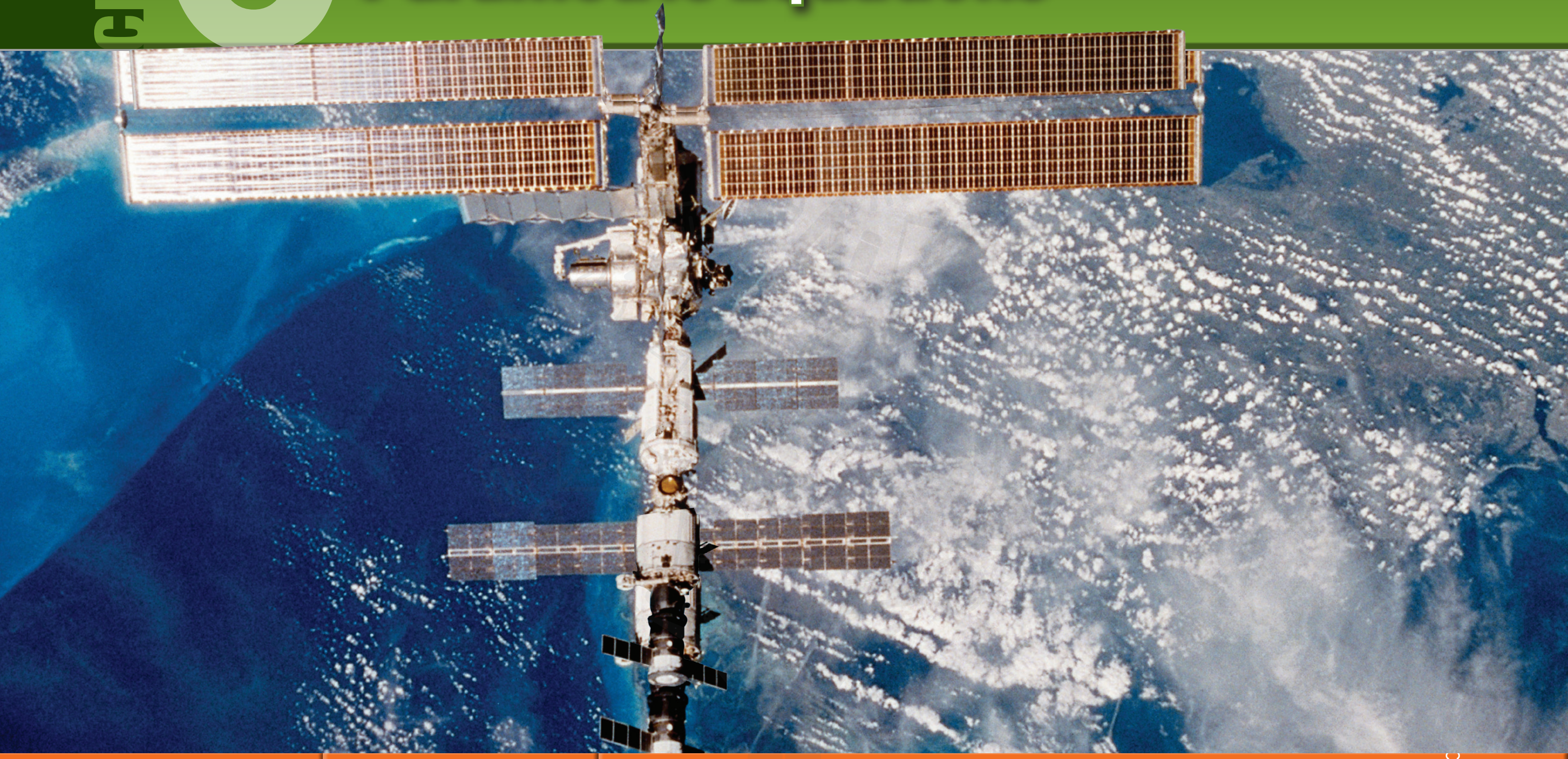
CHAPTER 6 Conic Sections and Parametric Equations



Then

You solved systems of linear equations algebraically and graphically.

Now

You will:

- Use the Midpoint and Distance Formulas.
- Write and graph equations of parabolas, circles, ellipses, and hyperbolas.
- Identify conic sections.
- Solve systems of quadratic equations and inequalities.

Why?

SPACE Conic sections are evident in many aspects of space. Equations of circles are used to pilot spacecraft and satellites in circular orbits around Earth and the Moon. Planets travel in elliptical paths, not circular ones as previously thought. Comets travel along one branch of a hyperbola, which can help us to predict when they will appear again.

Get Ready for the Chapter

1 Textbook Option Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Solve each equation by completing the square.

1. $x^2 + 8x + 7 = 0$

2. $x^2 + 5x - 6 = 0$

3. $x^2 - 8x + 15 = 0$

4. $x^2 + 2x - 120 = 0$

5. $2x^2 + 7x - 15 = 0$

6. $2x^2 + 3x - 5 = 0$

7. $x^2 - \frac{3}{2}x - \frac{23}{16} = 0$

8. $3x^2 - 4x = 2$

Solve each system of equations by using either substitution or elimination.

9. $y = x + 3$
$2x - y = -1$

10. $2x - 5y = -18$
$3x + 4y = 19$

11. $4y + 6x = -6$
$5y - x = 35$

12. $x = y - 8$
$4x + 2y = 4$

13. MONEY The student council paid AED 15 per registration for a conference. They also paid AED 10 for T-shirts for a total of AED 180. Last year, they spent AED 12 per registration and AED 9 per T-shirt for a total of AED 150 to buy the same number of registrations and T-shirts. Write and solve a system of two equations that represents the number of registrations and T-shirts bought each year.

QuickReview

Example 1

Solve $x^2 + 6x - 16 = 0$ by completing the square.

$x^2 + 6x = 16$

$x^2 + 6x + 9 = 16 + 9$

$(x + 3)^2 = 25$

$x + 3 = \pm 5$

$x + 3 = 5$ or $x + 3 = -5$

$x = 2$ $\quad$ $x = -8$

Example 2

Solve the system of equations algebraically.
$3y = x - 9$
$4x + 5y = 2$

Since x has a coefficient of 1 in the first equation, use the substitution method. First, solve that equation for x.

$3y = x - 9 \quad \rightarrow \quad x = 3y + 9$

$4(3y + 9) + 5y = 2$	Substitute $3y + 9$ for x.
$12y + 36 + 5y = 2$	Distributive Property
$17y = -34$	Combine like terms.
$y = -2$	Divide each side by 17.

To find x, use $y = -2$ in the first equation.

$3(-2) = x - 9$	Substitute -2 for y.
$-6 = x - 9$	Multiply.
$3 = x$	Add 9 to each side.

The solution is $(3, -2)$.

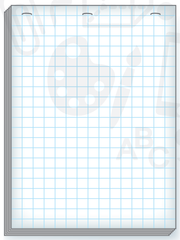
Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 6. To get ready, identify important terms and organize your resources.

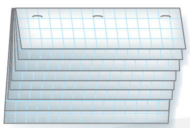
FOLDABLES® StudyOrganizer

Conic Sections Make this Foldable to help you organize your Chapter 6 notes about conic sections. Begin with eight sheets of grid paper.

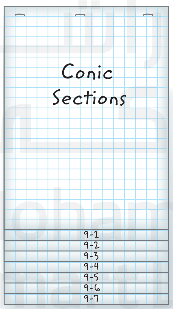
1 **Staple** the stack of grid paper along the top to form a booklet.



2 **Cut** seven lines from the bottom of the top sheet, six lines from the second sheet, and so on.



3 **Label** with lesson numbers as shown.



NewVocabulary

English

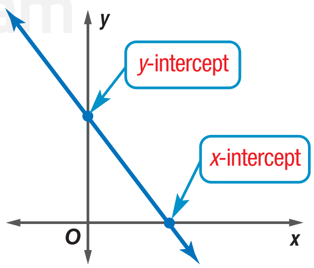
parabola
focus
directrix
circle
center of a circle
radius
ellipse
foci
major axis
minor axis
center of an ellipse
vertices
co-vertices
constant sum
hyperbola
transverse axis
conjugate axis
constant difference

ReviewVocabulary

quadratic equation an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$

system of equations a set of equations with the same variables

x- and y-intercepts the x- or y-coordinate of the point at which a graph crosses the x- or y-axis



LESSON 6-1

Parabolas

Then	Now	Why?
You graphed quadratic functions.	1 Write equations of parabolas in standard form. 2 Graph parabolas.	Satellite dishes can be used to send and receive signals and can be seen attached to residential homes and businesses. A satellite dish is a type of antenna constructed to receive signals from orbiting satellites. The signals are reflected off of the dish's parabolic surface to a common collection point.

NewVocabulary

parabola
focus
directrix
latus rectum
standard form
general form

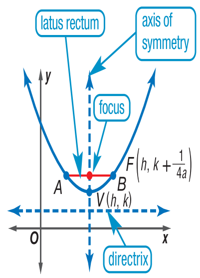
Mathematical Practices

1 Make sense of problems and persevere in solving them.

1 Equations of Parabolas

A **parabola** can be defined as the set of all points in a plane that are the same distance from a given point called the **focus** and a given line called the **directrix**.

The line segment through the focus of a parabola and perpendicular to the axis of symmetry is called the **latus rectum**. The endpoints of the latus rectum lie on the parabola.



KeyConcept Equations of Parabolas

Form of Equation	$y = a(x - h)^2 + k$	$x = a(y - k)^2 + h$
Direction of Opening	upward if $a > 0$, downward if $a < 0$	right if $a > 0$, left if $a < 0$
Vertex	(h, k)	(h, k)
Axis of Symmetry	$x = h$	$y = k$
Focus	$\left(h, k + \frac{1}{4a}\right)$	$\left(h + \frac{1}{4a}, k\right)$
Directrix	$y = k - \frac{1}{4a}$	$x = h - \frac{1}{4a}$
Length of Latus Rectum	$\left\lvert\frac{1}{a}\right\rvert$ units	$\left\lvert\frac{1}{a}\right\rvert$ units

The **standard form** of the equation of a parabola with vertex (h, k) and axis of symmetry $x = h$ is $y = a(x - h)^2 + k$.

- If $a > 0$, k is the minimum value of the related function and the parabola opens upward.
- If $a < 0$, k is the maximum value of the related function and the parabola opens downward.

An equation of a parabola in the form $y = ax^2 + bx + c$ is the **general form**. Any equation in general form can be written in standard form. The shape of a parabola and the distance between the focus and directrix depend on the value of a in the equation.

ReviewVocabulary

Completing the Square rewriting a quadratic expression as a perfect square trinomial

Example 1 Analyze the Equation of a Parabola

Write $y = 2x^2 - 12x + 6$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

$y = 2x^2 - 12x + 6$	Original equation
$= 2(x^2 - 6x) + 6$	Factor 2 from the x- and x^2-terms.
$= 2(x^2 - 6x + \blacksquare) + 6 - 2(\blacksquare)$	Complete the square on the right side.
$= 2(x^2 - 6x + 9) + 6 - 2(9)$	The 9 added when you complete the square is multiplied by 2.
$= 2(x - 3)^2 - 12$	Factor.

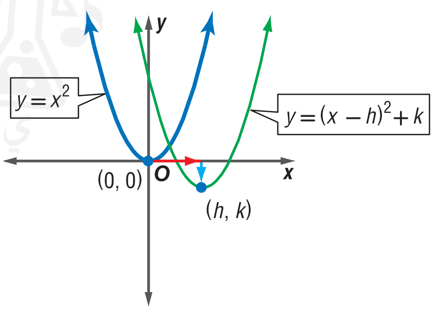
The vertex of this parabola is located at $(3, -12)$, and the equation of the axis of symmetry is $x = 3$. The parabola opens upward.

GuidedPractice

1. Write $y = 4x^2 + 16x + 34$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

2 Graph Parabolas

Previously you learned that the graph of the quadratic equation $y = a(x - h)^2 + k$ is a transformation of the parent graph of $y = x^2$ translated h units horizontally and k units vertically, and reflected and/or dilated depending on the value of a.



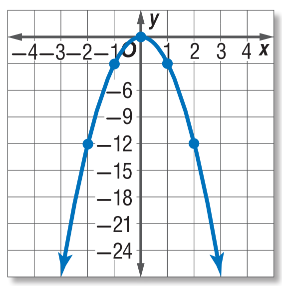
Example 2 Graph Parabolas

Graph each equation.

a. $y = -3x^2$

For this equation, $h = 0$ and $k = 0$. The vertex is at the origin. Since the equation of the axis of symmetry is $x = 0$, substitute some small positive integers for x and find the corresponding y-values.

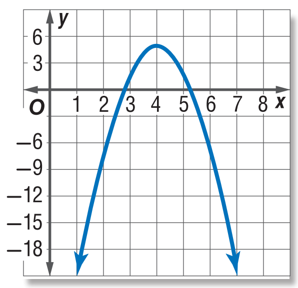
x	y
1	−3
2	−12
3	−27



Since the graph is symmetric about the y-axis, the points at $(-1, -3)$, $(-2, -12)$, and $(-3, -27)$ are also on the parabola. Use all of these points to draw the graph.

b. $y = -3(x - 4)^2 + 5$

The equation is of the form $y = a(x - h)^2 + k$, where $h = 4$ and $k = 5$. The graph of this equation is the graph of $y = -3x^2$ in part **a** translated 4 units to the right and up 5 units. The vertex is now at $(4, 5)$.



GuidedPractice

2A. $y = 2x^2$

2B. $y = 2(x - 1)^2 - 4$

WatchOut!

Structure Carefully examine the values for h and k before beginning to graph an equation.

- If h is positive, translate the graph h units to the right.
- If h is negative, translate the graph $|h|$ units to the left.
- If k is positive, translate the graph k units up.
- If k is negative, translate the graph $|k|$ units down.

StudyTip

Graphing When graphing these functions, it may be helpful to sketch the graph of the parent function.

Equations of parabolas with vertical axes of symmetry have the parent function $y = x^2$ and are of the form $y = a(x - h)^2 + k$. These are functions. Equations of parabolas with horizontal axes of symmetry are of the form $x = a(y - k)^2 + h$ and are not functions. The parent graph for these equations is $x = y^2$.

Example 3 Graph an Equation in General Form

Graph each equation.

a. $2x - y^2 = 4y + 10$

Step 1 Write the equation in the form $x = a(y - k)^2 + h$.

$2x - y^2 = 4y + 10$	Original equation
$2x = y^2 + 4y + 10$	Add y^2 to each side to isolate the x-term.
$2x = (y^2 + 4y + \blacksquare) + 10 - \blacksquare$	Complete the square.
$2x = (y^2 + 4y + 4) + 10 - 4$	Add and subtract 4, since $\left(\frac{4}{2}\right)^2 = 4$.
$2x = (y + 2)^2 + 6$	Factor and subtract.
$x = \frac{1}{2}(y + 2)^2 + 3$	$(h, k) = (3, -2)$

Step 2 Use the equation to find information about the graph. Then draw the graph based on the parent graph, $x = y^2$.

vertex: $(3, -2)$

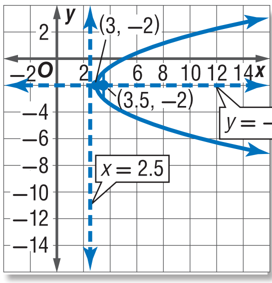
axis of symmetry: $y = -2$

focus: $\left(3 + \frac{1}{4\left(\frac{1}{2}\right)}, -2\right)$ or $(3.5, -2)$

directrix: $x = 3 - \frac{1}{4\left(\frac{1}{2}\right)}$ or 2.5

direction of opening: right, since $a > 0$

length of latus rectum: $\left|\frac{1}{\left(\frac{1}{2}\right)}\right|$ or 2 units



ReadingMath

latus rectum from the Latin *latus*, meaning side, and *rectum*, meaning straight

b. $y + 2x^2 + 32 = -16x - 1$

Step 1

$y + 2x^2 + 32 = -16x - 1$	Original equation
$y = -2x^2 - 16x - 33$	Solve for y.
$y = -2(x^2 + 8x + \blacksquare) - 33 - \blacksquare$	Complete the square.
$y = -2(x^2 + 8x + 16) - 33 - (-32)$	Add and subtract −32.
$y = -2(x + 4)^2 - 1$	Factor and simplify.

Step 2 vertex: $(-4, -1)$

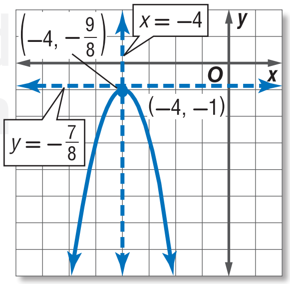
axis of symmetry: $x = -4$

focus: $\left(-4, -\frac{9}{8}\right)$

directrix: $y = -\frac{7}{8}$

length of latus rectum: $\frac{1}{2}$ unit

opens downward



GuidedPractice

3A. $3x - y^2 = 4x + 25$

3B. $y = x^2 + 6x - 4$

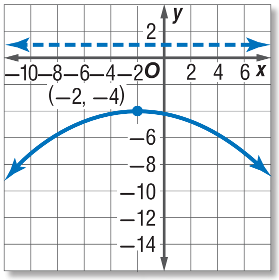
You can use specific information about a parabola to write an equation and draw a graph.

Example 4 Write an Equation of a Parabola

Write an equation for a parabola with vertex at $(-2, -4)$ and directrix $y = 1$. Then graph the equation.

The directrix is a horizontal line, so the equation of the parabola is of the form $y = a(x - h)^2 + k$. Find a, h, and k.

- The vertex is at $(-2, -4)$, so $h = -2$ and $k = -4$.
- Use the equation of the directrix to find a.



$y = k - \frac{1}{4a}$ Equation of directrix

$1 = -4 - \frac{1}{4a}$ Replace y with 1 and k with −4.

$5 = -\frac{1}{4a}$ Add 4 to each side.

$20a = -1$ Multiply each side by $4a$.

$a = -\frac{1}{20}$ Divide each side by 20.

So, the equation of the parabola is $y = -\frac{1}{20}(x + 2)^2 - 4$.

Guided Practice

Write an equation for each parabola described below. Then graph the equation.

4A. vertex (1, 3), focus (1, 5)

4B. focus (5, 6), directrix $x = -2$

Parabolas are often used in the real world.

Real-World Link

In California's Mojave Desert, parabolic mirrors are used to heat oil that flows through tubes placed at the focus. The heated oil is used to produce electricity.

Source: Solel

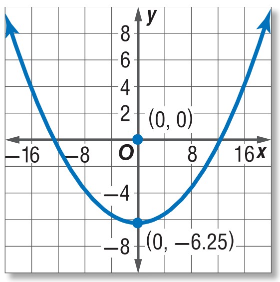
Real-World Example 5 Write an Equation for a Parabola

ENVIRONMENT **Solar energy may be harnessed by using parabolic mirrors. The mirrors reflect the rays from the Sun to the focus of the parabola. The focus of each parabolic mirror at the facility described at the left is 1.9 meters above the vertex. The latus rectum is 7.6 meters long.**

a. Assume that the focus is at the origin. Write an equation for the parabola formed by each mirror.

In order for the mirrors to collect the Sun's energy, the parabola must open upward. Therefore, the vertex must be below the focus.

focus: (0, 0) vertex: $(0, -1.9)$



The measure of the latus rectum is 7.6. So $7.6 = \left|\frac{1}{a}\right|$, and $a = \frac{1}{7.6}$.

Using the form $y = a(x - h)^2 + k$, an equation for the parabola formed by each mirror is $y = \frac{1}{7.6}x^2 - 1.9$.

b. Graph the equation.

Now use all of the information to draw a graph.

Guided Practice

5. Write and graph an equation for a parabolic mirror that has a focus 1.4 meters above the vertex and a latus rectum that is 5.5 meters long, when the focus is at the origin.

Check Your Understanding

Example 1 **Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.**

1. $y = 2x^2 - 24x + 40$
2. $y = 3x^2 - 6x - 4$
3. $x = y^2 - 8y - 11$
4. $x + 3y^2 + 12y = 18$

Examples 2–3 **Graph each equation.**

5. $y = (x - 4)^2 - 6$
6. $y = 4(x + 5)^2 + 3$
7. $y = -3x^2 - 4x - 8$
8. $x = 3y^2 - 6y + 9$

Example 4 **Write an equation for each parabola described below. Then graph the equation.**

9. vertex (0, 2), focus (0, 4)
10. vertex (−2, 4), directrix $x = -1$
11. focus (3, 2), directrix $y = 8$
12. vertex (−1, −5), focus (−5, −5)

Example 5

13. **ASTRONOMY** Consider a parabolic mercury mirror like the one described at the beginning of the lesson. The focus is 1.8 meters above the vertex and the latus rectum is 7.3 meters long.
    a. Assume that the focus is at the origin. Write an equation for the parabola formed by the parabolic microphone.
    b. Graph the equation.

Practice and Problem Solving

Example 1 **Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.**

14. $y = x^2 - 8x + 13$
15. $y = 3x^2 + 42x + 149$
16. $y = -6x^2 - 36x - 8$
17. $y = -3x^2 - 9x - 6$
18. $x = \frac{1}{3}y^2 - 3y + 4$
19. $x = \frac{2}{3}y^2 - 4y + 12$

Examples 2–3 **Graph each equation.**

20. $y = \frac{1}{3}x^2$
21. $y = -2x^2$
22. $y = -2(x - 2)^2 + 3$
23. $y = 3(x - 3)^2 - 5$
24. $x = \frac{1}{2}y^2$
25. $4x - y^2 = 2y + 13$

Example 4 **Write an equation for each parabola described below. Then graph the equation.**

26. vertex (0, 1), focus (0, 4)
27. vertex (1, 8), directrix $y = 3$
28. focus (−2, −4), directrix $x = -6$
29. focus (2, 4), directrix $x = 10$
30. vertex (−6, 0), directrix $x = 2$
31. vertex (9, 6), focus (9, 5)

Example 5

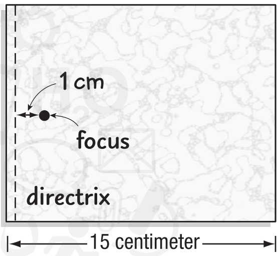
32. **BASEBALL** When a ball is thrown, the path it travels is a parabola. Suppose a baseball is thrown from ground level, reaches a maximum height of 15.2 meters, and hits the ground 61 meters from where it was thrown. Assuming this situation could be modeled on a coordinate plane with the focus of the parabola at the origin, find the equation of the parabolic path of the ball. Assume the focus is on ground level.

33. **PERSEVERANCE** Ground antennas and satellites are used to relay signals between the NASA Mission Operations Center and the spacecraft it controls. One such parabolic dish is 146 feet in diameter. Its focus is 48 feet from the vertex.
    a. Sketch two options for the dish, one that opens up and one that opens left.
    b. Write two equations that model the sketches in part **a**.
    c. If you wanted to find the depth of the dish, does it matter which equation you use? Why or why not?

34. **UMBRELLAS** A beach umbrella has an arch in the shape of a parabola that opens downward. The umbrella spans 1.8 meters across and is 0.45 meters high. Write an equation of a parabola to model the arch, assuming that the origin is at the point where the pole and umbrella meet at the vertex of the arch.

35. **AUTOMOBILES** An automobile headlight contains a parabolic reflector. The light coming from the source bounces off the parabolic reflector and shines out the front of the headlight. The equation of the cross section of the reflector is $y = \frac{1}{12}x^2$. How far from the vertex should the filament for the high beams be placed?

36. **MULTIPLE REPRESENTATIONS** Start with a sheet of wax paper that is about 15 centimeters long and 12 centimeters wide.



a. **Concrete** Make a line that is perpendicular to the sides of the sheet by folding the sheet near one end. Open up the paper again. This line is the directrix. Mark a point about midway between the sides of the sheet so that the distance from the directrix is about 1 inch. This is the focus.

b. **Concrete** Start with a new sheet of wax paper. Form another outline of a parabola with a focus that is about 3 centimeter from the directrix.

c. **Concrete** On a new sheet of a wax paper, form a third outline of a parabola with a focus that is about 5 centimeter from the directrix.

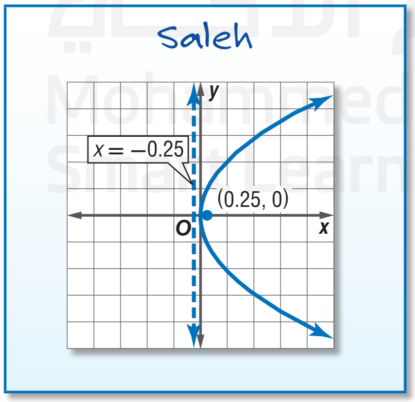
d. **Verbal** Compare the shapes of the three parabolas. How does the distance between the focus and the directrix affect the shape of a parabola?

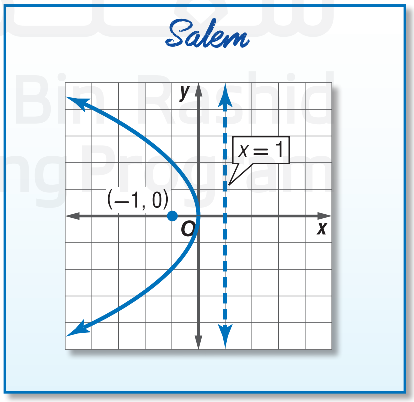
H.O.T. Problems Use Higher-Order Thinking Skills

37. **REASONING** How do you change the equation of the parent function $y = x^2$ to shift the graph to the right?

38. **OPEN ENDED** Two different parabolas have their vertex at $(-3, 1)$ and contain the point with coordinates $(-1, 0)$. Write two possible equations for these parabolas.

39. **CRITIQUE** Saleh and Salem are graphing $\frac{1}{4}y^2 + x = 0$. Is either of them correct? Explain your reasoning.





40. **WRITING IN MATH** Why are parabolic shapes used in the real world?

Standardized Test Practice

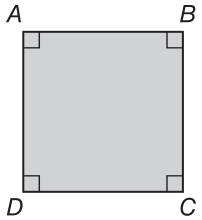
41. A gardener is placing a fence around a 1320-square-meter rectangular garden. He ordered 148 meters of fencing. If he uses all the fencing, what is the length of the longer side of the garden?

A 30 m
B 34 m
C 44 m
D 46 m

42. SAT/ACT When a number is divided by 5, the result is 7 more than the number. Find the number.

F $-\frac{35}{4}$
G $-\frac{35}{6}$
H $\frac{35}{6}$
J $\frac{28}{4}$
K $\frac{35}{4}$

43. GEOMETRY What is the area of the following square, if the length of $\overline{BD}$ is $2\sqrt{2}$?



A 1
B 2
C 3
D 4

44. SHORT RESPONSE The measure of the smallest angle of a triangle is two thirds the measure of the middle angle. The measure of the middle angle is three sevenths of the measure of the largest angle. Find the largest angle's measure.

Spiral Review

45. GEOMETRY Find the perimeter of a triangle with vertices at (2, 4), (−1, 3), and (1, −3). (Lesson 6-1)

46. WORK A worker can powerwash a wall of a certain size in 5 hours. Another worker can do the same job in 4 hours. If the workers work together, how long would it take to do the job? Determine whether your answer is reasonable.

Solve each equation or inequality. Round to the nearest ten-thousandth.

47. $\ln(x + 1) = 1$

48. $\ln(x - 7) = 2$

49. $e^x > 1.6$

50. $e^{5x} \geq 25$

Simplify.

51. $\sqrt{0.25}$

52. $\sqrt[3]{-0.064}$

53. $\sqrt[4]{z^8}$

54. $-\sqrt[6]{x^6}$

List all of the possible rational zeros of each function.

55. $h(x) = x^3 + 8x + 6$

56. $p(x) = 3x^3 - 5x^2 - 11x + 3$

57. $h(x) = 9x^6 - 5x^3 + 27$

Skills Review

Simplify each expression.

58. $\sqrt{24}$

59. $\sqrt{45}$

60. $\sqrt{252}$

61. $\sqrt{512}$

EXPLORE 6-2

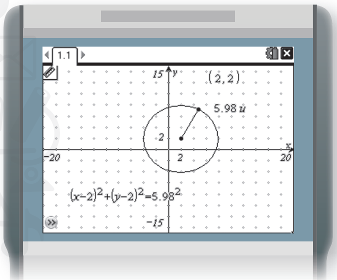
Graphing Technology Lab
Equations of Circles

You can use graphing technology to examine characteristics of circles and the relationship with an equation of the circle.

Activity

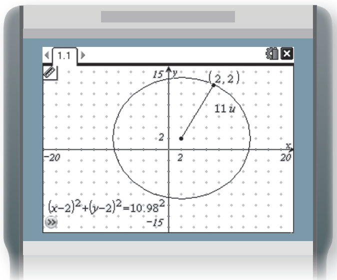
Step 1 Draw a circle.

- Add a new **Graphs** page. Select **Window/Zoom** menu and use the **Windows Setting** tool to adjust the window size as shown. From the **View** menu, select **Show Grid**. Then from the **Shapes** menu, select **Circle**. Place the pointer at the point (2, 2) and press **enter** to set the center of the circle. Move the pointer out, creating a circle like the one shown.
- Use the **Point On** tool from the **Points & Line** menu to place a point on the circle.
- Use the **Segment** tool from the **Points & Line** menu to draw the radius.



Step 2 Add labels.

- From the **Actions** menu, select **Coordinates and Equations**. Use the pointer to select the center of the circle to display its coordinate. Then select the circle to display its equation. Move each display outside the circle.
- Use the **Length** tool from the **Measurement** menu to display the length of the radius.



Step 3 Change the radius.

Move the pointer so that a point on the circle is highlighted, then press and hold the center of the touchpad until it is selected. Examine the equation of the circle. Then move the edge of the circle in. Make note of changes in the equation.

Step 4 Move the center of the circle.

Move the pointer so that the center of the circle is highlighted, then press and hold the center of the touchpad until it is selected. Move the center of the circle. Again, examine the equation of the circle.

Analyze the Results

1. How does moving the edge of the circle in or out affect the equation of the circle?
2. What effect does moving the center of the circle have on the equation?
3. Repeat the activity by placing the center of a circle in Quadrant II. Move the center to each of the other two quadrants. How does the equation change?
4. **MAKE A CONJECTURE** Without graphing, write an equation of each circle.
   a. center: (4, 2), radius: 3
   b. center: (−1, 1), radius: 8
   c. center: (−6, −5), radius: 2.5
   d. center: (h, k), radius: r

LESSON 6-2 Circles

Then

- You graphed and wrote equations of parabolas.

Now

1. Write equations of circles.
2. Graph circles.

Why?

- When an object is thrown into water, ripples move out from the center forming concentric circles. If the point where the object entered the water is assigned coordinates, each ripple can be modeled by an equation of a circle.

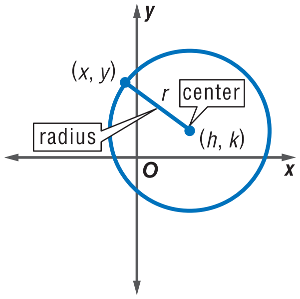
NewVocabulary
circle
center
radius

Mathematical Practices
4 Model with mathematics.

1 Equations of Circles

A **circle** is the set of all points in a plane that are equidistant from a given point in the plane, called the **center**. Any segment with endpoints at the center and a point on the circle is a **radius** of the circle.

Assume that (x, y) are the coordinates of a point on the circle at the right. The center is at (h, k), and the radius is r. You can find an equation of the circle by using the Distance Formula.



$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d$$ Distance Formula

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$ $(x_1, y_1) = (h, k)$, $(x_2, y_2) = (x, y)$, $d = r$

$$(x - h)^2 + (y - k)^2 = r^2$$ Square each side.

KeyConcept Equations of Circles

Standard Form of Equation	$x^2 + y^2 = r^2$	$(x - h)^2 + (y - k)^2 = r^2$
Center	(0, 0)	(h, k)
Radius	r	r

You can use the standard form of the equation of a circle to write an equation for a circle given the center and the radius or diameter.

Real-World Example 1 Write an Equation Given the Radius

DELIVERY Appliances + More offers free delivery within 35 kilometers of the store. The Abu Dhabi store is located 100 kilometers north and 45 kilometers east of the corporate office. Write an equation to represent the delivery boundary of the Abu Dhabi store if the origin of the coordinate system is the corporate office.

Since the corporate office is at (0, 0), the Abu Dhabi store is at (45, 100). The boundary of the delivery region is the circle centered at (45, 100) with radius 35 kilometers.

$$(x - h)^2 + (y - k)^2 = r^2$$ Equation of a circle

$$(x - 45)^2 + (y - 100)^2 = 35^2$$ $(h, k) = (45, 100)$ and $r = 35$

$$(x - 45)^2 + (y - 100)^2 = 1225$$ Simplify.

GuidedPractice

1. **WI-FI** A certain wi-fi phone has a range of 30 kilometers in any direction. If the phone is 4 kilometers south and 3 kilometers west of headquarters, write an equation to represent the area within which the phone can operate via the Wi-Fi system.

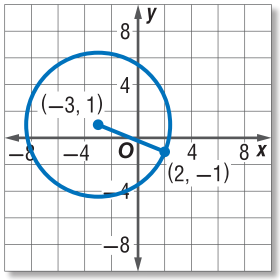
You can write the equation of a circle when you know the location of the center and a point on the circle.

StudyTip

Center-Radius Form Standard form is sometimes referred to as *center-radius form* because the center and radius of the circle are apparent in the equation.

Example 2 Write an Equation from a Graph

Write an equation for the graph.

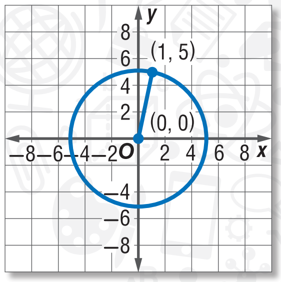


$(x - h)^2 + (y - k)^2 = r^2$	Standard form
$(2 + 3)^2 + (-1 - 1)^2 = r^2$	$x = 2, y = -1, h = -3, k = 1$
$(5)^2 + (-2)^2 = r^2$	Simplify.
$25 + 4 = r^2$	Evaluate the exponents.
$29 = r^2$	Add.

So, the equation of the circle is $(x + 3)^2 + (y - 1)^2 = 29$.

GuidedPractice

2A.



