = q\vec{E}$. Experiments such as the one shown in Figure 7.11 show that a magnetic field does not exert a force on a charge at rest but only on a *moving* charge.

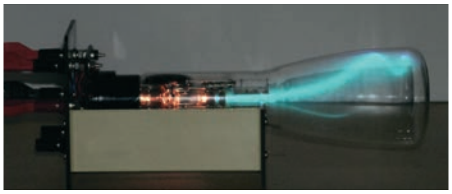


FIGURE 7.11 A beam of electrons (bluish green), made visible by a small amount of gas in an evacuated tube, is bent by a magnet (at the right edge of picture).

A magnetic field is defined in terms of the force exerted by the field on a moving charged particle. The magnetic force exerted by a magnetic field on a moving charged particle with charge q moving with velocity $\vec{v}$ is given by

$$\vec{F}_B = q\vec{v} \times \vec{B}. \tag{7.2}$$

The direction of the force is perpendicular to both the velocity of the moving charged particle and the magnetic field (Figure 7.12). This statement is right-hand rule 1. The right-hand rule gives the force direction on a positive charge given the known velocity and field directions. However, for a negative charge the force will be in the opposite direction.

The magnitude of the magnetic force on a moving charged particle is

$$F_B = |q| vB \sin\theta, \tag{7.3}$$

where θ is the angle between the velocity of the charged particle and the magnetic field. (The angle θ is always between 0° and 180°, and therefore, $\sin\theta \geq 0$.) You can see that no magnetic force acts on a charged particle moving parallel to a magnetic field because in that case $\theta = 0°$. If a charged particle is moving perpendicularly to the magnetic field, $\theta = 90°$ and (for fixed values of v and B) the magnitude of the magnetic force has its maximum value of

$$F_B = |q| vB \quad \left(\text{for } \vec{v} \perp \vec{B}\right). \tag{7.4}$$

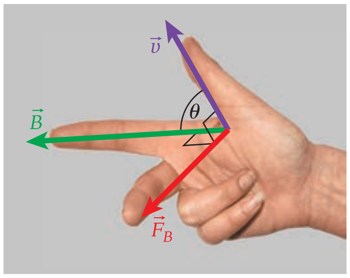


FIGURE 7.12 Right-hand rule 1 for the force exerted by a magnetic field, $\vec{B}$, on a particle with charge q moving with velocity $\vec{v}$. To find the direction of the magnetic force, point your thumb in the direction of the velocity of the moving charged particle and your index finger in the direction of the magnetic field, and your middle finger then gives you the direction of the magnetic force.

Magnetic Force and Work

Equation 7.2 established that the magnetic force is the vector product of the velocity vector and magnetic field vector and thus is perpendicular to both vectors. This implies that $\vec{F}_B \bullet \vec{v} = 0$ and, since the force is the product of mass and acceleration, also that $\vec{a} \bullet \vec{v} = 0$. We saw that this condition means that the direction of the velocity vector can change but the magnitude of the velocity vector, the speed, remains the same. Therefore, the kinetic energy, $\frac{1}{2}mv^2$, remains constant for a particle subjected to a magnetic force, and the magnetic force *does no work* on the moving particle.

This is a profound result: A constant magnetic field cannot be used to do work on a particle. The kinetic energy of a particle moving in a constant magnetic field remains constant, even though the direction of the particle's velocity vector can change as a function of time while the particle is moving through the magnetic field. An electric field, on the other hand, can easily be used to do work on a particle.

Units of Magnetic Field Strength

To discuss the motion of charges in magnetic fields, we need to know what units are used to measure the magnetic field strength. Solving equation 7.4 for the field strength and inserting the units of the other quantities gives

$$[F_B]=[q][v][B] \Rightarrow [B]=\frac{[F_B]}{[q][v]}=\frac{\text{N s}}{\text{C m}}.$$

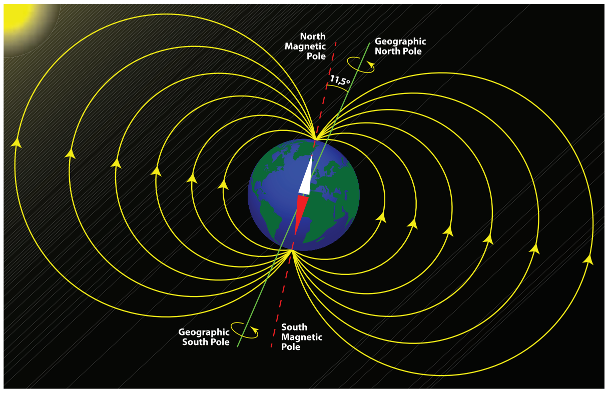


FIGURE 7.13 Global map of the strength of Earth's magnetic field.

Because the ampere (A) is defined as 1 C/s, (N s)/(C m) = N/(A m). The unit of magneticfield strength has been named the **tesla** (T), in honor of Croatian-born American physicist and inventor Nikola Tesla (1856–1943):

$$1 \text{ T}=1\ \frac{\text{N s}}{\text{C m}}=1\ \frac{\text{N}}{\text{A m}}.$$

A tesla is a rather large amount of magnetic field strength. Sometimes magnetic field strength is given in gauss (G), which is not an SI unit:

$$1 \text{ G}=10^{-4} \text{ T}.$$

For example, the strength of Earth's magnetic field at Earth's surface is on the order of 0.5 G (5×10^{-5} T). It varies with location from 0.2 G to 0.6 G, as illustrated in Figure 7.13.

SOLVED PROBLEM 7.1 Cathode Ray Tube

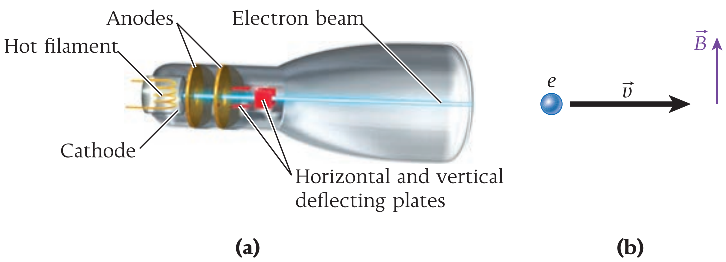


FIGURE 7.14 (a) A cathode ray tube. (b) Electrons moving with velocity $\vec{v}$ enter a constant magnetic field.

PROBLEM

Consider a cathode ray tube similar to the one shown in Figure 7.11. In this tube, a potential difference of ΔV = 111 V accelerates electrons horizontally (starting essentially from rest) in an electron gun, as shown in Figure 7.14a. The electron gun has a specially coated filament that emits electrons when heated. A negatively charged cathode controls the number of electrons emitted. Positively charged anodes focus and accelerate the electrons into a beam. Downstream from the anodes are horizontal and vertical deflecting plates. Beyond the electron gun is a constant magnetic field with magnitude $B = 3.40 \times 10^{-4}$ T. The direction of the magnetic field is upward, perpendicular to the initial velocity of the electrons. What is the magnitude of the acceleration of the electrons due to the magnetic field? (The mass of an electron is 9.11×10^{-31} kg.)

SOLUTION

THINK The electrons gain kinetic energy in the electron gun of the cathode ray tube. The gain in kinetic energy of each electron is equal to the charge of the electron times the potential difference. The speed of the electrons can be found from the definition of kinetic energy. The magnetic force on an electron can be found from the electron charge, the electron velocity, and the strength of the magnetic field, and it is equal to the mass of the electron times its acceleration.

SKETCH Figure 7.14b shows an electron, moving with velocity $\vec{v}$, entering a constant magnetic field that is perpendicular to the electron path.

RESEARCH The change in kinetic energy, ΔK, of the electrons plus the change in potential energy of the electrons is equal to zero:

$$\Delta K + \Delta U = \tfrac{1}{2}mv^2 + q\Delta V = 0.$$

Since, in this case, $q = -e$, we see that

$$e\Delta V = \tfrac{1}{2}mv^2, \qquad \text{(i)}$$

where ΔV is the magnitude of the potential difference across which the electrons were accelerated and m is the mass of an electron. We can solve equation (i) for the speed of the electrons:

$$v = \sqrt{\frac{2e\Delta V}{m}}. \qquad \text{(ii)}$$

The magnitude of the force exerted by the magnetic field on the electrons is given by equation 7.3:

$$F_B = evB \sin 90° = evB,$$

where $-e$ is the charge of an electron and B is the magnitude of the magnetic field. According to Newton's Second Law, $F_{\text{net}} = ma$. Since the only force present is the magnetic one, we have

$$F_B = ma = evB, \qquad \text{(iii)}$$

where a is the magnitude of the acceleration of the electrons.

SIMPLIFY We can rearrange equation (iii) and substitute the expression for the speed of the electrons from equation (ii) to obtain the acceleration of the electrons:

$$a = \frac{evB}{m} = \frac{eB\sqrt{\frac{2e\Delta V}{m}}}{m} = B\sqrt{2\Delta V \frac{e^3}{m^3}}.$$

CALCULATE Putting in the numerical values gives us

$$a = \left(3.40 \times 10^{-4} \text{ T}\right)\sqrt{2(111 \text{ V})\frac{\left(1.602 \times 10^{-19} \text{ C}\right)^3}{\left(9.11 \times 10^{-31} \text{ kg}\right)^3}} = 3.7357 \times 10^{14} \text{ m/s}^2.$$

ROUND We report our result to three significant figures:

$$a = 3.74 \times 10^{14} \text{ m/s}^2.$$

DOUBLE-CHECK The calculated acceleration is tremendously large, almost 40 trillion times the Earth's gravitational acceleration. So we certainly want to double-check. We first calculate the speed of the electrons:

$$v = \sqrt{\frac{2e\Delta V}{m}} = \sqrt{\frac{2\left(1.602 \times 10^{-19} \text{ C}\right)(111 \text{ V})}{9.11 \times 10^{-31} \text{ kg}}} = 6.25 \times 10^6 \text{ m/s}.$$

A speed of 6250 km/s may seem large, but it is reasonable for electrons because it is only 2% of the speed of light. The magnetic force on each electron is then

$$F_B = evB = \left(1.602 \times 10^{-19} \text{ C}\right)\left(6.25 \times 10^6 \text{ m/s}\right)\left(3.40 \times 10^{-4} \text{ T}\right) = 3.40 \times 10^{-16} \text{ N}.$$

The acceleration is very large because the mass of an electron is very small.

Concept Check 7.1

In what direction will the electron in Figure 7.14b be deflected as it enters the constant magnetic field?

a) into the page

b) out of the page

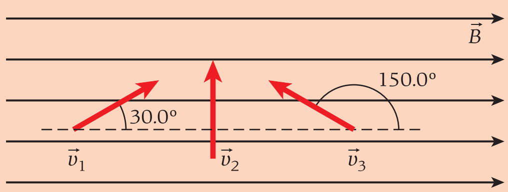
c) upward

d) downward

e) no deflection

Self-Test Opportunity 7.1

Three particles, each with charge $q = 6.15$ μC and speed $v = 465$ m/s, enter a uniform magnetic field with magnitude $B = 0.165$ T (see the figure). What is the magnitude of the magnetic force on each of the particles?



7.3 Motion of Charged Particles in a Magnetic Field

The fact that the force due to a magnetic field acting on a moving charged particle is perpendicular to both the field and the particle's velocity makes this force different from any we've considered so far. However, the tools we use to analyze this force–Newton's laws and the laws of conservation of energy, momentum, and angular momentum–are the same.

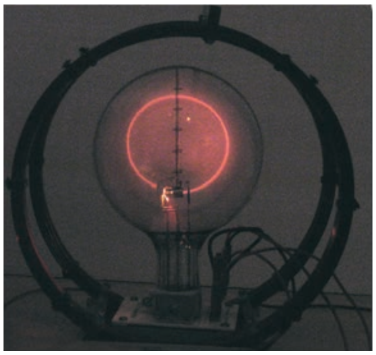


FIGURE 7.15 Electron beam bent into a circular path by the magnetic field generated by two coils.

Paths of Moving Charged Particles in a Constant Magnetic Field

Suppose you drive your car at constant speed around a circular track. The friction between the tires and the road provides the centripetal force that keeps the car moving in a circle. This force always points toward the center of the circle and creates a centripetal acceleration. A similar physical situation occurs when a particle with charge q and mass m moves with velocity $\vec{v}$ perpendicular to a uniform magnetic field, $\vec{B}$, as illustrated in Figure 7.15.

In this situation, the particle moves in a circle with constant speed v and the magnetic force of magnitude $F_B = |q|vB$ supplies the centripetal force that keeps the particle moving in a circle. Particles with opposite charges and the same mass will orbit in opposite directions at the same orbital radius. For example, electrons and positrons are elementary particles with the same mass; the electron has a negative charge, and the positron has a positive charge. Figure 7.16 is a bubble chamber photograph showing two electron-positron pairs. A bubble chamber is a device that can track charged particles moving in a constant magnetic field. Pair 1 has an electron and a positron that have the same relatively low speed. The particles initially travel in a circle. However, as they move through the bubble chamber, they slow down. (This slowdown is not due to the magnetic force but to collisions of the particles with the molecules of the gas in the bubble chamber.) Thus, the radius of the circle gets smaller and smaller, creating a spiral. The electron and positron in pair 2 have a much higher speed. Their tracks are curved but do not form a complete circle before the particles exit the bubble chamber.

If the velocity of a charged particle is parallel (or antiparallel) to the magnetic field, the particle experiences no magnetic force and continues to travel in a straight line.

For motion perpendicular to a magnetic field, as in Figure 7.15, the force required to keep a particle moving with speed v in a circle with radius r is the centripetal force:

$$F = \frac{mv^2}{r}.$$

Setting this expression for the centripetal force equal to that for the magnetic force, we obtain

$$vB|q| = \frac{mv^2}{r}.$$

Rearranging gives an expression for the radius of the circle in which the particle is traveling:

$$r = \frac{mv}{|q|B}. \tag{7.5}$$

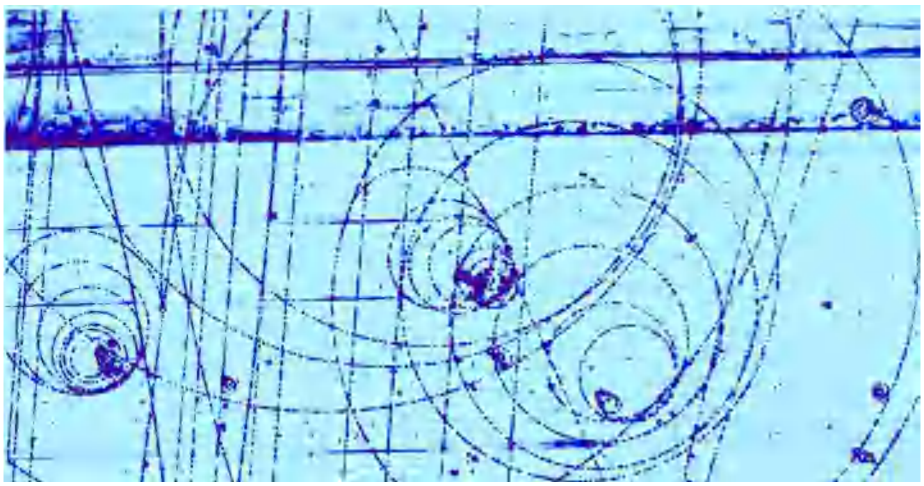


FIGURE 7.16 Bubble chamber photograph showing two electron-positron pairs. The bubble chamber is located in a constant magnetic field pointing directly out of the page.

A common way to express this relationship is in terms of the magnitude of the momentum of the particle:

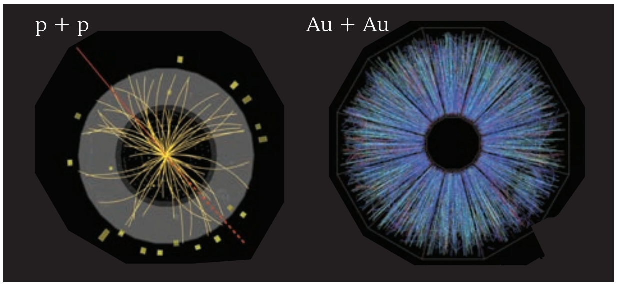
$$Br = \frac{p}{|q|}. \tag{7.6}$$

If the velocity $\vec{v}$ is not perpendicular to $\vec{B}$, then the velocity component perpendicular to $\vec{B}$ causes circular motion while the parallel velocity component is unaffected by $\vec{B}$ and drags this orbit into a helical shape.

Time Projection Chamber

Particle physicists create new elementary particles by colliding larger particles at the highest energies. In these collisions, many particles stream away from the interaction point at high speeds. A simple particle detector is not sufficient to identify these particles. A device that can help physicists study these collisions is a time projection chamber (TPC). The STAR TPC was described in Example 2.4.

Figure 7.17 shows collisions of two protons and two gold nuclei. The proton-proton collision creates dozens of particles; the gold-gold collision creates thousands of particles. Each charged particle leaves a track in the chamber. The color assigned by a computer to the track represents the ionization density of the track as particles pass through the gas of the chamber. As they pass through the gas, the particles ionize the atoms of the gas, releasing free electrons. The gas allows the free electrons to drift without recombining with positive ions. Electric fields applied between the center of the chamber and the end caps of the cylinder exert an electric force on the free electrons, making them drift toward the end caps, where they are recorded electronically. Using the drift time and the recording positions, computer software reconstructs the trajectories that the particles took through the chamber. The particles produced in the collisions have a velocity component that is perpendicular to the chamber's magnetic field and thus have circular trajectories.



(a) (b)

FIGURE 7.17 Curved tracks left by the motion of charged particles produced in collisions of (a) two protons, each with kinetic energy of 100 GeV, and (b) two gold nuclei, each with kinetic energy of 100 GeV.

EXAMPLE 7.1 Transverse Momentum of a Particle Seen by the ATLAS

One track of a moving charged particle from Figure 7.17a is shown in Figure 7.18. The radius of the circular trajectory this particle is following is $r = 2.300$ m. The magnitude of the magnetic field in the TPC is $B = 0.5000$ T. We can assume that the particle has charge $|q| = 1.602177 \times 10^{-19}$ C.

PROBLEM

What is the component of the particle's momentum that is perpendicular to the magnetic field?

SOLUTION

We'll call this component the transverse momentum of the particle, p_t. We use equation 7.6, replacing p with p_t, because the magnetic force depends only on p_t, and not on the component of the momentum that is parallel to $\vec{B}$:

$$Br = \frac{p_t}{|q|}.$$

We can express the magnitude of the transverse momentum of the particle in terms of the magnitude of the TPC's magnetic field and the absolute value of the charge of the particle:

$$p_t = |q|Br = \left(1.602177\times10^{-19}\ \text{C}\right)(0.5000\ \text{T})(2.300\ \text{m}) = 1.843\times10^{-19}\ \text{kg m/s}.$$

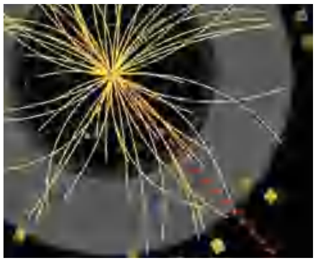


FIGURE 7.18 Circle fitted to the trajectory of one of the charged particles produced from the proton-proton collision in the ATLAS detector at CERN shown in Figure 7.17a.

– Continued

Instead of these SI units for momentum, particle physicists often use MeV/c (recall Example 7.5). Since 1 MeV $= 1.602 \times 10^{-13}$ J, the conversion between kg m/s and MeV/c is

$$1 \text{ MeV}/c = \frac{1.60218 \times 10^{-13} \text{J}}{2.99792 \times 10^{8} \text{m/s}} = 5.3443 \times 10^{-22} \text{kg m/s} \Leftrightarrow 1 \text{ kg m/s} = 1.87115 \times 10^{21} \text{MeV}/c.$$

Therefore, our result for the transverse momentum is

$$p_{\text{t}} = 344.8 \text{ MeV}/c.$$

The analysis of a particle's transverse momentum carried out here can be done by automated computer algorithms on up to approximately 5000 charged particles created in a single collision of two gold nuclei. This very complex task takes about 30 seconds for a computer (3-GHz processor) to finish. In contrast, the TPC can record up to 1000 events per second, which corresponds to 1 millisecond per event.

EXAMPLE 7.2 The Solar Wind and Earth's Magnetic Field

Section 7.1 discussed the Van Allen radiation belts that trap particles emitted from the Sun. The Sun throws approximately 1 million tons of matter into space every second. This matter is mostly protons traveling at a speed of around 400. km/s.

PROBLEM
If protons from the Sun are incident perpendicular to Earth's magnetic field (which has a magnitude of 50.0 μT at the Equator), what is the radius of the orbit of the protons? The mass of a proton is 1.67×10^{-27} kg.

SOLUTION
Equation 7.5 relates the magnitude of the magnetic field, B, the radius of a circular orbit, r, and the speed, v, of a particle with mass m and charge q traveling perpendicular to a magnetic field:

$$r = \frac{mv}{|q|B}.$$

Putting in the numerical values, we get

$$r = \frac{\left(1.67 \times 10^{-27} \text{ kg}\right)\left(400. \times 10^{3} \text{ m/s}\right)}{\left(1.602 \times 10^{-19} \text{ C}\right)\left(50.0 \times 10^{-6} \text{ T}\right)} = 83.5 \text{ m}.$$

Thus, the protons of the solar wind orbit around the Earth's magnetic field lines at the Equator in circles of radius 83.5 m. Protons that are incident on the Earth's magnetic field away from the Equator are not traveling perpendicular to the magnetic field, so their orbital radius is larger. However, the magnetic field lines are closer together, meaning that the field is stronger toward the poles. The protons thus spiral along the field lines as they approach the poles. The shape of the Earth's magnetic field forces these protons traveling toward the poles to reverse and travel back toward the Equator, trapping the protons in the Van Allen radiation belts. Thus, the Earth's magnetic field completely blocks the solar wind from reaching the Earth's surface. This is vital, because the blocked cosmic radiation would otherwise make it impossible for higher organisms to live on Earth by ionizing (removing electrons from) atoms and destroying large molecules, for example, DNA.

Cyclotron Frequency

If a particle performs a complete circular orbit inside a uniform magnetic field–for example, like the electrons in the beam shown in Figure 7.15–then the period of revolution, T, of the particle is the circumference of the circle divided by the speed:

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{|q|B}. \tag{7.7}$$

The frequency, f, of the motion of the charged particle is the inverse of the period:

$$f = \frac{1}{T} = \frac{|q|B}{2\pi m}. \tag{7.8}$$

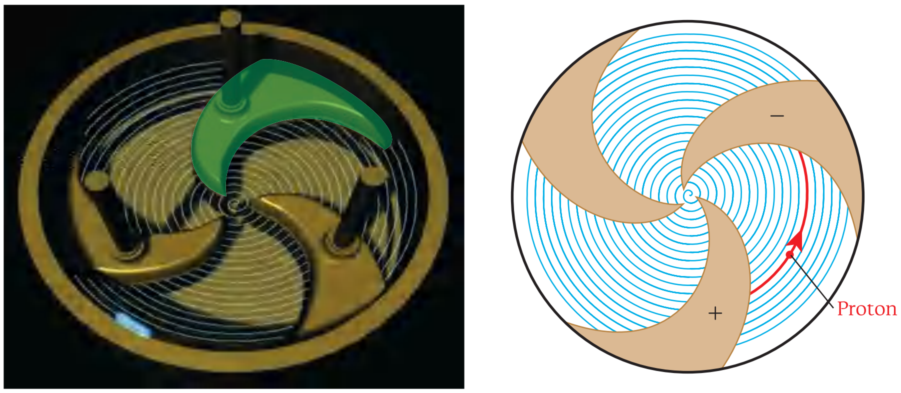
The angular speed, ω, of the motion is

$$\omega = 2\pi f = \frac{|q|B}{m}. \tag{7.9}$$

Thus, the frequency and the angular speed of the particle's motion are independent of the particle's speed and thus independent of the particle's kinetic energy. This fact is used in cyclotrons, which is why ω as given by equation 7.9 is referred to as the **cyclotron frequency.** In a cyclotron, particles are accelerated to higher and higher kinetic energies, and the fact that the cyclotron frequency is independent of the kinetic energy makes designing a cyclotron much easier.

EXAMPLE 7.3 Energy of a Cyclotron

A cyclotron is a particle accelerator (Figure 7.19). The golden horn-shaped pieces of metal shown in the figure (historically called *dees*) have alternating electric potentials applied to them, so a positively charged particle always has a negatively charged dee ahead when it emerges from under any dee, which is now positively charged. The resulting electric field accelerates the particle. Because the cyclotron sits in a strong magnetic field, the particle's trajectory is curved. The radius of the trajectory is proportional to the magnitude of the particle's momentum, according to equation 7.6, so the accelerated particle spirals outward until it reaches the edge of the magnetic field (where its path is no longer bent by the field) and is extracted. According to equation 7.9, the angular frequency is independent of the particle's momentum or energy, so the frequency with which the polarity of the dees is changed does not have to be adjusted as the particle is accelerated. (This holds true only as long as the speed of the accelerated particles does not approach a sizable fraction of the speed of light. To compensate for relativistic effects, the magnetic field of a cyclotron increases with the orbital radius of the accelerated particles.)



(a) (b)

FIGURE 7.19 (a) Computer-generated drawing of the central section of the K500 superconducting cyclotron at the National Superconducting Cyclotron Laboratory at Michigan State University, with the spiral trajectory of an accelerated particle superimposed. One of the three dees of the cyclotron is highlighted in green. (b) Top view of the K500, showing a proton being accelerated between two dees.

PROBLEM

What is the kinetic energy, in mega-electron-volts (MeV), of a proton extracted from a cyclotron with radius $r = 1.81$ m, if the magnetic field of the cyclotron is uniform and has magnitude $B = 0.851$ T? The mass of a proton is 1.67×10^{-27} kg.

SOLUTION

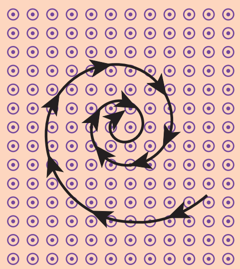
We can solve equation 7.5 for the speed, v, of the proton:

$$v = \frac{r|q|B}{m}.$$

– Continued

Self-Test Opportunity 7.2

A uniform magnetic field is directed out of the page (the standard notation of a dot within a circle represents the tip of the arrowhead of a field line). A charged particle is traveling in the plane of the page, as shown by the arrows in the figure.



a) Is the charge of the particle positive or negative?

b) Is the particle slowing down, speeding up, or moving at constant speed?

c) Is the magnetic field doing work on the particle?

Concept Check 7.2

Cosmic rays would continuously bombard Earth's surface if most of them were not deflected by Earth's magnetic field. Given that Earth is approximately a magnetic dipole (see Figure 7.6), the intensity of cosmic rays incident on its surface is greatest at the

a) North and South Poles.

b) Equator.

c) middle latitudes.

We substitute this expression for v into the equation for kinetic energy:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\frac{r|q|B}{m}\right)^2 = \frac{r^2q^2B^2}{2m}.$$

Putting in the given numbers, we get the kinetic energy in joules:

$$K = \frac{(1.81\text{ m})^2\left(1.602\times10^{-19}\text{ C}\right)^2(0.851\text{ T})^2}{2\left(1.67\times10^{-27}\text{ kg}\right)} = 1.82\times10^{-11}\text{ J}.$$

Since 1 eV = 1.602×10^{-19} J and 1 MeV = 10^6 eV, we have

$$K = 1.82\times10^{-11}\text{ J}\left(\frac{1\text{ eV}}{1.602\times10^{-19}\text{ J}}\right)\left(\frac{1\text{ MeV}}{10^6\text{ eV}}\right) = 114\text{ MeV}.$$

Mass Spectrometer

One application of the motion of charged particles in a magnetic field is a **mass spectrometer,** which allows precise determination of atomic and molecular masses and can be useful for carbon dating and the analysis of unknown chemical compounds. A mass spectrometer operates by ionizing the atoms or molecules to be studied and accelerating them through an electric potential. The ions are then passed through a velocity selector (described further in Solved Problem 7.2), which allows only ions with a given velocity to pass through and blocks the remaining ions. The ions then enter a region of constant magnetic field. In the magnetic field, the radius of curvature of the orbit of each ion is given by equation 7.5: $r = mv/|q|B$. Assuming that all the atoms or molecules are singly ionized (have a charge of $+1$ or -1), the radius of curvature is proportional to the mass of the ion. A schematic diagram of a mass spectrometer is shown in Figure 7.20.

Ions with different masses will have orbits with different radii in the constant magnetic field. For example, in Figure 7.20, ions with orbital radius r_1 have a smaller mass than ions with orbital radius r_2. The particle detector measures the distances from the entrance point, d_1 and d_2, which can be related to the orbital radii and thus the mass of the ions.

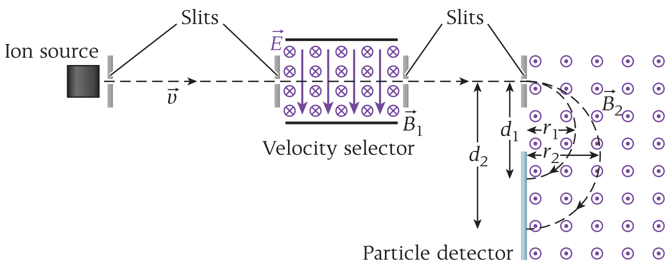


FIGURE 7.20 Schematic diagram of a mass spectrometer showing an ion source, a velocity selector consisting of crossed electric and magnetic fields (see Solved Problem 7.2), a region of constant magnetic field, and a particle detector.

SOLVED PROBLEM 7.2 Velocity Selector

Protons are accelerated from rest through an electric potential difference of $\Delta V = 14.0$ kV. The protons enter a velocity selector, consisting of a parallel plate capacitor in a constant magnetic field, directed perpendicularly into the plane of the page in Figure 7.21a. The electric field between the plates of the parallel plate capacitor is $\vec{E} = 4.30 \times 10^5$ V/m, directed along the plane of the page and downward in Figure 7.21a. This arrangement of perpendicular electric and magnetic fields is referred to as *crossed fields.*

PROBLEM

What magnetic field is required for the protons to move through the velocity selector without being deflected?

SOLUTION

THINK For a proton to move on a straight line without deflection requires that the net force on the proton be zero. Since the proton has a certain velocity and since the magnetic force depends on the velocity, it is plausible that this condition of zero net force cannot be realized for arbitrary speeds of the proton–hence the name *velocity selector*.

SKETCH Figure 7.21b shows the electric and magnetic forces on the protons as they pass through the velocity selector. Note that the two forces point in opposite directions.

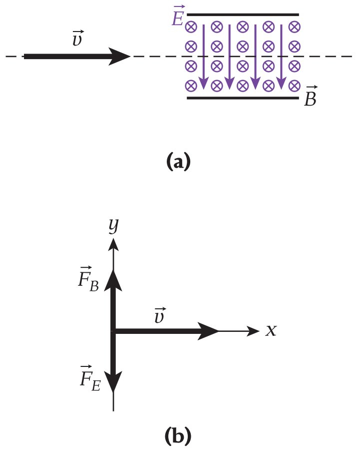


FIGURE 7.21 (a) A proton entering a velocity selector, consisting of crossed electric and magnetic fields. (b) Electric and magnetic forces on a proton passing through the combined fields.

RESEARCH The change in kinetic energy of the protons plus the change in electric potential energy is equal to zero, which can be expressed as

$$\Delta K = -\Delta U = \tfrac{1}{2}mv^2 = e\Delta V,$$

where m is the mass of a proton, v is the speed of the proton after acceleration, e is the charge of the proton, and ΔV is the electric potential difference across which the protons were accelerated. The speed of the protons after acceleration is

$$v = \sqrt{\frac{2e\Delta V}{m}}. \tag{i}$$

When the protons enter the velocity selector, the direction of the electric force is in the direction of the electric field, which is downward (negative y-direction). The magnitude of the electric force is

$$F_E = eE, \tag{ii}$$

where E is the magnitude of the electric field in the velocity selector. Right-hand rule 1 gives the direction of the magnetic force: With your thumb in the direction of the velocity of the protons (positive x-direction) and your index finger in the direction of the magnetic field (into the page), your middle finger points up (positive y-direction). Thus, the direction of the magnetic force on the protons is upward. The magnitude of the magnetic force is given by

$$F_B = evB, \tag{iii}$$

where B is the magnitude of the magnetic field in the velocity selector.

SIMPLIFY The condition that allows the protons to pass though the velocity selector without being deflected is that the electric force balances the magnetic force, or $F_E = F_B$. Using equations (ii) and (iii), we can express this condition as

$$eE = evB.$$

Solving for the magnetic field, B, and substituting for v from equation (i), we obtain

$$B = \frac{E}{\sqrt{\dfrac{2e\Delta V}{m}}} = E\sqrt{\frac{m}{2e\Delta V}}.$$

CALCULATE Putting in the numerical values gives us

$$B = \left(4.30\times10^5\ \text{V/m}\right)\sqrt{\frac{1.67\times10^{-27}\ \text{kg}}{2\left(1.602\times10^{-19}\ \text{C}\right)\left(14.0\times10^3\ \text{V}\right)}} = 0.262371\ \text{T}.$$

ROUND We report our result to three significant figures:

$$B = 0.262\ \text{T}.$$

DOUBLE-CHECK We verify that the electric force is equal to the magnetic force. The electric force is

$$F_E = eE = \left(1.602\times10^{-19}\ \text{C}\right)\left(4.30\times10^5\ \text{V/m}\right) = 6.89\times10^{-14}\ \text{N}.$$

To calculate the magnitude of the magnetic force, we need to find the speed of the protons:

$$v = \sqrt{\frac{2e\Delta V}{m}} = \sqrt{\frac{2\left(1.602\times10^{-19}\ \text{C}\right)\left(14.0\times10^3\ \text{V}\right)}{1.67\times10^{-27}\ \text{kg}}} = 1.64\times10^6\,\text{m/s}.$$

– Continued

This speed is 0.55% of the speed of light, which is not totally impossible. The magnitude of the magnetic force is then

$$F_B = evB = \left(1.602\times10^{-19}\ \text{C}\right)\left(1.64\times10^{6}\ \text{m/s}\right)\left(0.262\ \text{T}\right) = 6.88\times10^{-14}\ \text{N},$$

which agrees with the value of the electric force within rounding error. Thus, our result seems reasonable.

Magnetic Levitation

An interesting application of magnetic force is **magnetic levitation,** a situation in which an upward magnetic force on an object balances the downward gravitational force, achieving static equilibrium with no need for direct contact of surfaces. But if you try to balance a magnet over another magnet by orienting the north poles (or south poles) toward each other, you'll see right away that this is not possible. Instead, one of the magnets will simply flip over, and then the opposite poles will point toward each other, and the attractive force between them will cause the two magnets to snap together. A stable equilibrium requires a local minimum of the potential energy, which does not exist for the pure repulsive interaction of two like magnetic poles.

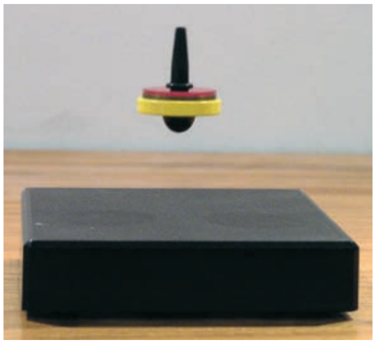


FIGURE 7.22 The Levitron, a toy demonstrating the magnetic levitation of a spinning magnet above a base magnet.

Figure 7.22 shows a toy called the Levitron demonstrating the principle of magnetic levitation. The magnetic top is spun on a plate and then lifted to the proper height and released. The top can remain suspended for several minutes. How does this toy work, considering the requirement for stable equilibrium just mentioned? The answer is that the rapid rotation of the top provides a sufficiently large angular momentum and creates a potential energy barrier that prevents the magnet from flipping over.

Of course, there are other ways to create stable magnetic levitation systems, all of them involving multiple magnets attached rigidly to each other. Magnetic levitation has real-world applications in magnetic levitation (maglev) trains. These trains have several advantages over trains that run on normal steel rails: There are no moving parts to wear out, there is less vibration, and reduced friction means that high speeds are possible. Several maglev trains are already in service around the world and more are being planned. One example is the Shanghai Maglev Train (Figure 7.1), which operates between the Shanghai Pudong Airport and downtown Shanghai and reaches speeds of up to 430 km/h (267 mph).

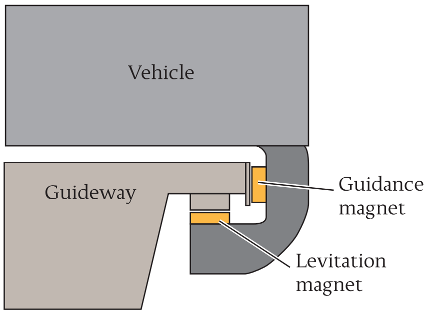


FIGURE 7.23 Cross section of one side of a train car of the Shanghai Maglev Train. The levitation magnets lift the cars 15 cm off the guideway, and the guidance magnets keep the cars centered on the guideway. The magnets are all mounted on the moving vehicle.

The Shanghai Maglev Train operates using magnets attached to the cars (Figure 7.23). These are normal, non-superconducting magnetic coils with electronic feedback to produce stable levitation and guidance. The train cars are held 15 cm above the guideway to allow clearance of any objects that may be on the guideway. The levitation and guidance magnets are held at a distance of 10 mm from the guideway, which is constructed of a magnetic material. The propulsion of the train is provided by magnetic fields built into the guideway. The train propulsion system operates like an electric motor (see Section 7.5) whose circular loops have been unwrapped to a linear configuration.

Maglev trains that use superconducting magnets have been tested, but some technical problems have yet to be resolved, including the maintenance of the superconducting coils and the exposure of the passengers to high magnetic fields.

7.4 Magnetic Force on a Current-Carrying Wire

Consider a wire carrying a current, i, in a constant magnetic field, $\vec{B}$ (Figure 7.24a). The magnetic field exerts a force on the moving charges in the wire. The charge, q, flowing past a point in the wire in a given time, t, is $q = ti$. During this time, the charge occupies a length, L, of wire given by $L = v_\text{d}t$, where v_d is the drift speed (the magnitude of the drift velocity) of the charge carriers in the wire. Thus, we obtain

$$q = ti = \frac{L}{v_\text{d}}i. \tag{7.10}$$

The magnitude of the magnetic force is then

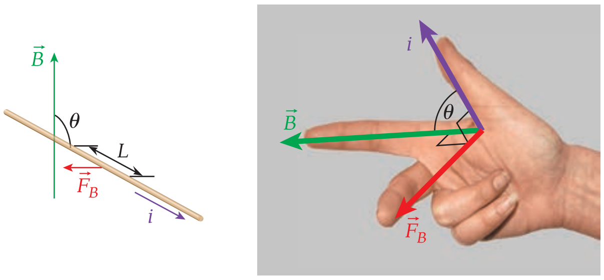
$$F_B = qv_d B\sin\theta = \left(\frac{L}{v_d}i\right)v_d B\sin\theta = iLB\sin\theta, \quad (7.11)$$

where θ is the angle between the direction of the current flow and the direction of the magnetic field. The direction of the force is perpendicular to both the current and the magnetic field and is given by a variant of right-hand rule 1, with the current in the direction of the velocity of a charged particle, as illustrated in Figure 7.24b. This variant of right-hand rule 1 takes advantage of the fact that current can be thought of as charges in motion.

Equation 7.11 can be expressed as a vector product:

$$\vec{F}_B = i\vec{L} \times \vec{B}, \quad (7.12)$$

where the notation $i\vec{L}$ represents the current in a length of wire. Equation 7.12 is simply a reformulation of equation 7.2 for the case in which the moving charges make up a current flowing in a wire. Since physical situations involving currents are far more common than those involving the motion of an isolated charged particle, equation 7.12 is the most useful form for determining the magnetic force in practical applications.

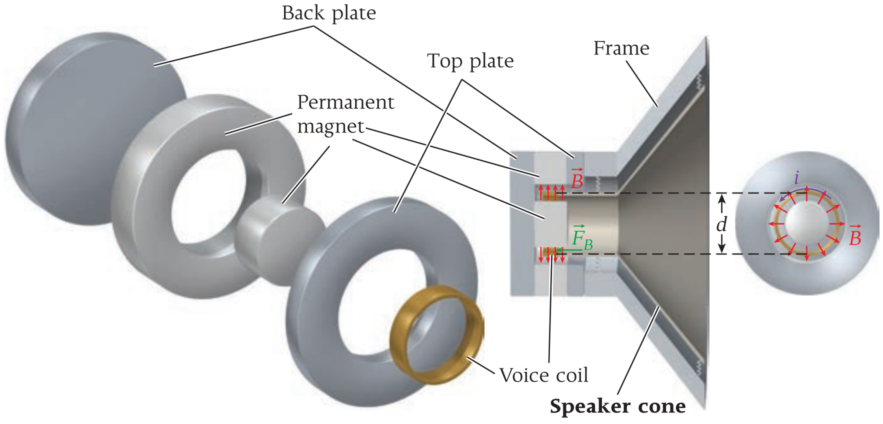


(a) **(b)**

FIGURE 7.24 (a) Magnetic force on a current-carrying wire. (b) A variant of right-hand rule 1 giving the direction of the magnetic force on a current-carrying wire. To determine the direction of the force on a current-carrying wire using your right hand, point your thumb in the direction of the current and your index finger in the direction of the magnetic field; then your middle finger will point in the direction of the force.

EXAMPLE 7.4 Force on the Voice Coil of a Loudspeaker

A loudspeaker produces sound by exerting a magnetic force on a voice coil in a magnetic field, as shown in Figure 7.25. The movable voice coil is connected to a speaker cone that actually produces the sounds. The magnetic field is produced by the two permanent magnets as shown. The magnitude of the magnetic field is $B = 1.50$ T. The voice coil is composed of $n = 100$ turns of wire carrying a current, $i = 1.00$ mA. The diameter of the voice coil is $d = 2.50$ cm.



(a) **(b)** **(c)**

FIGURE 7.25 Schematic diagram of a loudspeaker: (a) an exploded three-dimensional view of the driver end of the loudspeaker; (b) a cross-sectional side view of the loudspeaker; (c) a front view of the driver end of the loudspeaker.

PROBLEM

What is the magnetic force exerted by the magnetic field on the loudspeaker's voice coil?

SOLUTION

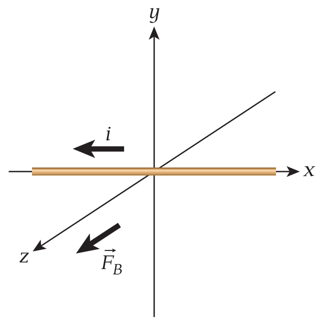
The magnitude of the magnetic force on the voice coil is given by equation 7.11:

$$F = iLB\sin\theta,$$

– *Continued*

Concept Check 7.3

The figure shows a wire lying along the x-axis with a current, i, flowing in the negative x-direction. The wire is in a uniform magnetic field. The magnetic force, $\vec{F}_B$, acts on the wire in the positive z-direction. The magnetic field is oriented so that the force is maximum. What is the direction of the magnetic field?



a) positive y-direction

b) negative x-direction

c) negative y-direction

d) positive z-direction

e) negative z-direction

where L is the length of wire carrying current i in the magnetic field with magnitude B. The wire makes an angle θ with the magnetic field. In this case, the wire is always perpendicular to the magnetic field, so $\theta = 90°$. The length of wire in the voice coil is given by the number of turns, n, times the circumference, πd, of each turn

$$L = n\pi d.$$

Thus, the force on the voice coil is

$$F = i(n\pi d)B(\sin 90°) = n\pi idB.$$

Putting in the numerical values, we get

$$F = n\pi idB = (100)(\pi)\left(1.00\times10^{-3}\text{ A}\right)\left(2.50\times10^{-2}\text{ m}\right)(1.50\text{ T}) = 0.0118\text{ N}.$$

From the right-hand rule 1 illustrated in Figure 7.24b, the direction of the force exerted by the magnetic field on the voice coil is toward the left in Figure 7.25b and perpendicularly into the page in Figure 7.25c.

If the current in the voice coil is reversed, the force will be in the opposite direction. If the current is proportional to the amplitude of a sound wave, sound waves can be reproduced in the cone of the loudspeaker. This basic idea is used in most speakers and headphones.

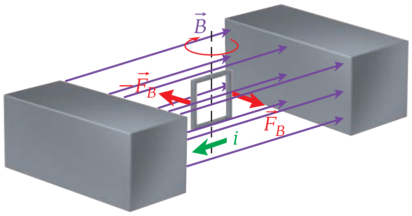


FIGURE 7.26 A primitive element of an electric motor consisting of a current-carrying loop in a magnetic field.

7.5 Torque on a Current-Carrying Loop

Electric motors rely on the magnetic force exerted on a current-carrying wire. This force is used to create a torque that turns a shaft. Let's consider a simple electric motor, consisting of a single square loop carrying a current, i, in a constant magnetic field, $\vec{B}$. The loop is oriented so that its horizontal sections are parallel to the magnetic field and its vertical sections are perpendicular to the magnetic field, as shown in Figure 7.26. The magnitude of the magnetic force on the two vertical sections of the loop is given by equation 7.11 with $\theta = 90°$:

$$F = iLB.$$

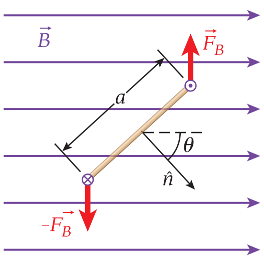


FIGURE 7.27 Bottom view of a current-carrying loop in a magnetic field, showing the forces acting on the loop.

The direction of the magnetic force is given by the variant of right-hand rule 1 illustrated in Figure 7.24b. The two magnetic forces, $\vec{F}_B$ and $-\vec{F}_B$, shown in Figure 7.26 have equal magnitudes and opposite directions. These forces create a torque that tends to rotate the loop around a vertical axis of rotation. These two forces sum to zero. The two horizontal sections of the loop are parallel to the magnetic field and thus experience no magnetic force. Thus, no net force acts on the coil, even though a torque is produced.

Now we consider the case where the loop rotates about its center. As the loop turns in the magnetic field, the forces on the vertical sides of the loop, perpendicular to the direction of the field, do not change. The forces on the square loop, with side length a, are illustrated in Figure 7.27, which shows a bottom view of the coil. In Figure 7.27, θ is the angle between a unit vector, $\hat{n}$, normal to the plane of the coil, and the magnetic field, $\vec{B}$. The unit normal vector is perpendicular to the plane of the wire loop and points in a direction given by right-hand rule 2 (Figure 7.28), based on the current flowing around the loop.

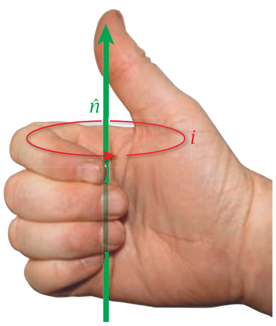


FIGURE 7.28 Right-hand rule 2 gives the direction of the unit normal vector for a current-carrying loop. According to the rule, if you curl the fingers of your right hand in the direction of the current in the loop, your thumb points in the direction of the unit normal vector.

In Figure 7.27, the current is flowing out of the page in the right side of the loop, indicated by the dot in a circle (representing the tip of an arrowhead) and flowing into the page in the left side of the loop, indicated by the cross in a circle (representing the tail of an arrow). The magnitude of the force on each of these vertical segments is

$$F = iaB.$$

The forces on the two horizontal segments of the loop are parallel or antiparallel to the axis of rotation and do not cause a torque, and these two forces sum to zero. Therefore, there is no net force on the loop.

The sum of the torques on the two vertical segments of the square loop gives the net torque exerted on the loop about its center:

$$\tau_1 = (iaB)\left(\frac{a}{2}\right)\sin\theta + (iaB)\left(\frac{a}{2}\right)\sin\theta = ia^2B\sin\theta = iAB\sin\theta, \tag{7.13}$$

where the index 1 on τ_1 indicates that it is the torque on a single loop and $A = a^2$ is the area of the loop. The reason that the loop continues to rotate and doesn't stop at $\theta = 0°$ is that it is connected to a device called a **commutator,** which causes the current to change directions as the coil rotates. This commutator consists of a split ring, with one end of the loop connected to each half of the loop, as shown in Figure 7.29. The current in the loop switches direction two times for every complete rotation of the loop.

If the single loop is replaced with a coil consisting of many loops wound closely together, the torque on the coil is found by multiplying the torque on a loop, τ_1 from equation 7.13, by the number of windings (loops in the coil), N:

$$\tau = N\tau_1 = NiAB\sin\theta. \qquad (7.14)$$

Does this expression for the torque hold for other shapes with area A, other than squares? The answer is yes, though we will not prove it here.

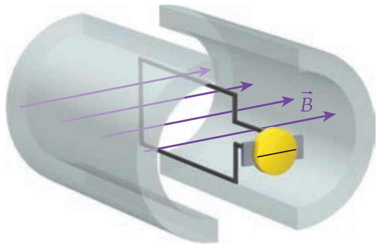


FIGURE 7.29 A wire loop connected to a source of current through a commutator ring.

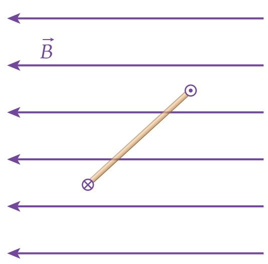
Concept Check 7.4

The top view of a current-carrying loop in a constant magnetic field is shown in the figure. The torque on the loop will cause it to rotate

a) clockwise.

b) counterclockwise.

c) not at all.



7.6 Magnetic Dipole Moment

A current-carrying coil can be described with one parameter, which contains information about a key characteristic of the coil in a magnetic field. The magnitude of the **magnetic dipole moment,** $\vec{\mu}$, of a current-carrying coil of wire is defined to be

$$\mu = NiA, \qquad (7.15)$$

where N is the number of windings, i is the current through the wire, and A is the area of the loops. The direction of the magnetic dipole moment is given by right-hand rule 2 and is the direction of the unit normal vector, $\hat{n}$. Using equation 7.15, we can rewrite equation 7.14 as

$$\tau = (NiA)B\sin\theta = \mu B\sin\theta. \qquad (7.16)$$

The torque on a magnetic dipole is given by

$$\vec{\tau} = \vec{\mu} \times \vec{B}. \qquad (7.17)$$

That is, the torque on a current-carrying coil is the vector product of the magnetic dipole moment of the coil and the magnetic field.

SOLVED PROBLEM 7.3 Torque on a Rectangular Current-Carrying Loop

A rectangular loop with height $h = 6.50$ cm and width $w = 4.50$ cm is in a uniform magnetic field of magnitude $B = 0.250$ T, which points in the negative y-direction (Figure 7.30a). The loop makes an angle of $\theta = 33.0°$ with the y-axis, as shown in the figure. The loop carries a current of magnitude $i = 9.00$ A in the direction indicated by the arrows.

PROBLEM

What is the magnitude of the torque on the loop around the z-axis?

– Continued

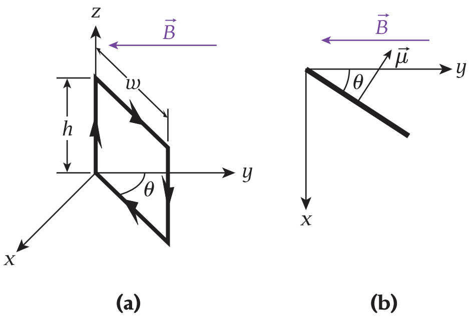


FIGURE 7.30 (a) A rectangular loop carrying a current in a magnetic field. (b) View of the rectangular loop looking down on the *xy*-plane. The magnetic dipole moment is perpendicular to the plane of the loop, with a direction determined by right-hand rule 2.

SOLUTION

THINK The torque on the loop is equal to the vector cross product of the magnetic dipole moment and the magnetic field. The magnetic dipole moment is perpendicular to the plane of the loop, with the direction given by right-hand rule 2.

SKETCH Figure 7.30b is a view of the loop looking down on the *xy*-plane.

RESEARCH The magnitude of the magnetic dipole moment of the loop is

$$\mu = NiA = iwh. \tag{i}$$

The magnitude of the torque on the loop is

$$\tau = \mu B \sin\theta_{\mu B}, \tag{ii}$$

where $\theta_{\mu B}$ is the angle between the magnetic dipole moment and the magnetic field. From Figure 7.30b, we can see that

$$\theta_{\mu B} = \theta + 90°. \tag{iii}$$

SIMPLIFY We can combine equations (i), (ii), and (iii) to obtain

$$\tau = iwhB\sin(\theta + 90°).$$

CALCULATE Putting in the numerical values, we get

$$\tau = (9.00 \text{ A})\left(4.50\times10^{-2} \text{ m}\right)\left(6.50\times10^{-2} \text{ m}\right)(0.250 \text{ T})\left[\sin(33.0° + 90°)\right]$$
$$= 0.0055195 \text{ N m}.$$

ROUND We report our result to three significant figures:

$$\tau = 5.52\times10^{-3} \text{ N m}.$$

DOUBLE-CHECK The magnitude of the force on each of the vertical segments of the loop is

$$F_B = ihB = (9.00 \text{ A})\left(6.50\times10^{-2} \text{ m}\right)(0.250 \text{ T}) = 0.146 \text{ N}.$$

The magnitude of the torque is then the magnitude of the force on the vertical segment that is not along the *z*-axis times the moment arm (which is *w*) times the sine of the angle between the force and the moment arm:

$$\tau = Fw\sin(33.0° + 90°) = 0.146 \text{ N}\left(4.50\times10^{-2} \text{ m}\right)\left[\sin(33.0° + 90°)\right] = 5.52\times10^{-3} \text{ N m}.$$

This is the same as the result calculated above.

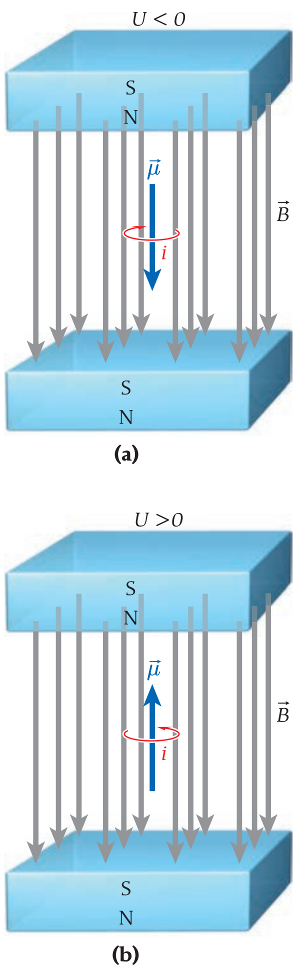


FIGURE 7.31 Magnetic dipole moment vector in an external magnetic field: (a) magnetic dipole and external magnetic field are parallel, resulting in a negative potential energy; (b) magnetic dipole and external magnetic field are antiparallel, resulting in a positive potential energy.

A magnetic dipole has potential energy in an external magnetic field. If the dipole moment is aligned with the magnetic field, the dipole has its minimum potential energy. If the dipole moment is oriented in a direction opposite to the external field, the dipole has its maximum potential energy. The work done by a torque is

$$W = \int_{\theta_0}^{\theta} \tau(\theta')\, d\theta'. \tag{7.18}$$

Using the work-energy theorem and equation 7.16 and setting $\theta_0 = 90°$, we can express the magnetic potential energy, *U*, of a magnetic dipole in an external magnetic field, $\vec{B}$, as

$$W = \int_{\theta_0}^{\theta} \tau(\theta')\, d\theta' = \int_{\theta_0}^{\theta} \mu B \sin\theta'\, d\theta' = -\mu B \cos\theta' \Big|_{\theta_0}^{\theta} = U(\theta) - U(90°),$$

or

$$U(\theta) = -\mu B \cos\theta = -\vec{\mu} \bullet \vec{B}, \tag{7.19}$$

where θ is the angle between the magnetic dipole moment and the external magnetic field.

The lowest value, $-\mu B$, of the potential energy of a magnetic dipole in an external magnetic field is achieved when the dipole's magnetic moment vector is parallel to the external magnetic field vector, and the highest value, $+\mu B$, results when the two vectors are antiparallel (see Figure 7.31). This dependence of potential energy on orientation occurs in diverse physical situations, for which magnetic dipoles in external magnetic fields are a simple model. So far, the only magnetic dipoles we have discussed are current-carrying loops. However, other types of magnetic dipoles exist, including bar magnets and even the Earth. In addition, elementary charged particles such as protons have intrinsic magnetic dipole moments.

Self-Test Opportunity 7.3

What is the maximum difference in magnetic potential energy between two orientations of a loop with area $0.100\ \mathrm{m}^2$ carrying a current of 2.00 A in a constant magnetic field of magnitude 0.500 T?

7.7 Hall Effect

Consider a conductor carrying a current, i, flowing in a direction perpendicular to a magnetic field, $\vec{B}$ (Figure 7.32a). The electrons in the conductor are moving with velocity $\vec{v}_{\mathrm{d}}$ in the direction opposite to the current. The moving electrons experience a force perpendicular to their velocity, causing them to move toward one edge of the conductor. After some time, many electrons have moved to one edge of the conductor, creating a net negative charge on that edge and leaving a net positive charge on the opposite edge of the conductor. This charge distribution creates an electric field, $\vec{E}$, which exerts a force on the electrons in a direction opposite to that exerted by the magnetic field. When the magnitude of the force exerted on the electrons by the electric field is equal to the magnitude of the force exerted on them by the magnetic field, the net number of electrons on the edges of the conductor no longer changes with time. This result is called the **Hall effect.** The potential difference, ΔV_{H}, between the edges of the conductor when equilibrium is reached is termed the **Hall potential difference,** given by

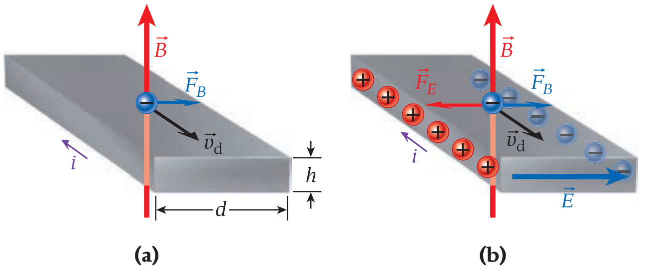


FIGURE 7.32 (a) A conductor carrying a current in a magnetic field. The charge carriers are electrons. (b) The electrons have drifted to one side of the conductor, leaving a net positive charge on the opposite side. This distribution of charges creates an electric field. The potential difference across the conductor is the Hall potential difference.

$$\Delta V_{\mathrm{H}} = Ed, \tag{7.20}$$

where d is the width of the conductor and E is the magnitude of the created electric field.

The Hall effect can be used to demonstrate that the charge carriers in metals are negatively charged. If the charge carriers in a metal were positive and moving in the direction of the current shown in Figure 7.32a, those positive charges would collect on the same edge of the conductor as the electrons in Figure 7.32b, giving an electric field with the opposite sign. Thus, the charge carriers in conductors are negatively charged and must be electrons. The Hall effect also establishes that in some semiconductors the charge carriers are electron holes (missing electrons), which appear to be positively charged carriers.

The Hall effect can also be used to determine a magnetic field by measuring the current flowing through the conductor and the resulting potential difference across the conductor. To obtain the formula for the magnetic field, we start with the equilibrium condition of the Hall effect, that the magnitudes of the magnetic and electric forces are equal:

$$F_E = F_B \Rightarrow eE = v_{\mathrm{d}}Be \Rightarrow B = \frac{E}{v_{\mathrm{d}}} = \frac{\Delta V_{\mathrm{H}}}{v_{\mathrm{d}}d}, \tag{7.21}$$

where substitution for E from equation 7.20 is used in the last step. In Chapter 5, we saw that the drift speed, v_{d}, of an electron in a conductor can be related to the magnitude of the current density, J, in the conductor:

$$J = \frac{i}{A} = nev_{\mathrm{d}},$$

where A is the cross-sectional area of the conductor and n is the number of electrons per unit volume in the conductor. As shown in Figure 7.32a, the cross-sectional area is given by $A = dh$, where d is the width and h is the height of the conductor. Solving $i/A = nev_{\mathrm{d}}$ for the drift speed and substituting hd for A gives

$$v_{\mathrm{d}} = \frac{i}{Ane} = \frac{i}{hdne}.$$

Substituting this expression for v_d into equation 7.21, we have

$$B = \frac{\Delta V_H}{v_d d} = \frac{\Delta V_H dhne}{id} = \frac{\Delta V_H hne}{i}. \tag{7.22}$$

Thus, equation 7.22 gives the magnetic field strength (magnitude) from a measured value of the Hall potential difference, ΔV_H, and the known height, h, and density of charge carriers, n, of the conductor. Equivalently, a rearranged form of equation 7.22 can be used to find the Hall voltage if the magnetic field strength is known:

$$\Delta V_H = \frac{iB}{neh}. \tag{7.23}$$

EXAMPLE 7.5 Hall Effect

Suppose we use a Hall probe to measure the magnitude of a constant magnetic field. The Hall probe is a strip of copper with a height, h, of 2.00 mm. We measure a voltage of 0.250 μV across the probe when we run a current of 1.25 A through it.

PROBLEM

What is the magnitude of the magnetic field?

SOLUTION

The magnetic field is given by equation 7.22:

$$B = \frac{\Delta V_H hne}{i}.$$

We have been given the values of V_H, h, and i, and we know e. The density of the electrons, n, is defined as the number of electrons per unit volume:

$$n = \frac{\text{number of electrons}}{\text{volume}}.$$

The density of copper is $\rho_{Cu} = 8.96\ \text{g/cm}^3 = 8960\ \text{kg/m}^3$, and 1 mole of copper has a mass of 63.5 g and $6.02 \cdot 10^{23}$ atoms. Each copper atom has one conduction electron. Thus, the density of the electrons is

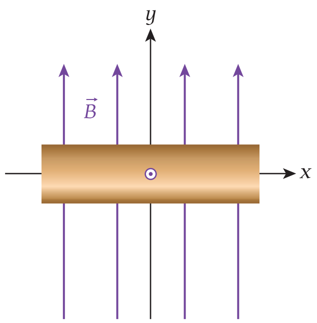
$$n = \left(\frac{1\ \text{electron}}{1\ \text{atom}}\right)\left(\frac{6.02\times10^{23}\ \text{atoms}}{63.5\ \text{g}}\right)\left(\frac{8.96\ \text{g}}{1\ \text{cm}^3}\right)\left(\frac{1.0\times10^{6}\ \text{cm}^3}{1\ \text{m}^3}\right) = 8.49\times10^{28}\ \frac{\text{electrons}}{\text{m}^3}$$

We can now calculate the magnitude of the magnetic field:

$$B = \frac{\left(0.250\times10^{-6}\ \text{V}\right)\left(0.002\ \text{m}\right)\left(8.49\times10^{28}\ \frac{\text{electrons}}{\text{m}^3}\right)\left(1.602\times10^{-19}\ \text{C}\right)}{1.25\ \text{A}} = 5.44\ \text{T}.$$

Concept Check 7.5

A conductor is carrying a current in a constant magnetic field, as shown in the figure. The resulting electric field due to the Hall effect is



a) in the positive x-direction.

b) in the negative x-direction.

c) in the positive y-direction.

d) in the negative y-direction.

WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- Magnetic field lines indicate the direction of a magnetic field in space. Magnetic field lines do not end on magnetic poles, but form closed loops instead.
- The magnetic force on a particle with charge q moving with velocity $\vec{v}$ in a magnetic field, $\vec{B}$, is given by $\vec{F} = q\vec{v} \times \vec{B}$ Right-hand rule 1 gives the direction of the force.
- For a particle with charge q moving with speed v perpendicular to a magnetic field of magnitude B, the magnitude of the magnetic force on the moving charged particle is $F = |q| vB$.
- The unit of magnetic field is the tesla, abbreviated T.
- The average magnitude of the Earth's magnetic field at the surface is approximately 0.5×10^{-4} T.
- A particle with mass m and charge q moving with speed v perpendicular to a magnetic field with magnitude B has a trajectory that is a circle with radius $r = mv / |q| B$.
- The cyclotron frequency, ω, of a particle with charge q and mass m moving in a circular orbit in a constant magnetic field of magnitude B is given by $\omega = |q| B/m$.

- The force exerted by a magnetic field, $\vec{B}$, on a length of wire, $\vec{L}$, carrying a current, i, is given by $\vec{F} = i\vec{L} \times \vec{B}$. The magnitude of this force is $F = iLB \sin \theta$, where θ is the angle between the direction of the current and the direction of the magnetic field.
- The magnitude of the torque on a loop carrying a current, i, in a magnetic field with magnitude B is $\tau = iAB \sin \theta$, where A is the area of the loop and θ is the angle between a unit vector normal to the loop and the direction of the magnetic field. Right-hand rule 2 gives the direction of the unit normal vector to the loop.
- The magnitude of the magnetic dipole moment of a coil carrying a current, i, is given by $\mu = NiA$, where N is the number of loops (windings) and A is the area of a loop. The direction of the dipole moment is given by right-hand rule 2 and is the direction in which the unit normal vector points.
- The Hall effect results when a current, i, flowing through a conductor with height h in a magnetic field of magnitude B produces a potential difference across the conductor (the Hall potential difference), given by $\Delta VH = iB/neh$, where n is the number of electrons per unit volume and e is the magnitude of an electron's charge.

ANSWERS TO SELF-TEST OPPORTUNITIES

7.1 Particle 1: $F_B = qvB \sin \theta =$
$(6.15 \times 10^{-6}\ \text{C})(465\ \text{m/s})(0.165\ \text{T})(\sin 30.0°) = 2.36 \times 10^{-4}\ \text{N}.$

Particle 2: $F_B = qvB \sin \theta =$
$(6.15 \times 10^{-6}\ \text{C})(465\ \text{m/s})(0.165\ \text{T})(\sin 90.0°) = 4.72 \times 10^{-4}\ \text{N}.$

Particle 3: $F_B = qvB \sin \theta =$
$(6.15 \times 10^{-6}\ \text{C})(465\ \text{m/s})(0.165\ \text{T})(\sin 150.0°) = 2.36 \times 10^{-4}\ \text{N}.$

7.2 a) positive

b) slowing down

c) no (Therefore, another force must be acting on the particle to slow it down.)

7.3 $\Delta U = U_{max} - U_{min} = 2\mu B = 2iAB =$
$2(2.00\ \text{A})(0.100\ \text{m}^2)(0.500\ \text{T}) = 0.200\ \text{J}.$

PROBLEM-SOLVING GUIDELINES

1. When working with magnetic fields and forces, you need to sketch a clear diagram of the problem situation in three dimensions. Often, a separate sketch of the velocity and magnetic field vectors (or of the length of wire and the field vectors) is useful to visualize the plane in which they lie, since the magnetic force will be perpendicular to that plane.

2. Remember that the right-hand rules apply for positive charges and currents. If a charge or a current is negative, you can use the right-hand rule, but the force will then be in the opposite direction.

3. A particle in both an electric and a magnetic field experiences an electric force, $\vec{F}_E = q\vec{E}$, and a magnetic force, $\vec{F}_B = q\vec{v} \times \vec{B}$. Be sure you take the vector sum of the individual forces.

MULTIPLE-CHOICE QUESTIONS

7.1 A magnetic field is oriented in a certain direction in a horizontal plane. An electron moves in a certain direction in the horizontal plane. For this situation, there

a) is one possible direction for the magnetic force on the electron.

b) are two possible directions for the magnetic force on the electron.

c) are infinite possible directions for the magnetic force on the electron.

7.2 A particle with charge q is at rest when a magnetic field is suddenly turned on. The field points in the z-direction. What is the direction of the net force acting on the charged particle?

a) in the x-direction
b) in the y-direction
c) The net force is zero.
d) in the z-direction

7.3 Which of the following has the largest cyclotron frequency?

a) an electron with speed v in a magnetic field with magnitude B

b) an electron with speed $2v$ in a magnetic field with magnitude B

c) an electron with speed $v/2$ in a magnetic field with magnitude B

d) an electron with speed $2v$ in a magnetic field with magnitude $B/2$

e) an electron with speed $v/2$ in a magnetic field with magnitude $2B$

7.4 In the Hall effect, a potential difference produced across a conductor of finite thickness in a magnetic field by a current flowing through the conductor is given by

a) the product of the density of electrons, the charge of an electron, and the conductor's thickness divided by the product of the magnitudes of the current and the magnetic field.

b) the reciprocal of the expression described in part (a).

c) the product of the charge on an electron and the conductor's thickness divided by the product of the density of electrons and the magnitudes of the current and the magnetic field.

d) the reciprocal of the expression described in (c).

e) none of the above.

7.5 An electron (with charge $-e$ and mass m_e) moving in the positive x-direction enters a velocity selector. The velocity selector consists of crossed electric and magnetic fields: $\vec{E}$ is directed in the positive y-direction, and $\vec{B}$ is directed in the positive z-direction. For a velocity v (in the positive x-direction), the net force on the electron is zero, and the electron moves straight through the velocity selector. With what velocity will a proton (with charge $+e$ and mass $m_p = 1836\ m_e$) move straight through the velocity selector?

a) v c) $v/1836$

b) $-v$ d) $-v/1836$

7.6 In which direction does a magnetic force act on an electron that is moving in the positive x-direction in a magnetic field pointing in the positive z-direction?

a) the positive y-direction c) the negative x-direction

b) the negative y-direction d) any direction in the xy-plane

7.7 A charged particle is moving in a constant magnetic field. Which of the following statements concerning the magnetic force exerted on the particle is (are) true? (Assume that the magnetic field is not parallel or antiparallel to the velocity.)

a) It does no work on the particle.

b) It may increase the speed of the particle.

c) It may change the velocity of the particle.

d) It can act only on the particle while the particle is in motion.

e) It does not change the kinetic energy of the particle.

7.8 An electron moves in a circular trajectory with radius r_i in a constant magnetic field. What is the final radius of the trajectory when the magnetic field is doubled?

a) $\frac{r_i}{4}$ c) r_i

b) $\frac{r_i}{2}$ d) $2r_i$

e) $4r_i$

7.9 Protons in the solar wind reach Earth's magnetic field with a speed of 400 km/s. If the magnitude of this field is $5.0 \cdot 10^{-5}$ T and the velocity of the protons is perpendicular to it, what is the cyclotron frequency of the protons after entering the field?

a) 122 Hz c) 321 Hz e) 763 Hz

b) 233 Hz d) 432 Hz

7.10 An isolated segment of wire of length $L = 4.50$ m carries a current of magnitude $i = 35.0$ A at an angle $\theta = 50.3°$ with respect to a constant magnetic field of magnitude $B = 6.70 \times 10^{-2}$ T (see the figure). What is the magnitude of the magnetic force on the wire?

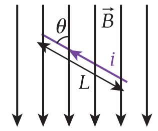
a) 2.66 N

b) 3.86 N

c) 5.60 N

d) 8.12 N

e) 11.8 N



7.11 A coil is composed of circular loops of radius $r = 5.13$ cm and has $N = 47$ windings. A current, $i = 1.27$ A, flows through the coil, which is inside a homogeneous magnetic field of magnitude 0.911 T. What is the maximum torque on the coil due to the magnetic field?

a) 0.148 N m

b) 0.211 N m

c) 0.350 N m

d) 0.450 N m

e) 0.622 N m

7.12 A copper conductor has a current, $i = 1.41$ A, flowing through it, perpendicular to a constant magnetic field, $B = 4.94$ T, as shown in the figure. The conductor is $d = 0.100$ m wide and $h = 2.00$ mm high. What is the electric potential difference between points 1 and 2?

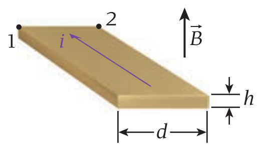
a) 2.56×10^{-9} V

b) 5.12×10^{-9} V

c) 7.50×10^{-8} V

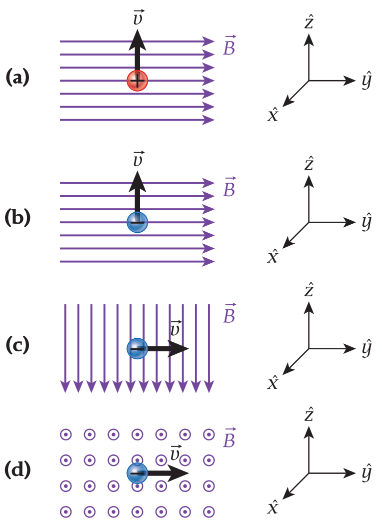
d) 2.56×10^{-7} V

e) 9.66×10^{-7} V



CONCEPTUAL QUESTIONS

7.13 Draw on the xyz-coordinate system and specify (in terms of the unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$) the direction of the magnetic force on each of the moving particles shown in the figures. *Note:* The positive y-axis is toward the right, the positive z-axis is toward the top of the page, and the positive x-axis is directed out of the page.



7.14 A particle with mass m, charge q, and velocity v enters a magnetic field of magnitude B and with direction perpendicular to the initial velocity of the particle. What is the work done by the magnetic field on the particle? How does this affect the particle's motion?

7.15 An electron is moving with a constant velocity. When it enters an electric field that is perpendicular to its velocity, the electron will follow a _______ trajectory. When the electron enters a magnetic field that is perpendicular to its velocity, it will follow a _______ trajectory.

7.16 A proton, moving in negative y-direction in a magnetic field, experiences a force of magnitude F, acting in the negative x-direction.

a) What is the direction of the magnetic field producing this force?

b) Does your answer change if the word "proton" in the statement is replaced by "electron"?

7.17 It would be mathematically possible, for a region with zero current density, to define a scalar magnetic potential analogous to the electrostatic potential: $V_B(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{B} \bullet d\vec{s}$, or $\vec{B}(\vec{r}) = -\nabla V_B(\vec{r})$. However, this has not been done. Explain why not.

7.18 A current-carrying wire is positioned within a large, uniform magnetic field, $\vec{B}$. However, the wire experiences no force. Explain how this might be possible.

7.19 A charged particle moves under the influence of an electric field only. Is it possible for the particle to move with a constant speed? What if the electric field is replaced with a magnetic field?

7.20 A charged particle travels with speed v, at an angle θ with respect to the z-axis. It enters at time $t = 0$ a region of space where there is a magnetic field of magnitude B in the positive z-direction. When does it emerge from this region of space?

7.21 An electron is traveling horizontally from the northwest toward the southeast in a region of space where the Earth's magnetic field is directed horizontally toward the north. What is the direction of the magnetic force on the electron?

7.22 At the Earth's surface, there is an electric field that points approximately straight down and has magnitude 150 V/m. Suppose you had a tuneable electron gun (you can release electrons with whatever kinetic energy you like) and a detector to determine the direction of motion of the electrons when they leave the gun. Explain how you could use the gun to find the direction toward the north magnetic pole. Specifically, what kinetic energy would the electrons need to have? (*Hint:* It might be easier to think about finding which direction is east or west.)

7.23 The work done by the magnetic field on a charged particle in motion in a cyclotron is zero. How, then, can a cyclotron be used as a particle accelerator, and what essential feature of the particle's motion makes it possible?

EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

Section 7.2

7.24 A proton moving with a speed of 4.00×10^5 m/s in the positive y-direction enters a uniform magnetic field of 0.400 T pointing in the positive x-direction. Calculate the magnitude of the force on the proton.

7.25 The magnitude of the magnetic force on a particle with charge $-2e$ moving with speed $v = 1.00 \times 10^5$ m/s is 3.00×10^{-18} N. What is the magnitude of the magnetic field component perpendicular to the direction of motion of the particle?

•7.26 A particle with a charge of $+10.0$ μC is moving at 300. m/s in the positive z-direction.

a) Find the minimum magnetic field required to keep it moving in a straight line at constant speed if there is a uniform electric field of magnitude 100. V/m pointing in the positive y-direction.

b) Find the minimum magnetic field required to keep the particle moving in a straight line at constant speed if there is a uniform electric field of magnitude 100. V/m pointing in the positive z-direction.

•7.27 A particle with a charge of 20.0 μC moves along the x-axis with a speed of 50.0 m/s. It enters a magnetic field given by $\vec{B} = 0.300\hat{y} + 0.700\hat{z}$ in teslas. Determine the magnitude and the direction of the magnetic force on the particle.

••7.28 The magnetic field in a region in space (where $x > 0$ and $y > 0$) is given by $\vec{B} = (x - az)\hat{y} + (xy - b)$ where a and b are positive constants. An electron moving with a constant velocity, $\vec{v} = v_0\hat{x}$, enters this region. What are the coordinates of the points at which the net force acting on the electron is zero?

Section 7.3

7.29 A proton is accelerated from rest by a potential difference of 400. V. The proton enters a uniform magnetic field and follows a circular path of radius 20.0 cm. Determine the magnitude of the magnetic field.

7.30 An electron with a speed of 4.00×10^5 m/s enters a uniform magnetic field of magnitude 0.0400 T at an angle of 35.0° to the magnetic field lines. The electron will follow a helical path.

a) Determine the radius of the helical path.

b) How far forward will the electron have moved after completing one circle?

7.31 A particle with mass m and charge q is moving within both an electric field and a magnetic field, $\vec{E}$ and $\vec{B}$. The particle has velocity $\vec{v}$, momentum $\vec{p}$, and kinetic energy, K. Find general expressions for $d\vec{p}/dt$ and dK/dt, in terms of these seven quantities.

7.32 The Earth is showered with particles from space known as *muons*. They have a charge identical to that of an electron but are many times heavier ($m = 1.88 \times 10^{-28}$ kg). Suppose a strong magnetic field is established in a lab ($B = 0.500$ T) and a muon enters this field with a velocity of 3.00×10^6 m/s at a right angle to the field. What will be the radius of the muon's resulting orbit?

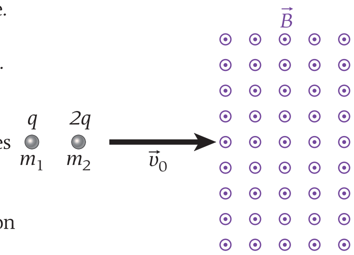
7.33 An electron in a magnetic field moves counterclockwise on a circle in the xy-plane, with a cyclotron frequency of $\omega = 1.20 \times 10^{12}$ Hz. What is the magnetic field, $\vec{B}$?

7.34 An electron with energy equal to 4.00×10^2 eV and an electron with energy equal to 2.00×10^2 eV are trapped in a uniform magnetic field and move in circular paths in a plane perpendicular to the magnetic field. What is the ratio of the radii of their orbits?

•7.35 A proton with an initial velocity given by $(1.00\hat{x} + 2.00\hat{y} + 3.00\hat{z})(10^5$ m/s) enters a magnetic field given by $(0.500$ T$)\hat{z}$. Describe the motion of the proton.

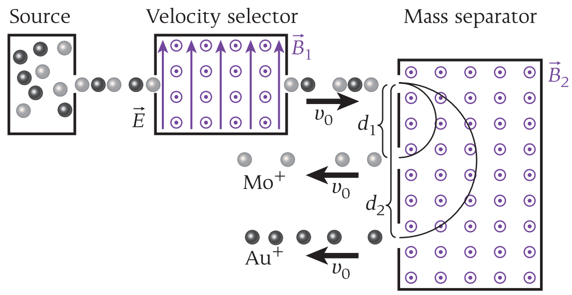
•7.36 Initially at rest, a small copper sphere with a mass of 3.00×10^{-6} kg and a charge of 5.00×10^{-4} C is accelerated through a 7000. V potential difference before entering a magnetic field of magnitude 4.00 T, directed perpendicular to its velocity. What is the radius of curvature of the sphere's motion in the magnetic field?

•7.37 Two particles with masses m_1 and m_2 and charges q and $2q$ travel with the same velocity, v, and enter a magnetic field of strength B at the same point, as shown in the figure. In the magnetic field, they move in semicircles with radii R and $2R$. What is the ratio of their masses? Is it possible to apply an electric field that would cause the particles to move in a straight line in the magnetic field? If yes, what would be the magnitude and the direction of the field?



•7.38 The figure shows a schematic diagram of a simple mass spectrometer, consisting of a velocity selector and a particle detector and being used to separate singly ionized atoms ($q = +e = 1.602 \times 10^{-19}$ C) of gold (Au) and molybdenum (Mo). The electric field inside the velocity selector has magnitude $E = 1.789 \times 10^4$ V/m and points toward the top of the page, and the magnetic field has magnitude $B_1 = 1.000$ T and points out of the page.

a)



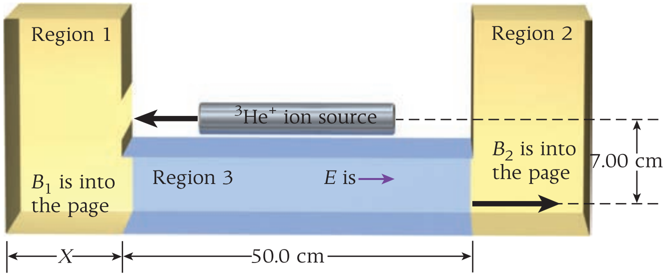
Draw the electric force vector, $\vec{F}_E$, and the magnetic force vector, $\vec{F}_B$, acting on the ions inside the velocity selector.

b) Calculate the velocity, v_0, of the ions that make it through the velocity selector (those that travel in a straight line). Does v_0 depend on the type of ion (gold versus molybdenum), or is it the same for both types of ions?

c) Write the equation for the radius of the semicircular path of an ion in the particle detector: $R = R(m, v_0, q, B_2)$.

d) The gold ions (represented by the dark gray spheres) exit the particle detector at a distance $d_2 = 40.00$ cm from the entrance slit, while the molybdenum ions (represented by the light gray spheres) exit the particle detector at a distance $d_1 = 19.81$ cm from the entrance slit. The mass of a gold ion is $m_{gold} = 3.271 \times 10^{-25}$ kg. Calculate the mass of a molybdenum ion.

••**7.39** A small particle accelerator for accelerating $^3He^+$ ions is shown in the figure. The $^3He^+$ ions exit the ion source with a kinetic energy of 4.00 keV. Regions 1 and 2 contain magnetic fields directed into the page, and region 3 contains an electric field directed from left to right. The $^3He^+$ ion beam exits the accelerator from a hole on the right that is 7.00 cm below the ion source, as shown in the figure.



a) If $B_1 = 1.00$ T and region 3 is 50.0 cm long with $E = 60.0$ kV/m, what value should B_2 have to cause the ions to move straight through the exit hole after being accelerated twice in region 3?

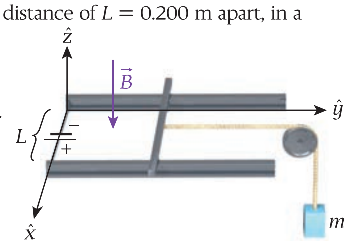
b) What minimum width X should region 1 have?

c) What is the velocity of the ions when they leave the accelerator?

Section 7.4

7.40 A straight wire of length 2.00 m carries a current of 24.0 A. It is placed on a horizontal tabletop in a uniform horizontal magnetic field. The wire makes an angle of 30.0° with the magnetic field lines. If the magnitude of the force on the wire is 0.500 N, what is the magnitude of the magnetic field?

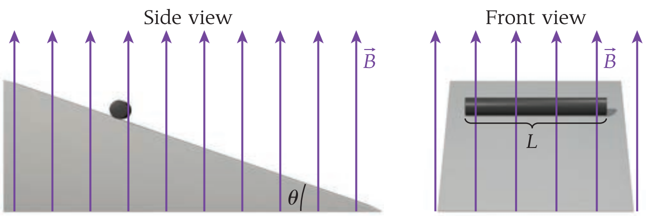
7.41 As shown in the figure, a straight conductor parallel to the x-axis can slide without friction on top of two horizontal conducting rails that are parallel to the y-axis and a distance of $L = 0.200$ m apart, in a vertical magnetic field of 1.00 T. A 20.0-A current is maintained through the conductor. If a string is connected exactly at the center of the conductor and passes over a frictionless pulley, what mass m suspended from the string allows the conductor to be at rest?



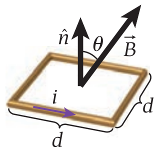
7.42 A copper wire of radius 0.500 mm is carrying a current at the Earth's Equator. Assuming that the magnetic field of the Earth has magnitude 0.500 G at the Equator and is parallel to the surface of the Earth and that the current in the wire flows toward the east, what current is required to allow the wire to levitate?

•**7.43** A copper sheet with length 1.00 m, width 0.500 m, and thickness 1.00 mm is oriented so that its largest surface area is perpendicular to a magnetic field of strength 5.00 T. The sheet carries a current of 3.00 A across its length. What is the magnitude of the force on this sheet? How does this magnitude compare to that of the force on a thin copper wire carrying the same current and oriented perpendicularly to the same magnetic field?

•**7.44** A conducting rod of length L slides freely down an inclined plane, as shown in the figure. The plane is inclined at an angle θ from the horizontal. A uniform magnetic field of strength B acts in the positive y-direction. Determine the magnitude and the direction of the current that would have to be passed through the rod to hold it in position on the inclined plane.



•**7.45** A square loop of wire, with side length $d =$ 8.00 cm, carries a current of magnitude $i = 0.150$ A and is free to rotate. It is placed between the poles of an electromagnet that produce a uniform magnetic field of 1.00 T. The loop is initially placed so that its normal vector, $\hat{n}$, is at a 35.0° angle relative to the direction of the magnetic field vector, with the angle θ defined as shown in the figure. The wire is copper (with a density of $\rho = 8960$ kg/m^3), and its diameter is 0.500 mm. What is the magnitude of the initial angular acceleration of the loop when it is released?



•**7.46** A rail gun accelerates a projectile from rest by using the magnetic force on a current-carrying wire. The wire has radius $r = 5.10 \times 10^{-4}$ m and is made of copper having a density of $\rho = 8960$ kg/m^3. The gun consists of rails of length $L = 1.00$ m in a constant magnetic field of magnitude $B = 2.00$ T, oriented perpendicular to the plane defined by the rails. The wire forms an electrical connection across the rails at one end of the rails. When triggered, a current of $1.00 \times \cdot 10^4$ A flows through the wire, which accelerates the wire along the rails. Calculate the final speed of the wire as it leaves the rails. (Neglect friction.)

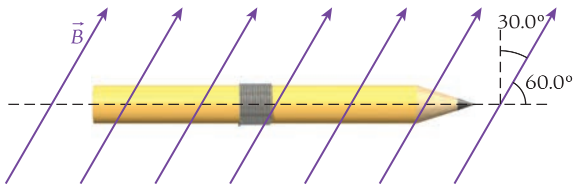
Sections 7.5 and 7.6

•**7.47** A square loop of wire of side length ℓ lies in the xy-plane, with its center at the origin and its sides parallel to the x- and y-axes. It carries a current, i, flowing in the counterclockwise direction, as viewed looking down the z-axis from the positive direction. The loop is in a magnetic field given by $\vec{B} = (B_0/a)(z\hat{x} + x\hat{z})$, where B_0 is a constant field strength, a is a constant with the dimension of length, and $\hat{x}$ and $\hat{z}$ are unit vectors in the positive x-direction and positive z-direction. Calculate the net force on the loop.

7.48 A rectangular coil with 20 windings carries a current of 2.00 mA flowing in the counterclockwise direction. It has two sides that are parallel to the y-axis and have length 8.00 cm and two sides that are parallel to the x-axis and have length 6.00 cm. A uniform magnetic field of 50.0 μT acts in the positive x-direction. What torque must be applied to the loop to hold it steady?

7.49 A coil consists of 120. circular loops of wire of radius 4.80 cm. A current of 0.490 A runs through the coil, which is oriented vertically and is free to rotate about a vertical axis (parallel to the z-axis). It experiences a uniform horizontal magnetic field in the positive x-direction. When the coil is oriented parallel to the x-axis, a force of 1.20 N applied to the edge of the coil in the positive y-direction can keep it from rotating. Calculate the strength of the magnetic field.

7.50 Twenty loops of wire are tightly wound around a round pencil that has a diameter of 6.00 mm. The pencil is then placed in a uniform 5.00 T magnetic field, as shown in the figure. If a 3.00 A current is present in the coil of wire, what is the magnitude of the torque on the pencil?

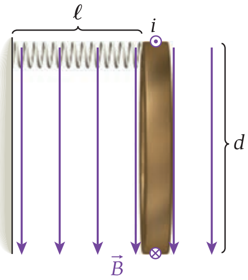


•**7.51** A copper wire with density $\rho = 8960$ kg/m^3 is formed into a circular loop of radius 50.0 cm. The cross-sectional area of the wire is $1.00 \times \cdot 10^{-5}$ m^2, and a potential difference of 0.0120 V is applied to the wire. What is the maximum angular acceleration of the loop when it is placed in a magnetic field of magnitude 0.250 T? The loop rotates about an axis in the plane of the loop that corresponds to a diameter.

7.52 A simple galvanometer is made from a coil that consists of N loops of wire of area A. The coil is attached to a mass, M, by a light rigid rod of length L. With no current in the coil, the mass hangs straight down, and the coil lies in a horizontal plane. The coil is in a uniform magnetic field of magnitude B that is oriented horizontally. Calculate the angle from the vertical of the rigid rod as a function of the current, i, in the coil.

7.53 Show that the magnetic dipole moment of an electron orbiting in a hydrogen atom is proportional to its angular momentum, L: $\mu = -eL/(2m)$, where $-e$ is the charge of the electron and m is its mass.

•**7.54** The figure shows a top view of a current-carrying ring of wire having a diameter $d = 8.00$ cm, which is suspended from the ceiling via a thin string. A 1.00A current flows in the ring in the direction indicated in the figure. The ring is connected to one end of a spring with a spring constant of 100. N/m. When the ring is in the position shown in the figure, the spring is at its equilibrium length, ℓ. Determine the extension of the spring when a magnetic field of magnitude $B = 2.00$ T is applied parallel to the plane of the ring, as shown in the figure.



•**7.55** A coil of wire consisting of 40 rectangular loops, with width 16.0 cm and height 30.0 cm, is placed in a constant magnetic field given by $\vec{B} = 0.0650\text{T}\hat{x} + 0.250\text{T}\hat{z}$. The coil is hinged to a fixed thin rod along the y-axis (along segment da in the figure) and is originally located in the xy-plane. A current of 0.200 A runs through the wire.

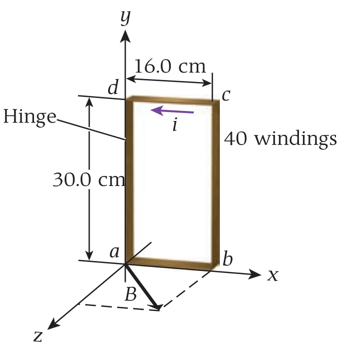
a) What are the magnitude and the direction of the force, $\vec{F}_{ab}$, that $\vec{B}$ exerts on segment ab of the coil?

b) What are the magnitude and the direction of force, $\vec{F}_{bc}$, that $\vec{B}$ exerts on segment bc of the coil?

c) What is the magnitude of the net force, F_{net}, that $\vec{B}$ exerts on the coil?

d) What are the magnitude and the direction of the torque, $\vec{\tau}$, that $\vec{B}$ exerts on the coil?

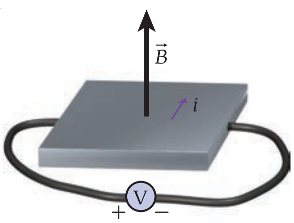
e) In what direction, if any, will the coil rotate about the y-axis (viewed from above and looking down that axis)?



Section 7.7

7.56 A high electron mobility transistor (HEMT) controls large currents by applying a small voltage to a thin sheet of electrons. The density and mobility of the electrons in the sheet are critical for the operation of the HEMT. HEMTs consisting of AlGaN/GaN/Si are being studied because they promise better performance at higher currents, temperatures, and frequencies than conventional silicon HEMTs can achieve. In one study, the Hall effect was used to measure the density of electrons in one of these new HEMTs. When a current of 10.0 μA flows through the length of the electron sheet, which is 1.00 mm long, 0.300 mm wide, and 10.0 nm thick, a magnetic field of 1.00 T perpendicular to the sheet produces a voltage of 0.680 mV across the width of the sheet. What is the density of electrons in the sheet?

•**7.57** The figure shows schematically a setup for a Hall effect measurement using a thin film of zinc oxide of thickness 1.50 μm. The current, i, across the thin film is 12.3 mA, and the Hall potential, V_{H}, is −20.1 mV when a magnetic field of magnitude $B = 0.900$ T is applied perpendicular to the current flow.



a) What are the charge carriers in the thin film? [*Hint:* They can be either electrons with charge $-e$ or electron holes (missing electrons) with charge $+e$.]

b) Calculate the density of charge carriers in the thin film.

Additional Exercises

7.58 A cyclotron in a magnetic field of 9.00 T is used to accelerate protons to 50.0% of the speed of light. What is the cyclotron frequency of these protons? What is the radius of their trajectory in the cyclotron? What are the cyclotron frequency and the trajectory radius of the same protons in the Earth's magnetic field? Assume that the Earth's magnetic field is 0.500 G.

7.59 A straight wire carrying a current of 3.41 A is placed at an angle of 10.0° to the horizontal between the pole tips of a magnet producing a field of 0.220 T upward. The poles' tips each have a 10.0 cm diameter. The magnetic force causes the wire to move out of the space between the poles. What is the magnitude of that force?

7.60 An electron is moving at $v = 6.00 \cdot 10^7$ m/s perpendicular to the Earth's magnetic field. If the field strength is $0.500 \cdot 10^{-4}$ T, what is the radius of the electron's circular path?

7.61 A straight wire with a constant current running through it is in Earth's magnetic field, at a location where the magnitude is 0.430 G. What is the minimum current that must flow through the wire for a 10.0-cm length of it to experience a force of 1.00 N?

7.62 A circular coil with a radius of 10.0 cm has 100 turns of wire and carries a current, $i = 100$. mA. It is free to rotate in a region with a constant horizontal magnetic field given by $\vec{B} = (0.0100\text{ T})\hat{x}$. If the unit normal vector to the plane of the coil makes an angle of 30.0° with the horizontal, what is the magnitude of the net torque acting on the coil?

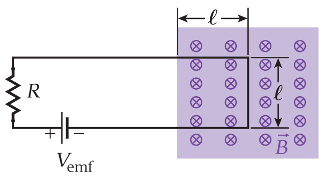
7.63 At $t = 0$ an electron crosses the positive y-axis (so $x = 0$) at 60.0 cm from the origin with velocity $2.00 \cdot 10^5$ m/s in the positive x-direction. It is in a uniform magnetic field.

a) Find the magnitude and the direction of the magnetic field that will cause the electron to cross the x-axis at $x = 60.0$ cm.

b) What work is done on the electron during this motion?

c) How long will the trip from y-axis to x-axis take?

•**7.64** A 12.0V battery is connected to a 3.00Ω resistor in a rigid rectangular loop of wire measuring 3.00 m by 1.00 m. As shown in the figure, a length $\ell = 1.00$ m of wire at the end of the loop extends into a 2.00 m by 2.00 m region with a magnetic field of magnitude 5.00 T, directed into the page. What is the net force on the loop?



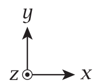
7.65 An alpha particle ($m = 6.64 \times 10^{-27}$ kg, $q = +2e$) is accelerated by a potential difference of 2700. V and moves in a plane perpendicular to a constant magnetic field of magnitude 0.340 T, which curves the trajectory of the alpha particle. Determine the radius of curvature and the period of revolution.

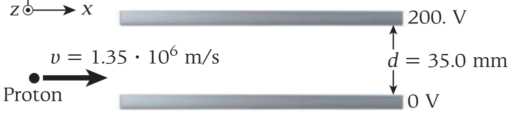
•**7.66** In a certain area, the electric field near the surface of the Earth is given by $\vec{E} = (-150.\text{ N/C})\hat{z}$, and the Earth's magnetic field is given by $\vec{B} = (50.0\ \mu\text{T})\hat{r}_{\text{N}} - (20.0\ \mu\text{T})\hat{z}$, where $\hat{z}$ is a unit vector pointing vertically upward and $\hat{r}_{\text{N}}$ is a horizontal unit vector pointing due north. What velocity, $\vec{v}$, will allow an electron in this region to move in a straight line at constant speed?

•**7.67** A helium leak detector uses a mass spectrometer to detect tiny leaks in a vacuum chamber. The chamber is evacuated with a vacuum pump and then sprayed with helium gas on the outside. If there is any leak, the helium molecules pass through the leak and into the chamber, whose volume is sampled by the leak detector. In the spectrometer, helium ions are accelerated and released into a tube, where their motion is perpendicular to an applied magnetic field, $\vec{B}$, and they follow a circular path of radius r and then hit a detector. Estimate the velocity required if the radius of the ions' circular path is to be no more than 5.00 cm, the magnetic field is 0.150 T, and the mass of a helium-4 atom is about 6.64×10^{-27} kg. Assume that each ion is singly ionized (has one electron less than the neutral atom). By what factor does the required velocity change if helium-3 atoms, which have about $\frac{3}{4}$ as much mass as helium-4 atoms, are used?

•7.68 In your laboratory, you set up an experiment with an electron gun that emits electrons with energy of 7.50 keV toward an atomic target. What deflection (magnitude and direction) will Earth's magnetic field (0.300 G) produce in the beam of electrons if the beam is initially directed due east and covers a distance of 1.00 m from the gun to the target? (*Hint:* First calculate the radius of curvature, and then determine how far away from a straight line the electron beam has deviated after 1.00 m.)

•7.69 A proton enters the region between the two plates shown in the figure moving in the x-direction with a speed $v = 1.35 \cdot 10^6$ m/s. The potential of the top plate is 200. V, and the potential of the bottom plate is 0 V. What are the direction and the magnitude of the magnetic field, $\vec{B}$, that is required between the plates for the proton to continue traveling in a straight line in the x-direction?





•7.70 An electron moving at a constant velocity, $\vec{v} = v_0\hat{x}$, enters a region in space where a magnetic field is present. The magnetic field, $\vec{B}$, is constant and points in the z-direction. What are the magnitude and the direction of the magnetic force acting on the electron? If the width of the region where the magnetic field is present is d, what is the minimum velocity the electron must have in order to escape this region?

•7.71 A 30-turn square coil with a mass of 0.250 kg and a side length of 0.200 m is hinged along a horizontal side and carries a 5.00 A current. It is placed in a magnetic field pointing vertically downward and having a magnitude of 0.00500 T. Determine the angle that the plane of the coil makes with the vertical when the coil is in equilibrium. Use $g = 9.81$ m/s^2.

•7.72 A semicircular loop of wire of radius R is in the xy-plane, centered about the origin. The wire carries a current, i, counterclockwise around the semicircle, from $x = -R$ to $x = +R$ on the x-axis. A magnetic field, $\vec{B}$, is pointing out of the plane, in the positive z-direction. Calculate the net force on the semicircular loop.

•7.73 A proton moving at speed $v = 1.00 \cdot 10^6$ m/s enters a region in space where a magnetic field given by $\vec{B} = (-0.500\text{ T})\hat{z}$ exists. The velocity vector of the proton is at an angle $\theta = 60.0°$ with respect to the positive z-axis.

a) Analyze the motion of the proton and describe its trajectory (in qualitative terms only).

b) Calculate the radius, r, of the trajectory projected onto a plane perpendicular to the magnetic field (in the xy-plane).

c) Calculate the period, T, and frequency, f, of the motion in that plane.

d) Calculate the pitch of the motion (the distance traveled by the proton in the direction of the magnetic field in 1 period).

MULTI-VERSION EXERCISES

7.74 A small aluminum ball with a mass of 5.063 g and a charge of 11.03 C is moving northward at 3079 m/s. You want the ball to travel in a horizontal circle with a radius of 2.137 m and in a clockwise direction when viewed from above. Ignoring gravity, what is the magnitude of the magnetic field that must be applied to the aluminum ball to cause it to move in this way?

7.75 A small aluminum ball with a charge of 11.17 C is moving northward at 3131 m/s. You want the ball to travel in a horizontal circle with a radius of 2.015 m and in a clockwise direction when viewed from above. Ignoring gravity, the magnitude of the magnetic field that must be applied to the aluminum ball to cause it to move in this way is $B = 0.8000$ T. What is the mass of the ball?

7.76 A small charged aluminum ball with a mass of 3.435 g is moving northward at 3183 m/s. You want the ball to travel in a horizontal circle with a radius of 1.893 m and in a clockwise direction when viewed from above. Ignoring gravity, the magnitude of the magnetic field that must be applied to the aluminum ball to cause it to move in this way is $B = 0.5107$ T. What is the charge on the ball?

7.77 A velocity selector is used in a mass spectrometer to produce a beam of charged particles with uniform velocity. Suppose the fields in a selector are given by $\vec{E} = (1.749 \times 10^4\text{ V/m})\hat{x}$ and $\vec{B} = (46.23\text{ mT})\hat{y}$. Find the speed with which a charged particle can travel through the selector in the z-direction without being deflected.

7.78 A velocity selector is used in a mass spectrometer to produce a beam of charged particles with uniform velocity. Suppose the fields in a selector are given by $\vec{E} = (2.207 \times 10^4\text{ V/m})\hat{x}$ and $\vec{B} = B_y\hat{y}$. The speed with which a charged particle can travel through the selector in the z-direction without being deflected is 4.713×10^5 m/s. What is the value of B_y?

7.79 A velocity selector is used in a mass spectrometer to produce a beam of charged particles of uniform velocity. Suppose the fields in a selector are given by $\vec{E} = E_x\hat{x}$ and $\vec{B} = (47.45\text{ mT})\hat{y}$. The speed with which a charged particle can travel through the selector in the z-direction without being deflected is 5.616×10^5 m/s. What is the value of E_x?

8 Magnetic Fields of Moving Charges

FIGURE 8.1 This array of 96 iron atoms holds 1 byte (8 bits) of information in the magnetic fields of the atoms—an impressive feat of data storage.

We saw in Chapter 7 that a magnetic field can affect the path of a charged particle or the flow of a current. In this chapter, we consider magnetic fields *caused* by electric currents. Any charged particle generates a magnetic field, if it is moving. Various magnetic fields are caused by different distributions of current. The powerful electromagnets used in industry, in physics research, in medical diagnostics, and in other applications are primarily solenoids–coils of current-carrying wire with hundreds or thousands of loops. We'll see in this chapter why solenoids generate particularly useful magnetic fields. We will also see that an individual atom can have a permanent magnetic field. Figure 8.1 shows how the magnetic fields of iron atoms (shown in red and blue) can be used to store information. This technology is still in its infancy (the image was produced in 2012). However, if it can be scaled up, it will allow 10 terabytes of data to be stored on a square centimeter of iron surface, a density that is higher than what is currently achievable by a factor of more than 1000.

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WHAT WE WILL LEARN

- Moving charges (currents) create magnetic fields.
- The magnetic field created by a current flowing in a long, straight wire varies inversely with the distance from the wire.
- Two parallel wires carrying current in the same direction attract each other. Two parallel wires carrying current in opposite directions repel each other.
- Ampere's Law is used to calculate the magnetic field caused by certain symmetrical current distributions, just as Gauss's Law is useful in calculating electric fields in situations having spatial charge symmetry.
- The magnetic field inside a long, straight wire varies linearly with the distance from the center of the wire.
- A solenoid is an electromagnet that can be used to produce a constant magnetic field with a large volume.
- Some atoms can be thought of as small magnets created by motion of electrons in the atom.
- Materials can exhibit three kinds of intrinsic magnetism: diamagnetism, paramagnetism, and ferromagnetism.
- Superconducting magnets can be used to produce very strong magnetic fields.

8.1 Biot-Savart Law

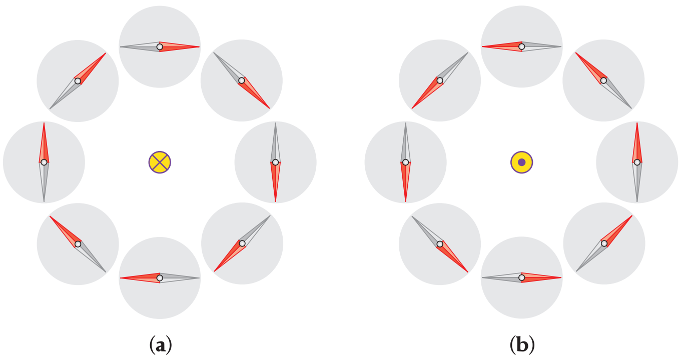


FIGURE 8.2 Wire (yellow circle) with current running through it: (a) into the page (indicated by the cross); (b) out of the page (indicated by the dot). The orientation of a compass needle placed close to the wire is shown at different locations around the wire.

Chapter 7 introduced magnetic fields and field lines by showing how a compass needle orients itself near a permanent magnet. In a similar demonstration, a strong current is run through a (very long, straight) wire (with or without insulation). If a compass needle is then brought close to the wire, the compass needle orients itself relative to the wire in the way shown in Figure 8.2. This observation was first made by the Danish physicist Hans Oersted (1777–1851) in 1819 while conducting a demonstration for students during a lecture. We conclude that the current in the wire produces a magnetic field. Since the direction of the compass needle indicates the direction of the magnetic field, we further conclude that the magnetic field lines form circles around this current-carrying wire. Note the difference between parts (a) and (b) of Figure 8.2: When the direction of the current is reversed, the orientation of the compass needle is also reversed. Although the figure doesn't show this, if the compass needle is moved farther and farther away from the wire, eventually it again orients itself in the direction of the magnetic field of the Earth. This indicates that the magnetic field produced by the wire gets weaker as a function of increasing distance from the wire. The next question we need to answer is this: How can we determine the magnetic field produced by a moving charge?

To describe the electric field in terms of the electric charge, we showed that:

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{|dq|}{r^2},$$

where dq is a charge element. The electric field points in a radial direction (inward toward or outward from the electric charge, depending on the sign of the charge), so

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r}.$$

The situation is slightly more complicated for a magnetic field because a current element, $i\,d\vec{s}$, that produces a magnetic field has a direction, as opposed to a nondirectional point charge that produces an electric field. As a result of a long series of experiments involving tests similar to that depicted in Figure 8.2 and conducted in the early 19th century, the French scientists Jean-Baptiste Biot (1774–1862) and Felix Savart (1791–1841) established that the magnetic field produced by a current element, $i\,d\vec{s}$, is given by

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i d\vec{s}\times\vec{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{i d\vec{s}\times\hat{r}}{r^2}. \quad (8.1)$$

Here $i\,d\vec{s}$, is a vector of differential length *ds* pointing in the direction in which the current flows along the conductor and $\vec{r}$ is the position vector measured *from* the current element *to* the point at which the field is to be determined. Figure 8.3 depicts the physical situation described by this formula, which is called the **Biot-Savart Law.**

The constant μ_0 in equation 8.1 is called the **magnetic permeability of free space** and has the value

$$\mu_0 = 4\pi \times 10^{-7} \ \frac{\text{T m}}{\text{A}} \tag{8.2}$$

From equation 8.1 and Figure 8.3, you can see that the direction of the magnetic field produced by the current element $i\,d\vec{s}$ is perpendicular to both the position vector and the current element. The magnitude of the magnetic field is given by

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}, \tag{8.3}$$

where θ (with possible values between 0° and 180°) is the angle between the direction of the position vector and the current element. The direction of the magnetic field is given by a variant of right-hand rule 1, introduced in Chapter 7. To determine the direction of the magnetic field using your right hand, point your thumb in the direction of the differential current element and your index finger in the direction of the position vector, and your middle finger will point in the direction of the differential magnetic field.

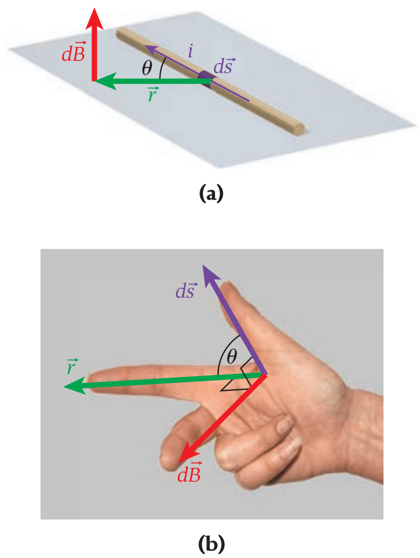


FIGURE 8.3 (a) Three-dimensional depiction of the Biot-Savart Law. The differential magnetic field is perpendicular to both the differential current element and the position vector. (b) Right-hand rule 1 applied to the quantities involved in the Biot-Savart Law.

8.2 Magnetic Fields due to Current Distributions

Chapter 7 addressed the superposition principle for magnetic fields. Using this superposition principle, we can compute the magnetic field at any point in space as the sum of the differential magnetic fields described by the Biot-Savart Law. This section examines the magnetic fields generated by the most common configurations of current-carrying wires.

Magnetic Field from a Long, Straight Wire

Let's first examine the magnetic field from an infinitely long, straight wire carrying a current, *i*. We consider the magnetic field, $d\vec{B}$, at a point *P* that is a perpendicular distance $r_\perp$ from the wire (Figure 8.4). The magnitude of the field, *dB*, at that point due to the current element *ids* is given by equation 8.3; the direction of the field is given by $d\vec{s} \times \vec{r}$ and is out of the page. We find the magnetic field from the right half of the wire and multiply by 2 to get the magnetic field from the whole wire. Thus, the magnitude of the magnetic field at a perpendicular distance $r_\perp$ from the wire is given by

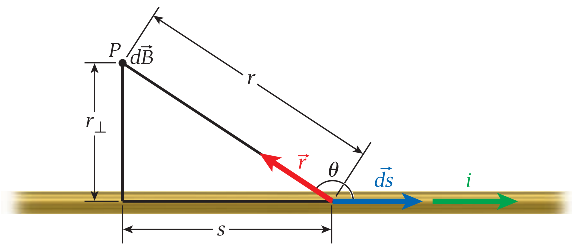


FIGURE 8.4 Magnetic field from a long, straight current-carrying wire.

$$B = 2\int_0^\infty dB = 2\int_0^\infty \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2} = \frac{\mu_0 i}{2\pi}\int_0^\infty \frac{ds\sin\theta}{r^2}.$$

We can relate r and θ to $r_\perp$ and s (r, s, and θ) by $r = \sqrt{s^2 + r_\perp^2}$ and $\sin\theta = \sin(\pi - \theta)$ $= r_\perp / \sqrt{s^2 + r_\perp^2}$ (see Figure 8.4). Substituting for r and $\sin\theta$ in the preceding expression for B gives

$$B = \frac{\mu_0 i}{2\pi}\int_0^\infty \frac{r_\perp\,ds}{(s^2 + r_\perp^2)^{3/2}}.$$

Evaluating this definite integral, we find

$$B = \frac{\mu_0 i}{2\pi}\left[\frac{1}{r_\perp^2}\frac{r_\perp s}{(s^2 + r_\perp^2)^{1/2}}\right]_0^\infty = \frac{\mu_0 i}{2\pi r_\perp}\left[\frac{s}{(s^2 + r_\perp^2)^{1/2}}\right]_{s\to\infty} - 0.$$

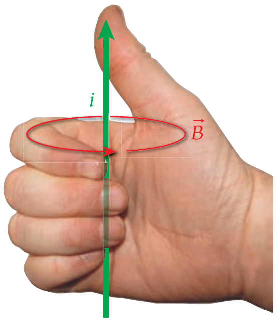


FIGURE 8.5 Right-hand rule 3 for the magnetic field from a current-carrying wire.

For $s \gg r_\perp$, the term in the brackets approaches the value 1. Therefore, the magnitude of the magnetic field at a perpendicular distance $r_\perp$ from a long, straight wire carrying a current, *i*, is

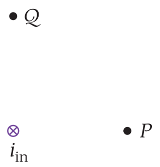
$$B = \frac{\mu_0 i}{2\pi r_\perp}. \tag{8.4}$$

The direction of the magnetic field at any point is found by applying right-hand rule 1 to the current element and position vectors shown in Figure 8.4. This results in a new right-hand rule, called *right-hand rule 3*, which can be used to determine the direction of the magnetic field due to a current-carrying wire. If you grab the wire with your right hand so that your thumb points in the direction of the current, your fingers will curl in the direction of the magnetic field (Figure 8.5).

Looking along a current-carrying wire would reveal that the magnetic field lines form concentric circles (Figure 8.6). Notice from the distance between the field lines that the field is strongest near the wire and drops off in proportion to $1/r_\perp$, as indicated by equation 8.4.

Concept Check 8.1

A wire is carrying a current, i_{in}, into the page as shown in the figure. In which direction does the magnetic field point at points *P* and *Q*?



a) to the right at *P* and upward (toward the top of the page) at *Q*

b) upward at *P* and to the right at *Q*

c) downward at *P* and to the right at *Q*

d) upward at *P* and to the left at *Q*

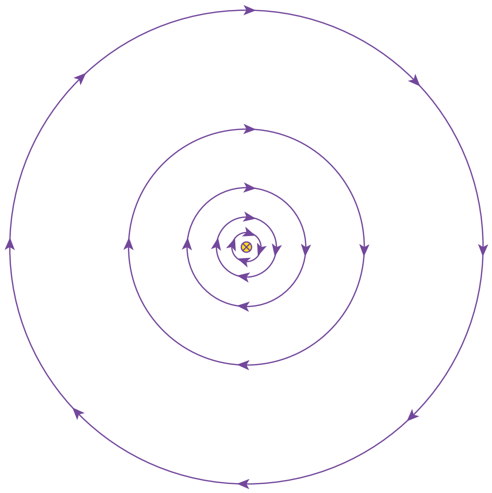
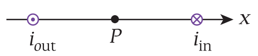


FIGURE 8.6 Magnetic field lines around a long, straight wire (yellow circle at the center) that is perpendicular to the page and carries a current flowing into the page, signified by the cross.

Concept Check 8.2

Wire 1 has a current flowing out of the page, i_{out}, as shown in the figure. Wire 2 has a current flowing into the page, i_{in}. What is the direction of the magnetic field at point *P*?



a) upward in the plane of the page

b) to the right

c) downward in the plane of the page

d) to the left

e) The magnetic field at point *P* is zero.

Two Parallel Wires

Let's examine the case in which two parallel wires are carrying current. The two wires exert magnetic forces on each other because the magnetic field of one wire exerts a force on the moving charges in the second wire. The magnitude of the magnetic field created by a current-carrying wire is given by equation 8.4. This magnetic field is always perpendicular to the wire with a direction given by right-hand rule 3 (Figure 8.5).

Let's first consider wire 1 carrying a current, i_1, toward the right, as shown in Figure 8.7a. The magnitude of the magnetic field a perpendicular distance *d* from wire 1 is

$$B_1 = \frac{\mu_0 i_1}{2\pi d}. \tag{8.5}$$

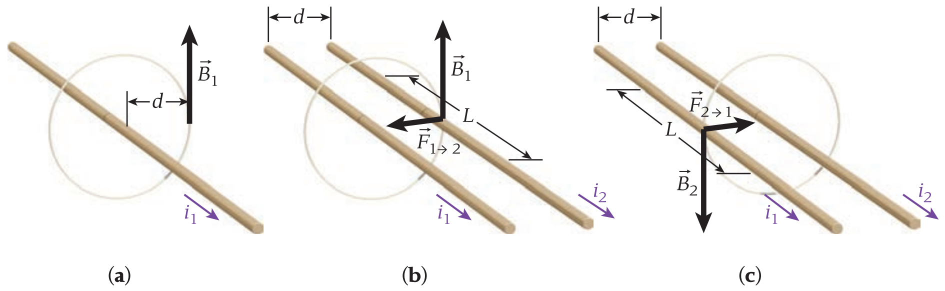


FIGURE 8.7 (a) Magnetic field line from one current-carrying wire. (b) Magnetic field created by the current in one wire exerting a force on a second current-carrying wire. (c) Magnetic field created by the current in the second wire exerting a force on the first current-carrying wire.

The direction of $\vec{B}_1$ is given by right-hand rule 3 and is shown for a particular point in Figure 8.7a.

Now consider wire 2 carrying a current, i_2, in the same direction as i_1 and placed parallel to wire 1 at a distance d from it (Figure 8.7b). The magnetic field due to wire 1 exerts a magnetic force on the moving charges in the current flowing in wire 2. In Chapter 7, we saw that the magnetic force on a current-carrying wire is given by

$$\vec{F} = i\vec{L} \times \vec{B}.$$

The magnitude of the magnetic force on a length, L, of wire 2 is then

$$F = iLB\sin\theta = i_2 LB_1, \tag{8.6}$$

because $\vec{B}_1$ is perpendicular to wire 2 and thus $\theta = 90°$. Substituting for B_1 from equation 8.5 into equation 8.6, we find the magnitude of the force exerted by wire 1 on a length L of wire 2:

$$F_{1\to2} = i_2 L\left(\frac{\mu_0 i_1}{2\pi d}\right) = \frac{\mu_0 i_1 i_2 L}{2\pi d}. \tag{8.7}$$

According to right-hand rule 1, $\vec{F}_{1\to2}$ points toward wire 1 and is perpendicular to both wires. An analogous calculation allows us to deduce that the force from wire 2 on a length, L, of wire 1 has the same magnitude and opposite direction: $\vec{F}_{2\to1} = -\vec{F}_{1\to2}$. This result is shown in Figure 8.7c and is a simple consequence of Newton's Third Law.

Concept Check 8.4

Two parallel wires are near each other, as shown in the figure. Wire 1 carries a current i, and wire 2 carries a current $2i$. Which statement about the magnetic forces that the two wires exert on each other is correct?

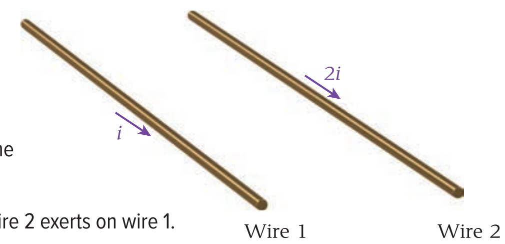
a) The two wires exert no forces on each other.

b) The two wires exert attractive forces of the same magnitude on each other.

c) The two wires exert repulsive forces of the same magnitude on each other.

d) Wire 1 exerts a stronger force on wire 2 than wire 2 exerts on wire 1.

e) Wire 2 exerts a stronger force on wire 1 than wire 1 exerts on wire 2.



SOLVED PROBLEM 8.1 Magnetic Field from Four Wires

Four wires are each carrying a current of magnitude $i = 1.00$ A. The wires are located at the four corners of a square with side $a = 3.70$ cm. Two of the wires are carrying current into the page, and the other two are carrying current out of the page (Figure 8.8).

PROBLEM

What is the y-component of the magnetic field at the center of the square?

– *Continued*

Concept Check 8.3

In Figure 8.2, compass needles show the magnetic field around a current-carrying wire. In the figure, the north-pointing end of the compass needle corresponds to

a) the red end.

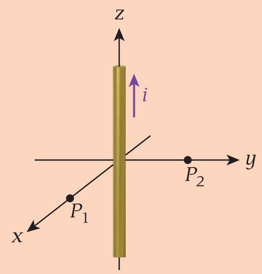
b) the gray end.

c) either the red end or the gray end, depending on how the compass is moved toward the wire.

d) The end cannot be identified from the information contained in the figure.

Self-Test Opportunity 8.1

The wire in the figure is carrying a current i in the positive z-direction. What is the direction of the resulting magnetic field at point P_1? What is the direction of the resulting magnetic field at point P_2?



Self-Test Opportunity 8.2

Consider two parallel wires carrying the same current in the same direction. Is the force between the two wires attractive or repulsive? Now consider two parallel wires carrying current in opposite directions. What is the force between the two wires?

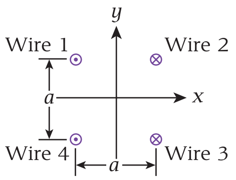


FIGURE 8.8 Four wires located at the corners of a square. Two of the wires are carrying current into the page, and the other two are carrying current out of the page.

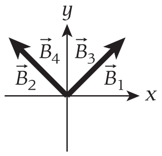


FIGURE 8.9 The magnetic fields from the four current-carrying wires.

SOLUTION

THINK The magnetic field at the center of the square is the vector sum of the magnetic fields from the four current-carrying wires. The magnitude of the magnetic field due to each of the four wires is the same. The direction of the magnetic field from each wire is determined using right-hand rule 3.

SKETCH Figure 8.9 shows the magnetic fields from the four wires: $\vec{B}_1$ is the magnetic field from wire 1, $\vec{B}_2$ is the magnetic field from wire 2, $\vec{B}_3$ is the magnetic field from wire 3, and $\vec{B}_4$ is the magnetic field from wire 4. Note that $\vec{B}_2$ and $\vec{B}_4$ are equal and $\vec{B}_1$ and $\vec{B}_3$ are equal.

RESEARCH The magnitude of the magnetic field from each of the four wires is given by

$$B = \frac{\mu_0 i}{2\pi r} = \frac{\mu_0 i}{2\pi\left(a/\sqrt{2}\right)},$$

where $a/\sqrt{2}$ is the distance from each wire to the center of the square.

Right-hand rule 3 gives us the directions of the magnetic fields, which are shown in Figure 8.9. The y-component of each of the magnetic fields is given by

$$B_y = B\sin 45°.$$

SIMPLIFY The sum of the y-components of the four magnetic fields is

$$B_{y,\text{sum}} = 4B_y = 4B\sin 45° = 4\frac{\mu_0 i}{2\pi\left(a/\sqrt{2}\right)}\left(\frac{1}{\sqrt{2}}\right) = \frac{2\mu_0 i}{\pi a},$$

where we have used $\sin 45° = 1/\sqrt{2}$.

CALCULATE Putting in the numerical values gives us

$$B_{y,\text{sum}} = \frac{2\mu_0 i}{\pi a} = \frac{2\left(4\pi\times 10^{-7}\text{ T m/A}\right)(1.00\text{ A})}{\pi\left(3.70\times 10^{-2}\text{ m}\right)} = 2.16216\times 10^{-5}\text{ T}.$$

ROUND We report our result to three significant figures:

$$B_{y,\text{sum}} = 2.16\times 10^{-5}\text{ T}$$

DOUBLE-CHECK To double-check our result, we calculate the magnitude of the magnetic field at the center of the square due to one of the wires:

$$B = \frac{\mu_0 i}{2\pi\left(a/\sqrt{2}\right)} = \frac{\left(4\pi\times 10^{-7}\text{ T m/A}\right)(1.00\text{ A})\sqrt{2}}{2\pi\left(3.70\times 10^{-2}\text{ m}\right)} = 7.64\times 10^{-6}\text{ T}.$$

The sum of the y-components is then

$$B_{y,\text{sum}} = \frac{4\left(7.64\times 10^{-6}\text{ T}\right)}{\sqrt{2}} = 2.16\times 10^{-5}\text{ T},$$

which agrees with our result.

Definition of the Ampere

The force $F_{1\to 2}$ described by equation 8.7 is used in the SI definition of the ampere: An *ampere* (A) is the constant current that, if maintained in two straight, parallel conductors of infinite length and negligible circular cross section, placed 1 m apart in vacuum, would produce between these conductors a force of 2×10^{-7} N per meter of length. This physical situation is described by equation 8.7 for the force between two parallel current-carrying wires with $i_1 = i_2 =$ exactly 1 A, $d =$ exactly 1 m, and $F_{1\to 2} =$ exactly 2×10^{-7} N. We can solve equation 8.7 for μ_0:

$$\mu_0 = \frac{(2\pi d)F_{1\to 2}}{i_1 i_2 L} = \frac{2\pi(1\text{ m})\left(2\times 10^{-7}\text{ N}\right)}{(1\text{ A})(1\text{ A})(1\text{ m})} = \text{exactly } 4\pi\times 10^{-7}\,\frac{\text{T m}}{\text{A}},$$

which indicates that the magnetic permeability of free space is *defined* to be exactly $\mu_0 = 4\pi\times 10^{-7}$ T m/A (see equation 8.2).

When Coulomb's Law was introduced, the value of the electric permittivity of free space, ϵ_0, was given: $\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/(N\ m^2)}$. Since 1 A = 1 C/s and 1 T = 1 (N s)/(C m) (see Chapter 7), the product of the two constants, ϵ_0 and μ_0, is

$$\mu_0\varepsilon_0 = \left(4\pi \times 10^{-7}\ \frac{\mathrm{T\ m}}{\mathrm{A}}\right)\left(8.85\times 10^{-12}\ \frac{\mathrm{C^2}}{\mathrm{N\ m^2}}\right) = 1.11\times 10^{-17}\ \frac{\mathrm{s^2}}{\mathrm{m^2}},$$

which has the units of the inverse of the square of a speed. Thus, $1/\sqrt{\mu_0\varepsilon_0}$ gives the value of this speed as that of the speed of light, $c = 3.00 \times 10^8$ m/s. This is by no means a coincidence, as we'll see in later chapters. For now, it is sufficient to state the empirical finding:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}.$$

Since the magnetic permeability of free space is defined to be exactly $\mu_0 = 4\pi\times 10^{-7}$ T m/A and the speed of light is defined as exactly $c = 299{,}792{,}458$ m/s the expression $c = 1/\sqrt{\mu_0\varepsilon_0}$ also fixes the value of the electric permittivity of free space.

EXAMPLE 8.1 Force on a Loop

A long, straight wire is carrying a current of magnitude $i_1 = 5.00$ A toward the right (Figure 8.10). A square loop with sides of length $a = 0.250$ m is placed with its sides parallel and perpendicular to the wire at a distance $d = 0.100$ m from the wire. The square loop carries a current of magnitude $i_2 = 2.20$ A in the counterclockwise direction.

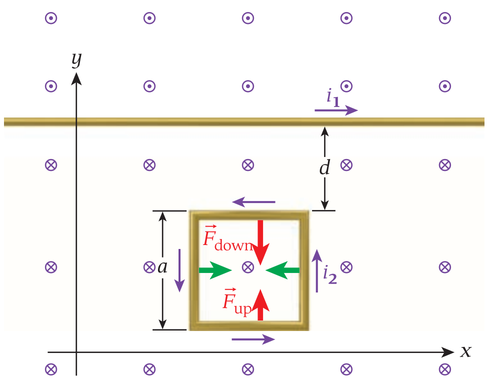


FIGURE 8.10 A current-carrying wire and a square loop.

PROBLEM

What is the net magnetic force on the square loop?

SOLUTION

The force on the square loop is due to the magnetic field created by the current flowing in the straight wire. Right-hand rule 3 tells us that the magnetic field from the current flowing in the wire is directed into the page in the region where the loop is located (see Figure 8.10). The right-hand rule and equation 8.4 tell us that the resulting force on the left side of the loop is toward the right, and the force on the right side of the loop is toward the left. In Figure 8.10, these two forces are represented by green arrows. The two forces are equal in magnitude and opposite in direction, so they sum to zero. The force on the top side of the loop is downward (red arrow in Figure 8.10, pointing in the negative y-direction), and its magnitude is given by equation 8.7:

$$F_{\mathrm{down}} = \frac{\mu_0 i_1 i_2 a}{2\pi d},$$

where a is the length of the top side of the loop. The force on the bottom side of the loop is upward (the other red arrow in Figure 8.10, pointing in the positive y-direction), and its magnitude is given by

$$F_{\mathrm{up}} = \frac{\mu_0 i_1 i_2 a}{2\pi(d+a)}.$$

Thus, we can express the net magnetic force on the loop as

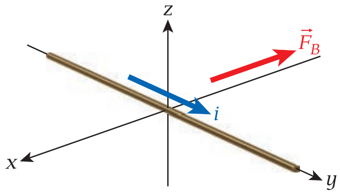
$$\vec{F} = (F_{\mathrm{up}} - F_{\mathrm{down}})\hat{y}.$$

Putting in the numbers results in

$$\vec{F} = \frac{(4\pi\times 10^{-7}\ \mathrm{T\ m/A})(5.00\ \mathrm{A})(2.20\ \mathrm{A})(0.250\ \mathrm{m})}{2\pi}\left(\frac{1}{0.350\ \mathrm{m}} - \frac{1}{0.100\ \mathrm{m}}\right)\hat{y} = (-3.93\times 10^{-6})\ \hat{y}\ \mathrm{N}.$$

Concept Check 8.5

A wire is carrying a current, i, in the positive y-direction, as shown in the figure. The wire is located in a uniform magnetic field, $\vec{B}$, oriented in such a way that the magnetic force on the wire is maximized. The magnetic force acting on the wire, $\vec{F}_B$, is in the negative x-direction. What is the direction of the magnetic field?



a) the positive x-direction

b) the negative x-direction

c) the negative y-direction

d) the positive z-direction

e) the negative z-direction

SOLVED PROBLEM 8.2 Electromagnetic Rail Accelerator

Electromagnetic rail accelerators are being developed for accelerating fuel pellets in fusion experiments and for launching spacecraft into orbit. The U.S. Navy is experimenting with electromagnetic rail accelerators that can launch projectiles at very high speeds, such as the

– Continued

FIGURE 8.11 A new electromagnetic railgun.

rail gun shown in Figure 8.11. In this rail gun, currents are run through two parallel conducting rails that are connected by a movable conductor oriented perpendicular to the rails. The projectile is attached to the movable conductor. For this example, we'll assume that the rail gun consists of two parallel rails of cross-sectional radius $r = 5.00$ cm, whose centers are separated by a distance $d = 25.0$ cm and which have a length $L = 5.00$ m, and that the rail gun accelerates the projectile to a kinetic energy of $K = 32.0$ MJ. The projectile also functions as the movable conductor.

PROBLEM

How much current is required to accelerate the projectile?

SOLUTION

THINK The currents in the two rails are in opposite directions. The current flowing through the movable conductor is perpendicular to the two currents in the rails. The magnetic fields from the two rails are in the same direction and exert forces on the movable conductor in the same direction. The force from the magnetic field of each rail depends on the distance from the rail. Thus, we must integrate the force along the distance between the two rails to obtain the total force. The total force on the movable conductor is twice the force from the magnetic field from one rail. The kinetic energy gained by the projectile is the total force exerted by the magnetic fields of the two rails times the distance over which the force acts.

SKETCH Figure 8.12 shows top and cross-sectional views of the rails and the movable conductor.

RESEARCH The current-carrying movable conductor, which completes the circuit between the two rails, is also the projectile and is accelerated by the magnetic forces produced by the two rails. The force exerted on the projectile depends on the distance, x, from the center of a rail, as illustrated in Figure 8.12b. Thus, to calculate the total force on the projectile, we must integrate over the length of the projectile. We use equation 8.4 to find the magnitude of the magnetic field, B_1, from current i flowing in rail 1 at a distance x from the center of the rail:

$$B_1 = \frac{\mu_0 i}{2\pi x}.$$

According to equation 8.6, the magnitude of the differential force, dF_1, exerted on a differential length, dx, of the projectile by the magnetic field from rail 1 is

$$dF_1 = i(dx)B_1 = i(dx)\left(\frac{\mu_0 i}{2\pi x}\right).$$

The direction of the force is given by right-hand rule 3, which tells us that the force is upward in the plane of the page in Figure 8.12a and into the page in Figure 8.12b. The magnitude of the force on the projectile is given by integrating dF_1 over the length of the projectile:

$$F_1 = \int_r^{d-r} dF_1 = \int_r^{d-r} \frac{\mu_0 i^2}{2\pi}\frac{dx}{x} = \frac{\mu_0 i^2}{2\pi}[\ln x]_r^{d-r} = \frac{\mu_0 i^2}{2\pi}(\ln(d-r) - \ln r) = \frac{\mu_0 i^2}{2\pi}\ln\left(\frac{d-r}{r}\right). \quad \text{(i)}$$

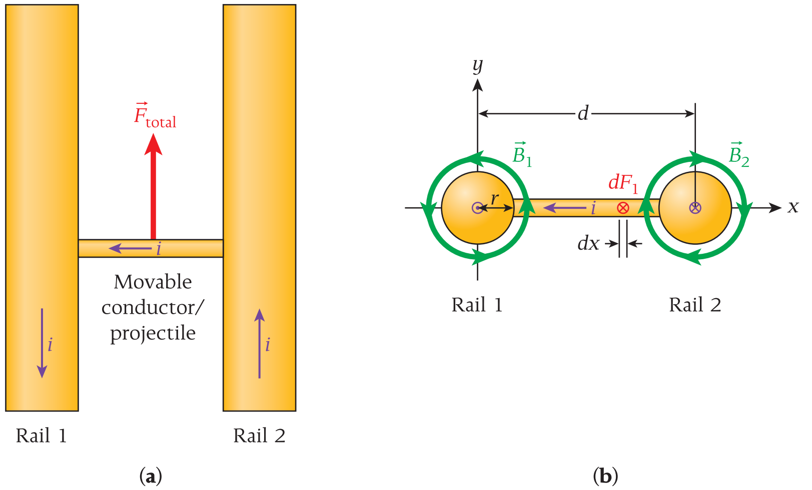


FIGURE 8.12 Schematic diagram of a rail gun: (a) top view; (b) cross-sectional view.

Because the magnetic field from rail 2 is in the same direction as the magnetic field from rail 1, the force exerted by the magnetic field from rail 2 on the projectile is the same as that from rail 1. Thus, the magnitude of the total force exerted on the projectile is

$$F_{\text{total}} = 2F_1. \tag{ii}$$

The kinetic energy gained by the projectile is equal to the magnitude of the force exerted times the distance over which the force acts

$$K = F_{\text{total}}L. \tag{iii}$$

SIMPLIFY We can combine equations (i), (ii), and (iii) to obtain

$$K = 2F_1L = 2\left[\frac{\mu_0 i^2}{2\pi}\ln\left(\frac{d-r}{r}\right)\right]L = \frac{\mu_0 L i^2}{\pi}\ln\left(\frac{d-r}{r}\right).$$

Solving this equation for the current gives us

$$i = \sqrt{\frac{K\pi}{\mu_0 L \ln\left(\frac{d-r}{r}\right)}}.$$

CALCULATE Putting in the numerical values, we get

$$i = \sqrt{\frac{K\pi}{\mu_0 L \ln\left(\frac{d-r}{r}\right)}} = \sqrt{\frac{\left(32.0\times 10^6 \text{ J}\right)\pi}{\left(4\pi\times 10^{-7} \frac{\text{T m}}{\text{A}}\right)(5.00 \text{ m})\ln\left(\frac{25.0 \text{ cm} - 5.00 \text{ cm}}{5.00 \text{ cm}}\right)}} = 3.397287\times 10^6 \text{ A}$$

ROUND We report our result to three significant figures:

$$i = 3.40\times 10^6 \text{ A} = 3.40 \text{ MA}.$$

DOUBLE-CHECK Let's double-check our result by looking at the force exerted on the projectile. Putting our result for the current into equation (i) for the magnitude of the force exerted by the magnetic field of rail 1 gives

$$F_1 = \frac{\mu_0 i^2}{2\pi}\ln\left(\frac{d-r}{r}\right) = \frac{\left(4\pi\times 10^{-7} \frac{\text{T m}}{\text{A}}\right)\left(3.40\times 10^6 \text{ A}\right)^2}{2\pi}\ln\left(\frac{25.0 \text{ cm} - 5.00 \text{ cm}}{5.00 \text{ cm}}\right) = 3.20\times 10^6 \text{ N}$$

We can approximate the magnitude of this force by assuming that the force is constant along the conductor and is equal to the value at $x = d/2$:

$$F_1 = i(d-2r)B = i\left(\frac{\mu_0 i(d-2r)}{2\pi\left(\frac{d}{2}\right)}\right) = \frac{\mu_0 i^2}{\pi}\frac{d-2r}{d}$$

$$= \frac{\left(4\pi\times 10^{-7} \frac{\text{T m}}{\text{A}}\right)\left(3.40\times 10^6 \text{ A}\right)^2}{\pi}\frac{[25.0 \text{ cm} - 2(5.00 \text{ cm})]}{25.0 \text{ cm}} = 2.77\times 10^6 \text{ N}.$$

This value is within a factor of 2 of our calculated value for the force, which seems reasonable. However, just to be sure, let's calculate the kinetic energy of the projectile using the calculated value of the force:

$$K = F_{\text{total}}L = 2F_1L = 2\left(3.20\times 10^6 \text{ N}\right)(5.00 \text{ m}) = 32.0\times 10^6 \text{ J}.$$

This result agrees with the 32.0 MJ assumed in the problem situation.

Note that if a projectile of mass $m = 5.00$ kg were given a kinetic energy of 32.0 MJ by this rail gun, its speed would be

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2\left(32.0\times 10^6 \text{ J}\right)}{5.00 \text{ kg}}} = 3.58 \text{ km/s}.$$

The rail gun would be capable of launching a projectile at 10 times the speed of sound, which is much larger than the typical speed of a bullet, about 3 times the speed of sound.

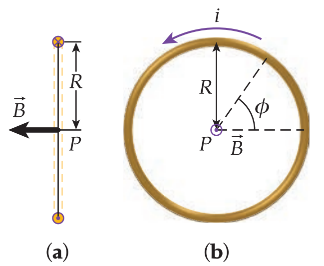


FIGURE 8.13 A circular loop of radius R carrying a current, i: (a) side view; (b) front view. The cross on the upper yellow circle in part (a) signifies that the current at the top of the loop is into the page, and the dot on the lower yellow circle in part (a) signifies that the current at the bottom of the loop is out of the page. Point P is located at the center of the loop.

Magnetic Field due to a Wire Loop

Now let's find the magnetic field at the center of a circular current-carrying loop of wire. Figure 8.13a shows a cross section of a circular loop with radius R carrying a current, i. Applying equation 8.3, $dB = \mu_0 i ds \sin\theta/(4\pi r^2)$, to this case, we can see that $r = R$ and $\theta = 90°$ for every current element $i ds$ along the loop. For the magnitude of the magnetic field at the center of the loop from each current element, we get

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds \sin 90°}{R^2} = \frac{\mu_0}{4\pi}\frac{i\,ds}{R^2}.$$

Going around the loop in Figure 8.13b, we can relate the angle ϕ to the current element by $ds = R\,d\phi$, allowing us to calculate the magnitude of the magnetic field at the center of the loop:

$$B = \int dB = \int_0^{2\pi} \frac{\mu_0}{4\pi}\frac{iR\,d\phi}{R^2} = \frac{\mu_0 i}{2R}. \tag{8.8}$$

Keep in mind that equation 8.8 only gives the magnitude of the magnetic field at the center of the loop, where the magnitude is $B(r = 0) = \frac{1}{2}\mu_0 i/R$. To determine the direction of the magnetic field, we again use a variant of right-hand rule 1. Using your right hand, point your thumb in the direction of the current element (into the page for the upper circle in Figure 8.13a marked with a cross), and your index finger in the direction of the radial vector from the current element (down); your middle finger then points to the left. Using right-hand rule 3 (Figure 8.5), we also find that the current shown in Figure 8.13 produces a magnetic field $\vec{E}$ directed toward the left.

Now let's find the magnetic field from the loop along the axis of the loop rather than at the center (Figure 8.14). We set up a coordinate system such that the axis of the loop lies along the x-axis and the center of the loop is located at $x = 0$, $y = 0$, and $z = 0$. The radial vector $\vec{r}$ is the displacement to any point along the x-axis from a current element, $i\,d\vec{s}$, along the loop. The current element shown in Figure 8.14 lies in the negative z-direction. The radial vector $\vec{r}$ lies in the xy-plane and so is perpendicular to the current element. This situation is the same for any current element around the loop. Therefore, we can employ equation 8.3 with $\theta = 90°$ and obtain an expression for the magnitude of the differential magnetic field at any point along the x-axis:

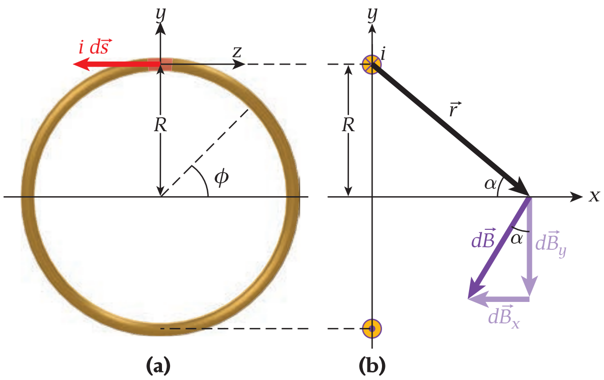


FIGURE 8.14 Geometry for calculating the magnetic field along the axis of a current-carrying loop: (a) front view, (b) side view.

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds \sin 90°}{r^2} = \frac{\mu_0}{4\pi}\frac{i\,ds}{r^2}.$$

A variant of right-hand rule 1 gives the direction of the differential magnetic field: Using your right hand, point your thumb in the direction of the differential current element (negative z-direction) and your index finger in the direction of the radial vector (positive x-direction and negative y-direction); the direction of the differential magnetic field will be given by your middle finger (negative x-direction and negative y-direction). The differential magnetic field is shown in Figure 8.14. To obtain the complete magnetic field, we need to integrate over the differential current element. From the symmetry of the situation, we can see that the y-component of the differential magnetic field, dB_y, will integrate to zero. The x-component of the differential magnetic field, dB_x, is given by

$$dB_x = dB \sin\alpha = \frac{\mu_0}{4\pi}\frac{i\,ds}{r^2}\sin\alpha,$$

where α is the angle between $\vec{r}$ and the x-axis (see Figure 8.14b). We can express the magnitude of $\vec{r}$ in terms of x and R as $r = \sqrt{x^2 + R^2}$ and express $\sin\alpha$ in terms of x and the radius of the loop R as $\sin\alpha = R/\sqrt{x^2 + R^2}$. We can then rewrite the expression for the x-component of the differential magnetic field as

$$dB_x = \frac{\mu_0}{4\pi}\frac{i\,ds}{x^2 + R^2}\frac{R}{\sqrt{x^2 + R^2}} = \frac{\mu_0 i\,ds}{4\pi}\frac{R}{\left(x^2 + R^2\right)^{3/2}}.$$

This expression for B_x is independent of the location of the current element, and so the integral to find the magnitude of the total field can be simplified to

$$B_x = \int dB_x = \frac{\mu_0 i R}{4\pi\left(x^2 + R^2\right)^{3/2}} \int ds.$$

Going around the loop, we can relate the angle ϕ to the current element by $ds = R\,d\phi$ (see Figure 8.14a), allowing us to calculate the magnetic field along the axis of the loop:

$$B_x = \frac{\mu_0 i R}{4\pi\left(x^2 + R^2\right)^{3/2}} \int_0^{2\pi} R\,d\phi = \frac{\mu_0 i 2\pi}{4\pi} \frac{R^2}{\left(x^2 + R^2\right)^{3/2}},$$

or

$$B_x = \frac{\mu_0 i}{2} \frac{R^2}{\left(x^2 + R^2\right)^{3/2}}. \quad (8.9)$$

From our earlier application of the variant of right-hand rule 1, we know that the magnetic field along the axis of the loop is in the negative x-direction, as shown in Figure 8.14. We can also apply right-hand rule 3 to obtain the direction of the magnetic field: At any point on the loop, point your thumb tangent to the loop in the direction of the current and your fingers will curl in a direction that indicates that the field inside the loop is in the negative x-direction.

Using more advanced techniques and with the aid of a computer, we can determine the magnetic field produced by a current-carrying loop at other points in space. The magnetic field lines from a wire loop are shown in Figure 8.15. The value for the magnetic field given by equation 8.8 is valid only at the center point of Figure 8.15. The value for the magnetic field given by equation 8.9 is valid only along the axis of the loop.

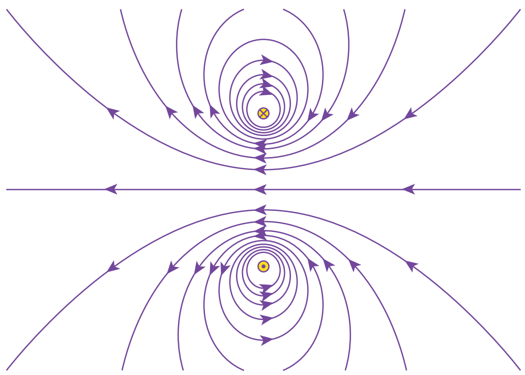


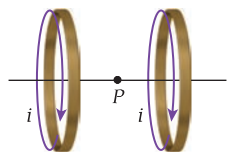
FIGURE 8.15 Magnetic field lines from a loop of wire carrying a current, looking at the loop edge-on. The upper yellow circle with a cross indicates current directed into the page, and the lower yellow circle with a dot indicates current directed out of the page.

Self-Test Opportunity 8.3

Show that equation 8.9 for the magnitude of the magnetic field along the axis of a current-carrying loop reduces to equation 8.8 for the magnitude of the magnetic field at the center of a current-carrying loop.

Concept Check 8.6

Two identical wire loops carry the same current, *i*, as shown in the figure. What is the direction of the magnetic field at point *P*?



a) upward (toward the top of the page)

b) toward the right

c) downward

d) toward the left

e) The magnetic field at point *P* is zero.

SOLVED PROBLEM 8.3 Field from a Wire Containing a Loop

A loop with radius $r = 8.30$ mm is formed in the middle of a long, straight insulated wire carrying a current of magnitude $i = 26.5$ mA (Figure 8.16a).

PROBLEM

What is the magnitude of the magnetic field at the center of the loop?

SOLUTION

THINK The magnetic field at the center of the loop is equal to the vector sum of the magnetic fields from the long, straight wire and from the loop.

SKETCH The magnetic field from the long, straight wire, $\vec{B}_{\text{wire}}$, and the magnetic field from the loop, $\vec{B}_{\text{loop}}$, are shown in Figure 8.16b.

– *Continued*

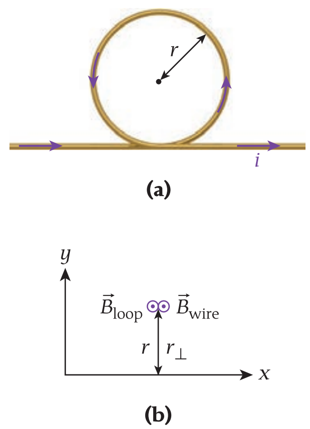


FIGURE 8.16 (a) A loop with radius *r* in a long, straight insulated wire carrying a current, *i*. (b) The magnetic field from the wire and the magnetic field from the loop, displaced slightly for clarity.

RESEARCH Using right-hand rule 3, we find that both magnetic fields point out of the page at the center of the loop, as illustrated in Figure 8.16b. Thus, we can add the magnitudes of the magnetic field produced by the wire and the magnetic field produced by the loop. The magnetic field produced by the wire at the center of the loop has a magnitude given by equation 8.4:

$$B_{\text{wire}} = \frac{\mu_0 i}{2\pi r_\perp},$$

where $r_\perp$ is the perpendicular distance from the wire, which is equal to r, the radius of the loop. The magnitude of the magnetic field produced by the loop at its center is given by equation 8.8:

$$B_{\text{loop}} = \frac{\mu_0 i}{2r}.$$

SIMPLIFY We add the magnitudes of the two magnetic fields since the vectors are in the same direction:

$$B = B_{\text{wire}} + B_{\text{loop}} = \frac{\mu_0 i}{2\pi r} + \frac{\mu_0 i}{2r} = \frac{\mu_0 i}{2r}\left(\frac{1}{\pi} + 1\right).1$$

CALCULATE Putting in the numerical values, we get

$$B = \frac{\mu_0 i}{2r}\left(\frac{1}{\pi} + 1\right) = \frac{\left(4\pi \times 10^{-7}\ \text{T m/A}\right)\left(26.5 \times 10^{-3}\ \text{A}\right)}{2\left(8.30 \times 10^{-3}\ \text{m}\right)}\left(\frac{1}{\pi} + 1\right) = 2.64463 \times 10^{-6}\ \text{T}.$$

ROUND We report our result to three significant figures and note the direction of the field:

$$\vec{B} = 2.64 \times 10^{-6}\ \text{T, out of the page.}$$

DOUBLE-CHECK To double-check our result, we calculate the magnitudes of the magnetic fields from the wire and from the loop separately. The magnitude of the magnetic field from the wire is

$$B_{\text{wire}} = \frac{\mu_0 i}{2\pi r} = \frac{\left(4\pi \times 10^{-7}\ \text{T m/A}\right)\left(26.5\quad 10^{-3}\ \text{A}\right)}{2\pi\left(8.30 \times 10^{-3}\ \text{m}\right)} = 6.385 \times 10^{-7}\ \text{T}.$$

The magnitude of the magnetic field from the loop is

$$B_{\text{loop}} = \frac{\mu_0 i}{2r} = \frac{\left(4\pi \times 10^{-7}\ \text{T m/A}\right)\left(26.5\quad 10^{-3}\ \text{A}\right)}{2\left(8.30 \times 10^{-3}\ \text{m}\right)} = 2.006 \times 10^{-6}\ \text{T}.$$

The sum of these two magnitudes matches our result:

$$6.385 \times 10^{-7}\ \text{T} + 2.006 \times 10^{-6}\ \text{T} = 2.64 \times 10^{-6}\ \text{T}.$$

8.3 Ampere's Law

Calculating the electric field resulting from a distribution of electric charge can require evaluating a difficult integral. However, if the charge distribution has cylindrical, spherical, or planar symmetry, we can apply Gauss's Law and obtain the electric field in an elegant manner. Similarly, calculating the magnetic field due to an arbitrary distribution of current elements using the Biot-Savart Law (equation 8.1) may involve the evaluation of a difficult integral. Alternatively, we can avoid using the Biot-Savart Law and instead apply **Ampere's Law** to calculate the magnetic field from a distribution of current elements when the distribution has cylindrical or other symmetry. Often, problems can be solved with much less effort in this way than by using a direct integration. The mathematical statement of Ampere's Law is

$$\oint \vec{B} \bullet d\vec{s} = \mu_0 i_{\text{enc}}. \tag{8.10}$$

The symbol $\oint$ means that the integrand, $\vec{B} \bullet d\vec{s}$, is integrated over a closed loop, called an **Amperian loop.** This loop is chosen so that the integral in equation 8.10 is not difficult to evaluate, a procedure similar to that used in applying Gauss's Law. The total current enclosed in this loop is i_{enc}, which is also similar to Gauss's Law, where the chosen closed surface encloses a total net charge.

As an example of how Ampere's Law is used, consider the five currents shown in Figure 8.17, which are all perpendicular to the plane. An Amperian loop, represented by the red line, encloses currents i_1, i_2, and i_3 and excludes currents i_4 and i_5. By Ampere's Law, the closed-loop integral over the magnetic field resulting from these three currents is given by

$$\oint \vec{B} \bullet d\vec{s} = \oint B\cos\theta\, ds = \mu_0\left(i_1 - i_2 + i_3\right),$$

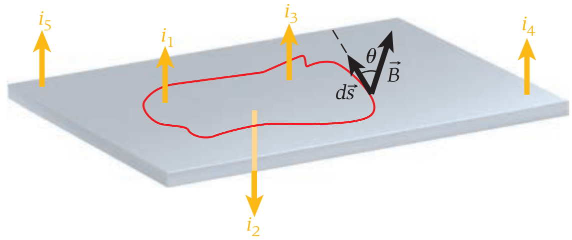
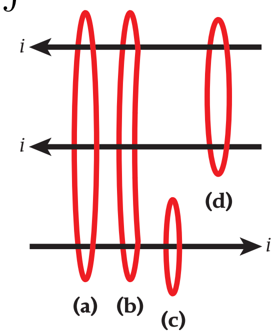


FIGURE 8.17 Five currents and an Amperian loop.

where θ is the angle between the direction of the magnetic field, $\vec{B}$, and the direction of the element of length, $d\vec{s}$, at each point along the Amperian loop. The integration over the Amperian loop can be done in either direction. Figure 8.17 indicates a direction of integration from the direction of $d\vec{s}$, along with the resulting magnetic field. The sign of the contributing currents can be determined using a right-hand rule: Curl your fingers in the direction of integration, and then currents in the same direction as your thumb are positive. Two of the three currents in the Amperian loop are positive, and one is negative. Adding the three currents is simple, but the integral $\oint B\cos\theta ds$ cannot be easily evaluated. However, let's examine some special situations in which Amperian loops contain symmetrical distributions of current that can be exploited to carry out the integration.

Concept Check 8.7

Three wires are carrying currents of the same magnitude, *i*, in the directions shown in the figure. Four Amperian loops (a), (b), (c), and (d) are shown. For which Amperian loop is the magnitude of $\oint \vec{B} \bullet d\vec{s}$ the greatest?



a) loop a
b) loop b
c) loop c
d) loop d
e) All four loops yield the same value of $\oint \vec{B} \bullet d\vec{s}$

Magnetic Field inside a Long, Straight Wire

Figure 8.18 shows a current, *i*, flowing out of the page in a wire with a circular cross section of radius *R*. This current is uniformly distributed over the cross-sectional area of the wire. To find the magnetic field due to this current, we use an Amperian loop with radius $r_\perp$, represented by the red circle. If $\vec{B}$ had an outward (or inward) component, by symmetry, it would have an outward (or inward) component at all points around the loop, and the corresponding magnetic field line could never be closed. Therefore, $\vec{B}$ must be tangential to the Amperian loop. Thus, we can rewrite the integral of Ampere's Law as

$$\oint \vec{B} \bullet d\vec{s} = B\oint d\vec{s} = B2\pi r_\perp.$$

We can calculate the enclosed current from the ratio of the area of the Amperian loop to the cross-sectional area of the wire:

$$i_{\text{enc}} = i\frac{A_{\text{loop}}}{A_{\text{wire}}} = i\frac{\pi r_\perp^2}{\pi R^2}.$$

Thus, we obtain

$$2\pi B r_\perp = \mu_0 i \frac{\pi r_\perp^2}{\pi R^2},$$

or

$$B = \left(\frac{\mu_0 i}{2\pi R^2}\right) r_\perp. \tag{8.11}$$

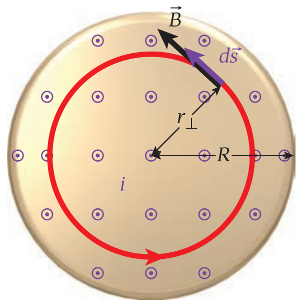


FIGURE 8.18 Using Ampere's Law to find the magnetic field produced inside a long, straight wire.

Let's compare the expressions for the magnitudes of the magnetic field outside and inside the wire–equations 8.4 and 8.11. First, substituting R for $r_\perp$ in both expressions, we obtain the same result for the magnetic field magnitude at the surface of the wire in both cases: $B(R) = \mu_0 i/2\pi R$. Both equations provide the same solution at the wire's surface. Inside the wire, we find that the magnetic field magnitude rises linearly with $r_\perp$ up to the value of $B(R) = \mu_0 i/(2\pi R)$ and from there falls off with the inverse of $r_\perp$. Figure 8.19 shows this dependence in the graph at the bottom. The upper part of the figure depicts the cross section through the wire (golden area), the magnetic field lines (black circles, spaced to indicate the strength of the magnetic field), and the magnetic field vectors at selected points in space (red arrows).

8.4 Magnetic Fields of Solenoids and Toroids

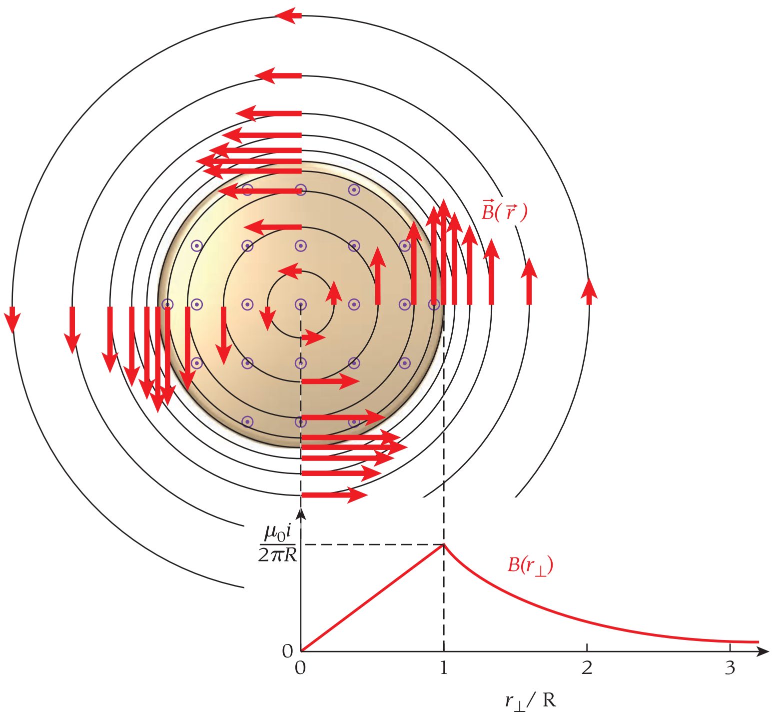


FIGURE 8.19 Radial dependence of the magnetic field for a wire with current flowing out of the page.

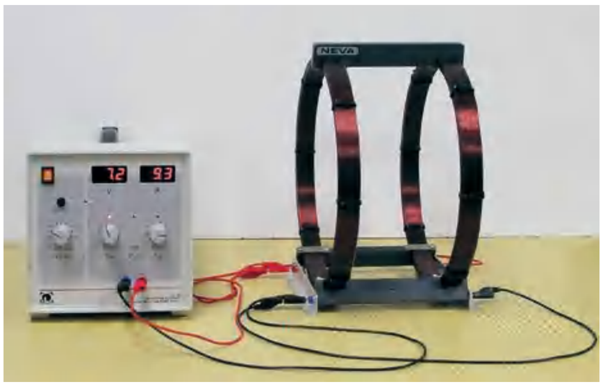
We have seen that current flowing through a single loop of wire produces a magnetic field that is not uniform, as illustrated in Figure 8.15. However, real-world applications often require a uniform magnetic field. A device commonly used to produce a uniform magnetic field is the **Helmholtz coil** (Figure 8.20a). A Helmholtz coil consists of two coaxial wire loops. Each coaxial loop consists of multiple loops (windings or turns) of a single wire, and therefore acts magnetically like a single loop.

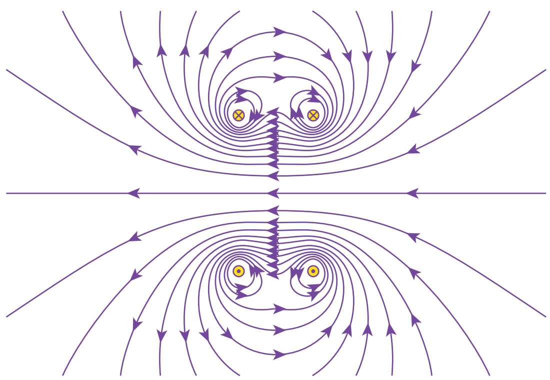
The magnetic field lines from a Helmholtz coil are shown in Figure 8.20b. You can see that there is a region of uniform magnetic field (characterized by horizontal parallel segments of the field lines) in the center between the loops, in contrast to the field from a single loop shown in Figure 8.15. Again, these field lines were calculated with the aid of a computer to provide a qualitative understanding of the geometry of magnetic fields.