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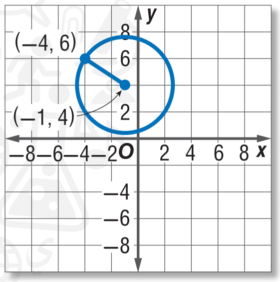


You can use the Midpoint and Distance Formulas when you know the endpoints of the radius or diameter of a circle.

Example 3 Write an Equation Given a Diameter

Write an equation for a circle if the endpoints of a diameter are at (7, 6) and (−1, −8).

Step 1 Find the center.

$(h, k) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	Midpoint Formula
$= \left(\frac{7 + (-1)}{2}, \frac{6 + (-8)}{2}\right)$	$(x_1, y_1) = (7, 6), (x_2, y_2) = (-1, -8)$
$= \left(\frac{6}{2}, \frac{-2}{2}\right)$	Add.
$= (3, -1)$	Simplify.

Step 2 Find the radius.

$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{(3 - 7)^2 + (-1 - 6)^2}$	$(x_1, y_1) = (7, 6), (x_2, y_2) = (3, -1)$
$= \sqrt{(-4)^2 + (-7)^2}$	Subtract.
$= \sqrt{65}$	Simplify.

The radius of the circle is $\sqrt{65}$ units, so $r^2 = 65$. Substitute h, k, and r^2 into the standard form of the equation of a circle. An equation of the circle is $(x - 3)^2 + (y + 1)^2 = 65$.

GuidedPractice

3. Write an equation for a circle if the endpoints of a diameter are at (3, −3) and (1, 5).

StudyTip

Axis of Symmetry Every diameter in a circle is an axis of symmetry. There are infinitely many axes of symmetry in a circle.

2 Graph Circles

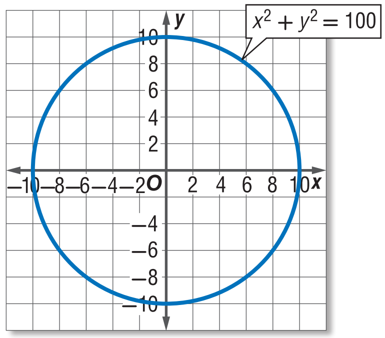
You can use symmetry to help you graph circles.

Example 4 Graph an Equation in Standard Form

Find the center and radius of the circle with equation $x^2 + y^2 = 100$. Then graph the circle.

- The center of the circle is at (0, 0), and the radius is 10.
- The table lists some integer values for x and y that satisfy the equation.
- Because the circle is centered at the origin, it is symmetric about the y-axis. Therefore, the points at $(-6, 8)$, $(-8, 6)$, and $(-10, 0)$ lie on the graph.
- The circle is also symmetric about the x-axis, so the points $(-6, -8)$, $(-8, -6)$, $(0, -10)$, $(6, -8)$, and $(8, -6)$ lie on the graph.
- Plot all of these points and draw the circle that passes through them.

x	y
0	10
6	8
8	6
10	0



GuidedPractice

4. Find the center and radius of the circle with equation $x^2 + y^2 = 81$. Then graph the circle.

Circles with centers that are not (0, 0) can be graphed by using translations. The graph of $(x - h)^2 + (y - k)^2 = r^2$ is the graph of $x^2 + y^2 = r^2$ translated h units horizontally and k units vertically.

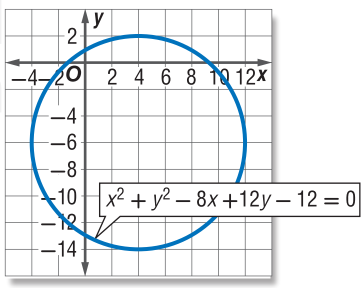
Example 5 Graph an Equation Not in Standard Form

Find the center and radius of the circle with equation $x^2 + y^2 - 8x + 12y - 12 = 0$. Then graph the circle.

Complete the squares.

$$x^2 + y^2 - 8x + 12y - 12 = 0$$
$$x^2 - 8x + \blacksquare + y^2 + 12y + \blacksquare = 12 + \blacksquare + \blacksquare$$
$$x^2 - 8x + 16 + y^2 + 12y + 36 = 12 + 16 + 36$$
$$(x - 4)^2 + (y + 6)^2 = 64$$

The center of the circle is at $(4, -6)$, and the radius is 8. The graph of $(x - 4)^2 + (y + 6)^2 = 64$ is the same as $x^2 + y^2 = 64$ translated 4 units to the right and down 6 units.



GuidedPractice

5. Find the center and radius of the circle with equation $x^2 + y^2 + 4x - 10y - 7 = 0$. Then graph the circle.

Check Your Understanding

Example 1

1. **WEATHER** On average, the eye of a tornado is about 200 feet across. Suppose the center of the eye is at the point (72, 39). Write an equation to represent the boundary of the eye.

Write an equation for each circle given the center and radius.

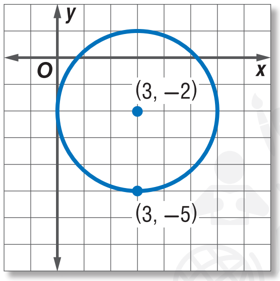
2. center: $(-2, -6)$, $r = 4$ units

3. center: $(1, -5)$, $r = 3$ units

Example 2

Write an equation for each graph.

4.



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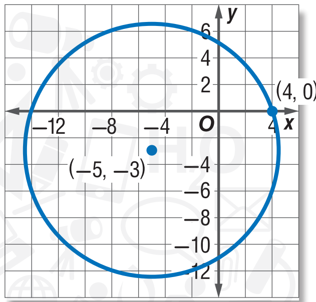


Example 3

Write an equation for each circle given the endpoints of a diameter.

6. $(-1, -7)$ and $(0, 0)$

7. $(4, -2)$ and $(-4, -6)$

Examples 4–5 **Find the center and radius of each circle. Then graph the circle.**

8. $x^2 + y^2 = 16$

9. $x^2 + (y - 7)^2 = 9$

10. $(x - 4)^2 + (y - 4)^2 = 25$

11. $x^2 + y^2 - 4x + 8y - 5 = 0$

Practice and Problem Solving

Example 1

Write an equation for each circle given the center and radius.

12. center: $(4, 9)$, $r = 6$

13. center: $(-3, 1)$, $r = 4$

14. center: $(-7, -3)$, $r = 13$

15. center: $(-2, -1)$, $r = 9$

16. center: $(1, 0)$, $r = \sqrt{15}$

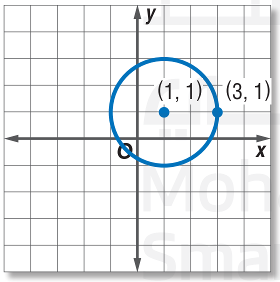
17. center: $(0, -6)$, $r = \sqrt{35}$

18. **MODELING** The radar for an airport control tower is located at (5, 10) on a map. It can detect a plane up to 20 kilometers away. Write an equation for the outer limits of the detection area.

Example 2

Write an equation for each graph.

19.



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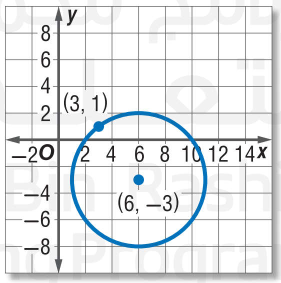

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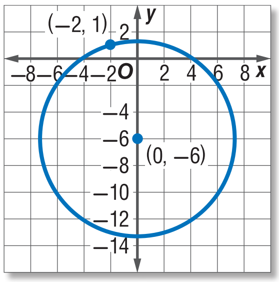

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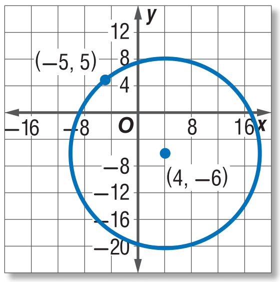


Example 3 **Write an equation for each circle given the endpoints of a diameter.**

23. (2, 1) and (2, −4)
24. (−4, −10) and (4, −10)
25. (5, −7) and (−2, −9)
26. (−6, 4) and (4, 8)
27. (2, −5) and (6, 3)
28. (18, 11) and (−19, −13)

29. **LAWN CARE** A sprinkler waters a circular section of lawn.

a. Write an equation to represent the boundary of the sprinkler area if the endpoints of a diameter are at (−12, 16) and (12, −16).

b. What is the area of the lawn that the sprinkler waters?

30. **SPACE** Apollo 8 was the first manned spacecraft to orbit the Moon at an average altitude of 185 kilometers above the Moon's surface. Write an equation to model a single circular orbit of the command module if the endpoints of a diameter of the Moon are at (1740, 0) and (−1740, 0). Let the center of the Moon be at the origin of the coordinate system measured in kilometers.

Examples 4–5 **Find the center and radius of each circle. Then graph the circle.**

31. $x^2 + y^2 = 75$
32. $(x - 3)^2 + y^2 = 4$
33. $(x - 1)^2 + (y - 4)^2 = 34$
34. $x^2 + (y - 14)^2 = 144$
35. $(x - 5)^2 + (y + 2)^2 = 16$
36. $x^2 + y^2 = 256$
37. $(x - 4)^2 + y^2 = \frac{8}{9}$
38. $\left(x + \frac{2}{3}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{16}{25}$
39. $x^2 + y^2 + 4x = 9$
40. $x^2 + y^2 - 6y + 8x = 0$
41. $x^2 + y^2 + 2x + 4y = 9$
42. $x^2 + y^2 - 3x + 8y = 20$
43. $x^2 + y^2 + 6y = -50 - 14x$
44. $x^2 - 18x + 53 = 18y - y^2$
45. $2x^2 + 2y^2 - 4x + 8y = 32$
46. $3x^2 + 3y^2 - 6y + 12x = 24$

47. **SPACE** A satellite is in a circular orbit 25,000 miles above Earth.

a. Write an equation for the orbit of this satellite if the origin is at the center of Earth. Use 8000 miles as the diameter of Earth.

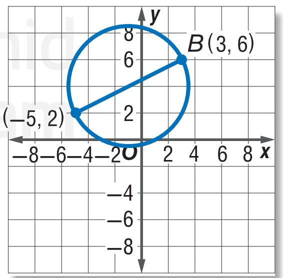
b. Draw a sketch of Earth and the orbit to scale. Label your sketch.

48. **SENSE-MAKING** Suppose an unobstructed radio station broadcast could travel 120 kilometers. Assume the station is centered at the origin.

a. Write an equation to represent the boundary of the broadcast area with the origin as the center.

b. If the transmission tower is relocated 40 kilometers east and 10 kilometers south of the current location, and an increased signal will transmit signals an additional 80 kilometers, what is an equation to represent the new broadcast area?

49. **GEOMETRY** Concentric circles are circles with the same center but different radii. Refer to the graph at the right where $\overline{AB}$ is a diameter of the circle.



a. Write an equation of the circle concentric with the circle at the right, with radius 4 units greater.

b. Write an equation of the circle concentric with the circle at the right, with radius 2 units less.

c. Graph the circles from parts a and b on the same coordinate plane.

50. **EARTHQUAKES** A stadium is located about 35 kilometers west and 40 kilometers north of a city. Suppose an earthquake occurs with its epicenter about 55 kilometers from the stadium. Assume that the origin of a coordinate plane is located at the center of the city. Write an equation for the set of points that could be the epicenter of the earthquake.

PRECISION Write an equation for the circle that satisfies each set of conditions.

51. center $(9, -8)$, passes through $(19, 22)$

52. center $\left(-\sqrt{15}, 30\right)$, passes through the origin

53. center at $(8, -9)$, tangent to y-axis

54. center at $(2, 4)$, tangent to x-axis

55. center in the first quadrant; tangent to $x = 5$, the x-axis, and the y-axis

56. center in the second quadrant; tangent to $y = 1$, $y = 5$, and the y-axis

57. **MULTIPLE REPRESENTATIONS** Graph $y = \sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$ on the same graphing calculator screen.

a. **Verbal** Describe the graph formed by the union of these two graphs.

b. **Algebraic** Write an equation for the union of the two graphs.

c. **Verbal** Most graphing calculators cannot graph the equation $x^2 + y^2 = 49$ directly. Describe a way to use a graphing calculator to graph the equation. Then graph the equation.

d. **Analytical** Solve $(x - 2)^2 + (y + 1)^2 = 4$ for y. Why do you need two equations to graph a circle on a graphing calculator?

e. **Verbal** Do you think that it is easier to graph the equation in part **d** using graph paper and a pencil or using a graphing calculator? Explain.

Find the center and radius of each circle. Then graph the circle.

58. $x^2 - 12x + 84 = -y^2 + 16y$

59. $4x^2 + 4y^2 + 36y + 5 = 0$

60. $\left(x + \sqrt{5}\right)^2 + y^2 - 8y = 9$

61. $x^2 + 2\sqrt{7}x + 7 + \left(y - \sqrt{11}\right)^2 = 11$

H.O.T. Problems Use Higher-Order Thinking Skills

62. **ERROR ANALYSIS** Hana says that $(x - 2)^2 + (y + 3)^2 = 36$ and $(x - 2) + (y + 3) = 6$ are equivalent equations. Samira says that the equations are *not* equivalent. Is either of them correct? Explain your reasoning.

63. **OPEN ENDED** Consider graphs with equations of the form $(x - 3)^2 + (y - a)^2 = 64$. Assign three different values for a, and graph each equation. Describe all graphs with equations of this form.

64. **REASONING** Explain why the phrase "in a plane" is included in the definition of a circle. What would be defined if the phrase were *not* included?

65. **OPEN ENDED** Concentric circles have the same center, but most often, not the same radius. Write equations of two concentric circles. Then graph the circles.

66. **REASONING** Assume that (x, y) are the coordinates of a point on a circle. The center is at (h, k), and the radius is r. Find an equation of the circle by using the Distance Formula.

67. **WRITING IN MATH** The circle with equation $(x - a)^2 + (y - b)^2 = r^2$ lies in the first quadrant and is tangent to both the x-axis and the y-axis. Sketch the circle. Describe the possible values of a, b, and r. Do the same for a circle in Quadrants II, III, and IV. Discuss the similarities among the circles.

Standardized Test Practice

68. GRIDDED RESPONSE Two circles, both with a radius of 6, have exactly one point in common. If A is a point on one circle and B is a point on the other circle, what is the maximum possible length for the line segment $\overline{AB}$?

69. In a movie theatre, there are 20% more girls than boys. If there are 180 girls, how many more girls than boys are there?

A 30
B 36
C 90
D 144

70. A AED 1,000 deposit is made at a bank that pays 2% compounded weekly. How much will you have in your account at the end of 10 years?

F AED 1,200.00 **H** AED 1,221.36
G AED 1,218.99 **J** AED 1,224.54

71. The mean of six numbers is 20. If one of the numbers is removed, the average of the remaining numbers is 15. What is the number that was removed?

A 42 **C** 45
B 43 **D** 48

Spiral Review

Graph each equation. (Lesson 6-2)

72. $y = -\frac{1}{2}(x - 1)^2 + 4$ **73.** $4(x - 2) = (y + 3)^2$ **74.** $(y - 8)^2 = -4(x - 4)$

Find the midpoint of the line segment with endpoints at the given coordinates. Then find the distance between the points. (Lesson 6-1)

75. $\left(-3, -\frac{2}{11}\right), \left(5, \frac{9}{11}\right)$ **76.** $\left(2\sqrt{3}, -5\right), \left(-3\sqrt{3}, 9\right)$ **77.** $(2.5, 4), (-2.5, 2)$

78. If y varies directly as x and $y = 8$ when $x = 6$, find y when $x = 15$.

79. If y varies jointly as x and z and $y = 80$ when $x = 5$ and $z = 8$, find y when $x = 16$ and $z = 2$.

80. If y varies inversely as x and $y = 16$ when $x = 5$, find y when $x = 20$.

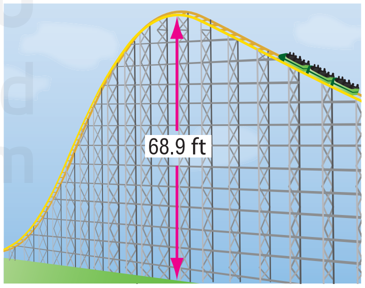
Evaluate each expression.

81. $\log_9 243$ **82.** $\log_2 \frac{1}{32}$ **83.** $\log_3 \frac{1}{81}$ **84.** $\log_{10} 0.001$

85. AMUSEMENT PARKS The velocity v in feet per second of a roller coaster at the bottom of a hill is related to the vertical drop h in feet and the velocity v_0 in feet per second of the coaster at the top of the hill by the formula $v_0 = \sqrt{v^2 - 64h}$.

a. Explain why $v_0 = v - 8\sqrt{h}$ is not equivalent to the given formula.

b. What velocity must the coaster have at the top of the hill to achieve a velocity of 38.1 feet per second at the bottom?



Skills Review

Solve each equation by completing the square.

86. $x^2 + 3x - 18 = 0$ **87.** $2x^2 - 3x - 3 = 0$ **88.** $x^2 + 2x + 6 = 0$

EXPLORE

6-3 Algebra Lab
Investigating Ellipses

Follow the steps below to construct a type of conic section.

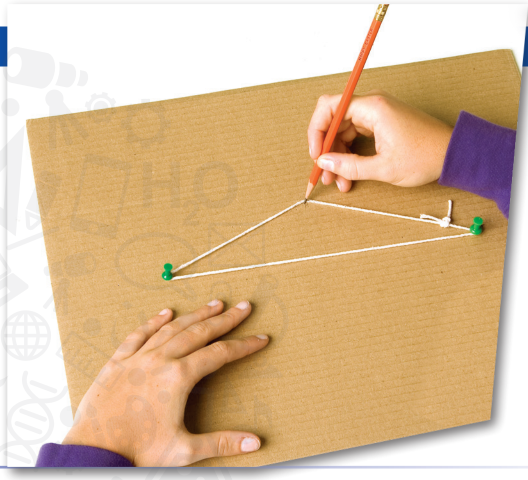
Mathematical Practices
5 Use appropriate tools strategically.

Activity Make an Ellipse

Step 1 Place two thumbtacks in a piece of cardboard, about 1 foot apart.

Step 2 Tie a knot in a piece of string and loop it around the thumbtacks. Place your pencil in the string.

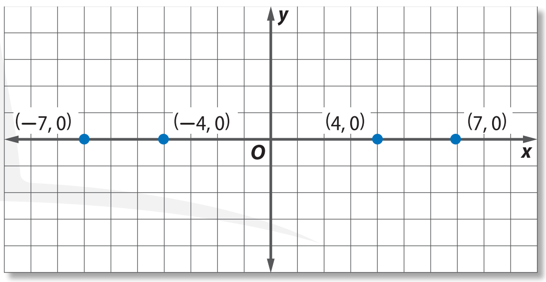
Step 3 Keep the string tight and draw a curve. Continue drawing until you return to your starting point.



The curve you have drawn is called an **ellipse**. The points where the thumbtacks are located are called the **foci** of the ellipse. *Foci* is the plural of *focus*.

Model and Analyze

Place a large piece of grid paper on a piece of cardboard.



1. Place the thumbtacks at (7, 0) and (−7, 0). Choose a string long enough to loop around both thumbtacks. Draw an ellipse.

2. Repeat Exercise 1, but place the thumbtacks at (4, 0) and (−4, 0). Use the same loop of string and draw an ellipse. How does this ellipse compare to the one in Exercise 1?

Place the thumbtacks at each set of points and draw an ellipse. You may change the length of the loop of string if you like.

3. (11, 0), (−11, 0) **4.** (3, 0), (−3, 0) **5.** (13, 3), (−9, 3)

Make a Conjecture

Describe what happens to the shape of an ellipse when each change is made.

6. The thumbtacks are moved closer together.

7. The thumbtacks are moved farther apart.

8. The length of the loop of string is increased.

9. The thumbtacks are arranged vertically.

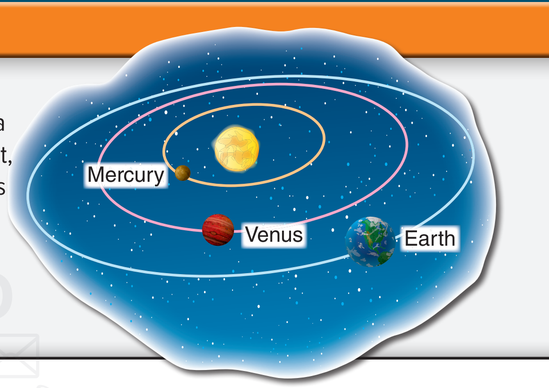
10. One thumbtack is removed, and the string is looped around the remaining thumbtack.

11. Pick a point on one of the ellipses you have drawn. Use a ruler to measure the distances from that point to the points where the thumbtacks were located. Add the distances. Repeat for other points on the same ellipse. What relationship do you notice?

12. Could this activity be done with a rubber band instead of a piece of string? Explain.

LESSON 6-3 Ellipses

:: Then	:: Now	:: Why?
• You graphed and wrote equations for circles.	• **1** Write equations of ellipses. **2** Graph ellipses.	• Mercury, like all of the planets of our solar system, does not orbit the Sun in a perfect circular path. At its farthest point, Mercury is about 69.2 million kilometers from the Sun. At its closest point, it is only about 45.9 million kilometers from the Sun. This orbit is in the shape of an ellipse with the Sun at a focus.



NewVocabulary

ellipse
foci
major axis
minor axis
center
vertices
co-vertices
constant sum

Mathematical Practices

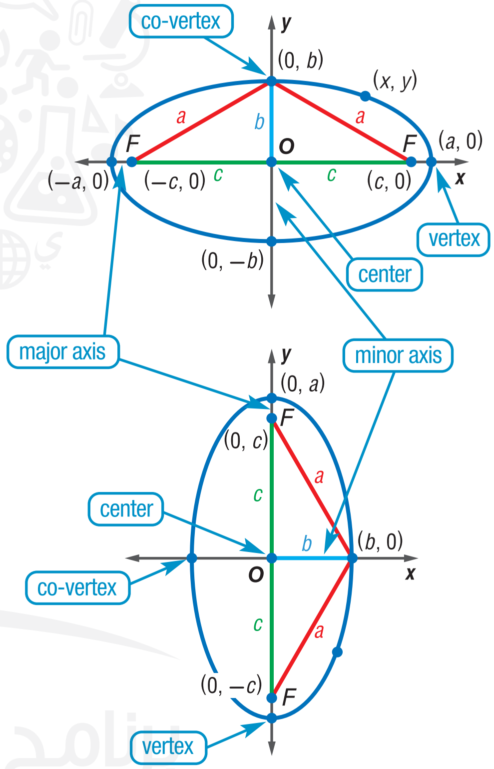
7 Look for and make use of structure.

1 Equations of Ellipses

An **ellipse** is the set of all points in a plane such that the sum of the distances from two fixed points is constant. These two points are called the **foci** of the ellipse.

Every ellipse has two axes of symmetry, the **major axis** and the **minor axis**. The axes are perpendicular at the **center** of the ellipse.

The foci of an ellipse always lie on the major axis. The endpoints of the major axis are the **vertices** of the ellipse and the endpoints of the minor axis are the **co-vertices** of the ellipse.



KeyConcept Equations of Ellipses Centered at the Origin

Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
Orientation	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Length of Major Axis	$2a$ units	$2a$ units
Length of Minor Axis	$2b$ units	$2b$ units

There are several important relationships among the many parts of an ellipse.

- The length of the major axis, $2a$ units, equals the sum of the distances from the foci to any point on the ellipse.
- The values of a, b, and c are related by the equation $c^2 = a^2 - b^2$.
- The distance from a focus to either co-vertex is a units.

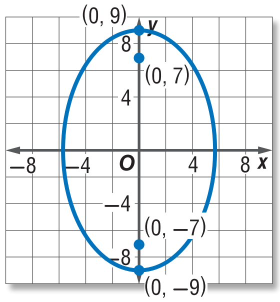
The sum of the distances from the foci to any point on the ellipse, or the **constant sum**, must be greater than the distance between the foci.

StudyTip

Major Axis In standard form, if the x^2-term has the greater denominator, then the major axis is horizontal. If the y^2-term has the greater denominator, then it is vertical.

Example 1 Write an Equation Given Vertices and Foci

Write an equation for the ellipse.



Step 1 Find the center.
The foci are equidistant from the center.
The center is at (0, 0).

Step 2 Find the value of a.
The vertices are (0, 9) and (0, −9),
so the length of the major axis is 18.
The value of a is 18 ÷ 2 or 9, and $a^2 = 81$.

Step 3 Find the value of b.
We can use $c^2 = a^2 - b^2$ to find b.
The foci are 7 units from the center, so $c = 7$.

$c^2 = a^2 - b^2$	Equation relating a, b, and c
$49 = 81 - b^2$	$a = 9$ and $c = 7$
$b^2 = 32$	Solve for b^2.

Step 4 Write the equation.
Because the major axis is vertical, a^2 goes with y and b^2 goes with x.
The equation for the ellipse is $\frac{y^2}{81} + \frac{x^2}{32} = 1$.

GuidedPractice

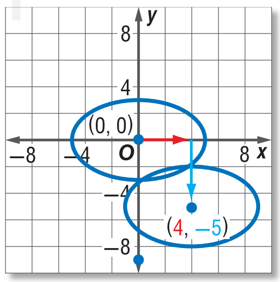
1. Write an equation for an ellipse with vertices at (−4, 0) and (4, 0) and foci at (2, 0) and (−2, 0).

Like other graphs, the graph of an ellipse can be translated. When the graph is translated h units right and k units up, the center of the translation is (h, k). This is equivalent to replacing x with $x - h$ and replacing y with $y - k$ in the parent function.

KeyConcept Equations of Ellipses Centered at (h, k)

Standard Form	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Orientation	horizontal	vertical
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Co-vertices	$(h, k \pm b)$	$(h \pm b, k)$

We can use this information to determine the equations for ellipses. The original ellipse at the right is horizontal and has a major axis of 10 units, so $a = 5$.



The length of the minor axis is 6 units, so $b = 3$.

The ellipse is translated 4 units right and 5 units down. So, the value of h is 4 and the value of k is −5.

The equation for the original ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

The equation for the translation is $\frac{(x-4)^2}{25} + \frac{(y+5)^2}{9} = 1$.

You can also determine the equation for an ellipse if you are given all four vertices.

Example 2 Write an Equation Given the Lengths of the Axes

Write an equation for the ellipse with vertices at (6, −8) and (6, 4) and co-vertices at (3, −2) and (9, −2).

The x-coordinate is the same for both vertices, so the ellipse is vertical.

The center of the ellipse is at $\left(\frac{6+6}{2}, \frac{-8+4}{2}\right)$ or (6, −2).

The length of the major axis is 4 − (−8) or 12 units, so $a = 6$.

The length of the minor axis is 9 − 3 or 6 units, so $b = 3$.

The equation for the ellipse is $\frac{(y+2)^2}{36} + \frac{(x-6)^2}{9} = 1$. $\quad a^2 = 36, b^2 = 9$

GuidedPractice

2. Write an equation for the ellipse with vertices at (−3, 8) and (9, 8) and co-vertices at (3, 12) and (3, 4).

Real-WorldCareer

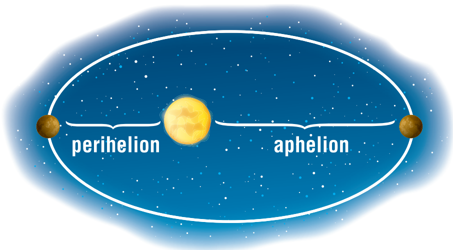
Aerospace Technician
Aerospace technicians work for NASA, helping engineers research and develop virtual reality and verbal communication between humans and computer systems. Although a bachelor's degree is desired, on-the-job training is available.

Source: NASA

Many real-world phenomena can be represented by ellipses.

Real-World Example 3 Write an Equation for an Ellipse

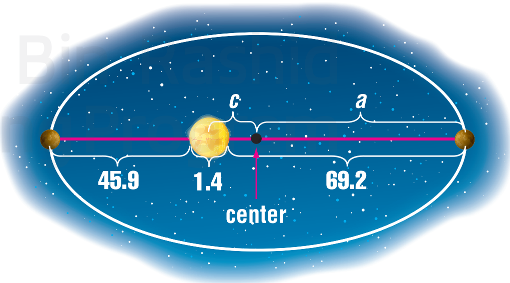
SPACE Refer to the application at the beginning of the lesson. Mercury's greatest distance from the Sun, or *aphelion*, is about 69.2 million kilometers. Mercury's closest distance, or *perihelion*, is about 45.9 million kilometers. The diameter of the Sun is about 1,400,129.3 kilometers. Use this information to determine an equation relating Mercury's elliptical orbit around the Sun in millions of kilometers.



Understand We need to determine an equation representing Mercury's orbit around the Sun.

Plan Including the diameter of the Sun, the sum of the perihelion and aphelion equals the length on the major axis of the ellipse. We can use this information to determine the values of a, b, and c.

Solve Find the value of a.
The value of a is one half the length of the major axis.
$a = 0.5(69.2 + 45.9 + 1.4)$ or 58.23



Find the value of c.
The value of c is the distance from the center of the ellipse to the focus.
This distance is equal to a minus the perihelion and the radius of the Sun.
$c = 58.23 - 45.9 - 0.7$or 11.67

Problem-SolvingTip

Sense-Making Draw a diagram when the problem situation involves spatial reasoning or geometric figures.

(continued on the next page)

Find the value of b.

$c^2 = a^2 - b^2$	Equation relating a, b, and c
$(11.67)^2 = (58.23)^2 - b^2$	$c = 7.25$ and $a = 36.185$
$136.1889 = 3390.7329 - b^2$	Simplify.
$b^2 = 3254.544$	Solve for b^2.
$b = 57.0486$	Take the square root of each side.

So, with the center of the orbit at the origin, the equation relating Mercury's orbit around the Sun can be modeled by $\frac{x^2}{3390.7329} + \frac{y^2}{3254.544} = 1$.

Check Use your answer to recalculate a, b, and c. Then determine the aphelion and perihelion based on your answer. Compare to the actual values.

Real-WorldLink

Earth's orbit around the Sun is nearly circular, with only about a 3% difference between perihelion and aphelion.

Source: *The Astronomer*

GuidedPractice

3. **SPACE** Pluto's distance from the Sun is 4.44 billion kilometers at perihelion and about 7.38 billion kilometers at aphelion. Determine an equation relating Pluto's orbit around the Sun in billions of kilometers with the center of the horizontal ellipse at the origin.

2 Graph Ellipses

When you are given an equation for an ellipse that is not in standard form, you can write it in standard form by completing the square for both x and y. Once the equation is in standard form, you can use it to graph the ellipse.

Example 4 Graph an Ellipse

Find the coordinates of the center and foci, and the lengths of the major and minor axes of an ellipse with equation $25x^2 + 9y^2 + 250x - 36y + 436 = 0$. Then graph the ellipse.

Step 1 Write in standard form. Complete the square for each variable to write this equation in standard form.

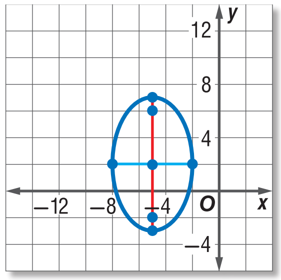
$25x^2 + 9y^2 + 250x - 36y + 436 = 0$	Original equation
$25x^2 + 250x + 9y^2 - 36y = -436$	Associative Property
$25(x^2 + 10x) + 9(y^2 - 4y) = -436$	Distributive Property
$25(x^2 + 10x + ■) + 9(y^2 - 4y + ■) = -436 + 25(■) + 9(■)$	Complete the squares.
$25(x^2 + 10x + 25) + 9(y^2 - 4y + 4) = -436 + 25(25) + 9(4)$	$5^2 = 25$ and $(-2)^2 = 4$
$25(x + 5)^2 + 9(y - 2)^2 = 225$	Write as perfect squares.
$\frac{(x+5)^2}{9} + \frac{(y-2)^2}{25} = 1$	Divide each side by 225.

Step 2 Find the center.
$h = -5$ and $k = 2$, so the center of the ellipse is at $(-5, 2)$.

Step 3 Find the lengths of the axes and graph.
The ellipse is vertical.
$a^2 = 25$, so $a = 5$. $b^2 = 9$, so $b = 3$.
The length of the major axis is $2 \cdot 5$ or 10.
The length of the minor axis is $2 \cdot 3$ or 6.
The vertices are at $(-5, 7)$ and $(-5, -3)$.
The co-vertices are at $(-2, 2)$ and $(-8, 2)$.

Step 4 Find the foci.
$c^2 = 25 - 9$ or 16, so $c = 4$.
The foci are at $(-5, 6)$ and $(-5, -2)$.

Step 5 Graph the ellipse.
Draw the ellipse that passes through the vertices and co-vertices.



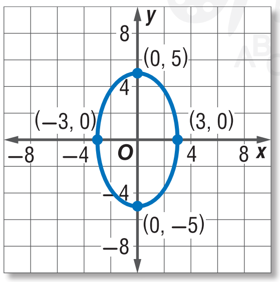
GuidedPractice

4. Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $x^2 + 4y^2 - 2x + 24y + 21 = 0$. Then graph the ellipse.

Check Your Understanding

Example 1 **Write an equation for each ellipse.**

1.



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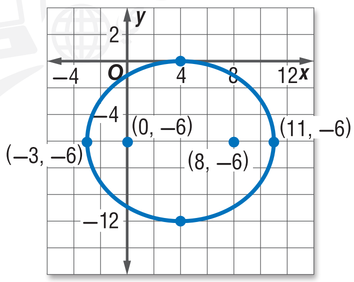


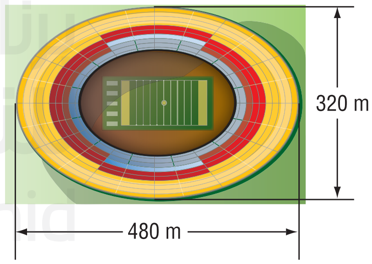
Example 2 **Write an equation for an ellipse that satisfies each set of conditions.**

3. vertices at $(-2, -6)$ and $(-2, 4)$, co-vertices at $(-5, -1)$ and $(1, -1)$

4. vertices at $(-2, 5)$ and $(14, 5)$, co-vertices at $(6, 1)$ and $(6, 9)$

Example 3

5. **SENSE-MAKING** An architectural firm sent a proposal to a city for building a coliseum, shown at the right.



a. Determine the values of a and b.

b. Assuming that the center is at the origin, write an equation to represent the ellipse.

c. Determine the coordinates of the foci.

6. **SPACE** Earth's orbit is about 147.1 million kilometers at perihelion and about 152.1 million kilometers at aphelion. Determine an equation relating Earth's orbit around the Sun in millions of miles with the center of the horizontal ellipse at the origin.

Example 4 **Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.**

7. $\frac{(y+1)^2}{64} + \frac{(x-5)^2}{28} = 1$

8. $\frac{(x+2)^2}{48} + \frac{(y-1)^2}{20} = 1$

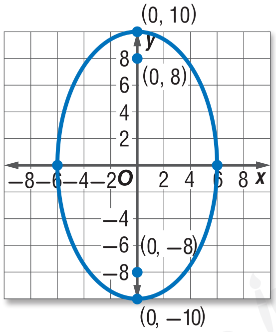
9. $4x^2 + y^2 - 32x - 4y + 52 = 0$

10. $9x^2 + 25y^2 + 72x - 150y + 144 = 0$

Practice and Problem Solving

Example 1 **Write an equation for each ellipse.**

11.



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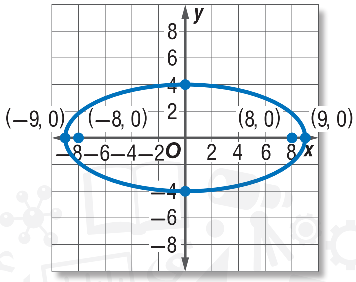

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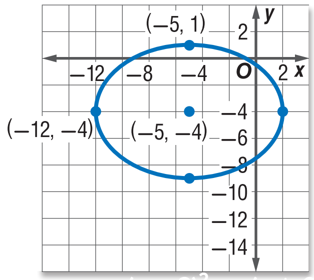

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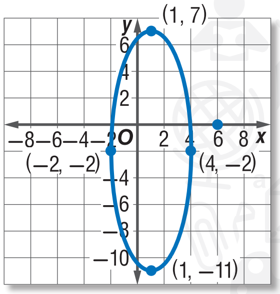

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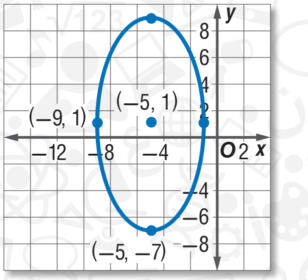

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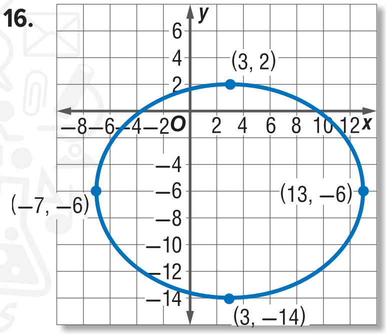


Example 2 **Write an equation for an ellipse that satisfies each set of conditions.**

17. vertices at $(-6, 4)$ and $(12, 4)$, co-vertices at $(3, 12)$ and $(3, -4)$

18. vertices at $(-1, 11)$ and $(-1, 1)$, co-vertices at $(-4, 6)$ and $(2, 6)$

19. center at $(-2, 6)$, vertex at $(-2, 16)$, co-vertex at $(1, 6)$

20. center at $(3, -4)$, vertex at $(8, -4)$, co-vertex at $(3, -2)$

21. vertices at $(4, 12)$ and $(4, -4)$, co-vertices at $(1, 4)$ and $(7, 4)$

22. vertices at $(-11, 2)$ and $(-1, 2)$, co-vertices at $(-6, 0)$ and $(-6, 4)$

Example 3

23. **MODELING** The opening of a tunnel in the mountains can be modeled by semiellipses, or halves of ellipses. If the opening is 14.6 meters wide and 8.6 meters high, determine an equation to represent the opening with the center at the origin.

Example 4 **Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.**

24. $\frac{(x-3)^2}{36} + \frac{(y-2)^2}{128} = 1$

25. $\frac{(x+6)^2}{50} + \frac{(y-3)^2}{72} = 1$

26. $\frac{x^2}{27} + \frac{(y-5)^2}{64} = 1$

27. $\frac{(x+4)^2}{16} + \frac{y^2}{75} = 1$

28. $3x^2 + y^2 - 6x - 8y - 5 = 0$

29. $3x^2 + 4y^2 - 18x + 24y + 3 = 0$

30. $7x^2 + y^2 - 56x + 6y + 93 = 0$

31. $3x^2 + 2y^2 + 12x - 20y + 14 = 0$

32. **SPACE** Like the planets, Halley's Comet travels around the Sun in an elliptical orbit. The aphelion is 5283.3 million kilometers and the perihelion is 88.3 million kilometers. Determine an equation relating the comet's orbit around the Sun in millions of kilometers with the center of the horizontal ellipse at the origin.

Write an equation for an ellipse that satisfies each set of conditions.

33. center at (−5, −2), focus at (−5, 2), co-vertex at (−8, −2)

34. center at (4, −3), focus at (9, −3), co-vertex at (4, −5)

35. foci at (−2, 8) and (6, 8), co-vertex at (2, 10)

36. foci at (4, 4) and (4, 14), co-vertex at (0, 9)

37. **GOVERNMENT** The Oval Office is located in the West Wing of the White House. It is an elliptical shaped room used as the main office by the President of the United States. The long axis is 10.9 meters long and the short axis is 8.8 meters long. Write an equation to represent the outer walls of the Oval Office. Assume that the center of the room is at the origin.

38. **SOUND** A whispering gallery is an elliptical room in which a faint whisper at one focus cannot be heard by other people in the room, but can easily be heard by someone at the other focus. Suppose an ellipse is 121.9 meters long and 36.6 meters wide. What is the distance between the foci?

39. **MULTIPLE REPRESENTATIONS** The *eccentricity* of an ellipse measures how circular the ellipse is.

a. **Graphical** Graph $\frac{x^2}{81} + \frac{y^2}{36} = 1$ and $\frac{x^2}{81} + \frac{y^2}{9} = 1$ on the same graph.

b. **Verbal** Describe the difference between the two graphs.

c. **Algebraic** The eccentricity of an ellipse is $\frac{c}{a}$. Find the eccentricity for each.

d. **Analytical** Make a conjecture about the relationship between the value of an ellipse's eccentricity and the shape of the ellipse as compared to a circle.

H.O.T. Problems Use Higher-Order Thinking Skills

40. **ERROR ANALYSIS** Shaima and Maha are determining the equation for an ellipse with foci at (−4, −11) and (−4, 5) and co-vertices at (2, −3) and (−10, −3). Is either of them correct? Explain your reasoning.

Shaima

$$\frac{(x-4)^2}{64} + \frac{(y+3)^2}{36} = 1$$

Maha

$$\frac{(x+4)^2}{100} + \frac{(y+3)^2}{36} = 1$$

41. **OPEN ENDED** Write an equation for an ellipse with a focus at the origin.

42. **CHALLENGE** When the values of a and b are equal, an ellipse is a circle. Use this information and your knowledge of ellipses to determine the formula for the area of an ellipse in terms of a and b.

43. **CHALLENGE** Determine an equation for an ellipse with foci at $(2, \sqrt{6})$ and $(2, -\sqrt{6})$ that passes through $(3, \sqrt{6})$.

44. **ARGUMENTS** What happens to the location of the foci as an ellipse becomes more circular? Explain your reasoning.

45. **REASONING** An ellipse has foci at (−7, 2) and (18, 2). If (2, 14) is a point on the ellipse, show that (2, −10) is also a point on the ellipse.

46. **WRITING IN MATH** Explain why the domain is $\{x \mid -a \le x \le a\}$ and the range is $\{y \mid -b \le y \le b\}$ for an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Standardized Test Practice

47. Multiply.

$$(2 + 3i)(4 + 7i)$$

A $8 + 21i$
B $-13 + 26i$
C $-6 + 10i$
D $13 + 12i$

48. The average lifespan of American women has been tracked, and the model for the data is $y = 0.2t + 73$, where $t = 0$ corresponds to 1960. What is the meaning of the y-intercept?

F In 2007, the average lifespan was 60.
G In 1960, the average lifespan was 58.
H In 1960, the average lifespan was 73.
J The lifespan is increasing 0.2 years every year.

49. **GRIDDED RESPONSE** If we decrease a number by 6 and then double the result, we get 5 less than the number. What is the number?

50. **SAT/ACT** The length of a rectangular prism is one inch greater than its width. The height is three times the length. Find the volume of the prism.

A $3x^3 + x^2 + 3x$
B $x^3 + x^2 + x$
C $3x^3 + 6x^2 + 3x$
D $3x^3 + 3x^2 + 3x$
E $3x^3 + 3x^2$

Spiral Review

Write an equation for the circle that satisfies each set of conditions. (Lesson 6-3)

51. center $(8, -9)$, passes through $(21, 22)$

52. center at $(4, 2)$, tangent to x-axis

53. center in the second quadrant; tangent to $y = -1$, $y = 9$, and the y-axis

54. **ENERGY** A parabolic mirror is used to collect solar energy. The mirrors reflect the rays from the Sun to the focus of the parabola. The focus of a particular mirror is 9.75 feet above the vertex, and the latus rectum is 39 feet long. (Lesson 6-2)

a. Assume that the focus is at the origin. Write an equation for the parabola formed by the mirror.

b. One foot is exactly 0.3048 meter. Rewrite the equation for the mirror in meters.

c. Graph one of the equations for the mirror.

d. Which equation did you choose to graph? Explain why.

Simplify each expression.

55. $\frac{6}{d^2 + 4d + 4} + \frac{5}{d + 2}$

56. $\frac{a}{a^2 - a - 20} + \frac{2}{a + 4}$

57. $\frac{x}{x + 1} + \frac{3}{x^2 - 4x - 5}$

Solve each equation.

58. $\log_{10} (x^2 + 1) = 1$

59. $\log_b 64 = 3$

60. $\log_b 121 = 2$

Simplify.

61. $-5ab^2(-3a^2b + 6a^3b - 3a^4b^4)$

62. $2xy(3xy^3 - 4xy + 2y^4)$

63. $(4x^2 - 3y^2 + 5xy) - (8xy + 3y^2)$

64. $(10x^2 - 3xy + 4y^2) - (3x^2 + 5xy)$

Skills Review

Write an equation of the line passing through each pair of points.

65. $(-2, 5)$ and $(3, 1)$

66. $(7, 1)$ and $(7, 8)$

67. $(-3, 5)$ and $(2, 2)$

CHAPTER 6

Mid-Chapter Quiz

Lessons 6-1 through 6-4

Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola. (Lesson 6-2)

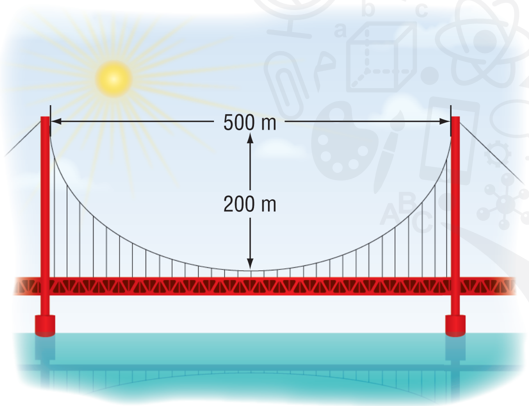
1. $y = 3x^2 - 12x + 21$

2. $x - 2y^2 = 4y + 6$

3. $y = \frac{1}{2}x^2 + 12x - 8$

4. $x = 3y^2 + 5y - 9$

5. **BRIDGES** Write an equation of a parabola to model the shape of the suspension cable of the bridge shown. Assume that the origin is at the lowest point of the cables. (Lesson 6-2)



Identify the coordinates of the vertex and focus, the equation of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum. (Lesson 6-2)

6. $y = x^2 + 6x + 5$

7. $x = -2y^2 + 4y + 1$

8. Find the center and radius of the circle with equation $(x - 1)^2 + y^2 = 9$. Then graph the circle. (Lesson 6-3)

9. Write an equation for a circle that has center at (3, −2) and passes through (3, 4). (Lesson 6-3)

10. Write an equation for a circle if the endpoints of a diameter are at (8, 31) and (32, 49). (Lesson 6-3)

11. **MULTIPLE CHOICE** What is the radius of the circle with equation $x^2 + 2x + y^2 + 14y + 34 = 0$? (Lesson 6-3)

A 2

B 4

C 8

D 16

Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with the given equation. Then graph the ellipse. (Lesson 6-3)

12. $\frac{(x + 4)^2}{16} + \frac{(y - 2)^2}{9} = 1$

13. $\frac{(x - 1)^2}{20} + \frac{(y + 2)^2}{4} = 1$

14. $4y^2 + 9x^2 + 16y - 90x + 205 = 0$

15. **MULTIPLE CHOICE** Which equation represents an ellipse with endpoints of the major axis at (−4, 10) and (−4, −6) and foci at about (−4, 7.3) and (−4, −3.3)? (Lesson 6-3)

F $\frac{(x - 2)^2}{36} + \frac{(y + 4)^2}{64} = 1$

G $\frac{(x + 4)^2}{64} + \frac{(y - 2)^2}{36} = 1$

H $\frac{(y - 2)^2}{64} + \frac{(x + 4)^2}{36} = 1$

J $\frac{(x - 2)^2}{64} + \frac{(y + 4)^2}{36} = 1$

LESSON 6-4 Hyperbolas

Then

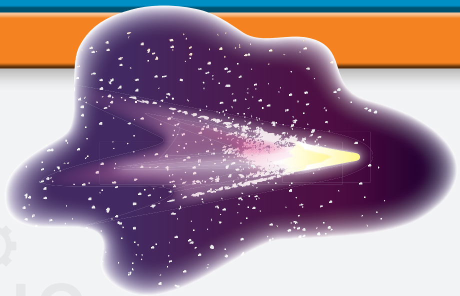
- You graphed and analyzed equations of ellipses.

Now

1. Write equations of hyperbolas.
2. Graph hyperbolas.

Why?

- Because Halley's Comet travels around the Sun in an elliptical path, it reappears in our sky. Other comets pass through our sky only once. Many of these comets travel in paths that resemble hyperbolas.



New Vocabulary

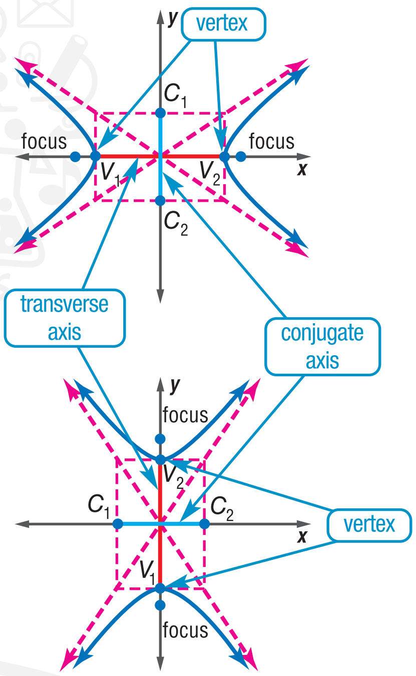
hyperbola
transverse axis
conjugate axis
foci
vertices
co-vertices
constant difference

Mathematical Practices

6 Attend to precision.

1 Equations of Hyperbolas

Similar to an ellipse, a **hyperbola** is the set of all points in a plane such that the absolute value of the differences of the distances from the foci is constant.



Every hyperbola has two axes of symmetry, the **transverse axis** and the **conjugate axis**. The axes are perpendicular at the center of the hyperbola.

The **foci** of a hyperbola always lie on the transverse axis. The **vertices** are the endpoints of the transverse axis. The **co-vertices** are the endpoints of the conjugate axis.

As a hyperbola recedes from the center, both halves approach asymptotes.

KeyConcept Equations of Hyperbolas Centered at the Origin

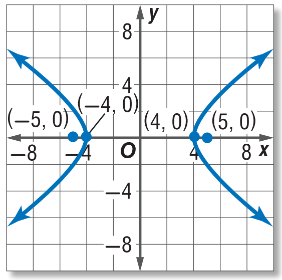
Standard Form	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Orientation	horizontal	vertical
Foci	$(\pm c, 0)$	$(0, \pm c)$
Length of Transverse Axis	$2a$ units	$2a$ units
Length of Conjugate Axis	$2b$ units	$2b$ units
Equations of Asymptotes	$y = \pm \frac{b}{a}x$	$y = \pm \frac{a}{b}x$

As with ellipses, there are several important relationships among the parts of hyperbolas.

- There are two axes of symmetry.
- The values of a, b, and c are related by the equation $c^2 = a^2 + b^2$.

Example 1 Write an Equation Given Vertices and Foci

Write an equation for the hyperbola shown in the graph.



Step 1 Find the center.

The vertices are equidistant from the center.

The center is at (0, 0).

Step 2 Find the values of a, b, and c.

The value of a is the distance between a vertex and the center, or 4 units.

The value of c is the distance between a focus and the center, or 5 units.

$c^2 = a^2 + b^2$ — Equation relating a, b, and c for a hyperbola

$5^2 = 4^2 + b^2$ — $c = 5$ and $a = 3$

$9 = b^2$ — Subtract 4^2 from each side.

Step 3 Write the equation.

The transverse axis is horizontal, so the equation is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

GuidedPractice

1. Write an equation for a hyperbola with vertices at (6, 0) and (−6, 0) and foci at (8, 0) and (−8, 0).

Math HistoryLink

Hypatia (415–370 B.C.) Hypatia was a mathematician, scientist, and philosopher in Alexandria, Egypt. She is considered the first woman to write on mathematical topics. Hypatia edited the book *On the Conics of Apollonius*, adding her own problems and examples to clarify the topic for her students. This book developed the ideas of hyperbolas, parabolas, and ellipses.

Hyperbolas can also be determined using the equations of their asymptotes.

Example 2 Write an Equation Given Asymptotes

The asymptotes for a vertical hyperbola are $y = \frac{5}{3}x$ and $y = -\frac{5}{3}x$ and the vertices are at (0, 5) and (0, −5). Write the equation for the hyperbola.

Step 1 Find the center.

The vertices are equidistant from the center.

The center of the hyperbola is at (0, 0).

Step 2 Find the values of a and b.

The hyperbola is vertical, so $a = 5$.

From the asymptotes, $b = 3$.

The value of c is not needed.

Step 3 Write the equation.

The equation for the hyperbola is $\frac{y^2}{25} - \frac{x^2}{9} = 1$.

ReadingMath

Standard Form In the standard form of a hyperbola, the squared terms are subtracted. For an ellipse, they are added.

GuidedPractice

2. The asymptotes for a horizontal hyperbola are $y = \frac{7}{9}x$ and $y = -\frac{7}{9}x$. The vertices are (9, 0) and (−9, 0). Write an equation for the hyperbola.

2 Graphs of Hyperbolas

Graphs of Hyperbolas Hyperbolas can be translated in the same manner as the other conic sections.

KeyConcept Equations of Hyperbolas Centered at (h, k)

Standard Form	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Orientation	horizontal	vertical
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Co-vertices	$(h, k \pm b)$	$(h \pm b, k)$
Equations of Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{a}{b}(x - h)$

StudyTip

Calculator You can graph a hyperbola on a graphing calculator by solving for y, and then graphing the two equations on the same screen.

Example 3 Graph a Hyperbola

Graph $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{16} = 1$. Identify the vertices, foci, and asymptotes.

Step 1 Find the center. The center is at $(3, -2)$.

Step 2 Find a, b, and c. From the equation, $a^2 = 4$ and $b^2 = 16$, so $a = 2$ and $b = 4$.

$c^2 = a^2 + b^2$ — Equation relating a, b, and c for a hyperbola

$c^2 = 2^2 + 4^2$ — $a = 2, b = 4$

$c^2 = 20$ — Simplify.

$c = \sqrt{20}$ or about 4.47 — Take the square root of each side.

Step 3 Identify the vertices and foci. The hyperbola is horizontal and the vertices are 2 units from the center, so the vertices are at $(1, -2)$ and $(5, -2)$.
The foci are about 4.47 units from the center.
The foci are at $(-1.47, -2)$ and $(7.47, -2)$.

Step 4 Identify the asymptotes.

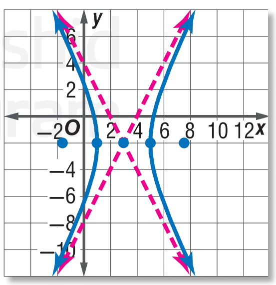
$y - k = \pm\frac{b}{a}(x - h)$ — Equation for asymptotes of a horizontal hyperbola

$y - (-2) = \pm\frac{4}{2}(x - 3)$ — $a = 2, b = 4, h = 3$, and $k = -2$

The equations for the asymptotes are $y = 2x - 8$ and $y = -2x + 4$.

Step 5 Graph the hyperbola. The hyperbola is symmetric about the transverse and conjugate axes. Use this symmetry to plot additional points for the hyperbola.

Use the asymptotes as a guide to draw the hyperbola that passes through the vertices and the other points.

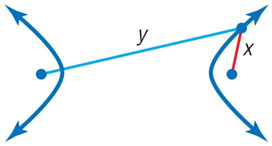


GuidedPractice

3. Graph $\frac{(y-4)^2}{9} - \frac{(x+3)^2}{25} = 1$. Identify the vertices, foci, and asymptotes.

In the equation for any hyperbola, the value of $2a$ represents the **constant difference**. This is the absolute value of the difference between the distances from any point on the hyperbola to the foci of the hyperbola.

Any point on the hyperbola at the right will have the same constant difference, $|y - x|$ or $2a$.



Real-WorldLink

Halley's Comet becomes visible to the unaided eye about every 76 years as it nears the Sun.

Source: NASA

Real-World Example 4 Write an Equation of a Hyperbola

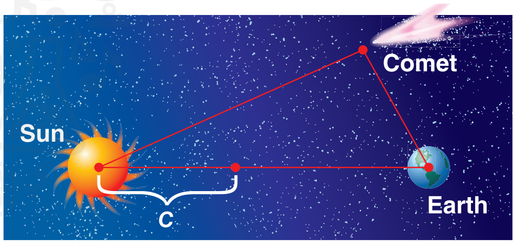
SPACE Earth and the Sun are 146 million kilometers apart. A comet follows a path that is one branch of a hyperbola. Suppose the comet is 30 million kilometers farther from the Sun than from Earth. Determine the equation of the hyperbola centered at the origin for the path of the comet.

Understand We need to determine the equation for the hyperbola.

Plan Find the center and the values of a and b. Once we have this information, we can determine the equation.

Solve The foci are Earth and the Sun, with the origin between them.

The value of c is $146 \div 2$ or 73.



The difference of the distances from the comet to each body is 30. Therefore, a is $30 \div 2$ or 15 million kilometers.

$c^2 = a^2 + b^2$	Equation relating a, b, and c for a hyperbola
$73^2 = 15^2 + b^2$	$a = 15$ and $c = 73$
$5104 = b^2$	Simplify.

The equation of the hyperbola is $\frac{x^2}{225} - \frac{y^2}{5104} = 1$.

Since the comet is farther from the Sun, it is located on the branch of the hyperbola near Earth.

Check (21, 70) is a point that satisfies the equation.

The distance between this point and the Sun (−73, 0) is

$\sqrt{[21 - (-73)]^2 + (70 - 0)^2}$ or 117.2 million kilometers.

The distance between this point and Earth (73, 0) is

$\sqrt{(21 - 73)^2 + (70 - 0)^2}$ or 87.2 million kilometers.

The difference between these distances is 30. ✓

GuidedPractice

4. **SEARCH AND RESCUE** Two receiving stations that are 150 kilometers apart receive a signal from a downed airplane. They determine that the airplane is 80 kilometers farther from station A than from station B. Determine the equation of the hyperbola centered at the origin on which the plane is located.

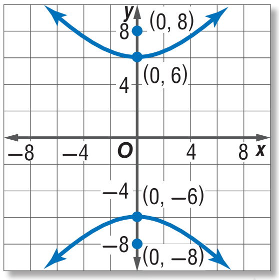
StudyTip

Exact Locations A third receiving station is necessary to determine the plane's exact location.

Check Your Understanding

Examples 1–2 **Write an equation for each hyperbola.**

1.



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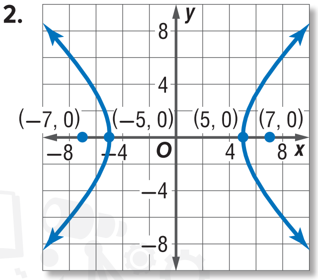

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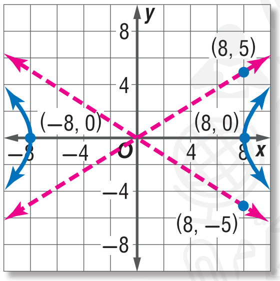

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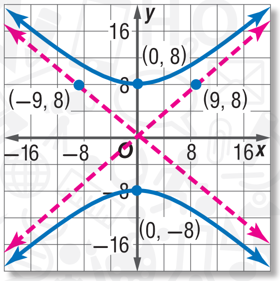


Example 3 **STRUCTURE** **Graph each hyperbola. Identify the vertices, foci, and asymptotes.**

5. $\frac{x^2}{64} - \frac{y^2}{49} = 1$

6. $\frac{y^2}{36} - \frac{x^2}{60} = 1$

7. $9y^2 + 18y - 16x^2 + 64x - 199 = 0$

8. $4x^2 + 24x - y^2 + 4y - 4 = 0$

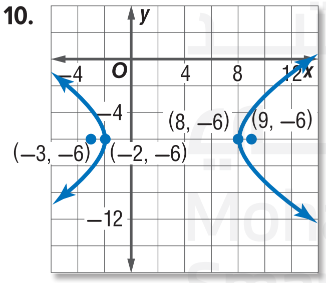
Example 4

9. **NAVIGATION** A ship determines that the difference of its distances from two stations is 60 nautical miles. Write an equation for a hyperbola on which the ship lies if the stations are at (−80, 0) and (80, 0).

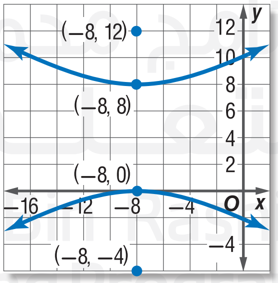
Practice and Problem Solving

Examples 1–2 **Write an equation for each hyperbola.**

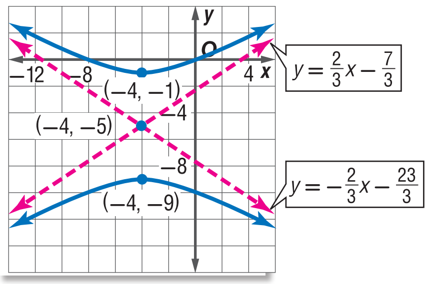
10.



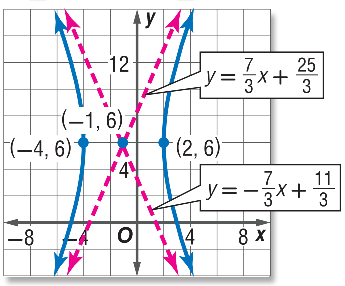
11.



12.



13.



Example 3 **Graph each hyperbola. Identify the vertices, foci, and asymptotes.**

14. $\frac{x^2}{36} - \frac{y^2}{4} = 1$

15. $\frac{y^2}{9} - \frac{x^2}{49} = 1$

16. $\frac{y^2}{36} - \frac{x^2}{25} = 1$

17. $\frac{x^2}{16} - \frac{y^2}{16} = 1$

18. $\frac{(x-3)^2}{16} - \frac{(y+1)^2}{4} = 1$

19. $\frac{(y+5)^2}{16} - \frac{(x+2)^2}{36} = 1$

20. $9y^2 - 4x^2 - 54y + 32x - 19 = 0$

21. $16x^2 - 9y^2 + 128x + 36y + 76 = 0$

22. $25x^2 - 4y^2 - 100x + 48y - 144 = 0$

23. $81y^2 - 16x^2 - 810y + 96x + 585 = 0$

Example 4

24. **NAVIGATION** A ship determines that the difference of its distances from two stations is 80 nautical miles. Write an equation for a hyperbola on which the ship lies if the stations are at $(-100, 0)$ and $(100, 0)$.

Determine whether the following equations represent ellipses or hyperbolas.

25. $4x^2 = 5y^2 + 6$

26. $8x^2 - 2x = 8y - 3y^2$

27. $-5x^2 + 4x = 6y + 3y^2$

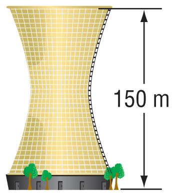
28. $7y - 2x^2 = 6x - 2y^2$

29. $6x - 7x^2 - 5y^2 = 2y$

30. $4x + 6y + 2x^2 = -3y^2$

31. **SPACE** Refer to the application at the beginning of the lesson. With the Sun as a focus and the center at the origin, a certain comet's path follows a branch of a hyperbola. If two of the coordinates of the path are (10, 0) and (30, 100) where the units are in millions of kilometers, determine the equation of the path.

32. **COOLING** Natural draft cooling towers are shaped like hyperbolas for more efficient cooling of power plants. The hyperbola in the tower at the right can be modeled by $\frac{x^2}{16} - \frac{y^2}{225} = 1$, where the units are in meters. Find the width of the tower at the top and at its narrowest point in the middle.



33. **MULTIPLE REPRESENTATIONS** Consider $xy = 16$.

a. **Tabular** Make a table of values for the equation for $-12 \le x \le 12$.

b. **Graphical** Graph the hyperbola represented by the equation.

c. **Logical** Determine and graph the asymptotes for the hyperbola.

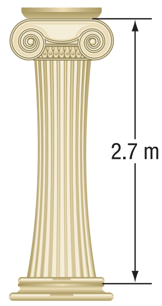
d. **Analytical** What special property do you notice about the asymptotes? Hyperbolas that represent this property are called *rectangular hyperbolas*.

e. **Analytical** Without any calculations, what do you think will be the coordinates of the vertices for $xy = 25$? for $xy = 36$?

34. **MODELING** Two receiving stations that are 250 kilometers apart receive a signal from a downed airplane. They determine that the airplane is 70 kilometers farther from station B than from station A. Determine the equation of the horizontal hyperbola centered at the origin on which the plane is located.

35. **WEATHER** Fatima and Ayesha live exactly 4000 feet apart. While on the phone at their homes, Fatima hears thunder out of her window and Ayesha hears it 3 seconds later out of hers. If sound travels 1100 feet per second, determine the equation for the horizontal hyperbola on which the lightning is located.

36. **ARCHITECTURE** Large pillars with cross sections in the shape of hyperbolas were popular in ancient Greece. The curves can be modeled by the equation $\frac{x^2}{0.16} - \frac{y^2}{4} = 1$, where the units are in meters. If the pillars are 2.7 meters tall, find the width of the top of each pillar and the width of each pillar at the narrowest point in the middle. Round to the nearest hundredth of a meter.



Write an equation for the hyperbola that satisfies each set of conditions.

37. vertices $(-8, 0)$ and $(8, 0)$, conjugate axis of length 20 units

38. vertices $(0, -6)$ and $(0, 6)$, conjugate axis of length 24 units

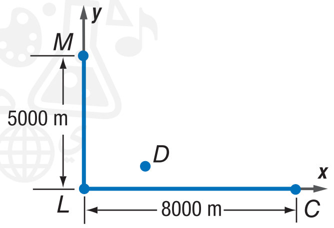
39. vertices $(6, -2)$ and $(-2, -2)$, foci $(10, -2)$ and $(-6, -2)$

40. vertices $(-3, 4)$ and $(-3, -8)$, foci $(-3, 9)$ and $(-3, -13)$

41. centered at the origin with a horizontal transverse axis of length 10 units and a conjugate axis of length 4 units

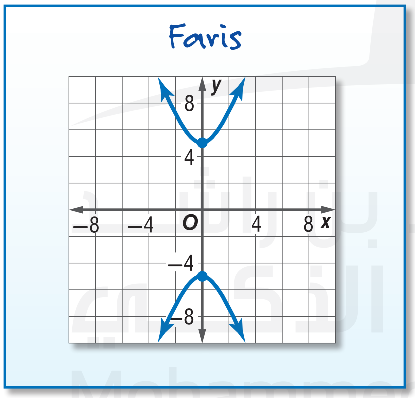
42. centered at the origin with a vertical transverse axis of length 16 units and a conjugate axis of length 12 units

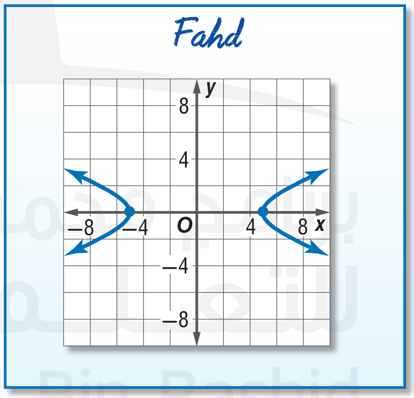
43. **TRIANGULATION** While looking for their lost cat in the woods, Ahmed, Mohammad, and Humaid hear a meow. Mohammad hears it 2 seconds after Ahmed and Humaid hears it 3 seconds after Ahmed. With Ahmed at the origin, determine the exact location of their cat if sound travels 1100 meters per second.



H.O.T. Problems Use Higher-Order Thinking Skills

44. **CRITIQUE** Faris and Fahd are graphing $\frac{y^2}{25} - \frac{x^2}{4} = 1$. Is either of them correct? Explain your reasoning.





45. **CHALLENGE** The origin lies on a horizontal hyperbola. The asymptotes for the hyperbola are $y = -x + 1$ and $y = x - 5$. Find the equation for the hyperbola.

46. **REASONING** What happens to the location of the foci of a hyperbola as the value of a becomes increasingly smaller than the value of b? Explain your reasoning.

47. **REASONING** Consider $\frac{y^2}{36} - \frac{x^2}{16} = 1$. Describe the change in the shape of the hyperbola and the locations of the vertices and foci if 36 is changed to 9. Explain why this happens.

48. **OPEN ENDED** Write an equation for a hyperbola with a focus at the origin.

49. **WRITING IN MATH** Why would you choose a conic section to model a set of data instead of a polynomial function?

Standardized Test Practice

50. You have 6 more 10 fils coins than 25 fils coins. You have a total of AED 5.15. How many 10 fils coins do you have?

A 13 **C** 19
B 16 **D** 25

51. How tall is a shrub that is 15 centimeters shorter than a pole three times as tall as the shrub?

F 24.5 cm
G 22.5 cm
H 21.5 cm
J 7.5 cm

52. SHORT RESPONSE A rectangle is 8 meters long and 6 meters wide. If each dimension is increased by the same number of meters, the area of the new rectangle formed is 32 square meters more than the area of the original rectangle. By how many meters was each dimension increased?