Taking multiple loops a step further, Figure 8.21 shows the magnetic field lines from four coaxial wire loops. The region of uniform magnetic field in the center of the loops is expanded, but note that the field is not uniform near the wires and near the two ends.

A strong uniform magnetic field is produced by a **solenoid,** consisting of many loops of a wire wound close together. Figure 8.22 shows the magnetic field lines from a solenoid with 600 turns, or loops. You can see that the magnetic field lines are very close together on the inside of the solenoid and far apart on the outside. Like that inside the Helmholtz coil (Figure 8.20b), the magnetic field is uniform inside the solenoid coil. The spacing of the field lines is a measure of the strength of the magnetic field, and you can see that the magnetic field is much stronger inside the solenoid than outside the solenoid .

An *ideal* solenoid has a magnetic field of zero outside and of a uniform constant finite value inside. To determine the magnitude of the magnetic field inside an ideal solenoid, we can apply Ampere's Law (equation 8.10) to a section of a solenoid far from its ends (Figure 8.23). To do so, we first choose an Amperian loop over which to carry out the integration. A judicious choice, shown by the red rectangle in Figure 8.23, encloses some current and





(a) (b)

FIGURE 8.20 (a) A typical Helmholtz coil used in physics labs generates a nearly uniform magnetic field in its interior. (b) Magnetic field lines for a Helmholtz coil.

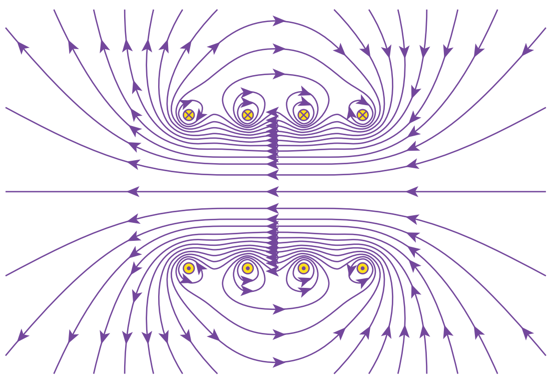


FIGURE 8.21 Magnetic field lines resulting from four coaxial wire loops with many windings.

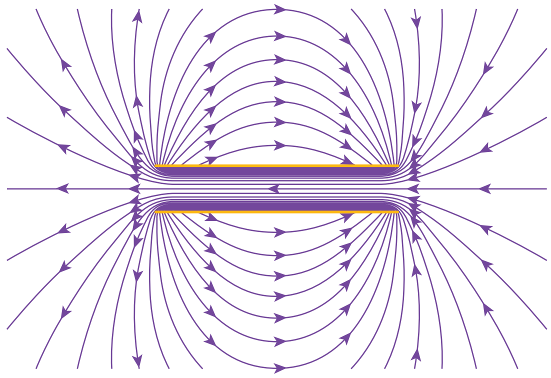


FIGURE 8.22 Magnetic field lines for a solenoid with 600 turns. The current along the top of the solenoid is flowing into the page, and the current along the bottom of the solenoid is flowing out of the page.

exploits the symmetry of the solenoid as well as simplifying the evaluation of the integral:

$$\oint \vec{B} \bullet d\vec{s} = \int_a^b \vec{B} \bullet d\vec{s} + \int_b^c \vec{B} \bullet d\vec{s} + \int_c^d \vec{B} \bullet d\vec{s} + \int_d^a \vec{B} \bullet d\vec{s}.$$

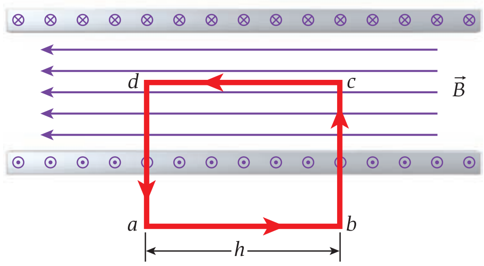


FIGURE 8.23 Amperian loop for determining the magnitude of the magnetic field of an ideal solenoid.

The value of the third integral on the right-hand side, between points c and d in the interior of the solenoid, is Bh. The values of the second and fourth integrals are zero because the magnetic field is perpendicular to the direction of integration. The first integral, between the points a and b in the exterior of the ideal solenoid, is zero because the magnetic field outside of an ideal solenoid is zero. Thus, the value of the integral over the entire Amperian loop is Bh.

The enclosed current is the current in the turns of the solenoid that are within the Amperian loop. The current is the same in each turn because the solenoid is made from one wire and the same current flows through each turn. Thus, the enclosed current is just the number of turns times the current:

$$i_{\text{enc}} = nhi,$$

where n is the number of turns per unit length. Therefore, according to Ampere's Law, we have

$$Bh = \mu_0 nhi.$$

Thus, the magnitude of the magnetic field inside an ideal solenoid is

$$B = \mu_0 ni. \tag{8.12}$$

Equation 8.12 is valid only away from the ends of the solenoid. Note that B does not depend on position inside the solenoid: An ideal solenoid creates a constant and uniform magnetic field inside itself and no field outside itself. A real-world solenoid, like the one shown in Figure 8.22, has fringe fields near its ends but can still produce a high-quality uniform magnetic field.

EXAMPLE 8.2 Solenoid

The solenoid of the STAR detector at the Brookhaven National Laboratory, New York, discussed in Chapter 7, has a magnetic field of magnitude 0.50 T when carrying a current of 0.40 kA. The solenoid is 8.0 m long.

PROBLEM

What is the number of turns in this solenoid, assuming that it is an ideal solenoid?

– Continued

SOLUTION

We use equation 8.12 to calculate the magnitude of the magnetic field of an ideal solenoid:

$$B = \mu_0 ni. \tag{i}$$

The number of turns per unit length is given by

$$n = \frac{N}{L}, \tag{ii}$$

where N is the number of turns and L is the length of the solenoid. Substituting for n from equation (ii) into equation (i), we get

$$B = \mu_0 i \frac{N}{L}. \tag{iii}$$

Solving equation (iii) for the number of turns, we obtain

$$N = \frac{BL}{\mu_0 i} = \frac{(0.50 \text{ T})(8.0 \text{ m})}{\left(4\pi \times 10^{-7} \ \frac{\text{T m}}{\text{A}}\right)\left(0.40 \quad 10^3 \text{ A}\right)} = 8.0 \times 10^3 \text{ turns.}$$

Concept Check 8.8

You have a solenoid with a fixed number of turns connected to a power supply that can supply a fixed amount of current. To double the field inside the solenoid, you can

a) double the radius of the solenoid.

b) halve the radius of the solenoid.

c) double the length of the solenoid.

d) halve the length of the solenoid.

If a solenoid is bent so that the two ends meet (Figure 8.24), it acquires a doughnut shape (a torus), with the wire forming a series of loops, each with the same current flowing through it. This device is called a *toroidal magnet*, or **toroid.** Just as for an ideal solenoid, the magnetic field outside the coils of an ideal toroidal magnet is zero. The magnitude of the magnetic field inside the toroid coil can be calculated by using Ampere's Law and choosing an Amperian loop in the form of a circle with radius r, such that $r_1 < r < r_2$, where r_1 and r_2 are the inner and outer radii of the toroid. The magnetic field is always tangential to the Amperian loop, so we have

$$\oint \vec{B} \bullet d\vec{s} = 2\pi rB.$$

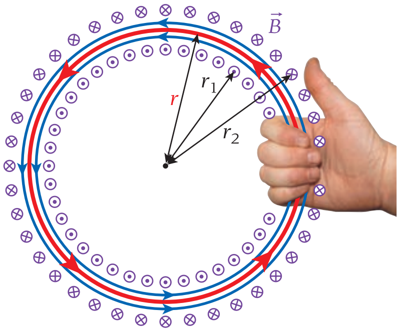


FIGURE 8.24 Toroidal magnet with Amperian loop (red) in the form of a circle with radius r. Right-hand rule 4 states that if you place the fingers of your right hand in the direction of the current flow, your thumb shows the direction of the magnetic field inside the toroid.

The enclosed current is the number of loops (or turns), N, in the toroid times the current, i, in the wire (in each loop); so, Ampere's law gives us

$$2\pi rB = \mu_0 Ni.$$

Therefore, the magnitude of the magnetic field inside of the toroid is given by

$$B = \frac{\mu_0 Ni}{2\pi r}. \tag{8.13}$$

Note that, unlike the magnetic field inside a solenoid, the magnitude of the magnetic field inside a toroid does depend on the radius. As the radius increases, the magnitude of the magnetic field decreases. The direction of the magnetic field can be obtained using *right-hand rule 4:* If you wrap the fingers of your right hand around the toroid in the direction of the current, as shown in Figure 8.24, your thumb points in the direction of the magnetic field inside the toroid.

SOLVED PROBLEM 8.4 Field of a Toroidal Magnet

A toroidal magnet is made from 202 m of copper wire that is capable of carrying a current of magnitude $i = 2.40$ A. The toroid has an average radius $R = 15.0$ cm and a cross-sectional diameter $d = 1.60$ cm (Figure 8.25a).

PROBLEM

What is the largest magnetic field that can be produced at the average toroidal radius, R?

SOLUTION

THINK The number of turns in the toroidal magnet is given by the length of the wire divided by the circumference of the cross-sectional area of the coil. With these parameters, the magnetic field of the toroidal magnet at $r = R$ can be calculated.

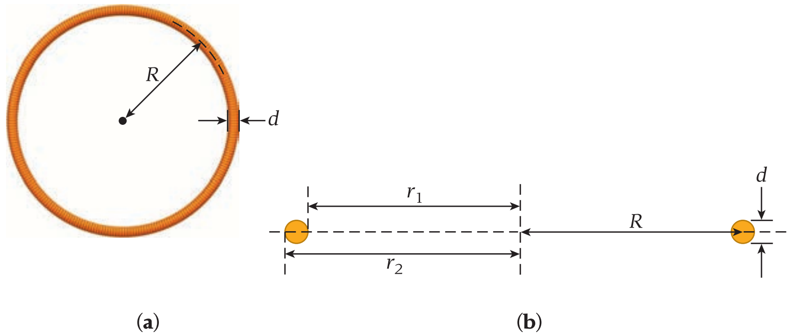


FIGURE 8.25 (a) A toroidal magnet. (b) Cross section of the toroidal magnet.

SKETCH Figure 8.25b shows a cross-sectional cut of the toroidal magnet.

RESEARCH The magnitude of the magnetic field of a toroidal magnet is given by equation 8.13:

$$B = \frac{\mu_0 N i}{2\pi R}, \qquad \text{(i)}$$

where N is the number of turns and R is the radius at which the magnetic field is measured. The number of turns, N, is given by the length, L, of the wire divided by the circumference of the cross-sectional area:

$$N = \frac{L}{\pi d}, \qquad \text{(ii)}$$

where d is the diameter of the cross-sectional area of the toroid.

SIMPLIFY We can combine equations (i) and (ii) to obtain an expression for B:

$$B = \frac{\mu_0 (L/\pi d) i}{2\pi R} = \frac{\mu_0 L i}{2\pi^2 R d}.$$

CALCULATE Putting in the numerical values gives us

$$B = \frac{\mu_0 L i}{2\pi^2 R d} = \frac{(4\pi \times 10^{-7} \text{ T m/A})(202 \text{ m})(2.40 \text{ A})}{2\pi^2 (15.0 \times 10^{-2} \text{ m})(1.60 \times 10^{-2} \text{ m})} = 0.0128597 \text{ T}.$$

ROUND We report our result to three significant figures:

$$B = 1.29 \times 10^{-2} \text{ T}$$

DOUBLE-CHECK As a double-check, we calculate the magnitude of the field inside a solenoid that has the same length as the circumference of the toroidal magnet. The number of turns per unit length is

$$n = \frac{L/\pi d}{2\pi R} = \frac{L}{2\pi^2 R d} = \frac{(202 \text{ m})}{2\pi^2 (15.0 \times 10^{-2} \text{ m})(1.60 \times 10^{-2} \text{ m})} = 4264 \text{ turns/m}.$$

The magnitude of the magnetic field inside a solenoid with that number of turns per unit length is

$$B = \mu_0 n i = (4\pi \times 10^{-7} \text{ T m/A})(4264 \text{ m}^{-1})(2.40 \text{ A}) = 1.29 \times 10^{-2} \text{ T}.$$

Thus, our answer for the magnitude of the field inside the toroid seems reasonable.

SOLVED PROBLEM 8.5 Electron Motion in a Solenoid

An ideal solenoid has 200. turns/cm. An electron inside the coil of the solenoid moves in a circle with radius r = 3.00 cm perpendicular to the solenoid's axis. The electron moves with a speed of v = 0.0500c, where c is the speed of light.

– Continued

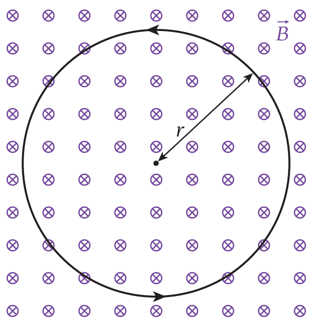


FIGURE 8.26 Electron traveling in a circular path inside a solenoid.

PROBLEM

What is the current in the solenoid?

SOLUTION

THINK The solenoid produces a uniform magnetic field, which is proportional to the current flowing in its coil. The radius of circular motion of the electron is related to the speed of the electron and the magnetic field inside the solenoid.

SKETCH Figure 8.26 shows the circular path of the electron in the uniform magnetic field of the solenoid.

RESEARCH The magnitude of the magnetic field inside the solenoid is given by

$$B = \mu_0 n i, \tag{i}$$

where i is the current in the solenoid and n is the number of turns per unit length. The magnetic force provides the centripetal force needed for the electron to move in a circle and so the radius of the electron's path can be related to B:

$$r = \frac{mv}{eB}, \tag{ii}$$

where m is the electron's mass, v is its speed, and e is the magnitude of the electron's charge.

SIMPLIFY Combining equations (i) and (ii), we have

$$r = \frac{mv}{e(\mu_0 n i)}.$$

Solving this equation for the current in the solenoid, we obtain

$$i = \frac{mv}{er\mu_0 n}. \tag{iii}$$

CALCULATE The speed of the electron was specified in terms of the speed of light:

$$v = 0.0500c = 0.0500\left(3.00 \times 10^8 \text{ m/s}\right) = 1.50 \times 10^7 \text{ m/s}.$$

Putting this and the other numerical values into equation (iii), we get

$$i = \frac{mv}{er\mu_0 n} = \frac{\left(9.11 \times 10^{-31} \text{ kg}\right)\left(1.50 \times 10^7 \text{ m/s}\right)}{\left(1.602 \times 10^{-19} \text{ C}\right)\left(3.00 \times 10^{-2} \text{ m}\right)\left(4\pi \times 10^{-7} \text{ T m/A}\right)\left(200 \times 10^2 \text{ m}^{-1}\right)}$$

$$= 0.113132 \text{ A}.$$

ROUND We report our result to three significant figures:

$$i = 0.113 \text{ A}.$$

DOUBLE-CHECK To double-check our result, we use it to calculate the magnitude of the magnetic field inside the solenoid:

$$B = \mu_0 n i = \left(4\pi \times 10^{-7} \text{ T m/A}\right)\left(200 \times 10^2 \text{ m}^{-1}\right)(0.113 \text{ A}) = 0.00284 \text{ T}.$$

This magnitude of magnetic field seems reasonable. Thus, our calculated value for the current in the solenoid seems reasonable.

8.5 Atoms as Magnets

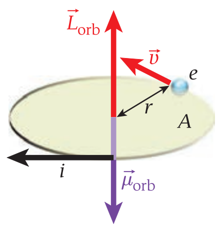


FIGURE 8.27 An electron moving with constant speed in a circular orbit in an atom.

The atoms that make up all matter contain moving electrons, which form current loops that produce magnetic fields. In most materials, these current loops are randomly oriented and produce no net magnetic field. Some materials have a fraction of these current loops aligned. These materials, called *magnetic materials* (Section 8.6), do produce a net magnetic field. Other materials can have their current loops aligned by an external magnetic field and become magnetized.

Let's consider a highly simplified model of the atom: An electron moving at constant speed v in a circular orbit with radius r (Figure 8.27). We can think of the moving charge of the electron as a current, i. Current is defined as the charge passing a particular point per unit of time. For this case, the charge is the charge of the electron, with magnitude e, and the time is related to the period, T, of the electron's orbit. Thus, the magnitude of the current is given by

$$i = \frac{e}{T} = \frac{e}{2\pi r/v} = \frac{ve}{2\pi r}.$$

The magnitude of the magnetic dipole moment of the orbiting electron is given by

$$\mu_{\text{orb}} = iA = \frac{ve}{2\pi r}\left(\pi r^2\right) = \frac{ver}{2}. \tag{8.14}$$

The magnitude of the orbital angular momentum of the electron is

$$L_{\text{orb}} = rp = rmv,$$

where m is the mass of the electron. Solving equation 8.14 for v and substituting that expression into the expression for the orbital angular momentum gives us

$$L_{\text{orb}} = rm\left(\frac{2\mu_{\text{orb}}}{er}\right) = \frac{2m\mu_{\text{orb}}}{e}.$$

Because the magnetic dipole moment and the angular momentum are vector quantities, we can write

$$\vec{\mu}_{\text{orb}} = -\frac{e}{2m}\vec{L}_{\text{orb}}, \tag{8.15}$$

where the negative sign is needed because the current is defined in terms of the direction of the flow of positive charge.

EXAMPLE 8.3 Orbital Magnetic Moment of the Hydrogen Atom

Assume that the hydrogen atom consists of an electron moving with speed v in a circular orbit with radius r around a stationary proton. Also assume that the centripetal force keeping the electron moving in a circle is the electrostatic force between the proton and the electron. The radius of the orbit of the electron is $r = 5.29 \times 10^{-11}$ m.)

PROBLEM

What is the magnitude of the orbital magnetic moment of the hydrogen atom?

SOLUTION

The magnitude of the orbital magnetic moment is

$$|\mu_{\text{orb}}| = \frac{e}{2m}L_{\text{orb}} = \frac{e}{2m}(rmv) = \frac{erv}{2}. \tag{i}$$

Equating the magnitude of the centripetal force keeping the electron moving in a circle with that of the electrostatic force between the electron and the proton gives us

$$\frac{mv^2}{r} = k\frac{e^2}{r^2},$$

where k is the Coulomb constant. We can solve this equation for the speed of the electron:

$$v = e\sqrt{\frac{k}{mr}}. \tag{ii}$$

Substituting for v from equation (ii) into equation (i) gives us

$$|\mu_{\text{orb}}| = \frac{er}{2}\left(e\sqrt{\frac{k}{mr}}\right) = \frac{e^2}{2}\sqrt{\frac{kr}{m}}.$$

When we put in the various numerical values, we get

$$|\mu_{\text{orb}}| = \frac{\left(1.602\times10^{-19}\,\text{C}\right)^2}{2}\sqrt{\frac{\left(8.99\times10^{9}\,\text{N m}^2/\text{C}^2\right)\left(5.29\times10^{-11}\,\text{m}\right)}{9.11\times10^{-31}\ \text{kg}}} = 9.27\times10^{-24}\,\text{A m}^2.$$

– *Continued*

This result agrees with experimental measurements of the orbital magnetic moment of the hydrogen atom. However, other predictions about the properties of hydrogen and other atoms that are based on the idea that electrons in atoms have circular orbits disagree with experimental observations. Thus, detailed description of the magnetic properties of atoms must incorporate phenomena described by quantum physics.

Spin

The magnetic dipole moment from the orbital motion of electrons is not the only contribution to the magnetic moment of atoms. Electrons and other elementary particles have their own, intrinsic magnetic moments, due to their spin. The phenomenon of spin will be covered thoroughly in the discussion of quantum physics, but some facts about spin and its connection to a particle's intrinsic angular momentum have been discovered experimentally and do not require an understanding of quantum mechanics. Electrons, protons, and neutrons all have a spin of magnitude $s = \frac{1}{2}$. The magnitude of the angular momentum of these particles is $S = \hbar\sqrt{s(s+1)}$, and the z-component of the angular momentum can have a value of either $S_z = -\frac{1}{2}\hbar$ or $S_z = +\frac{1}{2}\hbar$, where $\hbar$ is Planck's constant divided by 2π. This spin cannot be explained by orbital motion of some substructure in the particles. Electrons, for example, are apparently true point particles. Thus, spin is an intrinsic property, similar to mass or electric charge.

The magnetic character of bulk matter is determined largely by electron spin magnetic moments. The magnetic moment of a particle with spin $\vec{\mu}_s$ is related to its spin angular momentum, $\vec{S}$, via

$$\vec{\mu}_s = g\frac{q}{2m}\vec{S}, \tag{8.16}$$

where q is the charge of the particle and m is its mass. The quantity g is dimensionless and is called the *g-factor*. For the electron, its numerical value is $g = -2.0023193043622(15)$, one of the most precisely measured quantities in nature. If you compare this equation to equation 8.15 for the magnetic dipole moment due to the orbital angular momentum, you see that they are very similar.

8.6 Magnetic Properties of Matter

In Chapter 7, we saw that magnetic dipoles do not experience a net force in a homogeneous external magnetic field, but do experience a torque. This torque drives a single free dipole to an orientation in which it is antiparallel to the external field, because this is the state with the lowest magnetic potential energy. We've just seen in Section 8.5 that atoms can have magnetic dipoles. What happens when matter (which is composed of atoms) is exposed to an external magnetic field?

The dipole moments of the atoms in a material can point in different directions or in the same direction. The **magnetization,** $\vec{M}$, of a material is defined as the net dipole moment created by the dipole moments of the atoms in the material per unit volume. The magnetic field, $\vec{B}$, inside the material then depends on the external magnetic field, $\vec{B}_0$, and the magnetization $\vec{M}$:,

$$\vec{B} = \vec{B}_0 + \mu_0\vec{M}, \tag{8.17}$$

where μ_0 is again the magnetic permeability of free space. Instead of including the external magnetic field, $\vec{B}_0$, it is customary to use the **magnetic field strength,** $\vec{H}$:

$$\vec{H} = \frac{\vec{B}_0}{\mu_0}. \tag{8.18}$$

With this definition of the magnetic field strength, equation 8.17 can be written

$$\vec{B} = \mu_0(\vec{H} + \vec{M}). \tag{8.19}$$

Since the unit of magnetic field is $[B] = \mathrm{T}$ and the unit of the magnetic permeability is $[\mu_0] = \mathrm{T\,m/A}$, the units of magnetization and magnetic field strength are $[M] = [H] = \mathrm{A/m}$.

Diamagnetism and Paramagnetism

The question not yet answered is how the magnetization depends on the external magnetic field, $\vec{B}_0$, or, equivalently, on the magnetic field strength, $\vec{H}$. For most materials (not all!) this relationship is linear:

$$\vec{M} = \chi_m \vec{H}, \tag{8.20}$$

where the proportionality constant χ_m is called the **magnetic susceptibility** of the material (Table 8.1). But there are materials that do not obey the simple linear relationship of equation 8.20, and the most prominent among those are ferromagnets, which we'll discuss in the next subsection. Let's first examine diamagnetic and paramagnetic materials, for which equation 8.20 holds.

If $\chi_m < 0$, the dipoles inside the material tend to arrange themselves to oppose an external magnetic field, just like free dipoles. In this case, the magnetization vector points in the direction opposite to the magnetic field strength vector. Materials with $x_m < 0$ are said to be *diamagnetic*. Most materials exhibit **diamagnetism.** In diamagnetic materials, a weak magnetic dipole moment is induced by an external magnetic field in a direction opposite to the direction of the external field. The induced magnetic field disappears when the external field is removed. If the external field is nonuniform, interaction of the induced dipole moment of the diamagnetic material with the external field creates a force directed from a region of greater magnetic field strength to a region of lower magnetic field strength.

An example of biological material exhibiting diamagnetism is shown in Figure 8.28. Diamagnetic forces induced by a nonuniform external magnetic field of 16 T are levitating a live frog. (This experience apparently did not bother the frog.) The normally negligible diamagnetic force is large enough in this case to overcome gravity.

If the magnetic susceptibility in equation 8.20 is greater than zero, $x_m > 0$, the magnetization of the material points in the same direction as the magnetic field strength. Note that x_m for a vacuum is 0. This property is **paramagnetism,** and materials that exhibit it are said to be *paramagnetic*. Materials containing certain transition elements (including actinides and rare earths) exhibit paramagnetism. Each atom of these elements has a permanent magnetic dipole, but normally these dipole moments are randomly oriented and produce no net magnetic field. However, in the presence of an external magnetic field, some of these magnetic dipole moments align in the same direction as the external field. When the external field is removed, the induced magnetic dipole moment disappears. If the external field is nonuniform, this induced magnetic dipole moment interacts with the external field to produce a force directed from a region of lower magnetic field strength to a region of higher magnetic field strength–just the opposite of the effect of diamagnetism.

Substituting the expression for $\vec{M}$ from equation 8.20 into equation 8.19 for the magnetic field, $\vec{B}$, inside a material gives

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(\vec{H} + \chi_m \vec{H}) = \mu_0(1 + \chi_m)\vec{H}. \tag{8.21}$$

In analogy to the relative electric permittivity the **relative magnetic permeability,** κ_m, is commonly defined as

$$\kappa_m = 1 + \chi_m. \tag{8.22}$$

Then, the magnetic permeability, μ, of a material can be expressed as

$$\mu = (1 + \chi_m)\mu_0 = \kappa_m \mu_0. \tag{8.23}$$

Replacing μ_0 with μ in the Biot-Savart Law (equation 8.1) and Ampere's Law (equation 8.10) enables us to use these laws for calculating the magnetic field in a particular material.

Finally for paramagnetic materials, the magnitude of the magnetization is temperature dependent. Conventionally, this temperature dependence is expressed via *Curie's Law:*

$$M = \frac{cB}{T}, \tag{8.24}$$

where c is Curie's constant, B is the magnitude of the magnetic field, and T is the temperature in kelvins.

Table 8.1 Values of Magnetic Susceptibility for Some Common Diamagnetic and Paramagnetic Materials

Material	Magnetic Susceptibility, x_m
Aluminum	$+2.2 \times 10^{-5}$
Bismuth	-1.66×10^{-4}
Diamond (carbon)	-2.1×10^{-5}
Graphite (carbon)	-1.6×10^{-5}
Hydrogen	-2.2×10^{-9}
Lead	-1.8×10^{-5}
Lithium	$+1.4 \times 10^{-5}$
Mercury	-2.9×10^{-5}
Oxygen	$+1.9 \times 10^{-6}$
Platinum	$+2.65 \times 10^{-4}$
Silicon	-3.7×10^{-6}
Sodium	$+7.2 \times 10^{-6}$
Sodium chloride (NaCl)	-1.4×10^{-5}
Tungsten	$+6.8 \times 10^{-5}$
Uranium	$+4.0 \times 10^{-4}$
Vacuum	0
Water	-9×10^{-6}

FIGURE 8.28 A live frog being levitated by a strong magnetic field at the High Field Magnet Laboratory, Radboud University Nijmegen, The Netherlands.

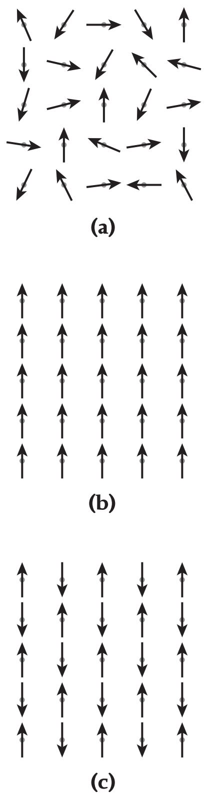


FIGURE 8.29 Magnetic domains: (a) randomly oriented; (b) perfect ferromagnetic order; (c) perfect antiferromagnetic order. (Note that this illustration only shows the directions of individual magnetic domains in an idealized scenario. The magnetic domains also can, and usually have, very different magnitudes.)

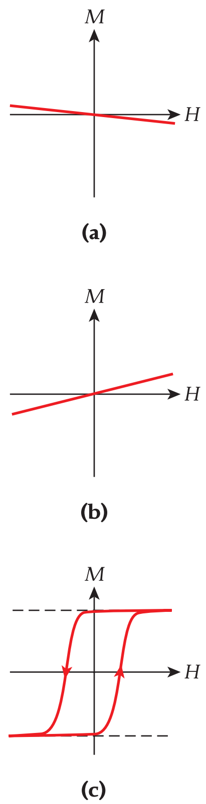


FIGURE 8.30 Magnetization as a function of the magnetic field strength: (a) for diamagnetic materials; (b) for paramagnetic materials; (c) the hysteresis loop for ferromagnetic materials.

Ferromagnetism

The elements iron, nickel, cobalt, gadolinium, and dysprosium–and alloys containing these elements–exhibit **ferromagnetism.** A ferromagnetic material shows long-range ordering at the atomic level, which causes the dipole moments of atoms to line up with each other in a limited region called a **domain.** Within a domain, the magnetic field can be strong. However, in bulk samples of the material, domains are randomly oriented, leaving no net magnetic field. Figure 8.29a shows randomly oriented magnetic dipole moments in a domain, and Figure 8.29b shows perfect ferromagnetic order. Figure 8.29c illustrates the interesting case of perfect antiferromagnetic order, in which the interaction between neighboring magnetic dipole moments causes them to be oriented in opposite directions. This ordering can be realized only at very low temperatures.

An external magnetic field can align domains as shown in Figure 8.29b, as a result of the interaction between the magnetic dipole moments of the domain and the external field. As a result, a ferromagnetic material retains all or some of its induced magnetism when the external magnetic field is removed, since the domains stay aligned. In addition, the magnetic field produced by a current in a solenoid or a toroid will be larger if a ferromagnetic material is present in the device. But in contrast to diamagnetic and paramagnetic materials, ferromagnetic materials do not obey the simple linear relationship given in equation 8.20. The domains retain their orientations, and thus the material exhibits a nonzero magnetization even in the absence of an external magnetic field. (This is why permanent magnets exist.)

Figure 8.30 illustrates the dependence of the magnetization on the magnetic field strength for the three types of materials we've discussed. Figure 8.30a and Figure 8.30b show the linear dependence according to equation 8.20 for diamagnetic and paramagnetic materials, respectively. Figure 8.30c shows the typical hysteresis loop obtained for ferromagnetic materials. The arrows on the red curve show the direction in which the magnetization process develops, and the dashed lines represent the maximum magnetization (positive and negative) possible. For any point on this hysteresis loop, the magnetization can be expressed in terms of an effective value of the magnetic permeability, μ, of the ferromagnetic material, similar to what is given in equation 8.23; however, this permeability is *not* a constant but depends on the applied magnetic field strength and even on the path by which that value of the field strength was attained. Regardless, the values of the effective permeability, μ, for ferromagnetic materials can be much larger than those for paramagnetic materials (greater by a factor of up to 10^4).

Ferromagnetism exhibits temperature dependence. At a certain temperature, called the *Curie temperature*, ferromagnetic materials cease to exhibit ferromagnetism. At this point, the ferromagnetic order due to the interaction of the dipole moments in these materials is overwhelmed by the thermal motion. For iron, the Curie temperature is 768 °C. Figure 8.31 shows a simple demonstration in which heating a permanent ferromagnet (Figure 8.31b)

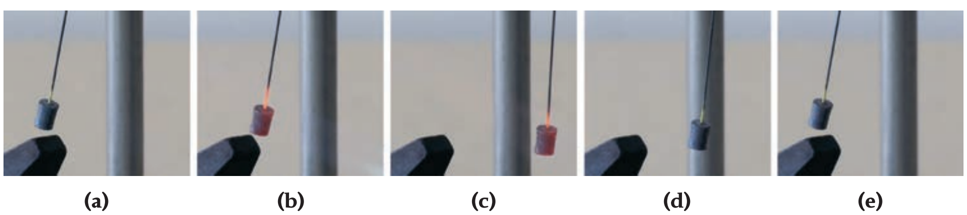


FIGURE 8.31 Demonstration of the temperature dependence of ferromagnetism: (a) A permanent magnet forms the bob of a pendulum and is deflected from the vertical by another magnet (lower left corner of each frame); (b) the magnet is heated and its temperature begins to increase; (c) the temperature of the magnet is high enough that its magnetic field is diminished and it hangs at a much smaller angle; (d) as the magnet cools, it begins to recover its original magnetic field; and (e) it returns to its original position.

diminishes the attraction between it and another magnet (Figure 8.31c). As the magnet subsequently cools (Figure 8.31d), it again becomes a permanent magnet (Figure 8.31e).

8.7 Magnetism and Superconductivity

Magnets for industrial applications and scientific research can be constructed using ordinary resistive wire with current flowing through it. A typical magnet of this type is a large solenoid. The current flowing through the wire of the magnet produces resistive heating, and the heat is usually removed by low-conductivity water flowing through hollow conductors. (Low-conductivity water has been purified so that it does not conduct electricity.) These room-temperature magnets typically produce magnetic fields with strengths up to 1.5 T and are usually relatively inexpensive to construct but expensive to operate because of the high cost of electricity.

Some applications, such as magnetic resonance imaging (MRI), require magnetic fields of the highest possible magnitude to ensure the best signal-to-noise ratio in the measurements. To achieve these fields, magnets are constructed using superconducting coils rather than resistive coils. Such a magnet can produce a stronger field than a room-temperature magnet, with a magnitude of 10 T or higher. Materials such as mercury and lead exhibit superconductivity at liquid helium temperatures, but some metals that are good conductors at room temperature, such as copper and gold, never become superconducting. The disadvantage of a superconducting magnet is that the conductor must be kept at the temperature of liquid helium, which is approximately 4 K (although recent discoveries described later in this section are easing this limitation). Thus, the magnet must be enclosed in a cryostat filled with liquid helium to keep it cold. An advantage of a superconducting magnet is the fact that once the current is established in the coil of the magnet, it will continue to flow until it is removed by external means. However, the energy saving realized by having no resistive loss in the coil is at least partially offset by the expenditure of energy required to keep the superconducting coil cold.

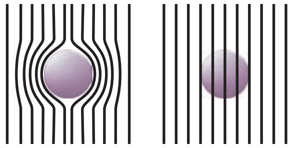


FIGURE 8.32 The Meissner effect, in which a material excludes external magnetic fields from its interior when it is cooled below the critical temperature at which it becomes superconducting.

When current flows through superconducting mercury or lead, the magnetic field inside the material becomes zero. The reduction to zero of the magnetic field inside a material cooled sufficiently that it becomes a superconductor is called the *Meissner effect*. Above the critical temperature, T_c, for the transition to superconductivity, the Meissner effect disappears, and the material becomes a normal conductor (Figure 8.32).

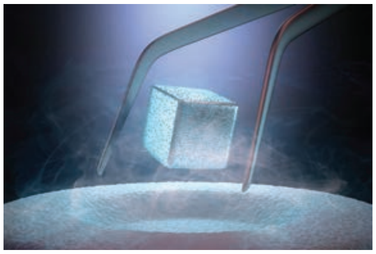


FIGURE 8.33 Through the Meissner effect, a superconductor expels the magnetic field of a permanent magnet, which thus hovers above it.

Figure 8.33 shows an impressive demonstration of the Meissner effect: A piece of a superconductor (cooled to a temperature below its critical temperature) causes a permanent magnet to float above it by expelling the magnet's intrinsic magnetic field. It achieves this because superconducting currents on its surface produce a magnetic field opposed to the applied field, which yields a net field of zero inside the superconductor and repulsion between the fields above the superconductor.

The conductor used in a superconducting magnet is specially designed to overcome the Meissner effect. Modern superconductors are constructed from filaments of niobium-titanium alloy embedded in solid copper. The niobium-titanium filaments have microscopic domains in which a magnetic field can exist without being excluded. The copper serves as a mechanical support and can take over the current load should the superconductor become normally conducting. This type of superconductor can produce magnetic fields with magnitudes as high as 15 T.

FIGURE 8.34 A high voltage current transformer in a power substation.

During the last two decades, physicists and engineers have discovered new materials that are superconducting at temperatures well above 4 K. Critical temperatures of up to 160 K have been reported for these *high-temperature superconductors*, which means that they can be made superconducting by cooling them with liquid nitrogen. Many researchers around the world are looking for materials that are superconducting at room temperature. These materials would revolutionize many areas of industry, in particular, transportation and the power grid. One major future project using superconducting wires for high-current power transmission of up to 5 gigawatts in each cable is the Tres Amigas SuperStation in Clovis, New Mexico. It will connect the two main North American power grids, the Eastern Interconnection and the Western Interconnection, with the Texas Interconnection.

WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- The magnetic permeability of free space, μ_0, is given by $4\pi \times 10^{-7}$ T m/A.
- The Biot-Savart Law, $d\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{i\, d\vec{s} \times \hat{r}}{r^2}$, describes the differential magnetic field, $d\vec{B}$, caused by a current element, $i\, d\vec{s}$, at a position $\vec{r}$ relative to the current element.
- The magnitude of the magnetic field at a distance $r_\perp$ from a long, straight wire carrying a current i is $B = \mu_0 i/2\pi r_\perp$.
- The magnetic field magnitude at the center of a loop with radius R carrying a current i is $B = \mu_0 i/2R$.
- Ampere's Law is given by $\oint \vec{B} \bullet d\vec{s} = \mu_0 i_{\text{enc}}$, where $d\vec{s}$ is the integration path and i_{enc} is the current enclosed in a chosen Amperian loop.
- The magnitude of the magnetic field inside a solenoid carrying a current i and having n turns per unit length is $B = \mu_0 n i$.
- The magnitude of the magnetic field inside a toroid having N turns and a radius r and carrying a current is given by $B = \mu_0 N i/2\pi r$.
- For an electron with charge $-e$ and mass m moving in a circular orbit, the magnetic dipole moment can be related to the orbital angular momentum through $\vec{\mu}_{\text{orb}} = -\dfrac{e}{2m}\vec{L}_{\text{orb}}$.
- For diamagnetic and paramagnetic materials, the magnetization is proportional to the magnetic field strength: $\vec{M} = \chi_{\text{m}}\vec{H}$. Ferromagnetic materials follow a hysteresis loop and thus deviate from this linear relationship.
- The magnetic field inside a diamagnetic or paramagnetic material is due to the external magnetic field strength and the magnetization: $\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(\vec{H} + \chi_{\text{m}}\vec{H}) = \mu_0(1 + \chi_{\text{m}})\vec{H} = \mu_0\kappa_{\text{m}}\vec{H} = \mu\vec{H}$, where k_{m} is the relative magnetic permeability.
- The four right-hand rules related to magnetic fields are shown in Figure 8.35. Right-hand rule 1 gives the direction of the magnetic force on a charged particle moving in a magnetic field. Right-hand rule 2 gives the direction of the unit normal vector for a current-carrying loop. Right-hand rule 3 gives the direction of the magnetic field from a current-carrying wire. Right-hand rule 4 gives the direction of the magnetic field inside a toroidal magnet.

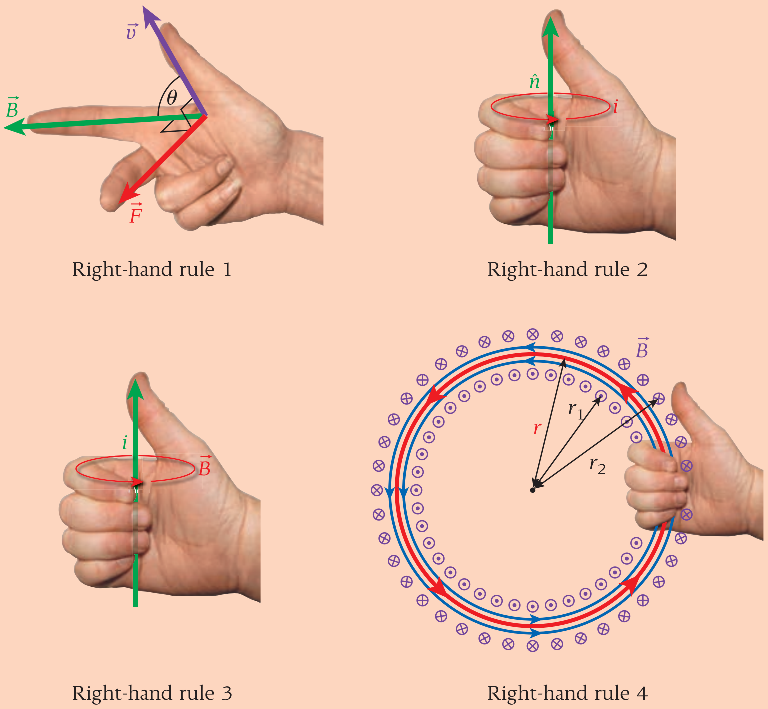


FIGURE 8.35 Four right-hand rules related to magnetic fields.

ANSWERS TO SELF-TEST OPPORTUNITIES

8.1 The magnetic field at point P_1 points in the positive y-direction. The magnetic field at point P_2 points in the negative x-direction.

8.2 Two parallel wires carrying current in the same direction attract each other. Two parallel wires carrying current in opposite directions repel each other.

8.3 $B_x = \dfrac{\mu_0 i}{2}\dfrac{R^2}{\left(0^2 + R^2\right)^{3/2}} = \dfrac{\mu_0 i}{2}\dfrac{R^2}{R^3} = \dfrac{\mu_0 i}{2R}.$

PROBLEM-SOLVING GUIDELINES

1. When using the Biot-Savart Law, you should always draw a diagram of the situation, with the current element highlighted. Check for simplifying symmetries before proceeding with calculations; you can save yourself a significant amount of work.

2. When applying Ampere's Law, choose an Amperian loop that has some geometrical symmetry, in order to simplify the evaluation of the integral. Often, you can use right-hand rule 3 to choose the direction of integration along the loop: Point your thumb in the direction of the net current through the loop and your fingers curl in the direction of integration. This method will also remind you to sum the currents through the Amperian loop to determine the enclosed current.

3. Remember the superposition principle for magnetic fields: The net magnetic field at any point in space is the vector sum of the individual magnetic fields generated by different objects. Make sure you do not simply add the magnitudes. Instead, you generally need to add the spatial components of the different sources of magnetic field separately.

4. All of the principles governing the motion of charged particles in magnetic fields and all of the problem-solving guidelines presented in Chapter 7 still apply. It does not matter if the magnetic field is due to a permanent magnet or an electromagnet.

5. In order to calculate the magnetic field in a material, you can use the formulas derived from Ampere's Law and Biot-Savart's Law, but you have to replace μ_0 with $\mu = (1+\chi_m)\mu_0 = \kappa_m \mu_0$.

MULTIPLE-CHOICE QUESTIONS

8.1 Two long, straight wires are parallel to each other. The wires carry currents of different magnitudes. If the amount of current flowing in each wire is doubled, the magnitude of the force between the wires will be

a) twice the magnitude of the original force.

b) four times the magnitude of the original force.

c) the same as the magnitude of the original force.

d) half of the magnitude of the original force.

8.2 A current element produces a magnetic field in the region surrounding it. At any point in space, the magnetic field produced by this current element points in a direction that is

a) radial from the current element to the point in space.

b) parallel to the current element.

c) perpendicular to the current element and to the radial direction.

8.3 The number of turns in a solenoid is doubled, and its length is halved. How does its magnetic field change?

a) It doubles. c) It quadruples.

b) It is halved. d) It remains unchanged.

8.4 Consider two parallel current-carrying wires. The magnetic fields cause attractive forces between the wires, so it appears that the magnetic field due to one wire is doing work on the other wire. How is this explained?

a) The magnetic force can do no work on isolated charges; this says nothing about the work it can do on charges confined in a conductor.

b) Since only an electric field can do work on charges, it is actually the electric fields doing the work here.

c) This apparent work is due to another type of force.

8.5 In a solenoid in which the wires are wound such that each loop touches the adjacent ones, which of the following will increase the magnetic field inside the magnet?

a) making the radius of the loops smaller

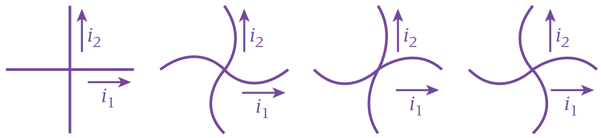
b) increasing the radius of the wire

c) increasing the radius of the solenoid

d) decreasing the radius of the wire

e) immersing the solenoid in gasoline

8.6 Two insulated wires cross at a 90° angle. Currents are sent through the two wires. Which one of the figures best represents the configuration of the wires if the current in the horizontal wire flows in the positive *x*-direction and the current in the vertical wire flows in the positive *y*-direction?



(a) **(b)** **(c)** **(d)**

8.7 What is a good rule of thumb for designing a simple magnetic coil? Specifically, given a circular coil of radius ~1 cm, what is the approximate magnitude of the magnetic field, in gausses per amp per turn? (*Note:* 1 G = 0.0001 T.)

a) 0.0001 G/(A-turn) c) 1 G/(A-turn)

b) 0.01 G/(A-turn) d) 100 G/(A-turn)

8.8 A solid cylinder carries a current that is uniform over its cross section. Where is the magnitude of the magnetic field the greatest?

a) at the center of the cylinder's cross section

b) in the middle of the cylinder

c) at the surface

d) none of the above

8.9 Two long, straight wires have currents flowing in them in the same direction, as shown in the figure. The force between the wires is

a) attractive. b) repulsive. c) zero.

8.10 In a magneto-optic experiment, a liquid sample in a 10-mL spherical vial is placed in a highly uniform magnetic field, and a laser beam is directed through the sample. Which of the following should be used to create the uniform magnetic field required by the experiment?

a) a 5 cm diameter flat coil consisting of one turn of 4-gauge wire

b) a 10 cm-diameter, 20 turn, single-layer, tightly wound coil made of 18-gauge wire

c) a 2cm diameter, 10 cm long, tightly wound solenoid made of 18 gauge wire

d) a set of two coaxial 10 cm diameter coils at a distance of 5 cm apart, each consisting of one turn of 4 gauge wire

8.11 Assume that a lightning bolt can be modeled as a long, straight line of current. If 15.0 C of charge passes by a point in 1.50×10^{-3} s, what is the magnitude of the magnetic field at a distance of 26.0 m from the lightning bolt?

a) 7.69×10^{-5} T c) 4.21×10^{-2} T e) 2.22×10^{2} T
b) 9.22×10^{-3} T d) 1.11×10^{-1} T

8.12 Two solenoids have the same length, but solenoid 1 has 15 times more turns and $\frac{1}{9}$ as large a radius and carries 7 times as much current as solenoid 2. Calculate the ratio of the magnitude of the magnetic field inside solenoid 1 to that of the magnetic field inside solenoid 2.

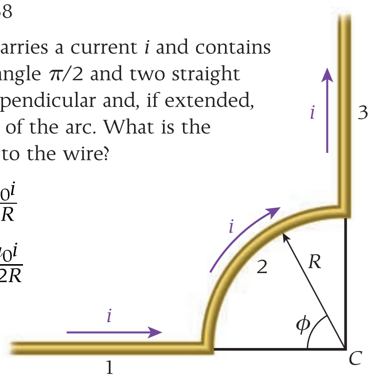
a) 105 c) 144 e) 197
b) 123 d) 168

8.13 The wire in the figure carries a current i and contains a circular arc of radius R and angle $\pi/2$ and two straight sections that are mutually perpendicular and, if extended, would intersect the center, C, of the arc. What is the magnetic field at point C due to the wire?

a) $B = \dfrac{\mu_0 i}{2R}$ d) $B = \dfrac{\mu_0 i}{8R}$

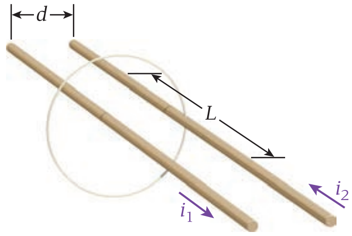
b) $B = \dfrac{\mu_0 i}{4R}$ e) $B = \dfrac{\mu_0 i}{12R}$

c) $B = \dfrac{\mu_0 i}{6R}$



8.14 Wire 1 carries a current i_1, and wire 2 carries a current i_2 in the opposite direction, as shown in the figure. What is the direction of the force exerted by wire 1 on a length L of wire 2?

a) toward wire 1
b) away from wire 1
c) Wire 1 does not exert a force on wire 2 in this situation.



CONCEPTUAL QUESTIONS

8.15 Many electrical applications use twisted-pair cables in which the ground and signal wires spiral about each other. Why?

8.16 Discuss how the accuracy of a compass needle in showing the true direction of north can be affected by the magnetic field due to currents in wires and appliances in a residential building.

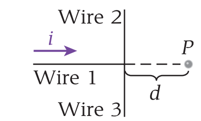
8.17 Can an ideal solenoid, one with no magnetic field outside the solenoid, exist? If not, does that invalidate the derivation of the magnetic field inside the solenoid (Section 8.4)?

8.18 Conservative forces tend to act on objects in such a way as to minimize a system's potential energy. Use this principle to explain the direction of the force on the current-carrying loop described in Example 8.1.

8.19 Two particles, each with charge q and mass m, are traveling in a vacuum on parallel trajectories a distance d apart and at a speed v (much less than the speed of light). Calculate the ratio of the magnitude of the magnetic force that each exerts on the other to the magnitude of the electric force that each exerts on the other: F_m/F_e.

8.20 A long, straight, cylindrical tube of inner radius a and outer radius b carries a total current i uniformly across its cross section. Determine the magnitude of the magnetic field due to the tube at the midpoint between the inner and outer radii.

8.21 Three identical straight wires are connected in a T, as shown in the figure. If current i flows into the junction, what is the magnetic field at point P, a distance d from the junction?



8.22 In a certain region, there is a constant and uniform magnetic field, $\vec{B}$. Any electric field in the region is also unchanging in time. Find the current density, $\vec{J}$, in this region.

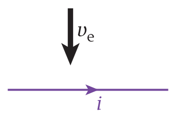
8.23 The magnetic character of bulk matter is determined largely by electron spin magnetic moments, rather than by orbital dipole moments. (Nuclear contributions are negligible, as the proton's spin magnetic moment is about 658 times smaller than that of the electron.) If the atoms or molecules of a substance have unpaired electron spins, the associated magnetic moments give rise to paramagnetism or to ferromagnetism if the interactions between atoms or molecules are strong enough to align them in domains. If the atoms or molecules have no net unpaired spins, then magnetic perturbations of electrons' orbits give rise to diamagnetism.

a) Molecular hydrogen gas (H_2) is weakly diamagnetic. What does this imply about the spins of the two electrons in the hydrogen molecule?

b) What would you expect the magnetic behavior of atomic hydrogen gas (H) to be?

8.24 Exposed to sufficiently high magnetic fields, materials *saturate*, or approach a maximum magnetization. Would you expect the saturation (maximum) magnetization of paramagnetic materials to be much less than, roughly the same as, or much greater than that of ferromagnetic materials? Explain why.

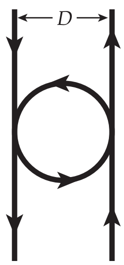
8.25 A long, straight wire carries a current, as shown in the figure. A single electron is shot directly toward the wire from above. The trajectory of the electron and the wire are in the same plane. Will the electron be deflected from its initial path, and if so, in which direction?



8.26 A square loop, with sides of length L, carries current i. Find the magnitude of the magnetic field from the loop at the center of the loop, as a function of i and L.

8.27 A current of constant density, J_0, flows through a very long cylindrical conducting shell with inner radius a and outer radius b. What is the magnetic field in the regions $r < a$, $a < r < b$, and $r > b$? Does $B_{a<r<b} = B_{r>b}$ for $r = b$?

8.28 Parallel wires, a distance D apart, carry a current, i, in opposite directions as shown in the figure. A circular loop, of radius $R = D/2$, has the same current flowing in a counterclockwise direction. Determine the magnitude and the direction of the magnetic field from the loop and the parallel wires at the center of the loop, as a function of i and R.



8.29 The current density in a cylindrical conductor of radius R varies as $J(r) = J_0 e^{-r/R}$ (in the region from zero to R). Express the magnitude of the magnetic field in the regions $r < R$ and $r > R$. Produce a sketch of the radial dependence, $B(r)$.

EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

Sections 8.1 and 8.2

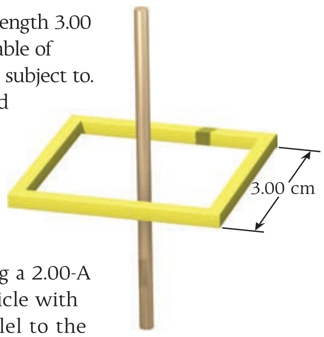
8.30 Two long parallel wires are separated by 3.0 mm. The current flowing in one of the wires is twice that in the other wire. If the magnitude of the force on a 1.0 m length of one of the wires is 7.0 μN, what are the magnitudes of the two currents?

8.31 An electron is shot from an electron gun with a speed of 4.0×10^5 m/s and moves parallel to and at a distance of 5.0 cm above a long, straight wire carrying a current of 15 A. Determine the magnitude and the direction of the acceleration of the electron the instant it leaves the electron gun.

8.32 An electron moves in a straight line at a speed of 5.00×10^6 m/s. What are the magnitude and the direction of the magnetic field created by the moving electron at a distance d = 5.00 m ahead of it on its line of motion? How does the answer change if the moving particle is a proton?

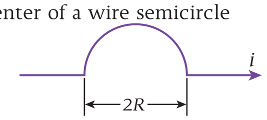
8.33 Suppose that the magnetic field of the Earth were due to a single current moving in a circle of radius 2.00×10^3 km through the Earth's molten core. The strength of the Earth's magnetic field on the surface near a magnetic pole is about 6.00×10^{-5} T. About how large a current would be required to produce such a field?

8.34 A square ammeter has sides of length 3.00 cm. The sides of the ammeter are capable of measuring the magnetic field they are subject to. When the ammeter is clamped around a wire carrying a direct current, as shown in the figure, the average value of the magnetic field measured in the sides is 3.00 G. What is the current in the wire?

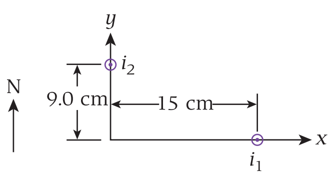


•8.35 A long, straight wire carrying a 2.00-A current lies along the x-axis. A particle with charge $q = -3.00$ μC moves parallel to the y-axis through the point $(x,y,z) = (0,2,0)$. Where in the xy-plane should another long, straight wire be placed so that there is no magnetic force on the particle at the point where it crosses the plane?

•8.36 Find the magnetic field in the center of a wire semicircle like that shown in the figure, with radius R = 10.0 cm, if the current in the wire is i = 12.0 A.



•8.37 Two very long wires run parallel to the z-axis, as shown in the figure. They each carry a current, $i_1 = i_2 = 25.0$ A, flowing in the direction of the positive z-axis. The magnetic field of the Earth is given by $\vec{B} = (2.60 \times 10^{-5})\hat{y}$ T (in the xy-plane and pointing due north). A magnetic compass needle is placed at the origin. Determine the angle θ between the compass needle and the x-axis. (*Hint:* The compass needle will align its axis along the direction of the net magnetic field.)

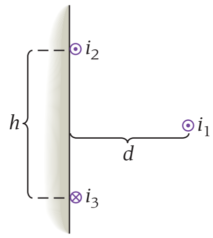


•8.38 Two identical coaxial coils of wire of radius 20.0 cm are directly on top of each other, separated by a 2.00 mm gap. The lower coil is on a flat table and has a current i in the clockwise direction; the upper coil carries an identical current and has a mass of 0.0500 kg. Determine the magnitude and the direction that the current in the upper coil has to have to keep it levitated at the distance 2.00 mm above the lower coil.

•8.39 A long, straight wire lying along the x-axis carries a current, i, flowing in the positive x-direction. A second long, straight wire lies along the y-axis and has a current i in the positive y-direction. What are the magnitude and the direction of the magnetic field at point $z = b$ on the z-axis?

•8.40 A square loop of wire with a side length of 10.0 cm carries a current of 0.300 A. What is the magnetic field in the center of the square loop?

•8.41 The figure shows the cross section through three long wires with a linear mass distribution of 100. g/m. They carry currents i_1, i_2, and i_3 in the directions shown. Wires 2 and 3 are 10.0 cm apart and are attached to a vertical surface, and each carries a current of 600. A. What current, i_1, will allow wire 1 to "float" at a perpendicular distance d = 10.0 cm from the vertical surface? (Neglect the thickness of the wires.)



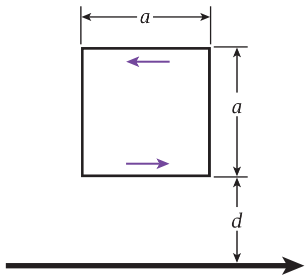
•8.42 A hairpin configuration is formed of two semi-infinite straight wires that are 2.00 cm apart and joined by a semicircular piece of wire (whose radius must be 1.00 cm and whose center is at the origin of xyz-coordinates). The top wire lies along the line y = 1.00 cm, and the bottom wire lies along the line y = −1.00 cm; these two wires are in the left side ($x < 0$) of the xy-plane. The current in the hairpin is 3.00 A, and it is directed toward the right in the top wire, clockwise around the semicircle, and to the left in the bottom wire. Find the magnetic field at the origin of the coordinate system.

•8.43 A long, straight wire is located along the x-axis ($y = 0$ and $z = 0$). The wire carries a current of 7.00 A in the positive x-direction. What are the magnitude and the direction of the force on a particle with a charge of 9.00 C located at (+1.00 m,+2.00 m,0), when it has a velocity of 3000. m/s in each of the following directions?

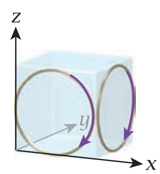
a) the positive x-direction
b) the positive y-direction
c) the negative z-direction

•8.44 A long, straight wire has a 10.0-A current flowing in the positive x-direction, as shown in the figure. Close to the wire is a square loop of copper wire that carries a 2.00 A current in the direction shown. The near side of the loop is d = 0.500 m away from the wire. The length of each side of the square is a = 1.00 m.

a) Find the net force between the two current-carrying objects.
b) Find the net torque on the loop.



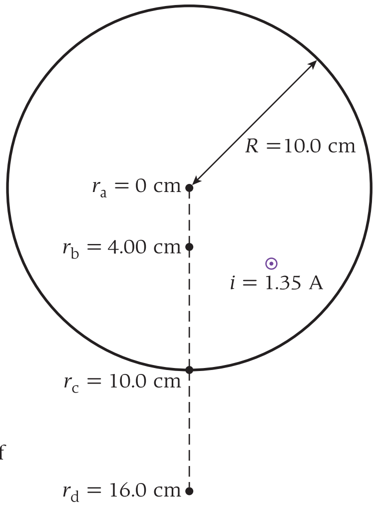
8.45 A square box with sides of length 1.00 m has one corner at the origin of a coordinate system, as shown in the figure. Two coils are attached to the outside of the box. One coil is on the box face that is in the xz-plane at $y = 0$, and the second is on the box face in the yz-plane at x = 1.00 m. Each of the coils has a diameter of 1.00 m and contains 30.0 turns of wire carrying a current of 5.00 A in each turn. The current in each coil is clockwise when the coil is viewed from outside the box. What are the magnitude and the direction of the magnetic field at the center of the box?



Section 8.3

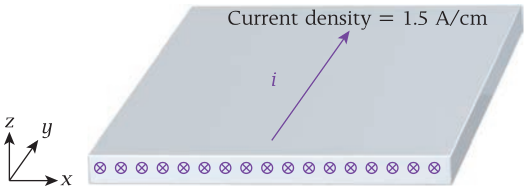
8.46 The current density in a cylindrical conductor of radius R varies as $J(r) = J_0 r/R$ (in the region from zero to R). Express the magnitude of the magnetic field in the regions $r < R$ and $r > R$. Produce a sketch of the radial dependence, $B(r)$.

8.47 The figure shows a cross section across the diameter of a long, solid, cylindrical conductor. The radius of the cylinder is $R = 10.0$ cm. A current of 1.35 A is uniformly distributed throughout the conductor and is flowing out of the page. Calculate the direction and the magnitude of the magnetic field at positions $r_a = 0.0$ cm, $r_b = 4.00$ cm, $r_c = 10.0$ cm, and $r_d = 16.0$ cm.



••8.48 A coaxial wire consists of a copper core of radius 1.00 mm surrounded by a copper sheath of inside radius 1.50 mm and outside radius 2.00 mm. A current, *i*, flows in one direction in the core and in the opposite direction in the sheath. Graph the magnitude of the magnetic field as a function of the distance from the center of the wire.

•8.49 A very large sheet of conducting material located in the *xy*-plane, as shown in the figure, has a uniform current flowing in the *y*-direction. The current density is 1.5 A/cm. Use Ampere's Law to calculate the direction and the magnitude of the magnetic field just above the center of the sheet (not close to any edges).



Section 8.4

8.50 A current of 2.00 A is flowing through a 1000 turn solenoid of length $L = 40.0$ cm. What is the magnitude of the magnetic field inside the solenoid?

8.51 Solenoid A has twice the diameter, three times the length, and four times the number of turns of solenoid B. The two solenoids have currents of equal magnitudes flowing through them. Find the ratio of the magnitude of the magnetic field in the interior of solenoid A to that of solenoid B.

8.52 A long solenoid (diameter of 6.00 cm) is wound with 1000 turns per meter of thin wire through which a current of 0.250 A is maintained. A wire carrying a current of 10.0 A is inserted along the axis of the solenoid. What is the magnitude of the magnetic field at a point 1.00 cm from the axis?

8.53 A long, straight wire carries a current of 2.5 A.

a) What is the strength of the magnetic field at a distance of 3.9 cm from the wire?

b) If the wire still carries 2.5 A, but is used to form a long solenoid with 32 turns per centimeter and a radius of 3.9 cm, what is the strength of the magnetic field inside the solenoid?

8.54 Figure 8.20a shows a Helmholtz coil used to generate uniform magnetic fields. Suppose the Helmholtz coil consists of two sets of coaxial wire loops with 15 turns of radius $R = 75.0$ cm, which are separated by R, and each coil carries a current of 0.123 A flowing in the same direction. Calculate the magnitude and the direction of the magnetic field in the center between the coils.

•8.55 A particle detector utilizes a solenoid that has 550 turns of wire per centimeter. The wire carries a current of 22 A. A cylindrical detector that lies within the solenoid has an inner radius of 0.80 m. Electron and positron beams are directed into the solenoid parallel to its axis. What is the minimum momentum perpendicular to the solenoid axis that a particle can have if it is to be able to enter the detector?

Sections 8.5 through 8.7

8.56 An electron has a spin magnetic moment of magnitude $\mu = 9.285 \times 10^{-24}$ A m^2. Consequently, it has energy associated with its orientation in a magnetic field. If the difference between the energy of an electron that is "spin up" in a magnetic field of magnitude B and the energy of one that is "spin down" in the same magnetic field (where "up" and "down" refer to the direction of the magnetic field) is 9.460×10^{-25} J, what is the field magnitude, B?

8.57 When a magnetic dipole is placed in a magnetic field, it has a natural tendency to minimize its potential energy by aligning itself with the field. If there is sufficient thermal energy present, however, the dipole may rotate so that it is no longer aligned with the field. Using $k_B T$ as a measure of the thermal energy, where k_B is Boltzmann's constant and T is the temperature in kelvins, determine the temperature at which there is sufficient thermal energy to rotate the magnetic dipole associated with a hydrogen atom from an orientation parallel to an applied magnetic field to one that is antiparallel to the applied field. Assume that the strength of the field is 0.15 T.

8.58 Aluminum becomes superconducting at a temperature around 1.0 K if exposed to a magnetic field of magnitude less than 0.0105 T. Determine the maximum current that can flow in an aluminum superconducting wire with radius $R = 1.0$ mm.

8.59 If you want to construct an electromagnet by running a current of 3.00 A through a solenoid with 500. windings and length 3.50 cm and you want the magnetic field inside the solenoid to have the magnitude $B = 2.96$ T, you can insert a ferrite core into the solenoid. What value of the relative magnetic permeability should this ferrite core have in order to make this work?

8.60 What is the magnitude of the magnetic field inside a long, straight tungsten wire of circular cross section with diameter 2.4 mm and carrying a current of 3.5 A, at a distance of 0.60 mm from its central axis?

•8.61 You charge up a small rubber ball of mass 200. g by rubbing it over your hair. The ball acquires a charge of 2.00 μC. You then tie a 1.00m long string to it and swing it in a horizontal circle, providing a centripetal force of 25.0 N. What is the magnetic moment of the system?

•8.62 Consider a model of the hydrogen atom in which an electron orbits a proton in the plane perpendicular to the proton's spin angular momentum (and magnetic dipole moment) at a distance equal to the Bohr radius, $a_0 = 5.292 \times 10^{-11}$ m. (This is an oversimplified classical model.) The spin of the electron is allowed to be either parallel to the proton's spin or antiparallel to it; the orbit is the same in either case. But since the proton produces a magnetic field at the electron's location, and the electron has its own intrinsic magnetic dipole moment, the energy of the electron differs depending on its spin. Calculate the energy difference between the two electron-spin configurations. Consider only the interaction between the magnetic dipole moment associated with the electron's spin and the field produced by the proton's spin.

••8.63 Consider an electron to be a uniformly dense sphere of charge, with a total charge of $-e = -1.602 \times 10^{-19}$ C, spinning at an angular frequency, ω.

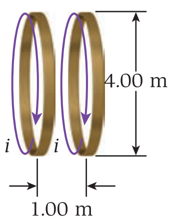
a) Write an expression for its classical angular momentum of rotation, L.

b) Write an expression for its magnetic dipole moment, μ.

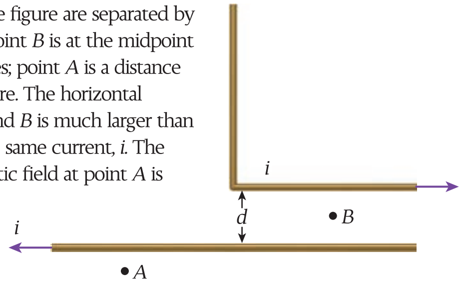
c) Find the ratio, $\gamma_e = \mu/L$, known as the *gyromagnetic ratio*.

Additional Exercises

8.64 Two 50 turn coils, each with a diameter of 4.00 m, are placed 1.00 m apart, as shown in the figure. A current of 7.00 A is flowing in the wires of both coils; the direction of the current is clockwise for both coils when viewed from the left. What is the magnitude of the magnetic field in the center between the two coils?



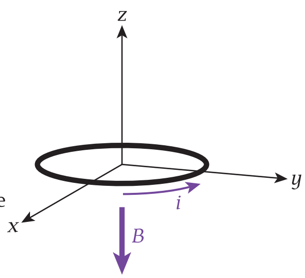
8.65 The wires in the figure are separated by a vertical distance d. Point B is at the midpoint between the two wires; point A is a distance $d/2$ from the lower wire. The horizontal distance between A and B is much larger than d. Both wires carry the same current, i. The strength of the magnetic field at point A is 2.00 mT. What is the strength of the field at point B?



8.66 You are standing at a spot where the magnetic field of the Earth is horizontal, points due northward, and has magnitude 40.0 μT. Directly above your head, at a height of 12.0 m, a long, horizontal cable carries a steady direct current of 500. A due northward. Calculate the angle θ by which your magnetic compass needle is deflected from true magnetic north by the effect of the cable. Don't forget the *sign* of θ–is the deflection eastward or westward?

8.67 The magnetic dipole moment of the Earth is approximately 8.0×10^{22} A m^2. The source of the Earth's magnetic field is not known; one possibility might be the circulation of ions in the Earth's molten outer core. Assume that the circulating ions move a circular loop of radius 2500 km. What "current" must they produce to yield the observed field?

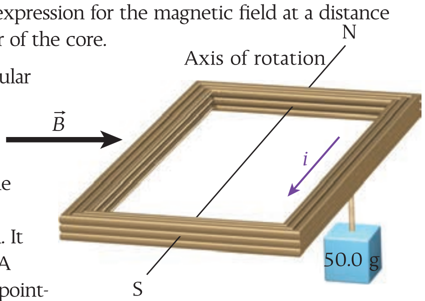
8.68 A circular wire loop has a radius $R = 0.12$ m and carries a current $i = 0.10$ A. The loop is placed in the xy-plane in a uniform magnetic field given by $\vec{B} = -1.5\hat{z}$ T, as shown in the figure. Determine the direction and the magnitude of the loop's magnetic moment and calculate the potential energy of the loop in the position shown. If the wire loop can move freely, how will it orient itself to minimize its potential energy, and what is the value of the lowest potential energy?



8.69 A 0.90 m long solenoid has a radius of 5.0 mm. When the wire carries a 0.20 A current, the magnetic field inside the solenoid is 5.0 mT. How many turns of wire are there in the solenoid?

8.70 In a coaxial cable, the solid core carries a current i. The sheath also carries a current i but in the opposite direction, and it has an inner radius a and an outer radius b. The current density is equally distributed over each conductor. Find an expression for the magnetic field at a distance $a < r < b$ from the center of the core.

•**8.71** A 50 turn rectangular coil of wire with dimensions 10.0 cm by 20.0 cm lies in a horizontal plane, as shown in the figure. The axis of rotation of the coil is aligned north and south. It carries a current $i = 1.00$ A and is in a magnetic field pointing from west to east. A mass of 50.0 g hangs from one side of the coil. Determine the strength the magnetic field has to have to keep the coil in the horizontal orientation.



•**8.72** Two long, straight parallel wires are separated by a distance of 20.0 cm. Each wire carries a current of 10.0 A in the same direction. What is the magnitude of the resulting magnetic field at a point that is 12.0 cm from each wire?

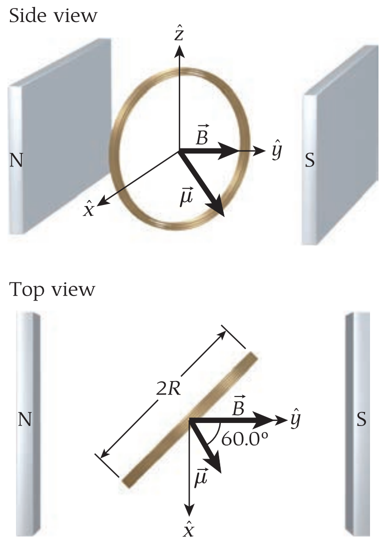
•**8.73** A particle with a mass of 1.00 mg and a charge q is moving at a speed of 1000. m/s along a horizontal path 10.0 cm below and parallel to a straight current-carrying wire. Determine q if the magnitude of the current in the wire is 10.0 A.

•**8.74** A conducting coil consisting of n turns of wire is placed in a uniform magnetic field given by $\vec{B} = 2.00\hat{y}$ T, as shown in the figure. The radius of the coil is $R = 5.00$ cm, and the angle between the magnetic field vector and the unit normal vector to the coil is $\theta = 60.0°$. The current through the coil is $i = 5.00$ A.

a) Specify the direction of the current in the coil, given the direction of the magnetic dipole moment, $\vec{\mu}$, shown in the figure.

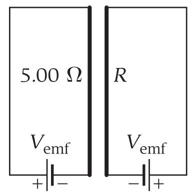
b) Calculate the number of turns, n, the coil must have for the torque on the loop to be 3.40 N m.

c) If the radius of the loop is decreased to $R = 2.50$ cm, what should the number of turns, N, be for the torque to remain unchanged? Assume that i, B, and θ stay the same.



•**8.75** A loop of wire of radius $R = 25.0$ cm has a smaller loop of radius $r = 0.900$ cm at its center, with the planes of the two loops perpendicular to each other. When a current of 14.0 A is passed through both loops, the smaller loop experiences a torque due to the magnetic field produced by the larger loop. Determine this torque, assuming that the smaller loop is sufficiently small that the magnetic field due to the larger loop is the same across its entire surface.

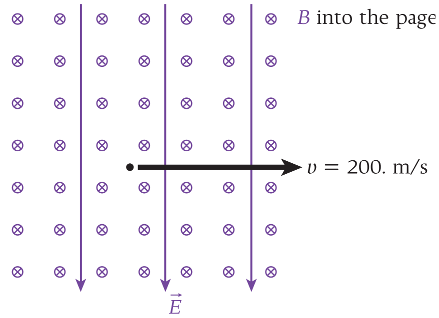
•**8.76** Two wires, each 25.0 cm long, are connected to two separate 9.00-V batteries, as shown in the figure. The resistance of the first wire is 5.00 V, and that of the other wire is unknown (R). If the separation between the wires is 4.00 mm, what value of R will produce a force of magnitude 4.00×10^{-5} N between them? Is the force attractive or repulsive?



•**8.77** A proton is moving under the combined influence of an electric field ($E = 1000.$ V/m) and a magnetic field ($B = 1.20$ T), as shown in the figure.

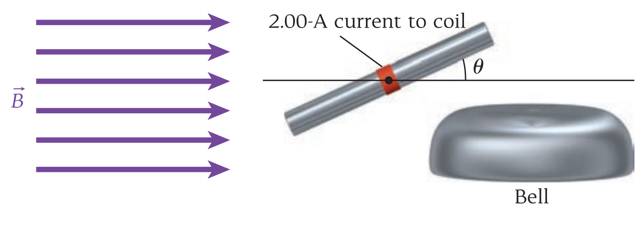
a) What is the acceleration of the proton at the instant it enters the crossed fields?

b) What would the acceleration be if the direction of the proton's motion were reversed?



•**8.78** A toy airplane of mass 0.175 kg, with a charge of 36 mC, is flying with a speed of 2.8 m/s at a height of 17.2 cm above and parallel to a wire, which is carrying a 25-A current. The airplane experiences some acceleration. Determine this acceleration.

•**8.79** An electromagnetic doorbell has been constructed by wrapping 70 turns of wire around a long, thin rod, as shown in the figure. The rod has a mass of 30.0 g, a length of 8.00 cm, and a cross-sectional area of 0.200 cm^2. The rod is free to pivot about an axis through its center, which is also



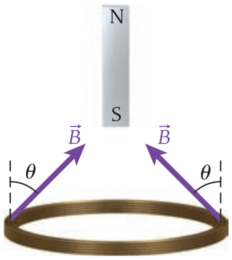
the center of the coil. Initially, the rod makes an angle of $\theta = 25.0°$ with the horizontal. When $\theta = 0.00°$, the rod strikes a bell. A uniform magnetic field of 900. G is directed at an angle $\theta = 0.00°$.

a) If a current of 2.00 A is flowing in the coil, what is the torque on the rod when $\theta = 25.0°$?

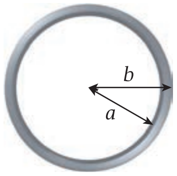
b) What is the angular velocity of the rod when it strikes the bell?

•8.80 Two long, parallel wires separated by a distance d carry currents in opposite directions. If the left-hand wire carries a current $i/2$ and the right-hand wire carries a current i, determine where the magnetic field is zero.

•8.81 A horizontally oriented coil of wire of radius 5.00 cm that carries a current, i, is being levitated by the south pole of a vertically oriented bar magnet suspended above its center, as shown in the figure. If the magnetic field on all parts of the coil makes an angle $\theta = 45.0°$ with the vertical, determine the magnitude and the direction of the current needed to keep the coil floating in midair. The magnitude of the magnetic field is $B = 0.0100$ T, the number of turns in the coil is $N = 10.0$, and the total coil mass is 10.0 g.



•8.82 As shown in the figure, a long, hollow, conducting cylinder of inner radius a and outer radius b carries a current that is flowing out of the page. Suppose that $a = 5.00$ cm, $b = 7.00$ cm, and the current $i = 100.$ mA, uniformly distributed over the cylinder wall (between a and b). Find the magnitude of the magnetic field at each of the following distances r from the center of the cylinder:



a) $r = 4.00$ cm

b) $r = 6.50$ cm

c) $r = 9.00$ cm

•8.83 A wire of radius R carries a current i. The current density is given by $J = J_0(1 - r/R)$, where r is measured from the center of the wire and J_0 is a constant. Use Ampere's Law to find the magnetic field inside the wire at a distance $r < R$ from the central axis.

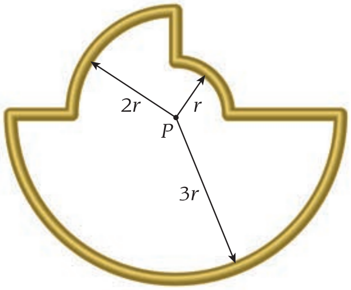
•8.84 A circular wire of radius 5.0 cm has a current of 3.0 A flowing in it. The wire is placed in a uniform magnetic field of 5.0 mT.

a) Determine the maximum torque on the wire.

b) Determine the range of the magnetic potential energy of the wire.

MULTI-VERSION EXERCISES

8.85 The loop shown in the figure is carrying a current of 3.857 A, and the distance $r = 1.411$ m. What is the magnitude of the magnetic field at point P inside the loop?



8.86 The loop shown in the figure is carrying a current of 3.961 A. The magnitude of the magnetic field at point P inside the loop is 7.213×10^{-7} T. What is the value of r?

8.87 In the current-carrying loop shown in the figure, the distance $r = 2.329$ m. The magnitude of the magnetic field at point P inside the loop is 5.937×10^{-7} T. What is the current in the loop?

8.88 A toroidal magnet has an inner radius of 1.895 m and an outer radius of 2.075 m. When the wire carries a 33.45 A current, the magnetic field at a distance of 1.985 m from the center of the toroid is 66.78 mT. How many turns of wire are there in the toroid?

8.89 A toroidal magnet has an inner radius of 1.121 m and an outer radius of 1.311 m. The magnetic field at a distance of 1.216 m from the center of the toroid is 78.30 mT. There are 22,381 turns of wire in the toroid. What is the current in the toroid?

8.90 A toroidal magnet has an inner radius of 1.351 m and an outer radius of 1.541 m. The wire carries a 49.13 A current, and there are 24,945 turns in the toroid. What is the magnetic field at a distance of 1.446 m from the center of the toroid?

9 Electromagnetic Induction

FIGURE 9.1 The Grand Coulee Dam on the Columbia River in the state of Washington is the largest single producer of electricity in the United States. Shown here are the giant generators, which apply the physical principle of induction to produce electricity.

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Most of us take electric power for granted—we flip a switch and we have power for lighting, heating, and entertainment. But the vast network that supplies this power—called *the grid*—depends on large generators that convert mechanical energy into electrical energy (Figure 9.1). The physical principles that enable this conversion are the subject of this chapter.

In Chapter 7, we saw that a magnetic field can affect the path of charged particles, or electric currents, and in Chapter 8, we saw that an electric current generates a magnetic field. In this chapter, we'll see that a changing magnetic field generates an electric current, and thus an electric field. Note the word "changing" here; just as a magnetic field is generated only when electrical charges are in motion, an electric field is generated only when a magnetic field is in motion (relative to a conductor) or otherwise changes as a function of time. This symmetry will turn out to be a key part of the unified description of electricity and magnetism presented in Chapter 11.

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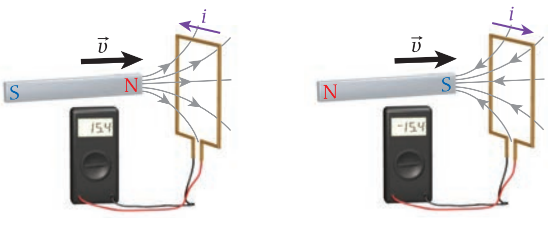
WHAT WE WILL LEARN

- A changing magnetic field inside a conducting loop induces a current in the loop.
- A changing current in a loop induces a current in a nearby loop.
- Faraday's Law of Induction states that a potential difference is induced in a loop when there is a change in the magnetic flux through the loop.
- Magnetic flux is the product of the magnitude of the average magnetic field and the perpendicular area that it penetrates.
- Lenz's Law states that the current induced in a loop by a changing magnetic flux produces a magnetic field that opposes this change in magnetic flux.
- A changing magnetic field induces an electric field.
- The inductance of a device is a measure of its opposition to changes in current flowing through it.
- Electric motors and electric generators are everyday applications of magnetic induction.
- A simple single-loop circuit with an inductor and a resistor has a characteristic time constant given by the inductance divided by the resistance.
- Energy is stored in a magnetic field.