53. SAT/ACT When the equation $y = 4x^2 - 5$ is graphed in the coordinate plane, the graph is which of the following?

A line **D** hyperbola
B circle **E** parabola
C ellipse

Spiral Review

Write an equation for an ellipse that satisfies each set of conditions. (Lesson 6-4)

54. endpoints of major axis at (2, 2) and (2, −10), endpoints of minor axis at (0, −4) and (4, −4)

55. endpoints of major axis at (0, 10) and (0, −10), foci at (0, 8) and (0, −8)

Find the center and radius of the circle with the given equation. Then graph the circle. (Lesson 6-3)

56. $(x - 3)^2 + y^2 = 16$ **57.** $x^2 + y^2 - 6y - 16 = 0$ **58.** $x^2 + y^2 + 9x - 8y + 4 = 0$

59. BASKETBALL Wafa plays basketball for her high school. So far this season, she has made 6 out of 10 free throws. She is determined to improve her free throw percentage. If she can make x consecutive free.

a. Graph the function.

b. What part of the graph is meaningful in the context of the problem?

c. Describe the meaning of the y-intercept.

d. What is the equation of the horizontal asymptote? Explain its meaning with respect to Wafa's shooting percentage.

Solve each equation.

60. $\left(\frac{1}{7}\right)^{y-3} = 343$ **61.** $10^{x-1} = 100^{2x-3}$ **62.** $36^{2p} = 216^{p-1}$

Graph each inequality.

63. $y \geq \sqrt{5x - 8}$ **64.** $y \geq \sqrt{x - 3} + 4$ **65.** $y < \sqrt{6x - 2} + 1$

Skills Review

66. Write an equation for a parabola with vertex at the origin that passes through (2, −8)

67. Write an equation for a parabola with vertex at (−3, −4) that opens up and has y-intercept 8.

LESSON 6-5 Identifying Conic Sections

Then

- You analyzed different conic sections.

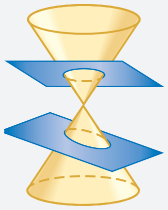
Now

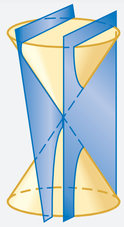
1. Write equations of conic sections in standard form.
2. Identify conic sections from their equations.

Why?

- Parabolas, circles, ellipses, and hyperbolas are called conic sections because they are the cross sections formed when a double cone is sliced by a plane.


Parabola


Circle and Ellipse


Hyperbola

Mathematical Practices

3 Construct viable arguments and critique the reasoning of others.

8 Look for and express regularity in repeated reasoning.

1 Conics in Standard Form

The equation for any conic section can be written in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, and C are not all zero. This general form can be converted to the standard forms below by completing the square.

ConceptSummary Standard Forms of Conic Sections

Conic Section	Standard Form of Equation	
Circle	$(x-h)^2 + (y-k) = r^2$	
	Horizontal Axis	Vertical Axis
Parabola	$y = a(x-h)^2 + k$	$x = a(y-k)^2 + h$
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Example 1 Rewrite an Equation of a Conic Section

Write $16x^2 - 25y^2 - 128x - 144 = 0$ in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

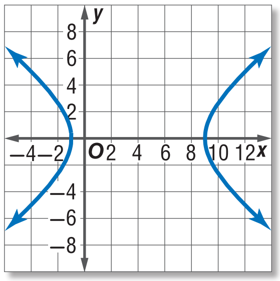
$16x^2 - 25y^2 - 128x - 144 = 0$ Original equation

$16(x^2 - 8x + \blacksquare) - 25y^2 = 144 + 16(\blacksquare)$ Isolate terms.

$16(x^2 - 8x + 16) - 25y^2 = 144 + 16(16)$ Complete the square.

$16(x-4)^2 - 25y^2 = 400$ Perfect square

$\frac{(x-4)^2}{25} - \frac{y^2}{16} = 1$ Divide each side by 400.



The graph is a hyperbola with its center at (4, 0).

GuidedPractice

1. Write $4x^2 + y^2 - 16x + 8y - 4 = 0$ in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

ReviewVocabulary

discriminant the expression $b^2 - 4ac$ from the Quadratic Formula

2 Identify Conic Sections

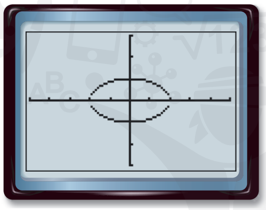
You can determine the type of conic without having to write $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ in standard form. When there is an xy-term ($B \neq 0$), you can use the discriminant to identify the conic. $B^2 - 4AC$ is the discriminant of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

ConceptSummary Classify Conics with the Discriminant

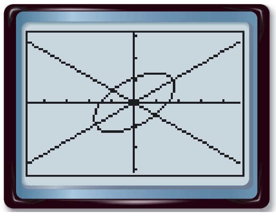
Discriminant	Conic Section
$B^2 - 4AC < 0$; $B = 0$ and $A = C$	circle
$B^2 - 4AC < 0$; either $B \neq 0$ or $A \neq C$	ellipse
$B^2 - 4AC = 0$	parabola
$B^2 - 4AC > 0$	hyperbola

When $B = 0$, the conic will be either vertical or horizontal. When $B \neq 0$, the conic will be neither vertical nor horizontal.

Horizontal Ellipse: $B = 0$



Rotated Ellipse: $B \neq 0$



StudyTip

Identifying Conics When there is no xy-term ($B = 0$), use A and C.
Parabola: A or $C = 0$ but not both.
Circle: $A = C$
Ellipse: A and C have the same sign but are not equal.
Hyperbola: A and C have opposite signs.

Example 2 Analyze an Equation of a Conic Section

Without writing in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

a. $y^2 + 4x^2 - 3xy + 4x - 5y - 8 = 0$

$A = 4$, $B = -3$, and $C = 1$

The discriminant is $(-3)^2 - 4(4)(1)$ or -7.

Because the discriminant is less than 0 and $B \neq 0$, the conic is an ellipse.

b. $3x^2 - 6x + 4y - 5y^2 + 2xy - 4 = 0$

$A = 3$, $B = 2$, and $C = -5$

The discriminant is $2^2 - 4(3)(-5)$ or 64.

Because the discriminant is greater than 0, the conic is a hyperbola.

c. $4y^2 - 8x + 6y - 14 = 0$

$A = 0$, $B = 0$, and $C = 4$

The discriminant is $0^2 - 4(0)(4)$ or 0.

Because the discriminant equals 0, the conic is a parabola.

GuidedPractice

2A. $8y^2 - 6x^2 + 4xy - 6x + 2y - 4 = 0$

2B. $3xy + 4x^2 - 2y + 9x - 3 = 0$

2C. $3x^2 + 16x - 12y + 2y^2 - 6 = 0$

Check Your Understanding

Example 1 **Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.**

1. $x^2 + 4y^2 - 6x + 16y - 11 = 0$
2. $x^2 + y^2 + 12x - 8y + 36 = 0$
3. $9y^2 - 16x^2 - 18y - 64x - 199 = 0$
4. $6y^2 - 24y + 28 - x = 0$

Example 2 **Without writing in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.**

5. $4x^2 + 6y^2 - 3x - 2y = 12$
6. $5y^2 = 2x + 6y - 8 + 3x^2$
7. $8x^2 + 8y^2 + 16x + 24 = 0$
8. $4x^2 - 6y = 8x + 2$
9. $4x^2 - 3y^2 + 8xy - 12 = 2x + 4y$
10. $5xy - 3x^2 + 6y^2 + 12y = 18$
11. $8x^2 + 12xy + 16y^2 + 4y - 3x = 12$
12. $16xy + 8x^2 + 8y^2 - 18x + 8y = 13$

13. **MODELING** A military jet performs for an air show. The path of the plane during one maneuver can be modeled by a conic section with equation $24x^2 + 1000y - 31{,}680x - 45{,}600 = 0$, where distances are represented in feet.
    a. Identify the shape of the curved path of the jet. Write the equation in standard form.
    b. If the jet begins its path upward, or ascent, at $x = 0$, what is the horizontal distance traveled by the jet from the beginning of the ascent to the end of the descent?
    c. What is the maximum height of the jet?

Practice and Problem Solving

Example 1 **Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.**

14. $3x^2 - 2y^2 + 18x + 8y - 35 = 0$
15. $3x^2 + 24x + 4y^2 - 40y + 52 = 0$
16. $x^2 + y^2 = 16 + 6y$
17. $32x + 28 = y - 8x^2$
18. $7x^2 - 8y = 84x - 2y^2 - 176$
19. $x^2 + 8y = 11 + 6x - y^2$
20. $4y^2 = 24y - x - 31$
21. $112y + 64x = 488 + 7y^2 - 8x^2$
22. $28x^2 + 9y^2 - 188 = 56x - 36y$
23. $25x^2 + 384y - 64y^2 + 200x = 1776$

Example 2 **Without writing in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.**

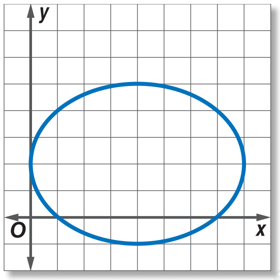
24. $4x^2 - 5y = 9x - 12$
25. $4x^2 - 12x = 18y - 4y^2$
26. $9x^2 + 12y = 9y^2 + 18y - 16$
27. $18x^2 - 16y = 12x - 4y^2 + 19$
28. $12y^2 - 4xy + 9x^2 = 18x - 124$
29. $5xy + 12x^2 - 16x = 5y + 3y^2 + 18$
30. $19x^2 + 14y = 6x - 19y^2 - 88$
31. $8x^2 + 20xy + 18 = 4y^2 - 12 + 9x$
32. $5x - 12xy + 6x^2 = 8y^2 - 24y - 9$
33. $18x - 24y + 324xy = 27x^2 + 3y^2 - 5$

34. **LIGHT** A lamp standing near a wall throws an arc of light in the shape of a conic section. Suppose the edge of the light can be represented by the equation $3y^2 - 2y - 4x^2 + 2x - 8 = 0$. Identify the shape of the edge of the light and graph the equation.

Match each graph with its corresponding equation.

a. $x^2 + y^2 - 8x - 4y = -4$ b. $9x^2 - 16y^2 - 72x + 64y = 64$ c. $9x^2 + 16y^2 = 72x + 64y - 64$

35.



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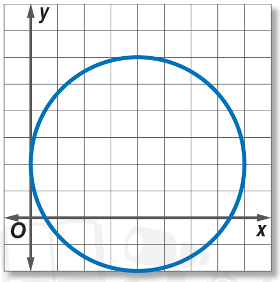

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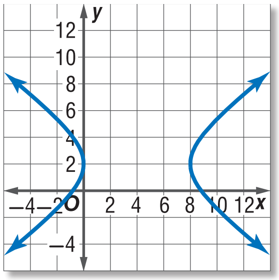


For Exercises 38–41, match each situation with an equation that could be used to represent it.

a. $47.25x^2 - 9y^2 + 18y + 33.525 = 0$

b. $25x^2 + 100y^2 - 1900x - 2200y + 45{,}700 = 0$

c. $16x^2 - 90x + y - 0.25 = 0$

d. $x^2 + y^2 - 18x - 30y - 14{,}094 = 0$

38. **COMPUTERS** the boundary of a wireless network with a range of 120 feet

39. **FITNESS** the oval path of your foot on an exercise machine

40. **COMMUNICATIONS** the position of a cell phone between two cell towers

41. **SPORTS** the height of a ball above the ground after being kicked

42. **SENSE-MAKING** The shape of the cables in a suspension bridge is approximately parabolic. If the towers for a planned bridge are 1000 meters apart and the lowest point of the suspension cables is 200 meters below the top of the towers, write the equation in standard form with the origin at the vertex.

43. **MULTIPLE REPRESENTATIONS** Consider an ellipse with center $(3, -2)$, vertex $M(-1, -2)$, and co-vertex $N(3, -4)$.

a. **Analytical** Determine the standard form of the equation of the ellipse.

b. **Algebraic** Convert part **a** to $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ form.

c. **Graphical** Graph the ellipse.

d. **Analytical** If the ellipse is rotated such that M is moved to $(3, -6)$, determine the location of N and the angle of rotation.

H.O.T. Problems Use Higher-Order Thinking Skills

44. **CHALLENGE** When a plane passes through the vertex of a cone, a *degenerate* conic is formed.

a. Determine the type of conic represented by $4x^2 + 8y^2 = 0$.

b. Graph the conic.

c. Describe the difference between this degenerate conic and a standard conic of the same type with $A = 4$ and $B = 8$.

45. **REASONING** Is the following statement *sometimes*, *always*, or *never* true? Explain.

When a conic is vertical and $A = C$, it is a circle.

46. **OPEN ENDED** Write an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A = 9C$, that represents a parabola.

47. **WRITING IN MATH** Compare and contrast the graphs of the four types of conics and their corresponding equations.

Standardized Test Practice

48. SAT/ACT A class of 25 students took a science test. Ten students had a mean score of 80. The other students had an average score of 60. What is the average score of the whole class?

A 66
B 68
C 70
D 72
E 78

49. Six times a number minus 11 is 43. What is the number?

F 12
G 11
H 10
J 9

50. EXTENDED RESPONSE The amount of water remaining in a storage tank as it is drained can be represented by the equation $L = -4t^2 - 10t + 130$, where L represents the number of liters of water remaining and t represents the number of minutes since the drain was opened. How many liters of water were in the tank initially? Determine to the nearest tenth of a minute how long it will take for the tank to drain completely.

51. Ahmed has a square piece of paper with sides 4 centimeters long. He rolled up the paper to form a cylinder. What is the volume of the cylinder?

A $\frac{4}{\pi}$
B $\frac{16}{\pi}$
C 4π
D 16π

Spiral Review

52. ASTRONOMY Suppose a comet's path can be modeled by a branch of the hyperbola with equation $\frac{y^2}{225} - \frac{x^2}{400} = 1$. Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola. Then graph the hyperbola. (Lesson 6-5)

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse. (Lesson 6-4)

53. $\frac{y^2}{18} + \frac{x^2}{9} = 1$

54. $4x^2 + 8y^2 = 32$

55. $x^2 + 25y^2 - 8x + 100y + 91 = 0$

Graph each function.

56. $f(x) = \frac{3}{x}$

57. $f(x) = \frac{-2}{x + 5}$

58. $f(x) = \frac{6}{x - 2} - 4$

59. SPACE A radioisotope is used as a power source for a satellite. The power output P (in watts) is given by $P = 50e^{-\frac{t}{250}}$, where t is the time in days.

a. Is the formula for power output an example of exponential *growth* or *decay*? Explain your reasoning.

b. Find the power available after 100 days.

c. Ten watts of power are required to operate the equipment in the satellite. How long can the satellite continue to operate?

Skills Review

Solve each system of equations.

60. $6g - 8h = 50$
$6h = 22 - 4g$

61. $3u + 5v = 6$
$2u - 4v = -7$

62. $10m - 9n = 15$
$5m - 4n = 10$

EXTEND

6-5 Graphing Technology Lab
Analyzing Quadratic Relations

You can use graphing technology to analyze quadratic relations.

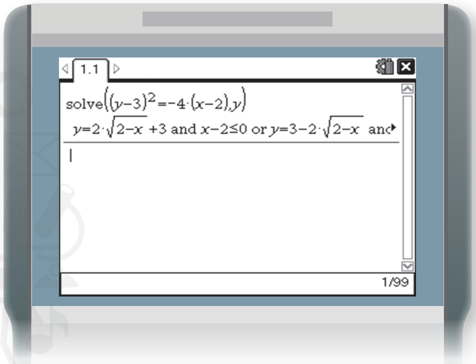
Activity 1 Characteristics of a Parabolic Relation

Graph $f(x) = 9x^2 + 1$, $g(x) = -x^2 + 3x - 4$, and $(y - 3)^2 = -4(x - 2)$. Identify the maxima, minima, and axes of symmetry.

Step 1 Add a new **Graphs** and **Calculator** page.

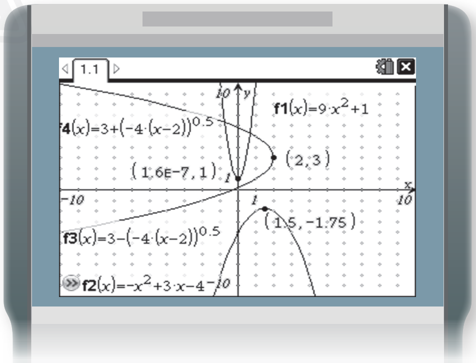
Step 2 Enter $f(x)$ into **f1** and $g(x)$ into **f2**.

Step 3 On the **Calculator** page, use **solve** to solve $(y - 3)^2 = -4(x - 2)$ for y. Hold the **shift** and use ▶ to highlight one equation; then press **ctrl** C. Press **ctrl** ▶ to go to the **Graphs** page; then press **tab ctrl** V. Repeat to copy the other equation into **f4**.



Step 4 Use **Analyze Graph**, **Maximum**, **Minimum**, and **Intersection** to find the coordinates of the extrema. For $f(x)$, the minimum is at (0, 1); for $g(x)$, the maximum is at (1.5, −1.75); for the relation, the vertex is at (2, 3).

Step 5 You can use the coordinates of the extrema to find the equations of the axes of symmetry. For $f(x)$, the equation of the axis of symmetry is $x = 1.5$; for $g(x)$, the equation of the axis of symmetry is $x = 0$; for the relation, the equation of the axis of symmetry is $y = 3$.



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You can also use a graphing calculator to determine the equation of a parabola.

Activity 2 Write an Equation for a Parabola

Given that $f(x)$ has zeros at −2 and 4 and $f(x)$ opens downward, write an equation for the parabola.

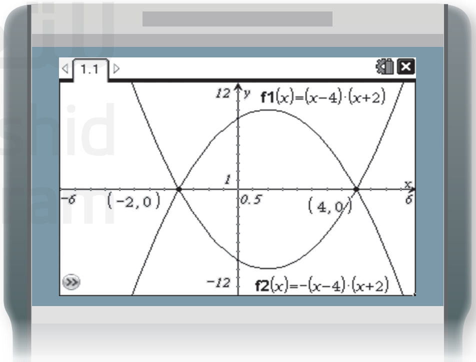
Step 1 Add a new **Graphs** and **Calculator** page.

Step 2 Because the zeros are $x = -2$ and $x = 4$, the factors of the quadratic equation are $(x + 2)$ and $(x - 4)$.

Step 3 Use the **expand** command on the **Calculator** page to multiply $(x + 2)$ and $(x - 4)$. So, $y = x^2 - 2x - 8$.

Step 4 On the **Graphs** page, graph $y = x^2 - 2x - 8$. Verify the roots and direction of opening.

Step 5 The function in the graph opens upward, not downward, so multiply $x^2 - 2x - 8$ by −1. Thus, $y = -x^2 + 2x + 8$. Graph this function.



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An equation for the parabola that has zeros at −2 and 4 and opens downward is $y = -x^2 + 2x + 8$.

(continued on the next page)

Analyzing Quadratic Relations *Continued*

You can use a graphing calculator to determine an equation of a quadratic relation.

Activity 3 Write an Equation for an Ellipse

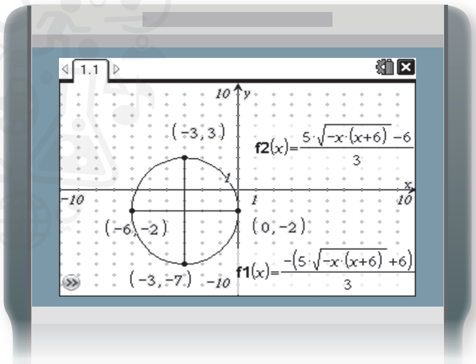
Write an equation for an ellipse that has vertices at (−3, 3) and (−3, −7) and co-vertices at (0, −2) and (−6, −2).

Step 1 Add a new **Graphs** and **Calculator** page.

Step 2 Turn on the grid from **View, Show Axis.** Then use **Points, Points On** to graph the four points. Use **Actions, Coordinates and Equations** to show the coordinates of the points.

Step 3 Use the graph to identify the intersection point of the segment formed by the vertices and the segment formed by the co-vertices. The center is at (−3, −2).

Step 4 Identify other important characteristics. The ellipse is oriented vertically. The length of the major axis is 10, so $a = 5$. The length of the minor axis is 6, so $b = 3$.



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Step 5 Write the equation in standard form.

$$\frac{[y-(-2)]^2}{5^2} + \frac{[x-(-3)]^2}{3^2} = 1 \text{ or}$$

$$\frac{(y+2)^2}{25} + \frac{(x+3)^2}{9} = 1$$

Step 6 Check the equation by using **Solve** under the **Algebra** menu on the **Calculator** page to solve for y. Copy and paste the two equations into the **Graph** page to graph the ellipse.

Exercises

Graph each function and relation. Identify the maxima, minima, and axes of symmetry.

1. $f(x) = -(x-3)^2 + 12$, $g(x) = x^2 + x - 12$, and $(y+5)^2 = -12(x-2)$.

2. $f(x) = -0.25x^2 - 3x - 6$, $g(x) = 2x^2 + 2x + 4$, and $(y+1)^2 = 2(x+6)$.

3. Given that $f(x)$ has zeros at −1 and 3 and $f(x)$ opens upward, write an equation for the parabola.

4. Given that $f(x)$ has zeros at −3 and −1 and $f(x)$ opens downward, write an equation for the parabola.

5. Write an equation for an ellipse that has vertices at (−6, 2) and (−6, −8) and co-vertices at (−3, −3) and (−9, −3).

6. Write an equation for an ellipse that has vertices at (−13, 2) and (1, 2) and co-vertices at (−6, 4) and (−6, 0).

EXPLORE

6-6 Graphing Technology Lab
Linear-Nonlinear Systems

You can use a graphing application to solve linear-nonlinear systems by using the **Y=** menu to graph each equation on the same set of axes.

Example Linear-Quadratic System

Solve the system of equations.
$3y - 4x = -7$
$4x^2 + 3y^2 = 91$

Step 1 Solve each equation for y.

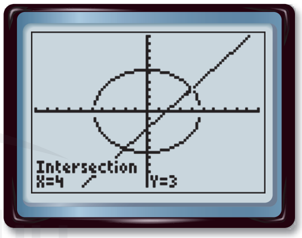
$$3y - 4x = -7 \qquad 4x^2 + 3y^2 = 91$$
$$3y = 4x - 7 \qquad 3y^2 = 91 - 4x^2$$
$$y = \frac{4}{3}x - \frac{7}{3} \qquad y = \pm\sqrt{\frac{91 - 4x^2}{3}}$$

Step 2 Enter $y = \frac{4}{3}x - \frac{7}{3}$ as **Y1**, $y = \sqrt{\frac{91 - 4x^2}{3}}$ as **Y2**, and $y = -\sqrt{\frac{91 - 4x^2}{3}}$ as **Y3**. Then graph the equations in a standard viewing window.

KEYSTROKES: [Y=] 4 [÷] 3 [X,T,θ,n] [−] 7 [÷] 3 [ENTER] [2nd] [x^2] [(] 91 [−] 4 [X,T,θ,n] [x^2] [)] [÷] 3 [)] [ENTER] [(−)] [2nd] [x^2] [(] 91 [−] 4 [X,T,θ,n] [x^2] [)] [÷] 3 [)] [ENTER] [ZOOM] 6

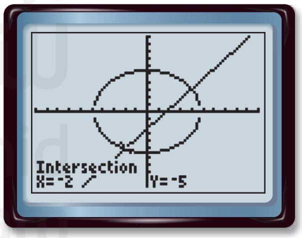
Step 3 Find the intersection of $y = \frac{4}{3}x - \frac{7}{3}$ with $y = \sqrt{\frac{91 - 4x^2}{3}}$.

KEYSTROKES: Press [2nd] [TRACE] 5 [ENTER] [ENTER] [ENTER]. The two graphs intersect at (4, 3).



Step 4 Find the intersection of $y = \frac{4}{3}x - \frac{7}{3}$ with $y = -\sqrt{\frac{91 - 4x^2}{3}}$.

KEYSTROKES: Press [2nd] [TRACE] 5 [ENTER] [▼] [ENTER]. Then use [◄] to move the cursor to the second intersection point. Press [ENTER]. The two graphs intersect at (−2, −5).



The solutions of the system are (4, 3) and (−2, −5).

Exercises

Use a graphing calculator to solve each system of equations.

1. $x^2 + y^2 = 100$
$x + y = 2$

2. $2y - x = 11$
$5x^2 + 2y^2 = 407$

3. $21x + 9y = -36$
$7x^2 + 9y^2 = 1152$

LESSON 6-6 Solving Linear-Nonlinear Systems

Then

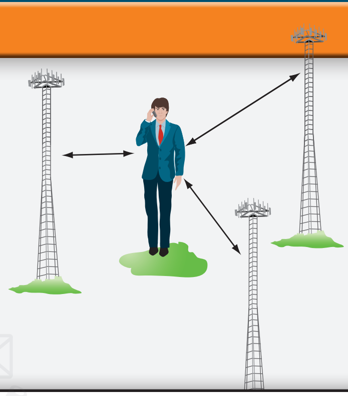
- You solved systems of linear equations.

Now

1. Solve systems of linear and nonlinear equations algebraically and graphically.
2. Solve systems of linear and nonlinear inequalities graphically.

Why?

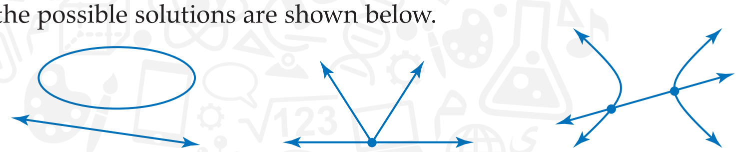
- Have you ever wondered how law enforcement agencies can track a cell phone user's location? A person using a cell phone can be located in respect to three cellular towers. The respective coordinates and distances each tower is from the caller are used to pinpoint the caller's location. This is accomplished using a system of quadratic equations.



Mathematical Practices
6 Attend to precision.

1 Systems of Equations

When a system of equations consists of a linear and a nonlinear equation, the system may have zero, one, or two solutions. Some of the possible solutions are shown below.



no solutions | **one solution** | **two solutions**

You can solve linear-quadratic systems by using graphical or algebraic methods.

Example 1 Linear-Quadratic System

Solve the system of equations.

$$9x^2 + 25y^2 = 225 \quad (1)$$
$$10y + 6x = 6 \quad (2)$$

Step 1 Solve the linear equation for y.

$10y + 6x = 6$ — Equation (2)

$y = -0.6x + 0.6$ — Solve for y.

Step 2 Substitute into the quadratic equation and solve for x.

$9x^2 + 25y^2 = 225$ — Quadratic equation

$9x^2 + 25(-0.6x + 0.6)^2 = 225$ — Substitute $-0.6x + 0.6$ for y.

$9x^2 + 25(0.36x^2 - 0.72x + 0.36) = 225$ — Simplify.

$9x^2 + 9x^2 - 18x + 9 = 225$ — Distribute.

$18x^2 - 18x - 216 = 0$ — Simplify.

$x^2 - x - 12 = 0$ — Divide each side by 18.

$(x - 4)(x + 3) = 0$ — Factor.

$x = 4$ or -3 — Zero Product Property

Step 3 Substitute x-values into the linear equation and solve for y.

$x = 4$		$x = -3$
$y = -0.6x + 0.6$	Equation (2)	$y = -0.6x + 0.6$
$= -0.6(4) + 0.6$	Substitute the x-values	$= -0.6(-3) + 0.6$
$= -1.8$	Simplify.	$= 2.4$

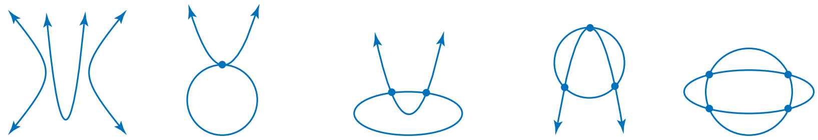
The solutions of the system are $(4, -1.8)$ and $(-3, 2.4)$.

Guided Practice

1A. $3y + x^2 - 4x - 17 = 0$
$3y - 10x + 38 = 0$

1B. $3(y - 4) - 2(x - 3) = -6$
$5x^2 + 2y^2 - 53 = 0$

If a quadratic system contains two conic sections, the system may have anywhere from zero to four solutions. Some graphical representations are shown below.



You can use elimination to solve quadratic-quadratic systems.

Example 2 Quadratic-Quadratic System

Solve the system of equations.

$x^2 + y^2 = 45$ (1)
$y^2 - x^2 = 27$ (2)

$y^2 + x^2 = 45$ Equation (1), Commutative Property
$(+)\ y^2 - x^2 = 27$ Equation (2)
$2y^2 = 72$ Add.
$y^2 = 36$ Divide each side by 2.
$y = \pm 6$ Take the square root of each side.

Substitute 6 and −6 into one of the original equations and solve for x.

$x^2 + y^2 = 45$	Equation (1)	$x^2 + y^2 = 45$
$x^2 + 6^2 = 45$	Substitute for y.	$x^2 + (-6)^2 = 45$
$x^2 = 9$	Subtract 36 from each side.	$x^2 = 9$
$x = \pm 3$	Take the square root of each side.	$x = \pm 3$

The solutions are (−3, −6), (−3, 6), (3, −6), and (3, 6).

StudyTip

Tools If you use **ZSquare** on the **ZOOM** menu, the graph of the first equation will look like a circle.

GuidedPractice

2A. $x^2 + y^2 = 8$
$x^2 + 3y = 10$

2B. $3x^2 + 4y^2 = 48$
$2x^2 - y^2 = -1$

2 Systems of Inequalities

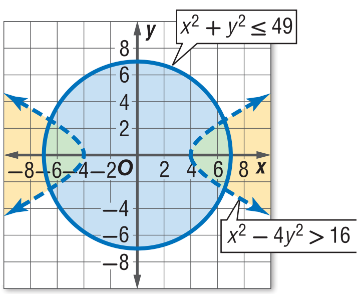
Systems of quadratic inequalities can be solved by graphing.

Example 3 Quadratic Inequalities

Solve the system of inequalities by graphing.

$x^2 + y^2 \le 49$
$x^2 - 4y^2 > 16$

The intersection of the graphs, shaded green, represents the solution of the system.



CHECK (6, 0) is in the shaded area. Use this point to check your solution.

$x^2 + y^2 \le 49$
$6^2 + 0^2 \overset{?}{\le} 49$
$36 \le 49$ ✓

$x^2 - 4y^2 > 16$
$62 - 4(0)^2 \overset{?}{>} 16$
$36 > 16$ ✓

GuidedPractice

3A. $5x^2 + 2y^2 \le 10$
$y \ge x^2 - 2x + 1$

3B. $x^2 - y^2 \le 8$
$x^2 + y^2 \ge 120$

Systems involving absolute value can also be solved by graphing.

StudyTip

Graphing Calculator Like linear inequalities, systems of quadratic and absolute value inequalities can be checked with a graphing calculator.

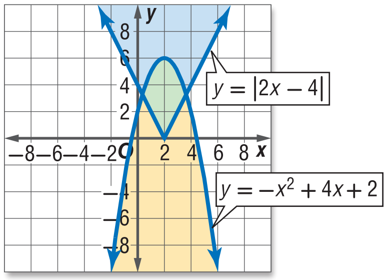
Example 4 Quadratics with Absolute Value

Solve the system of inequalities by graphing.

$$y \geq |2x - 4|$$
$$y \leq -x^2 + 4x + 2$$

Graph the boundary equations. Then shade appropriately.

The intersection of the graphs, shaded green, represents the solution to the system.



CHECK (2, 4) is in the shaded area. Use the point to check your solution.

$y \geq \lvert 2x - 4 \rvert$	$y \leq -x^2 + 4x + 2$
$4 \overset{?}{\geq} \lvert 2(2) - 4 \rvert$	$4 \overset{?}{\leq} -(2)^2 + 4(2) + 2$
$4 \geq 0$ ✓	$4 \leq 6$ ✓

GuidedPractice

4A. $y > |-0.5x + 2|$
$\frac{x^2}{16} + \frac{y^2}{36} \leq 1$

4B. $x^2 + y^2 \leq 49$
$y \geq |x^2 + 1|$

Check Your Understanding

Examples 1–2 **Solve each system of equations.**

1. $8y = -10x$
$y^2 = 2x^2 - 7$

2. $x^2 + y^2 = 68$
$5y = -3x + 34$

3. $y = 12x - 30$
$4x^2 - 3y = 18$

4. $6y^2 - 27 = 3x$
$6y - x = 13$

5. $x^2 + y^2 = 16$
$x^2 - y^2 = 20$

6. $y^2 - 2x^2 = 8$
$3y^2 + x^2 = 52$

7. $x^2 + 2y = 7$
$y^2 - x^2 = 8$

8. $4y^2 - 3x^2 = 11$
$3y^2 + 2x^2 = 21$

9. **PERSEVERANCE** Refer to the beginning of the lesson. A person using a cell phone can be located with respect to three cellular towers. In a coordinate system where one unit represents one kilometer, the location of the caller is determined to be 50 kilometers from the tower at the origin. The person is also 40 kilometers from a tower at (0, 30) and 13 kilometers from a tower at (35, 18). Where is the caller?

Examples 3–4 **Solve each system of inequalities by graphing.**

10. $6x^2 + 9(y - 2)^2 \leq 36$
$x^2 + (y + 3)^2 \leq 25$

11. $16x^2 + 4y^2 \leq 64$
$y \geq -x^2 + 2$

12. $4x^2 - 8y^2 \geq 32$
$y \geq |1.5x| - 8$

13. $x^2 + 8y^2 < 32$
$y < -|x - 2| + 2$

Practice and Problem Solving

Examples 1–2 **Solve each system of equations.**

14. $3x^2 - 2y^2 = -24$
$2y = -3x$

15. $5x^2 + 4y^2 = 20$
$5y = 7x + 35$

16. $x^2 + 3x = -4y - 2$
$y = -2x + 1$

17. $y = 2x$
$4x^2 - 2y^2 = -36$

18. $2y = x + 10$
$y^2 - 4y = 5x + 10$

19. $9y = 8x - 19$
$8x + 11 = 2y^2 + 5y$

20. $2y^2 + 5x^2 = 26$
$2x^2 - y^2 = 5$

21. $x^2 + y^2 = 16$
$x^2 - 4x + y^2 = 12$

22. $x^2 + y^2 = 8$
$5y^2 = 3x^2$

23. $y^2 - x^2 + 3y = 26$
$x^2 + 2y^2 = 34$

24. $x^2 - y^2 = 25$
$x^2 + y^2 + 7 = 0$

25. $x^2 - 10x + 2y^2 = 47$
$y^2 - 2x^2 = -14$

26. **FIREWORKS** Two fireworks are set off simultaneously but from different altitudes. The height y in feet of one is represented by $y = -16t^2 + 120t + 10$, where t is the time in seconds. The height of the other is represented by $y = -16t^2 + 60t + 310$.

a. After how many seconds are the fireworks the same height?

b. What is that height?