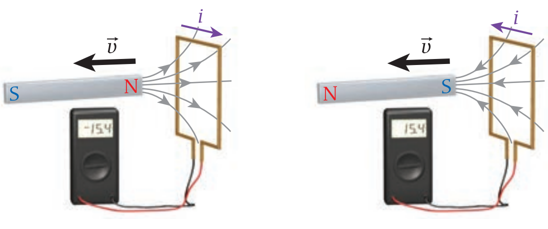
9.1 Faraday's Experiments

Some of the great discoveries about electricity and magnetism took place in the late 18th and early 19th centuries. In 1750, the American Benjamin Franklin showed that lightning is a form of electricity, with his famous kite-flying experiment. (Perhaps the most amazing aspect of that experiment was that he was not killed by the lightning strike.) In 1799, the Italian Alessandro Volta constructed the first battery, called a *voltaic pile* at the time. In 1820, the Danish physicist Hans C. Oersted demonstrated that an electric current could produce a magnetic field strong enough to deflect a compass needle. (He performed his experiment during a lecture to his students, making it one of the most productive lecture demonstrations in the history of science.)



(a) (b)

FIGURE 9.2 Moving a magnet toward a wire loop induces a current to flow in the loop. (a) With the north pole of the magnet pointing toward the loop, a positive current results. (b) With the south pole of the magnet pointing toward the loop, a negative current results.



(a) (b)

FIGURE 9.3 Moving a magnet away from a wire loop also induces a current to flow in the loop. (a) With the north pole of the magnet pointing toward the loop, a negative current results. (b) With the south pole of the magnet pointing toward the loop, a positive current results.

However, the experiments that are most relevant to this chapter were performed in the 1830s by the British chemist and physicist Michael Faraday and independently by the American physicist Joseph Henry. Their work demonstrated that a changing magnetic field could generate a potential difference in a conductor, strong enough to produce an electric current. This discovery is of basic importance to all the electrical and magnetic devices we use every day, from computers to cell phones, from televisions to credit cards, from the tiniest batteries to the largest electrical power grids. Both Faraday and Henry had fundamental electrical units named after them, and justifiably so.

To understand Faraday's experiments, consider a wire loop connected to an ammeter. A bar magnet is some distance from the loop with its north pole pointing toward the loop. While the magnet is stationary, no current flows in the loop. However, if the magnet is moved toward the loop (Figure 9.2a), a counterclockwise current flows in the loop as indicated by the positive current in the ammeter. If the magnet is moved toward the loop faster, a larger current is induced in the loop. If the magnet is reversed, so the south pole points toward the loop (Figure 9.2b), and moved toward the loop, current flows in the loop in the opposite direction. If the north pole of the magnet points toward the loop, and the magnet is then moved *away* from the loop (Figure 9.3a), a negative clockwise current, as indicated on the meter in Figure 9.3a, is induced in the loop. If the south pole of the magnet points toward the loop, and the magnet is moved away from the loop (Figure 9.3b), a positive current is induced.

The four results illustrated in Figures 9.2 and 9.3 can be replicated by holding the magnets stationary and moving the loops. For example, with the arrangement shown in Figure 9.2a, if the loop is moved toward the stationary magnet, a positive current flows in the loop.

Similar effects can be observed using two conducting loops (Figure 9.4). If a constant current is flowing through loop 1, no current is induced in loop 2. If the current in loop 1 is increased, a current is induced in loop 2 in the opposite direction. Thus, not only does the increasing current in the first loop induce a current in the second loop, but the induced current is in the opposite direction. Furthermore, if current is flowing in loop 1 in the same direction as before and is then *decreased* (Figure 9.5), the current induced in loop 2 flows in the *same* direction as the current in loop 1.

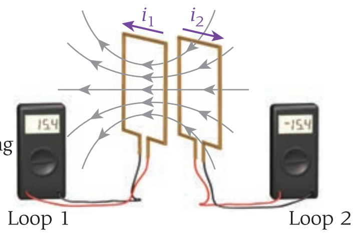


FIGURE 9.4 An increasing current in loop 1 induces a current in the opposite direction in loop 2. (The magnetic field lines shown are those produced by the current 1 flowing through loop 1.)

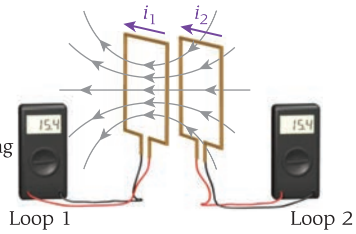
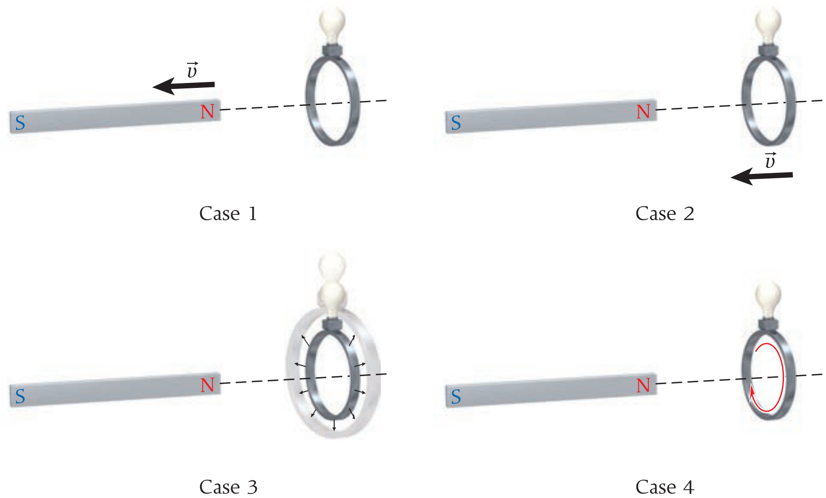


FIGURE 9.5 A decreasing current in loop 1 induces a current in the same direction in loop 2.

All of the phenomena illustrated in these four figures can be explained by Faraday's Law of Induction, discussed in Section 9.2, and by Lenz's Law, discussed in Section 9.3.

Concept Check 9.1

The four figures show a bar magnet and a low-voltage light bulb connected to the ends of a conducting loop. The plane of the loop is perpendicular to the dashed line. In case 1, the loop is stationary, and the magnet is moving away from the loop. In case 2, the magnet is stationary, and the loop is moving toward the magnet. In case 3, both the magnet and the loop are stationary, but the area of the loop is increasing. In case 4, the magnet is stationary, and the loop is rotating about its center. In which of these situations will the bulb light up?



a) case 1
b) cases 1 and 2
c) cases 1, 2, and 3
d) cases 1, 2, and 4
e) all four cases

9.2 Faraday's Law of Induction

From the observations in the preceding section, we see that a changing magnetic field through a loop induces a current in the loop. We can visualize the change in magnetic field as a change in the number of magnetic field lines passing through the loop. **Faraday's Law of Induction** in its qualitative form states:

> A potential difference is induced in a loop when the number of magnetic field lines passing through the loop changes with time.

The rate of change of the magnetic field lines determines the induced potential difference. The existence of this potential difference means that the changing magnetic field actually creates an electric field around the loop! Thus, there are two ways of producing an electric field: from electric charges and from a changing magnetic field. If the electric field arises from a charge, the resulting electric force on a test charge is conservative. Conservative forces do no work when they act on an object whose path starts and ends at the same point in space. In contrast, electric fields generated by changing magnetic fields give rise to electric forces that are *not* conservative. Thus, a test particle that moves around a circular loop once will have work done on it by this electric field. In fact, the amount of the work done is the induced potential difference times the charge of the test particle.

The magnetic field lines are quantified by the magnetic flux, in analogy with the electric flux. Chapter 2 introduced Gauss's Law for electric fields and defined the electric flux as the surface integral of an electric field passing through a differential element of area, dA. Mathematically, $\Phi_E = \iint \vec{E} \bullet d\vec{A}$, where $d\vec{A}$ is a vector of magnitude dA that is perpendicular to the differential area. By analogy, for a magnetic field, **magnetic flux** is defined as the surface integral of the magnetic field passing through a differential element of area:

$$\Phi_B = \iint \vec{B} \bullet d\vec{A}, \tag{9.1}$$

where $\vec{B}$ is the magnetic field at each differential area element, $d\vec{A}$, of a closed surface. The loop in the symbol for a surface integral means that the integration is over a closed surface. The two integrals signify integration over two variables. The differential area element, $d\vec{A}$, must be described by two spatial variables, such as x and y in Cartesian coordinates or θ and ϕ in spherical coordinates. With a closed surface, the differential area vector, $d\vec{A}$, always points out of the enclosed volume and is perpendicular to the surface everywhere.

Integration of the electric flux over a closed surface (see Chapter 2) yields Gauss's Law: $\oint\!\!\oint \vec{E} \bullet d\vec{A} = q/\varepsilon_0$. That is, the integral of the electric flux over a closed surface is equal to the enclosed electric charge, q, divided by the electric permittivity of free space, ε_0. Integration of the magnetic flux over a *closed* surface yields zero:

$$\oint\!\!\oint \vec{B} \bullet d\vec{A} = 0. \tag{9.2}$$

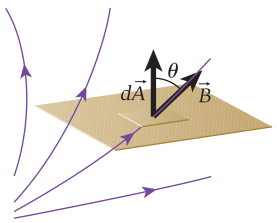


FIGURE 9.6 A nonuniform magnetic field $\vec{B}$ passing through a differential area, $d\vec{A}$.

This result is often termed **Gauss's Law for Magnetic Fields.** You might think that the integral of the magnetic flux over a closed surface would be equal to the enclosed "magnetic charge" divided by the magnetic permeability of free space. However, there are no free magnetic charges, no magnetic monopoles, no separate north poles or separate south poles. Magnetic poles are always found in pairs. Thus, Gauss's Law for Magnetic Fields is another way of stating that magnetic monopoles do not exist. (Extensive searches for magnetic monopoles have been conducted since the 1980s, but not successfully. However, several string theories and grand unified theories predict that magnetic monopoles exist.) Another way to state Gauss's Law for Magnetic Fields is that magnetic field lines have no beginning or end but form a continuous loop.

Figure 9.6 shows a nonuniform magnetic field, $\vec{B}$, passing through a differential area element, $d\vec{A}$. A portion of the closed surface is also shown. The angle between the magnetic field and the differential area vector is θ.

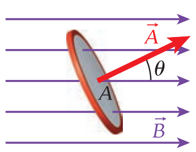


FIGURE 9.7 A flat loop of area A in a constant magnetic field, $\vec{B}$. The magnetic field makes an angle θ with respect to the surface normal vector of the loop.

Consider the special case of a flat loop of area A in a constant magnetic field, as illustrated in Figure 9.7. For this case, we can rewrite equation 9.1 as

$$\Phi_B = BA\cos\theta, \tag{9.3}$$

where B is the magnitude of the constant magnetic field, A is the area of the loop, and θ is the angle between the surface normal vector to the plane of the loop and the magnetic field lines. Thus, if the magnetic field is perpendicular to the plane of the loop, $\theta = 0°$ and $\Phi_B = BA$. If the magnetic field is parallel to the plane of the loop, $\theta = 90°$ and $\Phi_B = 0$.

The unit of magnetic flux is $[\Phi_B] = [B][A] = \mathrm{T\ m}^2$. This unit has received a special name, the **weber** (Wb):

$$1\ \mathrm{Wb} = 1\ \mathrm{T\ m}^2. \tag{9.4}$$

Faraday's Law of Induction is stated quantitatively, in terms of the magnetic flux, as follows:

The magnitude of the potential difference, ΔV_{ind}, induced in a conducting loop is equal to the time rate of change of the magnetic flux through the loop.

Faraday's Law of Induction is thus expressed by the equation

$$\Delta V_{\text{ind}} = -\frac{d\Phi_B}{dt}. \tag{9.5}$$

The negative sign in equation 9.5 is necessary because the induced potential difference establishes an induced current whose magnetic field tends to oppose the flux change. Section 9.3 on Lenz's Law discusses this phenomenon in detail.

The magnetic flux can be changed in several ways, including changing the magnitude of the magnetic field, changing the area of the loop, or changing the angle the loop makes with respect to the magnetic field. In all situations that involve some form of motion of a conductor relative to the source of a magnetic field, the induced potential difference is called a **motional emf.**

Induction in a Flat Loop inside a Magnetic Field

Let's apply equation 9.5 to a flat wire loop inside a uniform magnetic field, where *uniform* means that the field has the same value (same magnitude and same direction) at all points in space at a given time but can vary in time. This arrangement is the simplest case we can address. According to equation 9.3, the magnetic flux in this case is given by $\Phi_B = BA\cos\theta$. According to equation 9.5, the induced potential difference is then

$$\Delta V_{\text{ind}} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(BA\cos\theta). \tag{9.6}$$

We can use the product rule from calculus to expand this derivative:

$$\Delta V_{\text{ind}} = -A\cos\theta\frac{dB}{dt} - B\cos\theta\frac{dA}{dt} + AB\sin\theta\frac{d\theta}{dt}. \tag{9.7}$$

Because the time derivative of the angular displacement is the angular velocity, $d\theta/dt = \omega$, the induced potential difference in a flat loop inside a uniform magnetic field is

$$\Delta V_{\text{ind}} = -A\cos\theta\frac{dB}{dt} - B\cos\theta\frac{dA}{dt} + \omega AB\sin\theta. \tag{9.8}$$

Holding two of the three variables in equation 9.8 (A, B, and θ) constant results in the following *three special* cases:

1. Holding the area of the loop and its orientation relative to the magnetic field constant but varying the magnetic field in time yields

$$A \text{ and } \theta \text{ constant:}\quad \Delta V_{\text{ind}} = -A\cos\theta\frac{dB}{dt}. \tag{9.9}$$

2. Holding the magnetic field as well as the orientation of the loop relative to the magnetic field constant but changing the area of the loop that is exposed to the magnetic field yields

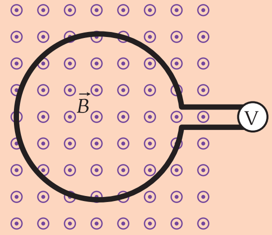
$$B \text{ and } \theta \text{ constant:}\quad \Delta V_{\text{ind}} = -B\cos\theta\frac{dA}{dt}. \tag{9.10}$$

3. Holding the magnetic field constant and keeping the area of the loop fixed but allowing the angle between the two to change as a function of time yields

$$A \text{ and } B \text{ constant:}\quad \Delta V_{\text{ind}} = \omega AB\sin\theta. \tag{9.11}$$

Self-Test Opportunity 9.1

The plane of the circular loop shown in the figure is perpendicular to a magnetic field with magnitude $B = 0.500$ T. The magnetic field goes to zero at a constant rate in 0.250 s. The induced voltage in the loop is 1.24 V during that time. What is the radius of the loop?



The following examples illustrate the first two cases. Section 9.4 addresses the third case, which has the most useful technical applications, leading directly to electric motors and generators.

EXAMPLE 9.1 Potential Difference Induced by a Changing Magnetic Field

A current of 600 mA is flowing in an ideal solenoid, resulting in a magnetic field of 0.025 T inside the solenoid. Then the current increases with time, t, according to

$$i(t) = i_0\left[1 + (2.4\ \text{s}^{-2})t^2\right].$$

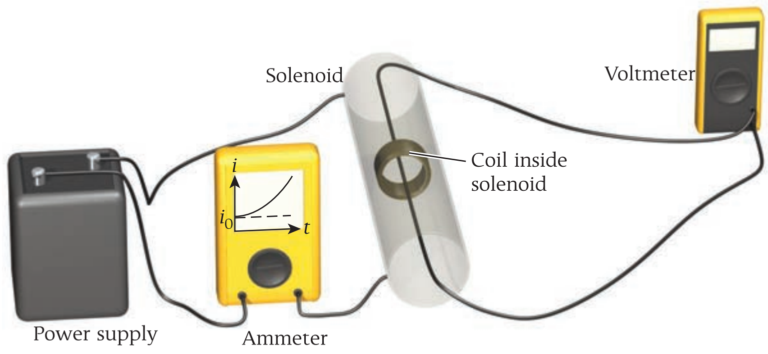


FIGURE 9.8 A time-varying current applied to a solenoid induces a potential difference in a coil.

PROBLEM

If a circular coil of radius 3.4 cm with $N = 200$ windings is located inside the solenoid with its normal vector parallel to the magnetic field (Figure 9.8), what is the induced potential difference in the coil at $t = 2.0$ s?

SOLUTION

First, we compute the area of the coil. Since it is circular, its area is πR^2. However, it has N windings, and thus the area is that of N loops of area πR^2. The net effect is that the number of windings acts as a simple multiplier for the loop area, and the total effective area of the coil is

$$A = N\pi R^2 = 200\pi(0.034\ \text{m})^2 = 0.73\ \text{m}^2. \qquad \text{(i)}$$

The magnetic field inside an ideal solenoid is $B = \mu_0 ni$, where n is the number of windings per unit length, and i is the current (see Chapter 8). Because the magnetic field is proportional to the current, we immediately obtain for the time dependence of the magnetic field in this case

$$B(t) = B_0\left[1 + (2.4\ \text{s}^{-2})t^2\right],$$

with $B_0 = \mu_0 n i_0 = 0.025$ T, according to the problem statement.

Further, in this case, the area of the coil and the angle between each loop and the magnetic field (which is zero) are constant. Therefore, equation 9.9 applies. We then find for the induced potential difference, where the area A already accounts for the number of windings, as shown in equation (i):

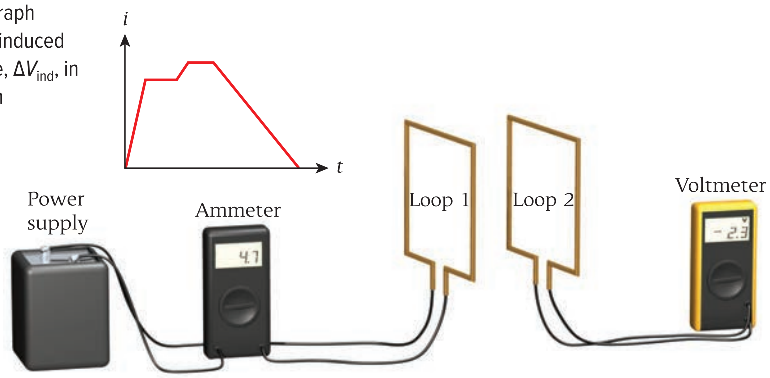
$$\begin{aligned}
\Delta V_{\text{ind}} &= -A\cos\theta\frac{dB}{dt}\\
&= -A\cos\theta\frac{d}{dt}\left[B_0(1 + \left(2.4\ \text{s}^{-2}\right)t^2)\right]\\
&= -AB_0\cos\theta(2\left(2.4\ \text{s}^{-2}\right)t)\\
&= -(0.73\ \text{m}^2)(0.025\ \text{T})(\cos 0°)\left(4.8\ \text{s}^{-2}\right)t\\
&= (-0.088\ \text{V/s})t.
\end{aligned}$$

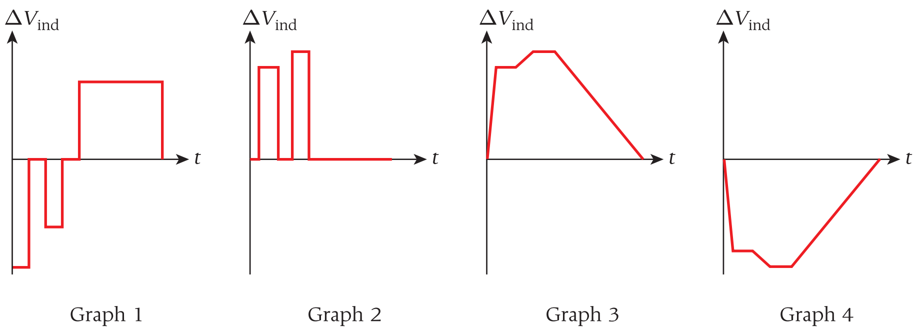
At time $t = 2.0$ s, the induced potential difference in the coil is $\Delta V_{\text{ind}} = -0.18$ V.

An important general point is made by Example 9.1: The potential difference induced in a coil with N windings and area A is simply N times the potential difference induced in a single loop of area A. Equations 9.8 through 9.11 are valid for multiloop coils, and the only way that the number of windings enters the calculations is as a multiplier in determining the effective area of the coil.

Concept Check 9.2

A power supply is connected to loop 1 and an ammeter as shown in the figure. Loop 2 is close to loop 1 and is connected to a voltmeter. A graph of the current *i* through loop 1 as a function of time, *t*, is also shown in the figure. Which graph best describes the induced potential difference, ΔV_{ind}, in loop 2 as a function of time, *t?*





a) graph 1 b) graph 2 c) graph 3 d) graph 4

EXAMPLE 9.2 Potential Difference Induced by a Moving Loop

A rectangular wire loop of width $w = 3.1$ cm and depth $d_0 = 4.8$ cm is pulled out of the gap between two permanent magnets. A magnetic field of magnitude $B = 0.073$ T is present throughout the gap (Figure 9.9).

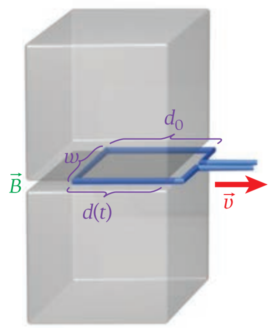


FIGURE 9.9 A wire loop (blue) is pulled out of a gap between two magnets.

PROBLEM

If the loop is removed at a constant speed of 1.6 cm/s, what is the induced voltage in the loop as a function of time?

SOLUTION

This situation corresponds to the special case of induction due to an area change, governed by equation 9.10. The magnetic field and the orientation of the loop relative to the field remain constant. We assume that the angle between the magnetic field vector and the area vector is zero. What changes is the area of the loop that is exposed to the magnetic field. With a gap as narrow as that shown in Figure 9.9, very little field occurs outside the gap, so the effective area of the loop exposed to the field is $A(t) = (w)(d(t))$, where $d(t) = d_0 - vt$ is the depth of the part of the loop inside the magnetic field at time t. While the entire loop is still inside the gap, no voltage is produced. Letting the time of arrival of the right edge of the loop at the right end of the gap be $t = 0$, we have

$$A(t) = (w)\big(d(t)\big) = w(d_0 - vt).$$

– *Continued*

This formula holds until the left edge of the loop reaches the right end of the gap, after which the area of the loop exposed to the magnetic field is zero. The left edge arrives at time $t_f = d/v = (4.8 \text{ cm})/(1.6 \text{ cm/s}) = 3.0$ s, and $A(t > t_f) = 0$. From equation 9.10, we find

$$\begin{aligned}\Delta V_{\text{ind}} &= -B\cos\theta \frac{dA}{dt} \\ &= -B\cos\theta \frac{d}{dt}\left[w(d_0 - vt)\right] \\ &= wvB\cos\theta \\ &= (0.031 \text{ m})(0.016 \text{ m/s})(0.073 \text{ T})\cos 0° \\ &= 3.6 \times 10^{-5} \text{ V}.\end{aligned}$$

During the time interval between 0 and 3 s, a constant potential difference of 36 μV is induced, and no potential difference is induced outside this time interval.

Concept Check 9.3

A long wire carries a current, *i,* as shown in the figure. A square loop moves in the same plane as the wire, as indicated. In which cases will the loop have an induced current?

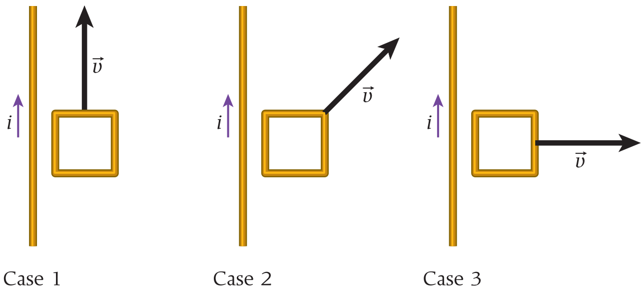
a) cases 1 and 2

b) cases 1 and 3

c) cases 2 and 3

d) None of the loops will have an induced current.

e) All of the loops will have an induced current.



9.3 Lenz's Law

Lenz's Law provides a rule for determining the direction of an induced current in a loop. An induced current will have a direction such that the magnetic field *due* to the induced current *opposes* the change in the magnetic flux that *induces* the current. The direction of the induced current can be used to determine the locations of higher and lower potential.

Let's apply Lenz's Law to the situations described in Section 9.1. The physical situation shown in Figure 9.2a involves moving a magnet toward a loop with the north pole pointed toward the loop. In this case, the magnetic field lines point away from the north pole of the magnet. As the magnet moves toward the loop, the magnitude of the magnetic field within the loop, in the direction pointing toward the loop, increases as depicted in Figure 9.10a. Lenz's Law states that the current induced in the loop tends to oppose the change in magnetic flux. The induced magnetic field, $\vec{B}_{\text{ind}}$, then points in the opposite direction from that of the field due to the magnet.

In Figure 9.2b, a magnet is moved toward a loop with the south pole pointed toward the loop. In this case, the magnetic field lines point toward the south pole of the magnet. As the magnet moves toward the loop, the magnitude of the field in the direction pointing toward the south pole increases, as depicted in Figure 9.10b. Lenz's Law states that the induced current creates a magnetic field that tends to oppose the increase in magnetic flux. This induced field points in the opposite direction from that of the field lines due to the magnet.

Similarly, Figure 9.10c and Figure 9.10d represent the physical situations depicted in Figure 9.3a and Figure 9.3b, respectively. In these two cases, the magnitude of the magnetic flux is decreasing, and a current is induced that produces a magnetic field opposing this decrease. In both cases, a current is induced in the loop that creates a magnetic field pointing in the same direction as the magnetic field from the magnet.

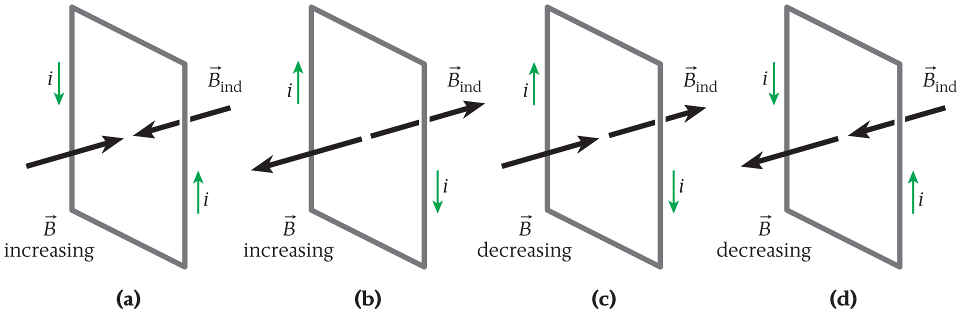
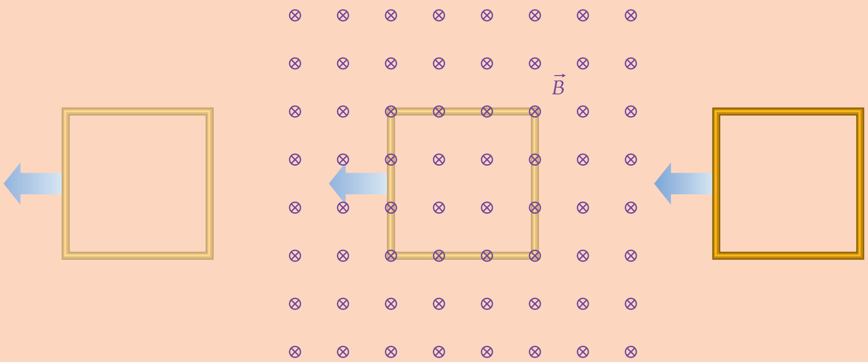


FIGURE 9.10 Relationship of an external magnetic field, $\vec{B}$, the induced current, i, and the magnetic field, $\vec{B}_{ind}$ resulting from that induced current: (a) An increasing magnetic field pointing to the right induces a current that creates a magnetic field pointing to the left. (b) An increasing magnetic field pointing to the left induces a current that creates a magnetic field pointing to the right. (c) A decreasing magnetic field pointing to the right induces a current that creates a magnetic field pointing to the right. (d) A decreasing magnetic field pointing to the left induces a current that creates a magnetic field pointing to the left.

For two loops with one having a changing current, Lenz's Law is applied in the same way. The increasing current in loop 1 in Figure 9.4 induces a current in loop 2 that creates a magnetic field opposing the increase in magnetic flux, as depicted in Figure 9.10b. The decreasing current in loop 1 in Figure 9.5 induces a current in loop 2 that creates a magnetic field opposing the decrease in magnetic flux, as depicted in Figure 9.10d.

Self-Test Opportunity 9.2

A square conducting loop with very small resistance is moved at constant speed from a region with no magnetic field through a region of constant magnetic field and then into a region with no magnetic field, as shown in the figure. As the loop enters the magnetic field, what is the direction of the induced current? As the loop leaves the magnetic field, what is the direction of the induced current?



Self-Test Opportunity 9.3

Suppose Lenz's Law instead stated that the induced magnetic field augments the magnetic flux, meaning that Faraday's Law of Induction would be written as $\Delta V_{ind} = +d\Phi_B/dt$, that is, with a positive instead of a negative sign. What would be the consequences? Can you explain why this would lead to a contradiction?

Eddy Currents

Let's consider two pendulums, each with a nonmagnetic conducting metal plate at the end that is designed to pass through the gap between strong permanent magnets (Figure 9.11). One metal plate is solid, and the other has slots cut in it. The pendulums are pulled to one side and released. The pendulum with the solid metal plate stops in the gap, while the slotted plate passes through the magnetic field, only slowing slightly. This demonstration illustrates the very important phenomenon of induced **eddy currents.** As the pendulum with the solid plate enters the magnetic field between the magnets, Lenz's Law says that the changing magnetic flux induces currents that tend to oppose the change in flux. These currents produce induced magnetic fields opposing the external field that created the currents. These induced magnetic fields interact with the external magnetic field (via their spatial gradients) to stop the pendulum. Larger induced currents produce larger induced magnetic fields and thus lead to more rapid deceleration of the pendulum. In the slotted plate, the

FIGURE 9.11 Two pendulums, one consisting of an arm and a solid metal plate and a second consisting of an arm and a slotted metal plate. The five frames are in time sequence from left to right, with the two pendulums starting their motion together in the second frame from the left. The pendulum with the solid plate stops in the gap, while the pendulum with the slotted plate passes through the gap.

induced eddy currents are broken up by the slots, and the slotted plate passes through the magnetic field, only slowing slightly. Eddy currents are not like the undirected and uniform current induced in the loop in Example 9.2 but are instead swirling eddies like those seen in turbulent flowing water.

Where does the energy contained in the motion of the pendulum with the solid plate in Figure 9.11 go–in other words, how do the eddy currents stop the pendulum? The answer is that the eddy currents disperse heat in the metal because of its finite resistance, as discussed in Chapter 5. The stronger the induced eddy currents are, the more rapidly energy is converted from the pendulum's motion into heat. This is why the slotted plate, with its much smaller induced eddy currents, is only slowed slightly as it passes through the gap between the magnets (although the slowing will stop it eventually).

Eddy currents are often undesirable, forcing equipment designers to minimize them by segmenting or laminating electrical devices that must operate in an environment of changing magnetic fields. However, eddy currents can also be useful and are employed in certain practical applications, such as the brakes of train cars.

Metal Detector

Passing through metal detectors, especially at airports, is an unavoidable part of life these days. A metal detector works by using electromagnetic induction, often called *pulse induction.* A metal detector has a transmitter coil and a receiver coil. An alternating current is applied to the transmitter coil, which then produces an alternating magnetic field. (*Alternating* means varying as a function of time between positive and negative values. Chapter 10 will provide more precise definitions, physical consequences, and mathematical details concerning alternating current.) As the magnetic field of the transmitter coil increases and decreases, it induces a current in the receiving coil that tends to counteract the change in the magnetic flux produced by the transmitter coil. The induced current in the receiver coil is measured when nothing but air is between the coils. If a conductor in the form of a metal object passes between transmitter and receiver coils, a current will be induced in the metal object in the form of eddy currents. These eddy currents will act to counter the increases and decreases of the changing magnetic field produced by the transmitter coil, which in turn induces a current in the receiver coil that tends to counter the increase in current in the metal. The measured current in the receiver coil will be less when any metal object is present between the two coils.

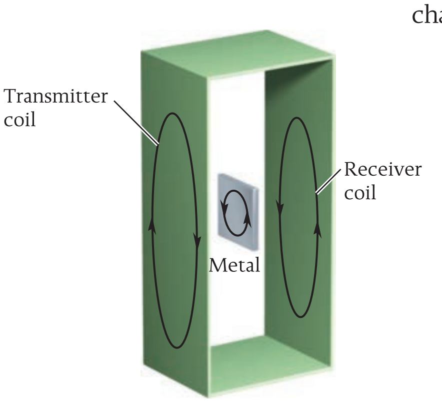


FIGURE 9.12 Schematic diagram of an airport metal detector.

A schematic diagram of an airport metal detector is shown in Figure 9.12. A transmitter coil and a receiver coil are located on opposite sides of an entry door. The person or object to be scanned passes through the door between the two coils. Suppose that the current in the transmitter coil is flowing in the direction shown and increasing. A current will be induced in the metal plate in the opposite direction and will tend to oppose the increase in the current in the transmitter coil. The increasing current in the metal plate will induce a current in the receiver coil that is in the opposite direction and tends to oppose the increase in the current in the metal plate (not shown in the diagram). Thus, the metal plate induces a current in the receiver coil that flows in the same direction as the current in the transmitter coil. Without the metal plate, the increasing current in the transmitter coil induces a current in the opposite direction in the receiver coil that tends to oppose the increase in the current in the transmitter coil (as shown in the diagram). Thus, the overall effect of the metal plate in the metal detector is to decrease the observed current in the receiver coil. The metal object does not have to be a flat plate; any piece of metal, provided it is large enough, will have currents induced in it that can be detected by measuring the induced current in the receiver coil.

Metal detectors are also used to control traffic lights. In this application, a rectangular wire loop, which serves as both transmitter and receiver coil, is embedded in the road surface. A pulse of current is passed through the loop, which induces eddy currents in any metal near the loop. The current in the loop is measured after the current pulse is completed. When a car moves onto the road surface above the loop, eddy currents induced in the metal of the car cause a different current to be measured between pulses, which then triggers the traffic light to switch to green (after an appropriate delay to allow other vehicles to clear the intersection, of course!). On older road surfaces, you can often see rectangular-shaped scars in the asphalt resulting from retrofitting intersections with these induction loops.

Induced Potential Difference on a Wire Moving in a Magnetic Field

Consider a conducting wire of length ℓ moving with constant velocity $\vec{v}$ perpendicular to a constant magnetic field, $\vec{B}$, directed into the page (Figure 9.13). The wire is oriented so that it is perpendicular to the velocity and to the magnetic field. The magnetic field exerts a force, $\vec{F}_B$, on the conduction electrons in the wire, causing them to move downward. This motion of the electrons produces a net negative charge at the bottom end of the wire and a net positive charge at the top end of the wire. This charge separation produces an electric field, $\vec{E}$, which exerts on the conduction electrons a force, $\vec{F}_E$, that tends to cancel the magnetic force. After some time, the two forces become equal in magnitude (but opposite in direction) producing a zero net force:

$$F_B = evB = F_E = eE. \tag{9.12}$$

Thus, the induced electric field can be expressed by

$$E = vB. \tag{9.13}$$

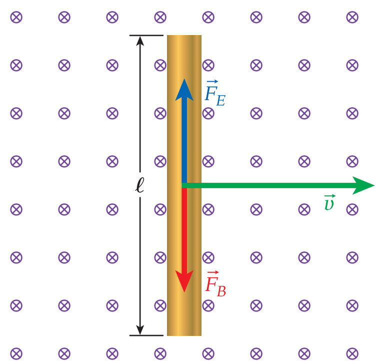
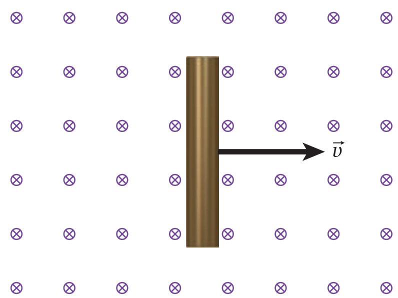


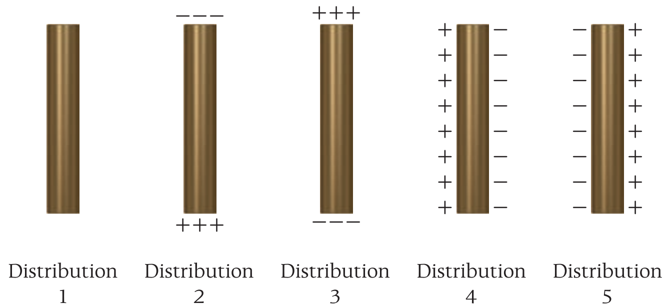
FIGURE 9.13 A moving conductor in a constant magnetic field. The magnetic and electric forces on the conduction electrons are shown.

Concept Check 9.4

A metal bar is moving with constant velocity $\vec{v}$ through a uniform magnetic field pointing into the page, as shown in the figure.



Which of the following most accurately represents the charge distribution on the surface of the metal bar?



a) distribution 1

b) distribution 2

c) distribution 3

d) distribution 4

e) distribution 5

Because the electric field is constant in the wire, it produces a potential difference between the two ends of the wire given by

$$E = \frac{\Delta V_{\text{ind}}}{\ell} = vB. \tag{9.14}$$

The induced potential difference between the ends of the wire is then

$$\Delta V_{\text{ind}} = v\ell B. \tag{9.15}$$

This is one form of motional emf, mentioned in Section 9.2.

FIGURE 9.14 An artist's conception of the Space Shuttle *Columbia* and the tethered satellite.

EXAMPLE 9.3 Satellite Tethered to a Space Shuttle

In 1996, the Space Shuttle *Columbia* deployed a tethered satellite on a wire out to a distance of 20. km (Figure 9.14). The wire was oriented perpendicular to the Earth's magnetic field at that point, and the magnitude of the field was $B = 5.1 \times 10^{-5}$ T. *Columbia* was traveling at a speed of 7.6 km/s.

PROBLEM

What was the potential difference induced between the ends of the wire?

SOLUTION

We can use equation 9.15 to determine the induced potential difference between the ends of the wire. The length of the wire is $L = 20.0$ km, and the speed of the wire through the magnetic field of the Earth ($B = 5.1 \times 10^{-5}$ T) is the same as the speed of the Space Shuttle, which is $v = 7.6$ km/s. Thus, we have

$$\Delta V_{\text{ind}} = vLB = (7.6 \times 10^3 \text{ m/s})(20.0 \times 10^3 \text{ m})(5.1 \times 10^{-5} \text{ T}) = 7.8 \text{ kV}.$$

The astronauts on the Space Shuttle measured a current of about 0.5 A at a voltage of 3.5 kV. The circuit consisted of the deployed wire and ionized atoms in space as the return path for the current. The wire broke just as the deployment length reached 20 km, but the generation of electric current from the motion of a spacecraft had been demonstrated.

Solved Problem 8.2 concerned an electromagnetic rail accelerator. The next example focuses on the phenomenon of induction in a similar system.

EXAMPLE 9.4 Pulled Conducting Rod

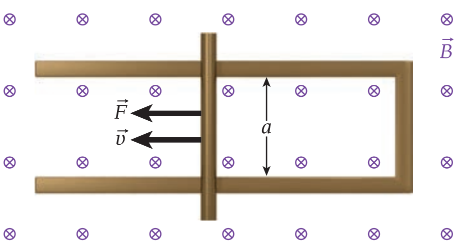


FIGURE 9.15 A conducting rod is pulled along two conducting rails with a constant velocity in a constant magnetic field directed into the page.

A conducting rod is pulled horizontally by a constant force of magnitude, $F = 5.00$ N, along a set of conducting rails separated by a distance $a = 0.500$ m (Figure 9.15). The two rails are connected, and no friction occurs between the rod and the rails. A uniform magnetic field with magnitude $B = 0.500$ T is directed into the page. The rod moves at constant speed, $v = 5.00$ m/s.

PROBLEM

What is the magnitude of the induced potential difference in the loop formed by the connected rails and the moving rod?

SOLUTION

The induced potential difference is given by equation 9.10, which applies to a loop in a magnetic field when the angle and magnetic field are held constant and the area of the loop changes with time:

$$\Delta V_{\text{ind}} = -B\cos\theta \frac{dA}{dt}.$$

In this case, $\theta = 0$ and $B = 0.500$ T. The area of the loop is increasing with time. We can express the area of the loop in terms of A_0, the area before the rod started moving, and an additional area given by the product of the speed of the loop and the time for which the loop has been moving times the distance, *a*, between the rails:

$$A = A_0 + a(vt) = A_0 + vta.$$

The change of the loop's area as a function of time is then

$$\frac{dA}{dt} = \frac{d}{dt}(A_0 + vta) = va.$$

Thus, the magnitude of the induced potential difference is

$$\Delta V_{\text{ind}} = \left| -B\cos\theta \frac{dA}{dt} \right| = vaB. \tag{i}$$

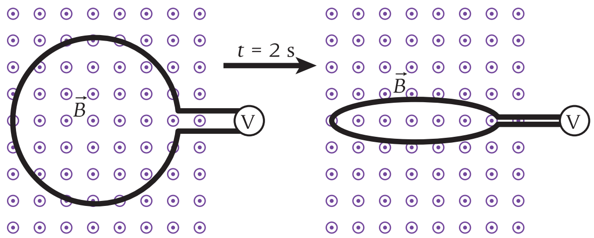
Inserting the numerical values, we obtain

$$\Delta V_{\text{ind}} = (5.00 \text{ m/s})(0.500 \text{ m})(0.500 \text{ T}) = 1.25 \text{ V}.$$

Note that equation (i), $\Delta V_{\text{ind}} = vaB$, which we derived from Faraday's Law of Induction, has the same form as equation 9.15 for the potential difference induced in a wire moving in a magnetic field, which was derived using the magnetic force on moving charges.

Concept Check 9.5

A wire loop is placed in a uniform magnetic field. Over a period of 2 s, the loop is shrunk. Which statement about the induced potential difference is correct?



a) There will be some induced potential difference.

b) There will be no induced potential difference because the loop changes size along one axis and not the other.

c) There will be no induced potential difference because the loop is not closed.

d) There will be no induced potential difference because the loop is shrinking.

SOLVED PROBLEM 9.1 Power from a Rotating Rod

A conducting rod with length ℓ = 8.17 cm rotates around one of its ends in a uniform magnetic field that has a magnitude B = 1.53 T and is directed parallel to the rotation axis of the rod (Figure 9.16). The other end of the rod slides on a frictionless conducting ring. The rod makes 6.00 revolutions per second. A resistor, R = 1.63 mΩ, is connected between the rotating rod and the conducting ring.

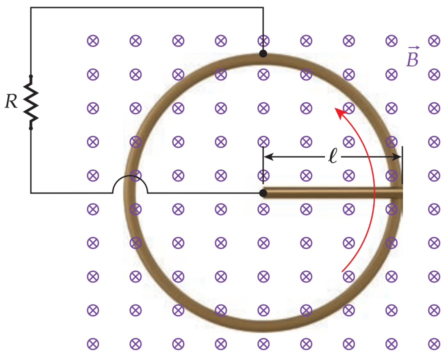


FIGURE 9.16 Conducting rod rotating in a constant magnetic field directed into the page.

PROBLEM

What is the power dissipated in the resistor due to magnetic induction?

SOLUTION

THINK We can calculate the potential difference induced in a conductor of length ℓ moving with speed v perpendicular to a magnetic field of magnitude B. However, the rotating rod has different speeds at different radii, $v(r)$. Therefore, we must calculate the potential difference induced in the rod by integrating $Bv(r)$ over the length of the rod. From the induced potential difference, we can calculate the power dissipated in the resistor.

SKETCH Figure 9.17 shows the speed as a function of radius for the conducting rod.

RESEARCH The potential difference, ΔV_{ind}, induced on a conductor of length ℓ moving with speed v perpendicular to a magnetic field of magnitude B is given by equation 9.15:

$$\Delta V_{\text{ind}} = v\ell B.$$

– Continued

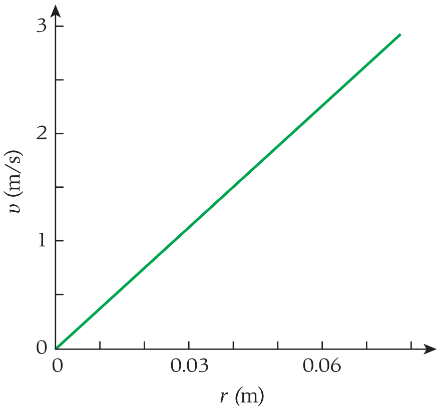


FIGURE 9.17 Speed as a function of radius for the conducting rod.

However, in this case, different parts of the conducting rod are moving at different speeds. We can express the speed of the different parts of the rod as a function of distance r from the axis of rotation:

$$v(r) = \frac{2\pi r}{T},$$

where $v(r)$ is the speed of the rod at distance r and T is the period of the rotation. We can then find the potential difference induced in the rotating rod over its length, ℓ:

$$\Delta V_{\text{ind}} = \int_0^{\ell} v(r) B dr. \tag{i}$$

The power dissipated in the resistor is given by

$$P = \frac{\Delta V_{\text{ind}}^2}{R}. \tag{ii}$$

SIMPLIFY Evaluating the definite integral in equation (i) gives

$$\Delta V_{\text{ind}} = \int_0^{\ell} \left(\frac{2\pi r}{T}\right) B dr = \frac{2\pi B}{T}\frac{\ell^2}{2} = \frac{\pi B \ell^2}{T}. \tag{iii}$$

Substituting the expression for ΔV_{ind} from equation (iii) into equation (ii) leads to an expression for the power dissipated in the resistor:

$$P = \frac{\left(\pi B \ell^2 / T\right)^2}{R} = \frac{\pi^2 B^2 \ell^4}{R T^2}.$$

CALCULATE The period is the inverse of the frequency. The frequency is $f = 6.00$ Hz, so the period is

$$T = \frac{1}{f} = \frac{1}{6.00} \text{ s}.$$

Putting in the numerical values gives us

$$P = \frac{\pi^2 B^2 \ell^4}{R T^2} = \frac{\pi^2 (1.53 \text{ T})^2 (0.0817 \text{ m})^4}{\left(1.63 \times 10^{-3}\,\Omega\right)\left(\frac{1}{6.00} \text{ s}\right)^2} = 22.7345 \text{ W}.$$

ROUND We report our result to three significant figures:

$$P = 22.7 \text{ W}.$$

DOUBLE-CHECK To double-check our result, we consider a conducting rod of the same length moving perpendicularly to the same magnetic field with a speed equal to the speed of the center of the rotating rod, which is

$$v(\ell/2) = \frac{2\pi(\ell/2)}{T} = \frac{2\pi L}{2T} = \frac{2\pi(0.0817 \text{ m})}{2\left(\frac{1}{6.00} \text{ s}\right)} = 1.54 \text{ m/s}.$$

The induced potential difference across the conducting rod moving perpendicularly would be

$$\Delta V_{\text{ind}} = v\ell B = (1.54 \text{ m/s})(0.0817 \text{ m})(1.53 \text{ T}) = 0.193 \text{ V}.$$

The power dissipated in the resistor would then be

$$P = \frac{\Delta V_{\text{ind}}^2}{R} = \frac{(0.193 \text{ V})^2}{1.63 \times 10^{-3}\,\Omega} = 22.9 \text{ W},$$

which is close to our result within rounding error. Thus, our result seems reasonable.

Finally, note that there is a possible additional source of potential difference between the two ends of the rod. Our solution assumed that the potential difference between the two ends is due exclusively to the magnetic induction. However, all charge carriers inside the rod are forced on a circular path due to the rotation. This requires a centripetal force, which should in principle reduce the potential difference between the two ends of the rod. However, for the small angular velocity of the rod in this problem, this effect is negligible.

9.4 Generators and Motors

The third special case of the basic induction process described in Section 9.2 is by far the most interesting technologically. In this case, the angle between the conducting loop and the magnetic field is varied over time, while keeping the area of the loop as well as the magnetic field strength constant. In this situation, equation 9.11 can be used to apply Faraday's Law of Induction to the generation and application of electric current. A device that produces electric current from mechanical motion is called an **electric generator.** A device that produces mechanical motion from electric current is called an **electric motor.** Figure 9.18 shows a very simple electric motor.

A simple generator consists of a loop forced to rotate in a fixed magnetic field. The force that causes the loop to rotate can be supplied by hot steam running over a turbine, as occurs in nuclear and coal-fired power plants. (Power plants actually use multiple loops in order to increase the power output.) On the other hand, the loop can be made to rotate by flowing water or wind to generate electricity in a pollution-free way.

Figure 9.19 shows two types of simple generators. In a direct-current generator, the rotating loop is connected to an external circuit through a split commutator ring, as illustrated in Figure 9.19a. As the loop turns, the connection is reversed twice per revolution, so the induced potential difference always has the same sign. Figure 9.19b shows a similar arrangement used to produce an alternating current. An alternating current is a current that varies in time between positive and negative values, with the variation often showing a sinusoidal form. Each end of the loop is connected to the external circuit through its own solid slip ring. Thus, this generator produces an induced potential difference that varies from positive to negative and back. A generator that produces alternating voltages and the resulting alternating current is also called an **alternator.** Figure 9.20 shows the induced potential difference as a function of time for each type of generator.

The devices in Figure 9.19 could also be used as motors by supplying current to the loop and using its resulting motion to do work.

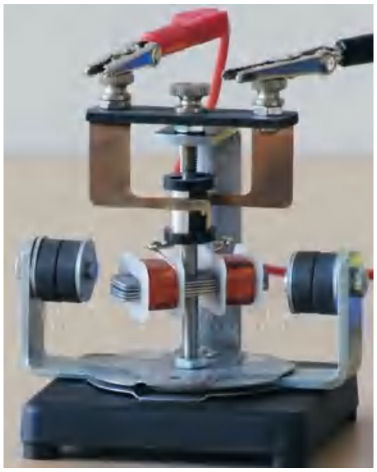


FIGURE 9.18 A very simple electric motor used for lecture demonstrations. It consists of a pair of permanent magnets on the outside and two solenoids, through which current is sent, on the inside.

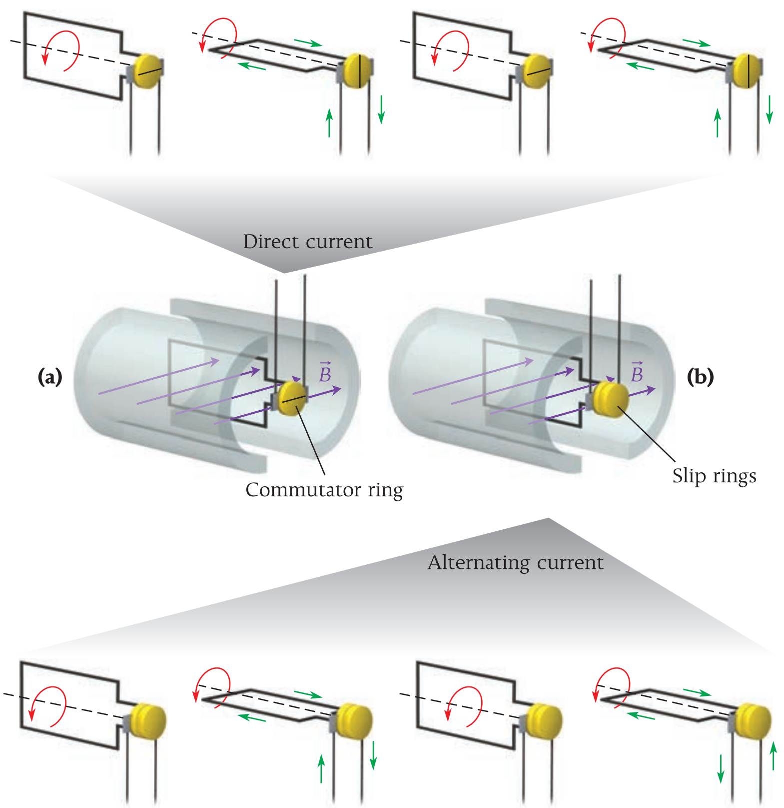


FIGURE 9.19 (a) A simple direct-current (DC) generator/motor. (b) A simple alternating-current (AC) generator/motor.

Self-Test Opportunity 9.4

A generator is operated by rotating a coil of N turns in a constant magnetic field of magnitude B at a frequency f. The resistance of the coil is R, and the cross-sectional area of the coil is A. Decide whether each of the following statements is true or false.

a) The average induced potential difference doubles if the frequency, f, is doubled.

b) The average induced potential difference doubles if the resistance, R, is doubled.

c) The average induced potential difference doubles if the magnetic field's magnitude, B, is doubled.

d) The average induced potential difference doubles if the area, A, is doubled.

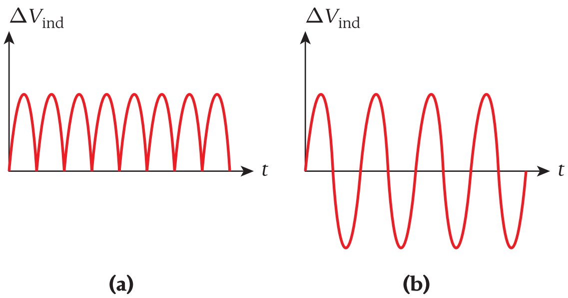


FIGURE 9.20 Induced potential difference as a function of time for (a) a simple direct-current generator; (b) a simple alternating-current generator.

Real-world generators and motors are considerably more complex than the simple examples in Figure 9.19. For example, instead of permanent magnets, current flowing in coils creates the required magnetic field. Many closely spaced loops are employed to make more efficient use of the rotational motion. Multiple loops also resolve the problem that a simple, one-loop motor could stop in a position where the current through the loop produces no torque. The magnetic field may also change with time in phase with the rotating loop. In some generators and motors, the loops (coils) are fixed and the magnet rotates.

FIGURE 9.21 Regenerative brakes on the rear wheels of a hybrid vehicle are used to slow the vehicle and convert the vehicle's kinetic energy into electrical energy that is stored in a lithium ion battery.

Regenerative Braking

Hybrid cars are propelled by a combination of gasoline power and electrical power. One attractive feature of a hybrid vehicle is that it is capable of **regenerative braking.** When the brakes are used to slow or stop a nonhybrid vehicle, the kinetic energy of the vehicle is turned into heat in the brake pads. This heat dissipates into the environment, and energy is lost. In a hybrid car, the brakes are connected to an electric motor (Figure 9.21), which functions as a generator, charging the car's battery. Thus, the kinetic energy of the car is partially recovered during braking, and this energy can later be used to propel the car, contributing to its efficiency and greatly increasing its gas mileage in stop-and-go driving.

9.5 Induced Electric Field

Faraday's Law of Induction states that a changing magnetic flux produces an induced potential difference, which can lead to an induced current. What are the consequences of this effect?

Consider a positive charge q moving in a circular path with radius r in an electric field, $\vec{E}$. The work done on the charge is equal to the integral of the scalar product of the force and the differential displacement vector. For now, let's assume that the electric field $\vec{E}$ is constant, that it has field lines that are circular, and that the charge moves along one of these lines. In one revolution of the charge, the work done on it is given by

$$\oint \vec{F} \bullet d\vec{s} = \oint q\vec{E} \bullet d\vec{s} = \oint q \cos 0° E ds = qE \oint ds = qE(2\pi r).$$

Since the work done by a constant electric field is $\Delta V_{ind} q$, we get

$$\Delta V_{ind} = 2\pi r E.$$

We can generalize this result by considering the work done on a charge q moving along an arbitrary closed path:

$$W = \oint \vec{F} \bullet d\vec{s} = q \oint \vec{E} \bullet d\vec{s}.$$

Again substituting $\Delta V_{\text{ind}}q$ for the work done, we obtain

$$\Delta V_{\text{ind}} = \oint \vec{E} \bullet d\vec{s}. \tag{9.16}$$

Now we can express the induced potential difference in a different way by combining equation 9.5 with equation 9.16:

$$\oint \vec{E} \bullet d\vec{s} = -\frac{d\Phi_B}{dt}. \tag{9.17}$$

Equation 27.17 states that a changing magnetic flux induces an electric field. This equation can be applied to any closed path in a changing magnetic field, even if no conductor exists in the path.

9.6 Inductance of a Solenoid

Consider a long solenoid with N turns carrying a current, i. This current creates a magnetic field in the center of the solenoid, resulting in a magnetic flux, Φ_B. The same magnetic flux goes through each of the N windings of the solenoid. It is customary to define the **flux linkage** as the product of the number of windings and the magnetic flux, or $N\Phi_B$. Equation 9.1 defined the magnetic flux as $\Phi_B = \iint \vec{B} \bullet d\vec{A}$. Inside the solenoid, the magnetic field vector and the surface normal vector, $d\vec{A}$, are parallel. And we saw in Chapter 8 that the magnitude of the magnetic field inside the solenoid is $B = \mu_0 ni$, where $\mu_0 = 4\pi \times 10^{-7}$ T m/A is the magnetic permeability of free space, i is the current, and n is the number of windings per unit length ($n = N/\ell$). Therefore, the magnetic flux in the interior of a solenoid is proportional to the current flowing through the solenoid, which trivially means that the flux linkage is also proportional to the current. We can express this proportionality as

$$N\Phi_B = Li, \tag{9.18}$$

using a proportionality constant, L, called the **inductance.** (*Note:* Use of the letter L to represent the inductance is the convention. Although L is also used for the physical quantity of length and the physical quantity of angular momentum, inductance is not connected to either of these in any way.)

Inductance is a measure of the flux linkage produced by a solenoid per unit of current. The unit of inductance is the **henry** (H), named after American physicist Joseph Henry (1797–1878) and given by

$$[L] = \frac{[\Phi_B]}{[i]} \Rightarrow 1 \text{ H} = \frac{1 \text{ T m}^2}{1 \text{ A}}. \tag{9.19}$$

The definition of the henry given in equation 9.19 means that the magnetic permeability of free space can also be given as $\mu_0 = 4\pi \times 10^{-7}$ H/m.

Now let's use equation 9.18 to find the inductance of a solenoid with cross-sectional area A and length ℓ. The flux linkage for this solenoid is

$$N\Phi_B = (n\ell)(BA), \tag{9.20}$$

where n is the number of turns per unit length and B is the magnitude of the magnetic field inside the solenoid. Thus, the inductance is given by

$$L = \frac{N\Phi_B}{i} = \frac{(n\ell)(\mu_0 ni)(A)}{i} = \mu_0 n^2 \ell A. \tag{9.21}$$

This expression for the inductance of a solenoid is good for long solenoids because fringe field effects at the ends of such a solenoid are small. You can see from equation 9.21 that the inductance of a solenoid depends only on the geometry (length, area, and number of turns) of the device. This dependence of inductance on geometry alone holds for all coils and solenoids, just as the capacitance of any capacitor depends only on its geometry.

Any solenoid has an inductance, and when a solenoid is used in an electric circuit, it is called an *inductor*, simply because its inductance is its most important property as far as the current flow is concerned.

9.7 Self-Induction and Mutual Induction

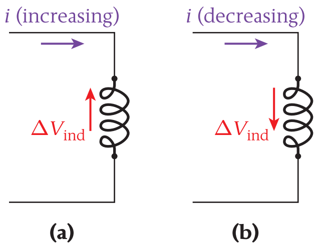


FIGURE 9.22 (a) Self-induced potential difference in an inductor when the current is increasing. (b) Self-induced potential difference in an inductor when the current is decreasing.

Consider the situation in which two coils, or inductors, are close to each other, and a changing current in the first coil produces a magnetic flux in the second coil. However, the changing current in the first coil also induces a potential difference in that coil, and thus the magnetic field from that coil also changes. This phenomenon is called **self-induction.** The resulting potential difference is termed the *self-induced potential difference*. Changing the current in the first coil also induces a potential difference in the second coil. This phenomenon is called **mutual induction.**

According to Faraday's Law of Induction, the self-induced potential difference for any inductor is given by

$$\Delta V_{\text{ind},L} = -\frac{d(N\Phi_B)}{dt} = -\frac{d(Li)}{dt} = -L\frac{di}{dt}, \tag{9.22}$$

where equation 9.18 allows us to substitute Li for $N\Phi_B$. Thus, in any inductor, a self-induced potential difference appears when the current changes with time. This self-induced potential difference depends on the time rate of change of the current and the inductance of the device.

Lenz's Law provides the direction of the self-induced potential difference. The negative sign in equation 9.22 provides the clue that the induced potential difference always opposes any change in current. For example, Figure 9.22a shows current flowing through an inductor and increasing with time. Thus, the self-induced potential difference will oppose the increase in current. In Figure 9.22b, the current flowing through an inductor is decreasing with time. Thus, a self-induced potential difference will oppose the decrease in current. We have assumed that these inductors are ideal inductors; that is, they have no resistance. All induced potential differences manifest themselves across the connections of the inductor. Inductors with resistance are treated in Section 9.8.

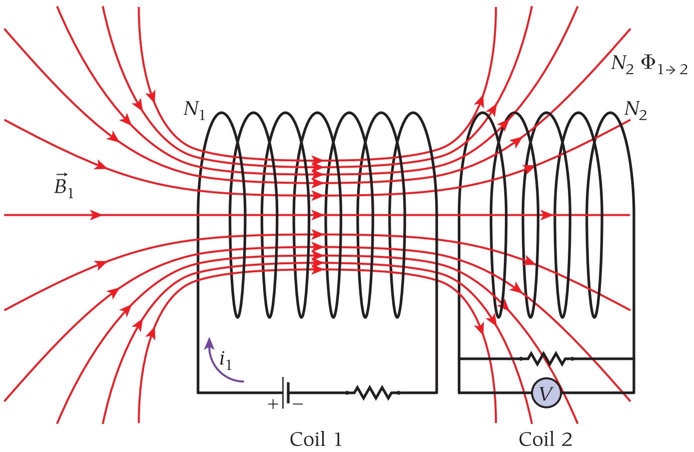


FIGURE 9.23 Coil 1 has current i_1. Coil 2 has a voltmeter capable of measuring small, induced potential differences.

Now let's consider two adjacent coils with their central axes aligned (Figure 9.23). Coil 1 has N_1 turns, and coil 2 has N_2 turns. The current in coil 1 produces a magnetic field, $\vec{B}_1$. The flux linkage in coil 2 resulting from the magnetic field in coil 1 is $N_2\Phi_{1\to2}$. The mutual inductance, $M_{1\to2}$, of coil 2 due to coil 1 is defined as

$$M_{1\to2} = \frac{N_2\Phi_{1\to2}}{i_1}. \tag{9.23}$$

Multiplying both sides of equation 9.23 by i_1 yields

$$i_1 M_{1\to2} = N_2\Phi_{1\to2}.$$

If the current in coil 1 changes with time, we can write

$$M_{1\to2}\frac{di_1}{dt} = N_2\frac{d\Phi_{1\to2}}{dt}.$$

The right-hand side of this equation is similar to the right-hand side of Faraday's Law of Induction (equation 9.5). Thus, we can write

$$\Delta V_{\text{ind},2} = -M_{1\to2}\frac{di_1}{dt}. \tag{9.24}$$

Now let's reverse the roles of the two coils (Figure 9.24). The current, i_2, in coil 2 produces a magnetic field, $\vec{B}_2$. The flux linkage in coil 1 resulting from the magnetic field in coil 2 is $N_1\Phi_{2\to1}$. Using the same analysis we applied to determine the mutual inductance of coil 2 due to coil 1, we find

$$\Delta V_{\text{ind},1} = -M_{2\to1}\frac{di_2}{dt}, \tag{9.25}$$

where $M_{2\to1}$ is the mutual inductance of coil 1 due to coil 2.

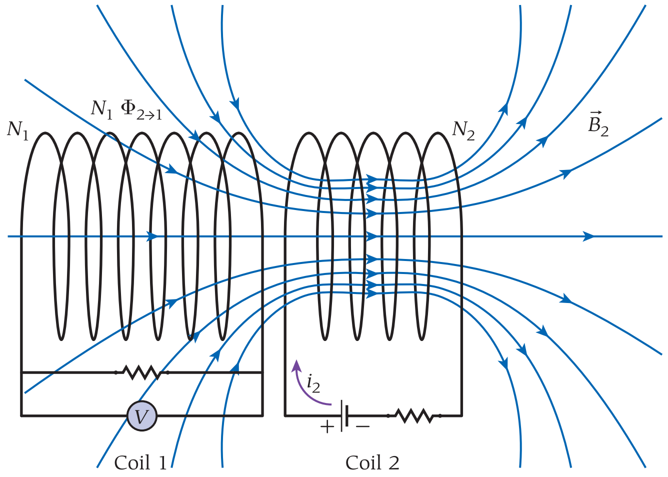


FIGURE 9.24 Coil 2 has current i_2. Coil 1 has a voltmeter capable of measuring small, induced potential differences.

We see that the potential difference induced in one coil is proportional to the change of current in the other coil. The proportionality constant is the mutual induction. If we switched the indexes 1 and 2 and repeated the entire analysis of the coils' effects on each other, we could show that

$$M_{1\to2} = M_{2\to1} = M.$$

We can then rewrite equations 9.24 and 9.25 as

$$\Delta V_{\text{ind},2} = -M\frac{di_1}{dt} \tag{9.26}$$

and

$$\Delta V_{\text{ind},1} = -M\frac{di_2}{dt}, \tag{9.27}$$

where M is the **mutual inductance** between the two coils. The SI unit of mutual inductance is the henry. One major application of mutual inductance is in transformers, which are discussed in Chapter 10.

SOLVED PROBLEM 9.2 Mutual Induction of a Solenoid and a Coil

A long solenoid with a circular cross section of radius $r_1 = 2.80$ cm and $n = 290$ turns/cm is inside and coaxial with a short coil that has a circular cross section of radius $r_2 = 4.90$ cm and $N = 31$ turns (Figure 9.25a). The current in the solenoid is increased at a constant rate from zero to $i = 2.20$ A over a time interval of 48.0 ms.

PROBLEM

What is the potential difference induced in the short coil while the current is changing?

SOLUTION

THINK The potential difference induced in the short coil is due to the changing current flowing in the solenoid. According to equation 9.23, the mutual inductance of the short

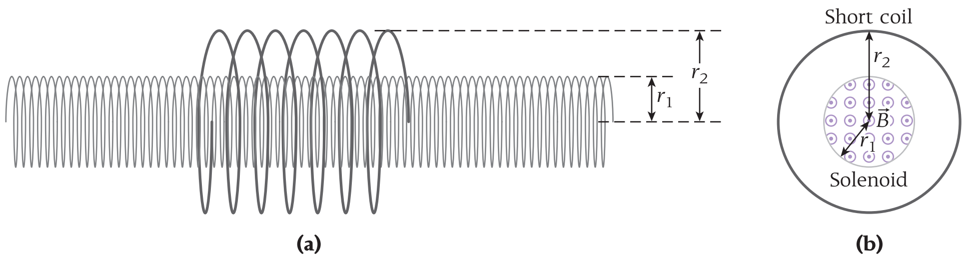


FIGURE 9.25 (a) A long solenoid of radius r_1 inside a short coil of radius r_2. (b) View of the two coils looking down the central axis.

– *Continued*

coil due to the solenoid is the number of turns in the short coil times the magnetic flux of the solenoid, divided by the current flowing in the solenoid. Having determined the mutual inductance, we can then calculate the potential difference induced in the short coil.

SKETCH Figure 9.25b shows a view of the two coils looking down their central axis.

RESEARCH We can formulate the mutual inductance between the coil and the solenoid as

$$M = \frac{N\Phi_{s\to c}}{i}, \tag{i}$$

where N is the number of turns in the short coil, $N\mathrm{F}_{s\to c}$ is the flux linkage in the coil resulting from the magnetic field of the solenoid, and i is the current in the solenoid. The flux can be expressed as

$$\Phi_{s\to c} = BA, \tag{ii}$$

where B is the magnitude of the magnetic field inside the solenoid and A is its cross-sectional area. Recall from Chapter 28 that for a solenoid, the magnitude of the magnetic field is

$$B = \mu_0 n i,$$

where n is the number of turns per unit length. The cross-sectional area of the solenoid is

$$A = \pi r_1^2. \tag{iii}$$

The potential difference induced in the short coil is then

$$\Delta V_{\text{ind}} = -M\frac{di}{dt}.$$

SIMPLIFY We can combine equations (i), (ii), and (iii) to obtain the mutual inductance between the two coils:

$$M = \frac{NBA}{i} = \frac{N(\mu_0 n i)(\pi r_1^2)}{i} = N\pi\mu_0 n r_1^2.$$

Then the potential difference induced in the short coil is

$$\Delta V_{\text{ind}} = -\left(N\pi\mu_0 n r_1^2\right)\frac{di}{dt}.$$

CALCULATE The change in the current is constant, so

$$\frac{di}{dt} = \frac{2.20 \text{ A}}{48.0\times 10^{-3} \text{ s}} = 45.8333 \text{ A/s}.$$

The mutual inductance between the two coils is

$$M = (31)\pi\left(4\pi\times 10^{-7} \text{ T m/A}\right)\left(290\times 10^2 \text{ m}^{-1}\right)\left(2.80\times 10^{-2} \text{ m}\right)^2 = 0.0027825 \text{ H}.$$

The potential difference induced in the short coil is then

$$\Delta V_{\text{ind}} = -(0.0027825 \text{ H})(45.8333 \text{ A/s}) = -0.127531 \text{ V}.$$

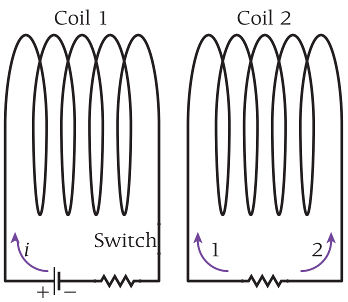
ROUND We report our result to three significant figures:

$$\Delta V_{\text{ind}} = -0.128 \text{ V}.$$

DOUBLE-CHECK The magnitude of the potential difference induced in the short outer coil is 128 mV, which is a magnitude that could be attained by moving a strong bar magnet in and out of a coil. Thus, our result seems reasonable.

Concept Check 9.6

Two identical coils are shown in the figure. Coil 1 has a current *i* flowing in the direction shown. When the switch in the circuit containing coil 1 is opened, what happens in coil 2?



a) A current is induced in coil 2 that flows in direction 1.

b) A current is induced in coil 2 that flows in direction 2.

c) No current is induced in coil 2.

9.8 RL Circuits

In Chapter 6, we saw that if a source of emf supplying a voltage, V_{emf}, is put into a single-loop circuit containing a resistor of resistance R and a capacitor of capacitance C, the charge, q, on the capacitor builds up over time according to

$$q = CV_{\text{emf}}\left(1 - e^{-t/\tau_{RC}}\right),$$

where the time constant of the circuit, $\tau_{RC} = RC$, is the product of the resistance and the capacitance. The same time constant governs the decrease of the initial charge, q_0, on the capacitor if the source of emf is suddenly removed and the circuit is short-circuited:

$$q = q_0 e^{-t/\tau_{RC}}.$$

If a source of emf is placed in a single-loop circuit containing a resistor with resistance R and an inductor with inductance L, called an **RL circuit,** a similar phenomenon occurs. Figure 9.26 shows a circuit in which a source of emf is connected to a resistor and an inductor in series. If the circuit included only the resistor and not the inductor, the current would increase almost instantaneously to the value given by Ohm's Law, $i = V_{emf}/R$, as soon as the switch was closed. However, in the circuit with both the resistor and the inductor, the increasing current flowing through the inductor creates a self-induced potential difference that tends to oppose the increase in current. As time passes, the change in current decreases, and the opposing self-induced potential difference also decreases. After a long time, the current becomes steady at the value V_{emf}/R.

We can use Kirchhoff's Loop Rule to analyze this circuit, assuming that the current, i, at any given time is flowing through the circuit in a counterclockwise direction. With current flowing counterclockwise around the circuit, the source of emf provides a gain in potential, $+V_{emf}$, and the resistor causes a drop in potential, $-iR$. The self-inductance of the inductor produces a drop in potential because it is opposing the increase in current. The drop in potential due to the inductor is proportional to the time rate of change of the current, as given by equation 9.22. Thus, we can write the sum of the potential drops around the circuit as

$$V_{emf} - iR - L\frac{di}{dt} = 0.$$

We can rewrite this equation as

$$L\frac{di}{dt} + iR = V_{emf}. \tag{9.28}$$

The solution to this differential equation is obtained in exactly the same way as the solution to the differential equation for the RC circuit was obtained in Chapter 6. The solution, which can be checked by substituting it into equation 9.28 is

$$i(t) = \frac{V_{emf}}{R}\left(1 - e^{-t/(L/R)}\right). \tag{9.9}$$

The quantity L/R is the time constant of the RL circuit:

$$\tau_{RL} = \frac{L}{R}. \tag{9.30}$$

This time dependence of the current in an RL circuit is shown in Figure 9.27a for three different values of the time constant.

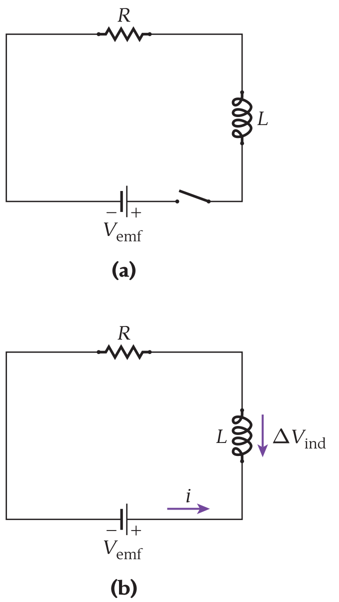


FIGURE 9.26 Single-loop circuit with a source of emf, a resistor, and an inductor: (a) switch open; (b) switch closed. When the switch is closed, the current flowing in the direction shown increases. A potential difference is induced across the inductor in the opposite direction, as shown.

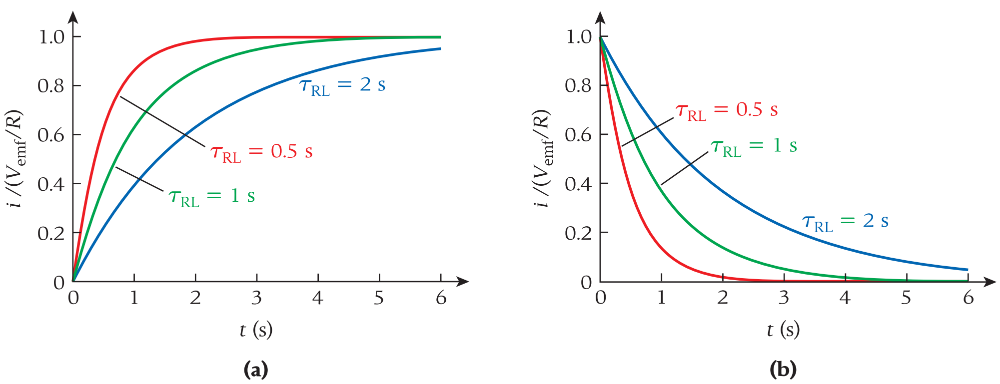


FIGURE 9.27 Time dependence of the current flowing through an RL circuit. (a) Current as a function of time when a resistor, an inductor, and a source of emf are connected in series. (b) Current as a function of time when the source of emf is suddenly removed from an RL circuit that has been connected for a long time.

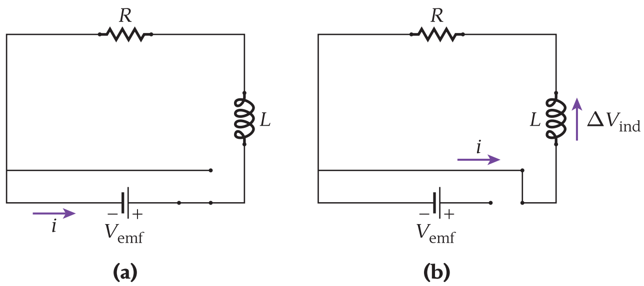
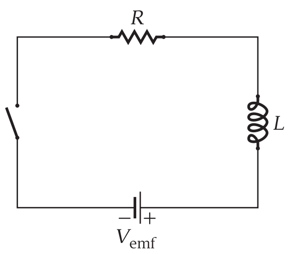


FIGURE 9.28 Single-loop circuit with a source of emf, a resistor, and an inductor. (a) The circuit with the source of emf connected. Current is flowing in the direction shown. (b) The source of emf is removed, and the resistor and the inductor are connected. Current flows in the same direction as before but is decreasing. A potential difference is induced across the inductor in the same direction as the current, as shown.

Concept Check 9.7

Consider the RL circuit shown in the figure. When the switch is closed, the current in the circuit increases exponentially to the value $i = V_{emf}/R$. If the inductor in this circuit is replaced with an inductor having three times the number of turns per unit length, the time required to reach a current of magnitude $0.9i$



a) increases.

b) decreases.

c) stays the same.

Looking at equation 9.9, you can see that for $t = 0$, the current is zero. For $t \to \infty$, the current is given by $i = V_{emf}/R$, which is as expected.

Now consider the circuit depicted in Figure 9.28, in which a source of emf was first connected and then suddenly removed. We can use equation 9.28 with $V_{emf} = 0$ to describe the time dependence of this circuit:

$$L\frac{di}{dt} + iR = 0. \tag{9.31}$$

The resistor causes a potential drop, and the inductor has a self-induced potential difference that tends to oppose the decrease in current. The solution of equation 9.31 is

$$i(t) = i_0 e^{-t/\tau_{RL}}. \tag{9.32}$$

The initial conditions when the source of emf is connected can be used to determine the initial current: $i_0 = V_{emf}/R$. Equation 9.32 describes a single-loop circuit with a resistor and an inductor that initially has a current i_0. The current drops exponentially with time with a time constant $\tau_{RL} = L/R$, and after a long time, the current in the circuit is zero. The current in this RL circuit as a function of time for three different values of the time constant is plotted in Figure 9.27b.

RL circuits can be used as timers to turn on devices at particular intervals and can also be used to filter out noise. However, these applications are usually handled with similar RC circuits because small capacitors are available in a wider range of capacitances than are inductors. The real value of inductors becomes apparent in circuits with all three components, resistors, capacitors, and inductors, which are covered in Chapter 10.

SOLVED PROBLEM 9.3 Work Done by a Battery

A series circuit contains a battery that supplies $V_{emf} = 40.0$ V, an inductor with $L = 2.20$ H, a resistor with $R = 160.0\ \Omega$, and a switch, connected as shown in Figure 9.9.

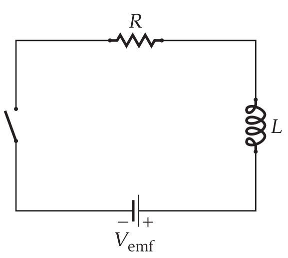


FIGURE 9.9 An RL circuit with a switch.