Examples 3–4 **TOOLS** **Solve each system of inequalities by graphing.**

27. $x^2 + y^2 \geq 36$
$x^2 + 9(y + 6)^2 \leq 36$

28. $-x > y^2$
$4x^2 + 14y^2 \leq 56$

29. $12x^2 - 4y^2 \geq 48$
$16(x - 4)^2 + 25y^2 < 400$

30. $8y^2 - 3x^2 \leq 24$
$2y > x^2 - 8x + 14$

31. $y > x^2 - 6x + 8$
$x \geq y^2 - 6y + 8$

32. $x^2 + y^2 \geq 9$
$25x^2 + 64y^2 \leq 1600$

33. $16(x - 3)^2 + 4y^2 \leq 64$
$y \leq -|x - 2| + 2$

34. $x^2 - 4x + y^2 + 6y \leq 23$
$y > |x - 2| - 6$

35. $2y - 4 \geq |x + 4|$
$12 - 2y > x^2 + 12x + 36$

36. $18y^2 - 3x^2 \leq 54$
$y \geq |2x| - 6$

37. $x^2 + y^2 < 16$
$y \geq |x - 2| + 6$

38. $x^2 < y - 2$
$y \leq |x + 8| - 7$

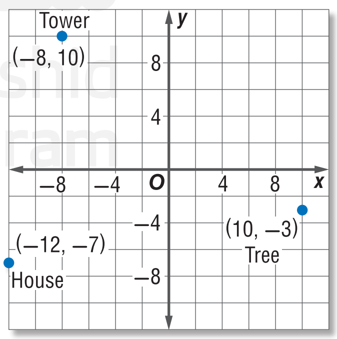
39. **SPACE** Two satellites are placed in orbit about Earth. The equations of the two orbits are $\frac{x^2}{(300)^2} + \frac{y^2}{(900)^2} = 1$ and $\frac{x^2}{(600)^2} + \frac{y^2}{(690)^2} = 1$, where distances are in kilometers and Earth is the center of each curve.

a. Solve each equation for y.

b. Use a graphing calculator to estimate the intersection points of the two orbits.

c. Compare the orbits of the two satellites.

40. **PETS** Asma's cat was missing one day. Fortunately, he was wearing an electronic monitoring device. If the cat is 10 units from the tree, 13 units from the tower, and 20 units from the house, determine the coordinates of his location.



41. **BASEBALL** In 1997, after Mark McGwire hit a home run, the claim was made that the ball would have traveled 538 feet if it had not landed in the stands. The path of the baseball can be modeled by $y = -0.0037x^2 + 1.77x - 1.72$ and the stands can be modeled by $y = \frac{3}{7}x - 128.6$. How far vertically and horizontally from home plate did the ball land in the stands?

42. **ADVERTISING** The corporate logo for an automobile manufacturer is shown at the right. Write a system of three equations to model this logo.

Write a system of equations that satisfies each condition.

43. a circle and an ellipse that intersect at one point

44. a parabola and an ellipse that intersect at two points

45. a hyperbola and a circle that do not intersect

46. an ellipse and a parabola that intersect at three points

47. an ellipse and a hyperbola that intersect at four points

48. **FINANCIAL LITERACY** Prices are often set on an equilibrium curve, where the supply of a certain product equals its corresponding demand by consumers. An economist represents the supply of a product with $y = p^2 + 10p$ and the corresponding demand with $y = -p^2 + 40p$, where p is the price. Determine the equilibrium price.

49. **PAINTBALL** The shape of a paintball field is modeled by $x^2 + 4y^2 = 10{,}000$ in yards where the center is at the origin. The teams are provided with short-range walkie-talkies with a maximum range of 80 yards. Are the teams capable of hearing each other anywhere on the field? Explain your reasoning graphically.

50. **MOVING** Laila is moving to a new city and needs for the location of her new home to satisfy the following conditions.

- It must be less than 10 kilometers from the office where she will work.
- Because of the terrible smell of the local paper mill, it must be at least 15 kilometers away from the mill.

If the paper mill is located 9.5 kilometers east and 6 kilometers north of Laila's office, write and graph a system of inequalities to represent the area(s) were she should look for a home.

H.O.T. Problems Use Higher-Order Thinking Skills

51. **CHALLENGE** Find all values of k for which the following system of equations has two solutions.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad x^2 + y^2 = k^2$$

52. **ARGUMENTS** When the vertex of a parabola lies on an ellipse, how many solutions can the quadratic system represented by the two graphs have? Explain your reasoning using graphs.

53. **OPEN ENDED** Write a system of equations, one a hyperbola and the other an ellipse, for which a solution is $(-4, 8)$.

54. **WRITING IN MATH** Explain how sketching the graph of a quadratic system can help you solve it.

Standardized Test Practice

55. SHORT RESPONSE Solve.

$$4x - 3y = 0$$
$$x^2 + y^2 = 25$$

56. You have 16 stamps. Some are postcard stamps that cost AED 0.23, and the rest cost AED 0.41. If you spent a total of AED 5.30 on the stamps, how many postcard stamps do you have?

A 7
B 8
C 9
D 10

57. Maysa received a promotion and a 7.2% raise. Her new salary is AED 53,600 a year. What was her salary before the raise?

F AED 50,000
G AED 53,600
H AED 55,000
J AED 57,500

58. SAT/ACT When a number is multiplied by $\frac{2}{3}$, the result is 188. Find the number.

A 292
B 282
C 272
D 262
E $125\frac{1}{3}$

Spiral Review

Match each equation with the situation that it could represent. (Lesson 6-6)

a. $9x^2 + 4y^2 - 36 = 0$
b. $0.004x^2 - x + y - 3 = 0$
c. $x^2 + y^2 - 20x + 30y - 75 = 0$

59. SPORTS the flight of a baseball

60. PHOTOGRAPHY the oval opening in a picture frame

61. GEOGRAPHY the set of all points 20 kilometers from a landmark

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola. (Lesson 6-5)

62. $\frac{y^2}{16} - \frac{x^2}{25} = 1$

63. $\frac{(y-3)^2}{25} - \frac{(x-2)^2}{16} = 1$

64. $6y^2 = 2x^2 + 12$

Simplify each expression.

65. $\frac{12p^2 + 6p - 6}{4(p+1)^2} \div \frac{6p - 3}{2p + 10}$

66. $\frac{x^2 + 6x + 9}{x^2 + 7x + 6} \div \frac{4x + 12}{3x + 3}$

67. $\frac{r^2 + 2r - 8}{r^2 + 4r + 3} \div \frac{r - 2}{3r + 3}$

Graph each function. State the domain and range.

68. $f(x) = -\left(\frac{1}{5}\right)^x$

69. $y = -2.5(5)^x$

70. $f(x) = 2\left(\frac{1}{3}\right)^x$

Skills Review

Solve each equation or formula for the specified variable.

71. $d = rt$, for r

72. $x = \frac{-b}{2a}$, for a

73. $V = \frac{1}{3}\pi r^2 h$, for h

74. $A = \frac{1}{2}h(a + b)$, for b

LESSON 6-7 Rotations of Conic Sections

Then

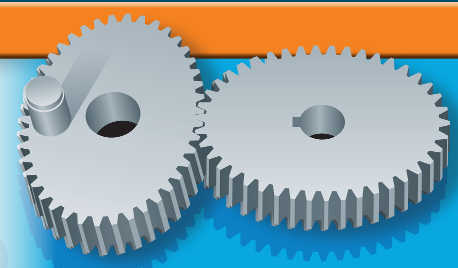
- You identified and graphed conic sections.

Now

1. Find rotation of axes to write equations of rotated conic sections.
2. Graph rotated conic sections.

Why?

- Elliptical gears are paired by rotating them about their foci. The driver gear turns at a constant speed, and the driven gear changes its speed continuously during each revolution.



1 Rotations of Conic Sections

In the previous lesson, you learned that when a conic section is vertical or horizontal with its axes parallel to the x- and y-axis, $B = 0$ in its general equation. The equation of such a conic does not contain an xy-term.

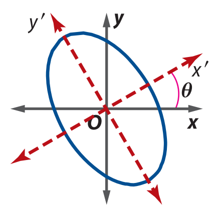
$Ax^2 + Cy^2 + Dx + Ey + F = 0$ — Axes of conic are parallel to coordinate axes.

In this lesson, you will examine conics with axes that are rotated and no longer parallel to the coordinate axes. In the general equation for such rotated conics, $B \neq 0$, so there is an xy-term.

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ — Axes of conic are rotated from coordinate axes.

If the xy-term were eliminated, the equation of the rotated conic could be written in standard form by completing the square. To eliminate this term, we rotate the coordinate axes until they are parallel to the axes of the conic.

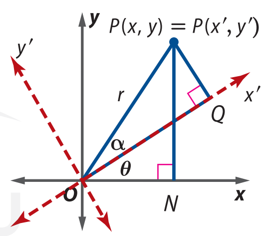
When the coordinate axes are rotated through an angle θ as shown, the origin remains fixed and new axes x' and y' are formed. The equation of the conic in the new $x'y'$-plane has the following general form.



$A(x')^2 + C(y')^2 + Dx' + Ey' + F = 0$ — Equation in $x'y'$-plane

Trigonometry can be used to develop formulas relating a point $P(x, y)$ in the xy-plane and $P(x', y')$ in the $x'y'$ plane.

Consider the figure at the right. Notice that in right triangle PNO, $OP = r$, $ON = x$, $PN = y$, and $m\angle NOP = \alpha + \theta$. Using $\triangle PNO$, you can establish the following relationships.



$x = r\cos(\alpha + \theta)$ — Cosine ratio
$\quad = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$ — Cosine Sum Identity

$y = r\sin(\alpha + \theta)$ — Sine ratio
$\quad = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$ — Sine Sum Identity

Using right triangle POQ, in which $OP = r$, $OQ = x'$, $PQ = y'$, and $m\angle QOP = \alpha$, you can establish the relationships $x' = r\cos\alpha$ and $y' = r\sin\alpha$. Substituting these values into the previous equations, you obtain the following.

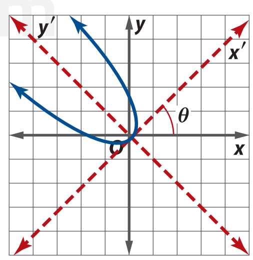
$x = x'\cos\theta - y'\sin\theta$ $\qquad$ $y = y'\cos\theta + x'\sin\theta$

KeyConcept Rotation of Axes of Conics

An equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ in the xy-plane can be rewritten as $A(x')^2 + C(y')^2 + Dx' + Ey' + F = 0$ in the rotated $x'y'$-plane.

The equation in the $x'y'$-plane can be found using the following equations, where θ is the angle of rotation.

$x = x'\cos\theta - y'\sin\theta$ $\qquad$ $y = x'\sin\theta + y'\cos\theta$



Example 1 Write an Equation in the $x'y'$-Plane

Use $\theta = \frac{\pi}{4}$ to write $6x^2 + 6xy + 9y^2 = 53$ in the $x'y'$-plane. Then identify the conic.

Find the equations for x and y.

$$x = x' \cos \theta - y' \sin \theta \qquad \text{Rotation equations for } x \text{ and } y \qquad y = x' \sin \theta + y' \cos \theta$$

$$= \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' \qquad \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \text{ and } \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

Substitute into the original equation.

$$6x^2 + 6xy + 9y^2 = 53$$

$$6\left(\frac{\sqrt{2}x' - \sqrt{2}y'}{2}\right)^2 + 6\left(\frac{\sqrt{2}x' - \sqrt{2}y'}{2}\right)\left(\frac{\sqrt{2}x' + \sqrt{2}y'}{2}\right) + 9\left(\frac{\sqrt{2}x' + \sqrt{2}y'}{2}\right)^2 = 53$$

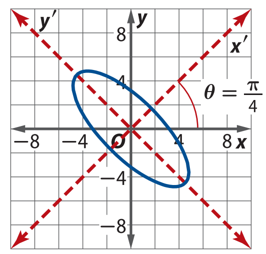
$$\frac{6[2(x')^2 - 4x'y' + 2(y')^2]}{4} + \frac{6[2(x')^2 - 2(y')^2]}{4} + \frac{9[2(x')^2 + 4x'y' + 2(y')^2]}{4} = 53$$

$$3(x')^2 - 6x'y' + 3(y')^2 + 3(x')^2 - 3(y')^2 + \frac{9}{2}(x')^2 + 9x'y' + \frac{9}{2}(y')^2 - 53 = 0$$

$$6(x')^2 - 12\,x'y' + 6(y')^2 + 6(x')^2 - 6(y')^2 + 9(x')^2 + 18x'y' + 9(y')^2 - 106 = 0$$

$$21(x')^2 + 6x'y' + 9(y')^2 - 106 = 0$$

The equation in the $x'y'$-plane is $21(x')^2 + 6x'y' + 9(y')^2 - 106 = 0$. For this equation, $B^2 - 4AC = 6^2 - 4(21)(9)$ or -720. Since $-720 < 0$, the conic is an ellipse as shown.

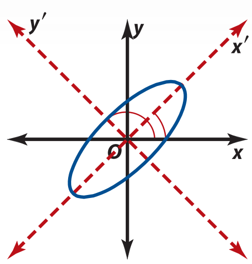


GuidedPractice

1. Use $\theta = \frac{\pi}{6}$ to write $7x^2 + 4\sqrt{3}xy + 3y^2 - 60 = 0$ in the $x'y'$-plane. Then identify the conic.

StudyTip

Angle of Rotation The angle of rotation θ is an acute angle due to the fact that either the x'-axis or the y'-axis will be in the first quadrant. For example, while the plane in the figure below could be rotated 123°, a 33° rotation is all that is needed to align the axes.



When the angle of rotation θ is chosen appropriately, the $x'y'$-term is eliminated from the general form equation, and the axes of the conic will be parallel to the axes of the $x'y'$-plane.

After substituting $x = x' \cos \theta - y' \sin \theta$ and $y = x' \sin \theta + y' \cos \theta$ into the general form of a conic, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, the coefficient of the $x'y'$-term is $B \cos 2\theta + (C - A) \sin 2\theta$. By setting this equal to 0, the $x'y'$-term can be eliminated.

$$B \cos 2\theta + (C - A) \sin 2\theta = 0 \qquad \text{Coefficient of } x'y'\text{-term}$$

$$B \cos 2\theta = -(C - A) \sin 2\theta \qquad \text{Subtract } (C - A) \sin 2\theta \text{ from each side.}$$

$$B \cos 2\theta = (A - C) \sin 2\theta \qquad \text{Distributive Property}$$

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{A - C}{B} \qquad \text{Divide each side by } B \sin 2\theta.$$

$$\cot 2\theta = \frac{A - C}{B} \qquad \frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta$$

KeyConcept Angle of Rotation Used to Eliminate *xy*-Term

An angle of rotation θ such that $\cot 2\theta = \frac{A - C}{B}$, $B \neq 0$, $0 < \theta < \frac{\pi}{2}$, will eliminate the xy-term from the equation of the conic section in the rotated $x'y'$-coordinate system.

Example 2 Write an Equation in Standard Form

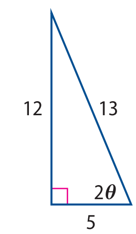
Using a suitable angle of rotation for the conic with equation $8x^2 + 12xy + 3y^2 = 4$, write the equation in standard form.

The conic is a hyperbola because $B^2 - 4AC > 0$. Find θ.

$$\cot 2\theta = \frac{A - C}{B} \qquad \text{Rotation of the axes}$$

$$= \frac{5}{12} \qquad A = 8,\ B = 12,\ \text{and } C = 3$$

The figure illustrates a triangle for which $\cot 2\theta = \frac{5}{12}$. From this, $\sin 2\theta = \frac{12}{13}$ and $\cos 2\theta = \frac{5}{13}$.



Use the half-angle identities to determine $\sin \theta$ and $\cos \theta$.

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}} \qquad \text{Half-Angle Identities} \qquad \cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}}$$

$$= \sqrt{\frac{1 - \frac{5}{13}}{2}} \qquad \cos 2\theta = \frac{5}{13} \qquad = \sqrt{\frac{1 + \frac{5}{13}}{2}}$$

$$= \frac{2\sqrt{13}}{13} \qquad \text{Simplify.} \qquad = \frac{3\sqrt{13}}{13}$$

Next, find the equations for x and y.

$$x = x' \cos \theta - y' \sin \theta \qquad \text{Rotation equations for } x \text{ and } y \qquad y = x' \sin \theta + y' \cos \theta$$

$$= \frac{3\sqrt{13}}{13}x' - \frac{2\sqrt{13}}{13}y' \qquad \sin \theta = \frac{2\sqrt{13}}{13} \text{ and } \cos \theta = \frac{3\sqrt{13}}{13} \qquad = \frac{2\sqrt{13}}{13}x' + \frac{3\sqrt{13}}{13}y'$$

$$= \frac{3\sqrt{13}x' - 2\sqrt{13}y'}{13} \qquad \text{Simplify.} \qquad = \frac{2\sqrt{13}x' + 3\sqrt{13}y'}{13}$$

Substitute these values into the original equation.

$$8x^2 \quad + \quad 12xy \quad + \quad 3y^2 \quad = 4$$

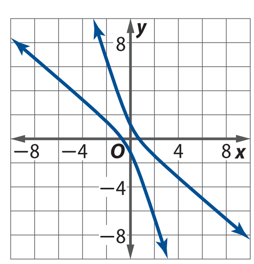
$$8\left(\frac{3\sqrt{13}x' - 2\sqrt{13}y'}{13}\right)^2 + 12\,\frac{3\sqrt{13}x' - 2\sqrt{13}y'}{13} \cdot \frac{2\sqrt{13}x' + 3\sqrt{13}y'}{13} + 3\left(\frac{2\sqrt{13}x' + 3\sqrt{13}y'}{13}\right)^2 = 4$$

$$\frac{72(x')^2 - 96x'y' + 32(y')^2}{13} + \frac{72(x')^2 + 60x'y' - 72(y')^2}{13} + \frac{12(x')^2 + 36x'y' + 27(y')^2}{13} = 4$$

$$\frac{156(x')^2 - 13(y')^2}{13} = 4$$

$$3(x')^2 - \frac{(y')^2}{4} = 1$$

The standard form of the equation in the $x'y'$-plane is $\frac{(x')^2}{\frac{1}{3}} - \frac{(y')^2}{4} = 1$. The graph of this hyperbola is shown.



StudyTip

***x'y'* Term** When you correctly substitute values of x' and y' in for x and y, the coefficient of the $x'y'$ term will become zero. If the coefficient of this term is not zero, then an error has occurred.

GuidedPractice

Using a suitable angle of rotation for the conic with each given equation, write the equation in standard form.

2A. $2x^2 - 12xy + 18y^2 - 4y = 2$

2B. $20x^2 + 20xy + 5y^2 - 12x - 36y - 200 = 0$

Two other formulas relating x' and y' to x and y can be used to find an equation in the xy-plane for a rotated conic.

KeyConcept Rotation of Axes of Conics

When an equation of a conic section is rewritten in the $x'y'$-plane by rotating the coordinate axes through θ, the equation in the xy-plane can be found using

$$x' = x \cos \theta + y \sin \theta, \text{ and } y' = y \cos \theta - x \sin \theta.$$

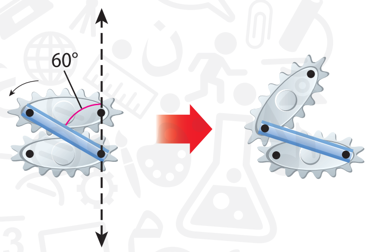
Real-WorldLink

In a system of gears where both gears spin, such as a bicycle, the speed of the gears in relation to each other is related to their size. If the diameter of one of the gears is $\frac{1}{2}$ of the diameter of the second gear, the first gear will rotate twice as fast as the second gear.

Source: *How Stuff Works*

Example 3 Write an Equation in the xy-Plane

PHYSICS **Elliptical gears can be used to generate variable output speeds. After a 60° rotation, the equation for the rotated gear in the $x'y'$-plane is $\frac{(x')^2}{36} + \frac{(y')^2}{18} = 1$. Write an equation for the ellipse formed by the rotated gear in the xy-plane.**



Use the rotation formulas for x' and y' to find the equation of the rotated conic in the xy-plane.

$x' = x \cos \theta + y \sin \theta$	Rotation equations for x' and y'	$y' = y \cos \theta - x \sin \theta$
$= x \cos 60° + y \sin 60°$	$\theta = 60°$	$= y \cos 60° - x \sin 60°$
$= \frac{1}{2}x + \frac{\sqrt{3}}{2}y$	$\sin 60° = \frac{1}{2}$ and $\cos 60° = \frac{\sqrt{3}}{2}$	$= \frac{1}{2}y - \frac{\sqrt{3}}{2}x$

Substitute these values into the original equation.

$$\frac{(x')^2}{36} + \frac{(y')^2}{18} = 1 \quad \text{Original equation}$$

$$(x')^2 + 2(y')^2 = 36 \quad \text{Multiply each side by 36.}$$

$$\left(\frac{x + \sqrt{3}y}{2}\right)^2 + 2\left(\frac{y - \sqrt{3}x}{2}\right)^2 = 36 \quad \text{Substitute.}$$

$$\frac{x^2 + 2\sqrt{3}xy + 3y^2}{4} + \frac{2y^2 - 4\sqrt{3}xy + 6x^2}{4} = 36 \quad \text{Simplify.}$$

$$\frac{7x^2 - 2\sqrt{3}xy + 5y^2}{4} = 36 \quad \text{Combine like terms.}$$

$$7x^2 - 2\sqrt{3}xy + 5y^2 = 144 \quad \text{Multiply each side by 4.}$$

$$7x^2 - 2\sqrt{3}xy + 5y^2 - 144 = 0 \quad \text{Subtract 144 from each side.}$$

The equation of the rotated ellipse in the xy-plane is $7x^2 - 2\sqrt{3}xy + 5y^2 - 144 = 0$.

GuidedPractice

3. If the equation for the gear after a 30° rotation in the $x'y'$-plane is $(x')^2 + 4(y')^2 - 40 = 0$, find the equation for the gear in the xy-plane.

2 Graph Rotated Conics

When the equations of rotated conics are given for the $x'y'$-plane, they can be graphed by finding points on the graph of the conic and then converting these points to the xy-plane.

Example 4 Graph a Conic Using Rotations

Graph $(x' - 2)^2 = 4(y' - 3)$ if it has been rotated 30° from its position in the xy-plane.

The equation represents a parabola, and it is in standard form. Use the vertex (2, 3) and axis of symmetry $x' = 2$ in the $x'y'$-plane to determine the vertex and axis of symmetry for the parabola in the xy-plane.

Find the equations for x and y for $\theta = 30°$.

$$x = x'\cos\theta - y'\sin\theta \qquad \text{Rotation equations for } x \text{ and } y \qquad y = x'\sin\theta + y'\cos\theta$$

$$= \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \qquad \sin 30° = \frac{1}{2} \text{ and } \cos 30° = \frac{\sqrt{3}}{2} \qquad = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$$

Use the equations to convert the $x'y'$-coordinates of the vertex into xy-coordinates.

$$x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \qquad \text{Conversion equation} \qquad y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$$

$$= \frac{\sqrt{3}}{2}(2) - \frac{1}{2}(3) \qquad x' = 2 \text{ and } y' = 3 \qquad = \frac{1}{2}(2) + \frac{\sqrt{3}}{2}(3)$$

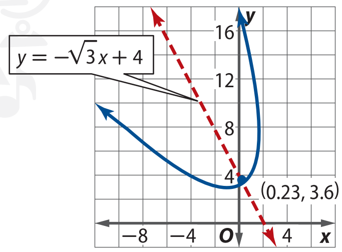
$$= \sqrt{3} - \frac{3}{2} \text{ or about } 0.23 \qquad \text{Multiply.} \qquad = 1 + \frac{3\sqrt{3}}{2} \text{ or about } 3.60$$

Find the equation for the axis of symmetry.

$$x' = x\cos\theta + y\sin\theta \qquad \text{Conversion equation}$$

$$2 = \frac{\sqrt{3}}{2}x + \frac{1}{2}y \qquad \sin 30° = \frac{1}{2} \text{ and } \cos 30° = \frac{\sqrt{3}}{2}$$

$$y = -\sqrt{3}x + 4 \qquad \text{Solve for } y.$$



The new vertex and axis of symmetry can be used to sketch the graph of the parabola in the xy-plane.

StudyTip

Graphing Convert other points on the conic from $x'y'$-coordinates to xy-coordinates. Then make a table of these values to complete the sketch of the conic.

GuidedPractice **Graph each equation at the indicated angle.**

4A. $\frac{(x')^2}{9} - \frac{(y')^2}{32} = 1;\ 60°$ **4B.** $\frac{(x')^2}{16} + \frac{(y')^2}{25} = 1;\ 30°$

One method of graphing conic sections with an xy-term is to solve the equation for y and graph with a calculator. Write the equation in quadratic form and then use the Quadratic Formula.

Example 5 Graph a Conic in Standard Form

Use a graphing calculator to graph the conic given by $4y^2 + 8xy - 60y + 2x^2 - 40x + 155 = 0$.

$$4y^2 + 8xy - 60y + 2x^2 - 40x + 155 = 0 \qquad \text{Original equation}$$

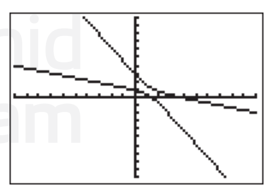
$$4y^2 + (8x - 60)y + (2x^2 - 40x + 155) = 0 \qquad \text{Quadratic form}$$

$$y = \frac{-(8x - 60) \pm \sqrt{(8x - 60)^2 - 4(4)(2x^2 - 40x + 155)}}{2(4)} \qquad a = 4,\ b = 8x - 60, \text{ and } c = 2x^2 - 40x + 155$$

$$= \frac{-8x + 60 \pm \sqrt{32x^2 - 320x + 1120}}{8} \qquad \text{Multiply and combine like terms.}$$

$$= \frac{-8x + 60 \pm 4\sqrt{2x^2 - 20x + 70}}{8} \qquad \text{Factor out } \sqrt{16}.$$

$$= \frac{-2x + 15 \pm \sqrt{2x^2 - 20x + 70}}{2} \qquad \text{Divide each term by 4.}$$



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Graphing both of these equations on the same screen yields the hyperbola shown.

StudyTip

Arranging Terms Arrange the terms in descending powers of y in order to convert the equation to quadratic form.

GuidedPractice

5. Use a graphing calculator to graph the conic given by $4x^2 - 6xy + 2y^2 - 60x - 20y + 275 = 0$.

Exercises

Write each equation in the $x'y'$-plane for the given value of θ. Then identify the conic. (Example 1)

1. $x^2 - y^2 = 9,\ \theta = \frac{\pi}{3}$
2. $xy = -8,\ \theta = 45°$
3. $x^2 - 8y = 0,\ \theta = \frac{\pi}{2}$
4. $2x^2 + 2y^2 = 8,\ \theta = \frac{\pi}{6}$
5. $y^2 + 8x = 0,\ \theta = 30°$
6. $4x^2 + 9y^2 = 36,\ \theta = 30°$
7. $x^2 - 5x + y^2 = 3,\ \theta = 45°$
8. $49x^2 - 16y^2 = 784,\ \theta = \frac{\pi}{4}$
9. $4x^2 - 25y^2 = 64,\ \theta = 90°$
10. $6x^2 + 5y^2 = 30,\ \theta = 30°$

Using a suitable angle of rotation for the conic with each given equation, write the equation in standard form. (Example 1)

11. $xy = -4$
12. $x^2 - xy + y^2 = 2$
13. $145x^2 + 120xy + 180y^2 = 900$
14. $16x^2 - 24xy + 9y^2 - 5x - 90y + 25 = 0$
15. $2x^2 - 72xy + 23y^2 + 100x - 50y = 0$
16. $x^2 - 3y^2 - 8x + 30y = 60$
17. $8x^2 + 12xy + 3y^2 + 4 = 6$
18. $73x^2 + 72xy + 52y^2 + 25x + 50y - 75 = 0$

Write an equation for each conic in the xy-plane for the given equation in $x'y'$ form and the given value of θ. (Example 3)

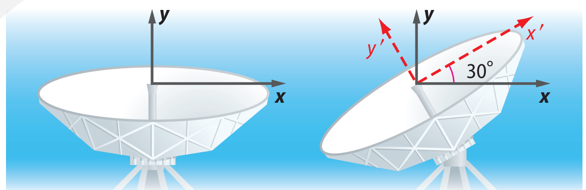
19. $(x')^2 + 3(y')^2 = 8,\ \theta = \frac{\pi}{4}$
20. $\frac{(x')^2}{25} - \frac{(y')^2}{225} = 1,\ \theta = \frac{\pi}{4}$
21. $\frac{(x')^2}{9} - \frac{(y')^2}{36} = 1,\ \theta = \frac{\pi}{3}$
22. $(x')^2 = 8y',\ \theta = 45°$
23. $\frac{(x')^2}{7} + \frac{(y')^2}{28} = 1,\ \theta = \frac{\pi}{6}$
24. $4x' = (y')^2,\ \theta = 30°$
25. $\frac{(x')^2}{64} - \frac{(y')^2}{16} = 1,\ \theta = 45°$
26. $(x')^2 = 5y',\ \theta = \frac{\pi}{3}$
27. $\frac{(x')^2}{4} - \frac{(y')^2}{9} = 1,\ \theta = 30°$
28. $\frac{(x')^2}{3} + \frac{(y')^2}{4} = 1,\ \theta = 60°$

29. **ASTRONOMY** Suppose $144(x')^2 + 64(y')^2 = 576$ models the shape in the $x'y'$-plane of a reflecting mirror in a telescope. (Example 4)
    a. If the mirror has been rotated 30°, determine the equation of the mirror in the xy-plane.
    b. Graph the equation.

Graph each equation at the indicated angle.

30. $\frac{(x')^2}{4} + \frac{(y')^2}{9} = 1;\ 60°$
31. $\frac{(x')^2}{25} - \frac{(y')^2}{36} = 1;\ 45°$
32. $(x')^2 + 6x' - y' = -9;\ 30°$
33. $8(x')^2 + 6(y')^2 = 24;\ 30°$
34. $\frac{(x')^2}{4} - \frac{(y')^2}{16} = 1;\ 45°$
35. $y' = 3(x')^2 - 2x' + 5;\ 60°$

36. **COMMUNICATION** A satellite dish tracks a satellite directly overhead. Suppose $y = \frac{1}{6}x^2$ models the shape of the dish when it is oriented in this position. Later in the day, the dish is observed to have rotated approximately 30°. (Example 4)
    a. Write an equation that models the new orientation of the dish.
    b. Use a graphing calculator to graph both equations on the same screen. Sketch this graph on your paper.



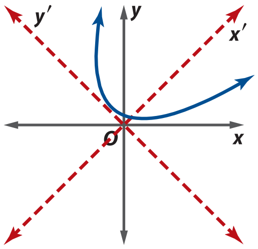
GRAPHING CALCULATOR **Graph the conic given by each equation.** (Example 5)

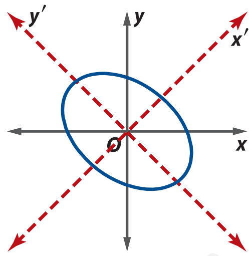
37. $x^2 - 2xy + y^2 - 5x - 5y = 0$
38. $2x^2 + 9xy + 14y^2 = 5$
39. $8x^2 + 5xy - 4y^2 = -2$
40. $2x^2 + 4\sqrt{3}xy + 6y^2 + 3x = y$
41. $2x^2 + 4xy + 2y^2 + 2\sqrt{2}x - 2\sqrt{2}y = -12$
42. $9x^2 + 4xy + 6y^2 = 20$
43. $x^2 + 10\sqrt{3}xy + 11y^2 - 64 = 0$
44. $x^2 + y^2 - 4 = 0$
45. $x^2 - 2\sqrt{3}xy - y^2 + 18 = 0$
46. $2x^2 + 9xy + 14y^2 - 5 = 0$

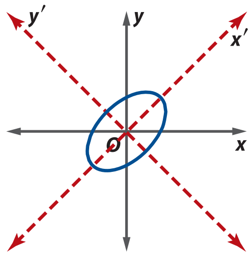
The graph of each equation is a degenerate case. Describe the graph.

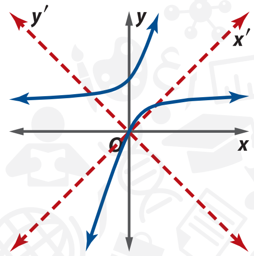
47. $y^2 - 16x^2 = 0$
48. $(x + 4)^2 - (x - 1)^2 = y + 8$
49. $(x + 3)^2 + y^2 + 6y + 9 - 6(x + y) = 18$

Match the graph of each conic with its equation.

50. 

51. 

52. 

53. 

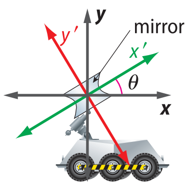
a. $x^2 - xy + y^2 = 2$

b. $145x^2 + 120xy + 180y^2 - 900 = 0$

c. $2x^2 - 72xy + 23y^2 + 100x - 50y = 0$

d. $16x^2 - 24xy + 9y^2 - 5x - 90y + 25 = 0$

54. **ROBOTICS** A hyperbolic mirror used in robotic systems is attached to the robot so that it is facing to the right. After it is rotated, the shape of its new position is represented by $51.75x^2 + 184.5\sqrt{3}xy - 132.75y^2 = 32{,}400$.



a. Solve the equation for y.

b. Use a graphing calculator to graph the equation.

c. Determine the angle θ through which the mirror has been rotated. Round to the nearest degree.

55. **INVARIANTS** When a rotation transforms an equation from the xy-plane to the $x'y'$-plane, the new equation is equivalent to the original equation. Some values are invariant under the rotation, meaning their values do not change when the axes are rotated. Use reasoning to explain how $A + C = A' + C'$ is a rotation invariant.