PROBLEM

The switch is closed at time $t = 0$. How much work is done by the battery between $t = 0$ and $t = 1.65 \times 10^{-2}$ s?

SOLUTION

THINK When the switch is closed, current begins to flow and power is provided by the battery. Power is defined as the voltage times the current at any given time. Work is the integral of the power over the time during which the circuit operates.

SKETCH Figure 9.30 shows a plot of the current in the RL circuit as a function of time.

RESEARCH The power in the circuit at any time t after the switch is closed is given by

$$P(t) = V_{emf}\,i(t), \tag{i}$$

where $i(t)$ is the current in the circuit. The current as a function of time for this circuit is given by equation 9.9,

$$i(t) = \frac{V_{\text{emf}}}{R}\left(1 - e^{-t/\tau_{\text{RL}}}\right), \tag{ii}$$

where $\tau_{\text{RL}} = L/R$. The work done by the battery is the integral of the power over the time in which the circuit has been in operation:

$$W = \int_0^T P(t)dt, \tag{iii}$$

where T is the time after the switch is closed.

SIMPLIFY We can combine equations (i), (ii), and (iii) to obtain

$$W = \int_0^T \frac{V_{\text{emf}}^2}{R}\left(1 - e^{-t/\tau_{\text{RL}}}\right) dt.$$

Evaluating the definite integral gives us

$$W = \frac{V_{\text{emf}}^2}{R}\left[t + \tau_{\text{RL}} e^{-t/\tau_{\text{RL}}}\right]_0^T = \frac{V_{\text{emf}}^2}{R}\left[T + \tau_{\text{RL}}\left(e^{-T/\tau_{\text{RL}}} - 1\right)\right]. \tag{iv}$$

CALCULATE First, we calculate the value of the time constant:

$$\tau_{\text{RL}} = \frac{L}{R} = \frac{2.20 \text{ H}}{160.0 \ \Omega} = 1.375 \times 10^{-2} \text{ s}$$

Putting all the numerical values into equation (iv) then gives

$$W = \frac{(40.0 \text{ V})^2}{160.0 \ \Omega}\left[\left(1.65 \times 10^{-2} \text{ s}\right) + \left(1.375 \times 10^{-2} \text{ s}\right)\left(e^{-\left(1.65\times 10^{-2} \text{ s}\right)/\left(1.375\times 10^{-2} \text{ s}\right)} - 1\right)\right] = 0.0689142 \text{ J}.$$

ROUND We report our result to three significant figures:

$$W = 6.89 \times 10^{-2} \text{ J}.$$

DOUBLE-CHECK To double-check our result, we assume that the current in the circuit is constant in time and equal to half of the final current:

$$i_{\text{ave}} = \frac{i(T)}{2} = \frac{\left(V_{\text{emf}}/R\right)\left(1 - e^{-T/\tau_{\text{RL}}}\right)}{2} = \frac{(40.0 \text{ V}/160.0 \ \Omega)\left(1 - e^{-\left(1.65\times 10^{-2} \text{ s}\right)/\left(1.375\times 10^{-2} \text{ s}\right)}\right)}{2} = 0.0874 \text{ A}.$$

This current would correspond to the average current if the current increased linearly with time. The work done would then be

$$W = PT = i_{\text{ave}} V_{\text{emf}} T = (0.0874 \text{ A})(40.0 \text{ V})\left(1.65 \times 10^{-2} \text{ s}\right) = 5.77 \times 10^{-2} \text{ J}.$$

This value is less than, but close to, our calculated result. Thus, our result seems reasonable.

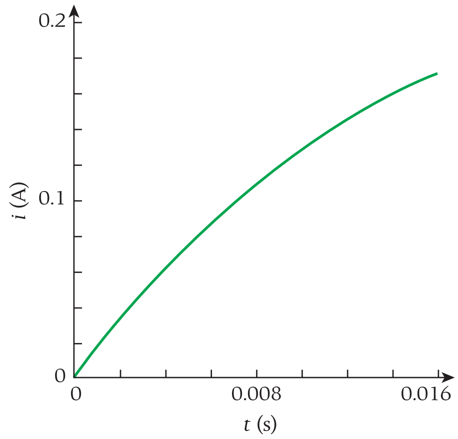


FIGURE 9.30 Current in the RL circuit as a function of time.

9.9 Energy and Energy Density of a Magnetic Field

We can think of an inductor as a device that can store energy in a magnetic field, similar to the way a capacitor can store energy in an electric field. The energy stored in the electric field of a capacitor is given by

$$U_E = \tfrac{1}{2}\frac{q^2}{C}.$$

Consider an inductor connected to a source of emf. Current begins to flow through the inductor, producing a self-induced potential difference opposing the increase in current. The instantaneous power provided by the source of emf is the product of the current and the voltage of the emf source, V_{emf}. Using equation 9.28 with $R = 0$, we can write

$$P = V_{\text{emf}} i = \left(L\frac{di}{dt}\right)i. \tag{9.33}$$

Integrating this power over the time it takes to reach a final current i in the circuit yields the energy provided by the source of emf. Since there are no resistive losses in this circuit, this amount of energy must be stored in the magnetic field of the inductor. Therefore,

$$U_B = \int_0^t P\,dt = \int_0^i Li'\,di' = \tfrac{1}{2}Li^2. \tag{9.34}$$

Equation 9.34 has a form similar to the analogous equation for a capacitor's electric field, with q replaced by i and $1/C$ replaced by L.

Now let's consider an ideal solenoid with length ℓ, cross-sectional area A, and n turns per unit length, carrying current i. The energy stored in the magnetic field of the solenoid according to equation 9.21 is

$$U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}\mu_0 n^2 \ell A i^2.$$

The magnetic field occupies the volume enclosed by the solenoid, which is given by ℓA. Thus, the energy density, u_B, of the magnetic field of the solenoid is

$$u_B = \frac{\tfrac{1}{2}\mu_0 n^2 \ell A i^2}{\ell A} = \tfrac{1}{2}\mu_0 n^2 i^2.$$

Since $B = \mu_0 n i$ for a solenoid, the energy density of the magnetic field of a solenoid can be expressed as

$$u_B = \frac{1}{2\mu_0}B^2. \tag{9.35}$$

Although we derived this expression for the special case of a solenoid, it applies to magnetic fields in general.

Concept Check 9.8

A long solenoid has a circular cross section of radius $r = 8.10$ cm, a length $\ell = 0.540$ m, and $n = 2.00 \times 10^4$ turns/m. The solenoid stores 42.5 mJ of energy when it carries a current i. If the current is doubled, to $2i$, the energy stored in the solenoid

a) decreases by a factor of 4.

b) decreases by a factor of 2.

c) remains the same.

d) increases by a factor of 2.

e) increases by a factor of 4.

9.10 Applications to Information Technology

Computers and many consumer electronics devices use magnetization and induction to store and retrieve information. Examples are computer hard drives, videotapes, audiotapes, and the magnetic strips on credit cards. During the last decade, the use of storage media based on other technologies, such as the optical storage of information on CDs and DVDs and the flash memory cards in digital cameras, has increased; however, magnetic storage devices are still a technological mainstay and the basis of a multibillion dollar industry.

Computer Hard Drive

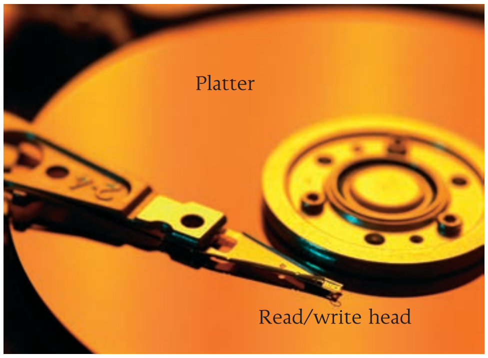


FIGURE 9.31 The read/write head and spinning platter inside a computer hard drive.

One device that stores information using magnetization and induction is the computer hard drive. The hard drive stores information in the form of *bits*, the binary code consisting of zeros and ones. Eight bits make a *byte*, which can represent a number or an alphanumeric character. A modern hard drive can hold up to 2 terabytes (2×10^{12} bytes) of information. A hard drive consists of one or more rotating platters with a ferromagnetic coating accessed by a movable read/write head, as shown in Figure 9.31.

The read/write head can be positioned to access any one of many tracks on the rotating platter. The operation of a read/write head in a conventional hard drive is illustrated in Figure 9.32a. As the coated platter moves below the read/write head, a pulse of current in one direction magnetizes the surface of the platter to represent a binary one, or a pulse of current in the opposite direction magnetizes the surface to represent a binary zero. In Figure 9.32a, a binary one is shown as a red arrow pointing to the right, and a binary zero is shown as a green arrow pointing to the left. In read mode, when the magnetized areas of the platter pass beneath the read sensor, a positive or negative current is induced, and the electronics of the hard drive can tell if the information is a zero or a one. The method used to encode and read back data shown in Figure 9.32a is called *longitudinal encoding* because the magnetic fields of the magnetized areas of the platter are parallel or antiparallel to the motion of the platter. The data storage capacity of hard drives was increased by

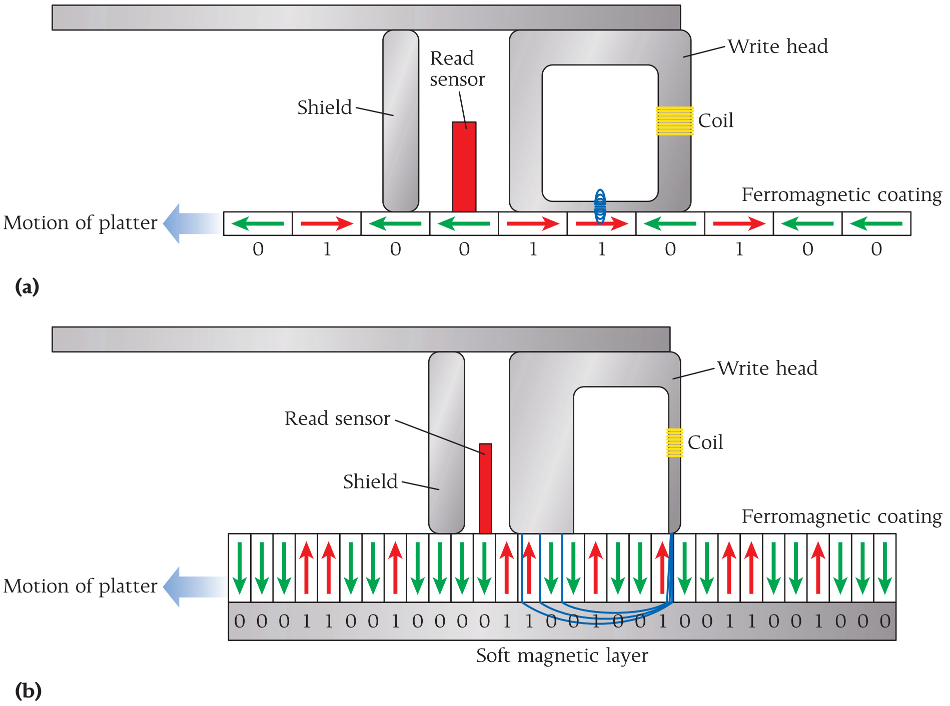


FIGURE 9.32 The read/write head of a computer hard drive. (a) Longitudinal encoding of information on the spinning platter. (b) Perpendicular encoding of information on the spinning platter.

making the magnetized areas smaller and by adding more platters and read/write heads. However, manufacturers found it difficult to construct hard drives holding more than 250 gigabytes (250×10^9 bytes) using this technique. As the manufacturers made the bits smaller, the bits interfered with each other, causing some bits to flip randomly and introduce errors in the stored data.

Recently, the technique of *perpendicular encoding* of data has been developed, illustrated in Figure 9.32b. Again, a read/write head is used above a spinning platter coated with a ferromagnetic substance. However, in this case, the magnetic fields are perpendicular to the surface of the platter, which allows a tighter packing of the bits and increases the capacity of the hard drive. The platter is constructed with a thicker ferromagnetic coating and a soft magnetic material on the bottom that acts to contain the magnetic field lines. Note that the magnetic field lines at the pointed end of the write head are very close together, whereas the magnetic field lines returning to the blunt end of the write head are widely spaced. Thus, the ferromagnetic coating of the platter is strongly magnetized in the up or down direction, depending on the direction of the current pulse through the coil of the write head, while the bits closer to the read sensor are not affected.

Hard drives using perpendicular encoding also incorporate the phenomenon called *giant magnetoresistance (GMR),* which allows the construction of a very small and sensitive read sensor. The French physicist Albert Fert and the German physicist Peter Grünberg received the Nobel Prize in Physics in 2007 for the discovery of this effect. Hard drives with information storage capacities of up to 2 terabytes or more that use perpendicular encoding and GMR read sensors are widely available. iPods with a storage capacity above 64 GB are an example of a device using this technology (iPod Touch and iPhones use a different storage technology that has no moving parts). The fact that you can watch full-length movies on your iPod and carry many thousands of songs in it as well is a direct result of physics research performed during the last two decades. And as research in nanoscience and nanotechnology continues to produce exciting results for technological applications, the amazing growth in the capabilities of consumer electronics devices will continue for the foreseeable future.

WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- According to Faraday's Law of Induction, the induced potential difference, ΔV_{ind}, in a conducting loop is given by the negative of the time rate of change of the magnetic flux passing through the loop: $\Delta V_{\text{ind}} = -\frac{d\Phi_B}{dt}$.
- The magnetic flux, Φ_B, is given by $\Phi_B = \iint \vec{B} \bullet d\vec{A}$, where $\vec{B}$ is the magnetic field and $d\vec{A}$ is the differential area element defined by a vector normal to the surface through which the magnetic field passes.
- For a constant magnetic field, $\vec{B}$, the magnetic flux, Φ_B, passing through an area, A, is given by $\Phi_B = BA\cos\theta$, where θ is the angle between the magnetic field vector and a normal vector to the area.
- Lenz's Law states that a changing magnetic flux through a conducting loop induces a current in the loop that opposes the change in magnetic flux.
- A magnetic field that is changing in time induces an electric field given by $\oint \vec{E} \bullet d\vec{s} = -\frac{d\Phi_B}{dt}$, where the integration is done over any closed path in the magnetic field.
- The inductance, L, of a device with conducting loops is the flux linkage (the product of the number of loops, N, and the magnetic flux, Φ_B) divided by the current, i: $L = \frac{N\Phi_B}{i}$.
- The inductance of a solenoid is given by $L = \mu_0 n^2 \ell A$, where n is the number of turns per unit length, ℓ is the length of the solenoid, and A is the cross-sectional area of the solenoid.
- The self-induced potential difference, $\Delta V_{\text{ind},L}$, for any inductor is given by $\Delta V_{\text{ind},L} = -L\frac{di}{dt}$, where L is the inductance of the device and $\frac{di}{dt}$ is the time rate of change of the current flowing through the inductor.
- A single-loop circuit with an inductance L and a resistance R has a characteristic time constant of $\tau_{\text{RL}} = \frac{L}{R}$.
- The energy, U_B, stored in the magnetic field of an inductor with an inductance L and carrying a current i is given by $U_B = \frac{1}{2}Li^2$.

ANSWERS TO SELF-TEST OPPORTUNITIES

9.1 $\frac{dB}{dt} = -\frac{0.500\ \text{T}}{0.250\ \text{s}} = -2.00\ \text{T/s}$

$$\Delta V_{\text{ind}} = -\frac{d\Phi_B}{dt} = -\frac{d(BA)}{dt} = -\pi r^2\frac{dB}{dt}$$

$$r = \sqrt{\frac{|\Delta V_{\text{ind}}|}{\pi|dB/dt|}} = \sqrt{\frac{1.24\ \text{V}}{\pi(2.00\ \text{T/s})}} = 0.444\ \text{m}.$$

9.2 As the loop enters the magnetic field, the magnetic flux is increasing. The current induced in the loop will be in the counterclockwise direction to oppose the increase in the flux. As the loop exits the magnetic field, the magnetic flux is decreasing. The current induced in the loop will be in a clockwise direction to oppose the decrease in the flux.

9.3 If the induced potential difference were equal to the change in the magnetic flux, then any increase in the flux going through a coil (perhaps from just a minute random fluctuation in the ambient magnetic field in the room) would lead to an induced potential difference, which would produce a current in the coil, which would act to increase the flux, which would lead to a larger induced potential difference, a larger current, and an even larger increase in flux. In other words, a runaway situation would result, which clearly violates energy conservation.

9.4 a) true b) false c) true d) true

PROBLEM-SOLVING GUIDELINES

1. To solve a problem involving electromagnetic induction, first ask: What is making the magnetic flux change? If the magnetic field is changing, you need to use equation 9.9; if the area through which the flux passes is changing, you need to use equation 9.10; and if the orientation between the magnetic field and the area is changing, you need to use equation 9.11. You don't need to memorize these equations, as long as you remember Faraday's Law (equation 9.5) and the definition of magnetic flux (equation 9.1).

2. Once you know which elements of the problem situation are constant and which are changing, use Lenz's Law to determine the direction of the induced current and the locations of higher and lower potential. Then, you can pick a direction for the differential area vector, $d\vec{A}$, of the flux and calculate the unknown quantities.

MULTIPLE-CHOICE QUESTIONS

9.1 A solenoid with 200 turns and a cross-sectional area of 60 cm^2 has a magnetic field of 0.60 T along its axis. If the field is confined within the solenoid and changes at a rate of 0.20 T/s, the magnitude of the induced potential difference in the solenoid will be

a) 0.0020 V. b) 0.02 V. c) 0.001 V. d) 0.24 V.

9.2 The rectangular loop of wire in Figure 9.9 is pulled with a constant acceleration from a region of zero magnetic field into a region of a uniform magnetic field. During this process, the current induced in the loop

a) will be zero.
b) will be some constant value that is not zero.
c) will increase linearly with time.
d) will increase exponentially with time.
e) will increase linearly with the square of the time.

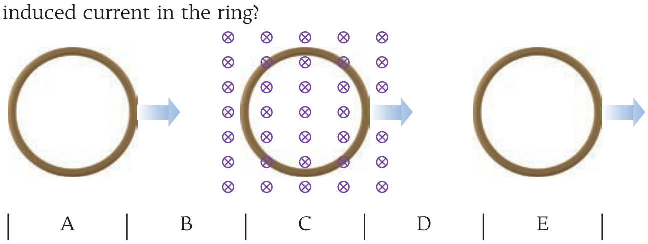
9.3 Which of the following will induce a current in a loop of wire in a uniform magnetic field?

a) decreasing the strength of the field
b) rotating the loop about an axis parallel to the field
c) moving the loop within the field
d) all of the above
e) none of the above

9.4 Faraday's Law of Induction states that

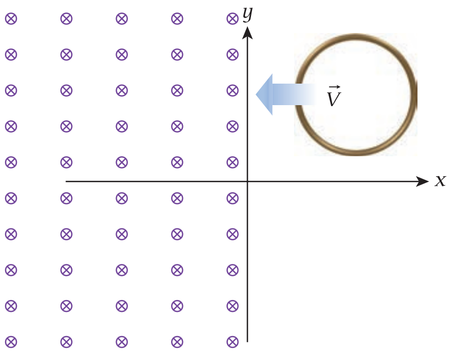
a) a potential difference is induced in a loop when there is a change in the magnetic flux through the loop.
b) the current induced in a loop by a changing magnetic field produces a magnetic field that opposes this change in magnetic field.
c) a changing magnetic field induces an electric field.
d) the inductance of a device is a measure of its opposition to changes in current flowing through it.
e) magnetic flux is the product of the average magnetic field and the area perpendicular to it that it penetrates.

9.5 A conducting ring is moving from left to right through a uniform magnetic field, as shown in the figure. In which region(s) is there an induced current in the ring?



a) regions B and D
b) regions B, C, and D
c) region C
d) regions A through E

9.6 A circular loop of wire moving in the *xy*-plane with a constant velocity in the negative *x*-direction enters a uniform magnetic field, which covers the region in which $x < 0$, as shown in the figure. The surface normal vector of the loop points in the direction of the magnetic field. Which of the following statements is correct?



a) The induced potential difference in the loop is at a maximum as the edge of the loop just enters the region with the magnetic field.
b) The induced potential difference in the loop is at a maximum when one fourth of the loop is in the region with the magnetic field.
c) The induced potential difference in the loop is at a maximum when the loop is halfway into the region with the magnetic field.
d) The induced potential difference in the loop is constant from the instant the loop starts to enter the region with the magnetic field.

9.7 Which of the following statements regarding self-induction is correct?

a) Self-induction occurs only when a direct current is flowing through a circuit.
b) Self-induction occurs only when an alternating current is flowing through a circuit.
c) Self-induction occurs when either a direct current or an alternating current is flowing through a circuit.
d) Self-induction occurs when either a direct current or an alternating current is flowing through a circuit as long as the current is varying.

9.8 You have a light bulb, a bar magnet, a spool of wire that you can cut into as many pieces as you want, and nothing else. How can you get the bulb to light up?

a) You can't. The bulb needs electricity to light it, not magnetism.
b) You cut a length of wire, connect the light bulb to the two ends of the wire, and pass the magnet through the loop that is formed.
c) You cut two lengths of wire and connect the magnet and the bulb in series.

9.9 Calculate the potential difference induced between the tips of the wings of a Boeing 747-400 with a wingspan of 64.67 m when it is in level flight at a speed of 913 km/h. Assume that the magnitude of the downward component of the Earth's magnetic field is $B = 5.00 \times 10^{-5}$ T.

a) 0.820 V c) 10.4 V e) 225 V
b) 2.95 V d) 30.1 V

9.10 A long solenoid with a circular cross section of radius $r_1 = 2.80$ cm and $n = 290$ turns/cm is inside of and coaxial with a short coil that has a circular cross section of radius $r_2 = 4.90$ cm and $N = 31$ turns. Suppose the current in the short coil is increased steadily from zero to $i = 2.80$ A in 18.0 ms. What is the magnitude of the potential difference induced in the solenoid while the current in the short coil is changing?

a) 0.0991 V c) 0.233 V e) 0.750 V
b) 0.128 V d) 0.433 V

9.11 A long solenoid has a circular cross section of radius $r = 8.10$ cm, a length $\ell = 0.540$ m, and $n = 2.00 \times 10^4$ turns/m. The solenoid is carrying a current of magnitude $i = 4.04 \times 10^{-3}$ A. How much energy is stored in the magnetic field of the solenoid?

a) 2.11×10^{-7} J c) 4.57×10^{-5} J e) 4.55×10^{-1} J
b) 8.91×10^{-6} J d) 6.66×10^{-3} J

9.12 Suppose the current in the short coil in Solved Problem 9.2 is increased steadily from zero to $i = 2.80$ A in 18.0 ms. What is the magnitude of the potential difference induced in the solenoid while the current in the short coil is changing?

a) 0.0991 V c) 0.233 V e) 0.750 V
b) 0.128 V d) 0.433 V

9.13 Suppose the length of the rotating rod in Solved Problem 9.1 is increased by a factor of 2. By what factor does the power dissipated in the resistor change?

a) $\frac{1}{2}$ c) 4 e) 16
b) 2 d) 8

9.14 Suppose the resistance of the resistor in Solved Problem 9.1 is increased by a factor of 2. By what factor does the power dissipated in the resistor change?

a) $\frac{1}{2}$ c) 4 e) 16
b) 2 d) 8

CONCEPTUAL QUESTIONS

9.15 When you plug a refrigerator into a wall socket, on occasion, a spark appears between the prongs. What causes this?

9.16 People with pacemakers or other mechanical devices as implants are often warned to stay away from large machinery or motors. Why?

9.17 Recall damped harmonic oscillators, in which the damping force is velocity dependent and always opposes the motion of the oscillator. One way of producing this type of force is to use a piece of metal, such as aluminum, that moves through a nonuniform magnetic field. Explain why this technique is capable of producing a damping force.

9.18 In a popular lecture demonstration, a cylindrical permanent magnet is dropped down a long aluminum tube as shown in the figure. Neglecting friction of the magnet against the inner walls of the tube and assuming that the tube is very long compared to the size of the magnet, will the magnet accelerate downward with an acceleration equal to g (free fall)? If not, describe the eventual motion of the magnet. Does it matter if the north pole or south pole of the magnet is on the lower side?

9.19 A popular demonstration of eddy currents involves dropping a magnet down a long metal tube and a long glass or plastic tube. As the magnet falls through a tube, the magnetic flux changes as the magnet moves toward or away from each part of the tube.

a) Which tube has the larger voltage induced in it?

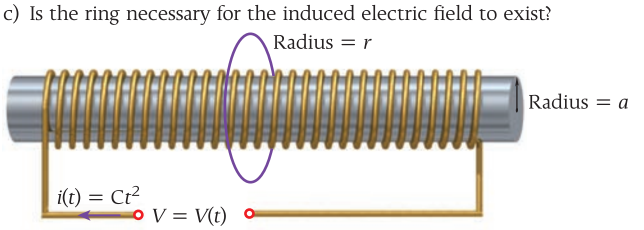
b) Which tube has the larger eddy currents induced in it?

9.20 The current in a very long, tightly wound solenoid with radius a and n turns per unit length varies over time according to the equation $i(t) = Ct^2$, where the current i is in amps and the time t is in seconds, and C is a constant with appropriate units. Concentric with the solenoid is a conducting ring of radius r, as shown in the figure.

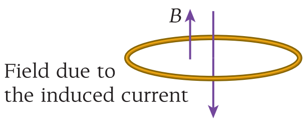
a) Write an expression for the potential difference induced in the ring.

b) Write an expression for the magnitude of the electric field induced at an arbitrary point on the ring.

c) Is the ring necessary for the induced electric field to exist?



9.21 A circular wire ring experiences an increasing magnetic field in the upward direction, as shown in the figure. What is the direction of the induced current in the ring?

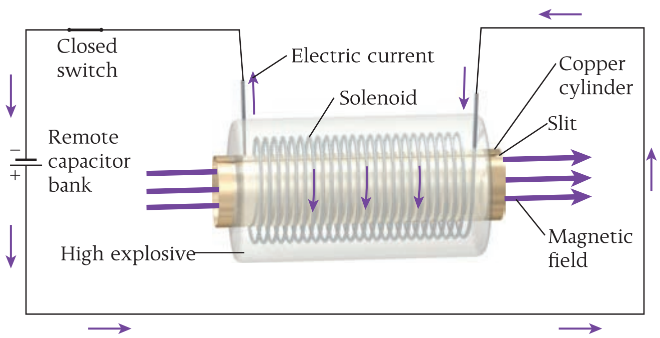


9.22 A square conducting loop with sides of length L is rotating at a constant angular speed, ω, in a uniform magnetic field of magnitude B. At time $t = 0$, the loop is oriented so that the direction normal to the loop is aligned with the magnetic field. Find an expression for the potential difference induced in the loop as a function of time.

9.23 A solid metal disk of radius R is rotating around its center axis at a constant angular speed of ω. The disk is in a uniform magnetic field of magnitude B that is oriented normal to the surface of the disk. Calculate the magnitude of the potential difference between the center of the disk and the outside edge.

9.24 Large electric fields are certainly a hazard to the human body, as they can produce dangerous currents, but what about large magnetic fields? A man 1.80 m tall walks at 2.00 m/s perpendicular to a horizontal magnetic field of 5.0 T; that is, he walks between the pole faces of a very big magnet. (Such a magnet can, for example, be found in the National Superconducting Cyclotron Laboratory at Michigan State University.) Given that his body is full of conducting fluids, estimate the potential difference induced between his head and feet.

9.25 At Los Alamos National Laboratories, one means of producing very large magnetic fields is the *EPFCG (explosively-pumped flux compression generator)*, which is used to study the effects of a high-power electromagnetic pulse (EMP) in electronic warfare. Explosives are packed and detonated in the space between a solenoid and a small copper cylinder coaxial with and inside the solenoid, as shown in the figure. The explosion occurs in a very short time and collapses the cylinder rapidly. This rapid change creates inductive currents that keep the magnetic flux constant while the cylinder's radius shrinks by a factor of r_i/r_f. Estimate the magnetic field produced, assuming that the radius is compressed by a factor of 14 and the initial magnitude of the magnetic field, B_i, is 1.0 T.



9.26 A metal hoop is laid flat on the ground. A magnetic field that is directed upward, out of the ground, is increasing in magnitude. As you look down on the hoop from above, what is the direction of the induced current in the hoop?

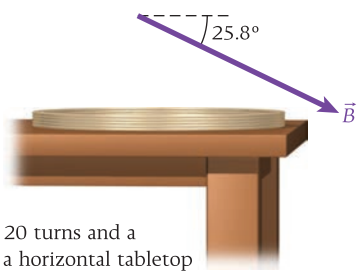
9.27 The wire of a tightly wound solenoid is unwound and then rewound to form another solenoid with double the diameter of the first solenoid. By what factor will the inductance change?

EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

Sections 9.1 and 9.2

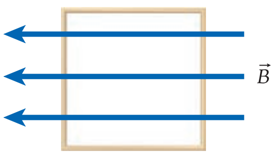
9.28 A circular coil of wire with 20 turns and a radius of 40.0 cm is laying flat on a horizontal tabletop as shown in the figure. There is a uniform magnetic field extending over the entire table with a magnitude of 5.00 T and directed to the north and downward, making an angle of 25.8° with the horizontal. What is the magnitude of the magnetic flux through the coil?



9.9 When a magnet in an MRI is abruptly shut down, the magnet is said to be *quenched*. Quenching can occur in as little as 20.0 s. Suppose a magnet with an initial field of 1.20 T is quenched in 20.0 s, and the final field is approximately zero. Under these conditions, what is the average induced potential difference around a conducting loop of radius 1.00 cm (about the size of a wedding ring) oriented perpendicular to the field?

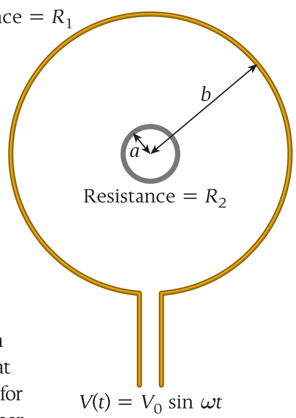
9.30 An 8-turn coil has square loops measuring 0.200 m along a side and a resistance of 3.00 Ω. It is placed in a magnetic field that makes an angle of 40.0° with the plane of each loop. The magnitude of this field varies with time according to $B = 1.50t^3$, where t is measured in seconds and B in teslas. What is the induced current in the coil at $t = 2.00$ s?

9.31 A metal loop has an area of 0.100 m^2 and is placed flat on the ground. There is a uniform magnetic field pointing due west, as shown in the figure. This magnetic field initially has a magnitude of 0.123 T, which decreases steadily to 0.075 T during a period of 0.579 s. Find the potential difference induced in the loop during this time.



•**9.32** A respiration monitor has a flexible loop of copper wire, which wraps about the chest. As the wearer breathes, the radius of the loop of wire increases and decreases. When a person in the Earth's magnetic field (assume 0.426×10^{-4} T) inhales, what is the average current in the loop, assuming that it has a resistance of 30.0 Ω and increases in radius from 20.0 cm to 25.0 cm over 1.00 s? Assume that the magnetic field is perpendicular to the plane of the loop.

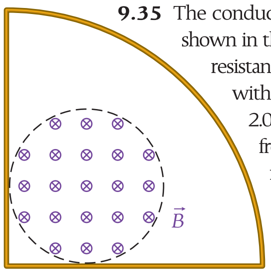
•**9.33** A circular conducting loop with radius a and resistance R_2 is concentric with a circular conducting loop with radius $b \gg a$ (b much greater than a) and resistance R_1. A time-dependent voltage is applied to the larger loop; its slow sinusoidal variation in time is given by $V(t) = V_0 \sin \omega t$, where V_0 and ω are constants with dimensions of voltage and inverse time, respectively. Assuming that the magnetic field throughout the inner loop is uniform (constant in space) and equal to the field at the center of the loop, derive expressions for the potential difference induced in the inner loop and the current i through that loop.



••**9.34** A long solenoid with cross-sectional area A_1 surrounds another long solenoid with cross-sectional area $A_2 < A_1$ and resistance R. Both solenoids have the same length and the same number of turns. A current given by $i = i_0 \cos \omega t$ is flowing through the outer solenoid. Find an expression for the magnetic field in the inner solenoid due to the induced current.

Section 9.3

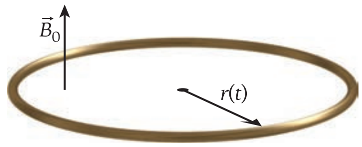
9.35 The conducting loop in the shape of a quarter-circle shown in the figure has a radius of 10.0 cm and a resistance of 0.200 Ω. The magnetic field strength within the dotted circle of radius 3.00 cm is initially 2.00 T. The magnetic field strength then decreases from 2.00 T to 1.00 T in 2.00 s. Find (a) the magnitude and (b) the direction of the induced current in the loop.



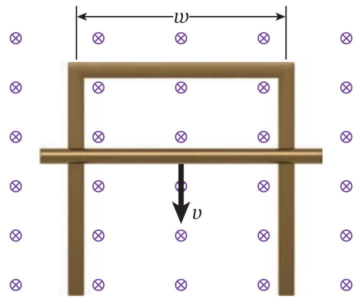
9.36 A supersonic aircraft with a wingspan of 10.0 m is flying over the north magnetic pole (in a magnetic field of magnitude 0.500 G oriented perpendicular to the ground) at a speed of three times the speed of sound (Mach 3). What is the potential difference between the tips of the wings? Assume that the wings are made of aluminum.

•**9.37** A helicopter hovers above the north magnetic pole in a magnetic field of magnitude 0.426 G and oriented perpendicular to the ground. The helicopter rotors are 10.0 m long, are made of aluminum, and rotate about the hub with a rotational speed of 1.00×10^4 rpm. What is the potential difference from the hub to the end of a rotor?

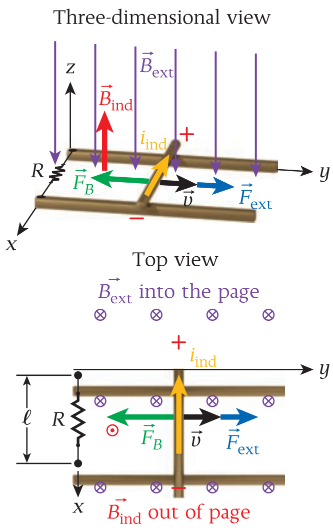
•**9.38** An elastic circular conducting loop expands at a constant rate over time such that its radius is given by $r(t) = r_0 + vt$, where $r_0 = 0.100$ m and $v = 0.0150$ m/s. The loop has a constant resistance of $R = 12.0\ \Omega$ and is placed in a uniform magnetic field of magnitude $B_0 = 0.750$ T, perpendicular to the plane of the loop, as shown in the figure. Calculate the direction and the magnitude of the induced current, i, at $t = 5.00$ s.



•**9.39** A rectangular frame of conducting wire has negligible resistance and width w and is held vertically in a magnetic field of magnitude B, as shown in the figure. A metal bar with mass m and resistance R is placed across the frame, maintaining contact with the frame. Derive an expression for the terminal velocity of the bar if it is allowed to fall freely along this frame starting from rest. Neglect friction between the wires and the metal bar.



•**9.40** Two parallel conducting rails with negligible resistance are connected at one end by a resistor of resistance R, as shown in the figure. The rails are placed in a magnetic field $\vec{B}_{ext}$, which is perpendicular to the plane of the rails. This magnetic field is uniform and time independent. The distance between the rails is ℓ. A conducting rod slides without friction on top of the two rails at constant velocity $\vec{v}$.



a) Using Faraday's Law of Induction, calculate the magnitude of the potential difference induced in the moving rod.

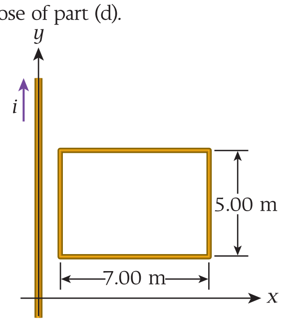
b) Calculate the magnitude of the induced current in the rod, i_{ind}.

c) Show that for the rod to move at a constant velocity as shown, it must be pulled with an external force, $\vec{F}_{ext}$ and calculate the magnitude of this force.

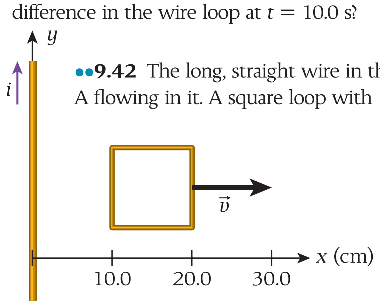
d) Calculate the work done, W_{ext}, and the power generated, P_{ext}, by the external force in moving the rod.

e) Calculate the power used (dissipated) by the resistor, P_R. Explain the correlation between this result and those of part (d).

•**9.41** A long, straight wire runs along the y-axis. The wire carries a current in the positive y-direction that is changing as a function of time according to $i = 2.00$ A $+ (0.300$ A/s$)t$. A loop of wire is located in the xy-plane near the y-axis, as shown in the figure. The loop has dimensions 7.00 m by 5.00 m and is 1.00 m away from the wire. What is the induced potential difference in the wire loop at $t = 10.0$ s?



••**9.42** The long, straight wire in the figure has a current $i = 1.00$ A flowing in it. A square loop with 10.0-cm sides and a resistance of 0.0200 Ω is positioned 10.0 cm away from the wire. The loop is then moved in the positive x-direction with a speed $v = 10.0$ cm/s.



a) Find the direction of the induced current in the loop.

b) Identify the directions of the magnetic forces acting on all sides of the square loop.

c) Calculate the direction and the magnitude of the net force acting on the loop at the instant it starts to move.

Section 9.4

9.43 A simple generator consists of a loop rotating inside a constant magnetic field (see Figure 9.19). If the loop is rotating with frequency f, the magnetic flux is given by $\Phi(t) = BA\cos(2\pi ft)$. If $B = 1.00$ T and $A = 1.00$ m^2, what must the value of f be for the maximum induced potential difference to be 110. V?

•9.44 A motor has a single loop inside a magnetic field of magnitude 0.870 T. If the area of the loop is 300. cm^2, find the maximum angular speed possible for this motor when connected to a source of emf providing 170. V.

•9.45 Your friend decides to produce electrical power by turning a coil of 1.00×10^5 circular loops of wire around an axis perpendicular to the Earth's magnetic field, which has a local magnitude of 0.300 G. The loops have a radius of 25.0 cm.

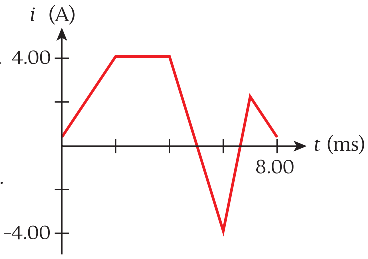
a) If your friend turns the coil at a frequency of 150. Hz, what peak current will flow in a resistor, $R = 1.50$ kΩ, connected to the coil?

b) The average current flowing in the coil will be 0.7071 times the peak current. What will be the average power obtained from this device?

Sections 9.6 and 9.7

9.46 Find the mutual inductance of the solenoid and the coil described in Example 9.1 and the induced potential difference in the coil at $t = 2.0$ s using the techniques described in Section 9.7. How do the results for the induced potential difference compare?

9.47 The figure shows the current through a 10.0-mH inductor over a time interval of 8.00 ms. Draw a graph showing the self-induced potential difference, $\Delta V_{\text{ind},L}$, for the inductor over the same interval.



•9.48 A short coil with radius $R = 10.0$ cm contains $N = 30.0$ turns and surrounds a long solenoid with radius $r = 8.00$ cm containing $n = 60$ turns per centimeter. The current in the short coil is increased at a constant rate from zero to $i = 2.00$ A in a time of $t = 12.0$ s. Calculate the induced potential difference in the long solenoid while the current is increasing in the short coil.

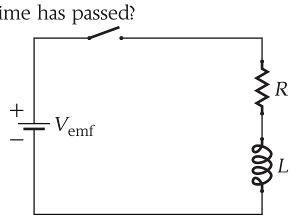
Section 9.8

9.49 Consider an RL circuit with resistance $R = 1.00$ MΩ and inductance $L = 1.00$ H, which is powered by a 10.0-V battery.

a) What is the time constant of the circuit?

b) If the switch is closed at time $t = 0$, what is the current just after that time? After 2.00 μs? When a long time has passed?

9.50 In the circuit in the figure, $R = 120.$ Ω, $L = 3.00$ H, and $V_{\text{emf}} = 40.0$ V. After the switch is closed, how long will it take the current in the inductor to reach 300. mA?



9.51 The current is increasing at a rate of 3.60 A/s in an RL circuit with $R = 3.25$ Ω and $L = 440.$ mH. What is the potential difference across the circuit at the moment when the current in the circuit is 3.00 A?

•9.52 In the circuit in the figure, a battery supplies $V_{\text{emf}} = 18.0$ V and $R_1 = 6.00$ Ω, $R_2 = 6.00$ Ω, and $L = 5.00$ H. Calculate each of the following *immediately* after the switch is closed:

a) the current flowing out of the battery

b) the current through R_1

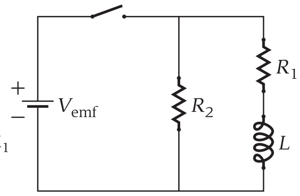
c) the current through R_2

d) the potential difference across R_1

e) the potential difference across R_2

f) the potential difference across L

g) the rate of current change across R_1



•9.53 In the circuit in the figure, a battery supplies $V_{\text{emf}} = 18.0$ V and $R_1 = 6.00$ Ω, $R_2 = 6.00$ Ω, and $L = 5.00$ H. Calculate each of the following a *long time* after the switch is closed:

a) the current flowing out of the battery

b) the current through R_1

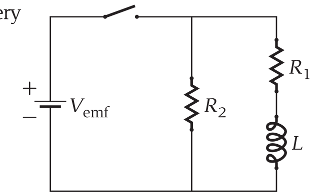
c) the current through R_2

d) the potential difference across R_1

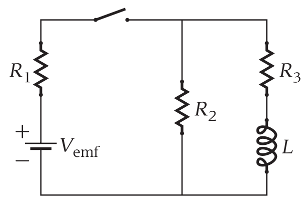
e) the potential difference across R_2

f) the potential difference across L

g) the rate of current change across R_1



•9.54 A circuit contains a battery, three resistors, and an inductor, as shown in the figure. What will be the current through each resistor (a) immediately after the switch is closed and (b) a long time after the switch is closed? (c) Suppose the switch is reopened a long time after it has been closed. What is the current in each resistor? After a long time?



Section 9.9

9.55 Having just learned that there is energy associated with magnetic fields, an inventor sets out to tap the energy associated with the Earth's magnetic field. What volume of space near Earth's surface contains 1.00 J of energy, assuming the strength of the magnetic field to be 5.00×10^{-5} T?

9.56 A clinical MRI (magnetic resonance imaging) superconducting magnet can be approximated as a solenoid with a diameter of 1.00 m, a length of 1.50 m, and a uniform magnetic field of 3.00 T. Determine (a) the energy density of the magnetic field and (b) the total energy in the solenoid.

9.57 A *magnetar* (magnetic neutron star) has a magnetic field near its surface of magnitude 4.00×10^{10} T.

a) Calculate the energy density of this magnetic field.

b) The Special Theory of Relativity associates energy with any mass m at rest according to $E_0 = mc^2$. Find the rest mass density associated with the energy density of part (a).

•9.58 An emf of 20.0 V is applied to a coil with an inductance of 40.0 mH and a resistance of 0.500 Ω.

a) Determine the energy stored in the magnetic field when the current reaches $\frac{1}{4}$ of its maximum value.

b) How long does it take for the current to reach this value?

•9.59 A student wearing a 15.0 g gold band with radius 0.750 cm (and with a resistance of 61.9 μΩ and a specific heat capacity of $c = 129$ J/kg °C) on her finger moves her finger from a region having a magnetic field of 0.0800 T, pointing along her finger, to a region with zero magnetic field in 40.0 ms. As a result of this action, thermal energy is added to the band due to the induced current, which raises the temperature of the band. Calculate the temperature rise in the band, assuming that all the energy produced is used in raising the temperature.

•9.60 A coil with N turns and area A, carrying a constant current, i, flips in an external magnetic field, $\vec{B}_{ext}$, so that its dipole moment switches from opposition to the field to alignment with the field. During this process, induction produces a potential difference that tends to reduce the current in the coil. Calculate the work done by the coil's power supply to maintain the constant current.

••**9.61** An electromagnetic wave propagating in vacuum has electric and magnetic fields given by $\vec{E}(\vec{x},t)=\vec{E}_0\cos(\vec{k}\bullet\vec{x}-\omega t)$ and $\vec{B}(\vec{x},t)=\vec{B}_0\cos(\vec{k}\bullet\vec{x}-\omega t)$, where $\vec{B}_r$ is given by $\vec{B}_0=\vec{k}\times\vec{E}_0/\omega$ and the wave vector $\vec{k}$ is perpendicular to both $\vec{E}_0$ and $\vec{B}_0$. The magnitude of $\vec{k}$ and the angular frequency ω satisfy the dispersion relation, $\omega/\left|\vec{k}\right|=(\mu_0\varepsilon_0)^{-1/}$ where μ_0 and ε_0 are the permeability and permittivity of free space, respectively. Such a wave transports energy in both its electric and magnetic fields. Calculate the ratio of the energy densities of the magnetic and electric fields, u_B/u_E, in this wave. Simplify your final answer as much as possible.

Additional Exercises

9.62 A wire of length $\ell = 10.0$ cm is moving with constant velocity in the xy-plane; the wire is parallel to the y-axis and moving along the x-axis. If a magnetic field of magnitude 1.00 T is pointing along the positive z-axis, what must the velocity of the wire be in order for a potential difference of 2.00 V to be induced across it?

9.63 The magnetic field inside the solenoid in the figure changes at the rate of 1.50 T/s. A conducting coil with 2000 turns surrounds the solenoid, as shown. The radius of the solenoid is 4.00 cm, and the radius of the coil is 7.00 cm. What is the potential difference induced in the coil?

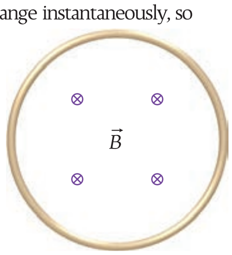
9.64 An ideal battery (with no internal resistance) supplies V_{emf} and is connected to a superconducting (no resistance!) coil of inductance L at time $t = 0$. Find the current in the coil as a function of time, $i(t)$. Assume that all connections also have zero resistance.

9.65 A 100-turn solenoid of length 8.00 cm and radius 6.00 mm carries a current of 0.400 A from right to left. The current is then reversed so that it flows from left to right. By how much does the energy stored in the magnetic field inside the solenoid change?

9.66 The electric field near the Earth's surface has a magnitude of 150. N/C, and the magnitude of the Earth's magnetic field near the surface is typically 50.0 μT. Calculate and compare the energy densities associated with these two fields. Assume that the electric and magnetic properties of air are essentially those of a vacuum.

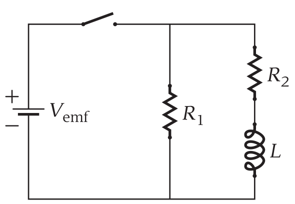
9.67 What is the inductance in a series RL circuit in which $R = 3.00$ kΩ if the current increases to $\frac{1}{2}$ of its final value in 20.0 μs?

9.68 A 100.-V battery is connected in series with a 500.-V resistor. According to Faraday's Law of Induction, current can never change instantaneously, so there is always some "stray" inductance. Suppose the stray inductance is 0.200 μH. How long will it take the current to build up to within 0.500% of its final value of 0.200 A after the resistor is connected to the battery?



9.69 A single loop of wire with an area of 5.00 m^2 is located in the plane of the page, as shown in the figure. A time-varying magnetic field in the region of the loop is directed into the page, and its magnitude is given by $B = 3.00\text{ T} + (2.00\text{ T/s})t$. At $t = 2.00$ s, what are the induced potential difference in the loop and the direction of the induced current?

9.70 A 9.00 V battery is connected through a switch to two identical resistors and an ideal inductor, as shown in the figure. Each of the resistors has a resistance of 100. Ω, and the inductor has an inductance of 3.00 H. The switch is initially open.



a) Immediately after the switch is closed, what is the current in resistor R_1 and in resistor R_2?

b) At 50.0 ms after the switch is closed, what is the current in resistor R_1 and in resistor R_2?

c) At 500. ms after the switch is closed, what is the current in resistor R_1 and in resistor R_2?

d) After a long time (>10.0 s), the switch is opened again. Immediately after the switch is opened, what is the current in resistor R_1 and in resistor R_2?

e) At 50.0 ms after the switch is opened, what is the current in resistor R_1 and in resistor R_2?

f) At 500. ms after the switch is opened, what is the current in resistor R_1 and in resistor R_2?

•**9.71** A long solenoid with length 3.00 m and $n = 290.$ turns/m carries a current of 3.00 A. It stores 2.80 J of energy. What is the cross-sectional area of the solenoid?

•**9.72** A rectangular conducting loop with dimensions a and b and resistance R is placed in the xy-plane. A magnetic field of magnitude B passes through the loop. The magnetic field is in the positive z-direction and varies in time according to $B = B_0(1 + c_1t^3)$, where c_1 is a constant with units of 1/s^3. What is the direction of the current induced in the loop, and what is its value at $t = 1$ s (in terms of a, b, R, B_0, and c_1)?

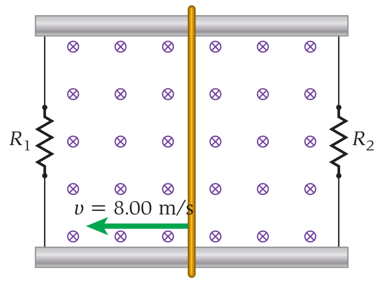
•**9.73** A circuit contains a 12.0-V battery, a switch, and a light bulb connected in series. When the light bulb has a current of 0.100 A flowing in it, it just starts to glow. This bulb draws 2.00 W when the switch has been closed for a long time. The switch is opened, and an inductor is added to the circuit, in series with the bulb. If the light bulb begins to glow 3.50 ms after the switch is closed again, what is the magnitude of the inductance? Ignore any time needed to heat the filament, and assume that you are able to observe a glow as soon as the current in the filament reaches the 0.100-A threshold.

•**9.74** A circular loop of area A is placed perpendicular to a time-varying magnetic field of magnitude $B(t) = B_0 + at + bt^2$, where B_0, a, and b are constants.

a) What is the magnetic flux through the loop at $t = 0$?

b) Derive an equation for the induced potential difference in the loop as a function of time.

c) What are the magnitude and the direction of the induced current if the resistance of the loop is R?



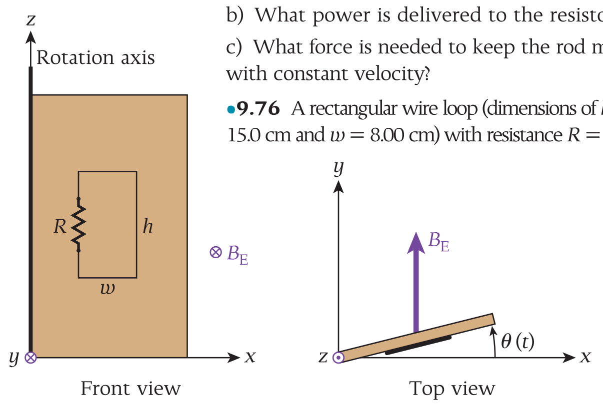
•**9.75** A conducting rod of length 50.0 cm slides over two parallel metal bars placed in a magnetic field with a magnitude of 1.00 kG, as shown in the figure. The ends of the rods are connected by two resistors, $R_1 = 100.$ Ω and $R_2 = 200.$ Ω. The conducting rod moves with a constant speed of 8.00 m/s.

a) What are the currents flowing through the two resistors?

b) What power is delivered to the resistors?

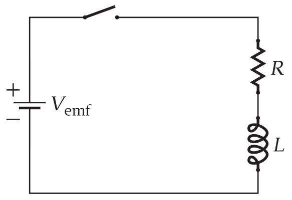
c) What force is needed to keep the rod moving with constant velocity?

•**9.76** A rectangular wire loop (dimensions of $h = 15.0$ cm and $w = 8.00$ cm) with resistance $R = 5.00$



Ω is mounted on a door, as shown in the figure. The Earth's magnetic field, $B_E = 2.60 \times 10^{-5}$ T, is uniform and perpendicular to the surface of the closed door (the surface is in the xz-plane). At time $t = 0$, the door is opened (right edge moves toward the y-axis) at a constant rate, with an opening angle of $\theta(t) = \omega t$, where $\omega = 3.50$ rad/s. Calculate the direction and the magnitude of the current induced in the loop, $i(t = 0.200\text{ s})$.

•**9.77** A steel cylinder with radius 2.50 cm and length 10.0 cm rolls without slipping down a ramp that is inclined at 15.0° above the horizontal and has a length (along the ramp) of 3.00 m. What is the induced potential difference between the ends of the cylinder as the cylinder leaves the bottom of the ramp, if the downward slope of the ramp points in the direction of the Earth's magnetic field at that location? (Use 0.426 G for the local strength of the Earth's magnetic field.)



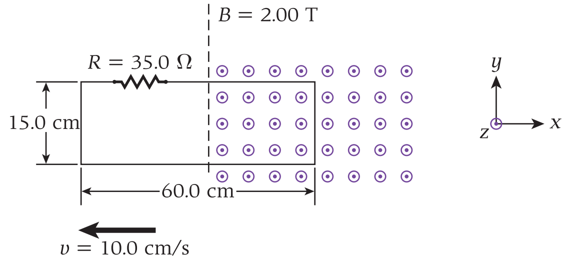
•**9.78** The figure shows a circuit in which a battery is connected to a resistor and an inductor in series.

a) What is the current in the circuit at any time t after the switch is closed?

b) Calculate the total energy provided by the battery from $t = 0$ to $t = L/R$.

c) Calculate the total energy dissipated in the resistor over the same time period.

d) Is energy conserved in this circuit?



•**9.79** As shown in the figure, a rectangular (60.0 cm long by 15.0 cm wide) circuit loop with resistance 35.0 Ω is held parallel to the xy-plane with one half inside a uniform magnetic field. A magnetic field given by $\vec{B} = 2.00\hat{z}$ is directed along the positive z-axis to the right of the dashed line; there is no external magnetic field to the left of the dashed line.

a) Calculate the magnitude of the force required to move the loop to the left at a constant speed of 10.0 cm/s while the right end of the loop is still in the magnetic field.

b) What power is expended to pull the loop out of the magnetic field at this speed?

c) What is the power dissipated by the resistor?

9.80 What is the resistance in an RL circuit with $L = 33.03$ mH if the time required for the current to reach 75% of its maximum value is 3.350 ms?

9.81 What is the inductance in an RL circuit with $R = 17.88\ \Omega$ if the time required for the current to reach 75% of its maximum value is 3.450 ms?

9.82 For an RL circuit with $R = 21.84\ \Omega$ and $L = 55.93$ mH, how long does it take the current to reach 75% of its maximum value?

9.83 A wedding ring (of diameter 1.95 cm) is tossed into the air and given a spin, resulting in an angular velocity of 13.3 rev/s. The rotation axis is a diameter of the ring. If the magnitude of the Earth's magnetic field at the ring's location is 4.77×10^{-5} T, what is the maximum induced potential difference in the ring?

9.84 A wedding ring is tossed into the air and given a spin, resulting in an angular velocity of 13.5 rev/s. The rotation axis is a diameter of the ring. The magnitude of the Earth's magnetic field is 4.97×10^{-5} T at the ring's location. If the maximum induced voltage in the ring is 1.446×10^{-6} V, what is the diameter of the ring?

9.85 A wedding ring of diameter 1.63 cm is tossed into the air and given a spin, resulting in an angular velocity of 13.7 rev/s. The rotation axis is a diameter of the ring. If the maximum induced voltage in the ring is 6.556×10^{-7} V, what is the magnitude of the Earth's magnetic field at this location?

10 Alternating Current Circuits

FIGURE 10.1 Almost all of the devices typically found on a student's desk are driven by alternating current.

Almost all the devices in our homes operate on electricity. For example, Figure 10.1 shows a small sample of the consumer electronics devices (a notebook computer, a laser printer, a desk lamp, a smartphone, and headphones) that make our lives easier and more enjoyable by turning electricity into light and sound and by allowing us to communicate with the world. Refrigerators, televisions, washing machines, microwave ovens, air conditioners, hair dryers, and many other devices on which we rely also operate on electricity.

Worldwide, over 2 terawatts (TW) of electric power are consumed, with one-quarter of the total used in North America. World electric power use has doubled in the last quarter-century, driven mostly by very strong growth in East Asia. Almost all of this electricity is delivered to consumers via alternating current, which is the main topic of this chapter.

In Chapters 4 through 6 and 9, we examined direct current (DC) circuits, in which current flows in one direction. In this chapter, we study alternating current (AC) circuits, which use time-varying emf. AC circuits contain the same circuit elements (resistors, capacitors, and inductors) as DC circuits but show interesting physical phenomena of tremendous importance for technological applications. In this chapter, we will learn what produces oscillations in currents and voltages, how the effective resistance in a circuit depends on the oscillation frequency, how this dependence is used in filters, and how alternating voltages and currents are transformed and rectified.

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WHAT WE WILL LEARN

- Voltages and currents in single-loop circuits containing an inductor and a capacitor oscillate with a characteristic frequency.
- Voltages and currents in single-loop circuits containing a resistor, an inductor, and a capacitor also oscillate with a characteristic frequency, but these oscillations are damped over time.
- A single-loop circuit containing a time-varying source of emf and a resistor has currents and voltages that are in phase and time-varying.
- A single-loop circuit containing a time-varying source of emf and a capacitor has a time-varying current and voltage that are out of phase by $+\pi/2$ rad ($+90°$), with the current leading the voltage. The voltage and the current in such a circuit are related by the capacitive reactance.
- A single-loop circuit containing a time-varying source of emf and an inductor has a time-varying current and voltage that are out of phase by $-\pi/2$ rad ($-90°$), with the voltage leading the current. The voltage and the current in such a circuit are related by the inductive reactance.
- A single-loop circuit containing a time-varying source of emf and a resistor, a capacitor, and an inductor has a time-varying current and voltage. The phase difference between the current and voltage depends on the values of the resistance, the capacitance, and the inductance and on the frequency of the emf source.
- A single-loop circuit containing a time-varying source of emf and a resistor, a capacitor, and an inductor has a resonant frequency determined by the values of the inductance and the capacitance.
- The impedance of an alternating-current circuit is similar to the resistance of a direct-current circuit, but the impedance depends on the frequency of the time-varying source of emf.
- Transformers can raise (or lower) alternating voltages while lowering (or raising) alternating currents.
- Rectifiers convert alternating current to direct current.

What are some devices in daily life which use sensors?

What kind of information could sensors collect?

10.1 LC Circuits

Previous chapters introduced three circuit elements: capacitors, resistors, and inductors. We have examined simple single-loop circuits containing resistors and capacitors (RC circuits) or resistors and inductors (RL circuits). Now we'll consider simple single-loop circuits containing inductors and capacitors: **LC circuits.** We'll see that LC circuits have currents and voltages that vary sinusoidally with time, rather than increasing or decreasing exponentially with time, like currents and voltages in RC and RL circuits. These variations of voltage and current in LC circuits are called **electromagnetic oscillations.**

To understand electromagnetic oscillations, consider a simple single-loop circuit containing an inductor and a capacitor (Figure 10.2). Recall that the energy stored in the electric field of a capacitor with capacitance C is given by (see Chapter 4)

$$U_E = \tfrac{1}{2}\frac{q^2}{C},$$

where q is the magnitude of the charge on the capacitor plates. The energy stored in the magnetic field of an inductor with inductance L is given by (see Chapter 9)

$$U_B = \tfrac{1}{2}Li^2,$$

where i is the current flowing through the inductor. Figure 10.2 shows how these energies vary with time in this LC circuit. Think of a mass oscillating on a spring showing potential energy and kinetic energy as a function of time for the mass oscillating on a spring. The resemblance is not coincidental! Much of this chapter's mathematical description of electromagnetic oscillations follows closely that developed for mechanical oscillations.

In Figure 10.2a, the capacitor is initially fully charged (with the positive charge on the bottom plate) and then connected to the circuit. At that time, the energy in the circuit is contained entirely in the electric field of the capacitor. The capacitor begins to discharge through the inductor in Figure 10.2b. At this point, current is flowing through the inductor, which generates a magnetic field. (A green arrow or label below each circuit diagram indicates the direction and the magnitude of the instantaneous current, *i*.) Now part of the

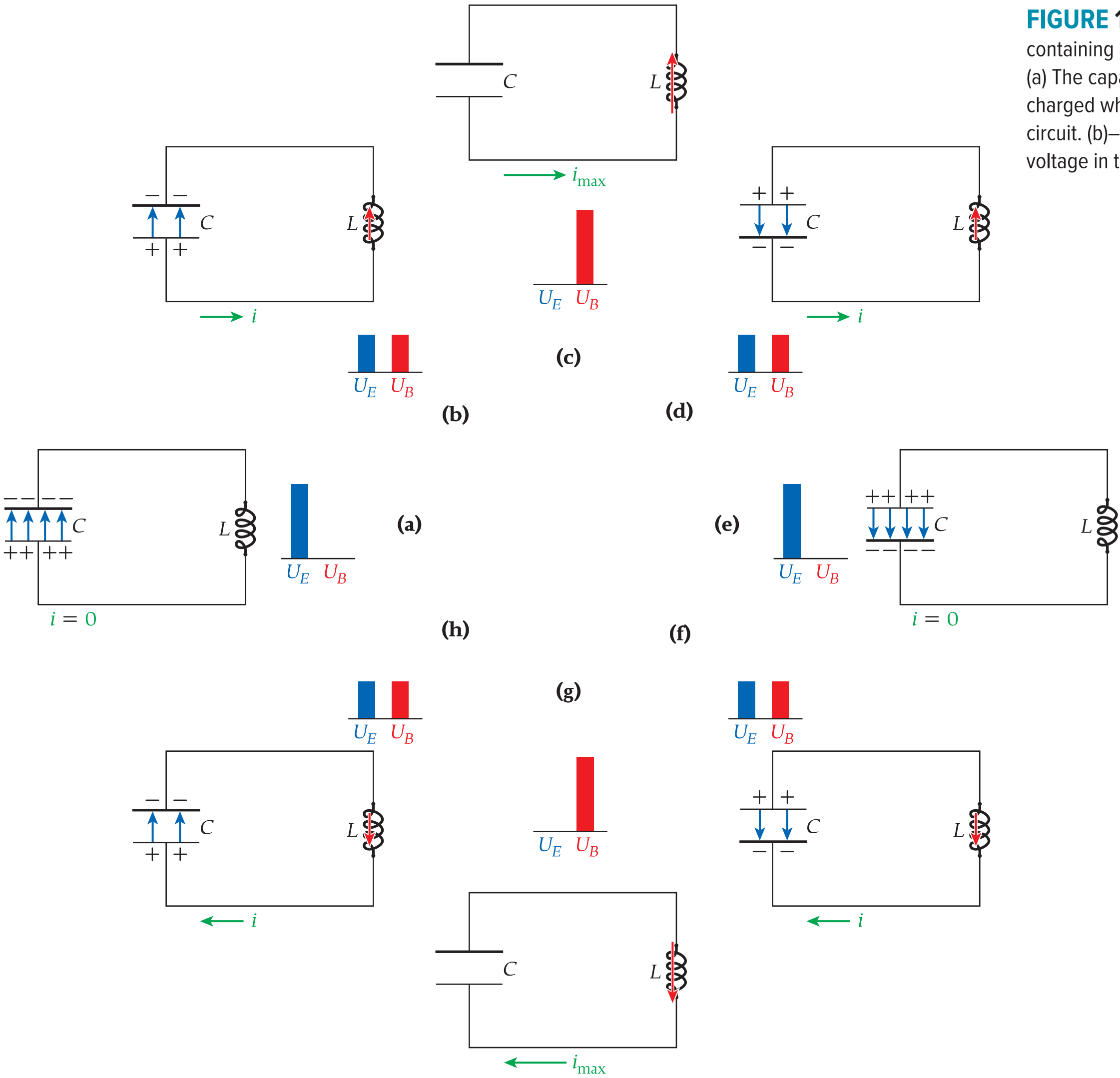


FIGURE 10.2 A single-loop circuit containing a capacitor and an inductor. (a) The capacitor is initially completely charged when it is connected to the circuit. (b)–(h) The current and the voltage in the circuit oscillate over time.

energy of the circuit is stored in the electric field of the capacitor and part in the magnetic field of the inductor. The current begins to level off as the inductor's increasing magnetic field induces an emf that opposes the current. In Figure 10.2c, the capacitor is completely discharged, and maximum current is flowing through the inductor. (When the magnitude of i has its maximum value, it is designated as i_{max} in the figure.) All the energy of the circuit is now stored in the magnetic field of the inductor. However, the current continues to flow, decreasing from its maximum value, which causes the magnetic field in the inductor to decrease. In Figure 10.2d, the capacitor begins to charge with the opposite polarity (positive charge on the top plate). Energy is again stored in the electric field of the capacitor, as well as in the magnetic field of the inductor.

In Figure 10.2e, the energy in the circuit is again entirely contained in the electric field of the capacitor. Note that the electric field now points in the opposite direction from the original field in Figure 10.2a. The current is zero, as is the magnetic field in the inductor. In Figure 10.2f, the capacitor begins to discharge again, producing a current flowing in the direction opposite to that in parts (b) through (d) of the figure; this current in turn creates a magnetic field in the opposite direction in the inductor. Again, part of the energy is stored in the electric field and part in the magnetic field. In Figure 10.2g, the energy is all stored in the magnetic field of the inductor, but with the magnetic field in the opposite direction from that in Figure 10.2c and with the maximum current in the

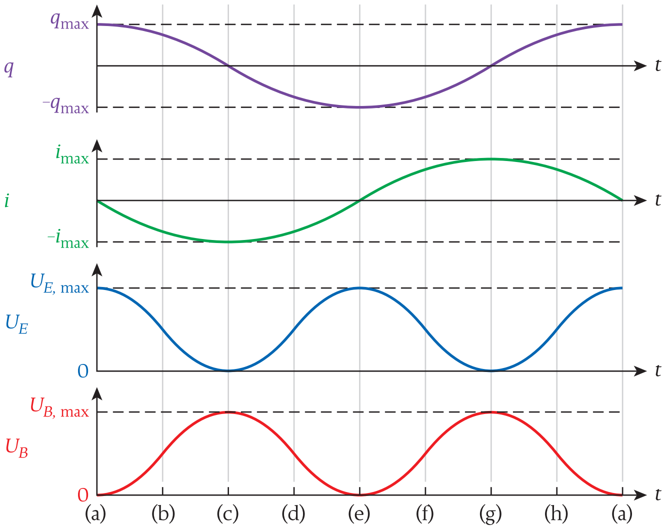


FIGURE 10.3 Variation of the charge, current, electric energy, and magnetic energy as a function of time for a simple, single-loop LC circuit. The letters along the bottom refer to the parts of Figure 10.2.

opposite direction from that in Figure 10.2c. In Figure 10.2h, the capacitor begins to charge again, meaning there is energy in both the electric and magnetic fields. The state of the circuit then returns to that shown in Figure 10.2a. The circuit continues to oscillate indefinitely because there is no resistor in it, and the electric and magnetic fields together conserve energy. A real circuit with a capacitor and an inductor does not oscillate indefinitely; instead, the oscillations die away with time because of small resistances in the circuit (covered in Section 10.3) or electromagnetic radiation (covered in Chapter 11).

The charge on either capacitor plate and the current in the LC circuit vary sinusoidally, as shown in Figure 10.3, where q_{max} refers to the maximum charge on the capacitor plate that is initially positively charged (the lower plate in Figure 10.2a). The energy in the electric field depends on the square of the charge on the capacitor, and the energy in the magnetic field depends on the square of the current in the inductor. Thus, the electric energy, U_E, and the magnetic energy, U_B, vary between zero and their respective maximum values as a function of time.

Concept Check 10.1

Figure 10.2a shows that the charge on the capacitor in an LC circuit is largest when the current is zero. What about the potential difference across the capacitor?

a) The potential difference across the capacitor is largest when the current is the largest.

b) The potential difference across the capacitor is largest when the charge is the largest.

c) The potential difference across the capacitor does not change.

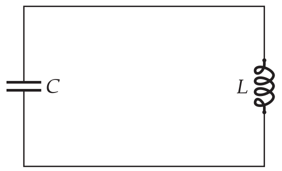


FIGURE 10.4 A single-loop LC circuit containing an inductor and a capacitor.

10.2 Analysis of LC Oscillations

Now, let's quantify the phenomena described in the preceding section. Consider a single-loop circuit containing a capacitor of capacitance C and an inductor of inductance L, but no resistor and no resistive losses in the circuit wire, as illustrated in Figure 10.4. We can write the total energy in the circuit, U, as the sum of the electric energy in the capacitor and the magnetic energy in the inductor:

$$U = U_E + U_B.$$

Using the expressions for the electric energy and the magnetic energy in terms of the charge and the current, $U_E = \frac{1}{2}(q^2/C)$ and $U_B = \frac{1}{2}Li^2$, we obtain

$$U = U_E + U_B = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2.$$

Because we have assumed zero resistance, no energy can be lost to heat, and the energy in the circuit will remain constant, because the electric field and the magnetic field together conserve energy. Thus, the derivative with respect to time of the energy in the circuit is zero:

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2\right) = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = 0.$$

By definition, the current is the time derivative of the charge, $i = dq/dt$, and therefore, the time derivative of the current is the second derivative of the charge:

$$\frac{di}{dt} = \frac{d}{dt}\left(\frac{dq}{dt}\right) = \frac{d^2q}{dt^2}.$$

With this expression for di/dt, the preceding equation for the time derivative of the total energy, dU/dt, becomes

$$\frac{q}{C}\frac{dq}{dt} + L\frac{dq}{dt}\frac{d^2q}{dt^2} = \frac{dq}{dt}\left(\frac{q}{C} + L\frac{d^2q}{dt^2}\right) = 0.$$

We can rewrite this equation as

$$\frac{d^2q}{dt^2} + \frac{q}{LC} = 0. \tag{10.1}$$

(We discard the solution $dq/dt = 0$ because it corresponds to the situation where there is initially no charge on the capacitor.) This differential equation has the same form as that for simple harmonic motion, which describes the position, x, of an object with mass m connected to a spring with spring constant k:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

We know the solution of this differential equation for the position as a function of time was a sinusoidal function: $x = x_{max}\cos(\omega_0 t + \phi)$, where ϕ is a phase constant and the angular frequency, ω_0, is given by $\omega_0 = \sqrt{k/m}$.

Simply substituting q for x and $1/LC$ for k/m in the differential equation for simple harmonic motion leads to the analogous solution for equation 10.1. Thus, the charge as a function of time in an LC circuit is given by

$$q = q_{max}\cos(\omega_0 t - \phi), \tag{10.2}$$

where q_{max} is the magnitude of the maximum charge in the circuit and ϕ is the phase constant, which is determined by the initial conditions for a given situation. (Note that the convention for electromagnetic oscillations is to use a negative sign in front of ϕ.) The angular frequency is given by

$$\omega_0 = \sqrt{\frac{1}{LC}} = \frac{1}{\sqrt{LC}}. \tag{10.3}$$

The current is given by the time derivative of equation 10.2:

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(q_{max}\cos(\omega_0 t - \phi)\right) = -\,\omega_0 q_{max}\sin(\omega_0 t - \phi).$$

Since the maximum current in the circuit is $i_{max} = \omega_0 q_{max}$, we get

$$i = -\,i_{max}\sin(\omega_0 t - \phi). \tag{10.4}$$

Equations 10.2 and 10.4 correspond to the top two curves in Figure 10.3 with $\phi = 0$. We can write expressions for the electric energy and the magnetic energy as functions of time:

$$U_E = \tfrac{1}{2}\frac{q^2}{C} = \tfrac{1}{2}\frac{\left[q_{max}\cos(\omega_0 t - \phi)\right]^2}{C} = \frac{q_{max}^2}{2C}\cos^2(\omega_0 t - \phi),$$

and

$$U_B = \tfrac{1}{2}Li^2 = \frac{L}{2}\left[-i_{max}\sin(\omega_0 t - \phi)\right]^2 = \frac{L}{2}i_{max}^2\sin^2(\omega_0 t - \phi).$$

Since $i_{max} = \omega_0 q_{max}$ and $\omega_0 = 1/\sqrt{LC}$, we can write

$$\frac{L}{2}i_{max}^2 = \frac{L}{2}\omega_0^2 q_{max}^2 = \frac{q_{max}^2}{2C}.$$

Thus, we can express the magnetic energy as a function of time as follows:

$$U_B = \frac{q_{max}^2}{2C}\sin^2(\omega_0 t - \phi).$$

Note that both the electric energy and the magnetic energy have a maximum value equal to $q_{max}^2/(2C)$ and a minimum of zero.

Concept Check 10.2

In Figure 10.3, suppose $t = 0$ at point (c). What is the phase constant in this case? (Define a clockwise current as positive.)

a) 0
b) $\pi/2$
c) π
d) $3\pi/2$
e) none of the above

We can obtain an expression for the total energy in the circuit, U, by summing the electric and magnetic energies and then using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$:

$$U = U_E + U_B = \frac{q_{max}^2}{2C}\cos^2(\omega_0 t - \phi) + \frac{q_{max}^2}{2C}\sin^2(\omega_0 t - \phi)$$
$$= \frac{q_{max}^2}{2C}\left[\sin^2(\omega_0 t - \phi) + \cos^2(\omega_0 t - \phi)\right]$$
$$= \frac{q_{max}^2}{2C} = \frac{L}{2}i_{max}^2.$$

Thus, the total energy in the circuit remains constant with time and is proportional to the square of the original charge put on the capacitor.

EXAMPLE 10.1 Characteristics of an LC Circuit

A circuit contains a capacitor, with $C = 1.50\ \mu F$, and an inductor, with $L = 3.50$ mH (Figure 10.4). The capacitor is fully charged using a 12.0-V battery and is then connected to the circuit.

PROBLEMS

What is the angular frequency of the circuit? What is the total energy in the circuit? What is the charge on the capacitor after $t = 2.50$ s?

SOLUTIONS

The angular frequency of the circuit is given by

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(3.50\times10^{-3}\ \text{H})(1.50\times10^{-6}\ \text{F})}} = 1.38\times10^4\ \text{rad/s}.$$

The total energy in the circuit is

$$U = \frac{q_{max}^2}{2C}.$$

The maximum charge on the capacitor is

$$q_{max} = CV_{emf} = (1.50\times10^{-6}\ \text{F})(12.0\ \text{V})$$
$$= 1.80\times10^{-5}\ \text{C}.$$

Thus, we can calculate the initial energy stored in the electric field of the capacitor, which is the same as the total energy in the circuit:

$$U = \frac{q_{max}^2}{2C} = \frac{(1.80\times10^{-5}\ \text{C})^2}{2(1.50\times10^{-6}\ \text{F})} = 1.08\times10^{-4}\ \text{J}.$$

The charge on the capacitor as a function of time is given by

$$q = q_{max}\cos(\omega_0 t - \phi).$$

To determine the constant ϕ, we remember that $q = q_{max}$ at $t = 0$, so

$$q(0) = q_{max} = q_{max}\cos[(\omega_0)(0) - \phi] = q_{max}\cos(-\phi) = q_{max}\cos\phi.$$

Thus, we see that $\phi = 0$, and we can write the charge as a function of time as follows:

$$q = q_{max}\cos\omega_0 t.$$

Putting in the values $q_{max} = 1.80\times10^{-5}$ C, $\omega_0 = 1.38\times10^4$ rad/s, and $t = 2.50$ s, we get

$$q = (1.80\times10^{-5}\ \text{C})\cos\left[(1.38\times10^4\ \text{rad/s})(2.50\ \text{s})\right] = 1.02\times10^{-5}\ \text{C}.$$

Self-Test Opportunity 10.1

The frequency of oscillation of an LC circuit is 200.0 kHz. At $t = 0$, the capacitor has its maximum positive charge on the lower plate. Decide whether each of the following statements is true or false.

a) At $t = 2.50\ \mu s$, the charge on the lower plate has its maximum negative value.

b) At $t = 5.00\ \mu s$, the current in the circuit is at its maximum value.

c) At $t = 2.50\ \mu s$, the energy in the circuit is stored completely in the inductor.

d) At $t = 1.25\ \mu s$, half the energy in the circuit is stored in the capacitor and half the energy is stored in the inductor.

10.3 Damped Oscillations in an RLC Circuit

Now let's consider a single-loop circuit that has a capacitor and an inductor but also a resistor–an **RLC circuit,** as shown in Figure 10.5. We saw in the preceding section that the energy of a circuit with a capacitor and an inductor remains constant and that the energy is transformed from electric to magnetic and back again with no losses. However, if a resistor is present in the circuit, the current flow produces ohmic losses, which show up as thermal energy. Thus, the energy of the circuit decreases because of these losses. The rate of energy loss is given by

$$\frac{dU}{dt} = -i^2R.$$

We can rewrite the change in the energy in the circuit as a function of time:

$$\frac{dU}{dt} = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = -i^2R.$$

Again, since $i = dq/dt$ and $di/dt = d^2q/dt^2$, we can write

$$\frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} + i^2R = \frac{q}{C}\frac{dq}{dt} + L\frac{dq}{dt}\frac{d^2q}{dt^2} + \left(\frac{dq}{dt}\right)^2 R = 0,$$

or

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0. \quad (10.5)$$

The solution of this differential equation (for small damping, meaning sufficiently small values of the resistance) is

$$q = q_{max}e^{-Rt/2L}\cos\omega t, \quad (10.6)$$

where

$$\omega = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2} \quad (10.7)$$

and $\omega_0 = 1/\sqrt{LC}$.

The calculus used in arriving at the solution in equation 10.6 is not shown. You can verify that the solution satisfies equation 10.5 by straightforward substitution from equations 10.6 and 10.7 into 10.5. You can also refer back to where it was shown that the equation of motion for a weakly damped (or underdamped) mechanical oscillator has a similar solution.

If the capacitor in the single-loop RLC circuit of Figure 10.5 is charged and then connected in the circuit, the charge on the capacitor will vary sinusoidally with time while decreasing in amplitude (Figure 10.6). Taking the derivative of equation 10.6 shows that the current, $i = dq/dt$, has an amplitude that is damped at the same rate that the charge is damped and that this amplitude also varies sinusoidally with time. After some time, no current remains in the circuit.

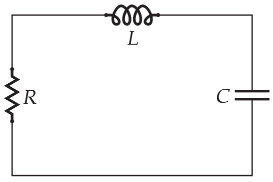


FIGURE 10.5 A single-loop RLC circuit containing a resistor, an inductor, and a capacitor.

Self-Test Opportunity 10.2

Compare equation 10.5 for the charge on the capacitor as a function of time to the differential equation for the position of a mass on a spring

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0.$$

Which quantity in the RLC circuit plays the role of the mass *m*, which that of the spring constant *k*, and which that of the damping constant *b*?