GRAPHING CALCULATOR Graph each pair of equations and find any points of intersection. If the graphs have no points of intersection, write *no solution*.

56. $x^2 + 2xy + y^2 - 8x - y = 0$
$8x^2 + 3xy - 5y^2 = 15$

57. $9x^2 + 4xy + 5y^2 - 40 = 0$
$x^2 - xy - 2y^2 - x - y + 2 = 0$

58. $x^2 + \sqrt{3}xy - 3 = 0$
$16x^2 - 20xy + 9y^2 = 40$

59. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate angles of rotation that produce the original graphs.

a. TABULAR For each equation in the table, identify the conic and find the minimum angle of rotation needed to transform the equation so that the rotated graph coincides with its original graph.

Equation	Conic	Minimum Angle of Rotation
$x^2 - 5x + 3 - y = 0$		
$6x^2 + 10y^2 = 15$		
$2xy = 9$		

b. VERBAL Describe the relationship between the lines of symmetry of the conics and the minimum angles of rotation needed to produce the original graphs.

c. ANALYTICAL A noncircular ellipse is rotated 50° about the origin. It is then rotated again so that the original graph is produced. What is the second angle of rotation?

H.O.T. Problems Use Higher-Order Thinking Skills

60. **ERROR ANALYSIS** Mahmoud and Ahmed are describing the graph of $x^2 + 4xy + 6y^2 + 3x - 4y = 75$. Mahmoud says that it is an ellipse. Ahmed thinks it is a parabola. Is either of them correct? Explain your reasoning.

61. **CHALLENGE** Show that a circle with the equation $x^2 + y^2 = r^2$ remains unchanged under any rotation θ.

62. **REASONING** *True* or *false*: Every angle of rotation θ can be described as an acute angle. Explain.

63. **PROOF** Prove $x' = x \cos\theta + y \sin\theta$ and $y' = y \cos\theta - x \sin\theta$. (*Hint:* Solve the system $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$ by multiplying one equation by $\sin\theta$ and the other by $\cos\theta$.)

64. **REASONING** The angle of rotation θ can also be defined as $\tan 2\theta = \frac{B}{A - C}$, when $A \neq C$, or $\theta = \frac{\pi}{4}$, when $A = C$. Why does defining the angle of rotation in terms of cotangent not require an extra condition with an additional value for θ?

65. **WRITING IN MATH** The discriminant can be used to classify a conic $A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$ in the $x'y'$-plane. Explain why the values of A' and C' determine the type of conic. Describe the parameters necessary for an ellipse, a circle, a parabola, and a hyperbola.

66. **REASONING** *True* or *false*: Whenever the discriminant of an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is equal to zero, the graph of the equation is a parabola. Explain.

Spiral Review

Graph the hyperbola given by each equation.

67. $\frac{x^2}{9} - \frac{y^2}{64} = 1$

68. $\frac{y^2}{25} - \frac{x^2}{49} = 1$

69. $\frac{(x-3)^2}{64} - \frac{(y-7)^2}{25} = 1$

Determine the eccentricity of the ellipse given by each equation.

70. $\frac{(x+17)^2}{39} + \frac{(y+7)^2}{30} = 1$

71. $\frac{(x-6)^2}{12} + \frac{(y+4)^2}{15} = 1$

72. $\frac{(x-10)^2}{29} + \frac{(y+2)^2}{24} = 1$

73. **INVESTING** Mansour has a total of AED 5000 in his savings account and in a certificate of deposit. His savings account earns 3.5% interest annually. The certificate of deposit pays 5% interest annually if the money is invested for one year. Mansour calculates that his interest earnings for the year will be AED 227.50.

a. Write a system of equations for the amount of money in each investment.

b. Use Cramer's Rule to determine how much money is in Mansour's savings account and in the certificate of deposit.

74. **OPTICS** The amount of light that a source provides to a surface is called the *illuminance*. The illuminance E in foot candles on a surface that is R feet from a source of light with intensity I candelas is $E = \frac{I \cos \theta}{R^2}$, where θ is the measure of the angle between the direction of the light and a line perpendicular to the surface being illuminated. Verify that $E = \frac{I \cot \theta}{R^2 \csc \theta}$ is an equivalent formula.

Solve each equation.

75. $\log_4 8n + \log_4 (n-1) = 2$

76. $\log_9 9p + \log_9 (p+8) = 2$

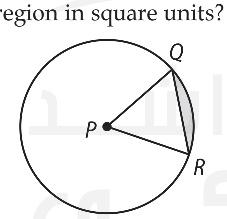
Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$.

77. $f(x) = x^4 - x^3 - 16x^2 + 4x + 48$; $(x-4)$, $(x-2)$

78. $f(x) = 2x^4 + 9x^3 - 23x^2 - 81x + 45$; $(x+5)$, $(x+3)$

Skills Review for Standardized Tests

79. **SAT/ACT** P is the center of the circle and $PQ = QR$. If $\triangle PQR$ has an area of $9\sqrt{3}$ square units, what is the area of the shaded region in square units?



A $24\pi - 9\sqrt{3}$

B $9\pi - 9\sqrt{3}$

C $18\pi - 9\sqrt{3}$

D $6\pi - 9\sqrt{3}$

E $12\pi - 9\sqrt{3}$

80. **REVIEW** Which is NOT the equation of a parabola?

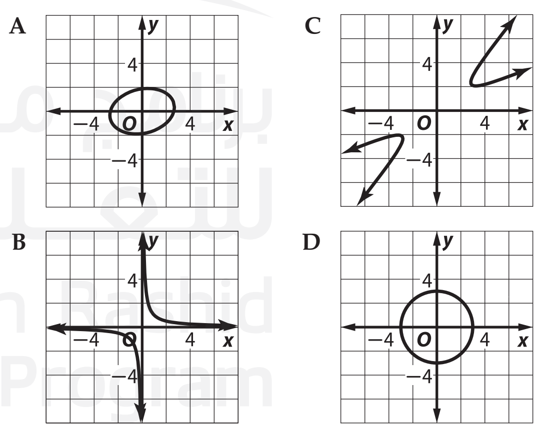
F $y = 2x^2 + 4x - 9$

G $3x + 2y^2 + y + 1 = 0$

H $x^2 + 2y^2 + 8y = 8$

J $x = \frac{1}{2}(y-1)^2 + 5$

81. Which is the graph of the conic given by the equation $4x^2 - 2xy + 8y^2 - 7 = 0$?



82. **REVIEW** How many solutions does the system $\frac{x^2}{5^2} - \frac{y^2}{3^2} = 1$ and $(x-3)^2 + y^2 = 9$ have?

F 0

G 1

H 2

J 4

EXTEND

6-7 Graphing Technology Lab
Systems of Nonlinear Equations and Inequalities

Objective

- Use a graphing calculator to approximate solutions to systems of nonlinear equations and inequalities.

Graphs of conic sections represent a nonlinear system. Solutions of systems of nonlinear equations can be found algebraically. However, approximations can be found by using your graphing calculator. Graphing calculators can only graph functions. To graph a conic section that is not a function, solve the equation for *y*.

Activity 1 Nonlinear System

Solve the system by graphing.

$x^2 + y^2 = 13$

$xy + 6 = 0$

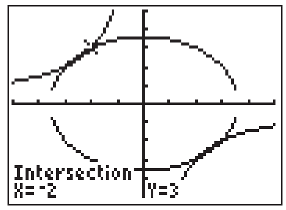
Step 1 Solve each equation for y.

$$y = \sqrt{13 - x^2} \text{ and } y = -\sqrt{13 - x^2} \qquad y = -\frac{6}{x}$$

Step 2 Graph the equations in the appropriate window.

Step 3 Use the intersect feature from the CALC menu to find the four points of intersection.

The solutions are $(-3, 2)$, $(-2, 3)$, $(2, -3)$, and $(3, -2)$.



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Exercises

Solve each system of equations by graphing. Round to the nearest tenth.

1. $xy = 2$
$x^2 - y^2 = 3$

2. $49 = y^2 + x^2$
$x = 1$

3. $x = 2 + y$
$x^2 + y^2 = 100$

4. $25 - 4x^2 = y^2$
$2x + y + 1 = 0$

5. $y^2 = 9 - 3x^2$
$x^2 = 10 - 2y^2$

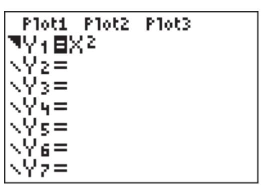
6. $y = -1 - x$
$4 + x = (y - 1)^2$

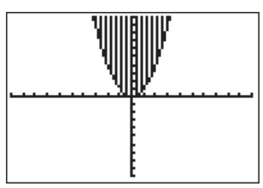
7. **CHALLENGE** A palace contains two square rooms, the family room and the den. The total area of the two rooms is 468 square meters, and the den is 180 square meters smaller than the family room.

a. Write a system of second-degree equations that models this situation.

b. Graph the system found in part **a**, and estimate the length of each room.

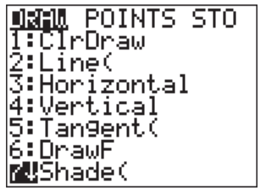
Systems of nonlinear inequalities can also be solved using a graphing calculator. Recall that inequalities can be graphed by using the *greater than* and *less than* commands from the **Y=** menu. An inequality symbol is found by scrolling to the left of the equal sign and pressing ENTER until the shaded triangles are flashing. The triangle above represents *greater than* and the triangle below represents *less than*. The graph of $y \geq x^2$ is shown below.





[−10, 10] scl: 1 by [−10, 10] scl: 1

Inequalities with conic sections that are not functions, such as ellipses, circles, and some hyperbolas, can be graphed by using the Shade(command from the DRAW menu. The restrictive information required is Shade(*lowerfunc, upperfunc, Xleft, Xright, 3, 4*).



This command draws the lower function *lowerfunc* and the upper function *upperfunc* in terms of x. It then shades the area that is above *lowerfunc* and below *upperfunc* between the left and right boundaries *Xleft* and *Xright*. The final two entries *3* and *4* specify the type of shading and can remain constant.

TechnologyTip

Clear Screen To clear any drawings from the calculator screen, select ClrDraw from the DRAW menu.

Activity 2 Nonlinear System of Inequalities

Solve the system of inequalities by graphing.

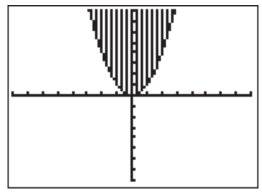
$x^2 + y^2 \leq 36$

$y - x^2 > 0$

Step 1 Solve each inequality for y.

$y \leq \sqrt{36 - x^2}$ and $y \geq -\sqrt{36 - x^2}$ $\qquad y > x^2$

Step 2 Graph $y > x^2$, and shade the correct region. Make each inequality symbol by scrolling to the left of the equal sign and selecting ENTER until the shaded triangles are flashing.



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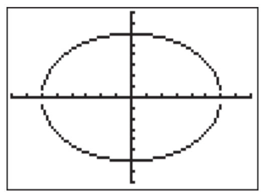
StudyTip

Left and Right Boundaries If the left and right boundaries are not apparent, enter window values that exceed both boundaries. For example, if the boundaries should be $x = -5$ and $x = 5$, entering −10 and 10 will still produce the correct graph.

Step 3 To graph $x^2 + y^2 \leq 36$, the lower boundary is $y = -\sqrt{36 - x^2}$ and the upper boundary is $y = \sqrt{36 - x^2}$. The two halves of the circle meet at $x = -6$ and $x = 6$ as shown.



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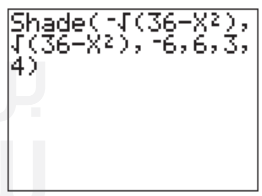
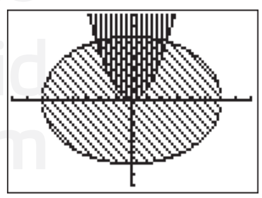
Step 4 From the DRAW menu, select 7: Shade. Enter Shade($-\sqrt{36 - x^2}$, $\sqrt{36 - x}$, −6, 6, 3, 4).



The solution of the system is represented by the double-shaded area.



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Exercises

Solve each system of inequalities by graphing.

8. $2y^2 \leq 32 - 2x^2$
$x + 4 \geq y^2$

9. $y + 5 \geq x^2$
$9y^2 \leq 36 + x^2$

10. $x^2 + 4y^2 \leq 32$
$4x^2 + y^2 \leq 32$

LESSON

6-8 Parametric Equations

Then

- You modeled motion using quadratic functions.

Now

1. Graph parametric equations.
2. Solve problems related to the motion of projectiles.

Why?

- You have used quadratic functions to model the paths of projectiles such as a tennis ball. Parametric equations can also be used to model and evaluate the trajectory and range of projectiles.

NewVocabulary
parametric equation
parameter
orientation
parametric curve

1 Graph Parametric Equations

So far in this text, you have represented the graph of a curve in the xy-plane using a single equation involving two variables, x and y. In this lesson you represent some of these same graphs using two equations by introducing a third variable.

Consider the graphs below, each of which models different aspects of what happens when a certain object is thrown into the air. Figure 6.5.1 shows the vertical distance the object travels as a function of time, while Figure 6.5.2 shows the object's horizontal distance as a function of time. Figure 6.5.3 shows the object's vertical distance as a function of its horizontal distance.

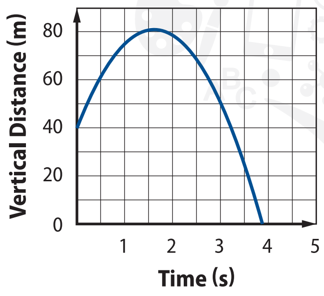


Figure 6.5.1

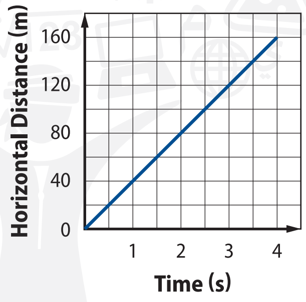


Figure 6.5.2

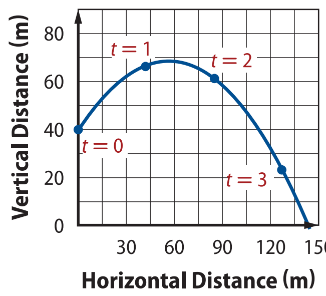


Figure 6.5.3

Each of these graphs and their equations tells part of what is happening in this situation, but not the whole story. To express the position of the object, both horizontally and vertically, as a function of time we can use **parametric equations**. The equations below both represent the graph shown in Figure 6.5.3.

Rectangular Equation

$y = -\frac{2}{225}x^2 + x + 40$

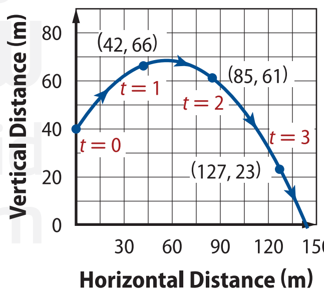
Parametric Equations

$x = 30\sqrt{2}t$ — Horizontal component

$y = -16t^2 + 30\sqrt{2}t + 40$ — Vertical component

From the parametric equations, we can now determine where the object was at a given time by evaluating the horizontal and vertical components for t. For example, when $t = 0$, the object was at $(0, 40)$. The variable t is called a **parameter**.

The graph shown is plotted over the time interval $0 \leq t \leq 4$. Plotting points in the order of increasing values of t traces the curve in a specific direction called the **orientation** of the curve. This orientation is indicated by arrows on the curve as shown.



KeyConcept Parametric Equations

If f and g are continuous functions of t on the interval I, then the set of ordered pairs $(f(t), g(t))$ represent a **parametric curve**. The equations

$$x = f(t) \text{ and } y = g(t)$$

are parametric equations for this curve, t is the parameter, and I is the parameter interval.

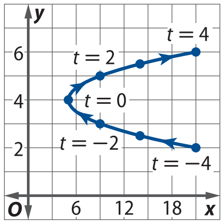
Example 1 Sketch Curves with Parametric Equations

Sketch the curve given by each pair of parametric equations over the given interval.

a. $x = t^2 + 5$ **and** $y = \frac{t}{2} + 4;\ -4 \le t \le 4$

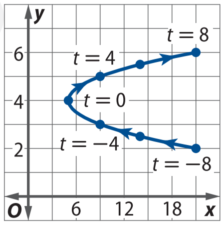
Make a table of values for $-4 \le t \le 4$. Then, plot the (x, y) coordinates for each t-value and connect the points to form a smooth curve. The arrows in the graph indicate the orientation of the curve as t moves from -4 to 4.

t	x	y	t	x	y
−4	21	2	1	6	4.5
−3	14	2.5	2	9	5
−2	9	3	3	14	5.5
−1	6	3.5	4	21	6
0	5	4			



b. $x = \frac{t^2}{4} + 5$ **and** $y = \frac{t}{4} + 4;\ -8 \le t \le 8$

t	x	y	t	x	y
−8	21	2	2	6	4.5
−6	14	2.5	4	9	5
−4	9	3	6	14	5.5
−2	6	3.5	8	21	6
0	5	4			



Guided Practice

1A. $x = 3t$ and $y = \sqrt{t} + 6;\ 0 \le t \le 8$

1B. $x = t^2$ and $y = 2t + 3;\ -10 \le t \le 10$

> **Study Tip**
> **Plane Curves** Parametric equations can be used to represent curves that are not functions, as shown in Example 1.

Notice that the two different sets of parametric equations in Example 1 trace out the same curve. The graphs differ in their *speeds* or how rapidly each curve is traced out. If t represents time in seconds, then the curve in part **b** is traced in 16 seconds, while the curve in part **a** is traced out in 8 seconds.

Another way to determine the curve represented by a set of parametric equations is to write the set of equations in rectangular form. This can be done using substitution to eliminate the parameter.

Example 2 Write in Rectangular Form

Write $x = -3t$ **and** $y = t^2 + 2$ **in rectangular form.**

To eliminate the parameter t, solve $x = -3t$ for t. This yields $t = -\frac{1}{3}x$. Then substitute this value for t in the equation for y.

$y = t^2 + 2$ — Equation for y

$= \left(-\frac{1}{3}x\right)^2 + 2$ — Substitute $-\frac{1}{3}x$ for t.

$= \frac{1}{9}x^2 + 2$ — Simplify.

This set of parametric equations yields the parabola $y = \frac{1}{9}x^2 + 2$.

Guided Practice

2. Write $x = t^2 - 5$ and $y = 4t$ in rectangular form.

> **Study Tip**
> **Eliminating a Parameter** When you are eliminating a parameter to convert to rectangular form, you can solve either of the parametric equations first.

In Example 2, notice that a parameter interval for t was not specified. When not specified, the implied parameter interval is all values for t which produce real number values for x and y.

Sometimes the domain must be restricted after converting from parametric to rectangular form.

Example 3 Rectangular Form with Domain Restrictions

Write $x = \frac{1}{\sqrt{t}}$ and $y = \frac{t+1}{t}$ in rectangular form. Then graph the equation. State any restrictions on the domain.

To eliminate t, square each side of $x = \frac{1}{\sqrt{t}}$. This yields $x^2 = \frac{1}{t}$, so $t = \frac{1}{x^2}$. Substitute this value for t in parametric equation for y.

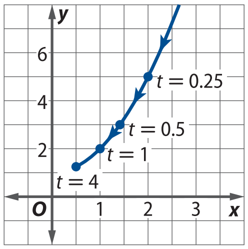
$y = \frac{t+1}{t}$ — Parametric equation for y

$= \frac{\frac{1}{x^2} + 1}{\frac{1}{x^2}}$ — Substitute $\frac{1}{x^2}$ for t.

$= \frac{\frac{x^2+1}{x^2}}{\frac{1}{x^2}}$ — Simplify the numerator.

$= x^2 + 1$ — Simplify.

While the rectangular equation is $y = x^2 + 1$, the curve is only defined for $t > 0$. From the parametric equation $x = \frac{1}{\sqrt{t}}$, the only possible values for x are values greater than zero. As shown in the graph, the domain of the rectangular equation needs to be restricted to $x > 0$.



GuidedPractice

3. Write $x = \sqrt{t + 4}$ and $y = \frac{1}{t}$ in rectangular form. Graph the equation. State any restrictions on the domain.

The parameter in a parametric equation can also be an angle, θ.

Example 4 Rectangular Form with θ as Parameter

Write $x = 2\cos\theta$ and $y = 4\sin\theta$ in rectangular form. Then graph the equation.

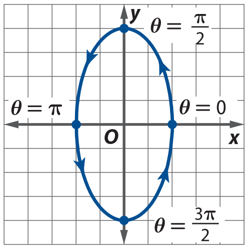
To eliminate the angular parameter θ, first solve the equations for $\cos\theta$ and $\sin\theta$ to obtain $\cos\theta = \frac{x}{2}$ and $\sin\theta = \frac{y}{4}$. Then use the Pythagorean Identity to eliminate the parameter θ.

$\cos^2\theta + \sin^2\theta = 1$ — Pythagorean Identity

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$ — $\cos\theta = \frac{x}{2}$ and $\sin\theta = \frac{y}{4}$

$\frac{x^2}{4} + \frac{y^2}{16} = 1$ — Simplify.

You should recognize this equation as that of an ellipse centered at the origin with vertices at (0, 4) and (0, −4) and covertices at (2, 0) and (−2, 0) as shown. As θ varies from 0 to 2π, the ellipse is traced out counterclockwise.



TechnologyTip

Parameters When graphing parametric equations on a calculator, θ and t are interchangeable.

GuidedPractice

4. Write $x = 3\sin\theta$ and $y = 8\cos\theta$ in rectangular form. Then sketch the graph.

As you saw in Example 1, parametric representations of rectangular graphs are not unique. By varying the definition for the parameter, you can obtain parametric equations that produce graphs that vary only in speed and/or orientation.

StudyTip

Parametric Form The easiest method of converting an equation from rectangular to parametric form is to use $x = t$. When this is done, the other parametric equation is the original equation with t replacing x.

Example 5 Write Parametric Equations from Graphs

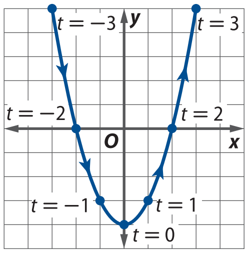
Use each parameter to write the parametric equations that can represent $y = x^2 - 4$. Then graph the equation, indicating the speed and orientation.

a. $t = x$

$y = x^2 - 4$ — Original equation

$= t^2 - 4$ — Substitute for x in original equation.

The parametric equations are $x = t$ and $y = t^2 - 4$. The associated speed and orientation are indicated on the graph.



b. $t = 4x + 1$

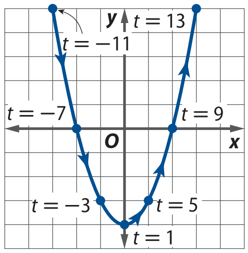
$x = \frac{t - 1}{4}$ — Solve for x.

$y = \left(\frac{t - 1}{4}\right)^2 - 4$ — Substitute for x in original equation.

$= \frac{t^2}{16} - \frac{t}{8} - \frac{63}{16}$ — Simplify.

$x = \frac{t - 1}{4}$ and $y = \frac{t^2}{16} - \frac{t}{8} - \frac{63}{16}$ are the parametric equations.

Notice that the speed is much slower than part **a**.



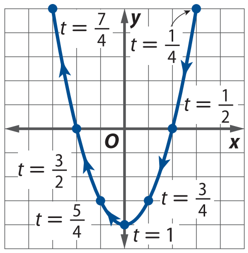
c. $t = 1 - \frac{x}{4}$

$4 - 4t = x$ — Solve for x.

$y = (4 - 4t)^2 - 4$ — Substitute for x in original equation.

$= 16t^2 - 32t + 12$ — Simplify.

The parametric equations are $x = 4 - 4t$ and $y = 16t^2 - 32t + 12$. Notice that the speed is much faster than part **a**. The orientation is also reversed, as indicated by the arrows.



GuidedPractice

Use each parameter to determine the parametric equations that can represent $x = 6 - y^2$. Then graph the equation, indicating the speed and orientation.

5A. $t = x + 1$ **5B.** $t = 3x$ **5C.** $t = 4 - 2x$

2 Projectile Motion

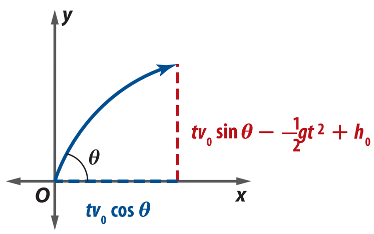
Parametric equations are often used to simulate projectile motion. The path of a projectile launched at an angle other than 90° with the horizontal can be modeled by the following parametric equations.

KeyConcept Projectile Motion

For an object launched at an angle θ with the horizontal at an initial velocity v_0, where g is the gravitational constant, t is time, and h_0 is the initial height:

Horizontal Distance $x = tv_0 \cos \theta$

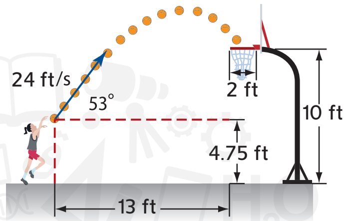
Vertical Position $y = tv_0 \sin \theta - \frac{1}{2}gt^2 + h_0$



Real-World Example 6 Projectile Motion

BASKETBALL Khadija is practicing free throws for an upcoming basketball game. She releases the ball with an initial velocity of 24 feet per second at an angle of 53° with the horizontal. The horizontal distance from the free throw line to the front rim of the basket is 13 feet. The vertical distance from the floor to the rim is 10 feet. The front of the rim is 2 feet from the backboard. She releases the shot 4.75 feet from the ground. Does Khadija make the basket?

Make a diagram of the situation.



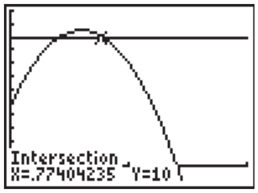
> **StudyTip**
>
> **Gravity** At the surface of Earth, the acceleration due to gravity is 9.8 meters per second squared or 32 feet per second squared. When solving problems, be sure to use the appropriate value for gravity based on the units of the velocity and position.

To determine whether she makes the shot, you need the horizontal distance that the ball has traveled when the height of the ball is 10 feet. First, write a parametric equation for the vertical position of the ball.

$y = tv_0 \sin \theta - \frac{1}{2}gt^2 + h_0$ — Parametric equation for vertical position

$\quad = t(24) \sin 53 - \frac{1}{2}(32)t^2 + 4.75$ — $v_0 = 24$, $\theta = 53°$, $g = 32$, and $h_0 = 4.75$

Graph the equation for the vertical position and the line $y = 10$. The curve will intersect the line in two places. The second intersection represents the ball as it is moving down toward the basket. Use 5: intersect on the CALC menu to find the second point of intersection with $y = 10$. The value is about 0.77 second.



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Determine the horizontal position of the ball at 0.77 second.

$x = tv_0 \cos \theta$ — Parametric equation for horizontal position

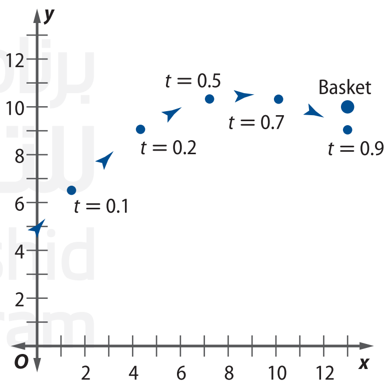
$\quad = 0.77(24) \cos 53$ — $v_0 = 24$, $\theta = 53°$, and $t \approx 0.77$

$\quad \approx 11.1$ — Use a calculator.

Because the horizontal position is less than 13 feet when the ball reaches 10 feet for the second time, the shot is short of the basket. Khadija does not make the free throw.

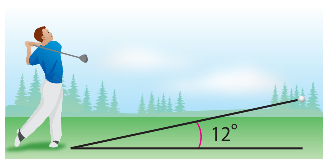
CHECK You can confirm the results of your calculation by graphing the parametric equations and determining the path of the ball in relation to the basket.

t	x	y	t	x	y
0	0	4.75	0.5	7.22	10.33
0.1	1.44	6.51	0.6	8.67	10.49
0.2	2.89	7.94	0.7	10.11	10.32
0.3	4.33	9.06	0.8	11.55	9.84
0.4	5.78	9.86	0.9	13.00	9.04



GuidedPractice

6. **GOLF** Saeed drives a golf ball with an initial velocity of 56 meters per second at an angle of 12° down a flat driving range. How far away will the golf ball land?



> **Real-WorldLink**
>
> In April 2007, Morgan Pressel became the youngest woman ever to win a major LPGA championship.
>
> **Source:** *LPGA*

Exercises

Sketch the curve given by each pair of parametric equations over the given interval. (Example 1)

1. $x = t^2 + 3$ and $y = \frac{t}{4} - 5$; $-5 \leq t \leq 5$
2. $x = \frac{t^2}{2}$ and $y = -4t$; $-4 \leq t \leq 4$
3. $x = -\frac{5t}{2} + 4$ and $y = t^2 - 8$; $-6 \leq t \leq 6$
4. $x = 3t + 6$ and $y = \sqrt{t} + 1$; $0 \leq t \leq 9$
5. $x = 2t - 1$ and $y = -\frac{t^2}{2} + 7$; $-4 \leq t \leq 4$
6. $x = -2t^2$ and $y = \frac{t}{3} - 6$; $-6 \leq t \leq 6$
7. $x = \frac{t}{2}$ and $y = -\sqrt{t} + 5$; $0 \leq t \leq 8$
8. $x = t^2 - 4$ and $y = 3t - 8$; $-5 \leq t \leq 5$

Write each pair of parametric equations in rectangular form. Then graph the equation and state any restrictions on the domain. (Examples 2 and 3)

9. $x = 2t - 5$, $y = t^2 + 4$
10. $x = 3t + 9$, $y = t^2 - 7$
11. $x = t^2 - 2$, $y = 5t$
12. $x = t^2 + 1$, $y = -4t + 3$
13. $x = -t - 4$, $y = 3t^2$
14. $x = 5t - 1$, $y = 2t^2 + 8$
15. $x = 4t^2$, $y = \frac{6t}{5} + 9$
16. $x = \frac{t}{3} + 2$, $y = \frac{t^2}{6} - 7$

17. **MOVIE STUNTS** During the filming of a movie, a stunt double leaps off the side of a building. The pulley system connected to the stunt double allows for a vertical fall modeled by $y = -16t^2 + 15t + 100$, and a horizontal movement modeled by $x = 4t$, where x and y are measured in feet and t is measured in seconds. Write and graph an equation in rectangular form to model the stunt double's fall for $0 \leq t \leq 3$. (Example 3)

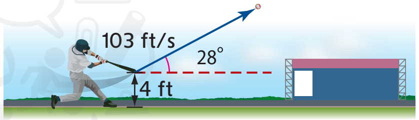
Write each pair of parametric equations in rectangular form. Then graph the equation. (Example 4)

18. $x = 3 \cos \theta$ and $y = 5 \sin \theta$
19. $x = 7 \sin \theta$ and $y = 2 \cos \theta$
20. $x = 6 \cos \theta$ and $y = 4 \sin \theta$
21. $x = 3 \cos \theta$ and $y = 3 \sin \theta$
22. $x = 8 \sin \theta$ and $y = \cos \theta$
23. $x = 5 \cos \theta$ and $y = 6 \sin \theta$
24. $x = 10 \sin \theta$ and $y = 9 \cos \theta$
25. $x = \sin \theta$ and $y = 7 \cos \theta$

Use each parameter to write the parametric equations that can represent each equation. Then graph the equations, indicating the speed and orientation. (Example 5)

26. $t = 3x - 2$; $y = x^2 + 9$
27. $t = 8x$; $y^2 = 9 - x^2$
28. $t = 2 - \frac{x}{3}$; $y = \frac{x^2}{12}$
29. $t = \frac{x}{5} + 4$; $y = 10 - x^2$
30. $t = 4x + 7$; $y = \frac{x^2 - 1}{2}$
31. $t = \frac{1 - x}{2}$; $y = \frac{3 - x^2}{4}$

32. **BASEBALL** A baseball player hits the ball at a 28° angle with an initial speed of 103 feet per second. The bat is 4 feet from the ground at the time of impact. Assuming that the ball is not caught, determine the distance traveled by the ball. (Example 6)



33. **PLAY BALL** Obaid attempts a 43-yard goal. He kicks the ball at a 41° angle with an initial speed of 70 feet per second. The goal is 15 feet high. Is the kick long enough to make the goal? (Example 6)

Write each pair of parametric equations in rectangular form. Then state the restriction on the domain.

34. $x = \sqrt{t} + 4$, $y = 4t + 3$
35. $x = \log t$, $y = t + 3$
36. $x = \sqrt{t - 7}$, $y = -3t - 8$
37. $x = \log (t - 4)$, $y = t$
38. $x = \frac{1}{\sqrt{t + 3}}$, $y = t$
39. $x = \frac{1}{\log (t + 2)}$, $y = 2t - 4$

40. **TENNIS** Mazen hits a tennis ball 55 centimeters above the ground at an angle of 15° with the horizontal. The ball has an initial speed of 18 meters per second.
    a. Use a graphing calculator to graph the path of the tennis ball using parametric equations.
    b. How long does the ball stay in the air before hitting the ground?
    c. If Mazen is 10 meters from the net and the net is 1.5 meters above the ground, will the tennis ball clear the net? If so, by how many meters? If not, by how many meters is the ball short?

Write a set of parametric equations for the line or line segment with the given characteristics.

41. line with a slope of 3 that passes through (4, 7)
42. line with a slope of −0.5 that passes through (3, −2)
43. line segment with endpoints (−2, −6) and (2, 10)
44. line segment with endpoints (7, 13) and (13, 11)

Match each set of parametric equations with its graph.

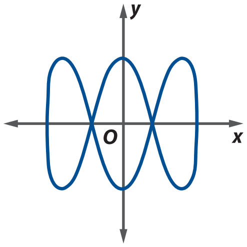
45. $x = \cos 2t$, $y = \sin 4t$

46. $x = \cos 3t$, $y = \sin t$

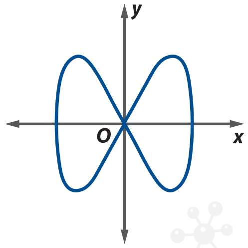
47. $x = \cos t$, $y = \sin 3t$

48. $x = \cos 4t$, $y = \sin 3t$

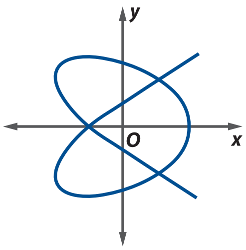
a.