Concept Check 10.3

What is the condition for small damping that needs to be fulfilled for equation 10.6 to be a solution for equation 10.5? (*Hint:* You can find this by analogy with the damped oscillation of a mass on a spring, for which the differential equation is $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$ and the condition for small damping is $b < 2\sqrt{mk}$. Alternatively, you can use dimensional analysis.)

a) $R < 2\sqrt{L/C}$

b) $R < \sqrt{2C/L}$

c) $R < \sqrt{2LC}$

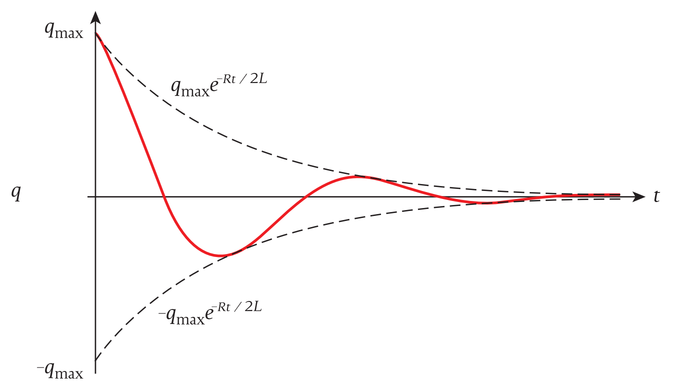


FIGURE 10.6 Graph of the charge on the capacitor as a function of time in a circuit containing a capacitor, an inductor, and a resistor.

We can analyze the energy in the circuit as a function of time by calculating the energy stored in the electric field of the capacitor:

$$U_E = \frac{1}{2}\frac{q^2}{C} = \frac{q_{\max}^2}{2C} e^{-Rt/L} \cos^2 \omega t.$$

Thus, U_E and U_B both decrease exponentially in time, and therefore so does the total energy in the circuit, $U_E + U_B$.

10.4 Driven AC Circuits

So far, we have been studying circuits that contain a source of constant emf or that start with a constant charge and contain energy that then oscillates between electric and magnetic fields. However, many interesting effects occur in a circuit in which the current oscillates continuously. Let's investigate some of these effects, starting with a time-varying source of emf and then considering in turn a resistor, a capacitor, and an inductor connected to this source.

Alternating Driving emf

A source of emf can be capable of producing a time-varying voltage, as opposed to the sources of constant emf considered in previous chapters. We'll assume that the source of time-varying emf provides a sinusoidal voltage as a function of time, the *driving emf*, given by

$$V_{\text{emf}} = V_{\max} \sin \omega t, \tag{10.8}$$

where ω is the angular frequency of the emf and $V_{\max}$ is the maximum amplitude or value of the emf.

The current induced in a circuit containing a source of time-varying emf will also vary sinusoidally with time. This time-varying current is called **alternating current (AC).** However, the alternating current may not always remain in phase with the time-varying emf. The current, i, as a function of time is given by

$$i = I \sin(\omega t - \phi), \tag{10.9}$$

where I is the amplitude of the current and the angular frequency of the time-varying current is the same as that of the driving emf, but the phase constant ϕ is not zero. Note that, as is the convention, the phase constant is preceded by a negative sign.

Circuit with a Resistor

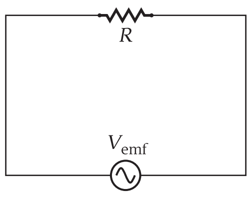


FIGURE 10.7 Single-loop circuit with a resistor and a source of time-varying emf.

Let's begin our analysis of RLC circuits with alternating current by considering a circuit containing only a resistor and a source of time-varying emf (Figure 10.7). Applying Kirchhoff's Loop Rule to this circuit, we obtain

$$V_{\text{emf}} - v_R = 0,$$

where v_R is the voltage drop across the resistor. Substituting v_R for V_{emf} in equation 10.8, we get

$$v_R = V_{\max} \sin \omega t = V_R \sin \omega t,$$

where V_R is the maximum voltage drop across the resistor. Note that the voltage as a function of time is represented by a lowercase v and the amplitude of the voltage with uppercase V. According to Ohm's Law, $V = iR$, so we can write

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega t = I_R \sin \omega t. \tag{10.10}$$

Thus, the current amplitude and the voltage amplitude are related as follows:

$$V_R = I_R R. \tag{10.11}$$

Figure 10.8a shows the voltage across the resistor and the current through it as functions of time. The time-varying current can be represented by a phasor, $\vec{I}_R$, and the

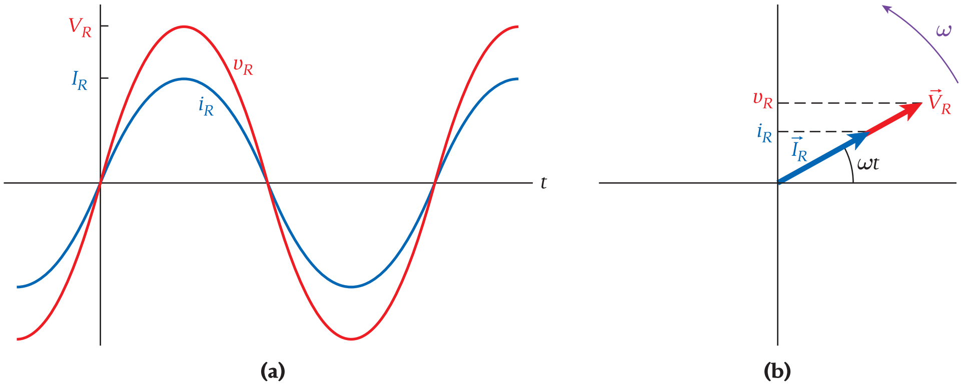


FIGURE 10.8 Alternating voltage and current for a single-loop circuit containing a source of time-varying emf and a resistor: (a) voltage and current as functions of time; (b) phasors representing voltage and current, showing that they are in phase.

time-varying voltage by a phasor, $\vec{V}_R$ (Figure 10.8b). A **phasor** is a counterclockwise-rotating vector (with its tail fixed at the origin) whose projection on the vertical axis represents the sinusoidal variation of the particular quantity in time. The angular velocity of the phasors in Figure 10.8b is ω of equation 10.10. The current flowing through the resistor and the voltage across the resistor are in phase, which means that the phase difference between the current and the voltage is zero.

Circuit with a Capacitor

Now let's examine a circuit that contains a capacitor and a source of time-varying emf (Figure 10.9). The voltage across the capacitor is given by Kirchhoff's Loop Rule,

$$V_{\text{emf}} - v_C = 0,$$

where v_C is the voltage drop across the capacitor. Thus, we have

$$v_C = V_{\max} \sin \omega t = V_C \sin \omega t,$$

where V_C is the maximum voltage across the capacitor. Since $q = CV$ for a capacitor, we can write

$$q = Cv_C = CV_C \sin \omega t.$$

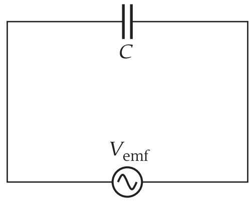


FIGURE 10.9 Single-loop circuit with a capacitor and a source of time-varying emf.

However, we want an expression for the current (rather than the charge) as a function of time. Therefore, we take the derivative with respect to time of the preceding equation:

$$i_C = \frac{dq}{dt} = \frac{d\left(CV_C \sin \omega t\right)}{dt} = \omega CV_C \cos \omega t.$$

This equation can be written in a form comparable to that of equation 10.10 by defining a quantity that is similar to resistance, called the **capacitive reactance,** X_C:

$$X_C = \frac{1}{\omega C}. \tag{10.12}$$

This definition allows us to express the current, i_C, as

$$i_C = \frac{V_C}{X_C} \cos \omega t,$$

or, with $I_C = V_C/X_C$, as

$$i_C = I_C \cos \omega t.$$

We can use $\cos\theta = \sin(\theta + \pi/2)$ to express this result in a form analogous to that of equation 10.10:

$$i_C = I_C \sin\left(\omega t + \pi/2\right). \tag{10.13}$$

This expression for the current flowing in a circuit with only a capacitor is similar to the expression for the current flowing in a circuit with only a resistor, except that current and voltage are out of phase by $\pi/2$ rad (90°). Figure 10.10a shows the voltage and the current as functions of time.

Concept Check 10.4

A circuit containing a capacitor (Figure 10.9) has a source of time-varying emf that provides a voltage given by $v_C = V_C \sin \omega t$. What is the current, i_C, through the capacitor when the potential difference across it is largest ($v_C = V_{\max}$)?

a) $i_C = 0$

b) $i_C = +I_{\max}$

c) $i_C = -I_{\max}$

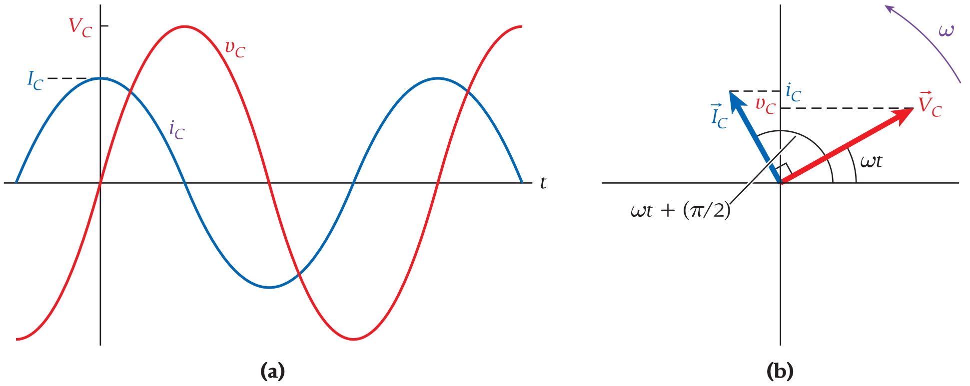


FIGURE 10.10 Alternating voltage and current for a single-loop circuit containing a source of time-varying emf and a capacitor: (a) voltage and current as functions of time; (b) phasors representing voltage and current, showing that they are out of phase by $\pi/2$ rad (90°).

The corresponding phasors, $\vec{I}_C$ and $\vec{V}_C$, shown in Figure 10.10b, indicate that *for a circuit with only a capacitor, the current leads the voltage*. The amplitude of the voltage across the capacitor and the amplitude of the current through the capacitor are related by

$$V_C = I_C X_C. \tag{10.14}$$

This equation resembles Ohm's Law with the capacitive reactance replacing the resistance. One major difference between the capacitive reactance and the resistance is that the capacitive reactance depends on the angular frequency of the time-varying emf.

Circuit with an Inductor

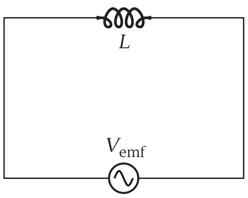


FIGURE 10.11 Single-loop circuit with an inductor and a source of time-varying emf.

Now let's consider a circuit with a source of time-varying emf and an inductor (Figure 10.11). We again apply Kirchhoff's Loop Rule to this circuit to obtain the voltage across the inductor:

$$v_L = V_{\max} \sin \omega t = V_L \sin \omega t,$$

where V_L is the maximum voltage across the inductor. A changing current in an inductor induces an emf given by

$$v_L = L\frac{di_L}{dt}.$$

Note that for positive di/dt the voltage drop across the inductor is positive because the direction of current is the direction of decreasing potential. Thus, we can write

$$L\frac{di_L}{dt} = V_L \sin \omega t,$$

or

$$\frac{di_L}{dt} = \frac{V_L}{L}\sin \omega t.$$

We are interested in the current rather than its time derivative, so we integrate the preceding equation:

$$i_L = \int \frac{di_L}{dt}dt = \int \frac{V_L}{L}\sin \omega t\, dt = -\frac{V_L}{\omega L}\cos \omega t.$$

Here we set the constant of integration to zero because we are not interested in solutions that contain both an oscillating and a constant current. The **inductive reactance,** which, like the capacitive reactance, is similar to resistance, is defined as

$$X_L = \omega L. \tag{10.15}$$

Using the inductive reactance, we can express i_L as

$$i_L = -\frac{V_L}{X_L}\cos \omega t = -I_L \cos \omega t,$$

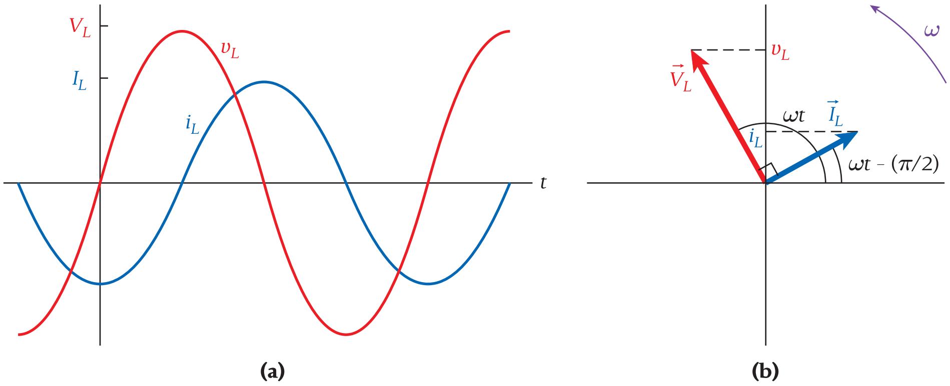


FIGURE 10.12 Alternating voltage and current for a single-loop circuit containing a source of time-varying emf and an inductor: (a) voltage and current as functions of time; (b) phasors representing voltage and current, showing that they are out of phase by $-\pi/2$ rad ($-90°$).

where I_L is the maximum current. Thus,

$$V_L = I_L X_L,$$

which again resembles Ohm's Law except that the inductive reactance depends on the angular frequency of the time-varying emf.

Because $-\cos\theta = \sin(\theta - \pi/2)$, we can rewrite $i_L = -I_L \cos\omega t$ as follows:

$$i_L = I_L \sin(\omega t - \pi/2). \tag{10.16}$$

Thus, the current flowing in a circuit with an inductor and a source of time-varying emf is out of phase with the voltage by $-\pi/2$ rad. Figure 10.12a shows voltage and current as functions of time. The corresponding phasors, $\vec{I}_L$ and $\vec{V}_L$, are shown in Figure 10.12b, which shows that *for a circuit with an inductor, the voltage leads the current.*

10.5 Series RLC Circuit

Now we're ready to consider a single-loop circuit that has all three circuit elements, along with a source of time-varying emf (Figure 10.13). We will not present a full mathematical analysis of this RLC circuit but will use phasors to analyze the important aspects.

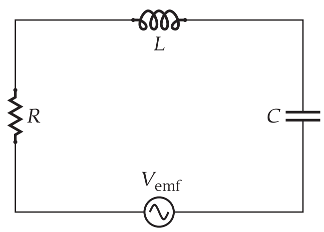


FIGURE 10.13 A single-loop circuit containing a source of time-varying emf, a resistor, an inductor, and a capacitor.

The time-varying current in the simple RLC circuit can be described by a phasor, $\vec{I}_{\text{m}}$ (Figure 10.14). The projection of $\vec{I}_{\text{m}}$ on the vertical axis represents the current i flowing in the circuit as a function of time, t, where the angle of the phasor is given by $\omega t - \phi$ such that

$$i = I_{\text{m}} \sin(\omega t - \phi).$$

The current i and the voltages across the circuit components have different phases with respect to the time-varying emf, as we saw in the previous section:

- *For the resistor,* the voltage v_R and the current i are in phase with each other, and the voltage phasor, $\vec{V}_R$, is in phase with $\vec{I}_{\text{m}}$.
- *For the capacitor,* the current i leads the voltage v_C by $\pi/2$ rad (90°), so the voltage phasor, $\vec{V}_C$, has an angle that is $\pi/2$ rad (90°) less than the angles of $\vec{I}_{\text{m}}$ and $\vec{V}_R$.
- *For the inductor,* the current i lags behind the voltage v_L by $\pi/2$ rad (90°), so the voltage phasor, $\vec{V}_L$, has an angle that is $\pi/2$ rad (90°) greater than the angles of $\vec{I}_{\text{m}}$ and $\vec{V}_R$.

The voltage phasors for the RLC circuit are shown in Figure 10.15. The instantaneous voltage across each component is represented by the projection of the respective phasor on the vertical axis.

The total voltage drop across all components, V, is given by

$$V = v_R + v_C + v_L. \tag{10.17}$$

The total voltage, V, can be thought of as the projection on the vertical axis of the phasor $\vec{V}_{\text{m}}$, representing the time-varying emf in the circuit (Figure 10.16). The phasors in Figure 10.15 rotate together, so equation 10.17 holds at any time. The voltage phasors must sum as vectors to

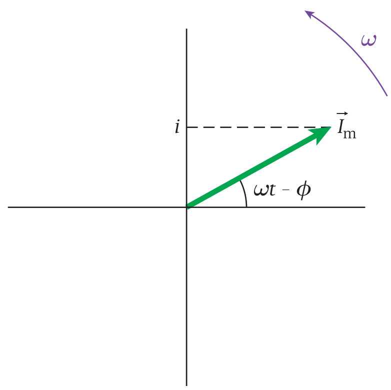


FIGURE 10.14 Phasor $\vec{I}_m$ representing the current i flowing in an RLC circuit.

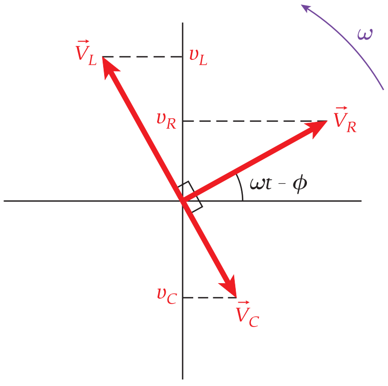


FIGURE 10.15 Voltage phasors for an RLC series circuit. The phasor $\vec{V}_l$ is in phase with the phasor $\vec{I}_m$, representing the current in the circuit.

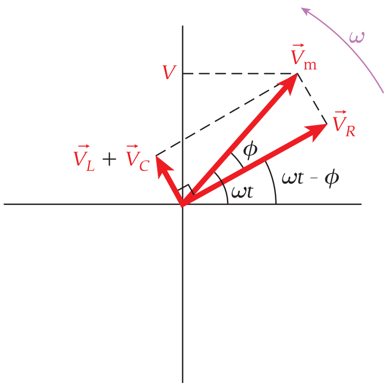


FIGURE 10.16 Sum of the voltage phasors in an RLC series circuit.

Concept Check 10.5

A circuit like that shown in Figure 10.13 containing a capacitor, an inductor, and a resistor connected in series with a source of time-varying emf has $V_{emf} = V_m \sin \omega t$. At a point in time when V_{emf} is increasing, how is the current in the circuit behaving?

a) The current is increasing.

b) The current is decreasing.

c) The current is not changing.

d) The current may be increasing or decreasing.

match $\vec{V}_m$ in order to satisfy equation 10.17 at all times. This vector sum is shown in Figure 10.16. In this figure, the sum of the two phasors $\vec{V}_L$ and $\vec{V}_C$ has been replaced with $\vec{V}_L + \vec{V}_C$. The vector sum of $\vec{V}_L + \vec{V}_C$ and $\vec{V}_R$ must equal $\vec{V}_m$. Thus, we can write

$$V_m^2 = V_R^2 + (V_L - V_C)^2, \tag{10.18}$$

because the vectors $\vec{V}_L$ and $\vec{V}_C$ always point in opposite directions, and $\vec{V}_m$ is perpendicular to both. Now we can substitute our previously derived expressions for V_R, V_L, and V_C into equation 10.18, taking the amplitude of the current in all three components to be I_m because they are connected in series:

$$V_m^2 = (I_m R)^2 + (I_m X_L - I_m X_C)^2.$$

We can then solve for the amplitude of the current in the circuit:

$$I_m = \frac{V_m}{\sqrt{R^2 + (X_L - X_C)^2}}.$$

The denominator of the term on the right-hand side is called the **impedance,** Z:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}. \tag{10.19}$$

The impedance of a circuit depends on the frequency of the time-varying emf. This time dependence is expressed explicitly when substitutions are made for the capacitive reactance, X_C, and the inductive reactance, X_L:

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}. \tag{10.20}$$

The impedance of an AC circuit has the unit ohm (Ω), just like the resistance in a DC circuit. We can then write

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_m}{Z}. \tag{10.21}$$

The current flowing in an AC circuit depends on the difference between the inductive reactance and the capacitive reactance and is called the *total reactance*. The phase constant, ϕ, can be expressed in terms of this difference. The phase constant is defined as the phase difference between the voltage phasors $\vec{V}_R$ and $\vec{V}_m$ depicted in Figure 10.16. Thus, we can express the phase constant as

$$\phi = \tan^{-1}\left(\frac{V_L - V_C}{V_R}\right).$$

Since $V_L = X_L I_m$, $V_C = X_C I_m$, and $V_R = R I_m$, this can be rewritten as follows:

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right).$$

Using $X_C = 1/\omega C$ and $X_L = \omega L$, we can then obtain the frequency dependence of the phase constant:

$$\phi = \tan^{-1}\left(\frac{\omega L - (\omega C)^{-1}}{R}\right). \tag{10.22}$$

The current in the RLC circuit can now be written as

$$i = I_m \sin(\omega t - \phi), \tag{10.23}$$

where I_m is the magnitude of the phasor $\vec{I}_m$. The voltage across all the components in the circuit is given by the time-varying source of emf:

$$V = V_{emf}(t) = V_m \sin \omega t, \tag{10.24}$$

where V_m is the magnitude of the phasor $\vec{V}_m$.

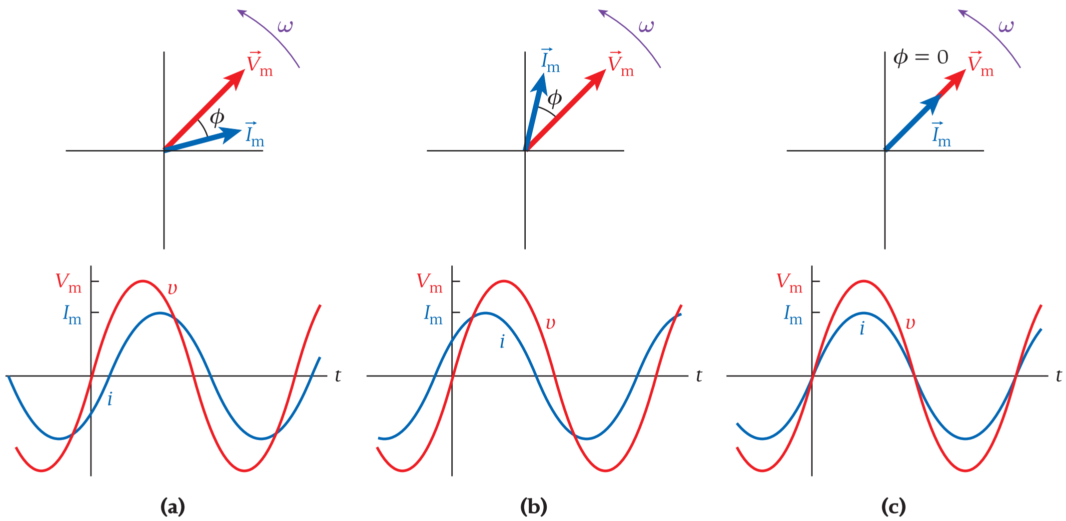


FIGURE 10.17 Current and voltage as functions of time for an RLC circuit with: (a) $X_L > X_C$; (b) $X_L < X_C$; (c) $X_L = X_C$.

Thus, three conditions are possible for an AC circuit containing a resistor, a capacitor, and an inductor connected in series:

- For $X_L > X_C$, ϕ is positive, and the current in the circuit lags behind the voltage in the circuit. This circuit is similar to a circuit with only an inductor, except that the phase constant is not necessarily $\pi/2$ rad (90°), as illustrated in Figure 10.17a.
- For $X_L < X_C$, ϕ is negative, and the current in the circuit leads the voltage in the circuit. This circuit is similar to a circuit with only a capacitor, except that the phase constant is not necessarily $-\pi/2$ rad (−90°), as illustrated in Figure 10.17b.
- For $X_L = X_C$, ϕ is zero, and the current in the circuit is in phase with the voltage in the circuit. This circuit is similar to a circuit with only a resistance, as illustrated in Figure 10.17c. When $\phi = 0$, the circuit is said to be in **resonance.**

The current amplitude, I_m, in the series RLC circuit depends on the frequency of the time-varying emf, as well as on L and C. Inspection of equation 10.21 shows that the maximum current occurs when

$$\omega L - \frac{1}{\omega C} = 0,$$

which corresponds to $\phi = 0$ and $X_L = X_C$. The angular frequency, ω_0, at which the maximum current occurs, called the **resonant angular frequency,** is

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Self-Test Opportunity 10.3

Consider a series RLC circuit like the one shown in Figure 10.13. The circuit is driven at an angular frequency ω by the time-varying emf. The resonant angular frequency is ω_0. Decide whether each of the following statements is true or false.

a) If $\omega = \omega_0$, the voltage and the current are in phase.

b) If $\omega < \omega_0$, the voltage lags behind the current.

c) If $\omega > \omega_0$, then $X_C > X_L$.

A Practical Example

Now let's look at a real circuit (Figure 10.18). The diagram for this circuit is shown in Figure 10.13. The circuit has a source of time-varying emf with $V_m = 7.5$ V and also has $L = 8.2$ mH, $C = 100$ μF, and $R = 10\ \Omega$. The maximum current, I_m, was measured as a function of the ratio of the angular frequency of the time-varying emf to the resonant angular frequency, ω/ω_0 (Figure 10.19). Red circles indicate the results of the measurements. The maximum value of the current occurs, as expected, at the resonant angular frequency. However, with $R = 10\ \Omega$, the relationship between I_m and ω/ω_0, given by equation 10.21, results in the green curve, which does not reproduce the measured results. To better describe the current, we must remember that in a real circuit, the inductor has a resistance, even at the resonant frequency. The black curve in Figure 10.19 corresponds to equation 10.21 with $R = 15.4\ \Omega$.

The resonant behavior of an RLC circuit resembles the response of a damped mechanical oscillator. Figure 10.20 shows the calculated maximum current, I_m, as

FIGURE 10.18 Real series circuit containing an 8.2-mH inductor, a 10-Ω resistor, and a 100-μF capacitor.

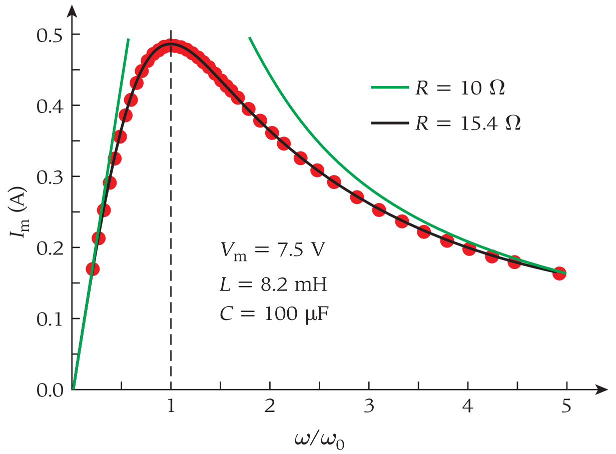


FIGURE 10.19 Graph of the maximum current, I_m, versus the ratio of the angular frequency, ω, of the time-varying emf to the resonance frequency, ω_0, for an RLC circuit. Red dots represent measurements. The text explains the green and black curves.

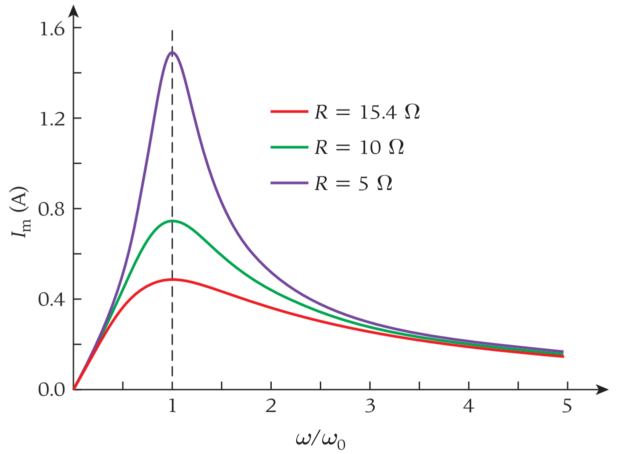


FIGURE 10.20 Graph of the maximum current, I_m, versus the ratio of the angular frequency, ω, of the time-varying emf to the resonance frequency, ω_0, for three series RLC circuits, with $L = 8.2$ mH, $C = 100\ \mu$F, and three different resistances.

a function of the ratio of the angular frequency of the time-varying emf to the resonant angular frequency, ω/ω_0, for a series RLC circuit with $V_m = 7.5$ V, $L = 8.2$ mH, $C = 100\ \mu$F, and three different resistances. You can see that as the resistance is lowered, the maximum current at the resonant angular frequency increases, producing a more pronounced peak.

Self-Test Opportunity 10.4

Consider a series RLC circuit like the one shown in Figure 10.13. Decide whether each of the following statements is true or false.

a) The current through the resistor is the same as the current through the inductor at all times.

b) In an ideal scenario, energy is dissipated in the resistor but not in the capacitor or in the inductor.

c) The voltage drop across the resistor is the same as the voltage drop across the inductor at all times.

EXAMPLE 10.2 Characterizing an RLC Circuit

Suppose an RLC series circuit like the one shown in Figure 10.13 has $R = 91.0\ \Omega$, $C = 6.00\ \mu$F, and $L = 60.0$ mH. The source of time-varying emf has an angular frequency of $\omega = 64.0$ rad/s.

PROBLEM

What is the impedance of this circuit?

SOLUTION

Normally, we would solve this problem by obtaining an expression for the impedance in terms of the quantities provided. However, instead we'll calculate several intermediate numerical answers to gain insight into the characteristics of this circuit.

The impedance is given by $Z = \sqrt{R^2 + (X_L - X_C)^2}$. To see which of the quantities on the right-hand side has the greatest impact on the impedance, we calculate the quantities individually. The inductive reactance is

$$X_L = \omega L = (64.0 \text{ rad/s})\left(60.0 \times 10^{-3} \text{ H}\right) = 3.84\ \Omega\,.$$

The capacitive reactance is

$$X_C = \frac{1}{\omega C} = \frac{1}{(64.0 \text{ rad/s})\left(6.00 \times 10^{-6} \text{ F}\right)} = 2.60 \text{ k}\Omega.$$

We see that the impedance of this circuit is dominated by the capacitive reactance at the given value of the angular frequency. This type of circuit is called *capacitive.*

Putting in our results for the capacitive and inductive reactances, we calculate the impedance:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(91.0\ \Omega)^2 + \left(3.84\ \Omega - 2.60 \cdot 10^3\ \Omega\right)^2} = 2.60 \text{ k}\Omega.$$

That is, the inductive reactance and the resistance are completely negligible within rounding error. For comparison, the impedance of this circuit when it is in resonance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(91.0\ \Omega)^2 + 0} = 91.0\ \Omega.$$

This result implies that the circuit as described in the problem statement is far from resonance, which is consistent with the very different values we obtained for the capacitive and inductive reactances. (Remember that at resonance these two reactances have the same value!)

Frequency Filters

We have been analyzing circuits that have a time-varying emf with a single frequency. However, many applications involve time-varying emfs that reflect a superposition of many frequencies. In some situations, certain frequencies need to be filtered out of this kind of circuit. (Series RLC circuits can be used as frequency filters.) One example of such a circuit can be found in DSL (digital subscriber line) filters for making connections to the Internet over a household telephone line. A typical DSL filter is shown in Figure 10.21.

FIGURE 10.21 A typical band-pass filter for phones connected to a household circuit that has a DSL Internet connection.

A DSL Internet connection operates at high frequencies and is connected to a household's regular phone line. The high operating frequency of the DSL connection causes noise on the phones in the house. Therefore, a **band-pass filter** is normally installed on all the phones in the house to filter out the high-frequency noise created by the DSL Internet connection. Frequency filters can be designed to pass low frequencies and block high frequencies (*low-pass filter*) or to pass high frequencies and block low frequencies (*high-pass filter*). A low-pass filter can be combined with a high-pass filter to allow a range of frequencies to pass (*band-pass filter*) and block the frequencies outside that range.

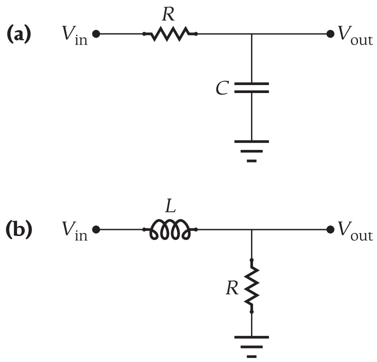


FIGURE 10.22 Two low-pass filters: (a) RC version; (b) RL version.

Figure 10.22 shows two examples of a low-pass filter, where V_{in} is a time-varying emf with many frequencies. A low-pass filter is essentially a voltage divider. Part of the original voltage passes through the circuit, while part goes to ground. For the RC version of the low-pass filter, shown in Figure 10.22a, low frequencies essentially have an open circuit, while high frequencies are preferentially sent to ground. Thus, only signals with low frequencies will pass through the filter. This behavior makes sense because current going to ground must pass through a capacitor that essentially blocks the flow of current for low frequencies because the capacitor plates are charging, while rapidly changing current does not allow charge to build up on the plates of the capacitor, allowing current to flow. For the RL version, shown in Figure 10.22b, low frequencies easily pass through the inductor while high frequencies are blocked. This effect arises because the self-induced emf in an inductor opposes rapid changes in current, effectively blocking current through the inductor at high frequencies, while a slow change in current produces a much smaller opposing emf, allowing current to flow.

To quantify the performance of the low-pass filter in Figure 10.22a, we define the input section of the circuit to be the resistor and the capacitor. The impedance of this section is $Z_{in} = \sqrt{R^2 + X_C^2}$. The impedance of the output section is just $Z_{out} = X_C$. The ratio of the emf into the filter and the emf emerging from the filter is

$$\frac{V_{out}}{V_{in}} = \frac{Z_{out}}{Z_{in}}. \tag{10.25}$$

The ratio of the emfs can then be written as

$$\frac{V_{out}}{V_{in}} = \frac{X_C}{\sqrt{R^2 + X_C^2}} = \frac{1}{\sqrt{\left(\frac{R}{X_C}\right)^2 + 1}} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}. \tag{10.26}$$

For the RL version of the low-pass filter, shown in Figure 10.22b, $Z_{in} = \sqrt{R^2 + X_L^2}$ and $Z_{out} = R$, allowing us to write

$$\frac{V_{out}}{V_{in}} = \frac{R}{\sqrt{R^2 + X_L^2}} = \frac{1}{\sqrt{1 + \left(\omega^2 L^2 / R^2\right)}}. \tag{10.27}$$

The *breakpoint frequency*, ω_B, between the responses to low and high frequencies is the frequency at which the ratio V_{out}/V_{in} is $1/\sqrt{2} = 0.707$. At that frequency for the RC version, we have

$$\frac{1}{\sqrt{1 + \omega_B^2 R^2 C^2}} = \frac{1}{\sqrt{2}},$$

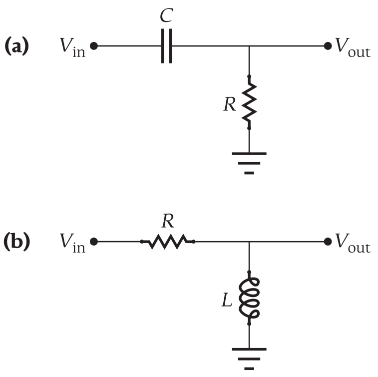


FIGURE 10.23 Two high-pass filters: (a) RC version; (b) RL version.

from which we can solve for the breakpoint frequency:

$$\omega_B = \frac{1}{RC}. \tag{10.28}$$

For the RL version of the low-pass filter, the breakpoint frequency is obtained from equation 10.27:

$$\omega_B = \frac{R}{L}. \tag{10.29}$$

Figure 10.23 shows two examples of a high-pass filter. A high-pass filter is also a voltage divider. For the RC version of the high-pass filter, shown in Figure 10.23a, signals with low frequencies cannot pass the capacitor while signals with high frequencies pass through easily. This behavior makes sense because the signal must pass through a capacitor that essentially blocks the flow of current for low frequencies because the capacitor plates are charging, while rapidly changing current does not allow charge to build up on the plates of the capacitor, allowing current to flow. For the RL version, shown in Figure 10.23b, signals with low frequency have essentially an open circuit to ground, while signals with high frequencies are blocked from reaching ground. Thus, only signals with high frequencies will be passed through the filter. This effect arises because the self-induced emf in an inductor opposes rapid changes in current, effectively blocking current through the inductor at high frequencies, while a slow change in current produces a much smaller opposing emf, allowing current to flow.

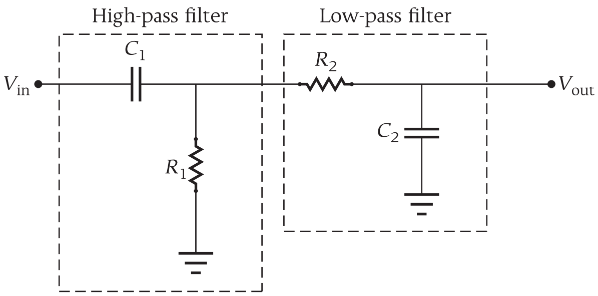


FIGURE 10.24 A band-pass filter consisting of a high-pass filter connected in series with a low-pass filter.

For the RC version of the high-pass filter in Figure 10.23a, the impedance of the input section is $Z_{in} = \sqrt{R^2 + X_C^2}$, while the impedance of the output section is $Z_{out} = R$. The ratio of the output emf to the input emf is then

$$\frac{V_{out}}{V_{in}} = \frac{R}{\sqrt{R^2 + X_C^2}} = \frac{1}{\sqrt{1 + X_C^2/R^2}} = \frac{1}{\sqrt{1 + \dfrac{1}{\omega^2 R^2 C^2}}}. \tag{10.30}$$

For the RL version of the high-pass filter shown in Figure 10.23b, the ratio of the output emf to the input emf is

$$\frac{V_{out}}{V_{in}} = \frac{X_L}{\sqrt{R^2 + X_L^2}} = \frac{1}{\sqrt{\dfrac{R^2}{X_L^2} + 1}} = \frac{1}{\sqrt{1 + \dfrac{R^2}{\omega^2 L^2}}}. \tag{10.31}$$

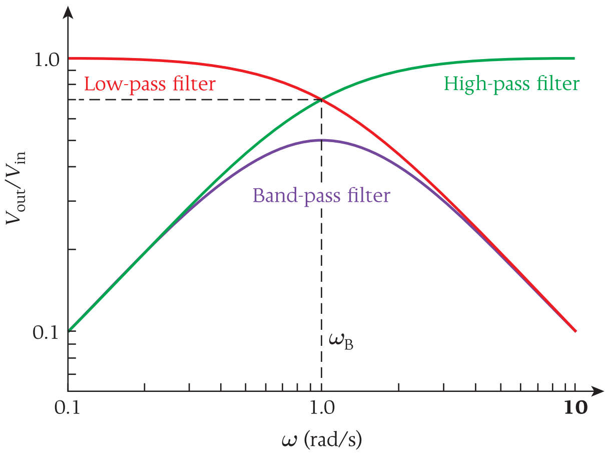


FIGURE 10.25 The frequency response of a low-pass filter, a high-pass filter, and a band-pass filter.

For these high-pass filters, as the frequency increases, the ratio of output emf to input emf approaches 1, while for low frequencies, the ratio of output emf to input emf goes to zero. The breakpoint frequencies for the high-pass filters are the same as for the low-pass filters: $\omega_B = 1/(RC)$ for the RC version and $\omega_B = R/L$ for the RL version.

An example of a band-pass filter is shown in Figure 10.24. The band-pass filter consists of a high-pass filter in series with a low-pass filter. Thus, both high and low frequencies are suppressed, and a narrow band of frequencies is allowed to pass through the filter.

Figure 10.25 shows the frequency response of a low-pass filter and a high-pass filter with $R = 50.0\ \Omega$ and $C = 20.0$ mF. For this combination of resistance and capacitance, the breakpoint frequency is

$$\omega_B = \frac{1}{RC} = \frac{1}{(50.0\ \Omega)\left(20.0 \times 10^{-3}\ \text{F}\right)} = 1.00\ \text{rad/s}.$$

Also shown in Figure 10.25 is the frequency response of a band-pass filter with $R_1 = R_2 = 50.0\ \Omega$ and $C_1 = C_2 = 20.0$ mF.

Concept Check 10.6

The frequency response for a band-pass filter plotted in Figure 10.25 is the ________ of the frequency responses for the low-pass and high-pass filters.

a) sum

b) product

c) difference

d) ratio

e) It is none of the above.

EXAMPLE 10.3 Crossover Circuit for Audio Speakers

One way to improve the performance of an audio system is to send high frequencies to a small speaker called a *tweeter* and low frequencies to a large speaker called a *woofer*. Figure 10.26 shows a simple crossover circuit that preferentially passes high frequencies to a tweeter and low frequencies to a woofer. The crossover circuit consists of an RC high-pass filter and an RL low-pass filter connected in parallel to the output of the audio amplifier. The speakers act as resistors, as shown in Figure 10.26. The capacitance and the resistance of this crossover circuit are $C = 10.0\ \mu\text{F}$ and $L = 10.0$ mH. The speakers each have a resistance of $R = 8.00\ \Omega$.

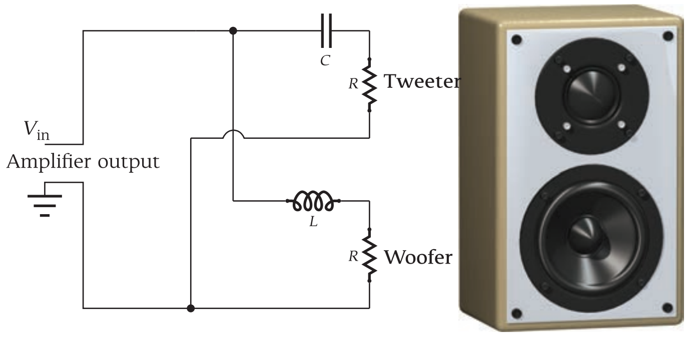


FIGURE 10.26 Crossover circuit for audio speakers.

PROBLEM

What is the crossover frequency for this crossover circuit?

SOLUTION

We can use equation 10.27 for the response of the RL low-pass filter and equation 10.30 for the response of the RC high-pass filter and equate the two responses:

$$\frac{V_{out}}{V_{in}} = \frac{R}{\sqrt{R^2 + X_L^2}} = \frac{R}{\sqrt{R^2 + X_C^2}}.$$

Thus, the responses of the low-pass filter and the high-pass filter are the same when

$$X_L = X_C. \tag{i}$$

We can rewrite equation (i) as

$$\omega_{crossover} L = \frac{1}{\omega_{crossover} C},$$

where $\omega_{crossover}$ is the crossover angular frequency. Thus, the crossover angular frequency is

$$\omega_{crossover} = \frac{1}{\sqrt{LC}}.$$

We want to determine the crossover frequency, and since $f = \omega/2\pi$, we have

$$f_{crossover} = \frac{\omega_{crossover}}{2\pi} = \frac{1}{2\pi\sqrt{LC}}.$$

Putting in the numerical values, we get

$$f_{crossover} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(10.0\ \text{mH})(10.0\ \mu\text{F})}} = 503.\ \text{Hz}.$$

Figure 10.27 shows the response of the crossover circuit as a function of frequency. The low-pass response and the high-pass response cross at $f_{crossover} = 503.$ Hz, sending higher frequencies predominantly to the tweeter and lower frequencies predominantly to the woofer.

This simple crossover circuit would not produce ideal audio performance over a broad range of frequencies and speaker designs. More sophisticated crossover circuits that have better performance deal with mid-range frequencies as well.

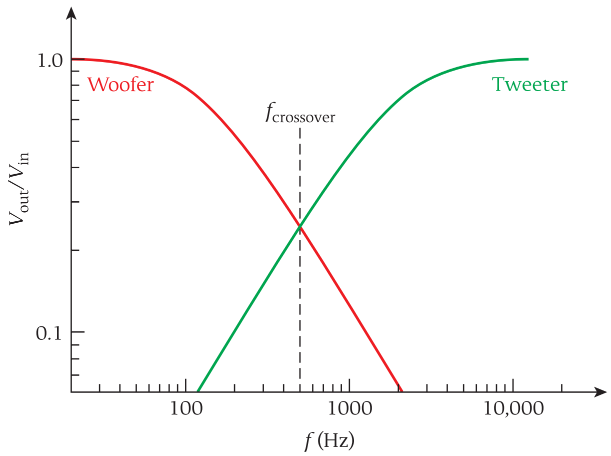


FIGURE 10.27 The response of the crossover circuit as a function of frequency.

10.6 Energy and Power in AC Circuits

When an RLC circuit is in operation, some of the energy in the circuit is stored in the electric field of the capacitor, some of the energy is stored in the magnetic field of the inductor, and some energy is dissipated in the form of heat in the resistor. In most applications, we are interested in the steady-state behavior of the circuit, behavior that occurs after initial (transient) effects die out. (A full mathematical analysis would also account for the transient effects, which die out exponentially in a way similar to that described by equation 10.6 for a single-loop RLC circuit with no emf source.) The sum of the energy stored in the capacitor and the inductor does not change in the steady state, as we saw in Section 10.2. Therefore, the energy transferred from the source of emf to the circuit is transferred to the resistor.

The rate at which energy is dissipated in the resistor is the power, P, given by

$$P = i^2R = \left[I\sin(\omega t - \phi)\right]^2 R = I^2R\sin^2(\omega t - \phi), \tag{10.32}$$

where the alternating current, i, is given by equation 10.9. We can express the average power, $\langle P \rangle$, using the fact that the average value of $\sin^2(\omega t - \phi)$ over a full oscillation is $\frac{1}{2}$:

$$\langle P \rangle = \tfrac{1}{2}I^2R = \left(\frac{I}{\sqrt{2}}\right)^2 R.$$

In calculations of power and energy, it is common to use the **root-mean-square (rms) current,** I_{rms}. (In general, *root-mean-square*, or *rms*, means the square root of the mean of the square of the specific quantity.) From equation 10.32, we have $i^2 = \left[I\sin(\omega t - \varphi)\right]^2$, and the mean (or average) of i^2 is $I^2/2$. Thus, $I_{rms} = I/\sqrt{2}$. We can then write the average power as

$$\langle P \rangle = I_{rms}^2 R. \tag{10.33}$$

In a similar way, we can define the root-mean-square values of other time-varying quantities, such as the voltage:

$$V_{rms} = \frac{V_m}{\sqrt{2}}. \tag{10.34}$$

The current and voltage values normally quoted for alternating currents and measured by AC ammeters and voltmeters are I_{rms} and V_{rms}. For example, wall sockets in the United States provide $V_{rms} = 110$ V, which corresponds to a maximum voltage of $\sqrt{2}\,(110\text{ V}) \approx 156$ V.

We can rewrite equation 10.21 in terms of root-mean-square values by multiplying both sides of the equation by $1/\sqrt{2}$:

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{V_{rms}}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}. \tag{10.35}$$

This form is used most often to describe the characteristics of AC circuits.

We can describe the average power dissipated in an AC circuit in a different way by starting with equation 10.33:

$$\langle P \rangle = I_{rms}^2 R = \frac{V_{rms}}{Z} I_{rms} R = I_{rms} V_{rms} \frac{R}{Z}. \tag{10.36}$$

From Figure 10.16, we see that the cosine of the phase constant is equal to the ratio of the maximum value of the voltage across the resistor to the maximum value of the time-varying emf:

$$\cos\phi = \frac{V_R}{V_m} = \frac{IR}{IZ} = \frac{R}{Z}. \tag{10.37}$$

We can thus rewrite equation 10.36 as follows:

$$\langle P \rangle = I_{rms} V_{rms} \cos\phi. \tag{10.38}$$

This expression gives the average power dissipated in an AC circuit, where the term $\cos\phi$ is called the **power factor.** You can see that for $\phi = 0$, maximum power is dissipated in the circuit; that is, the maximum power is dissipated in an AC circuit when the frequency of the time-varying emf matches the resonant frequency of the circuit.

We can combine equations 10.19, 10.35, and 10.36 to obtain an expression for the average power as a function of the angular frequency, the inductance, the resistance, and the resonant frequency:

$$\langle P \rangle = I_{rms} V_{rms} \frac{R}{Z} = \frac{V_{rms}}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} V_{rms} \frac{R}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}},$$

or simply

$$\langle P \rangle = \frac{V_{\text{rms}}^2 R}{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}.$$

Since $\omega_0 = 1/\sqrt{LC}$, we can write $C = 1/(L\omega_0^2)$, and thus, we find for the average power for a series RLC circuit in terms of the angular frequency:

$$\langle P \rangle = \frac{V_{\text{rms}}^2 R}{R^2 + \left(\omega L - \dfrac{L\omega_0^2}{\omega}\right)^2} = \frac{V_{\text{rms}}^2 R\omega^2}{R^2\omega^2 + L^2\left(\omega^2 - \omega_0^2\right)^2}. \tag{10.39}$$

Typically all voltages, currents, and powers in AC circuits are specified as root-mean-square values. For example, the typical 110-V AC wall circuit has $V_{\text{rms}} = 110$ V, and the ubiquitous 1000-W hair dryer uses $P_{\text{rms}} = 1000$ W.

SOLVED PROBLEM 10.1 Voltage Drop across an Inductor

A series RLC circuit has a source of time-varying emf that supplies $V_{\text{rms}} = 170.0$ V, a resistance $R = 820.0\ \Omega$, an inductance $L = 30.0$ mH, and a capacitance $C = 0.290$ mF. The circuit is operating at its resonant frequency.

PROBLEM

What is the root-mean-square voltage drop across the inductor?

SOLUTION

THINK At the resonant frequency, the impedance of the circuit is equal to the resistance. We can calculate the root-mean-square current in the circuit. The voltage drop across the inductor is then the product of the root-mean-square current in the circuit and the inductive reactance.

SKETCH A diagram of the series RLC circuit is shown in Figure 10.28.

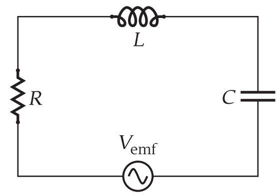


FIGURE 10.28 A series RLC circuit.

RESEARCH At resonance, the impedance of the circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = R.$$

At resonance, the root-mean-square current, I_{rms}, in the circuit is given by

$$V_{\text{rms}} = I_{\text{rms}} R.$$

The root-mean-square voltage drop across the inductor, V_L, at resonance is

$$V_L = I_{\text{rms}} X_L,$$

where the inductive reactance X_L is defined as

$$X_L = \omega L$$

and ω is the angular frequency at which the circuit is operating. The resonant angular frequency, ω_0, of the circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

SIMPLIFY Combining all these equations gives us the voltage drop across the inductor at resonance:

$$V_L = \left(\frac{V_{\text{rms}}}{R}\right)(\omega_0 L) = \frac{LV_{\text{rms}}}{R}\frac{1}{\sqrt{LC}} = \frac{V_{\text{rms}}}{R}\sqrt{\frac{L}{C}}.$$

CALCULATE Putting in the numerical values gives us

$$V_L = \frac{170.0\ \text{V}}{820.0\ \Omega}\sqrt{\frac{30.0 \times 10^{-3}\ \text{H}}{0.290 \times 10^{-3}\ \text{F}}} = 2.10861\ \text{V}.$$

– Continued

ROUND We report our result to three significant figures:

$$V_L = 2.11 \text{ V}.$$

DOUBLE-CHECK The root-mean-square voltage drop across the capacitor is

$$V_C = \left(\frac{V_{\text{rms}}}{R}\right)\left(\frac{1}{\omega_0 C}\right) = \frac{V_{\text{rms}}}{RC}\sqrt{LC} = \left(\frac{V_{\text{rms}}}{R}\right)\sqrt{\frac{L}{C}},$$

which is the same as the root-mean-square voltage drop across the inductor. At resonance, the instantaneous voltage drop across the inductor is the negative of the voltage drop across the capacitor. Thus, the rms voltage across the capacitor should be the same as the rms voltage across the inductor. Thus, our result seems reasonable.

SOLVED PROBLEM 10.2 Power Dissipated in an RLC Circuit

A series RLC circuit has a source of emf providing $V_{\text{rms}} = 120.0$ V at a frequency $f = 50.0$ Hz, as well as an inductor, $L = 0.500$ H, a capacitor, $C = 3.30$ μF, and a resistor, $R = 276.$ Ω.

PROBLEM

What is the average power dissipated in the circuit?

SOLUTION

THINK The average power dissipated in the circuit is the root-mean-square current times the root-mean-square voltage, but it depends on the angular frequency of the source of emf. The current in the circuit can be found using the impedance.

SKETCH A diagram of a series RLC circuit is shown in Figure 10.28.

RESEARCH The angular frequency, ω, of the source of emf is

$$\omega = 2\pi f.$$

The impedance, Z, of the circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2},$$

where the inductive reactance is given by

$$X_L = \omega L$$

and the capacitive reactance is given by

$$X_C = \frac{1}{\omega C}.$$

We can find the root-mean-square current, I_{rms}, in the circuit using the relationship

$$V_{\text{rms}} = I_{\text{rms}} Z.$$

The average power dissipated in the circuit, $\langle P \rangle$, is given by

$$\langle P \rangle = I_{\text{rms}} V_{\text{rms}} \cos\phi,$$

where ϕ is the phase constant between the voltage and the current in the circuit:

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right).$$

SIMPLIFY We can combine all these equations to obtain an expression for the average power dissipated in the circuit:

$$\langle P \rangle = \frac{V_{\text{rms}}}{Z} V_{\text{rms}} \cos\phi = \frac{V_{\text{rms}}^2}{\sqrt{R^2 + (X_L - X_C)^2}} \cos\phi.$$

CALCULATE First, we calculate the inductive reactance:

$$X_L = \omega L = 2\pi f L = 2\pi(50.0 \text{ Hz})(0.500 \text{ H}) = 157.1 \text{ Ω}.$$

Next, we calculate the capacitive reactance:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2\pi(50.0 \text{ Hz})(3.30 \times 10^{-6} \text{ F})} = 964.6 \ \Omega.$$

The phase constant is then

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{157.1 \ \Omega - 964.6 \ \Omega}{276 \ \Omega}\right) = -1.241 \text{ rad} = -71.13°.$$

We now calculate the average power dissipated in the circuit:

$$\langle P \rangle = \frac{(120.0 \text{ V})^2}{\sqrt{(276 \ \Omega)^2 + (157.1 \ \Omega - 964.6 \ \Omega)^2}} \cos-(-1.241 \text{ rad}) = 5.46477 \text{ W}.$$

ROUND We report our result to three significant figures:

$$\langle P \rangle = 5.46 \text{ W}.$$

DOUBLE-CHECK To double-check our result, we can calculate the power that would be dissipated in the circuit if it were operating at the resonant frequency. At the resonant frequency, the maximum power is dissipated in the circuit, and the impedance of the circuit is equal to the resistance of the resistor. Thus, the maximum average power is

$$\langle P \rangle_{\max} = \frac{V_{\text{rms}}^2}{R} = \frac{(120.0 \text{ V})^2}{276 \ \Omega} = 52.2 \text{ W}.$$

Our result for the power dissipated at $f = 50.0$ Hz is lower than the maximum average power, so it seems plausible.

Quality Factor

The **quality factor,** Q, of a series RLC circuit is defined as

$$Q = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}. \tag{10.40}$$

The quality factor is the ratio of total energy stored in the system divided by the energy dissipated per cycle of the oscillation. For a series RLC circuit, the quality factor characterizes the selectivity of the circuit. The higher the value of Q, the more selective the circuit, that is, the more precisely a given frequency can be isolated (as in an AM radio receiver, discussed next). The lower the value of Q, the less selective the circuit becomes.

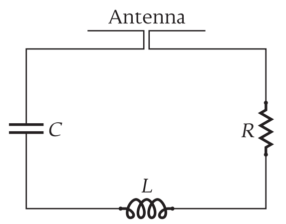


FIGURE 10.29 A series RLC circuit with the source of time-varying emf replaced by an antenna. This circuit can function as an AM radio receiver.

AM Radio Receiver

Let's look at a typical example of a selective series RLC circuit, an AM radio receiver. An AM radio receiver can be constructed using a series RLC circuit in which the time-varying emf is supplied by an antenna that picks up transmissions from a distant radio station broadcasting at a given frequency and converts those transmissions to voltage, as illustrated in Figure 10.29.

Figure 10.30 is a plot of the average power as a function of the frequency of the signal received on the antenna for the circuit shown in Figure 10.29, assuming that $R = 0.09111 \ \Omega$, $L = 5.000 \ \mu\text{H}$, $C = 6.693$ nF, and $V_{\text{rms}} = 3.500$ mV. The resonant angular frequency for this circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(5.000 \times 10^{-6} \text{ H})(6.693 \times 10^{-9} \text{ F})}} = 5.466 \times 10^6 \text{ rad/s}$$

which corresponds to a resonant frequency of

$$f_0 = \frac{\omega_0}{2\pi} = \frac{5.466 \times 10^6 \text{ rad/s}}{2\pi} = 870.0 \text{ kHz}.$$

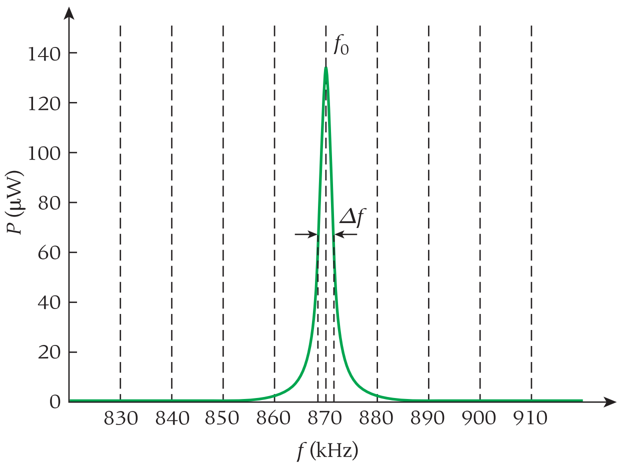


FIGURE 10.30 Power response of a series RLC circuit functioning as an AM radio receiver. The label Δf indicates the *full width at half maximum,* or the difference between the frequencies where the power has half the value that it has at its maximum value, at frequency f_0.

Concept Check 10.7

WiFi networks are installed in most coffee shops and many residences to provide access to the Internet. The most common WiFi standard is known as 802.11 g, which supports communication rates of up to 54 megabits per second. Wireless networks in the United States and Canada that follow this standard use a frequency around 2.4 GHz, in 14 different channels in the band between 2.401 GHz and 2.495 GHz. Each channel has a full width at half maximum of 22 MHz. What is the quality factor of these WiFi networks?

a) 0.1
b) 9.9
c) 33
d) 109
e) 300

The quality factor for this circuit is

$$Q = \frac{\omega_0 L}{R} = \frac{(5.466\times 10^6 \text{ rad/s})(5.000\times 10^{-6} \text{ H})}{0.09111\ \Omega} = 300.0.$$

A method for determining the approximate quality factor of a series RLC circuit uses the formula

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f},$$

where $\Delta\omega$ and Δf are the full widths at half maximum for the angular frequency and the frequency, respectively, on the power response curve. The higher the value of Q, the narrower the power response to frequency. In Figure 10.30, $\Delta f = 2.9$ kHz, which gives a quality factor of

$$Q = \frac{f_0}{\Delta f} = \frac{870.0 \text{ kHz}}{2.9 \text{ kHz}} = 300.0.$$

This is the same result obtained using the formula that defines the quality factor in equation 10.40. Note that these two formulas for the quality factor have the same results only for high Q!

The alternative formula for the quality factor of a series RLC circuit is similar to the expression given that for the quality of a weakly damped mechanical oscillator, $Q \approx \dfrac{\omega_0}{2\omega_\gamma}$, where ω_0 is the resonant angular frequency and ω_γ is the damping angular frequency.

In Figure 10.30, the frequencies of adjacent channels in the AM radio band are indicated by the vertical dashed lines located 10 kHz apart. The response of the series RLC circuit allows the AM receiver to tune in one station and exclude the adjacent channels.

SOLVED PROBLEM 10.3 Unknown Inductance in an RL Circuit

Consider a series RL circuit with a source of time-varying emf. In this circuit, $V_{\text{rms}} = 33.0$ V with $f = 7.10$ kHz and $R = 83.0\ \Omega$. A current $I_{\text{rms}} = 0.158$ A flows in the circuit.

PROBLEM

What is the magnitude of the inductance, L?

SOLUTION

THINK The specified voltage and current are implicitly root-mean-square values. We can relate the voltage and current through the impedance of the circuit. The impedance of this circuit depends on the resistance and the inductance, as well as the frequency of the source of emf.

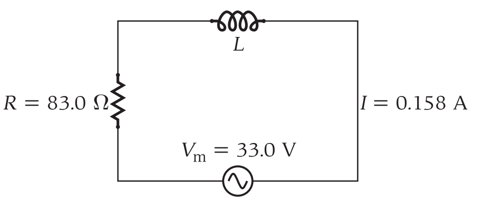


FIGURE 10.31 A series RL circuit.

SKETCH A diagram of the circuit is shown in Figure 10.31.

RESEARCH We can relate the time-varying emf, V_{m}, and the impedance Z in the circuit:

$$V_{\text{m}} = IZ. \qquad \text{(i)}$$

The impedance is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + X_L^2}, \qquad \text{(ii)}$$

where R is the resistance, X_L is the inductive reactance, and X_C is the capacitive reactance, which is zero. The angular frequency, ω, of the circuit is given by

$$\omega = 2\pi f,$$

where f is the frequency. We can express the inductive reactance as

$$X_L = \omega L. \qquad \text{(iii)}$$

SIMPLIFY We can combine equations (i), (ii), and (iii) to obtain

$$Z^2 = R^2 + X_L^2 = \left(\frac{V_{\text{m}}}{I}\right)^2 = R^2 + (\omega L)^2. \qquad \text{(iv)}$$

Rearranging equation (iv) gives

$$\omega L = \sqrt{\frac{V_{\text{m}}^2}{I^2} - R^2}.$$

And, finally, we find the unknown inductance:

$$L = \frac{1}{2\pi f}\sqrt{\frac{V_\mathrm{m}^2}{I^2} - R^2}.$$

CALCULATE Putting in the numerical values gives us

$$L = \frac{1}{2\pi\left(7.10\times10^3\ \mathrm{s}^{-1}\right)}\sqrt{\frac{(33.0\ \mathrm{V})^2}{(0.158\ \mathrm{A})^2} - (83.0\ \Omega)^2} = 0.0042963\ \mathrm{H}.$$

ROUND We report our result to three significant figures:

$$L = 4.30\times10^{-3}\ \mathrm{H} = 4.30\ \mathrm{mH}.$$

DOUBLE-CHECK To double-check our result for the inductance, we first calculate the inductive reactance:

$$X_L = 2\pi f L = 2\pi\left(7.10\times10^3\ \mathrm{s}^{-1}\right)\left(4.30\times10^{-3}\ \mathrm{H}\right) = 192.\ \Omega.$$

The impedance of the circuit is then

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{(83.0\ \Omega)^2 + (192\ \Omega)^2} = 209.\ \Omega.$$

We use this value of Z to calculate a value of V_m:

$$V_\mathrm{m} = IZ = (0.158\ \mathrm{A})(209.\ \Omega) = 33.0\ \mathrm{V},$$

which agrees with the value specified in the problem statement. Thus, our result is consistent.

10.7 Transformers

This section discusses the root-mean-square values of currents and voltages, rather than the maximum or instantaneous values. The result obtained for the power is always the average power when root-mean-square values are used. This practice is the convention normally followed by scientists, engineers, and electricians dealing with AC circuits.

In an AC circuit that has only a resistor, the phase constant is zero. Thus, we can express the power as

$$P = IV. \tag{10.41}$$

For a given power delivered to a circuit, the application dictates the choice of high current or high voltage. For example, to provide enough power to operate a computer or a vacuum cleaner, using a high voltage might be dangerous. The design of electric generators is complicated by the use of high voltages. Therefore, in these devices, lower voltages and higher currents are advantageous.

However, the transmission of electric power requires the opposite condition. The power dissipated in a transmission line is given by $P = I^2R$. Thus, the power lost in a line, like those in Figure 10.32a is proportional to the square of the current in the line. As an example, consider a power plant that produces 500. MW of power. If the power is transmitted at 350. kV, the current in the power lines will be

$$I = \frac{P}{V} = \frac{500.\ \mathrm{MW}}{350.\ \mathrm{kV}} = \frac{5.00\times10^8\ \mathrm{W}}{3.50\times10^5\ \mathrm{V}} = 1.43\ \mathrm{kA}.$$

If the total resistance of the power lines is 50. Ω, the power lost in the transmission lines is

$$P = I^2R = (1.43\ \mathrm{kA})^2(50.0\ \Omega) = 102.\ \mathrm{MW},$$

or about 20% of the generated power. A similar calculation would show that transmitting the power at 200. kV instead of 350. kV would increase the power loss by a factor of 3.1. Thus, approximately 60% of the power generated would be lost in transmission. This is why the transmission of electric power is always done at the highest possible voltage.

The ability to change voltage allows electric power to be generated and used at low, safe voltages but transmitted at the highest practical voltage. Alternating currents and voltages are transformed from high to low values by a device called, appropriately, a **transformer.** A transformer

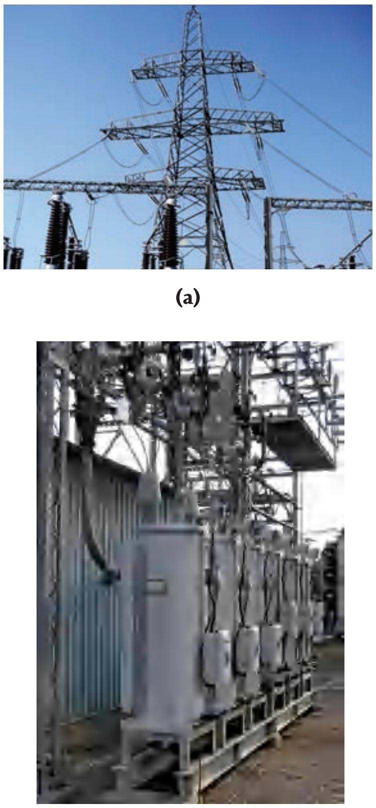
Concept Check 10.8

In the series RL circuit in Solved Problem 10.3, what is the magnitude of the phase difference between the time-varying emf and the current in the circuit?

a) 30.0°
b) 45.0°
c) 66.6°
d) 75.0°
e) 90.0°

Self-Test Opportunity 10.5

You might argue that power companies should simply reduce the resistance in their transmission lines to avoid the substantial power losses. Typical wires for electric power transmission lines are finger-thick. How big would they need to be to reduce the resistance by a factor of 100, assuming that all other parameters (material used, length) remain the same? (*Hint:* Consult Section 25.3.)



(a)

(b)

FIGURE 10.32 (a) High-voltage power lines; (b) transformers for residential power lines.

FIGURE 10.33 A transformer (black rectangle with yellow core) is the main component of a cell phone charger. (The charger also includes a rectifier, described in Section 10.8.)

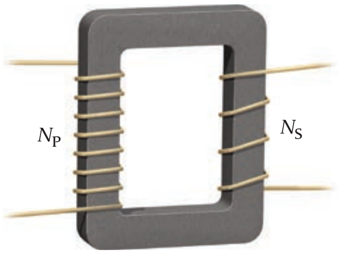


FIGURE 10.34 Transformer with N_P primary windings and N_S secondary windings.

that takes voltages from lower to higher values is called a *step-up transformer;* a transformer that takes voltages from higher to lower values is called a *step-down transformer*. Transformers are the main components of, for example, your cell phone charger (Figure 10.33) and the power supply for your MP3 player, your laptop, and pretty much every other consumer electronics device. Most of these devices require voltages of 12 V or less, but the grid delivers 110 V to outlets in the United States, necessitating the transformers that clutter your desk drawers.

A transformer consists of two sets of coils wrapped around an iron core (Figure 10.34). The primary coil, with N_P turns, is connected to a source of emf described by

$$V_{emf} = V_{max} \sin \omega t.$$

We'll assume that the primary coil acts as an inductor. The primary circuit has the current and voltage out of phase by $\pi/2$ rad (90°), so the power factor, $\cos\phi$, is zero. Thus, the source of emf is not delivering any power to the transformer if only the primary coil is connected. In other words, if the secondary coil is not connected to a closed circuit, the transformer does not draw any power. For example, your cell phone charger does not draw power if it is plugged into the wall socket but the other end is not connected to the cell phone. (This statement is not absolutely true; there is finite resistance in the wires in the primary coil, which this description is neglecting.)

The secondary coil of a transformer has N_S turns. The time-varying emf in the primary coil induces a time-varying magnetic field in the iron core. This core passes through the secondary coil. Thus, a time-varying voltage is induced in the secondary coil, as described by Faraday's Law of Induction:

$$V_{emf} = -N\frac{d\Phi_B}{dt},$$

where N is the number of turns and Φ_B is the magnetic flux. Because of the iron core, both the primary and secondary coils experience the same changing magnetic flux. Thus,

$$V_S = -N_S\frac{d\Phi_B}{dt}$$

and

$$V_P = -N_P\frac{d\Phi_B}{dt},$$

where V_S and V_P are the voltages across the secondary and primary windings, respectively. Dividing the first of these two equations by the other and rearranging gives

$$\frac{V_P}{N_P} = \frac{V_S}{N_S},$$

or

$$V_S = V_P\frac{N_S}{N_P}. \tag{10.42}$$

The transformer changes the voltage of the primary circuit to a secondary voltage, given by the ratio of the number turns in the secondary coil divided by the number of turns in the primary coil.

If a resistor, R, is connected across the secondary windings, a current, I_S, will begin to flow through the secondary coil. The power in the secondary circuit is then $P_S = I_S V_S$. This current induces a time-varying magnetic field that induces a voltage in the primary coil, so that the emf source then produces enough current, I_P, to maintain the original voltage. This current, I_P, is in phase with the voltage because of the resistor, so power can be transmitted to the transformer. Energy conservation requires that the power delivered to the primary coil be transferred to the secondary coil, so we can write

$$P_P = I_P V_P = P_S = I_S V_S.$$

Using equation 10.42, we can express the current in the secondary circuit as

$$I_S = I_P\frac{V_P}{V_S} = I_P\frac{N_P}{N_S}. \tag{10.43}$$

The current in the secondary circuit is equal to the current in the primary circuit multiplied by the ratio of the number of primary turns divided by the number of secondary turns.

When the secondary circuit begins to draw current, current must be supplied to the primary circuit. Since $V_S = I_S R$ in the secondary circuit, we can use equations 10.42 and 10.43 to write

$$I_P = \frac{N_S}{N_P} I_S = \frac{N_S}{N_P} \frac{V_S}{R} = \frac{N_S}{N_P}\left(V_P \frac{N_S}{N_P}\right)\frac{1}{R} = \left(\frac{N_S}{N_P}\right)^2 \frac{V_P}{R}. \tag{10.44}$$

The effective resistance of the primary circuit can be expressed in terms of $V_P = I_P R_P$, so that the effective resistance is

$$R_P = \frac{V_P}{I_P} = V_P \left(\frac{N_P}{N_S}\right)^2 \frac{R}{V_P} = \left(\frac{N_P}{N_S}\right)^2 R. \tag{10.45}$$

Note that we have assumed that there are no losses in the transformer, that the primary coil is only an inductor, that there are no losses in magnetic flux between the primary and secondary coils, and that the secondary circuit has the only resistance. Real transformers do have some losses. Part of these losses result from the fact that the alternating magnetic fields from the coils induce eddy currents in the iron core of the transformer. To counter this effect, transformer cores are constructed by laminating layers of metal to inhibit the formation of eddy currents. Modern transformers can transform voltages with very little loss.

Another application of transformers is **impedance matching.** The power transfer between a source of emf and a device that uses power is at a maximum when the impedance is the same in both. Often, the source of emf and the intended device do not have the same impedance. A common example is a stereo amplifier and its speakers. Usually, the amplifier has high impedance and the speakers have low impedance. A transformer placed between the amplifier and the speakers can help match the impedances of the devices, producing a more efficient power transfer.

10.8 Rectifiers

Many electronic devices require direct current rather than alternating current. However, many common sources of electrical power provide alternating current. Therefore, this current must be converted to direct current to operate electronic equipment. A **rectifier** is a device that converts alternating current to direct current. Most rectifiers use an electronic component that was described in Section 25.8–the diode. A diode is designed to allow current to flow in one direction and not in the other direction. The symbol for a diode is —▶|—, and the direction of the arrowhead signifies the direction in which the diode will conduct current.

Let's start with a simple circuit containing a source of time-varying emf, a resistor, and a diode, as shown in Figure 10.35b. The voltage provided by the source of emf is alternately positive and negative, as shown in Figure 10.35a. Note that both ends of the source of emf are connected simultaneously so that when one end produces a positive voltage, the other end produces a negative voltage. The circuit in Figure 10.35b produces current in the resistor that indeed flows in only one direction. However, the circuit blocks half the current, as illustrated in Figure 10.35c. Thus, this type of circuit is often termed a **halfwave rectifier.**

To allow all the current to flow in one direction, the type of circuit shown in Figure 10.36 is employed. Again the voltage alternates between positive and negative, as shown in Figure 10.36a. Two equivalent circuit diagrams are shown in Figure 10.36b and Figure 10.36c. All the current in the resistor flows in one direction, as illustrated in Figure 10.36d. This type of circuit is called a **fullwave rectifier.**

To illustrate how the fullwave rectifier works, Figure 10.37 shows instantaneous views of the circuit with positive and negative voltage. In Figure 10.37a, the voltage from the

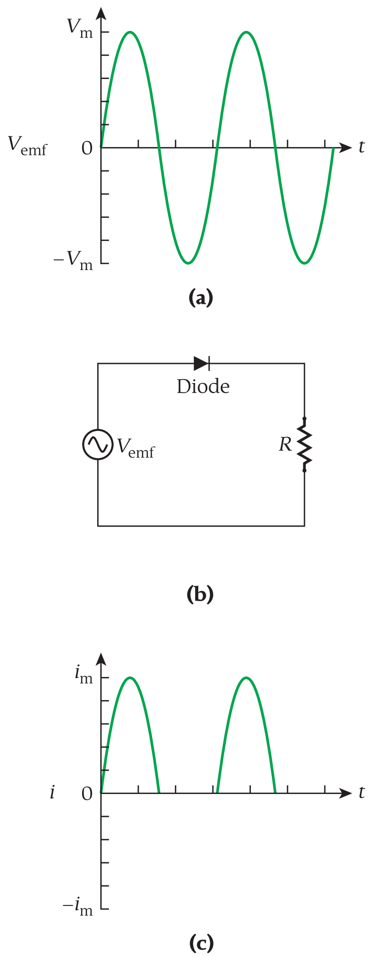


FIGURE 10.35 Circuit containing a source of time-varying emf, a resistor, and a diode, forming a halfwave rectifier: (a) the emf as a function of time; (b) the circuit diagram; (c) the current flowing through the circuit as a function of time.

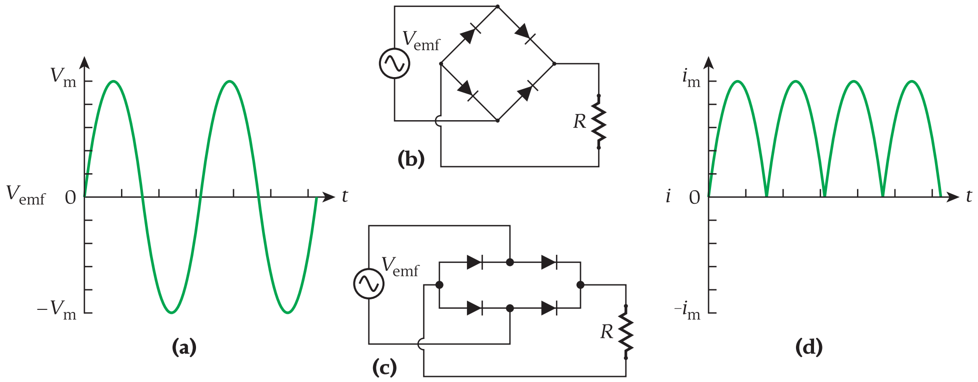


FIGURE 10.36 Circuit containing a source of time-varying emf, a resistor, and four diodes, forming a fullwave rectifier: (a) the emf as a function of time; (b) the circuit diagram; (c) alternative way of drawing the circuit diagram; (d) the current flowing through the circuit as a function of time.

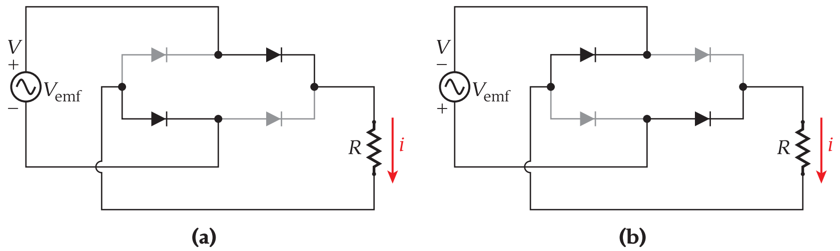


FIGURE 10.37 A fullwave rectifier with the diodes that are not conducting current at the given instant in gray. The current in the resistor always flows in the same direction. (a) Positive voltage. (b) Negative voltage.

Self-Test Opportunity 10.6

A typical alternator found in automobiles produces three-phase alternating current. Each phase is shifted by 120° from the next phase. Draw the circuit diagram for the fullwave rectifier for this alternator, based on the circuit diagram in Figure 10.36c but incorporating six diodes instead of four.

source of emf is positive. The black diodes are conducting current, while the gray diodes are not. The voltage is reversed in Figure 10.37b, and the current flows through the other pair of diodes; the current in the resistor is still in the same direction.

Although the fullwave rectifier does indeed convert alternating current to direct current, the resulting direct current varies with time. This variance, often called *ripple*, can be smoothed out by adding a capacitor to the output of the rectifier, creating an RC circuit with a time constant governed by the choice of R and C, as shown in Figure 10.38. In Figure 10.38a, the time-varying emf is shown, and the circuit diagram is presented in Figure 10.38b. The direct current is filtered by the added capacitor. The resulting current as a function of time is shown in Figure 10.38c. The current still varies with time but much less so than the current flowing out of the fullwave rectifier without a capacitor.

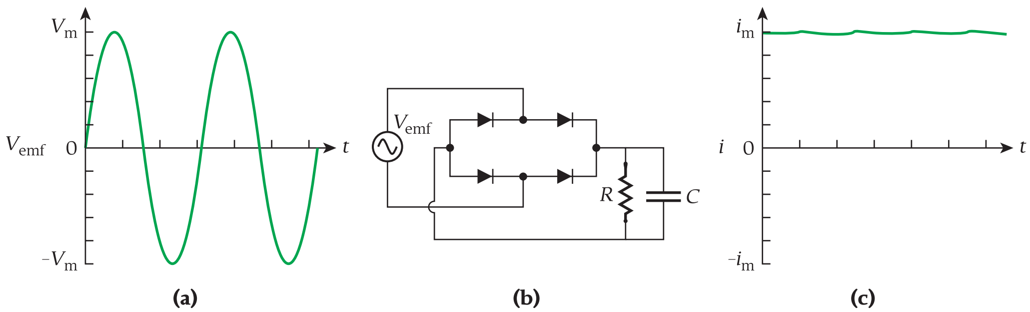


FIGURE 10.38 Circuit containing a source of time-varying emf, a resistor, a capacitor, and four diodes, forming a filtered fullwave rectifier: (a) the emf as a function of time; (b) the circuit diagram; (c) the current flowing through the circuit as a function of time.

WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- The energy stored in the electric field of a capacitor with capacitance C and charge q is given by $U_E = \frac{1}{2}(q^2/C)$; the energy stored in the magnetic field of an inductor with inductance L that is carrying current i is given by $U_B = \frac{1}{2}Li^2$.
- The current in a single-loop circuit containing an inductor and a capacitor (an LC circuit) oscillates with a frequency given by $\omega_0 = 1/\sqrt{LC}$.
- The current in a single-loop circuit containing a resistor, an inductor, and a capacitor (an RLC circuit) oscillates with a frequency given by $\omega = \sqrt{\omega_0^2 - (R/2L)^2}$, where $\omega_0 = 1/\sqrt{LC}$.
- The charge, q, on a capacitor in a single-loop RLC circuit oscillates and decreases exponentially with time according to $q = q_{max}\, e^{-Rt/2L}\cos(\omega t)$, where q_{max} is the original charge on the capacitor.
- For a single-loop circuit containing a source of time-varying emf and a resistor, R, $V_R = I_R R$, where V_R and I_R are the voltage and the current, respectively.
- For a single-loop circuit containing a source of time-varying emf that has frequency ω and a capacitor, $V_C = I_C X_C$, where V_C and I_C are the voltage and the current, respectively, and $X_C = 1/\omega C$ is the capacitive reactance.
- For a single-loop circuit containing a source of time-varying emf that has frequency ω and an inductor, $V_L = I_L X_L$, where V_L and I_L are the voltage and the current, respectively, and $X_L = \omega L$ is the inductive reactance.
- For a single-loop RLC circuit containing a source of time-varying emf that has frequency ω, $V = IZ$, where V and I are the voltage and the current, respectively, and $Z = \sqrt{R^2 + (X_L - X_C)^2}$ is the impedance.
- The phase constant, ϕ, between the current and the voltage in a single-loop RLC circuit containing a source of time-varying emf that has frequency ω is given by $\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$.
- The average power in a single-loop RLC circuit containing a source of time-varying emf that has frequency ω is given by $\langle P\rangle = I_{rms}V_{rms}\cos\phi$, where $I_{rms} = I_m/\sqrt{2}$ and $V_{rms} = V_m/\sqrt{2}$.
- All currents, voltages, and powers quoted for alternating-current (AC) circuits are typically root-mean-square values.
- A transformer with N_P windings in the primary coil and N_S windings in the secondary coil can convert a primary alternating voltage, V_P, to a secondary alternating voltage, V_S, given by $V_S = V_P\dfrac{N_S}{N_P}$, and a primary alternating current, I_P, to a secondary alternating current, I_S, given by $I_S = I_P\dfrac{N_P}{N_S}$.

ANSWERS TO SELF-TEST OPPORTUNITIES

10.1 For this circuit, $\omega_0 = 2\pi f = 2\pi(200 \text{ kHz})$ rad/s $= 4\pi \times 10^5$ rad/s; $\phi = 0$ since $q(0) = q_{max}$

a) true ($\cos\omega_0 t = -1$)

b) false ($\sin\omega_0 t = 0$)

c) false ($\cos\omega_0 t = -1$ and $\sin\omega_0 t = 0$)

d) false ($\cos\omega_0 t = 0$ and $\sin\omega_0 t = 1$)

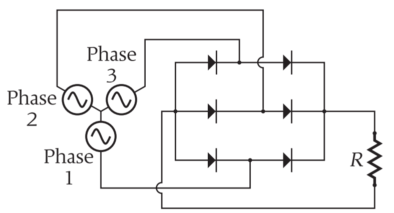
10.2 It can be seen that k/m corresponds to $1/LC$, and $b/2m$ corresponds to $R/2L$. Thus, the inductance, L, plays the role of the mass, m, the capacitance, C, corresponds to the inverse spring constant, $1/k$, and the resistance, R, has the function of the damping constant, b.

10.3 a) true b) true c) false

10.4 a) true b) true c) false

10.5 The resistance is inversely proportional to the inverse of the area of the wire and thus inversely proportional to the square of the radius. Therefore, a wire that was 10 times thicker would have a resistance that was 100 times lower.

10.6



Three-phase bridge rectifier

PROBLEM-SOLVING GUIDELINES

1. Most problems concerning AC circuits require you to calculate resistance, capacitive reactance, inductive reactance, or impedance. Be sure that you understand what each of these quantities is and how to use them in calculating currents and voltages.

2. You will often have to distinguish between the instantaneous current or voltage in a circuit and the root-mean-square or maximum value of current or voltage. The common convention is to use lowercase i and v for instantaneous values and uppercase I and V for constant values (with subscripts as necessary). Be sure you use notation that is clear so that you won't become confused during the calculations.

3. Remember the phase relations for AC circuits: For a resistor, current and voltage are in phase; for a capacitor, current leads voltage; for an inductor, current lags voltage.

4. Phasors add by vector operations, not by simple scalar arithmetic. Whenever you use phasors to determine current or voltage, check the results by checking the phase relationships given in the preceding guideline.

5. It is usually easier to work with angular frequency (ω) than with frequency (f) in analyzing AC circuits. Most often, you will be given an angular frequency in the problem statement, but if you are given a frequency, convert it to an angular frequency by multiplying it by 2π.

MULTIPLE-CHOICE QUESTIONS

10.1 A 200-Ω resistor, a 40.0-mH inductor and a 3.0-μF capacitor are connected in series with a source of time-varying emf that provides 10.0 V at a frequency of 1000 Hz. What is the impedance of the circuit?

a) 200 Ω
b) 228 Ω
c) 342 Ω
d) 282 Ω

10.2 For which values of f is $X_L > X_C$?

a) $f > 2\pi(LC)^{1/2}$
b) $f > (2\pi LC)^{-1}$
c) $f > (2\pi(LC)^{1/2})^{-1}$
d) $f > 2\pi LC$

10.3 Which statement about the phase relation between the electric and magnetic fields in an LC circuit is correct?

a) When one field is at its maximum, the other is also, and the same for the minimum values.

b) When one field is at maximum strength, the other is at minimum (zero) strength.

c) The phase relation, in general, depends on the values of L and C.

10.4 For the band-pass filter shown in Figure 10.25, how can the width of the frequency response be increased?

a) increase R_1
b) decrease C_1
c) increase R_2
d) increase C_2
e) do any of the above

10.5 The phase constant, ϕ, between the voltage and the current in an AC circuit depends on the ___.

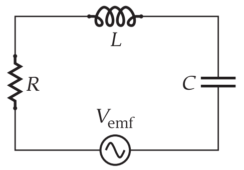
a) inductive reactance
b) capacitive reactance
c) resistance
d) all of the above

10.6 The AM radio band covers the frequency range from 520 kHz to 1610 kHz. Assuming a fixed inductance in a simple LC circuit, what ratio of capacitance is necessary to cover this frequency range? That is, what is the value of C_h/C_l, where C_h is the capacitance for the highest frequency and C_l is the capacitance for the lowest frequency?

a) 9.59
b) 0.104
c) 0.568
d) 1.76

10.7 In the RLC circuit in the figure, $R = 60\ \Omega$, $L = 3$ mH, $C = 4$ mF, and the source of time-varying emf has a peak voltage of 120 V. What should the angular frequency, ω, be to produce the largest current in the resistor?

a) 4.2 rad/s
b) 8.3 rad/s
c) 204 rad/s
d) 289 rad/s
e) 5000 rad/s
f) 20,000 rad/s



10.8 A standard North American wall socket plug is labeled 110 V. This label indicates the ________ value of the voltage.

a) average
b) maximum
c) root-mean-square (rms)
d) instantaneous

10.9 A circuit contains a source of time-varying emf, which is given by $V_{emf} = 120.0 \sin[(377\text{ rad/s})t]$ V, and a capacitor with capacitance $C = 5.00\ \mu$F.
What is the current in the circuit at $t = 1.00$ s?

a) 0.226 A
b) 0.451 A
c) 0.555 A
d) 0.750 A
e) 1.25 A

10.10 A source of time-varying emf supplies $V_{max} = 115.0$ V at $f = 60.0$ Hz in a series RLC circuit in which $R = 374\ \Omega$, $L = 0.310$ H, and $C = 5.50\ \mu$F. What is the impedance of this circuit?

a) 321 Ω
b) 523 Ω
c) 622 Ω
d) 831 Ω
e) 975 Ω

CONCEPTUAL QUESTIONS

10.11 What is the impedance of a series RLC circuit when the frequency of time-varying emf is set to the resonant frequency of the circuit?

10.12 Estimate the total energy stored in the 5.00 km of space above Earth's surface if the average magnitude of the magnetic field at Earth's surface is about 0.500×10^{-4} T.

10.13 In a DC circuit containing a capacitor, a current will flow through the circuit for only a very short time, while the capacitor is being charged or discharged. On the other hand, a steady alternating current will flow in a circuit containing the same capacitor but powered by a source of time-varying emf. Does it mean that charges are crossing the gap (dielectric) of the capacitor?

10.14 In an RL circuit with alternating current, the current lags behind the voltage. What does this mean, and how can it be explained qualitatively, based on the phenomenon of electromagnetic induction?

10.15 In Solved Problem 10.3, the voltage supplied by the source of time-varying emf is 33.0 V, the voltage across the resistor is $V_R = IR = 13.1$ V, and the voltage across the inductor is $V_L = IX_L = 30.3$ V. Does this circuit obey Kirchhoff's rules?

10.16 Why is rms power specified for an AC circuit, not average power?

10.17 Why can't we use a universal charger that plugs into a household outlet to charge all our electrical devices–cell phone, toy dog, can opener, and so on–rather than using a separate charger with its own transformer for each device?

10.18 If you use a parallel plate capacitor with air in the gap between the plates as part of a series RLC circuit in a generator, you can measure current flowing through the generator. Why is it that the air gap in the capacitor does not act like an open switch, blocking all current flow in the circuit?

10.19 A common configuration of wires has twisted pairs as opposed to straight, parallel wires. What is the technical advantage of using twisted pairs of wires versus straight, parallel pairs?

10.20 In a classroom demonstration, an iron core is inserted into a large solenoid connected to an AC power source. The effect of the core is to magnify the magnetic field in the solenoid by the relative magnetic permeability, k_m, of the core (where k_m is a dimensionless constant, substantially greater than unity for a ferromagnetic material, introduced in Chapter 8) or, equivalently, to replace the magnetic permeability of free space, μ_0, with the magnetic permeability of the core, $\mu = k_m\mu_0$.

a) The measured root-mean-square current drops from approximately 10 A to less than 1 A and remains at the lower value. Explain why.

b) What would happen if the power source were DC?

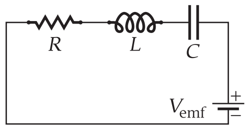
10.21 Along Capitol Drive in Milwaukee, Wisconsin, there are a large number of radio broadcasting towers. Contrary to expectation, radio reception there is terrible; unwanted stations often interfere with the one tuned in. Given that a car radio tuner is a resonant oscillator–its resonant frequency is adjusted to that of the desired station–explain this crosstalk phenomenon.

10.22 A series RLC circuit is in resonance when driven by a sinusoidal voltage at its resonant frequency, $\omega_0 = (LC)^{-1/2}$. But if the same circuit is driven by a *square-wave voltage* (which is alternately on and off for equal time intervals), it will exhibit resonance at its resonant frequency and at $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$, of this frequency. Explain why.

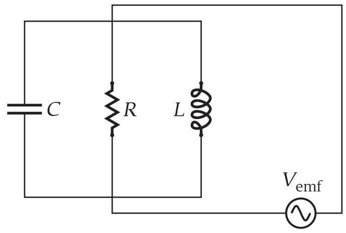
10.23 Is it possible for the voltage amplitude across the inductor in a series RLC circuit to exceed the voltage amplitude of the voltage supply? Why or why not?

10.24 Why can't a transformer be used to step up or step down the voltage in a DC circuit?

10.25 The figure shows a circuit with a source of constant emf connected in series to a resistor, an inductor, and a capacitor. What is the steady-state current flow through the circuit?



•**10.26** An RLC circuit has a capacitor, a resistor, and an inductor connected in parallel, as shown in the figure, and a source of time-varying emf providing V_{rms} at a frequency f. Find an expression for I_{rms} in terms of V_{rms}, f, L, C, and R.

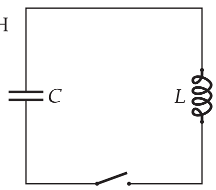


EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

Sections 10.1 and 10.2

10.27 For the LC circuit in the figure, $L = 32.0$ mH and $C = 45.0$ μF. The capacitor is charged to $q_0 = 10.0$ μC, and at $t = 0$, the switch is closed. At what time is the energy stored in the capacitor first equal to the energy stored in the inductor?



10.28 A 2.00-μF capacitor is fully charged by being connected to a 12.0 V battery. The fully charged capacitor is then connected to a 0.250-H inductor. Calculate (a) the maximum current in the inductor and (b) the frequency of oscillation of the LC circuit.

10.29 An LC circuit consists of a 1.00 mH inductor and a fully charged capacitor. After 2.10 ms, the energy stored in the capacitor is half of its original value. What is the capacitance?

10.30 The time-varying current in an LC circuit where $C = 10.0$ μF is given by $i(t) = (1.00\text{A})\sin(1200.t)$, where t is in seconds.

a) At what time after $t = 0$ does the current reach its maximum value?

b) What is the total energy of the circuit?

c) What is the inductance, L?

•**10.31** A 10.0 μF capacitor is fully charged by a 12.0 V battery and is then disconnected from the battery and allowed to discharge through a 0.200 H inductor. Find the first three times when the charge on the capacitor is 80.0 μC, taking $t = 0$ as the instant when the capacitor is connected to the inductor.

•**10.32** A 4.00 mF capacitor is connected in series with a 7.00 mH inductor. The peak current in the wires between the capacitor and the inductor is 3.00 A.

a) What is the total electric energy in this circuit?

b) Write an expression for the charge on the capacitor as a function of time, assuming the capacitor is fully charged at $t = 0$ s.

Section 10.3

10.33 A circuit contains a 4.50 nF capacitor and a 4.00-mH inductor. If some charge is placed initially on the capacitor, an oscillating current with angular frequency ω_0 is produced. By what factor does this angular frequency change if a 1.00 kΩ resistor is connected in series with the capacitor and the inductor?

•**10.34** An RLC oscillator circuit contains a 50.0 Ω resistor and a 1.00-mH inductor. What capacitance is necessary for the time constant of the circuit (the $1/e$ value) to be equal to the oscillation period? Plot the voltage across the resistor as a function of time.

•**10.35** A 2.00 μF capacitor was fully charged by being connected to a 12.0-V battery. The fully charged capacitor is then connected in series with a resistor and an inductor: $R = 50.0$ Ω and $L = 0.200$ H. Calculate the damped frequency of the resulting circuit.

•**10.36** An LC circuit consists of a capacitor, $C = 2.50$ μF, and an inductor, $L = 4.00$ mH. The capacitor is fully charged using a battery and then connected to the inductor. An oscilloscope is used to measure the frequency of the oscillations in the circuit. Next, the circuit is opened, and a resistor, R, is inserted in series with the inductor and the capacitor. The capacitor is again fully charged using the same battery and then connected to the circuit. The angular frequency of the damped oscillations in the

RLC circuit is found to be 20.0% less than the angular frequency of the oscillations in the LC circuit.

a) Determine the resistance of the resistor.

b) How long after the capacitor is reconnected in the circuit will the amplitude of the damped current through the circuit be 50.0% of the initial amplitude?

c) How many complete damped oscillations will have occurred in that time?

Section 10.4

10.37 At what frequency will a 10.0 μF capacitor have the reactance $X_C = 200.\ \Omega$?

10.38 A capacitor with capacitance $C = 5.00 \times 10^{-6}$ F is connected to an AC power source having a peak value of 10.0 V and $f = 100.$ Hz. Find the reactance of the capacitor and the maximum current in the circuit.

Section 10.5

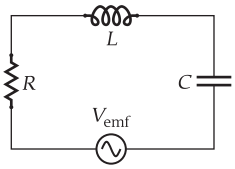
10.39 A series circuit contains a 100.0 Ω resistor, a 0.500 H inductor, a 0.400 μF capacitor, and a source of time-varying emf providing 40.0 V.

a) What is the resonant angular frequency of the circuit?

b) What current will flow through the circuit at the resonant frequency?

10.40 A variable capacitor used in an RLC circuit produces a resonant frequency of 5.0 MHz when its capacitance is set to 15 pF. What will the resonant frequency be when the capacitance is increased to 380 pF?

10.41 Determine the phase constant and the impedance of the RLC circuit shown in the figure when the frequency of the time-varying emf is 1.00 kHz, $C = 100.$ μF, $L = 10.0$ mH, and $R = 100.\ \Omega$.



10.42 What is the resonant frequency of the series RLC circuit of Problem 10.41 if $C = 4.00$ μF, $L = 5.00$ mH, and $R = 1.00$ kΩ? What is the maximum current in the circuit if $V_m = 10.0$ V at the resonant frequency?

•**10.43** In a series RLC circuit, $V = (12.0\text{ V})(\sin \omega t)$, $R = 10.0\ \Omega$, $L = 2.00$ H, and $C = 10.0$ μF. At resonance, determine the amplitude of the voltage across the inductor. Is the result reasonable, considering that the voltage supplied to the entire circuit has an amplitude of 12.0 V?

•**10.44** An AC power source with $V_m = 220.$ V and $f = 60.0$ Hz is connected in a series RLC circuit. The resistance, R, inductance, L, and capacitance, C, of this circuit are, respectively, 50.0 Ω, 0.200 H, and 0.0400 mF. Find each of the following quantities:

a) the inductive reactance

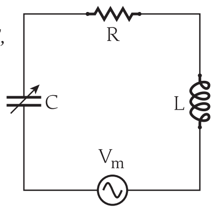
b) the capacitive reactance

c) the impedance of the circuit

d) the maximum current through the circuit at this frequency

e) the maximum potential difference across each circuit element

•**10.45** The series RLC circuit shown in the figure has $R = 2.20\ \Omega$, $L = 9.10$ mH, $C = 2.27$ mF, $V_m = 110.$ V, and $\omega = 377$ rad/s.



a) What is the maximum current, I_m, in this circuit?

b) What is the phase constant, ϕ, between the voltage and the current?

c) The capacitance, C, can be varied. What value of C will allow the largest current amplitude oscillations to occur, and what are the magnitudes of this current, I'_m, and the phase angle, ϕ', between the current and the voltage?

•**10.46** Design an RC high-pass filter that passes a signal with frequency 5.00 kHz, has a ratio $V_{out}/V_{in} = 0.500$, and has an impedance of 1.00 kΩ at very high frequencies.

a) What components will you use?

b) What is the phase of V_{out} relative to V_{in} at the frequency of 5.00 kHz?

••**10.47** Design an RC high-pass filter that rejects 60.0-Hz line noise from a circuit used in a detector. Your criteria are reduction of the amplitude of the line noise by a factor of 1000. and total impedance at high frequencies of 2.00 kΩ.

a) What components will you use?

b) What is the frequency range of the signals that will be passed with at least 90.0% of their amplitude?

Section 10.6

10.48 What is the maximum value of the AC voltage whose root-mean-square value is (a) 110 V or (b) 220 V?

10.49 The quality factor, Q, of a circuit can be defined by $Q = \omega_0(U_E + U_B)/P$. Express the quality factor of a series RLC circuit in terms of its resistance R, inductance L, and capacitance C.

10.50 A label on a hair dryer reads "110V 1250W." What is the peak current in the hair dryer, assuming that it behaves like a resistor?

10.51 A radio tuner has a resistance of 1.00 μΩ, a capacitance of 25.0 nF, and an inductance of 3.00 mH.

a) Find the resonant frequency of this tuner.

b) Calculate the power in the circuit if a signal at the resonant frequency produces an emf across the antenna of $V_{rms} = 1.50$ mV.

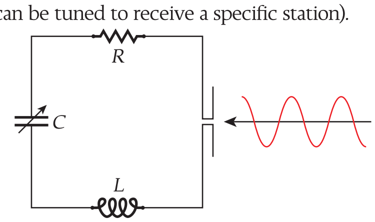
•**10.52** A circuit contains a 100 Ω resistor, a 0.0500 H inductor, a 0.400-μF capacitor, and a source of time-varying emf connected in series. The time-varying emf corresponds to $V_{rms} = 50.0$ V at a frequency of 2000. Hz.

a) Determine the current in the circuit.

b) Determine the voltage drop across each component of the circuit.

c) How much power is drawn from the source of emf?

•**10.53** The figure shows a simple FM antenna circuit in which $L = 8.22$ μH and C is variable (the capacitor can be tuned to receive a specific station). The radio signal from your favorite FM station produces a sinusoidal time-varying emf with an amplitude of 12.9 μV and a frequency of 88.7 MHz in the antenna.



a) To what value, C_0, should you tune the capacitor in order to best receive this station?

b) Another radio station's signal produces a sinusoidal time-varying emf with the same amplitude, 12.9 μV, but with a frequency of 88.5 MHz in the antenna. With the circuit tuned to optimize reception at 88.7 MHz, what should the value, R_0, of the resistance be in order to reduce by a factor of 2 (compared to the current if the circuit were optimized for 88.5 MHz) the current produced by the signal from this station?

Section 10.7

10.54 The transmission of electric power occurs at the highest possible voltage to reduce losses. By how much could the power loss be reduced by raising the voltage by a factor of 10.0?

10.55 Treat the solenoid and coil of Solved Problem 9.2 as a transformer.

a) Find the root-mean-square voltage in the coil if the solenoid has a root-mean-square voltage of 120 V and a frequency of 60. Hz. The length of the solenoid is 12.0 cm.

b) What is the voltage in the coil if the frequency is 0 Hz (DC current)?

10.56 A transformer has 800 turns in the primary coil and 40 turns in the secondary coil.

a) What happens if an AC voltage of 100. V is across the primary coil?

b) If the initial AC current is 5.00 A, what is the output current?

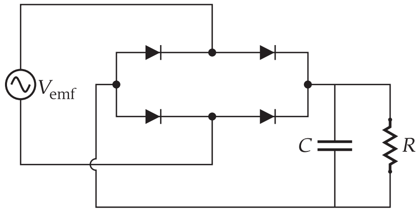
c) What happens if a DC current at 100. V flows into the primary coil?

d) If the initial DC current is 5.00 A, what is the output current?

10.57 A transformer contains a primary coil with 200 turns and a secondary coil with 120 turns. The secondary coil drives a current I through a 1.00 kΩ resistor. If an input voltage of $V_{rms} = 75.0$ V is applied across the primary coil, what is the power dissipated in the resistor?

Section 10.8

10.58 Consider the filtered fullwave rectifier shown in the figure. If the frequency of the source of time-varying emf is 60. Hz, what is the frequency of the resulting current?



•**10.59** A voltage $V_{rms} = 110$ V at a frequency of 60. Hz is applied to the primary coil of a transformer. The transformer has a ratio $N_P/N_S = 11$. The secondary coil is used as the source of V_{emf} for the filtered fullwave rectifier of Problem 10.58.

a) What is the maximum voltage in the secondary coil of the transformer?

b) What is the DC voltage provided to the resistor?

Additional Exercises

10.60 A vacuum cleaner motor can be viewed as an inductor with an inductance of 100. mH. For a 60.0 Hz AC voltage of $V_{rms} = 115.$ V, what capacitance must be in series with the motor to maximize the power output of the vacuum cleaner?

10.61 When you turn the dial on a radio to tune it, you are adjusting a variable capacitor in an LC circuit. Suppose you tune to an AM station broadcasting at a frequency of 1000. kHz, and there is a 10.0 mH inductor in the tuning circuit. When you have tuned in the station, what is the capacitance of the capacitor?

10.62 A series RLC circuit has a source of time-varying emf providing 12.0 V at a frequency f_0, with $L = 7.00$ mH, $R = 100.\ \Omega$, and $C = 0.0500$ mF.

a) What is the resonant frequency of this circuit?

b) What is the average power dissipated in the resistor at this resonant frequency?

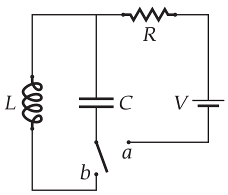
10.63 What are the maximum values of (a) current and (b) voltage when an incandescent 60 W light bulb (at 110 V) is connected to a wall plug labeled 110 V?

10.64 A 360 Hz source of emf is connected in a circuit consisting of a capacitor, a 25 mH inductor, and an 0.80-Ω resistor. For the current and the voltage to be in phase, what should the value of C be?

10.65 What is the resistance in an RLC circuit with $L = 65.0$ mH and $C = 1.00\ \mu$F if the circuit loses 3.50% of its total energy as thermal energy in each cycle?

10.66 A transformer with 400. turns in its primary coil and 20. turns in its secondary coil is designed to deliver an average power of 1200. W with a maximum voltage of 60.0 V. What is the maximum current in the primary coil?

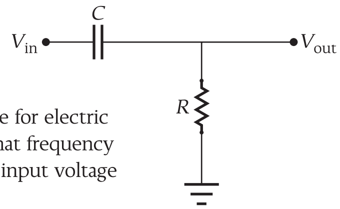
10.67 A 5.00 μF capacitor in series with a 4.00 Ω resistor is charged with a 9.00 V battery for a long time by closing the switch (position *a* in the figure). The capacitor is then discharged through an inductor ($L = 40.0$ mH) by closing the switch (position *b*) at $t = 0$.



a) Determine the maximum current through the inductor.

b) What is the first time at which the current is at its maximum?

•**10.68** In the RC high-pass filter shown in the figure, $R = 10.0$ kΩ and $C = 0.0470$ μF. What is the 3.00-dB frequency of this circuit (where dB means basically the same for electric current as for sound)? That is, at what frequency does the ratio of output voltage to input voltage satisfy $20\log(V_{out}/V_{in}) = -3.00$?



•**10.69** The discussion of RL, RC, and RLC circuits in this chapter has assumed a purely resistive resistor, one whose inductance and capacitance are exactly zero. While the capacitance of a resistor can generally be neglected, inductance is an intrinsic part of the resistor. Indeed, one of the most widely used resistors, the wire-wound resistor, is nothing but a solenoid made of highly resistive wire. Suppose a wire-wound resistor of unknown resistance is connected to a DC power supply. At a voltage of $V = 10.0$ V across the resistor, the current through the resistor is 1.00 A. Next, the same resistor is connected to an AC power source providing $V_{rms} = 10.0$ V at a variable frequency. When the frequency is 20.0 kHz, a current, $I_{rms} = 0.800$ A, is measured through the resistor.

a) Calculate the resistance of the resistor.

b) Calculate the inductive reactance of the resistor.

c) Calculate the inductance of the resistor.

d) Calculate the frequency of the AC power source at which the inductive reactance of the resistor exceeds its resistance.

•**10.70** In a certain RLC circuit, a 20.0 Ω resistor, a 10.0-mH inductor, and a 5.00 μF capacitor are connected in series with an AC power source for which $V_{rms} = 10.0$ V and $f = 100.$ Hz. Calculate

a) the amplitude of the current,

b) the phase between the current and the voltage, and

c) the maximum voltage across each component.

•**10.71** a) A loop of wire 5.00 cm in diameter is carrying a current of 2.00 A. What is the energy density of the magnetic field at its center?

b) What current has to flow in a straight wire to produce the same energy density at a point 4.00 cm from the wire?

•**10.72** A 75,000 W light bulb (yes, there are such things!) operates at $I_{rms} = 200.$ A and $V_{rms} = 440.$ V in a 60.0 Hz AC circuit. Find the resistance, R, and self-inductance, L, of this bulb. Its capacitive reactance is negligible.

•**10.73** Show that the power dissipated in a resistor connected to an AC power source with a frequency ω oscillates with a frequency 2ω.

•**10.74** A 300 Ω resistor is connected in series with a 4.00 μF capacitor and a source of time-varying emf providing $V_{rms} = 40.0$ V.

a) At what frequency will the potential drop across the capacitor equal that across the resistor?

b) What is the rms current through the circuit when this occurs?

•**10.75** An electromagnet consists of 200 loops and has a length of 10.0 cm and a cross-sectional area of 5.00 cm^2. Find the resonant frequency of this electromagnet when it is attached to the Earth (treat the Earth as a spherical capacitor).

•**10.76** Laboratory experiments with series RLC circuits require some care, as these circuits can produce large voltages at resonance. Suppose you have a 1.00 H inductor (not difficult to obtain) and a variety of resistors and capacitors. Design a series RLC circuit that will resonate at a frequency (not an angular frequency) of 60.0 Hz and will produce at resonance a magnification of the voltage across the capacitor or the inductor by a factor of 20.0 times the input voltage or the voltage across the resistor.

•**10.77** A particular RC low-pass filter has a breakpoint frequency of 200. Hz. At what frequency will the output voltage divided by the input voltage be 0.100?

MULTI-VERSION EXERCISES

10.78 An inductor with inductance $L = 42.1$ mH is connected to an AC power source that supplies $V_{\text{emf}} = 19.1$ V at $f = 605$ Hz. Find the reactance of the inductor.

10.79 An inductor with inductance $L = 52.5$ mH is connected to an AC power source that supplies $V_{\text{emf}} = 19.9$ V at $f = 669$ Hz. Find the maximum current in the circuit.

10.80 An inductor with inductance L is connected to an AC power source that supplies $V_{\text{emf}} = 20.7$ V at $f = 733$ Hz. If the reactance of the inductor is to be 81.52 Ω, what should the value of L be?

10.81 An inductor with inductance L is connected to an AC power source that supplies $V_{\text{emf}} = 21.5$ V at $f = 797$ Hz. If the maximum current in the circuit is to be 0.1528 A, what should the value of L be?

11 Electromagnetic Waves

FIGURE 11.1 The solar power plant Solúcar PS10, located near Seville, Spain, concentrates the power from the Sun's electromagnetic radiation to produce electricity.

In the last few chapters we have studied electric and magnetic fields and seen how they can change over time. The present chapter concludes our study of electromagnetism and focuses on how the interaction between electric and magnetic fields gives rise to electromagnetic waves.

Literally every place on Earth is bathed in electromagnetic waves, of which the most obvious type is probably visible light. Other types of electromagnetic waves that are familiar to you range from TV and radio waves to microwaves and X-rays. Our cell phones, wireless Internet connections, and the GPS system all utilize electromagnetic waves, as do laser pointers. In this chapter, we'll examine the nature of electromagnetic waves, including their similarities and differences and their transmission of energy and pressure to other objects. Solar power plants, like the one shown in Figure 11.1, are impressive demonstrations of how the energy contained in electromagnetic waves emitted by the Sun can be harnessed.

Most of the results in this chapter apply only to electromagnetic waves propagating through vacuum. For practical purposes, however, electromagnetic waves propagating through Earth's atmosphere can be treated in the same way. There are some significant differences for electromagnetic waves propagating through other media, but these are not addressed in this chapter.

The next chapter is the first to focus on *optics*–the properties and behavior of light–and many of those properties apply to other electromagnetic waves as well.

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WHAT WE WILL LEARN

- Changing electric fields induce magnetic fields, and changing magnetic fields induce electric fields.
- Maxwell's equations describe electromagnetic phenomena.
- Electromagnetic waves have both electric and magnetic fields.
- Solutions of Maxwell's equations can be expressed in terms of sinusoidally varying traveling waves.
- For an electromagnetic wave, the electric field is perpendicular to the magnetic field and both fields are perpendicular to the direction in which the wave is traveling.
- The speed of light can be expressed in terms of constants related to electric and magnetic fields.
- Light is an electromagnetic wave.
- Electromagnetic waves can transport energy and momentum.
- The intensity of an electromagnetic wave is proportional to the square of the root-mean-square magnitude of the electric field of the wave.
- The direction of the electric field of a traveling electromagnetic wave is called the *polarization direction*.

11.1 Maxwell's Law of Induction for Induced Magnetic Fields

In Chapter 9, we saw that a changing magnetic field induces an electric field. According to Faraday's Law of Induction,

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}, \tag{11.1}$$

where $\vec{E}$ is the electric field induced around a closed loop by the changing magnetic flux, Φ_B, through that loop. In a similar way, a changing electric field induces a magnetic field. **Maxwell's Law of Induction** (named for British physicist James Clerk Maxwell, 1831–1879) describes this phenomenon as follows:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}, \tag{11.2}$$

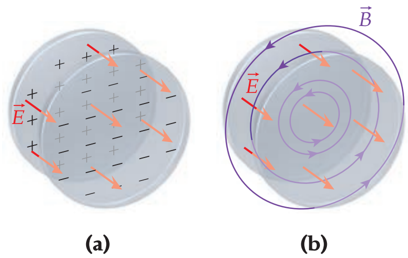


FIGURE 11.2 (a) A charged circular capacitor. The red arrows represent the electric field between the plates. (b) A capacitor with charge increasing with time. The red arrows represent the electric field, and the purple loops represent the induced magnetic field.

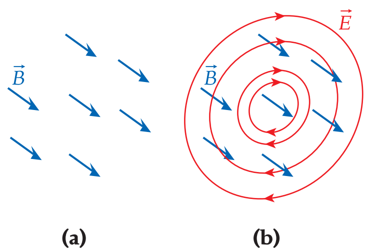


FIGURE 11.3 (a) A constant uniform magnetic field. (b) A uniform magnetic field increasing with time, which induces an electric field represented by the red loops.

where $\vec{B}$ is the magnetic field induced around a closed loop by a changing electric flux, Φ_E, through that loop. This equation is similar to equation 11.1 except for the constant $\mu_0\epsilon_0$ and the lack of a negative sign. The constant is a consequence of the SI units used for magnetic fields. The fact that the right-hand side of equation 11.2 does not have a negative sign implies that the induced magnetic field has the opposite sign from that of an induced electric field when both are induced under similar conditions, as we'll see shortly.

First, note that it is *not at all obvious* that equation 11.2 can be written in analogy to equation 11.1. When Maxwell first wrote this equation, it represented a major step forward in the unification of electricity and magnetism. Faraday discovered his law in 1831, but it took a quarter of a century for Maxwell to come up with the counterpart. What is stated here as simple fact is in reality the first great conceptual leap toward the unification of all physical forces of nature. This unification began with Maxwell's work one and a half centuries ago and continues in modern physics research today.

A circular capacitor can be used to illustrate an induced magnetic field (Figure 11.2). For the capacitor shown in Figure 11.2a, the charge is constant, and a constant electric field appears between the plates. There is no magnetic field. For the capacitor shown in Figure 11.2b, the charge is increasing with time. Thus, the electric flux between the plates is increasing with time. A magnetic field, $\vec{B}$, is induced, represented by the purple loops, which also indicate the direction of $\vec{B}$. Along each loop, the magnetic field vector has the same magnitude and is directed tangentially to the loop. When the charge stops increasing, the electric flux remains constant and the magnetic field disappears.

Next, consider a uniform magnetic field that is also constant in time, as in Figure 11.3a. In Figure 11.3b, the magnetic field is still uniform in space but is increasing with time, which induces an electric field, shown by the red loops.

The electric field vector has constant magnitude along each loop and is directed tangentially to the loops as shown. Note that this induced electric field points in the opposite direction from the induced magnetic field caused by an increasing electric field (Figure 11.2b).

Now recall Ampere's Law:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}, \tag{11.3}$$

which relates the integral around a loop of the dot product of the magnetic field and the differential displacement along the loop, $\vec{B} \cdot d\vec{s}$, to the current flowing through the loop. However, Maxwell realized that this equation is incomplete because it does not account for contributions to the magnetic field caused by changing electric fields. Equations 11.2 and 11.3 can be combined to produce a description of magnetic fields created by moving charges and by changing electric fields:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i_{\text{enc}}. \tag{11.4}$$

Equation 11.4 is called the **Maxwell-Ampere Law.** You can see that for the case of constant current, such as current flowing in a conductor, this equation reduces to Ampere's Law. For the case of a changing electric field without current flowing, such as the electric field between the plates of a capacitor, this equation reduces to Maxwell's Law of Induction. It is important to realize that the Maxwell-Ampere Law describes two different sources of magnetic field: the conventional current (as discussed in Chapter 8) and the time-varying electric flux (examined in more detail in the following subsection).

Displacement Current

Looking at the Maxwell-Ampere Law (equation 11.4), you can see that the expression $\varepsilon_0 d\Phi_E/dt$ on the right-hand side of the equation must have the units of current. Although no actual "current" is "displaced," this term is called the **displacement current,** i_d:

$$i_d = \varepsilon_0 \frac{d\Phi_E}{dt}. \tag{11.5}$$

With this definition, we can rewrite equation 11.4 as

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 (i_d + i_{\text{enc}}).$$

Again, let's consider a parallel plate capacitor with circular plates, now placed in a circuit in which a current, *i*, is flowing while the capacitor is charging (Figure 11.4). For a parallel plate capacitor, the charge, *q*, is related to the electric field between the plates, *E*, as follows (see Chapter 4)

$$q = \varepsilon_0 A E$$

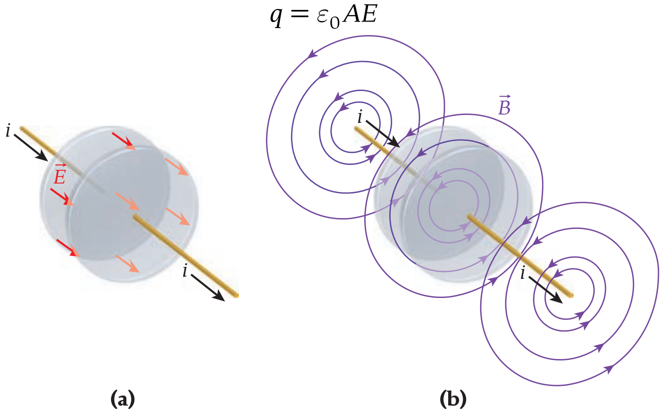
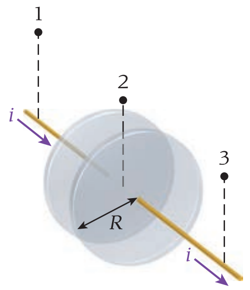


FIGURE 11.4 A parallel plate capacitor in a circuit being charged by a current, *i*: (a) the electric field between the plates at a given instant; (b) the magnetic field around the wires and between the plates of the capacitor.

Concept Check 11.1

The displacement current, i_d, for the charging circular capacitor with radius R shown in the figure is equal to the conduction current, i, in the wires. Points 1 and 3 are located a perpendicular distance r from the wires, and point 2 is located the same perpendicular distance r from the center of the capacitor, such that $r > R$.
Rank the magnetic fields at points 1, 2, and 3, from largest magnitude to smallest.



a) $B_1 > B_2 > B_3$

b) $B_3 > B_2 > B_1$

c) $B_1 = B_3 > B_2$

d) $B_2 > B_1 = B_3$

e) $B_1 = B_2 = B_3$

where A is the area of the plates. The current, i, in the circuit can be obtained by taking the time derivative of this equation:

$$i = \frac{dq}{dt} = \varepsilon_0 A \frac{dE}{dt}. \tag{11.6}$$

Assuming that the electric field between the plates of the capacitor is uniform, we can obtain an expression for the displacement current:

$$i_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d(AE)}{dt} = \varepsilon_0 A \frac{dE}{dt}. \tag{11.7}$$

Thus, the current in the circuit, i, given by equation 11.6, is equal to the displacement current, i_d, given by equation 11.7. Although no actual current is flowing between the plates of the capacitor, in the sense that no actual charges move across the capacitor gap from one plate to the other, the displacement current can be used to calculate the induced magnetic field.

To calculate the magnetic field between the two circular plates of the capacitor, we assume that the volume between the two plates can be replaced with a conductor of radius R carrying current i_d. In Chapter 8, we saw that the magnetic field at a perpendicular distance r from the center of the capacitor is given by

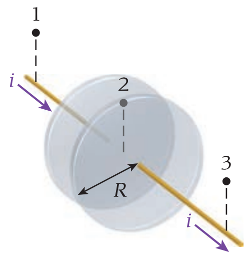
$$B = \left(\frac{\mu_0 i_d}{2\pi R^2}\right) r \quad (\text{for } r < R).$$

The system outside the capacitor can be treated as a current-carrying wire; so the magnetic field at a perpendicular distance r from the wire is

$$B = \frac{\mu_0 i_d}{2\pi r} \quad (\text{for } r > R).$$

Concept Check 11.2

The displacement current, i_d, for the charging circular capacitor with radius R shown in the figure is equal to the conduction current, i, in the wires. Points 1 and 3 are located a perpendicular distance r from the wires, and point 2 is located the same perpendicular distance r from the center of the capacitor, such that $r < R$. Rank the magnetic fields at points 1, 2, and 3, from largest magnitude to smallest.



a) $B_1 > B_2 > B_3$

b) $B_3 > B_2 > B_1$

c) $B_1 = B_3 > B_2$

d) $B_2 > B_1 = B_3$

e) $B_1 = B_2 = B_3$

Maxwell's Equations

The Maxwell-Ampere Law (equation 11.4) completes the set of four equations known as **Maxwell's equations,** which describe the interactions between electrical charges, currents, electric fields, and magnetic fields. These equations treat electricity and magnetism as two aspects of a unified force called **electromagnetism.** All of the results described previously for electricity and magnetism are still valid, but these equations show how electric and magnetic fields interact with each other, giving rise to a broad range of electromagnetic phenomena. This chapter focuses on electromagnetic waves. A summary of Maxwell's equations is given in Table 11.1. (Again, as a reminder, $\oint\!\!\!\oint d\vec{A}$ in the first two equations represents integration over a closed surface, and $\oint d\vec{s}$ in the last two equations indicates integration over a closed curve.)

If you scrutinize Maxwell's equations, you might notice a lack of symmetry between $\vec{E}$ and $\vec{B}$. This difference arises from the fact that electric charges exist in isolation and a corresponding current appears when charges move, but apparently no isolated, stationary magnetic charges occur in nature. Particles that hypothetically have a single magnetic charge (a north pole or a south pole, but not both) are called *magnetic monopoles,* but empirically

Table 11.1 Maxwell's Equations Describing Electromagnetic Phenomena

Name	Equation	Description
Gauss's Law for Electric Fields	$\oiint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$	The net electric flux through a closed surface is proportional to the net enclosed electric charge.
Gauss's Law for Magnetic Fields	$\oiint \vec{B} \cdot d\vec{A} = 0$	The net magnetic flux through a closed surface is zero (no magnetic monopoles exist).
Faraday's Law of Induction	$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$	An electric field is induced by a changing magnetic flux.
Maxwell-Ampere Law	$\oint \vec{B} \cdot d\vec{s} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i_{enc}$	A magnetic field is induced by a changing electric flux or by a current.

it has been found that magnetic poles always come in pairs, a north pole together with a south pole. There is no fundamental reason for the absence of magnetic monopoles, and many experiments have searched unsuccessfully for them. The most sensitive of these experiments was MACRO, which used a massive detector that operated for many years in a laboratory located deep under the Gran Sasso mountain in Italy. MACRO searched for magnetic monopoles in cosmic rays, without success.

We'll now begin our study of **electromagnetic waves.** Electromagnetic waves consist of electric and magnetic fields, can travel through vacuum without any supporting medium, and do not involve moving charges or currents. The existence of electromagnetic waves was first demonstrated in 1888 by the German physicist Heinrich Hertz (1857–1894). Hertz used an RLC circuit that induced a current in an inductor that drove a spark gap. A spark gap consists of two electrodes that, when a potential difference is applied across them, produce a spark by exciting the gas between the electrodes. Hertz placed a loop and a small spark gap several meters apart. He observed that sparks were induced in the remote loop in a pattern that correlated with the electromagnetic oscillations in the primary RLC circuit. Thus, electromagnetic waves were able to travel through space without any medium to support them. For this contribution and others, the basic unit of oscillation, cycles per second, was named the hertz (Hz) in his honor.

11.2 Wave Solutions to Maxwell's Equations

As Section 11.7 will show, it is possible to use advanced calculus to derive a general wave equation from Maxwell's equations starting with those equations in differential form. However, we'll first assume that electromagnetic waves propagating in vacuum (no moving charges or currents) have the form of a traveling wave and show that this form satisfies Maxwell's equations.

Proposed Solution

We assume the following equations express the electric and magnetic fields in a particular electromagnetic wave that happens to be traveling in the positive x-direction:

$$\vec{E}(\vec{r},t) = E_{max} \sin(\kappa x - \omega t)\hat{y}$$

and

$$\vec{B}(\vec{r},t) = B_{max} \sin(\kappa x - \omega t)\hat{z}, \tag{11.8}$$

where $\kappa = 2\pi/\lambda$ is the wave number and $\omega = 2\pi f$ is the angular frequency of a wave with wavelength λ and frequency f. Note that the magnitudes of both fields have no dependence on the y- or z-coordinates, only on the x-coordinate and time. This type of wave, in which the electric and magnetic field vectors lie in a plane, is called a **plane wave.**

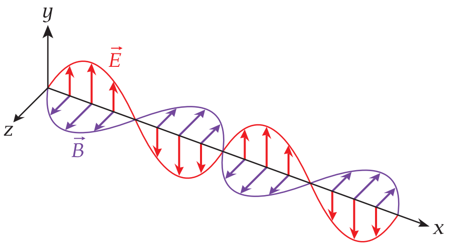


FIGURE 11.5 Representation of an electromagnetic wave traveling in the positive x-direction at a given instant.

Equation 11.8 indicates that this particular electromagnetic wave is traveling in the positive x-direction because, as the time t increases, the coordinate x has to increase to maintain the same value for the fields. The wave described by equation 11.8 is shown in Figure 11.5.

In the particular case illustrated in Figure 11.5, the electric field is completely in the y-direction and the magnetic field is completely in the z-direction; that is, both fields are perpendicular to the direction of wave propagation. It turns out that the electric field is always perpendicular to the direction the wave is traveling and is always perpendicular to the magnetic field. However, in general, for an electromagnetic wave that is propagating along the x-axis, the electric field can point anywhere in the yz-plane.

The representation of the wave in Figure 11.5 is an instantaneous abstraction. The vectors shown represent the magnitude and the direction for the electric and magnetic fields; however, you should realize that these fields are not solid objects. Nothing made of matter actually moves left and right or up and down as the wave travels. The vectors pointing left and right and up and down represent the electric and magnetic fields.

Showing that the traveling wave described by equation 11.8 satisfies all of Maxwell's equations involves quite a bit of vector calculus but also uses many of the concepts that have been developed in the preceding chapters. The following subsections work through this process in detail, one equation at a time.

Concept Check 11.3

An electromagnetic plane wave is traveling through vacuum. The electric field of the wave is given by $\vec{E} = E_{\text{max}} \cos(\kappa x - \omega t)\,\hat{y}$. Which of the following equations describes the magnetic field of the wave?

a) $\vec{B} = B_{\text{max}} \cos(\kappa x - \omega t)\,\hat{x}$

b) $\vec{B} = B_{\text{max}} \cos(\kappa y - \omega t)\,\hat{y}$

c) $\vec{B} = B_{\text{max}} \cos(\kappa z - \omega t)\,\hat{z}$

d) $\vec{B} = B_{\text{max}} \cos(\kappa y - \omega t)\,\hat{z}$

e) $\vec{B} = B_{\text{max}} \cos(\kappa x - \omega t)\,\hat{z}$

Gauss's Law for Electric Fields

Let's start with Gauss's Law for Electric Fields. For an electromagnetic wave in vacuum, there is no enclosed charge anywhere ($q_{\text{enc}} = 0$); thus, we must show that the proposed solution of equation 11.8 satisfies

$$\oiint \vec{E} \cdot d\vec{A} = 0. \qquad (11.9)$$

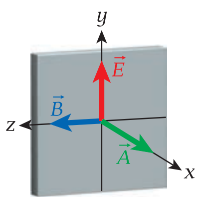


FIGURE 11.6 Gaussian surface (gray box) around a portion of the vector representation of an electromagnetic wave traveling in the positive x-direction. The area vector is shown for the front face of the Gaussian surface.

We choose a rectangular box as a Gaussian surface enclosing a portion of the vector representation of the wave (Figure 11.6). For the faces of the box in the yz-plane, $\vec{E} \cdot d\vec{A}$ is zero because the vectors $\vec{E}$ and $d\vec{A}$ are perpendicular to each other. The same is true for the faces in the xy-plane. The faces in the xz-plane contribute $+EA_1$ and $-EA_1$, where A_1 is the area of the top face and the bottom face. Thus, the integral is zero, and Gauss's Law for Electric Fields is satisfied.

If we analyzed the vector representation at different times, we would get a different electric field. However, because the electric field is always in the y-direction, the integral will always be zero.

Gauss's Law for Magnetic Fields

For Gauss's Law for Magnetic Fields, we must show that

$$\oiint \vec{B} \cdot d\vec{A} = 0. \qquad (11.10)$$

We again use the closed surface in Figure 11.6 for the integration. For the faces in the yz-plane and for those in the xz-plane, $\vec{B} \cdot d\vec{A}$ is zero because the vectors $\vec{B}$ and $d\vec{A}$ are perpendicular to each other. The faces in the xy-plane contribute $+BA_2$ and $-BA_2$, where A_2 is the area of each of the two faces in the xy-plane. Thus, the integral is zero, and Gauss's Law for Magnetic Fields is satisfied.

Faraday's Law of Induction

Now let's address Faraday's Law of Induction:

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}. \qquad (11.11)$$

To evaluate the integral on the left-hand side of this equation, we assume a closed loop in the xy-plane that has width dx and height h and goes from a to b to c to d and back to a, as depicted by the gray rectangle in Figure 11.7. The differential area $d\vec{A} = \hat{n}dA = \hat{n}hdx$

of this rectangle has its unit surface normal vector, $\hat{n}$, pointing in the positive z-direction. Note that the electric and magnetic fields change with distance along the x-axis. Thus, going from point x to point $x + dx$, the electric field changes from $\vec{E}(x)$ to $\vec{E}(x + dx) = \vec{E}(x) + d\vec{E}$.

To evaluate the integral of equation 11.11 over the closed loop, we split the loop into four pieces, integrating counterclockwise from a to b, b to c, c to d, and d to a. The contributions to the integral that are parallel to the x-axis, from integrating from b to c and from d to a, are zero because the electric field is always perpendicular to the direction of integration. For the integrations in the y-direction, a to b and c to d, the electric field is parallel or antiparallel to the direction of integration; therefore, the scalar product reduces to a conventional product. Because the electric field is independent of the y-coordinate, it can be taken out of the integral. Thus, the integral along each of the segments in the y-direction is a simple product of the integrand (the magnitude of the electric field at the corresponding x-coordinate) and the length of the integration interval (h), times -1 for the integration in the negative y-direction because $\vec{E}$ is antiparallel to the direction of integration. Thus, the integral of equation 11.11 evaluates to

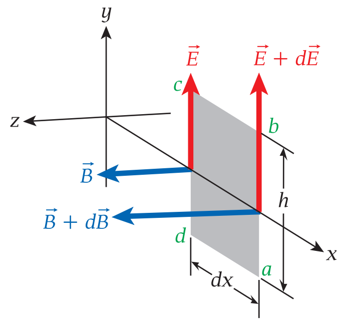


FIGURE 11.7 Two snapshots of the electric and magnetic fields in an electromagnetic wave. The gray area represents an integration loop for Faraday's Law.

$$\oint \vec{E} \cdot d\vec{s} = E\int_a^b ds - E\int_c^d ds = (E + dE)(h) - Eh = (dE)(h).$$

The right-hand side of equation 11.11 is given by

$$-\frac{d\Phi_B}{dt} = -A\frac{dB}{dt} = -(h)(dx)\frac{dB}{dt},$$

where dt is the time during which the wave travels a distance dx. Thus, we have

$$(h)(dE) = -(h)(dx)\frac{dB}{dt},$$

or

$$\frac{dE}{dx} = -\frac{dB}{dt}. \tag{11.12}$$

The derivatives dE/dx and dB/dt are each taken with respect to a single variable, although both E and B depend on both x and t. Thus, we can more appropriately write equation 11.12 using partial derivatives:

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}. \tag{11.13}$$

Using the assumed forms for the electric and magnetic fields (equation 11.8), we can expand the partial derivatives:

$$\frac{\partial E}{\partial x} = \frac{\partial}{\partial x}\left(E_{\max}\sin(\kappa x - \omega t)\right) = \kappa E_{\max}\cos(\kappa x - \omega t),$$

and

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial t}\left(B_{\max}\sin(\kappa x - \omega t)\right) = -\omega B_{\max}\cos(\kappa x - \omega t).$$

Substituting these expressions into equation 11.13 gives

$$\kappa E_{\max}\cos(\kappa x - \omega t) = -\left[-\omega B_{\max}\cos(\kappa x - \omega t)\right].$$

The angular frequency and wave number are related via

$$\frac{\omega}{\kappa} = \frac{2\pi f}{(2\pi/\lambda)} = f\lambda = c, \tag{11.14}$$

where c is the speed of the wave. (In general, we could use v for the speed of this wave. However, we choose to use c, because, as we'll see, all electromagnetic waves propagate in vacuum with a characteristic speed, the speed of light, which is conventionally represented by c.) Thus, we have

$$\frac{E_{\max}}{B_{\max}} = \frac{\omega}{\kappa} = c.$$

We can use equation 11.8 to rewrite this equation in terms of the ratio of the magnitudes of the fields at a fixed place and time as

$$\frac{E}{B}=\frac{\left|\vec{E}\left(\vec{r},t\right)\right|}{\left|\vec{B}\left(\vec{r},t\right)\right|}=\frac{E_{\max}\left|\sin\left(\kappa x-\omega t\right)\right|}{B_{\max}\left|\sin\left(\kappa x-\omega t\right)\right|}=c. \tag{11.15}$$

Thus, equation 11.8 satisfies Faraday's Law of Induction if the ratio of the electric and magnetic field magnitudes is c.

Maxwell-Ampere Law

Finally, we address the Maxwell-Ampere Law. For electromagnetic waves, in which no current flows, we can write

$$\oint \vec{B}\cdot d\vec{s}=\mu_0\varepsilon_0\frac{d\Phi_E}{dt}. \tag{11.16}$$

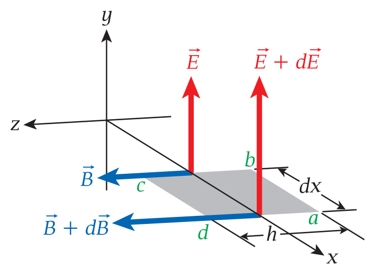


FIGURE 11.8 Representations of the electric and magnetic fields in an electromagnetic wave at a given instant. The gray area represents an integration loop for the Maxwell-Ampere Law.

To evaluate the integral on the left-hand side of this equation, we assume a closed loop in the xz-plane that has width dx and height h, represented by the gray rectangle in Figure 11.8. The differential area of this rectangle is oriented in the positive y-direction.

The integral around the loop in the counterclockwise direction (a to b to c to d to a) is given by

$$\oint \vec{B}\cdot d\vec{s}=Bh-\left(B+dB\right)\left(h\right)=-\left(dB\right)\left(h\right). \tag{11.17}$$

As before, the parts of the loop parallel to the x-axis do not contribute to the integral. The right-hand side of equation 11.16 can be written as

$$\mu_0\varepsilon_0\frac{d\Phi_E}{dt}=\mu_0\varepsilon_0\,A\frac{dE}{dt}=\mu_0\varepsilon_0\left(h\right)\,\left(dx\right)\frac{dE}{dt}, \tag{11.18}$$

where dt is the time during which the wave travels a distance dx. Substituting from equations 11.17 and 11.18 into equation 11.16, we get

$$-\left(dB\right)\left(h\right)=\mu_0\varepsilon_0\left(h\right)\left(dx\right)\frac{dE}{dt}.$$

Expressing this equation in terms of partial derivatives, as we did for equation 11.12, we get

$$-\frac{\partial B}{\partial x}=\mu_0\varepsilon_0\frac{\partial E}{\partial t}.$$

Now, using equation 11.8, we have

$$-\left[\kappa B_{\max}\cos\left(\kappa x-\omega t\right)\right]=-\mu_0\varepsilon_0\omega E_{\max}\cos\left(\kappa x-\omega t\right),$$

or

$$\frac{E_{\max}}{B_{\max}}=\frac{\kappa}{\mu_0\varepsilon_0\omega}=\frac{1}{\mu_0\varepsilon_0 c}.$$

We can express this equation in terms of the electric and magnetic field magnitudes as before:

$$\frac{E}{B}=\frac{1}{\mu_0\varepsilon_0 c}. \tag{11.19}$$

Equation 11.8 satisfies the Maxwell-Ampere Law if the ratio of the electric and magnetic field magnitudes is given by $1/\mu_0\varepsilon_0 c$.

The Speed of Light

From equations 11.15 and 11.19, we can conclude that

$$\frac{E}{B}=\frac{1}{\mu_0\varepsilon_0 c}=c,$$

which leads to

$$c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}. \tag{11.20}$$

Thus, the speed of an electromagnetic wave can be expressed in terms of the two fundamental constants related to electric and magnetic fields: the magnetic permeability and the electric permittivity of free space (vacuum). Putting the accepted values of these constants into equation 11.20 gives

$$c = \frac{1}{\sqrt{\left(4\pi\times10^{-7}\ \text{H/m}\right)\left(8.85\times10^{-12}\ \text{F/m}\right)}} = 3.00\times10^{8}\ \text{m/s}.$$

This calculated speed is equal to the measured speed of light. This equality means that all electromagnetic waves travel (in vacuum) at the speed of light and suggests that light is an electromagnetic wave.

Equation 11.15 states $E/B = c$. Even though c is a very large number, equation 11.15 does not mean that the electric field magnitude is much larger than the magnetic field magnitude. In fact, electric and magnetic fields are measured in different units, so a direct comparison is not possible.

The speed of light plays an important role in the theory of special relativity. The speed of light is always the same in any reference frame. Thus, if you send an electromagnetic wave out in a specific direction, any observer, regardless of whether that observer is moving toward you or away from you or in another direction, will see that wave moving at the speed of light. This amazing result, along with the plausible postulate that the laws of physics are the same for all inertial observers, leads to the theory of special relativity.

The speed of light can be measured extremely precisely, much more precisely than the meter could be determined from the original reference standard. Therefore, the speed of light is now defined to be precisely

$$c = 299{,}792{,}458\ \text{m/s}. \tag{11.21}$$

The definition of the meter is now simply the distance that light can traverse in vacuum in a time interval of 1/299,792,458 s.

Concept Check 11.4

What is the time required for laser light to travel from the Earth to the Moon and back again? The distance between the Earth and the Moon is $3.84 \cdot 10^8$ m.

a) 0.640 s
b) 1.28 s
c) 2.56 s
d) 15.2 s
e) 85.0 s

Self-Test Opportunity 11.1

The brightest star in the night sky is Sirius, which is at a distance of $8.30 \cdot 10^{16}$ m from Earth. When we see the light from this star, how far back in time (in years) are we looking?

11.3 The Electromagnetic Spectrum

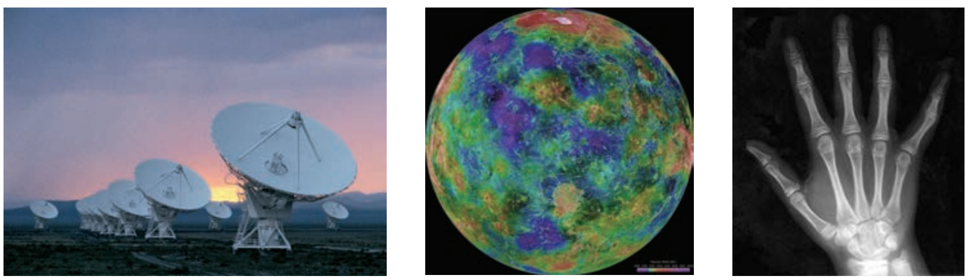
All electromagnetic waves travel at the speed of light. However, the wavelength and the frequency of electromagnetic waves vary dramatically. The speed of light, c, the wavelength, λ, and the frequency, f, are related by

$$c = \lambda f. \tag{11.22}$$

Examples of electromagnetic waves are light, radio waves, microwaves, X-rays, and gamma rays. Three applications of electromagnetic waves are shown in Figure 11.9.

The **electromagnetic spectrum** is illustrated in Figure 11.10, which includes electromagnetic waves with wavelengths ranging from 1000 m and longer to less than 10^{-12} m, with corresponding frequencies ranging from 10^5 to 10^{20} Hz. Electromagnetic waves with wavelengths (and frequencies) in certain ranges are identified by characteristic names:

- **Visible light** refers to electromagnetic waves that we can see with our eyes, with wavelengths from 400 nm (blue) to 700 nm (red). The response of the human eye peaks at around 550 nm (green) and drops off quickly away from



(a) (b) (c)

FIGURE 11.9 (a) Very Large Array radio telescope. (b) False color radar image of the surface of Venus. (c) X-ray image of a hand.

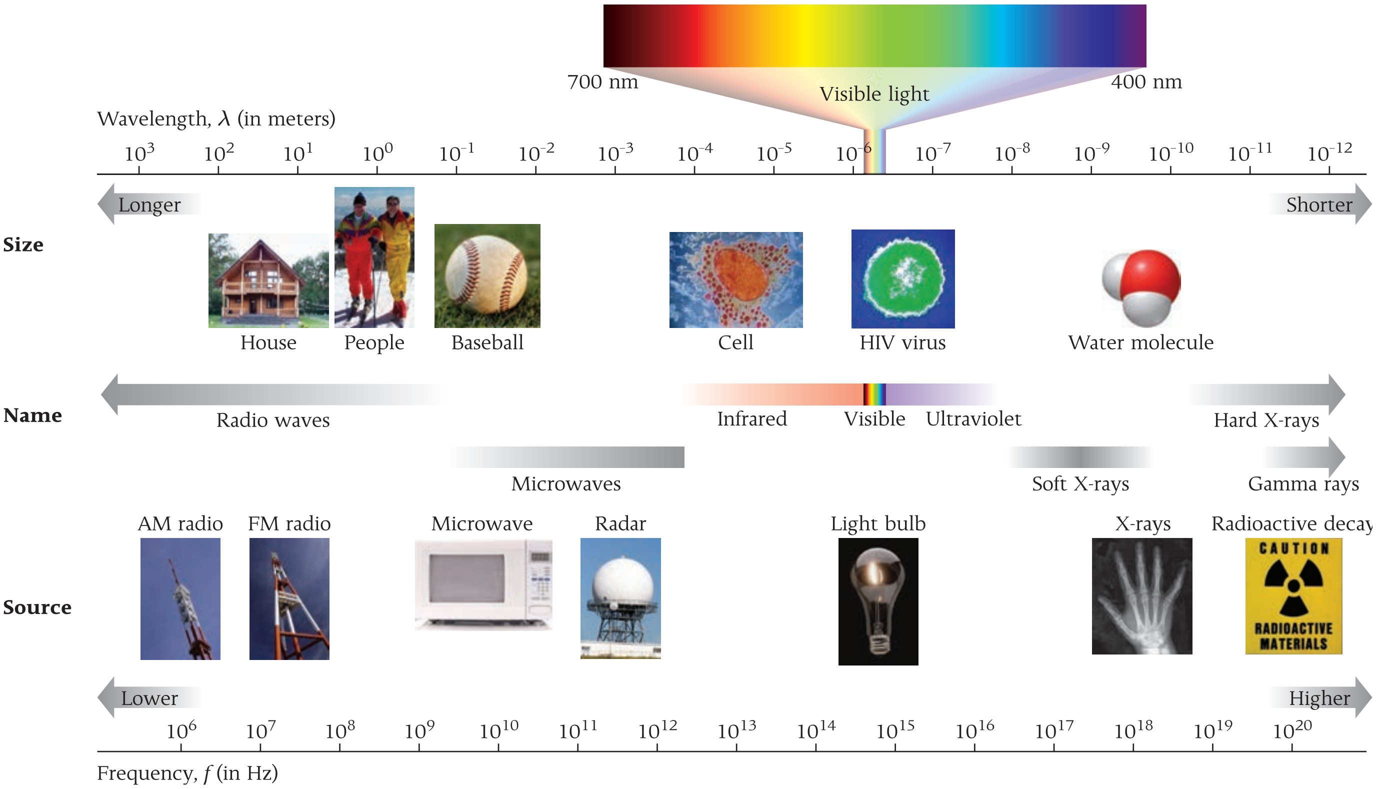


FIGURE 11.10 The electromagnetic spectrum.

that wavelength. Other wavelengths of electromagnetic waves are invisible to the human eye. However, we can detect them by other means.

- **Infrared waves**, with wavelengths just longer than visible light up to around 10^{-4} m, are felt as warmth. Detectors of infrared waves can be used to measure heat leaks in homes and offices, as well as locate brewing volcanoes. Many animals have developed the ability to see infrared waves, so they can see in the dark. Infrared beams are also used in automatic faucets in public restrooms and in remote control units for TV and DVD players.
- **Ultraviolet rays**, with wavelengths just shorter than visible light down to a few nanometers (10^{-9} m), can damage skin and cause sunburn. Fortunately, Earth's atmosphere, particularly its ozone layer, prevents most of the Sun's ultraviolet rays from reaching Earth's surface. Ultraviolet rays are used in hospitals to sterilize equipment and also produce optical properties such as fluorescence.
- **Radio waves** have frequencies ranging from several hundred kHz (AM radio) to 100 MHz (FM radio). They are also widely used in astronomy because they can pass through clouds of dust and gas that block visible light; the Very Large Array shown in Figure 11.9a is a collection of telescopes that utilize radio waves.
- **Microwaves,** used to pop popcorn in microwave ovens and transmit phone messages through relay towers or satellites, have frequencies around 10 GHz. Radar uses waves with wavelengths between those of radio waves and microwaves, which enable them to travel easily through the atmosphere and reflect off objects from the size of a baseball to the size of a storm cloud. Figure 11.9b shows a radar image of the surface of Venus, which is always obscured by clouds that block visible light.
- **X-rays** used to produce medical images, such the one shown in Figure 11.9c, have wavelengths on the order of 10^{-10} m. This length is about the same as the distance between atoms in a solid crystal, so X-rays are used to determine the detailed molecular structure of any material that can be crystallized.

Self-Test Opportunity 11.2

An FM radio station broadcasts at 90.5 MHz, and an AM radio station broadcasts at 870 kHz. What are the wavelengths of these electromagnetic waves?

- **Gamma rays** emitted in the decay of radioactive nuclei have very short wavelengths, on the order of 10^{-12} m, and can cause damage to human cells. They are often used in medicine to destroy cancer cells or other malignant tissues that are hard to reach.

Communication Frequency Bands

The frequency ranges assigned to radio and television broadcasts are shown in Figure 11.11. The range of frequencies assigned to AM (amplitude modulation) radio stations is from 535 kHz to 1705 kHz. FM (frequency modulation) radio stations use the frequencies between 88.0 MHz and 108.0 MHz. VHF (very high frequency) television operates in two ranges: 54.0 MHz to 88.0 MHz for channels 2 through 6, and 174.0 MHz to 216.0 MHz for channels 7 through 13. UHF (ultra-high frequency) television channels 14 through 51 broadcast in the range from 470.0 MHz to 698.0 MHz. Most high-definition television (HDTV) broadcasts use the UHF band and channels 14 through 51.

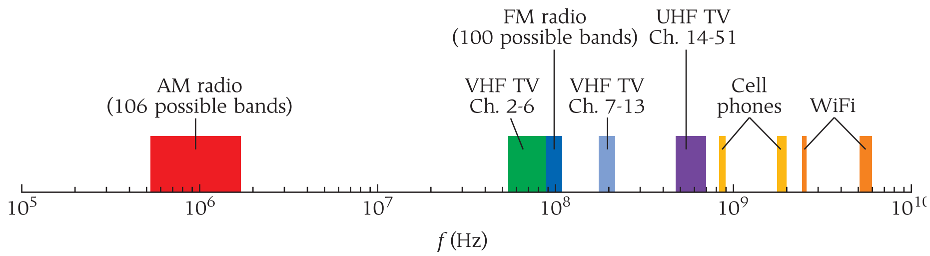


FIGURE 11.11 The frequency bands assigned to radio broadcasts, television broadcasts, cell phones, and WiFi computer connections in the United States.

A radio or television station transmits a carrier signal on a given frequency. The carrier signal is a sine wave with a frequency equal to the frequency of the broadcasting station. In the case of AM broadcasts, the amplitude of the carrier wave is modified by the information being transmitted, as illustrated in Figure 11.12a. The modulation of the amplitude of the carrier signal carries the transmitted message. Figure 11.12a shows a simple sine wave, indicating that a simple tone is being transmitted. The signal is received by a tuned RLC circuit whose resonant frequency is equal to the frequency of the carrier signal. The current induced in the circuit is proportional to the message being transmitted. AM transmission is vulnerable to noise and signal loss because the message is proportional to the amplitude of the signal, which can change if conditions vary.

For FM transmission, the frequency of the carrier signal is modified by the message to produce a modulated signal, as shown in Figure 11.12b. This type of transmission is much less affected by noise and signal loss because the message is extracted from frequency shifts of the carrier signal, rather than from changes in the amplitude of the carrier signal. FM radio receivers commonly use a *Foster-Seeley discriminator* to demodulate the FM signal.

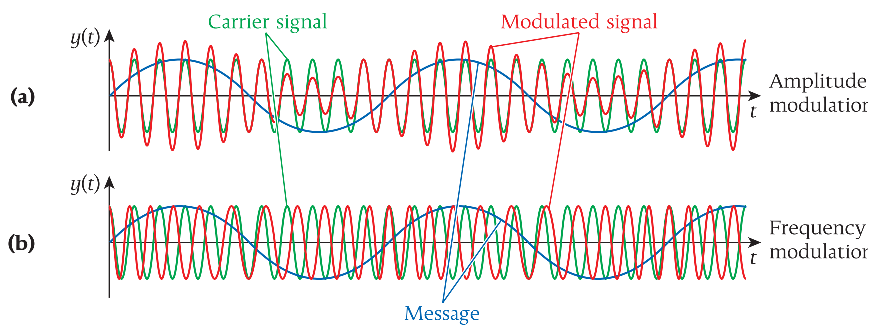


FIGURE 11.12 (a) Amplitude modulation. (b) Frequency modulation. For both cases, the green curve represents the carrier signal, the red curve represents the modulated signal, and the blue curve signifies the information being transmitted.

A Foster-Seeley discriminator uses an RLC circuit tuned to the frequency of the carrier signal and connected to two diodes, resembling the fullwave rectifier discussed in Chapter 10. If the input equals the carrier frequency, the two halves of the tuned circuit produce the same rectified voltage and the output is zero. As the frequency of the carrier signal changes, the balance between the two halves of the rectified circuit changes, resulting in a voltage proportional to the frequency deviation of the carrier signal. A Foster-Seeley discriminator is sensitive to amplitude variations and is usually coupled with a limiter amplifier stage to desensitize it to variations in the strength of the carrier wave by allowing lower power signals to pass through it unaffected while removing the peaks of the signals that exceed a certain power level.

HDTV transmitters broadcast information digitally in the form of zeros and ones. One byte of information contains eight bits, where a bit is a zero or a one. The screen is subdivided into picture elements (pixels) with digital representations of the red, green, and blue color of each pixel. Currently, the highest resolution for HDTV is 1080i, which has 1920 pixels in the horizontal direction and 1080 pixels in the vertical direction. Half the picture (every other horizontal line) is updated 60 times every second, and the two halves of the image are interlaced to form the complete image. (See the subsection "Applications of Polarization" in Section 11.6 for more information on video formats.) HDTV is broadcast using a compression-decompression (codec) technique, typically the standard known as MPEG-2, to reduce the amount of data that must be transmitted. A typical HDTV station broadcasts about 17 megabytes of information per second.

Cell phone transmissions occur in the frequency bands from 824 to 894 MHz and 1.85 to 1.99 GHz. WiFi data connections for computers operate in the ranges from 2.401 to 2.484 GHz (for the international standard; the U.S. standard band has an upper limit of 2.473 GHz) and from 5.15 to 5.85 GHz. These frequencies are in the microwave range, and some people worry about prolonged exposure to electromagnetic waves emitted by cell phones and WiFi. However, the relatively low power of these devices combined with the fact that the energy of these waves is much lower than that of other waves that are commonly encountered, such as visible light, argue that such exposure presents little danger. Quantum mechanics will discuss the energy of the photons associated with electromagnetic waves.

Traveling Electromagnetic Waves

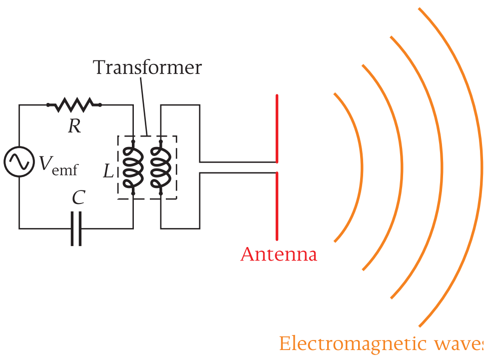


FIGURE 11.13 An RLC circuit coupled to an antenna that emits traveling electromagnetic waves.

Subatomic processes can produce electromagnetic waves such as gamma rays, X-rays, and light. Electromagnetic waves can also be produced by an RLC circuit connected to an antenna (Figure 11.13). The connection between the RLC circuit and the antenna occurs through a transformer. A dipole antenna is used to approximate an electric dipole. The voltage and current in the antenna vary sinusoidally with time and cause the flow of charge in the antenna to oscillate with the frequency, ω_0, of the RLC circuit. The accelerating charges create **traveling electromagnetic waves.** These waves travel from the antenna at speed c and with frequency $f = \omega_0/(2\pi)$.

The traveling electromagnetic waves propagate as wave fronts spreading out spherically from the antenna. However, at a large distance from the antenna, the wave fronts appear to be almost flat, or planar. Thus, such a traveling wave is described by equation 11.8. If a second RLC circuit tuned to the same frequency, ω_0, as the emitting circuit is placed in the path of these electromagnetic waves, voltage and current will be induced in this second circuit. These induced oscillations are the basis for radio transmission and reception. If the second circuit has $\omega = 1/\sqrt{LC}$, different from ω_0, much smaller voltages and currents will be induced. Only if the resonant frequency of the receiving circuit is the same as or very close to the transmitted frequency will any signal be induced in the receiving circuit. Thus, the receiver can select a transmission with a given frequency and reject all others.

The principle of transmission of electromagnetic waves was discovered by Heinrich Hertz in 1888, as described in Section 11.1, and was used by the Italian physicist Guglielmo Marconi (1874–1937) to transmit wireless signals.

11.4 Poynting Vector and Energy Transport

When you walk out into the sunlight, you feel warmth. If you stay out too long in the bright sunshine, you become sunburned. These phenomena are caused by electromagnetic waves emitted from the Sun. These electromagnetic waves carry energy that was generated in the nuclear reactions in the core of the Sun.

The rate at which energy is transported by an electromagnetic wave is usually defined in terms of a vector, $\vec{S}$, given by

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}. \tag{11.23}$$

This quantity is called the **Poynting vector,** after British physicist John Poynting (1852–1914), who first discussed its properties. The magnitude of $\vec{S}$ is related to the instantaneous rate at which energy is transported by an electromagnetic wave over a given area, or more simply, the instantaneous power per unit area:

$$S = \left|\vec{S}\right| = \left(\frac{\text{power}}{\text{area}}\right)_{\text{instantaneous}}. \tag{11.24}$$

The units of the Poynting vector are thus watts per square meter (W/m^2).

For an electromagnetic wave, where $\vec{E}$ is perpendicular to $\vec{B}$, equation 11.23 yields

$$S = \frac{1}{\mu_0}EB.$$

According to equation 11.15, the magnitudes of the electric field and the magnetic field are directly related via $E/B = c$. Thus, we can express the instantaneous power per unit area of an electromagnetic wave in terms of the magnitude of the electric field or that of the magnetic field. Since it is usually easier to measure an electric field than a magnetic field, the instantaneous power per unit area is given by

$$S = \frac{1}{c\mu_0}E^2. \tag{11.25}$$

We can now substitute a sinusoidal form for the electric field, $E = E_{\max}\sin(\kappa x - \omega t)$, and obtain an expression for the transmitted power per unit area. However, the usual means of describing the power per unit area in an electromagnetic wave is the intensity, I, of the wave, given by

$$I = S_{\text{ave}} = \left(\frac{\text{power}}{\text{area}}\right)_{\text{ave}} = \frac{1}{c\mu_0}\left[E_{\max}^2 \sin^2(\kappa x - \omega t)\right]_{\text{ave}}.$$

The units of intensity are the same as the units of the Poynting vector, W/m^2. The time-averaged value of $\sin^2(\kappa x - \omega t)$ is $\frac{1}{2}$, so we can express the intensity as

$$I = \frac{1}{c\mu_0}E_{\text{rms}}^2, \tag{11.26}$$

where $E_{\text{rms}} = E_{\max}/\sqrt{2}$.

Because the magnitudes of the electric and magnetic fields of an electromagnetic wave are related by $E = cB$ and c is such a large number, you might conclude that the energy transported by the electric field is much larger than the energy transported by the magnetic field. Actually these energies are *the same*. To see this, recall from Chapters 4 and 9 that the energy density of an electric field is given by

$$u_E = \tfrac{1}{2}\varepsilon_0 E^2,$$

and the energy density of a magnetic field is given by

$$u_B = \frac{1}{2\mu_0}B^2.$$

If we substitute $E = cB$ and $c = 1/\sqrt{\mu_0 \varepsilon_0}$ into the expression for the energy density of the electric field, we get

$$u_E = \tfrac{1}{2}\varepsilon_0 (cB)^2 = \tfrac{1}{2}\varepsilon_0 \left(\frac{B}{\sqrt{\mu_0 \varepsilon_0}} \right)^2 = \frac{1}{2\mu_0} B^2 = u_B. \qquad (11.27)$$

Thus, the energy density of the electric field is the same as the energy density of the magnetic field everywhere in the electromagnetic wave.

EXAMPLE 11.1 Using Solar Panels to Charge an Electric Car

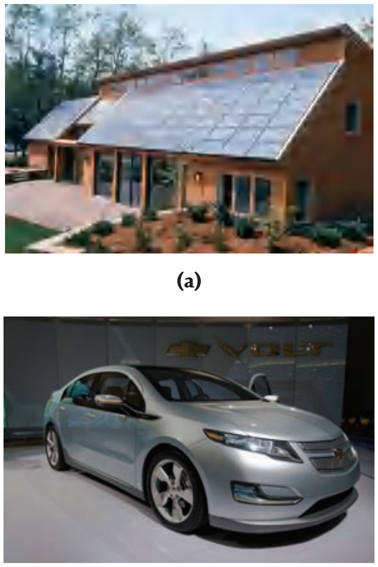


FIGURE 11.14 (a) Photovoltaic solar panels mounted on the roof of a house. (b) A plug-in electric car capable of driving 37 miles on a single charge.

Suppose that photovoltaic (solar power to electric power) solar panels (Figure 11.14a) can be mounted on the roof of your house at a cost per area of $\eta = \text{AED}1420/\text{m}^2$. You have an electric car (Figure 11.14b) that requires a charge corresponding to an energy of $U = 10.0$ kWh for a day of local driving. The solar panels convert solar power to electricity with an efficiency $\varepsilon = 14.1\%$ and have an area A. Suppose that sunlight is incident on your solar panels for $\Delta t = 4.00$ h a day with an average intensity of $S_{\text{ave}} = 600.$ W/m^2.