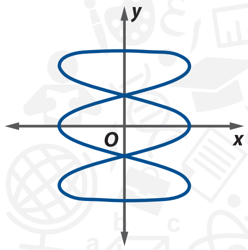
b.



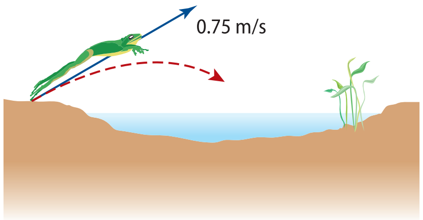
c.



d.



49. **BIOLOGY** A frog jumps off the bank of a creek with an initial velocity of 0.75 meter per second at an angle of 45° with the horizontal. The surface of the creek is 0.3 meter below the edge of the bank. Let g equal 9.8 meters per second squared.



a. Write the parametric equations to describe the position of the frog at time t. Assume that the surface of the water is located at the line $y = 0$.

b. If the creek is 0.5 meter wide, will the frog reach the other bank, which is level with the surface of the creek? If not, how far from the other bank will it hit the water?

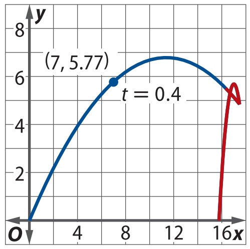
c. If the frog was able to jump on a lily pad resting on the surface of the creek 0.4 meter away and stayed in the air for 0.38 second, what was the initial speed of the frog?

50. **RACE** Hala and Hidaya are competing in a 100-meter dash. When the starter gun fires, Hala runs 8.0 meters per second after a 0.1 second delay from the point (0, 2) and Hidaya runs 8.1 meters per second after a 0.3 second delay from the point (0, 5).

a. Using the y-axis as the starting line and assuming that the women run parallel to the x-axis, write parametric equations to describe each runner's position after t seconds.

b. Who wins the race? If the women ran 200 meters instead of 100 meters, who would win? Explain your answer.

51. **FOOTBALL** The graph below models the path of a football kicked by one player and then headed back by another player. The path of the initial kick is shown in blue, and the path of the headed ball is shown in red.



a. If the ball is initially kicked at an angle of 50°, find the initial speed of the ball.

b. At what time does the ball reach the second player if the second player is standing about 17.5 feet away?

c. If the second player heads the ball at an angle of 75°, an initial speed of 8 feet per second, and at a height of 4.75 feet, approximately how long does the ball stay in the air from the time it is first kicked until it lands?

52. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate a *cycloid*, the curve created by the path of a point on a circle with a radius of 1 unit as it is rolled along the x-axis.

a. **GRAPHICAL** Use a graphing calculator to graph the parametric equations $x = t - \sin t$ and $y = 1 - \cos t$, where t is measured in radians.

b. **ANALYTICAL** What is the distance between x-intercepts? Describe what the x-intercepts and the distance between them represent.

c. **ANALYTICAL** What is the maximum value of y? Describe what this value represents and how it would change for circles of differing radii.

H.O.T. Problems Use Higher-Order Thinking Skills

53. **CHALLENGE** Consider a line ℓ with parametric equations $x = 2 + 3t$ and $y = -t + 5$. Write a set of parametric equations for the line m perpendicular to ℓ containing the point (4, 10).

54. **WRITING IN MATH** Explain why there are infinitely many sets of parametric equations to describe one line in the xy-plane.

55. **REASONING** Determine whether parametric equations for projectile motion can apply to objects thrown at an angle of 90°. Explain your reasoning.

56. **CHALLENGE** A line in three-dimensional space contains the points $P(2, 3, -8)$ and $Q(-1, 5, -4)$. Find two sets of parametric equations for the line.

57. **WRITING IN MATH** Explain the advantage of using parametric equations versus rectangular equations when analyzing the horizontal/vertical components of a graph.

Spiral Review

Graph each equation at the indicated angle.

58. $\frac{(x')^2}{9} - \frac{(y')^2}{4} = 1$ at a 60° rotation from the xy-axis

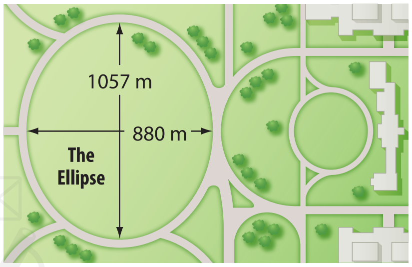
59. $(x')^2 - (y')^2 = 1$ at a 45° rotation from the xy-axis

Write an equation for the hyperbola with the given characteristics.

60. vertices (5, 4), (5, −8); conjugate axis length of 4

61. transverse axis length of 4; foci (3, 5), (3, −1)

62. **WHITE HOUSE** There is an open area known as The Ellipse. Write an equation to model The Ellipse. Assume that the origin is at its center.



Simplify each expression.

63. $\frac{\sin x}{\csc x - 1} + \frac{\sin x}{\csc x + 1}$

64. $\frac{1}{1 - \cos x} + \frac{1}{1 + \cos x}$

Use the properties of logarithms to rewrite each logarithm below in the form $a \ln 2 + b \ln 3$, where a and b are constants. Then approximate the value of each logarithm given that $\ln 2 \approx 0.69$ and $\ln 3 \approx 1.10$.

65. $\ln 54$

66. $\ln 24$

67. $\ln \frac{8}{3}$

68. $\ln \frac{9}{16}$

For each function, determine any asymptotes and intercepts. Then graph the function and state its domain.

69. $h(x) = \frac{x}{x + 6}$

70. $h(x) = \frac{x^2 + 6x + 8}{x^2 - 7x - 8}$

71. $f(x) = \frac{x^2 + 8x}{x + 5}$

72. $f(x) = \frac{x^2 + 4x + 3}{x^3 + x^2 - 6x}$

Solve each equation.

73. $\sqrt{3z - 5} - 3 = 1$

74. $\sqrt{5n - 1} = 0$

75. $\sqrt{2c + 3} - 7 = 0$

76. $\sqrt{4a + 8} + 8 = 5$

Skills Review for Standardized Tests

77. **SAT/ACT** With the exception of the shaded squares, every square in the figure contains the sum of the number in the square directly above it and the number in the square directly to its left. For example, the number 4 in the unshaded square is the sum of the 2 in the square above it and the 2 in the square directly to its left. What is the value of x?

0	1	2	3	4	5
1	2	4			
2					
3			x		
4					
5					

A 7 **B** 8 **C** 15 **D** 23 **E** 30

78. Saleh and Sultan are performing a physics experiment in which they will launch a model rocket. The rocket is supposed to release a parachute 91.5 meters in the air, 7 seconds after liftoff. They are firing the rocket at a 78° angle from the horizontal. To protect other students from the falling rockets, the teacher needs to place warning signs 45.7 meters from where the parachute is released. How far should the signs be from the point where the rockets are launched?

F 111.6 meters
G 116.2 meters
H 121.6 meters
J 126.2 meters

79. **FREE RESPONSE** An object moves along a curve according to $y = \frac{10\sqrt{3t} \pm \sqrt{496 - 2304t}}{62}$, $x = \sqrt{t}$.

a. Convert the parametric equations to rectangular form.

b. Identify the conic section represented by the curve.

c. Write an equation for the curve in the $x'y'$-plane, assuming it was rotated 30°.

d. Determine the eccentricity of the conic.

e. Identify the location of the foci in the $x'y'$-plane, if they exist.

EXTEND

6-8 Graphing Technology Lab
Modeling with Parametric Equations

Objective

- Use a graphing calculator to model functions parametrically.

As shown in Lesson 6-5, the independent variable t in parametric equations represents time. This parameter reflects the speed with which the graph is drawn. If one graph is completed for $0 \leq t \leq 5$, while an identical graph is completed for $0 \leq t \leq 10$, then the first graph is faster.

StudyTip

Setting Parameters Use the situation in the problem as a guide for setting the range of values for x, y, and t.

Activity Parametric Graph

PLAY BALL **Standing side by side, Noura and her brother Omar throw a ball at exactly the same time. Noura throws the ball with an initial velocity of 20 meters per second at 60°. Omar throws the ball 15 meters per second at 45°. Assuming that the balls were thrown from the same initial height, simulate the throws on a graphing calculator.**

Step 1 The parametric equations for each throw are as follows.

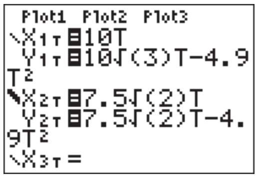
Neva: $x = 20t \cos 60$ $\quad$ $y = 20t \sin 60 - 4.9t^2$

$= 10t$ $\quad$ $= 10\sqrt{3}t - 4.9t^2$

Owen: $x = 15t \cos 45$ $\quad$ $y = 15t \sin 45 - 4.9t^2$

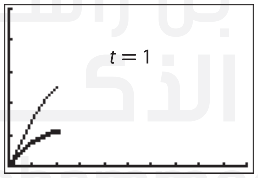
$= 7.5\sqrt{2}t$ $\quad$ $= 7.5\sqrt{2}t - 4.9t^2$

Step 2 Set the mode. In the MODE menu, select degree, par, and simul. This allows the equations to be graphed simultaneously. Enter the parametric equations. In parametric form, X,T,θ,n uses t instead of x. Set the second set of equations to shade dark to distinguish between the throws.

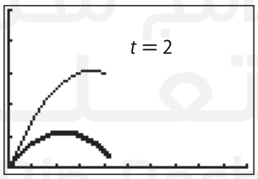


Step 3 Set the t-values to range from 0 to 8 as an estimate for the duration of the throws. Set tstep to 0.1 in order to watch the throws in the graph.

Step 4 Graph the equations.



[0, 50] scl: 5 by [0, 25] scl: 5



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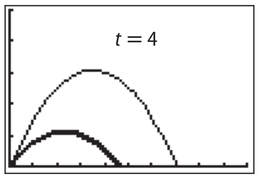


[0, 50] scl: 5 by [0, 25] scl: 5

Noura's throw goes higher and at a greater distance while Omar's lands first.

Exercises

1. **PLAY BALL** Omar's next throw is 21 meters per second at 50°. A second later, Noura throws her ball 24 meters per second at 35°. Simulate the throws on a graphing calculator and interpret the results.

[0, 60] scl: 5 by [0, 25] scl: 5; t [0, 8]; tscl: 0.1

Sample answer: At $t = 2.5$ seconds, the balls are about the same height. At about $t = 3.25$ seconds, Noura's throw has traveled the same horizontal distance as Omar's. At $t = 3.5$, Omar's has already landed while Noura's is still in the air.

2. **BASEBALL** Noura throws a baseball 27 meters per second at 82°. A second later, Omar hits a ball 45 meters per second at 20°. Assuming they are still side by side and the initial height of the hit is one meter lower, simulate the situation on a graphing calculator and interpret the results.

[0, 125] scl: 5 by [0, 50] scl: 5; t [0, 8]; tscl: 0.1

Sample answer: Noura's throw goes much higher than Omar's hit. Both the throw and hit reach their maximum height near $t = 2.5$. The hit lands a full second before the throw.

CHAPTER 6 Study Guide and Review

Study Guide

KeyConcepts

Midpoint and Distance Formulas (Lesson 6-1)

- $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Parabolas (Lesson 6-2)

- Standard Form: $y = a(x - h)^2 + k$
  $x = a(y - k)^2 + h$

Circles (Lesson 6-3)

- The equation of a circle with center (h, k) and radius r can be written in the form $(x - h)^2 + (y - k)^2 = r^2$.

Ellipses (Lesson 6-4)

- Standard Form: horizontal $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
  vertical $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$

Hyperbolas (Lesson 6-5)

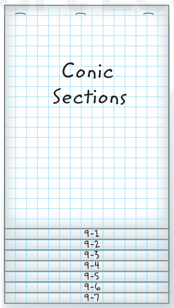
- Standard Form: horizontal $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
  vertical $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Solving Linear-Nonlinear Systems (Lesson 6-7)

- Systems of quadratic equations can be solved using substitution and elimination.
- A system of quadratic equations can have zero, one, two, three, or four solutions.

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.



KeyVocabulary

center (of a circle)
center (of an ellipse)
circle
conjugate axis
constant difference
constant sum
co-vertices (of a hyperbola)
co-vertices (of an ellipse)
directrix
ellipse
foci (of a hyperbola)
foci (of an ellipse)
focus
hyperbola
latus rectum
major axis
minor axis
parabola
radius
transverse axis
vertices (of a hyperbola)
vertices (of an ellipse)

VocabularyCheck

State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.

1. The set of all points in a plane that are equidistant from a given point in the plane, called the focus, forms a circle.
2. A(n) ellipse is the set of all points in a plane such that the sum of the distances from the two fixed points is constant.
3. The endpoints of the major axis of an ellipse are the foci of the ellipse.
4. The radius is the distance from the center of a circle to any point on the circle.
5. The line segment with endpoints on a parabola, through the focus of the parabola, and perpendicular to the axis of symmetry is called the latus rectum.
6. Every hyperbola has two axes of symmetry, the transverse axis and the major axis.
7. A directrix is the set of all points in a plane that are equidistant from a given point in the plane, called the center.
8. A hyperbola is the set of all points in a plane such that the absolute value of the sum of the distances from any point on the hyperbola to two given points is constant.
9. A parabola can be defined as the set of all points in a plane that are the same distance from the focus and a given line called the directrix.
10. The major axis is the longer of the two axes of symmetry of an ellipse.
11. The equation for a graph can be written using the variables x and y, or using ________ equations, generalluy using t or the angle θ.
12. The graph of $f(t) = (\sin t, \cos t)$ is a ________ with a shape that is circle traced clockwise.

Lesson-by-Lesson Review

6-1 Parabolas

Graph each equation.

13. $y = 3x^2 + 24x - 10$

14. $x = \frac{1}{2}y^2 - 4y + 3$

15. $3y - x^2 = 8x - 11$

16. $x = y^2 - 14y + 25$

Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

17. $y = -\frac{1}{2}x^2$

18. $x - 6y = y^2 + 4$

19. $y = 4x^2 - 16x + 9$

20. $x = y^2 + 14y + 20$

21. **SPORTS** When a football is kicked, the path it travels is shaped like a parabola. Suppose a football is kicked from ground level, reaches a maximum height of 50 feet, and lands 200 feet away. Assuming the football was kicked at the origin, write an equation of the parabola that models the flight of the football.

Example 1

Write $3y - x^2 = 4x + 7$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

Write the equation in the form $y = a(x - h)^2 + k$ by completing the square.

$3y = x^2 + 4x + 7$ — Isolate the terms with x.

$3y = (x^2 + 4x + \blacksquare) + 7 - \blacksquare$ — Complete the square.

$3y = (x^2 + 4x + 4) + 7 - 4$ — $\left(\frac{4}{2}\right)^2 = 4$

$3y = (x + 2)^2 + 3$ — $(x^2 + 4x + 4) = (x + 2)$

$y = \frac{1}{3}(x + 2)^2 + 1$ — Divide each side by 3.

Vertex: $(-2, 1)$; axis of symmetry: $x = -2$; direction of opening: upward since $a > 0$.

6-2 Circles

Write an equation for the circle that satisfies each set of conditions.

22. center $(-1, 6)$, radius 3 units

23. endpoints of a diameter $(2, 5)$ and $(0, 0)$

24. endpoints of a diameter $(4, -2)$ and $(-2, -6)$

Find the center and radius of each circle. Then graph the circle.

25. $(x + 5)^2 + y^2 = 9$

26. $(x - 3)^2 + (y + 1)^2 = 25$

27. $(x + 2)^2 + (y - 8)^2 = 1$

28. $x^2 + 4x + y^2 - 2y - 11 = 0$

29. **SOUND** A loudspeaker in a school is located at the point (65, 40). The speaker can be heard in a circle with a radius of 30.5 meters. Write an equation to represent the possible boundary of the loudspeaker sound.

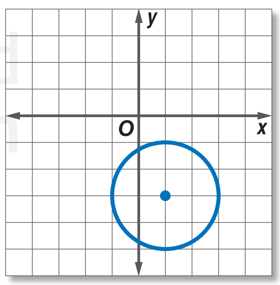
Example 2

Find the center and radius of the circle with equation $x^2 - 2x + y^2 + 6y + 6 = 0$. Then graph the circle.

Complete the squares.

$$x^2 - 2x + y^2 + 6y + 6 = 0$$
$$(x^2 - 2x + \blacksquare) + (y^2 + 6y + \blacksquare) = -6 + \blacksquare + \blacksquare$$
$$(x^2 - 2x + 1) + (y^2 + 6y + 9) = -6 + 1 + 9$$
$$(x - 1)^2 + (y + 3)^2 = 4$$

The center of the circle is at $(1, -3)$ and the radius is 2.



6-3 Ellipses

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

30. $\frac{x^2}{9} + \frac{y^2}{36} = 1$

31. $\frac{y^2}{10} + \frac{x^2}{5} = 1$

32. $\frac{x^2}{36} + \frac{(y-4)^2}{4} = 1$

33. $27x^2 + 9y^2 = 81$

34. $\frac{(x+1)^2}{25} + \frac{(y-2)^2}{16} = 1$

35. $9x^2 + 4y^2 + 54x - 8y + 49 = 0$

36. $9x^2 + 25y^2 - 18x + 50y - 191 = 0$

37. $7x^2 + 3y^2 - 28x - 12y = -19$

38. **LANDSCAPING** Saeed's family has a garden in their front yard that is shaped like an ellipse. The major axis is 16 meters and the minor axis is 10 meters. Write an equation to model the garden. Assume the origin is at the center of the garden and the major axis is horizontal.

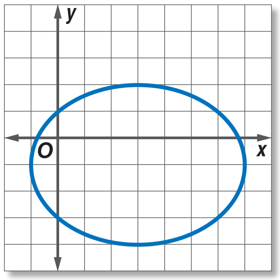
Example 3

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with equation $9x^2 + 16y^2 - 54x + 32y - 47 = 0$. Then graph the ellipse.

First, convert to standard form.

$$9x^2 + 16y^2 - 54x + 32y - 47 = 0$$
$$9(x^2 - 6x + ■) + 16(y^2 + 2y + ■) = 47 + 9(■) + 16(■)$$
$$9(x^2 - 6x + 9) + 16(y^2 + 2y + 1) = 47 + 9(9) + 16(1)$$
$$9(x-3)^2 + 16(y+1)^2 = 144$$
$$\frac{(x-3)^2}{16} + \frac{(y+1)^2}{9} = 1$$

The center of the ellipse is $(3, -1)$. The ellipse is horizontal. $a^2 = 16$, so $a = 4$. $b^2 = 9$, so $b = 3$. The length of the major axis is $2 \cdot 4$ or 8. The length of the minor axis is $2 \cdot 3$ or 6. To find the foci: $c^2 = 16 - 9$ or 7, so $c = \sqrt{7}$. The foci are $(3 + \sqrt{7}, -1)$ and $(3 - \sqrt{7}, -1)$.



6-4 Hyperbolas

Graph each hyperbola. Identify the vertices, foci, and asymptotes.

39. $\frac{y^2}{9} - \frac{x^2}{4} = 1$

40. $\frac{(x-3)^2}{1} - \frac{(y+2)^2}{4} = 1$

41. $\frac{(y+1)^2}{16} - \frac{(x-4)^2}{9} = 1$

42. $4x^2 - 9y^2 = 36$

43. $9y^2 - x^2 - 4x + 18y + 4 = 0$

44. **MIRRORS** A hyperbolic mirror is shaped like one branch of a hyperbola. It reflects light rays directed at one focus toward the other focus. Suppose a hyperbolic mirror is modeled by the upper branch of the hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$. A light source is located at $(-10, 0)$. Where should the light hit the mirror so that the light will be reflected to $(0, -5)$?

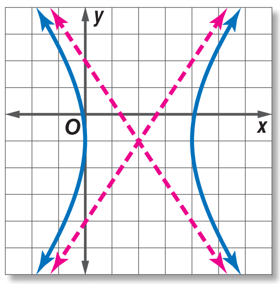
Example 4

Graph $9x^2 - 4y^2 - 36x - 8y - 4 = 0$. Identify the vertices, foci, and asymptotes.

Complete the square.

$$9x^2 - 4y^2 - 36x - 8y - 4 = 0$$
$$9(x^2 - 4x + ■) - 4(y^2 + 2y + ■) = 4 + 9(■) - 4(■)$$
$$9(x^2 - 4x + 4) - 4(y^2 + 2y + 1) = 4 + 9(4) - 4(1)$$
$$9(x-2)^2 - 4(y+1)^2 = 36$$
$$\frac{(x-2)^2}{4} - \frac{(y+1)^2}{9} = 1$$

The center is at $(2, -1)$. The vertices are at $(0, -1)$ and $(4, -1)$. The foci are at $(2 + \sqrt{13}, -1)$ and $(2 - \sqrt{13}, -1)$. The equations of the asymptotes are $y + 1 = \pm\frac{3}{2}(x - 2)$



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6-5 Identifying Conic Sections

Write each equation in standard form. State whether the graph of the equation is a *parabola, circle, ellipse,* or *hyperbola*. Then graph.

45. $3x^2 + 12x - y + 8 = 0$

46. $9x^2 + 16y^2 = 144$

47. $x^2 + y^2 - 8x - 2y + 8 = 0$

48. $-9x^2 + y^2 + 36x - 45 = 0$

Without writing the equation in standard form, state whether the graph of the equation is a *parabola, circle, ellipse,* or *hyperbola*.

49. $7x^2 + 9y^2 = 63$

50. $5y^2 + 2y + 4x - 13x^2 = 81$

51. $x^2 - 8x + 16 = 6y$

52. $x^2 + 4x + y^2 - 285 = 0$

53. **LIGHT** Suppose the edge of a shadow can be represented by the equation $16x^2 + 25y^2 - 32x - 100y - 284 = 0$.

a. What is the shape of the shadow?

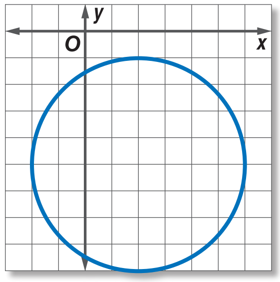
b. Graph the equation.

Example 5

Write $3x^2 + 3y^2 - 12x + 30y + 39 = 0$ in standard form. State whether the graph of the equation is a *parabola, circle, ellipse*, or *hyperbola*. Then graph the equation.

$$3x^2 + 3y^2 - 12x + 30y + 39 = 0$$
$$3(x^2 - 4x + \blacksquare) + 3(y^2 + 10y + \blacksquare) = -39 + 3(\blacksquare) + 3(\blacksquare)$$
$$3(x^2 - 4x + 4) + 3(y^2 + 10y + 25) = -39 + 3(4) + 3(25)$$
$$3(x - 2)^2 + 3(y + 5)^2 = 48$$
$$(x - 2)^2 + (y + 5)^2 = 16$$

In this equation $A = 3$ and $C = 3$. Since A and C are both positive and $A = C$, the graph is a circle. The center is at (2, −5), and the radius is 4.



6-6 Solving Linear-Nonlinear Systems

Solve each system of equations.

54. $x^2 + y^2 = 8$
$x + y = 0$

55. $x - 2y = 2$
$y^2 - x^2 = 2x + 4$

56. $y + x^2 = 4x$
$y + 4x = 16$

57. $3x^2 - y^2 = 11$
$x^2 + 4y^2 = 8$

58. $5x^2 + y^2 = 30$
$9x^2 - y^2 = -16$

59. $\frac{x^2}{30} + \frac{y^2}{6} = 1$
$x = y$

60. **PHYSICAL SCIENCE** Two balls are launched into the air at the same time. The heights they are launched from are different. The height y in feet of one is represented by $y = -16t^2 + 80t + 25$ where t is the time in seconds. The height of the other ball is represented by $y = -16t^2 + 30t + 100$.

a. After how many seconds are the balls at the same height?

b. What is this height?

61. **ARCHITECTURE** An architect is building the front entrance of a building in the shape of a parabola with the equation $y = -\frac{1}{10}(x - 10)^2 + 20$. While the entrance is being built, the construction team puts in two support beams with equations $y = -x + 10$ and $y = x - 10$. Where do the support beams meet the parabola?

Solve each system of inequalities by graphing.

62. $x^2 + y^2 < 64$
$x^2 + 16(y - 3)^2 < 16$

63. $x^2 + y^2 < 49$
$16x^2 - 9y^2 \geq 144$

64. $x + y < 4$
$9x^2 - 4y^2 \geq 36$

65. $x^2 + y^2 < 25$
$4x^2 - 9y^2 < 36$

66. $x^2 + y^2 < 36$
$4x^2 + 9y^2 > 36$

67. $y^2 < x$
$x^2 - 4y^2 < 16$

Example 6

Solve the system of equations.
$x^2 + y^2 = 100$
$3x - y = 10$

Use substitution to solve the system.
First, rewrite $3x - y = 10$ as $y = 3x - 10$.

$$x^2 + y^2 = 100$$
$$x^2 + (3x - 10)^2 = 100$$
$$x^2 + 9x^2 - 60x + 100 = 100$$
$$10x^2 - 60x + 100 = 100$$
$$10x^2 - 60x = 0$$
$$10x(x - 6) = 0$$
$$10x = 0 \quad \text{or} \quad x - 6 = 0$$
$$x = 0 \qquad\qquad x = 6$$

Now solve for y.

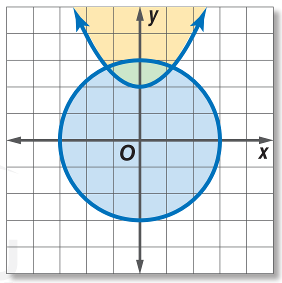
$y = 3x - 10 = 3(0) - 10 = -10$

$y = 3x - 10 = 3(6) - 10 = 8$

The solutions of the system are (0, −10) and (6, 8)

Example 7

Solve the system of inequalities by graphing.
$x^2 + y^2 \leq 9$
$2y \geq x^2 + 4$



The solution is the green shaded region.

6-7 Rotations of Conic Sections

Use a graphing calculator to graph the conic given by each equation.

68. $x^2 - 4xy + y^2 - 2y - 2x = 0$

69. $x^2 - 3xy + y^2 - 3y - 6x + 5 = 0$

70. $2x^2 + 2y^2 - 8xy + 4 = 0$

71. $3x^2 + 9xy + y^2 = 0$

72. $4x^2 - 2xy + 8y^2 - 7 = 0$

Write each equation in the $x'y'$-plane for the given value of θ. Then identify the conic.

73. $x^2 + y^2 = 4;\ \theta = \frac{\pi}{4}$

74. $x^2 - 2x + y = 5;\ \theta = \frac{\pi}{3}$

75. $x^2 - 4y^2 = 4;\ \theta = \frac{\pi}{2}$

76. $9x^2 + 4y^2 = 36;\ \theta = 90°$

Example 8

Use a graphing calculator to graph $x^2 + 2xy + y^2 + 4x - 2y = 0$.

$x^2 + 2xy + y^2 + 4x - 2y = 0$ Original equation

$1y^2 + (2x - 2)y + (x^2 + 4x) = 0$ Quadratic form

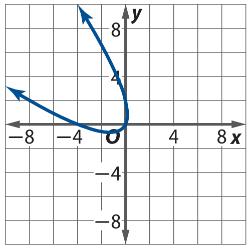
Use the Quadratic Formula.

$$y = \frac{-(2x-2) \pm \sqrt{(2x-2)^2 - 4(1)(x^2+4x)}}{2(1)}$$

$$= \frac{-2x + 2 \pm \sqrt{4x^2 - 8x + 4 - 4x^2 - 16x}}{2}$$

$$= \frac{-2x + 2 \pm 2\sqrt{1 - 6x}}{2}$$

$$= -x + 1 \pm \sqrt{1 - 6x}$$



Graph as

$y_1 = -x + 1 + \sqrt{1 - 6x}$ and

$y_2 = -x + 1 - \sqrt{1 - 6x}$.

6-8 Parametric Equations

Sketch the curve given by each pair of parametric equations over the given interval.

77. $x = \sqrt{t},\ y = 1 - t;\ 0 \leq t \leq 9$

78. $x = t + 2,\ y = t^2 - 4;\ -4 \leq t \leq 4$

Write each pair of parametric equations in rectangular form. Then graph the equation.

79. $x = t + 5$ and $y = 2t - 6$

80. $x = 2t$ and $y = t^2 - 2$

81. $x = t^2 + 3$ and $y = t^2 - 4$

82. $x = t^2 - 1$ and $y = 2t + 1$

Example 9

Write $x = 5 \cos t$ and $y = 9 \sin t$ in rectangular form. Then graph the equation.

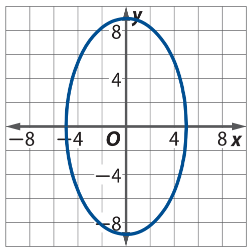
$x = 5 \cos t$ $\qquad y = 9 \sin t$

$\cos t = \frac{x}{5}$ $\qquad \sin t = \frac{y}{9}$ Solve for sin *t* and cos *t*

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{9}\right)^2 = 1$

$\frac{x^2}{25} + \frac{y^2}{81} = 1$



The parametric equations represent the graph of an ellipse.

CHAPTER 6 Practice Test

Find the midpoint of the line segment with endpoints at the given coordinates.

1. $(8, 3), (-4, 9)$
2. $\left(\frac{3}{4}, 0\right), \left(\frac{1}{2}, -1\right)$
3. $(-10, 0), (-2, 6)$

Find the distance between each pair of points with the given coordinates.

4. $(-5, 8), (4, 3)$
5. $\left(\frac{1}{3}, \frac{2}{3}\right), \left(-\frac{5}{6}, -\frac{11}{6}\right)$
6. $(4, -5), (4, 9)$

State whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

7. $y^2 = 64 - x^2$
8. $4x^2 + y^2 = 16$
9. $4x^2 - 9y^2 + 8x + 36y = 68$
10. $\frac{1}{2}x^2 - 3 = y$
11. $y = -2x^2 - 5$
12. $16x^2 + 25y^2 = 400$
13. $x^2 + 6x + y^2 = 16$
14. $\frac{y^2}{4} - \frac{x^2}{16} = 1$
15. $(x + 2)^2 = 3(y - 1)$
16. $4x^2 + 16y^2 + 32x + 63 = 0$
17. **MULTIPLE CHOICE** Which equation represents a hyperbola that has vertices at $(-3, -3)$ and $(5, -3)$ and a conjugate axis of length 6 units?

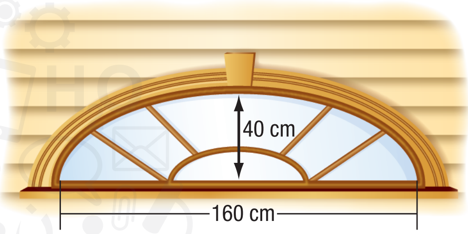
 A $\frac{(y-1)^2}{16} - \frac{(x+3)^2}{9} = 1$

 B $\frac{(x-1)^2}{16} - \frac{(y+3)^2}{9} = 1$

 C $\frac{(y+1)^2}{16} - \frac{(x-3)^2}{9} = 1$

 D $\frac{(x+1)^2}{16} - \frac{(y-3)^2}{9} = 1$

18. **CARPENTRY** Ayoub built a small window frame shaped like the top half of an ellipse. The window is 40 centimeters tall at its highest point and 160 centimeters wide at the bottom. What is the height of the window 20 centimeters from the center of the base?



Solve each system of equations.

19. $x^2 + y^2 = 100$
 $y = -x - 2$
20. $x^2 + 2y^2 = 11$
 $x + y = 2$
21. $x^2 + y^2 = 34$
 $y^2 - x^2 = 9$

Solve each system of inequalities.

22. $x^2 + y^2 \leq 9$
 $y > -x^2 + 2$
23. $\frac{(x-2)^2}{4} - \frac{(y-4)^2}{9} \geq 1$
 $x - 4y < 8$
24. **MULTIPLE CHOICE** Which is NOT the equation of a parabola?

 F $y = 3x^2 + 5x - 3$

 G $2y + 3x^2 + x - 9 = 0$

 H $x = 3(y + 1)^2$

 J $x^2 + 2y^2 + 6x = 10$

25. **FORESTRY** A forest ranger at an outpost in the Sam Houston National Forest and another ranger at the primary station both heard an explosion. The outpost and the primary station are 6 kilometers apart.

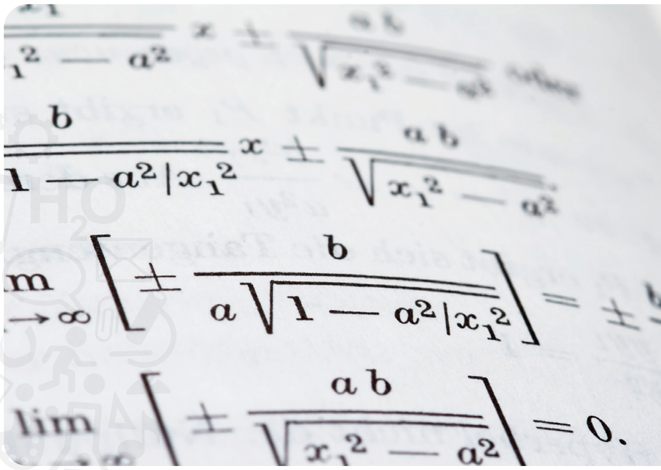
 a. If one ranger heard the explosion 6 seconds before the other, write an equation that describes all the possible locations of the explosion. Place the two ranger stations on the x-axis with the midpoint between the stations at the origin. The transverse axis is horizontal. (*Hint*: The speed of sound is about 0.35 kilometer per second.)

 b. Draw a sketch of the possible locations of the explosion. Include the ranger stations in the drawing.

CHAPTER 6 Preparing for Standardized Tests

Use a Formula

Sometimes it is necessary to use a formula to solve problems on standardized tests. In some cases you may even be given a sheet of formulas that you are permitted to reference while taking the test.



Strategies for Using a Formula

Step 1

Read the problem statement carefully.

Ask yourself:

- What am I being asked to solve?
- What information is given in the problem?
- Are there any formulas that I can use to help me solve the problem?

Step 2

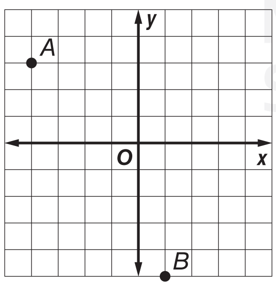
Solve the problem and check your solution.