PROBLEM

How much do you need to spend on solar panels to give your electric car its daily charge?

SOLUTION

We equate the total amount of energy produced by the solar panels to the energy required to charge the car:

$$U_{\text{produced}} = P\Delta t = U.$$

The amount of power produced by the solar panels is the average intensity of the sunlight times the area of the solar panels times the efficiency of the solar panels:

$$P = \varepsilon A S_{\text{ave}}.$$

Thus, the total area required is

$$A = \frac{P}{\varepsilon S_{\text{ave}}} = \frac{(U/\Delta t)}{\varepsilon S_{\text{ave}}} = \frac{U}{\varepsilon S_{\text{ave}} \Delta t}.$$

The total cost will then be

$$\text{Cost} = \eta A = \frac{\eta U}{\varepsilon S_{\text{ave}} \Delta t}.$$

Putting in the numerical values gives us

$$\text{Cost} = \frac{\eta U}{\varepsilon S_{\text{ave}} \Delta t} = \frac{\left(\text{AED}1420/\text{m}^2\right)(10.0 \text{ kWh})}{(0.141)\left(0.600 \text{ kW/m}^2\right)(4.00 \text{ h})} = \text{AED } 42\,000.$$

If you drove your electric car 59.2 kilometers per day every day for 10 years, that would work out to 19.4 fils per kilometer. In contrast, the cost would be 47 fils per kilometer for a gasoline-powered car with a 8.5 km/l rating and gas costing AED 4.00 per liter.

This is not a large cost savings over the 10-year period (AED 42 000 for solar versus AED 101 685 for gasoline power) considering an electric car may cost more than a gasoline-powered car. However, your solar-powered electric car would be a completely carbon-neutral, zero-emission mode of transportation (the same as riding your bicycle, without the exercise benefits). Material scientists are working intensely to increase the efficiency of commercially available solar cells, and mass production has dramatically lowered the cost of solar panels in recent years (more than 70% since the first edition of this textbook was published) and is expected to continue to do so in the future. So solar-powered electric cars could soon be a viable and attractive alternative to gasoline-powered cars.

EXAMPLE 11.2 The Root-Mean-Square Electric and Magnetic Fields from Sunlight

The average intensity of sunlight at the Earth's surface is approximately 1400 $\mathrm{W/m^2}$, if the Sun is directly overhead.

PROBLEM

What are the root-mean-square electric and magnetic fields of these electromagnetic waves?

SOLUTION

The intensity of sunlight can be related to the root-mean-square electric field using equation 11.26:

$$I = \frac{1}{c\mu_0} E_{\mathrm{rms}}^2.$$

Solving for the root-mean-square electric field gives us

$$E_{\mathrm{rms}} = \sqrt{Ic\mu_0} = \sqrt{\left(1400\ \mathrm{W/m^2}\right)\left(3.00\times10^8\ \mathrm{m/s}\right)\left(4\pi\times10^{-7}\ \mathrm{T\ m/A}\right)}$$
$$= 730\ \mathrm{V/m}.$$

In comparison, the root-mean-square electric field in a typical home is 5–10 V/m. Standing directly under an electric power transmission line, one would experience a root-mean-square electric field of 200–10,000 V/m depending on the conditions.

The root-mean-square magnetic field of sunlight is

$$B_{\mathrm{rms}} = \frac{E_{\mathrm{rms}}}{c} = \frac{730\ \mathrm{V/m}}{3.00\times10^8\ \mathrm{m/s}} = 2.4\ \mu\mathrm{T}.$$

In comparison, the root-mean-square value of the Earth's magnetic field is 50 μT, the root-mean-square magnetic field found in a typical home is 0.5 μT, and the root-mean-square magnetic field under a power transmission line is 2 μT.

11.5 Radiation Pressure

When you walk out into the sunlight, you feel warmth, but you do not feel any force from the sunlight. Sunlight is exerting a pressure on you, but that pressure is so small that you cannot notice it. Because the electromagnetic waves making up sunlight are radiated from the Sun and travel to the Earth, they are referred to as **radiation.** In nuclear physics, this type of radiation is not necessarily the same as radioactive radiation resulting from the decay of unstable nuclei. However, radio waves, infrared waves, visible light, and X-rays are all fundamentally the same electromagnetic radiation. (Which is not to say, however, that all kinds of electromagnetic radiation have the same effect on the human body. For example, UV light can give you sunburn and even trigger skin cancer, whereas there is no credible evidence that radiation emitted from cell phones can cause cancer.)

Let's calculate the magnitude of the pressure exerted by these radiated electromagnetic waves. Electromagnetic waves carry energy, U, as shown in Section 11.4. Electromagnetic waves also have linear momentum, $\vec{p}$. This concept is subtle because electromagnetic waves have no mass, and we saw that momentum is equal to mass multiplied by velocity. Maxwell showed that if a plane wave of radiation is totally absorbed by a surface (perpendicular to the direction of the plane wave) for a time interval, Δt, and an amount of energy, ΔU, is absorbed by the surface in that process, then the magnitude of the momentum transferred to that surface by the wave in that time interval is

$$\Delta p = \frac{\Delta U}{c}.$$

The topic of relativity shows that this relationship between energy and momentum holds for massless objects; for now, it is stated as a fact, without proof.

The magnitude of the force on the surface is then $F = \Delta p/Dt$ (Newton's Second Law). The total energy, ΔU, absorbed by an area A of the surface during the time interval Δt is equal to the product of the area, the time interval, and the radiation intensity, I (introduced in Section 11.4): $\Delta U = IA\Delta t$. Therefore, the magnitude of the force exerted by the electromagnetic wave on this area is

$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta U}{c\Delta t} = \frac{IA\Delta t}{c\Delta t} = \frac{IA}{c}.$$

Since pressure is defined as force (magnitude) per unit area, the radiation pressure, p_r, is

$$p_r = \frac{F}{A},$$

and, consequently,

$$p_r = \frac{I}{c} \quad \text{(for total absorption).} \tag{11.28}$$

Equation 11.28 states that the radiation pressure due to electromagnetic waves is simply the intensity divided by the speed of light, but only for the case of total absorption of the radiation by the surface.

The other limiting case is total reflection of the electromagnetic waves. In that case, the momentum transfer is *twice* as big as for total absorption, just like the momentum transfer from a ball to a wall is twice as big in a perfectly elastic collision as in a perfectly inelastic collision. In the perfectly elastic collision, the ball's initial momentum is reversed and $\Delta p = p_i - (-p_i) = 2p_i$, whereas for the totally inelastic collision, $\Delta p = p_i - 0 = p_i$, so, the radiation pressure for the case of perfect reflection of the electromagnetic waves off a surface is

$$p_r = \frac{2I}{c} \quad \text{(for perfect reflection).} \tag{11.29}$$

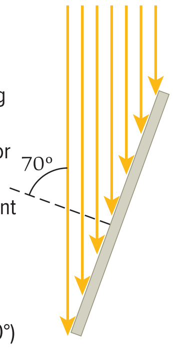
The radiation pressure from sunlight is comparatively small. The intensity of sunlight at the Earth's surface is at most 1400 W/m^2, when the Sun is directly overhead and there are no clouds in the sky. (This can happen only between the Tropics of Cancer and Capricorn, located at ±23° latitude relative to the Equator.) Thus, the maximum radiation pressure for sunlight that is totally absorbed is

$$p_r = \frac{I}{c} = \frac{1400 \text{ W/m}^2}{3\times10^8 \text{ m/s}} = 4.67\times10^{-6} \text{ N/m}^2 = 4.67 \text{ μPa}.$$

For comparison, atmospheric pressure is 101 kPa which is greater than the sunlight's radiation pressure on the surface of Earth by more than a factor of 20 billion. Another useful comparison is the lowest pressure difference that human hearing can detect, which is generally quoted as approximately 20 μPa for sounds in the 1-kHz frequency range, where the human ear is most sensitive.

Concept Check 11.5

What is the radiation pressure due to sunlight incident on a perfectly absorbing surface, whose surface normal vector is at an angle of 70° relative to the incident light?



a) (4.67 μPa)(cos 70°)

b) (4.67 μPa)(sin 70°)

c) (4.67 μPa)(tan 70°)

d) (4.67 μPa)(cot 70°)

Concept Check 11.6

What is the maximum radiation pressure due to sunlight incident on a perfectly reflecting surface?

a) 0

b) 2.34 μPa

c) 4.67 μPa

d) 9.34 μPa

EXAMPLE 11.3 Radiation Pressure from a Laser Pointer

A green laser pointer has a power of 1.00 mW. You shine the laser pointer perpendicularly on a mirror, which reflects the light. The spot of light on the mirror is 2.00 mm in diameter.

PROBLEM

What force does the light from the laser pointer exert on the mirror?

SOLUTION

The intensity of the light is given by

$$I = \frac{\text{power}}{\text{area}} = \frac{1.00\times10^{-3} \text{ W}}{\pi\left(1.00\times10^{-3} \text{ m}\right)^2} = 318 \text{ W/m}^2.$$

The radiation pressure for a perfectly reflecting surface is given by equation 11.29 and also is equal to the force exerted by the light divided by the area over which it acts:

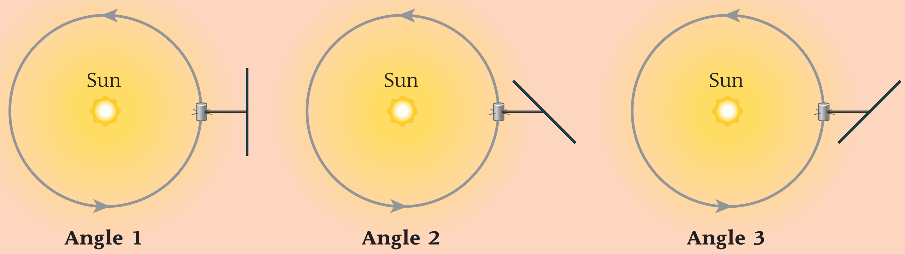
$$p_r = \frac{\text{force}}{\text{area}} = \frac{2I}{c}.$$

Thus, the force exerted on the mirror is

$$\text{Force} = (\text{area})\left(\frac{2I}{c}\right) = \pi\left(1.0\times10^{-3}\ \text{m}\right)^2 \frac{2\left(318\ \text{W/m}^2\right)}{3.00\times10^8\ \text{m/s}} = 6.66\times10^{-12}\ \text{N}.$$

Self-Test Opportunity 11.3

Suppose you have a satellite in orbit around the Sun, as shown in the figure. The orbit is in the counter-clockwise direction looking down on the north pole of the Sun. You want to deploy a solar sail consisting of a large, totally reflecting mirror, which can be oriented so that it is perpendicular to the light coming from the Sun or at an angle with respect to the light coming from the Sun. Describe the effect on the orbit of your satellite for the three deployment angles shown in the figure.



SOLVED PROBLEM 11.1 Solar Stationary Satellite

Suppose researchers want to place a satellite that will remain stationary above the north pole of the Sun in order to study the Sun's long-term rotational characteristics. The satellite will have a totally reflecting solar sail and be located at a distance of $1.50\cdot10^{11}$ m from the center of the Sun. The intensity of sunlight at that distance is 1400 W/m^2. The plane of the solar sail is perpendicular to a line connecting the satellite and the center of the Sun. The mass of the satellite and sail is 100.0 kg.

PROBLEM

What is the required area of the solar sail?

SOLUTION

THINK In an equilibrium position for the satellite, the area of the solar sail times the radiation pressure from the Sun produces a force that is balanced by the gravitational force between the satellite and the Sun. We can equate these two forces and solve for the area of the solar sail.

SKETCH Figure 11.15 is a diagram of the satellite with a solar sail near the Sun.

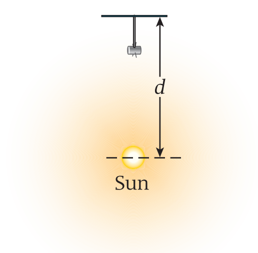


FIGURE 11.15 A satellite with a solar sail near the Sun.

RESEARCH The satellite will be stationary if the force of gravity, F_g, is balanced by the force from the radiation pressure of sunlight, F_{rp}:

$$F_g = F_{rp}.$$

The force corresponding to the radiation pressure from sunlight is equal to the radiation pressure, p_r, times the area of the solar sail, A:

$$F_{rp} = p_r A.$$

The radiation pressure can be expressed in terms of the intensity of the sunlight, I, incident on the totally reflecting solar sail:

$$p_r = \frac{2I}{c}.$$

The force of gravity between the satellite and the Sun is given by

$$F_g = G\frac{mm_{\text{Sun}}}{d^2},$$

– *Continued*

where G is the universal gravitational constant, m is the mass of the satellite and sail, m_{Sun} is the mass of the Sun, and d is the distance between the satellite and the Sun.

SIMPLIFY We can combine all these equations to obtain

$$\left(\frac{2I}{c}\right)A = G\frac{mm_{\text{Sun}}}{d^2}.$$

Solving for the area of the solar sail gives us

$$A = G\frac{cmm_{\text{Sun}}}{2Id^2}.$$

CALCULATE Putting in the numerical values, we get

$$A = \left(6.67\times10^{-11}\ \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2}\right)\frac{\left(3.00\times10^8\ \text{m/s}\right)\left(100.0\ \text{kg}\right)\left(1.99\times10^{30}\ \text{kg}\right)}{2\left(1400\ \text{W/m}^2\right)\left(1.50\times10^{11}\ \text{m}\right)^2} = 63{,}206.2\ \text{m}^2.$$

ROUND We report our result to three significant figures:

$$A = 6.32\times10^4\ \text{m}^2.$$

DOUBLE-CHECK If the solar sail were circular, the radius of the sail would be

$$R = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{6.32\times10^4\ \text{m}^2}{\pi}} = 142\ \text{m}.$$

We can relate the thickness of the sail, t, times the density, ρ, of the material from which the sail is constructed to the mass per unit area of the sail:

$$t\rho = \frac{m}{A}.$$

If the sail were composed of a sturdy material such as kapton ($\rho = 1420\ \text{kg/m}^3$) and had a mass of 75 kg, the thickness of the sail would be

$$t = \frac{75\ \text{kg}}{\left(1420\ \text{kg/m}^3\right)\left(6.32\times10^4\ \text{m}^2\right)} = 8.36\times10^{-7}\ \text{m} = 0.836\ \mu\text{m}.$$

Kapton is a polyimide film developed to remain stable in the wide range of temperatures found in space; from near absolute zero to over 600 K. Current production techniques cannot produce kapton this thin. However, the required areal mass density may be realizable using other materials in the future.

SOLVED PROBLEM 11.2 Laser-Powered Sailing

One idea for propelling long-range spacecraft involves using a high-powered laser beam, rather than sunlight, focused on a large totally reflecting sail. The spacecraft could then be propelled from Earth. Suppose a 10.0 GW laser could be focused at long distances. The spacecraft has a mass of 200.0 kg, and its reflecting sail is large enough to intercept all of the light emitted by the laser.

PROBLEM

Neglecting gravity, how long would it take the spacecraft to reach a speed of 30.0% of the speed of light, starting from rest?

SOLUTION

THINK The radiation pressure from the laser produces a constant force on the spacecraft's sail, resulting in a constant acceleration. Using the constant acceleration, we can calculate the time to reach the final speed starting from rest.

SKETCH Figure 11.16 is a diagram of a laser focusing light on the spacecraft with a totally reflecting sail.

FIGURE 11.16 A laser focusing its light on a spacecraft with a totally reflecting sail.

RESEARCH The radiation pressure, p_r, from the light with intensity I produced by the laser is

$$p_r = \frac{2I}{c}.$$

Pressure is defined as force, F, per unit area, A, of the spot the beam produces on the sail, so we can write

$$\frac{2I}{c} = \frac{F}{A}.$$

The intensity of the laser is given by the power, P, of the laser divided by the spot's area, A. Assuming that the sail of the spacecraft can intercept the entire laser beam, we can write

$$\frac{F}{A} = \frac{2(P/A)}{c}.$$

Solving for the force exerted by the laser beam on the sail and using Newton's Second Law, we can write

$$F = \frac{2P}{c} = ma. \qquad \text{(i)}$$

SIMPLIFY We can solve equation (i) for the acceleration:

$$a = \frac{2P}{mc}.$$

Assuming that all the power of the laser remains focused on the sail of the spacecraft, the spacecraft will experience a constant acceleration. Then, the final speed, v, of the spacecraft can be related to the time it takes to reach that speed through

$$v = at = 0.300c.$$

Solving for the time gives us

$$t = \frac{0.300c}{2P/mc} = \frac{0.300mc^2}{2P}.$$

CALCULATE Putting in the numerical values gives us

$$t = \frac{0.300mc^2}{2P} = \frac{0.300(200.0\text{ kg})(3.00\times10^8\text{ m/s})^2}{2(10.0\times10^9\text{ W})} = 270{,}000{,}000\text{ s}.$$

ROUND We report our result to three significant figures:

$$t = 270{,}000{,}000\text{ s} = 8.56\text{ yr}.$$

DOUBLE-CHECK To double-check our result, we calculate the acceleration of the spacecraft:

$$a = \frac{2P}{mc} = \frac{2(10.0\times10^9\text{ W})}{(200\text{ kg})(3.00\times10^8\text{ m/s})} = 0.333\text{ m/s}^2.$$

This acceleration is 3% of the acceleration due to gravity at the surface of the Earth. This acceleration is produced by a laser with 10 times the power of a typical power station, which must run continuously for 8.56 yr. The distance the spacecraft will travel during that time is

$$x = \tfrac{1}{2}at^2 = \tfrac{1}{2}(0.333\text{ m/s}^2)(2.70\times10^8\text{ s})^2 = 1.21\times10^{16}\text{ m} = 1.28\text{ light-years},$$

which is slightly more than $1\frac{1}{4}$ times the distance light travels in a year. The laser must remain focused on the spacecraft at this distance. Thus, although our calculations seem reasonable, the requirements for a laser-driven spacecraft with a reflecting sail seem difficult to achieve. We must modify this calculation because the speed involved is a significant fraction of the speed of light.

11.6 Polarization

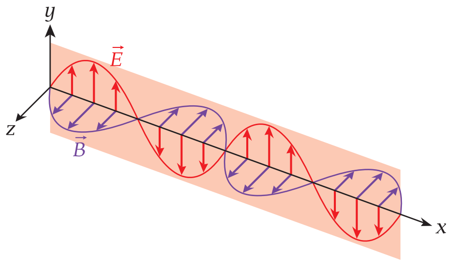


FIGURE 11.17 An electromagnetic wave with the plane of oscillation of the electric field shown in pink.

For the electromagnetic wave represented in Figure 11.5, the electric field always points along the y-axis. The direction in which the wave is traveling is the positive x-direction, so the electric field of the electromagnetic wave lies within a plane of oscillation (Figure 11.17).

We can visualize the polarization of an electromagnetic wave by looking at the electric field vector of the wave in the yz-plane, which is perpendicular to the direction in which the wave is traveling (Figure 11.18a). The electric field vector changes from the positive y-direction to the negative y-direction and back again as the wave travels. The electric field of the wave oscillates in the y-direction only, never changing its orientation. This type of wave is called a **plane-polarized wave** in the y-direction.

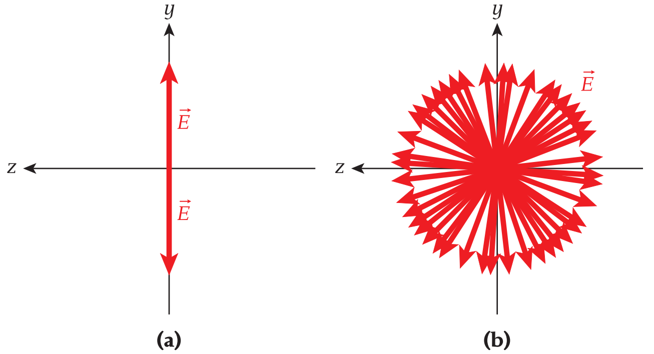


FIGURE 11.18 (a) Electric field vectors in the yz-plane, defining the plane of polarization to be the xy-plane. (b) Electric field vectors oriented at random angles.

The electromagnetic waves making up the light emitted by most common light sources, such as the Sun or an incandescent light bulb, have random polarizations. Each wave has its electric field vector oscillating in a different plane. Such light is called **unpolarized light.** Light from an unpolarized source can be represented by many vectors like the ones shown in Figure 11.18a, but with random orientations (Figure 11.18b). Unpolarized light can also be represented by summing the y-components and the z-components separately to produce the net y- and z-components. Unpolarized light has equal components in the y- and z-directions (Figure 11.19a). If there is less net polarization in the y-direction than in the z-direction, then the light is said to be partially polarized in the z-direction (Figure 11.19b).

Unpolarized light can be transformed to polarized light by passing the unpolarized light through a polarizer. A **polarizer** allows only one component of the electric field vectors of the light waves to pass through. One way to make a polarizer is to produce a material that consists of long parallel chains of molecules. This discussion will not go into the details of the molecular structure but will simply characterize each polarizer by a polarizing direction. Unpolarized light passing through a polarizer emerges polarized in the polarizing direction (Figure 11.20). The components of the light that have the same direction as the polarizer are transmitted, but the components of the light that are perpendicular to the polarizer are absorbed.

Now let's consider the intensity of the light that passes through a polarizer. Unpolarized light with intensity I_0 has equal components in the y- and z-directions. After passing

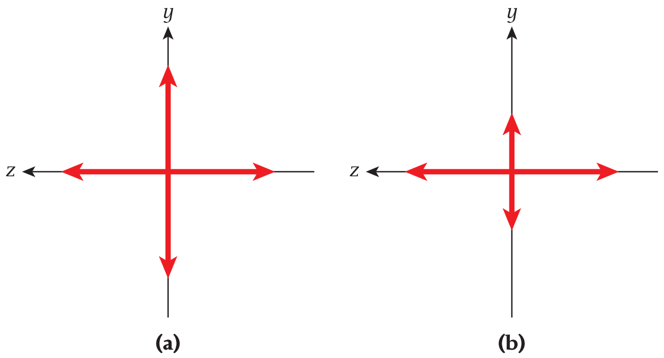


FIGURE 11.19 (a) Net components of the electric field for unpolarized light. (b) Net components of the electric field for partially polarized light.

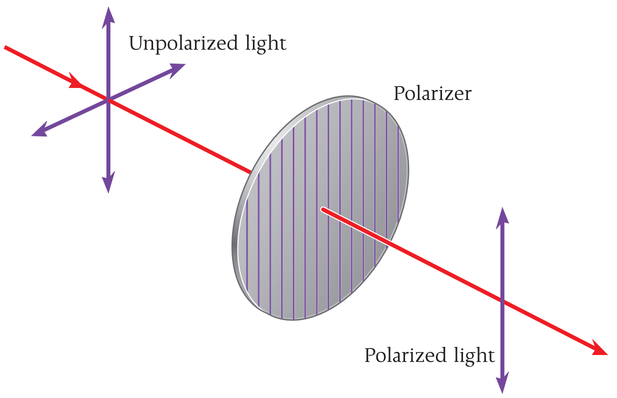


FIGURE 11.20 Unpolarized light passing through a vertical polarizer. After the light passes through the polarizer, it is vertically polarized.

through a vertical polarizer, only the y-component (or vertical component) remains. The intensity, I, of the light that passes through the polarizer is given by

$$I = \tfrac{1}{2} I_0, \tag{11.30}$$

because the unpolarized light had equal contributions from the y- and z-components and only the y-components are transmitted by the polarizer. The factor $\frac{1}{2}$ applies only to the case of unpolarized light passing through a polarizer.

Let's consider polarized light passing through a polarizer (Figure 11.21). If the polarizer axis is parallel to the polarization direction of the incident polarized light, all of the light will be transmitted with the original polarization (Figure 11.21a). If the polarizing angle of the polarizer is perpendicular to the polarization of polarized light, no light will be transmitted (Figure 11.21b).

What happens when polarized light is incident on a polarizer and the polarization of the light is neither parallel nor perpendicular to the polarizing angle of the polarizer (Figure 11.22)? Let's assume that the angle between the incident polarized light and the polarizing angle is θ. The magnitude of the transmitted electric field, E, is given by

$$E = E_0 \cos\theta,$$

where E_0 is the magnitude of the electric field of the incident polarized light. From equation 11.26, we can see that the intensity of the light before passing through the polarizer, I_0, is given by

$$I_0 = \frac{1}{c\mu_0} E_{\text{rms}}^2 = \frac{1}{2c\mu_0} E_0^2.$$

After the light passes through the polarizer, the intensity, I, is given by

$$I = \frac{1}{2c\mu_0} E^2.$$

We can express the transmitted intensity in terms of the initial intensity as follows:

$$I = \frac{1}{2c\mu_0} E^2 = \frac{1}{2c\mu_0} (E_0 \cos\theta)^2 = I_0 \cos^2\theta. \tag{11.31}$$

This equation is called the **Law of Malus.** It applies only to polarized light incident on a polarizer.

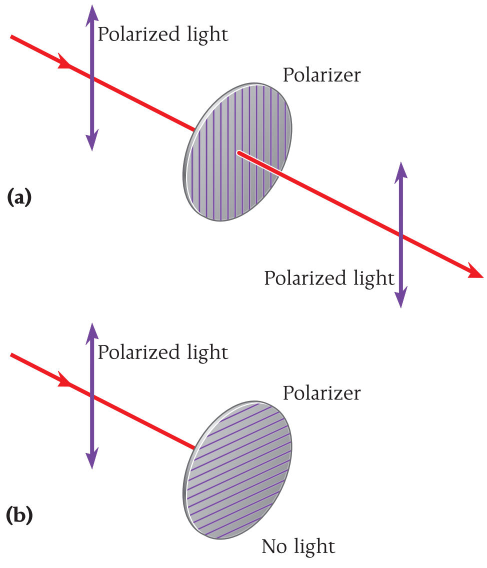


FIGURE 11.21 (a) Vertically polarized light incident on a vertical polarizer. (b) Vertically polarized light incident on a horizontal polarizer.

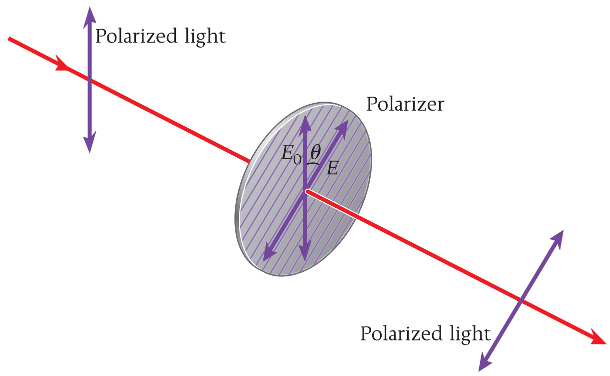


FIGURE 11.22 Polarized light passing through a polarizer whose polarizing angle is neither parallel nor perpendicular to the polarization of the incident light.

EXAMPLE 11.4 Three Polarizers

Suppose that unpolarized light with intensity I_0 is initially incident on the first of three polarizers in a line. The first polarizer has a polarizing direction that is vertical. The second polarizer has a polarizing angle of 45.0° with respect to the vertical. The third polarizer has a polarizing angle of 90.0° with respect to the vertical.

PROBLEM

What is the intensity of the light after passing through all three polarizers, in terms of the initial intensity?

SOLUTION

Figure 11.23 illustrates the light passing through the three polarizers. The intensity of the unpolarized light is I_0. The intensity of the light after passing through the first polarizer is

$$I_1 = \tfrac{1}{2} I_0.$$

The intensity of the light after passing through the second polarizer is

$$I_2 = I_1 \cos^2(45° - 0°) = I_1 \cos^2 45° = \tfrac{1}{2} I_0 \cos^2 45°.$$

– *Continued*

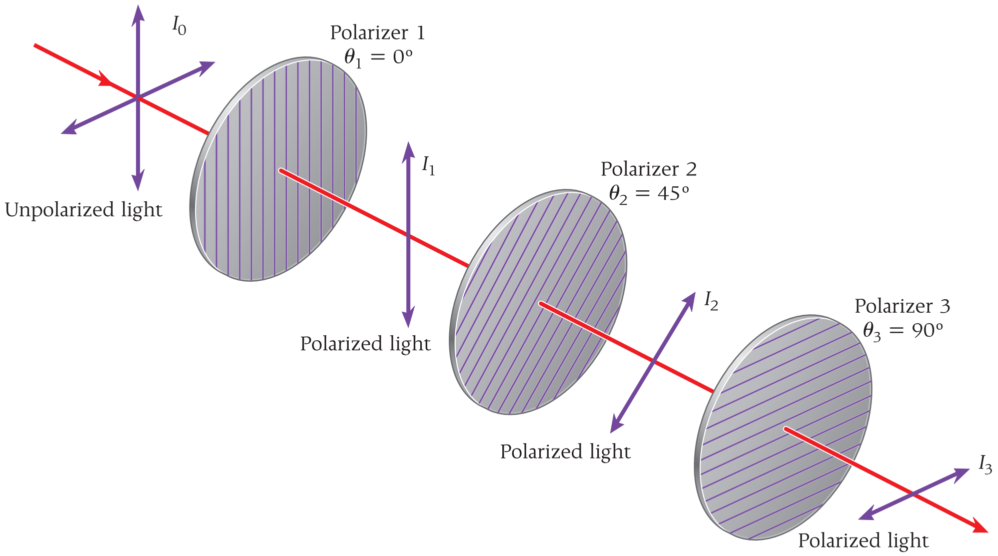


FIGURE 11.23 Unpolarized light passing through three polarizers.

The intensity of the light after passing through the third polarizer is

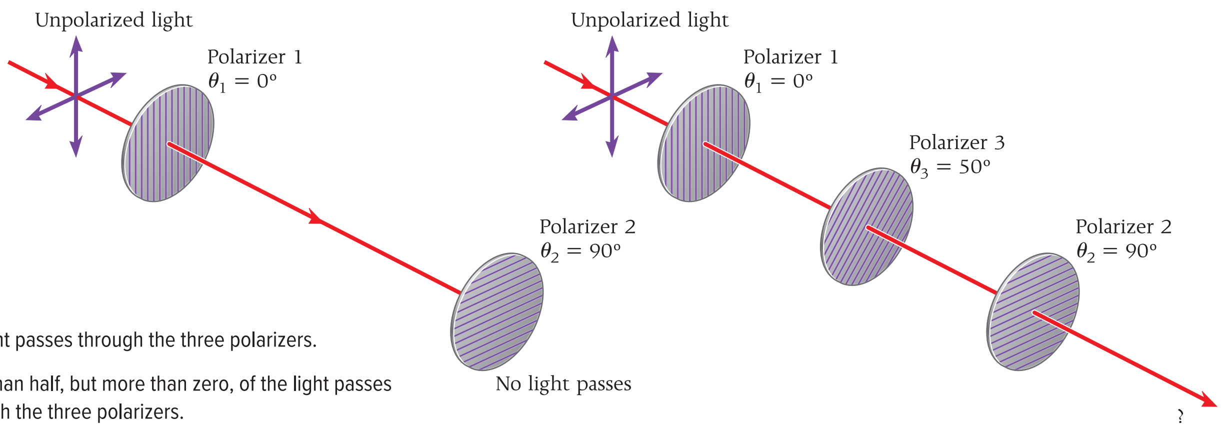
$$I_3 = I_2 \cos^2(90° - 45°) = I_2 \cos^2 45° = \tfrac{1}{2} I_0 \cos^4 45°,$$

or $I_3 = I_0/8$.

The fact that $\frac{1}{8}$ of the light's initial intensity is transmitted is somewhat surprising because polarizers 1 and 3 have polarizing angles that are perpendicular to each other. Acting by themselves, polarizers 1 and 3 would block all of the light. Yet by adding an additional obstacle (polarizer 2) between these two polarizers, $\frac{1}{8}$ of the original intensity gets through. A series of polarizers with small differences in their polarizing angles can thus be used to rotate the polarization direction of light with only modest losses in intensity.

Concept Check 11.7

The figure shows unpolarized light incident on polarizer 1 with polarizing angle $\theta_1 = 0°$ and then on polarizer 2 with polarizing angle $\theta_2 = 90°$, which results in no light passing through. If polarizer 3 with polarizing angle $\theta_3 = 50°$ is placed between polarizers 1 and 2, which of the following statements is true?



a) No light passes through the three polarizers.

b) Less than half, but more than zero, of the light passes through the three polarizers.

c) Exactly half of the light passes through the three polarizers.

d) More than half, but not all, of the light passes through the three polarizers.

e) All of the light passes through the three polarizers.

SOLVED PROBLEM 11.3 Multiple Polarizers

Suppose you have light polarized in the vertical direction and want to rotate the polarization to the horizontal direction ($\theta = 90.0°$). If you pass the vertically polarized light through a polarizer whose polarizing angle is horizontal, all the light will be blocked. If, instead, you use a series of ten polarizers, each of which has a polarizing angle u that is 9.00° more than that of the preceding one, with the first polarizer having $\theta = 9.00°$, you can rotate the polarization by 90.0° and still have light passing through.

PROBLEM

What fraction of the intensity of the incident light is transmitted through the ten polarizers?

SOLUTION

THINK Each polarizer is rotated 9.00° from the preceding polarizer. Thus, each polarizer transmits a fraction of the intensity equal to $f = \cos^2 9°$. The fraction transmitted is then f^{10}.

SKETCH Figure 11.24 shows the direction of the initial polarization and the orientations of the ten polarizers.

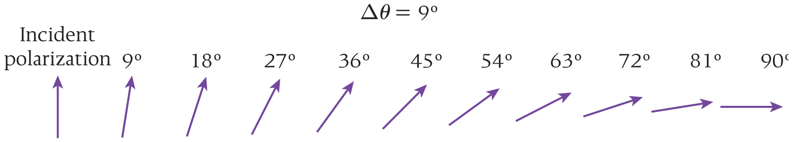


FIGURE 11.24 The direction of the polarization of the incident light and the direction of the polarizing angles of ten polarizers.

RESEARCH The intensity, I, of polarized light passing through a polarizer whose polarizing direction makes an angle θ with the polarization of the incident light is given by

$$I = I_0 \cos^2\theta,$$

where I_0 is the intensity of the incident polarized light. In this case, the polarizing direction of each successive polarizer is rotated $\Delta\theta = 9°$ relative to the polarizing direction of the preceding polarizer. Thus, each polarizer reduces the intensity by the factor

$$\frac{I}{I_0} = \cos^2\Delta\theta.$$

SIMPLIFY The reduction in intensity after the light has passed through ten polarizers, each with its polarization direction differing from that of the preceding polarizer by $\Delta\theta$ is

$$\frac{I_{10}}{I_0} = \left(\cos^2\Delta\theta\right)^{10}.$$

CALCULATE Putting in the numerical values, we get

$$\frac{I_{10}}{I_0} = \left(\cos^2 9°\right)^{10} = 0.780546.$$

ROUND We report our result to three significant figures:

$$\frac{I_{10}}{I_0} = 0.781 = 78.1\%.$$

DOUBLE-CHECK Using ten polarizers, each rotated by 9° more than the preceding one, the polarization of the incident polarized light was rotated by 90° and 78.1% of the light was transmitted, whereas using one polarizer rotated by 90° would have blocked all of the incident light. To see if our answer is reasonable, let's assume that instead of ten polarizers, we use n polarizers, each rotated by an angle $\Delta\theta = \theta_{max}/n$, where $\theta_{max} = 90°$. For each polarizer, the angle between its polarization direction and that of the preceding polarizer is small, so we can use the small-angle approximation for $\cos\Delta\theta$ to write

$$\frac{I_1}{I_0} \approx \left(1 - \frac{(\Delta\theta)^2}{2}\right)^2.$$

– Continued

The intensity of the light that passes through the n polarizers is then

$$\frac{I_n}{I_0} \approx \left(1 - \frac{(\theta_{\max}/n)^2}{2}\right)^{2n} = \left(1 - \frac{\theta_{\max}^2}{2n^2}\right)^{2n}.$$

For large n,

$$\frac{I_n}{I_0} \approx 1 = 100\%.$$

Using ten polarizers to rotate the polarization of the incident polarized light allowed 78.1% of the light to pass. Using more polarizers with smaller changes in the polarization direction would allow the transmission to approach 100%. Thus, our result seems reasonable.

Applications of Polarization

Polarization has many practical applications. Sunglasses often have a polarized coating that blocks reflected light, which is usually polarized. A computer's or television's liquid crystal display (LCD) has an array of liquid crystals sandwiched between two polarizers whose polarizing angles are rotated 90° with respect to each other. Normally, the liquid crystal rotates the polarization of the light between the two polarizers so that light passes through. An array of addressable electrodes applies a varying voltage across each of the liquid crystals, causing the liquid crystals to rotate the polarization less, darkening the area covered by the electrode. The television or computer monitor screen can then display a large number of picture elements, or *pixels*, that produce a high-resolution image.

Figure 11.25a shows a top view of the layers of an LCD screen. Unpolarized light is emitted by a backlight. This light passes through a vertical polarizer. The polarized light then passes through a transparent layer of conducting pixel pads. These pads are designed to put varying amounts of voltage across the next layer, which is composed of liquid crystals, with respect to the transparent common electrode. If no voltage is applied across the liquid crystals, they rotate the polarization of the incident light by 90°. This light with rotated polarization can then pass through the transparent common electrode, the color filter, the horizontal polarizer, and the screen cover. When voltage of varying magnitude is applied to the pixel pad, the liquid crystals rotate the polarization of the incident light by a varying amount. When the full voltage is applied to the pixel pad, the polarization of the incident light is not rotated, and the horizontal polarizer blocks any light transmitted through the transparent common electrode and the color filter.

Figure 11.25b shows a front view of a small segment of the LCD screen, illustrating how the screen produces an image. The image is created by an array of pixels. Each pixel is subdivided into three subpixels: one red, one green, and one blue. By varying the voltage across each subpixel, a superposition of red, green, and blue light is created, producing a color on that pixel. It is difficult to connect a single wire to each subpixel, however. A high-definition 1080p LCD screen has 1080 times 1920 times 3 subpixels,

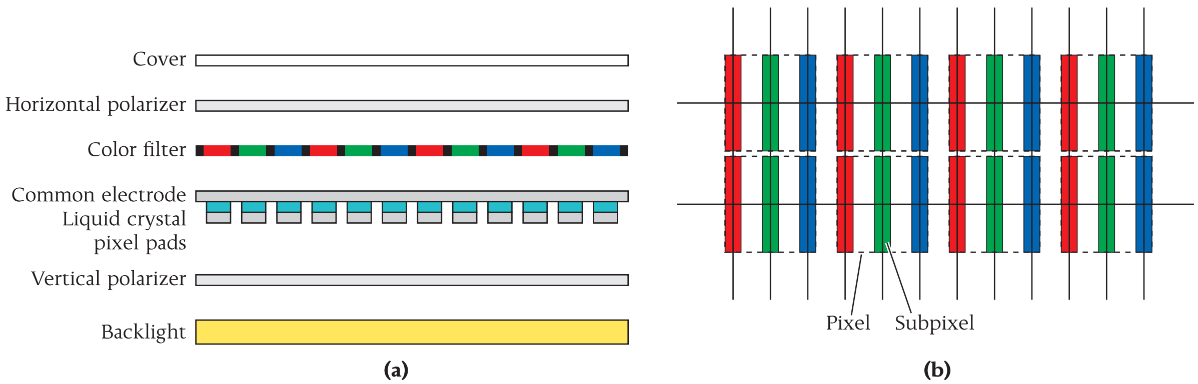


FIGURE 11.25 (a) Top view of the layers making up an LCD screen. (b) Front view of a subset of pixels and subpixels on an LCD screen.

or 6,220,800 subpixels. These subpixels are connected in columns and rows, as shown in Figure 11.25b. To turn on a subpixel, a voltage from both a column and a row must be applied. Thus, the subpixels are turned on one row at a time. With the voltage for one row on, the voltages for the subpixels in the desired columns are turned on. A small capacitor holds the voltage until the row is turned on again.

A high-definition 1080p LCD screen is scanned 60 times a second, producing a complete image in each scan. On a high-definition 1080i screen, every other row of the image is scanned 60 times a second and then the two images are interlaced. Another high-definition standard is 720p, which scans 720 rows 60 times a second with a horizontal resolution of 1280 pixels. The 720p and 1080i standards are in common use in television broadcasting. The standard resolution for a television image is 480i, with every other row being updated 60 times a second and producing 640 columns of pixels.

Viewing a 3D movie also involves the use of polarization filters. Moviegoers are equipped with glasses that have different polarization filters built into each lens. The projection equipment generates two different images with different polarizations on the screen. These images are also slightly offset from each other, and the viewer's brain constructs the 3D illusion by combining the two offset images. Older projection systems use linear polarization filters with polarization directions perpendicular to each other. However, this type of system yields diminished 3D effects when viewers tilt their heads sideways.

Modern 3D movie projection equipment uses circular polarization filters. Circular polarized light comes in two orthogonal varieties, left-circular and right-circular. Circular polarization filters work just like linear polarization filters, subject to a corresponding version of the Law of Malus. The 3D viewing glasses used with this modern equipment have one lens that passes left-circular and one lens that passes right-circular polarized light. This type of glasses produces 3D effects that are not affected when viewers tilt their heads.

There is one important difference between linearly polarized and circular polarized light: Linear polarized light remains in the same state of polarization after being reflected by a mirror, whereas circular polarized light changes its state from left-circular polarized to right-circular polarized, or vice versa. Figure 11.26 shows an interesting experiment you can do yourself: Hold a pair of 3D viewing glasses with circular polarization in front of a mirror and take a picture through one of the lenses. You can see that the light passing through the left lens, reflecting off the mirror, and then passing through the same lens again gets totally attenuated. In contrast, the light passing through the left lens, reflecting off the mirror, and then passing through the right lens is transmitted without being attenuated. If 3D glasses with linear polarization filters were used for this experiment instead, the light would pass through the lenses at half of its original intensity (equation 11.30), and the entire image of the camera phone could be seen in the mirror.

FIGURE 11.26 A camera phone takes a picture of its reflection in a mirror through 3D viewing glasses with circular polarization filters. One lens of the glasses polarizes light in a left-circular fashion, and the other polarizes it in a right-circular fashion.

11.7 Derivation of the Wave Equation

Table 11.1 lists the four equations known as Maxwell's equations in integral form. There are also equivalent differential versions of these equations, which is the way they are usually printed on T-shirts and posters:

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0},$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B},$$

$$\vec{\nabla} \bullet \vec{B} = 0,$$

and

$$\vec{\nabla} \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\vec{E} + \mu_0 \vec{j},$$

where ρ is the charge density (charge q per unit volume) and $\vec{j}$ is the current density. In vacuum and in the absence of charges, both are zero; $\rho = 0$ and $\vec{j} = 0$. The symbol $\vec{\nabla}$ is the gradient operator, which represents the vector with the partial derivatives in each spatial direction. In Cartesian coordinates, it is $\vec{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

In Section 11.2, we saw that electromagnetic waves as described by equation 11.8 are valid solutions to all the Maxwell equations in vacuum. However, strictly speaking, we have not yet written the wave equation that the electric field and the magnetic field obey. Now, with the aid of the differential form of Maxwell's equations, we can derive the wave equation for the electric field, which is

$$\frac{\partial^2}{\partial t^2}\vec{E} - c^2 \nabla^2 \vec{E} = 0. \tag{11.32}$$

The wave equation for the magnetic field is

$$\frac{\partial^2}{\partial t^2}\vec{B} - c^2 \nabla^2 \vec{B} = 0. \tag{11.33}$$

Self-Test Opportunity 11.4

Derive the wave equation for the magnetic field (equation 11.33) in the same way as Derivation 11.1 handles the wave equation for the electric field.

Self-Test Opportunity 11.5

Show that $\vec{E}(\vec{r},t) = E_{max} \sin(\kappa x - \omega t)\hat{y}$ and $\vec{B}(\vec{r},t) = B_{max} \sin(\kappa x - \omega t)\hat{z}$ are indeed solutions of the wave equation for the electric and magnetic fields.

DERIVATION 11.1 Wave Equation for the Electric Field in Vacuum

To derive the wave equation for the electric field in vacuum, we take the vector product of the second Maxwell equation and the gradient operator, $\vec{\nabla}$:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial}{\partial t}\vec{B}. \tag{i}$$

On the right-hand side of equation (i), we can interchange the order of the time derivative and the spatial derivative:

$$-\vec{\nabla} \times \frac{\partial}{\partial t}\vec{B} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t}\left(\varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right) = -\varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\vec{E}. \tag{ii}$$

The second step of this transformation makes use of the third Maxwell equation with $\vec{j} = 0$, which is appropriate in vacuum. The left-hand side of equation (i) is a double vector product. Here is the BAC-CAB rule for double vector products: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \bullet \vec{C}) - \vec{C}(\vec{A} \bullet \vec{B})$. Applying this rule, we find

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \bullet \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}, \tag{iii}$$

where the first Maxwell equation $\left(\text{in vacuum: } \vec{\nabla} \bullet \vec{E} = 0\right)$ is used in the second step. The symbol ∇^2 is the scalar product of the gradient operator with itself: $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. If we substitute from equations (ii) and (iii) into equation (i) and use the fact that the speed of light is $c = 1/\sqrt{\mu_0 \varepsilon_0}$ (equation 11.20) we obtain the desired wave equation:

$$\frac{\partial^2}{\partial t^2}\vec{E} - \frac{1}{\varepsilon_0 \mu_0}\nabla^2 \vec{E} = \frac{\partial^2}{\partial t^2}\vec{E} - c^2 \nabla^2 \vec{E} = 0.$$

This implies that electromagnetic waves moving at the speed of light are indeed a solution of Maxwell's equations, as discussed (but not exactly proven) in Section 11.2.

WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- When a capacitor is being charged, a displacement current can be visualized between the plates, given by $i_d = \varepsilon_0 d\Phi_E/dt$, where Φ_E is the electric flux.
- Maxwell's equations describe how electrical charges, currents, electric fields, and magnetic fields affect each other, forming a unified theory of electromagnetism.
  - Gauss's Law for Electric Fields, $\oiint \vec{E} \cdot d\vec{A} = q_{enc}/\varepsilon_0$, relates the net electric flux through a closed surface to the net enclosed electric charge.
  - Gauss's Law for Magnetic Fields, $\oiint \vec{B} \cdot d\vec{A} = 0$, states that the net magnetic flux through any closed surface is zero.
  - Faraday's Law of Induction, $\oint \vec{E} \cdot d\vec{s} = - d\Phi_B/dt$, relates the induced electric field to the changing magnetic flux.
  - The Maxwell-Ampere Law, $\oint \vec{B} \cdot d\vec{s} = \mu_0\varepsilon_0 \; d\Phi_E/dt + \mu_0 i_{enc}$, relates the induced magnetic field to the changing electric flux and to the current.
- For an electromagnetic wave traveling in the positive x-direction, the electric and magnetic fields can be described by $\vec{E}(\vec{r},t) = E_{max} \sin(\kappa x - \omega t)\hat{y}$ and $\vec{B}(\vec{r},t) = B_{max} \sin(\kappa x - \omega t)\hat{z}$, where $\kappa = 2\pi/\lambda$ is the wave number and $\omega = 2\pi f$ is the angular frequency.
- The magnitudes of the electric and magnetic fields of an electromagnetic wave at any fixed time and place are related by the speed of light, $E = cB$.
- The speed of light can be related to the two basic electromagnetic constants: $c = 1/\sqrt{\mu_0\varepsilon_0}$.
- The instantaneous power per unit area carried by an electromagnetic wave is the magnitude of the Poynting vector, $S = [1/(c\mu_0)]E^2$, where E is the magnitude of the electric field.
- The intensity of an electromagnetic wave is defined as the average power per unit area carried by the wave, $I = S_{ave} = [1/(c\mu_0)]E_{rms}^2$, where E_{rms} is the root-mean-square magnitude of the electric field.
- For an electromagnetic wave, the energy density carried by the electric field is $u_E = \frac{1}{2}\varepsilon_0 E^2$, and the energy density carried by the magnetic field is $u_B = [1/(2\mu_0)]B^2$. For any such wave, $u_E = u_B$.
- The radiation pressure exerted by electromagnetic waves of intensity I is given by $p_r = I/c$ if the electromagnetic waves are totally absorbed or by $p_r = 2I/c$ if the waves are perfectly reflected.
- The polarization of an electromagnetic wave is given by the direction of the electric field vector.
- The intensity of unpolarized light that has passed through a polarizer is $I = I_0/2$, where I_0 is the intensity of the unpolarized light incident on the polarizer.
- The intensity of polarized light that has passed through a polarizer is $I = I_0 \cos^2\theta$, where I_0 is the intensity of the polarized light incident on the polarizer and θ is the angle between the polarization of the incident polarized light and the polarizing angle of the polarizer.

ANSWERS TO SELF-TEST OPPORTUNITIES

11.1 $t = \dfrac{d}{c} = \dfrac{8.30\times10^{16}\ \text{m}}{3.00\times10^{8}\ \text{m/s}} = 2.77\times10^{8}\ \text{s} = 8.77\ \text{yr.}$

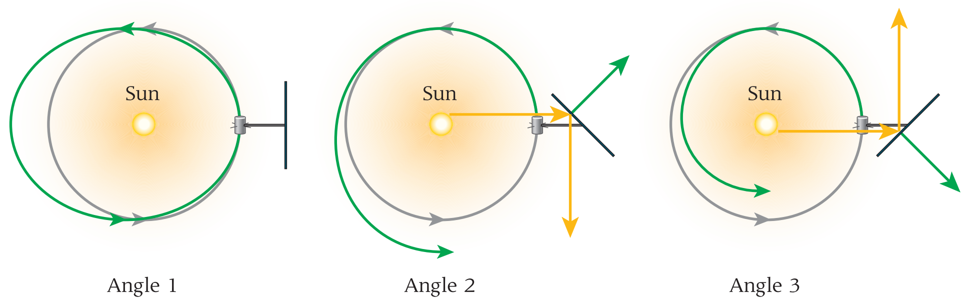
11.2 $c = \lambda f \Rightarrow \lambda = \dfrac{c}{f}$

$$\lambda_{FM} = \frac{3.00\times10^{8}\ \text{m}}{90.5\times10^{6}\ \text{Hz}} = 3.31\ \text{m}$$

$$\lambda_{AM} = \frac{3.00\times10^{8}\ \text{m}}{870\times10^{3}\ \text{Hz}} = 345\ \text{m.}$$

11.3 Deployment angle 1 will produce an elliptical orbit with the Sun at one focus. The force from radiation pressure depends on the inverse square of the distance, just as the force of gravity does. Thus, the orbit will become an ellipse, just as if the mass of the Sun or the mass of the object were suddenly reduced slightly. Because the force is perpendicular to the velocity of the satellite, the energy of the satellite is not affected.

Deployment angle 2 will result in a growing orbit. The resulting force from the reflected light produces a component of force that is in the same direction as the velocity of the spacecraft. Thus, the spacecraft gains energy, and the radius of the orbit increases. Note that the speed of the spacecraft decreases but its total energy increases.



Deployment angle 3 will result in a shrinking orbit. The resulting force from the reflected light produces a component of force that is in the opposite direction from the velocity of the spacecraft. Thus, the spacecraft loses energy, and the radius of the orbit decreases. Note that the speed of the spacecraft increases but its total energy decreases.

11.4 Take the vector product of the gradient operator $\vec{\nabla}$ and the fourth Maxwell equation: $\vec{\nabla}\times\vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\vec{\nabla}\times\frac{\partial}{\partial t}\vec{E}$.

The right-hand side of this equation is

$$\varepsilon_0\mu_0\vec{\nabla}\times\frac{\partial}{\partial t}\vec{E}=\varepsilon_0\mu_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})=\varepsilon_0\mu_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)=-\varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{B}.$$

The left-hand side is $\vec{\nabla}\times\vec{\nabla}\times\vec{B}=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B}=-\nabla^2\vec{B}$.

11.5 $\frac{\partial^2}{\partial t^2}\sin(\kappa x-\omega t)=-\omega^2\sin(\kappa x-\omega t)$

and

$$\frac{\partial^2}{\partial x^2}\sin(\kappa x-\omega t)=-\kappa^2\sin(\kappa x-\omega t).$$

Thus, this function is a solution for $c=\omega/\kappa$.

PROBLEM-SOLVING GUIDELINES

1. The same basic relationships that characterize any waves apply to electromagnetic waves. Remember that $c=\lambda f$ and $\omega=c\kappa$, where c is the speed of an electromagnetic wave.

2. It is often helpful to draw a diagram showing the direction of the wave motion and the orientation of both the electric and magnetic fields. Remember the relationships between $\vec{E}$ and $\vec{B}$ for both magnitude and direction, including $E/B=(\mu_0\varepsilon_0)^{-1/2}=c$ for electromagnetic waves.

MULTIPLE-CHOICE QUESTIONS

11.1 Which of the following phenomena can be observed for electromagnetic waves but not for sound waves?

a) interference
b) diffraction
c) polarization
d) absorption
e) scattering

11.2 Which of the following statements concerning electromagnetic waves are incorrect? (Select all that apply.)

a) Electromagnetic waves in vacuum travel at the speed of light.

b) The magnitudes of the electric field and the magnetic field are equal.

c) Only the electric field vector is perpendicular to the direction of the wave's propagation.

d) Both the electric field vector and the magnetic field vector are perpendicular to the direction of propagation.

e) An electromagnetic wave carries energy only when $E=B$.

11.3 The international radio station Voice of Slobbovia announces that it is "transmitting to North America on the 49-meter band." Which frequency is the station transmitting on?

a) 820 kHz
b) 6.12 MHz
c) 91.7 MHz
d) The information given tells nothing about the frequency.

11.4 Which of the following exerts the largest amount of radiation pressure?

a) a 1-mW laser pointer on a 2-mm-diameter spot 1 m away

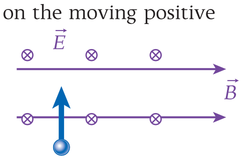
b) a 200-W light bulb on a 4-mm-diameter spot 10 m away

c) a 100-W light bulb on a 2-mm-diameter spot 4 m away

d) a 200-W light bulb on a 2-mm-diameter spot 5 m away

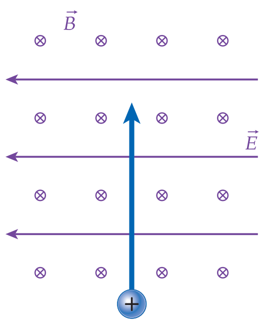
e) All of the above exert the same pressure.

11.5 What is the direction of the net force on the moving positive charge in the figure?



a) into the page
b) toward the right
c) out of the page
d) toward the left

11.6 A proton moves perpendicularly to crossed electric and magnetic fields as shown in the figure. What is the direction of the net force on the proton?



a) toward the left
b) toward the right
c) into the page
d) out of the page

11.7 It is speculated that isolated magnetic "charges" (magnetic monopoles) may exist somewhere in the universe. Which of Maxwell's equations, (1) Gauss's Law for Electric Fields, (2) Gauss's Law for Magnetic Fields, (3) Faraday's Law of Induction, and/or (4) the Maxwell-Ampere Law, would be altered by the existence of magnetic monopoles?

a) only (2)
b) (1) and (2)
c) (2) and (3)
d) only (3)

11.8 According to Gauss's Law for Magnetic Fields, all magnetic field lines form a complete loop. Therefore, the direction of the magnetic field $\vec{B}$ points from ____ pole to ____ pole outside of an ordinary bar magnet and from ____ pole to ____ pole inside the magnet.

a) north, south, north, south
b) north, south, south, north
c) south, north, south, north
d) south, north, north, south

11.9 Unpolarized light with intensity $I_{in}=1.87\ \mathrm{W/m^2}$ passes through two polarizers. The emerging polarized light has intensity $I_{out}=0.383\ \mathrm{W/m^2}$. What is the angle between the two polarizers?

a) 23.9°
b) 34.6°
c) 50.2°
d) 72.7°
e) 88.9°

11.10 The average intensity of sunlight at the Earth's surface is approximately $1400\ \mathrm{W/m^2}$, if the Sun is directly overhead. The average distance between the Earth and the Sun is 1.50×10^{11} m. What is the average power emitted by the Sun?

a) 99.9×10^{25} W
b) 4.0×10^{26} W
c) 6.3×10^{27} W
d) 4.3×10^{28} W
e) 5.9×10^{29} W

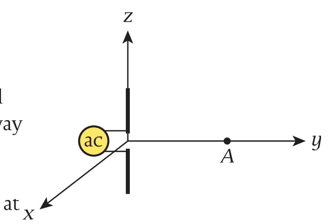
CONCEPTUAL QUESTIONS

11.11 In a polarized light experiment, a setup similar to the one in Figure 11.23 is used. Unpolarized light with intensity I_0 is incident on polarizer 1. Polarizers 1 and 3 are crossed (at a 90° angle), and their orientations are fixed during the experiment. Initially, polarizer 2 has its polarizing angle at 45°. Then, at time $t = 0$, polarizer 2 starts to rotate with angular velocity ω about the direction of propagation of light in a clockwise direction as viewed by an observer looking toward the light source. A photodiode is used to monitor the intensity of the light emerging from polarizer 3.

a) Determine an expression for this intensity as a function of time.

b) How would the expression from part (a) change if polarizer 2 were rotated about an axis parallel to the direction of propagation of the light but displaced by a distance $d < R$, where R is the radius of the polarizer?

11.12 A dipole antenna is located at the origin with its axis along the z-axis. As electric current oscillates up and down the antenna, polarized electromagnetic radiation travels away from the antenna along the positive y-axis. What are the possible directions of electric and magnetic fields at point A on the y-axis? Explain.



11.13 Does the information in Section 11.6 affect the answer to Example 11.2 regarding the root-mean-square magnitude of electric field at the Earth's surface from the Sun?

11.14 Maxwell's equations predict that there are no magnetic monopoles. If these monopoles existed, how would the motion of charged particles change as they approached such a monopole?

11.15 If two communication signals were sent at the same time to the Moon, one via radio waves and one via visible light, which one would arrive at the Moon first?

11.16 Show that Ampere's Law is not necessarily consistent if the surface through which the flux is to be calculated is a closed surface, but that the Maxwell-Ampere Law always is. (Hence, Maxwell's introduction of his law of induction and the displacement current are not optional; they are logically necessary.) Show also that Faraday's Law of Induction does not suffer from this consistency problem.

11.17 Maxwell's equations and Newton's laws of motion are mutually inconsistent; the great edifice of classical physics is fatally flawed. Explain why.

11.18 Practically everyone who has studied the electromagnetic spectrum has wondered how the world would appear if we could see over a range of frequencies comparable to the ten octaves over which we can hear rather than the less than one octave over which we can see. (An octave refers to a factor of 2 in frequency.) But this is practically impossible. Why?

11.19 Isotropic electromagnetic waves expand uniformly outward in all directions in three dimensions. Electromagnetic waves from a small, isotropic source are not plane waves, which have constant maximum amplitudes.

a) How does the maximum amplitude of the electric field of radiation from a small, isotropic source vary with distance from the source?

b) Compare this with the electrostatic field of a point charge.

11.20 A pair of sunglasses is held in front of a flat-panel computer monitor (which is on) so that the lenses are always parallel to the display. As the lenses are rotated, it is noticed that the intensity of light coming from the display and passing through the lenses is varying. Why?

11.21 Two polarizing filters are crossed at 90°, so when light is shined from behind the pair of filters, no light passes through. A third filter is inserted between the two, initially aligned with one of them. Describe what happens as the intermediate filter is rotated through an angle of 360°.

EXERCISES

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

Section 11.1

11.22 An electric field of magnitude 200.0 V/m is directed perpendicular to a circular planar surface with radius 6.00 cm. If the electric field increases at a rate of 10.0 V/(m s), determine the magnitude and the direction of the magnetic field at a radial distance 10.0 cm away from the center of the circular area.

•11.23 A wire of radius 1.00 mm carries a current of 20.0 A. The wire is connected to a parallel plate capacitor with circular plates of radius $R = 4.00$ cm and a separation between the plates of $s = 2.00$ mm. What is the magnitude of the magnetic field due to the changing electric field at a point that is a radial distance of $r = 1.00$ cm from the center of the parallel plates? Neglect edge effects.

•11.24 The current flowing in a solenoid that is 20.0 cm long and has a radius of 2.00 cm and 500. turns decreases from 3.00 A to 1.00 A in 0.100 s. Determine the magnitude of the induced electric field inside the solenoid 1.00 cm from its center.

11.25 A parallel plate capacitor has air between disk-shaped plates of radius 4.00 mm that are coaxial and 1.00 mm apart. Charge is being accumulated on the plates of the capacitor. What is the displacement current between the plates at an instant when the rate of charge accumulation on the plates is 10.0 μC/s?

•11.26 A parallel plate capacitor has circular plates of radius 10.0 cm that are separated by a distance of 5.00 mm. The potential across the capacitor is increased at a constant rate of 1.20 kV/s. Determine the magnitude of the magnetic field between the plates at a distance $r = 4.00$ cm from the center.

•11.27 The voltage across a cylindrical conductor of radius r, length L, and resistance R varies with time. The time-varying voltage causes a time-varying current, i, to flow in the cylinder. Show that the displacement current equals $\varepsilon_0 \rho\, di/dt$, where ρ is the resistivity of the conductor.

Section 11.2

11.28 The amplitude of the electric field of an electromagnetic wave is 250. V/m. What is the amplitude of the magnetic field of the electromagnetic wave?

11.29 Determine the distance in centimeters that light can travel in vacuum during 1.00 ns.

11.30 How long does it take light to travel from the Moon to the Earth? From the Sun to the Earth? From Jupiter to the Earth?

11.31 Omar made a telephone call from his home telephone in New York to his brother stationed in Baghdad, about 10,000 km away, and the signal was carried on a telephone cable. The following day, Omar called his brother again from work using his cell phone, and the signal was transmitted via a satellite 36,000 km above the Earth's surface, halfway between New York and Baghdad. Estimate the time taken for the signals sent by (a) the telephone cable and (b) via the satellite to reach Baghdad, assuming that the signal speed in both cases is the same as the speed of light, c. Would there be a noticeable delay in either case?

•11.32 Electric and magnetic fields in many materials can be analyzed using the relationships for these fields in vacuum, but substituting relative values of the permittivity and the permeability, $\varepsilon = \kappa\varepsilon_0$ and $\mu = \kappa_m\mu_0$, for their vacuum values, where κ is the dielectric constant and κ_m the relative permeability of the material. Calculate the ratio of the speed of electromagnetic waves in vacuum to their speed in such a material.

Section 11.3

11.33 The wavelength range for visible light is 400 nm to 700 nm (see Figure 11.10) in air. What is the frequency range of visible light?

11.34 The antenna of a cell phone is a straight rod 8.0 cm long. Calculate the operating frequency of the signal from this phone, assuming that the antenna length is $\frac{1}{4}$ of the wavelength of the signal.

•11.35 Suppose an RLC circuit in resonance is used to produce a radio wave of wavelength 150 m. If the circuit has a 2.0-pF capacitor, what size inductor is used?

•11.36 Three FM radio stations covering the same geographical area broadcast at frequencies 91.1, 91.3, and 91.5 MHz, respectively. What is the maximum allowable wavelength width of the band-pass filter in a radio receiver such that the FM station 91.3 can be played free of interference from FM 91.1 or FM 91.5? Use $c = 3.00 \cdot 10^8$ m/s, and calculate the wavelength to an uncertainty of 1 mm.

Section 11.4

11.37 A monochromatic point source of light emits 1.5 W of electromagnetic power uniformly in all directions. Find the Poynting vector at a point situated at each of the following locations:

a) 0.30 m from the source

b) 0.32 m from the source

c) 1.00 m from the source

11.38 Consider an electron in a hydrogen atom, which is 0.050 nm from the proton in the nucleus.

a) What electric field does the electron experience?

b) In order to produce an electric field whose root-mean-square magnitude is the same as that of the field in part (a), what intensity must a laser light have?

11.39 A 3.00 kW carbon dioxide laser is used in laser welding. If the beam is 1.00 mm in diameter, what is the amplitude of the electric field in the beam?

11.40 Suppose that charges on a dipole antenna oscillate slowly at a rate of 1.00 cycle/s, and the antenna radiates electromagnetic waves in a region of space. If someone measured the time-varying magnetic field in the region and found its maximum to be 1.00 mT, what would be the maximum electric field, E, in the region, in units of volts per meter? What is the period of the charge oscillation? What is the magnitude of the Poynting vector?

11.41 Calculate the average value of the Poynting vector, S_{ave}, for an electromagnetic wave having an electric field of amplitude 100. V/m.

a) What is the average energy density of this wave in J/m^3?

b) How large is the amplitude of the magnetic field?

•11.42 The most intense beam of light that can propagate through dry air must have an electric field whose maximum amplitude is no greater than the breakdown value for air: $E_{\max}^{\text{air}} = 3.0\times10^6$ V/m, assuming that this value is unaffected by the frequency of the wave.

a) Calculate the maximum amplitude the magnetic field of this wave can have.

b) Calculate the intensity of this wave.

c) What happens to a wave more intense than this?

••11.43 A continuous-wave (cw) argon-ion laser beam has an average power of 10.0 W and a beam diameter of 1.00 mm. Assume that the intensity of the beam is the same throughout the cross section of the beam (which is not true, as the actual distribution of intensity is a Gaussian function).

a) Calculate the intensity of the laser beam. Compare this with the average intensity of sunlight at Earth's surface (1400. W/m^2).

b) Find the root-mean-square electric field in the laser beam.

c) Find the average value of the Poynting vector over time.

d) If the wavelength of the laser beam is 514.5 nm in vacuum, write an expression for the instantaneous Poynting vector, where the instantaneous Poynting vector is zero at $t = 0$ and $x = 0$.

e) Calculate the root-mean-square value of the magnetic field in the laser beam.

••11.44 A voltage, V, is applied across a cylindrical conductor of radius r, length L, and resistance R. As a result, a current, i, is flowing through the conductor, which gives rise to a magnetic field, B. The conductor is placed along the y-axis, and the current is flowing in the positive y-direction. Assume that the electric field is uniform throughout the conductor.

a) Find the magnitude and the direction of the Poynting vector at the surface of the conductor of the static electric and magnetic fields.

b) Show that $\int \vec{S} \cdot d\vec{A} = i^2 R$.

Section 11.5

11.45 Radiation from the Sun reaches the Earth at a rate of 1.40 kW/m^2 above the atmosphere and at a rate of 1.00 kW/m^2 on an ocean beach.

a) Calculate the maximum values of E and B above the atmosphere.

b) Find the pressure and the force exerted by the radiation on a person lying flat on the beach who has an area of 0.750 m^2 exposed to the Sun.

11.46 Scientists have proposed using the radiation pressure of sunlight for travel to other planets in the Solar System. If the intensity of the electromagnetic radiation produced by the Sun is about 1.40 kW/m^2 near the Earth, what size would a sail have to be to accelerate a spaceship with a mass of 10.0 metric tons at 1.00 m/s^2?

a) Assume that the sail absorbs all the incident radiation.

b) Assume that the sail perfectly reflects all the incident radiation.

11.47 A solar sail is a giant circle (with a radius $R = 10.0$ km) made of a material that is perfectly reflecting on one side and totally absorbing on the other side. In deep space, away from other sources of light, the cosmic microwave background will provide the primary source of radiation incident on the sail. Assuming that this radiation is that of an ideal black body at $T = 2.725$ K, calculate the net force on the sail due to its reflection and absorption. Also assume that any heat transferred to the sail will be conducted away, and that the photons are incident perpendicular to the surface of the sail.

•11.48 Two astronauts are at rest in outer space, one 20.0 m from the Space Shuttle and the other 40.0 m from the shuttle. Using a 100.0-W laser, the astronaut located 40.0 m away from the shuttle decides to propel the other astronaut toward the Space Shuttle. He focuses the laser on a piece of totally reflecting fabric on the other astronaut's space suit. If his total mass with equipment is 100.0 kg, how long will it take him to reach the Space Shuttle?

•11.49 A laser that produces a spot of light that is 1.00 mm in diameter is shone perpendicularly on the center of a thin, perfectly reflecting circular (2.00 mm in diameter) aluminum plate mounted vertically on a flat piece of cork that floats on the surface of the water in a large beaker. The mass of this "sailboat" is 0.100 g, and it travels 2.00 mm in 63.0 s. Assuming that the laser power is constant in the region where the sailboat is located during its motion, what is the power of the laser? (Neglect air resistance and the viscosity of water.)

•11.50 A tiny particle of density 2000. kg/m^3 is at the same distance from the Sun as the Earth is ($1.50 \cdot 10^{11}$ m). Assume that the particle is spherical and perfectly reflecting. What would its radius have to be for the outward radiation pressure on it to be 1.00% of the inward gravitational attraction of the Sun? (Take the Sun's mass to be $2.00 \cdot 10^{30}$ kg.)

•11.51 Silica aerogel, an extremely porous, thermally insulating material made of silica, has a density of 1.00 mg/cm^3. A thin circular slice of aerogel has a diameter of 2.00 mm and a thickness of 0.10 mm.

a) What is the weight of the aerogel slice (in newtons)?

b) What are the intensity and the radiation pressure of a 5.00-mW laser beam of diameter 2.00 mm on the sample?

c) How many 5.00-mW lasers with a beam diameter of 2.00 mm would be needed to make the slice float in the Earth's gravitational field? Use $g = 9.81$ m/s^2.

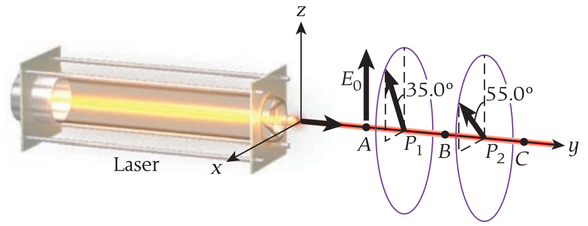
Section 11.6

11.52 Two polarizers are out of alignment by 30.0°. If light of intensity 1.00 W/m^2 and initially polarized halfway between the polarizing angles of the two filters passes through both filters, what is the intensity of the transmitted light?

11.53 A 10.0 mW vertically polarized laser beam passes through a polarizer whose polarizing angle is 30.0° from the horizontal. What is the power of the laser beam when it emerges from the polarizer?

•**11.54** Unpolarized light of intensity I_0 is incident on a series of five polarizers, with the polarization direction of each rotated 10.0° from that of the preceding one. What fraction of the incident light will pass through the series?

•**11.55** A laser produces light that is polarized in the vertical direction. The light travels in the positive *y*-direction and passes through two polarizers, which have polarizing angles of 35.0° and 55.0° from the vertical, as shown in the figure. The laser beam is collimated (neither converging nor expanding), has a circular cross section with a diameter of 1.00 mm, and has an average power of 15.0 mW at point *A*. At point *C*, what are the amplitudes of the electric and magnetic fields, and what is the intensity of the laser light?



Additional Exercises

11.56 A laser beam takes 50.0 ms to be reflected back from a totally reflecting sail on a spacecraft. How far away is the sail?

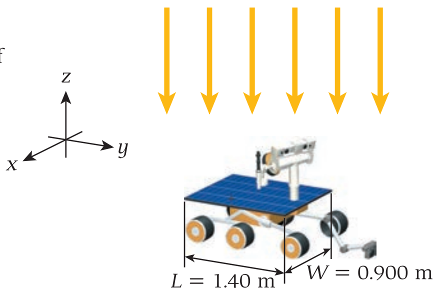
11.57 A house with a south-facing roof has photovoltaic panels on the roof. The photovoltaic panels have an efficiency of 10.0% and occupy an area with dimensions 3.00 m by 8.00 m. The average solar radiation incident on the panels is 300. W/m^2, averaged over all conditions for a year. How many kilowatt-hours of electricity will the solar panels generate in a 30-day month?

11.58 What is the radiation pressure due to Betelgeuse (which has a luminosity, or power output, 10,000 times that of the Sun) at a distance from it equal to that of Uranus's orbit from the Sun?

11.59 A 200.-W laser produces a beam with a cross-sectional area of 1.00 mm^2 and a wavelength of 628 m. What is the amplitude of the electric field in the beam?

11.60 What is the wavelength of the electromagnetic waves used for cell phone communications at 848.97 MHz?

11.61 As shown in the figure, sunlight is coming straight down (negative *z*-direction) on a solar panel (of length $L = 1.40$ m and width $W = 0.900$ m) on the Mars rover *Spirit*. The amplitude of the electric field in the solar radiation is 673 V/m and is uniform (the radiation has the same amplitude everywhere). If the solar panel converts solar radiation to electrical power with an efficiency of 18.0%, how much average power can the panel generate?



11.62 A 14.9μF capacitor, a 24.3 kΩ resistor, a switch, and a 25.0V battery are connected in series. What is the rate of change of the electric field between the plates of the capacitor at $t = 0.3621$ s after the switch is closed? The area of the plates is 1.00 cm^2.

11.63 A focused 300 W spotlight delivers 40% of its light within a circular area with a diameter of 2 m. What is the root-mean-square electric field in this illuminated area?

11.64 What is the electric field amplitude of an electromagnetic wave whose magnetic field amplitude is 5.00×10^{-3} T?

11.65 What is the distance between successive heating antinodes in a microwave oven's cavity? A microwave oven typically operates at a frequency of 2.4 GHz.

11.66 The solar constant measured by Earth satellites is roughly 1400 W/m^2.

a) Find the maximum electric field of the electromagnetic radiation from the Sun.

b) Find the maximum magnetic field of these electromagnetic waves.

•**11.67** The peak electric field at a distance of 2.25 m from a light bulb is 21.2 V/m.

a) What is the peak magnetic field there?

b) What is the power output of the bulb?

•**11.68** If the peak electric field due to a star whose radius is twice that of the Sun is 44.0 V/m at a distance of 15 AU, what is its temperature? Treat the star as a blackbody.

•**11.69** A 5.00 mW laser pointer has a beam diameter of 2.00 mm.

a) What is the root-mean-square value of the electric field in this laser beam?

b) Calculate the total electromagnetic energy in 1.00 m of this laser beam.

•**11.70** At the surface of the Earth, the Sun delivers an estimated 1.00 kW/m^2 of energy. Suppose sunlight hits a 10.0 m by 30.0 m roof at an angle of 90.0°.

a) Estimate the total power incident on the roof.

b) Find the radiation pressure on the roof.

•**11.71** The National Ignition Facility has the most powerful laser in the world; it uses 192 beams to aim 500. TW of power at a spherical pellet of diameter 2.00 mm. How fast would a pellet of density 2.00 g/cm^3 accelerate if only one of the laser beams hits it for 1.00 ns and 2.00% of the light is absorbed?

•**11.72** A resistor consists of a solid cylinder of radius *r* and length *L*. The resistor has resistance *R* and is carrying current *i*. Use the Poynting vector to calculate the power radiated out of the surface of the resistor.

•**11.73** A radio tower is transmitting 30.0 kW of power equally in all directions. Assume that the radio waves that hit the Earth are reflected.

a) What is the magnitude of the Poynting vector at a distance of 12.0 km from the tower?

b) What is the root-mean-square value of the electric force on an electron at this location?

•**11.74** Quantum theory says that electromagnetic waves actually consist of discrete packets–photons–each with energy $E = \hbar\omega$, where $\hbar = 1.054572\times10^{-34}$ J s is Planck's reduced constant and ω is the angular frequency of the wave.

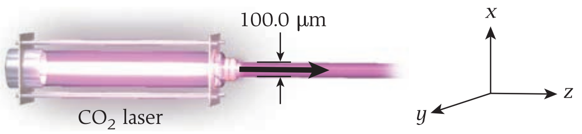
a) Find the momentum of a photon.

b) Find the magnitude of angular momentum of a photon. Photons are *circularly polarized;* that is, they are described by a superposition of two plane-polarized waves with equal field amplitudes, equal frequencies, and perpendicular polarizations, one-quarter of a cycle (90° or $\pi/2$ rad) out of phase, so the electric and magnetic field vectors at any fixed point rotate in a circle with the angular frequency of the waves. It can be shown that a circularly polarized wave of energy *U* and angular frequency v has an angular momentum of magnitude $L = U/\omega$. (The direction of the angular momentum is given by the thumb of the right hand, when the fingers are curled in the direction in which the field vectors circulate.)

c) The ratio of the angular momentum of a particle to $\hbar$ is its spin quantum number. Determine the spin quantum number of the photon.

•**11.75** A microwave operates at 250. W. Assuming that the waves emerge from a point source emitter on one side of the oven, how long does it take to melt an ice cube 2.00 cm on a side that is 10.0 cm away from the emitter if 10.0% of the photons that strike the cube are absorbed by it? How many photons of wavelength 10.0 cm hit the ice cube per second? Assume a cube density of 0.960 g/cm^3.

•**11.76** An industrial carbon dioxide laser produces a beam of radiation with average power of 6.00 kW at a wavelength of 10.6 μm. Such a laser can be used to cut steel up to 25 mm thick. The laser light is polarized in the x-direction, travels in the positive z-direction, and is collimated (neither diverging or converging) at a *constant* diameter of 100.0 μm. Write the equations for the laser light's electric and magnetic fields as a function of time and of position z along the beam. Recall that $\vec{E}$ and $\vec{B}$ are vectors. Leave the overall phase unspecified, but be sure to check the relative phase between $\vec{E}$ and $\vec{B}$.



MULTI-VERSION EXERCISES

11.77 During the testing of a new light bulb, a sensor is placed 11.9 cm from the bulb. It records an intensity of 182.9 W/m^2 for the radiation emitted by the bulb. What is the root-mean-square value of the electric field at the sensor's location?

11.78 During the testing of a new light bulb, a sensor is placed 42.1 cm from the bulb. It records an intensity of 191.4 W/m^2 for the radiation emitted by the bulb. What is the root-mean-square value of the magnetic field at the sensor's location?

11.79 During the testing of a new light bulb, a sensor is placed 52.5 cm from the bulb. It records a root-mean-square value of 9.142×10^{-7} T for the magnetic field of the radiation emitted by the bulb. What is the intensity of that radiation at the sensor's location?

11.80 During the testing of a new light bulb, a sensor is placed 17.7 cm from the bulb. It records a root-mean-square value of 279.9 V/m for the electric field of the radiation emitted from the bulb. What is the intensity of that radiation at the sensor's location?

11.81 To visually examine sunspots through a telescope, astronomers have to reduce the intensity of the sunlight to avoid harming their retinas. They accomplish this intensity reduction by mounting two polarizers on the telescope. The first polarizer has a polarizing angle of 28.1° relative to the horizontal, and the second has a polarizing angle of 88.6°. By what fraction is the intensity of the incident sunlight reduced by the polarizers?

11.82 To visually examine sunspots through a telescope, astronomers have to reduce the intensity of the sunlight to avoid harming their retinas. They accomplish this intensity reduction by mounting two polarizers on the telescope. The first polarizer has a polarizing angle of 38.3° relative to the horizontal. If the astronomers want to reduce the intensity of the sunlight by a factor of 0.7584, what polarizing angle should the second polarizer have with the horizontal? Assume that this angle is greater than that of the first polarizer.

11.83 To visually examine sunspots through a telescope, astronomers have to reduce the intensity of the sunlight to avoid harming their retinas. They accomplish this intensity reduction by mounting two linear polarizers on the telescope. The second polarizer has a polarizing angle of 110.6° relative to the horizontal. If the astronomers want to reduce the intensity of the sunlight by a factor of 0.7645, what polarizing angle should the first polarizer have with the horizontal? Assume that this angle is smaller than that of the second polarizer.