- Substitute the known quantities that are given in the problem statement into the formula.
- Simplify to solve for the unknown values in the formula.
- Check to make sure your answer makes sense. If time permits, check your answer.

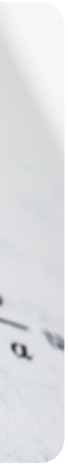
Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve. Show your work.

What is the distance between points A and B on the coordinate plane? Round your answer to the nearest tenth if necessary.



Scoring Rubric	
Criteria	**Score**
Full Credit: The answer is correct and a full explanation is provided that shows each step.	2
Partial Credit: • The answer is correct but the explanation is incomplete. • The answer is incorrect but the explanation is correct.	1
No Credit: Either an answer is not provided or the answer does not make sense.	0



Read the problem statement carefully. You are given the coordinates of two points on a coordinate plane and asked to find the distance between them. To solve this problem, you must use the **Distance Formula**.

Example of a 2-point response:

Use the Distance Formula to find the distance between points $A(-4, 3)$ and $B(1, -5)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[1 - (-4)]^2 + [(-5) - 3]^2}$$
$$= \sqrt{5^2 + (-8)^2}$$
$$= \sqrt{25 + 64}$$
$$= \sqrt{89} \text{ or about } 9.4$$

The distance between points A and B is about 9.4 units.

The steps, calculations, and reasoning are clearly stated. The student also arrives at the correct answer. So, this response is worth the full 2 points.

Exercises

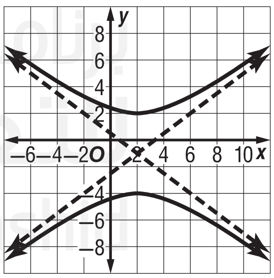
Read each problem. Identify what you need to know. Then use the information in the problem to solve. Show your work.

1. What is the midpoint of segment CD with endpoints $C(5, -12)$ and $D(-9, 4)$?

2. Huda is making a map of her hometown on a coordinate plane. She plots the school at $S(7, 3)$ and the park at $P(-4, 12)$. If the scale of the map is 1 unit = 250 meters, what is the actual distance between the school and the park? Round to the nearest meter.

3. Yousif is making a concrete table for his backyard. The tabletop will be circular with a diameter of 6 feet and a depth of 6 inches. How much concrete will Yousif need to make the top of the table? Round to the nearest cubic foot.

4. What is the equation, in standard form, of the hyperbola graphed below?



5. If the surface area of a cube is increased by a factor of 9, what is the change in the length of the sides of the cube?

   A The length is 2 times the original length.

   B The length is 3 times the original length.

   C The length is 6 times the original length.

   D The length is 9 times the original length.

CHAPTER 6

Standardized Test Practice

Cumulative, Chapters 1 through 6

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Which is the first *incorrect* step in simplifying $\log_3 \frac{3}{48}$?

Step 1: $\log_3 \frac{3}{48} = \log_3 3 - \log_3 48$

Step 2: $= 1 - 16$

Step 3: $= -15$

A Step 1

B Step 2

C Step 3

D Each step is correct.

2. Which is the equation for the parabola that has vertex $(-3, -23)$ and passes through the point $(1, 9)$?

F $y = x^2 + 10x + 7$

G $y = x^2 - 6x + 19$

H $y = 2x^2 + 12x - 5$

J $y = 2x^2 - 3x + 10$

3. What are the vertices of the ellipse with equation $\frac{(x-3)^2}{36} + \frac{(y-2)^2}{144} = 1$?

A $(-3, 2)$ and $(9, 2)$

B $(-2, 3)$ and $(10, 3)$

C $(3, -10)$ and $(3, 14)$

D $(2, -11)$ and $(4, 13)$

4. Hooke's law states that the force needed to keep a spring extended x units is proportional to x. If a force of 40 N is needed to keep a spring extended 5 centimeters, what is the force needed to keep the spring extended 14 cm?

F 8 N

G 19 N

H 112 N

G 1600 N

Test-Taking Tip

Question 2 You can check your answer by submitting 1 for *x* and making sure that the *y*-value is 9.

5. Yasmin is making a map of her backyard on a coordinate grid. She plots point $G(-4, -6)$ to represent her mom's garden and point $S(3, 7)$ to represent the rope swing hanging on an oak tree. If the scale of the map is 1 unit = 5 meters, what is the approximate distance between the garden and the rope swing?

A 74 meters

B 79 meters

C 82 meters

D 90 meters

6. If $\sqrt{x+5} + 1 = 4$, what is the value of x?

F 4 **G** 10 **H** 11 **J** 20

7. The area of the base of a rectangular suitcase measures $3x^2 + 5x - 4$ square units. The height of the suitcase measures $2x$ units. Which polynomial expression represents the volume of the suitcase?

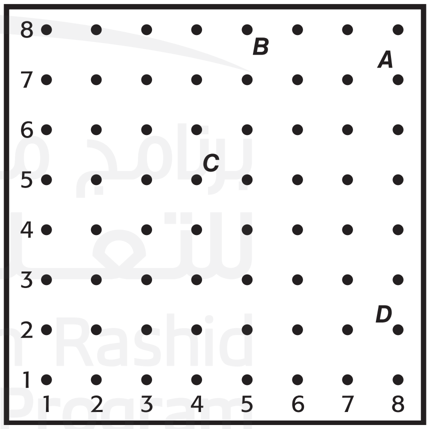
A $3x^3 + 5x^2 - 4x$

B $6x^2 + 10x - 8$

C $6x^3 + 10x^2 - 8x$

D $3x^3 + 10x^2 - 4$

8. Laila was given this geoboard to model the slope $-\frac{3}{4}$.



If the peg in the upper right-hand corner represents the origin on a coordinate plane, where could Laila place a rubber band to represent the given slope?

F from peg *A* to peg *B*

G from peg *A* to peg *C*

H from peg *B* to peg *D*

J from peg *C* to peg *D*

Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

9. A football player kicked the ball upwards at a velocity of 32 ft/s . How long will the ball take to hit the ground? Use the law $h(t) = v_0t - 16t^2$ where $h(t)$ represents an object's height in feet, v0 the initial velocity in meters per second, and t time in seconds.

10. GRIDDED RESPONSE

11. Tarek is preparing boxes of assorted chocolates. Chocolate-covered peanuts cost AED 7 per gram. Chocolate-covered caramels cost AED 6.50 per gram. The boxes of assorted candies contain five more grams of peanut candies than caramel candies. If the total amount sold was AED 575, how many grams of each candy were needed to make the boxes?

12. Amer went shopping and bought two shirts, three pairs of pants, one belt, and two pairs of shoes. The following matrix shows the prices for each item respectively.

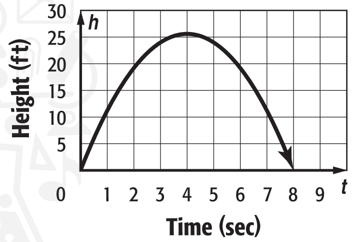
[AED 20.15 AED 32 AED 15 AED 25.99]

Use matrix multiplication to find the total amount of money Amer spent while shopping.

Extended Response

Record your answers on a sheet of paper. Show your work.

13. Zayed graphed the quadratic equation $h(t) = -16t^2 + 128t$ to model the flight of a firework. The parabola shows the height, in feet, of the firework t seconds after it was launched.



a. What is the vertex of the parabola?

b. What does the vertex of the parabola represent?

c. How long is the firework in the air before it lands?

14. The Sharjah Secondary School Yearbook Staff is selling yearbooks and chrome picture frames engraved with the year. The number of yearbooks and frames sold to members of each grade is shown in the table.

Sales for Each Class		
Grade	Yearbooks	Frames
9th	423	256
10th	464	278
11th	546	344
12th	575	497

a. Find the difference in the sales of yearbooks and frames made to the 10th and 11th grade classes.

b. Find the total number of yearbooks and frames sold.

c. A yearbook costs AED 48 and a frame costs AED 18. Find the sales of yearbooks and frames for each class.

CHAPTER 7 Vectors

Then

In **previous chapters,** you used trigonometry to solve triangles.

Now

In **this chapter,** you will:

- Represent and operate with vectors algebraically in the two- and three-dimensional coordinate systems.
- Find the projection of one vector onto another.
- Find cross products of vectors in space and find volumes of parallelepipeds.
- Find the dot products of and angles between vectors.

Why? ▲

ROWING Vectors are often used to model changes in direction due to water and air currents. For example, a vector can be used to determine the resultant speed and direction of a kayak that is traveling 12.9 kilometers per hour against a 4.8 mile-per-hour river current.

PREREAD Scan the lesson titles and Key Concept boxes in Chapter 7. Use this information to predict what you will learn in this chapter.

Get Ready for the Chapter

Take the Quick Check Below

QuickCheck

Find the distance between each given pair of points and the midpoint of the segment connecting the given points. (Prerequisite Skill)

1. $(1, 4), (-2, 4)$

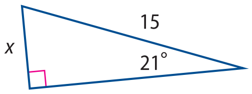
2. $(-5, 3), (-5, 8)$

3. $(2, -9), (-3, -7)$

4. $(-4, -1), (-6, -8)$

Find the value of x. Round to the nearest tenth if necessary.

5.



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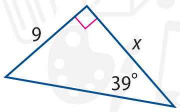

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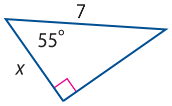

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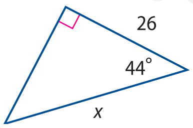


9. BALLOON A hot air balloon is being held in place by two people holding ropes and standing 35 meters apart. The angle formed between the ground and the rope held by each person is 40°. Determine the length of each rope to the nearest tenth of a meter.

Find all solutions for the given triangle, if possible. If no solution exists, write *no solution*. Round side lengths to the nearest tenth and angle measures to the nearest degree.

10. $a = 10, b = 7, A = 128°$

11. $a = 15, b = 16, A = 127°$

12. $a = 15, b = 18, A = 52°$

13. $a = 30, b = 19, A = 91°$

NewVocabulary

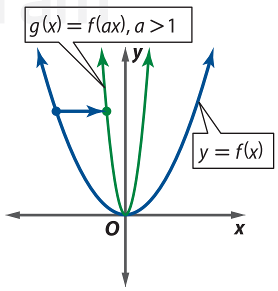
English

vector
initial point
terminal point
standard position
direction
magnitude
quadrant bearing
true bearing
parallel vectors
equivalent vectors
opposite vectors
resultant
zero vector
component form
unit vector
dot product
orthogonal
z-axis
octants
ordered triple
cross product
triple scalar product

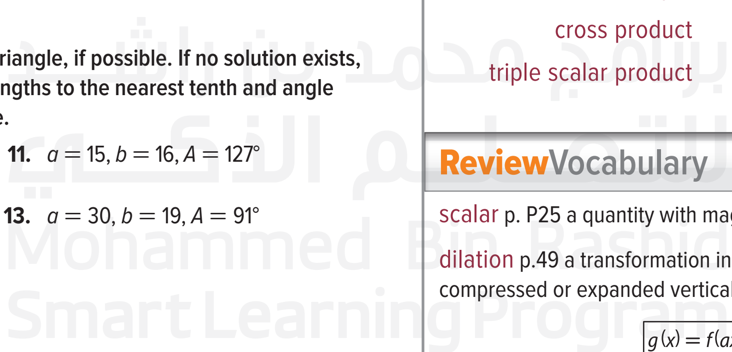
ReviewVocabulary

scalar p. P25 a quantity with magnitude only

dilation p.49 a transformation in which the graph of a function is compressed or expanded vertically or horizontally



Horizontal dilation



LESSON 7-1

Introduction to Vectors

Then

- You used trigonometry to solve triangles.

Now

1. Represent and operate with vectors geometrically.
2. Solve vector problems, and resolve vectors into their rectangular components.

Why?

- A successful goal attempt in football depends on several factors. While the speed of the ball after it is kicked is certainly important, the direction the ball takes is as well. We can describe both of these factors using a single quantity called a *vector*.

NewVocabulary

vector
initial point
terminal point
standard position
direction
magnitude
quadrant bearing
true bearing
parallel vectors
equivalent vectors
opposite vectors
resultant
triangle method
parallelogram method
zero vector
components
rectangular components

1 Vectors Many physical quantities, such as speed, can be completely described by a single real number called a *scalar*. This number indicates the *magnitude* or *size* of the quantity. A **vector** is a quantity that has both magnitude and *direction*. The velocity of a ball is a vector that describes both the speed and direction of the ball .

Example 1 Identify Vector Quantities

State whether each quantity described is a *vector* quantity or a *scalar* quantity.

a. a boat traveling at 15 kilometers per hour

This quantity has a magnitude of 15 kilometers per hour, but no direction is given. Speed is a scalar quantity.

b. a hiker walking 25 paces due west

This quantity has a magnitude of 25 paces and a direction of due west. This directed distance is a vector quantity.

c. a person's weight on a bathroom scale

Weight is a vector quantity that is calculated using a person's mass and the downward pull due to gravity. (Acceleration due to gravity is a vector.)

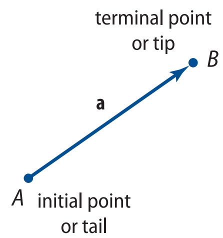
GuidedPractice

1A. a car traveling 60 kilometers per hour 15° east of south

1B. a parachutist falling straight down at 20.2 kilometers per hour

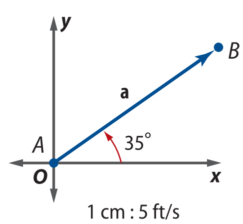
1C. a child pulling a sled with a force of 40 newtons

A vector can be represented geometrically by a directed line segment, or arrow diagram, that shows both magnitude and direction. Consider the directed line segment with an **initial point** A (also known as the *tail*) and **terminal point** B (also known as the *head* or *tip*) shown. This vector is denoted by $\overrightarrow{AB}$, $\vec{a}$, or **a**.

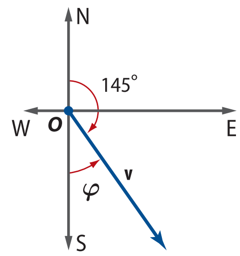


If a vector has its initial point at the origin, it is in **standard position**. The **direction** of a vector is the directed angle between the vector and the horizontal line that could be used to represent the positive x-axis. The direction of **a** is 35°.

The length of the line segment represents, and is proportional to, the **magnitude** of the vector. If the scale of the arrow diagram for **a** is 1 cm = 5 ft/s, then the magnitude of **a**, denoted |**a**|, is 2.6 × 5 or 13 feet per second.



The direction of a vector can also be given as a bearing. A **quadrant bearing** φ, or *phi*, is a directional measurement between 0° and 90° east or west of the north-south line. The quadrant bearing of vector **v** shown is 35° east of south or southeast, written S35°E.



A **true bearing** is a directional measurement where the angle is measured clockwise from north. True bearings are always given using three digits. So, a direction that measures 25° clockwise from north would be written as a true bearing of 025°.

StudyTip

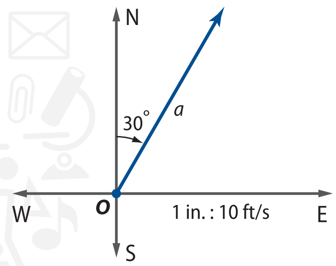
True Bearing When a degree measure is given without any additional directional components, it is assumed to be a true bearing. The true bearing of **v** is 145°.

Example 2 Represent a Vector Geometrically

Use a ruler and a protractor to draw an arrow diagram for each quantity described. Include a scale on each diagram.

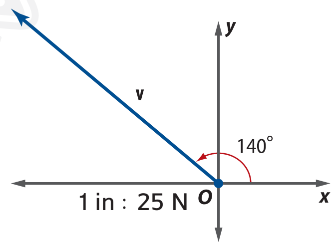
a. a = 20 feet per second at a bearing of 030°

Using a scale of 1 in: 10 ft/s, draw and label a 20 ÷ 10 or 2-inch arrow at an angle of 30° clockwise from the north.



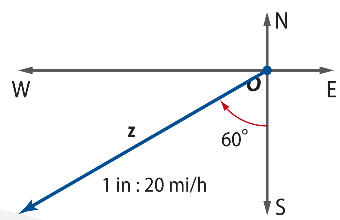
b. v = 75 Newtons of force at 140° to the horizontal

Using a scale of 1 in : 25 N, draw and label a 75 ÷ 25 or 3-inch arrow in standard position at a 140° angle to the *x*-axis.



c. z = 30 miles per hour at a bearing of S60°W

Using a scale of 1 in : 20 mi/h, draw and label a 30 ÷ 20 or 1.5-inch arrow 60° west of south.



GuidedPractice

2A. **t** = 20 meters per second at a bearing of 065°

2B. **u** = 15 kilometers per hour at a bearing of S25°E

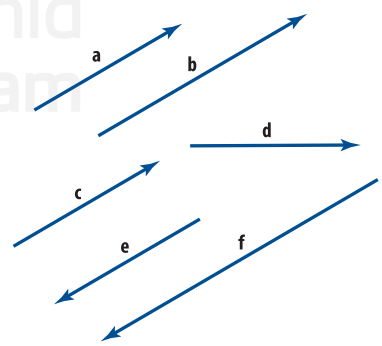
2C. **m** = 60 Newtons of force at 80° to the horizontal

WatchOut!

Magnitude The magnitude of a vector can represent distance, speed, or force. When a vector represents velocity, the length of the vector does not imply distance traveled.

In your operations with vectors, you will need to be familiar with the following vector types.

- **Parallel vectors** have the same or opposite direction but not necessarily the same magnitude. In the figure, **a** ∥ **b** ∥ **c** ∥ **e** ∥ **f**.
- **Equivalent vectors** have the same magnitude and direction. In the figure, **a** = **c** because they have the same magnitude and direction. Notice that **a** ≠ **b**, since |**a**| ≠ |**b**|, and **a** ≠ **d**, since **a** and **d** do not have the same direction.
- **Opposite vectors** have the same magnitude but opposite direction. The vector opposite **a** is written −**a**. In the figure, **e** = −**a**.



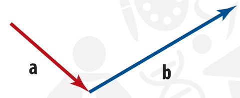
When two or more vectors are added, their sum is a single vector called the **resultant**. The resultant vector has the same effect as applying one vector after the other. Geometrically, the resultant can be found using either the **triangle method** or the **parallelogram method**.

KeyConcept Finding Resultants

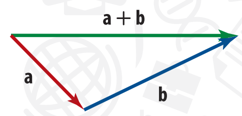
Triangle Method (Tip-to-Tail)

To find the resultant of **a** and **b**, follow these steps.

Step 1 Translate **b** so that the tail of **b** touches the tip of **a**.

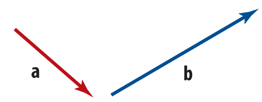


Step 2 The resultant is the vector from the tail of **a** to the tip of **b**.

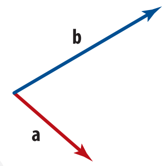


Parallelogram Method (Tail-to-Tail)

To find the resultant of **a** and **b**, follow these steps.

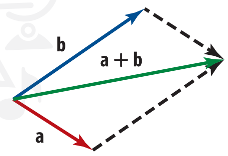


Step 1 Translate **b** so that the tail of **b** touches the tail of **a**.



Step 2 Complete the parallelogram that has **a** and **b** as two of its sides.

Step 3 The resultant is the vector that forms the indicated diagonal of the parallelogram.

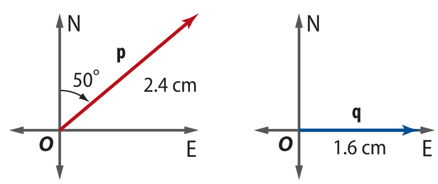


Real-World Example 3 Find the Resultant of Two Vectors

ORIENTEERING **In an orienteering competition, Amani walks N50°E for 120 meters and then walks 80 meters due east. How far and at what quadrant bearing is Amani from her starting position?**

Let **p** = walking 120 meters N50°E and **q** = walking 80 meters due east. Draw a diagram to represent **p** and **q** using a scale of 1 cm : 50 m.

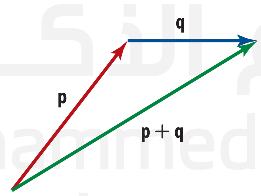
Use a ruler and a protractor to draw a 120 ÷ 50 or 2.4-centimeter arrow 50° east of north to represent **p** and an 80 ÷ 50 or 1.6-centimeter arrow due east to represent **q**.



1 cm : 50 m

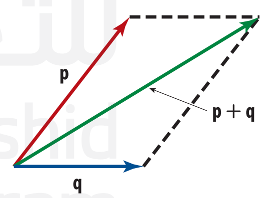
Method 1 Triangle Method

Translate **q** so that its tail touches the tip of **p**. Then draw the resultant vector **p** + **q** as shown.



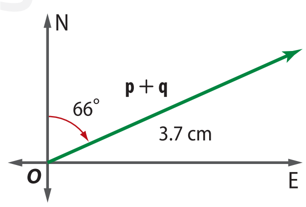
Method 2 Parallelogram Method

Translate **q** so that its tail touches the tail of **p**. Then complete the parallelogram and draw the diagonal, resultant **p** + **q**, as shown.



Both methods produce the same resultant vector **p** + **q**. Measure the length of **p** + **q** and then measure the angle this vector makes with the north-south line as shown.

The vector's length of approximately 3.7 centimeters represents 3.7 × 50 or 185 meters. Therefore, Tia is approximately 185 feet at a bearing of 66° east of north or N66°E from her starting position.



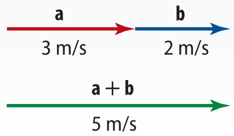
1 cm : 50 m

StudyTip

Resultants The parallelogram method must be repeated in order to find the resultant of more than two vectors. The triangle method, however, is easier to use when finding the resultant of three or more vectors. Continue to place the initial point of subsequent vectors at the terminal point of the previous vector.

StudyTip

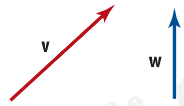
Parallel Vectors with Same Direction To add two or more parallel vectors with the *same direction*, add their magnitudes. The resultant has the same direction as the original vectors.



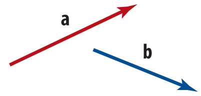
GuidedPractice

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest centimeter and its direction relative to the horizontal.

3A.

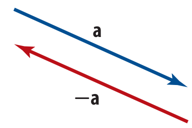


3B.



3C. **PINBALL** A pinball is struck by flipper and is sent 310° at a velocity of 7 centimeters per second. The ball then bounces off of a bumper and heads 055° at a velocity of 4 centimeters per second. Find the resulting direction and velocity of the pinball.

When you add two opposite vectors, the resultant is the **zero vector** or *null* vector, denoted by $\vec{0}$ or **0**, which has a magnitude of 0 and no specific direction. Subtracting vectors is similar to subtraction with integers. To find $\mathbf{p} - \mathbf{q}$, add the opposite of **q** to **p**. That is, $\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q})$.



$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

A vector can also be multiplied by a scalar.

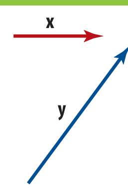
KeyConcept Multiplying Vectors by a Scalar

If a vector **v** is multiplied by a real number scalar k, the scalar multiple $k\mathbf{v}$ has a magnitude of $|k|\ |\mathbf{v}|$. Its direction is determined by the sign of k.

- If $k > 0$, $k\mathbf{v}$ has the same direction as **v**.
- If $k < 0$, $k\mathbf{v}$ has the opposite direction as **v**.

Example 4 Operations with Vectors

Draw a vector diagram of $3\mathbf{x} - \frac{3}{4}\mathbf{y}$.



Rewrite the expression as the addition of two vectors: $3\mathbf{x} - \frac{3}{4}\mathbf{y} = 3\mathbf{x} + \left(-\frac{3}{4}\mathbf{y}\right)$. To represent $3\mathbf{x}$, draw a vector 3 times as long as **x** in the same direction as **x** (Figure 7.1.1). To represent $-\frac{3}{4}\mathbf{y}$, draw a vector $\frac{3}{4}$ the length of **y** in the opposite direction from **y** (Figure 7.1.2). Then use the triangle method to draw the resultant vector (Figure 7.1.3).

Figure 7.1.1

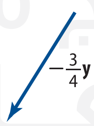


Figure 7.1.2

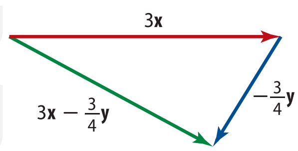
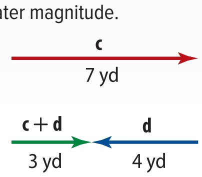


Figure 7.1.3

StudyTip

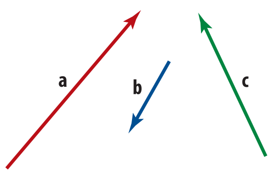
Parallel Vectors with Opposite Directions To add two parallel vectors with *opposite directions*, find the absolute value of the difference in their magnitudes. The resultant has the same direction as the vector with the greater magnitude.



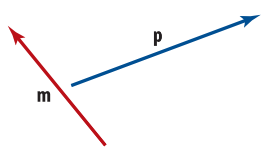
GuidedPractice

Draw a vector diagram of each expression.

4A. $\mathbf{a} - \mathbf{c} + 2\mathbf{b}$



4B. $\mathbf{m} - \frac{1}{4}\mathbf{p}$



2 Vector Applications

Vector addition and trigonometry can be used to solve vector problems involving triangles which are often oblique.

In navigation, a *heading* is the direction in which a vessel, such as an airplane or boat, is steered to overcome other forces, such as wind or current. The *relative velocity* of the vessel is the resultant when the heading velocity and other forces are combined.

Real-World Example 5 Use Vectors to Solve Navigation Problems

AVIATION An airplane is flying with an airspeed of 310 knots on a heading of 050°. If a 78-knot wind is blowing from a true heading of 125°, determine the speed and direction of the plane relative to the ground.

Step 1 Draw a diagram to represent the heading and wind velocities (Figure 7.1.4). Translate the wind vector as shown in Figure 7.1.5, and use the triangle method to obtain the resultant vector representing the plane's ground velocity **g**. In the triangle formed by these vectors (Figure 7.1.6), $\gamma = 125° - 50°$ or 75°.

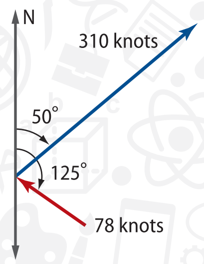


Figure 7.1.4

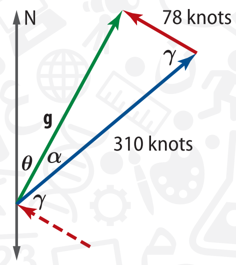


Figure 7.1.5

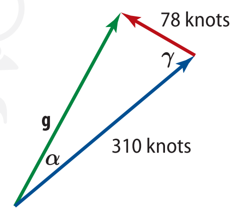


Figure 7.1.6

Step 2 Use the Law of Cosines to find $|\mathbf{g}|$, the plane's speed relative to the ground.

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$ Law of Cosines

$$|\mathbf{g}|^2 = 78^2 + 310^2 - 2(78)(310)\cos 75°$$ $c = |\mathbf{g}|$, $a = 78$, $b = 310$, and $\gamma = 75°$

$$|\mathbf{g}| = \sqrt{78^2 + 310^2 - 2(78)(310)\cos 75°}$$ Take the positive square root of each side.

$$\approx 299.4$$ Simplify.

The ground speed of the plane is about 299.4 knots.

Step 3 The heading of the resultant **g** is represented by angle θ, as shown in Figure 7.1.5. To find θ, first calculate α using the Law of Sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}$$ Law of Sines

$$\frac{\sin\alpha}{78} = \frac{\sin 75°}{299.4}$$ $c = |\mathbf{g}|$ or 299.4, $a = 78$, and $\gamma = 75°$

$$\sin\alpha = \frac{78\sin 75°}{299.4}$$ Solve for $\sin\alpha$.

$$\alpha = \sin^{-1}\frac{78\sin 75°}{299.4}$$ Apply the inverse sine function.

$$\approx 14.6°$$ Simplify.

The measure of θ is $50° - \alpha$, which is $50° - 14.6°$ or 35.4°.

Therefore, the speed of the plane relative to the ground is about 299.4 knots at about 035°.

GuidedPractice

5. **SWIMMING** Ali rows due east at a speed of 3.5 feet per second across a river directly toward the opposite bank. At the same time, the current of the river is carrying him due south at a rate of 2 feet per second. Find Ali's speed and direction relative to the shore.

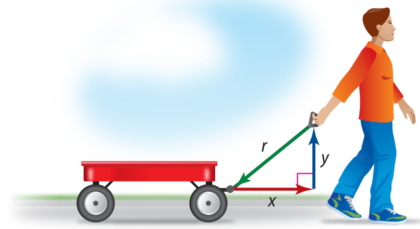
StudyTip

Alternate Interior Angles The translation of the tail of the wind vector to the tip of the vector representing the plane's heading produces two parallel vectors cut by a transversal. Since alternate interior angles of two parallel lines cut by a transversal are congruent, the angles made by these two vectors in both places in Figure 7.1.5 have the same measure.

WatchOut!

Wind Direction In Example 5, notice that the wind is blowing *from* a bearing of 125° and the vector is drawn so that the tip of the vector points *toward* the north-south line. Had the wind been blowing at a bearing of 125°, the vector would have pointed *away* from this line.

Two or more vectors with a sum that is a vector **r** are called **components** of **r**. While components can have any direction, it is often useful to express or *resolve* a vector into two perpendicular components. The **rectangular components** of a vector are horizontal and vertical.



In the diagram, the force **r** exerted to pull the wagon can be thought of as the sum of a horizontal component force **x** that moves the wagon forward and a vertical component force **y** that pulls the wagon upward.

Real-WorldLink

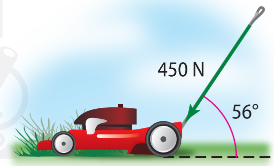
It takes a force of about 3 newtons to flip a light switch. The force due to gravity on a person is about 600 newtons. The force exerted by a weightlifter is about 2000 newtons.

Source: *Contemporary College Physics*

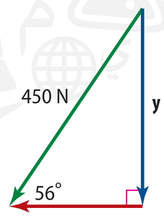
Real-World Example 6 Resolve a Force into Rectangular Components

LAWN CARE **Hala is pushing the handle of a lawn mower with a force of 450 newtons at an angle of 56° with the ground.**

a. Draw a diagram that shows the resolution of the force that Hala exerts into its rectangular components.



Hala's push can be resolved into a horizontal push **x** forward and a vertical push **y** downward as shown.



b. Find the magnitudes of the horizontal and vertical components of the force.

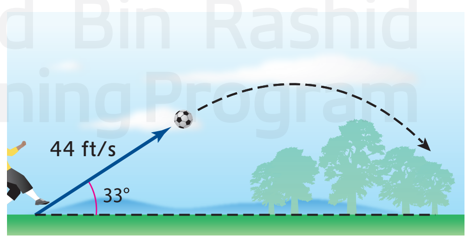
The horizontal and vertical components of the force form a right triangle. Use the sine or cosine ratios to find the magnitude of each force.

$\cos 56° = \frac{\lvert\mathbf{x}\rvert}{450}$	Right triangle definitions of cosine and sine	$\sin 56° = \frac{\lvert\mathbf{y}\rvert}{450}$
$\lvert\mathbf{x}\rvert = 450 \cos 56°$	Solve for *x* and *y*.	$\lvert\mathbf{y}\rvert = 450 \sin 56°$
$\lvert\mathbf{x}\rvert \approx 252$	Use a calculator.	$\lvert\mathbf{y}\rvert \approx 373$

The magnitude of the horizontal component is about 252 newtons, and the magnitude of the vertical component is about 373 newtons.

GuidedPractice

6. FOOTBALL A player kicks a football so that it leaves the ground with a velocity of 44 feet per second at an angle of 33° with the ground.



A. Draw a diagram that shows the resolution of this force into its rectangular components.

B. Find the magnitude of the horizontal and vertical components of the velocity.

Exercises

State whether each quantity described is a *vector* quantity or a *scalar* quantity. (Example 1)

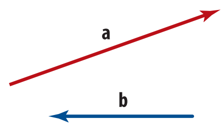
1. a box being pushed with a force of 125 newtons
2. wind blowing at 20 knots
3. a deer running 15 meters per second due west
4. a baseball thrown with a speed of 85 miles per hour
5. a 3.75-kilogram stone hanging from a rope
6. a rock thrown straight up at a velocity of 50 feet per second

Use a ruler and a protractor to draw an arrow diagram for each quantity described. Include a scale on each diagram. (Example 2)

7. $\mathbf{h}$ = 13 centimeters per second at a bearing of 205°
8. $\mathbf{g}$ = 6 kilometers per hour at a bearing of N70°W
9. $\mathbf{j}$ = 5 meters per minute at 300° to the horizontal
10. $\mathbf{k}$ = 28 kilometers at 35° to the horizontal
11. $\mathbf{m}$ = 40 meters at a bearing of S55°E
12. $\mathbf{n}$ = 32 meter per second at a bearing of 030°

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest tenth of a centimeter and its direction relative to the horizontal. (Example 3)

13.



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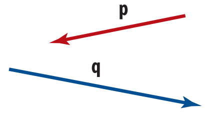

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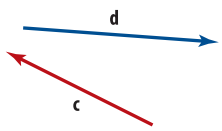
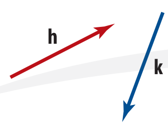


16.



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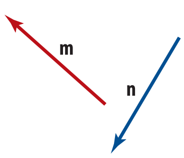

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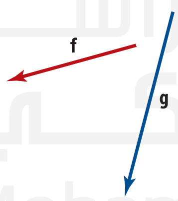
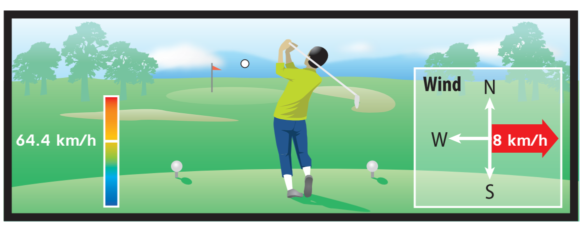


19. **GOLF** While playing a golf video game, Omar hits a ball 35° above the horizontal at a speed of 64.4-kilometer per hour with a 8 kilometers per hour wind blowing, as shown. Find the resulting speed and direction of the ball. (Example 3)



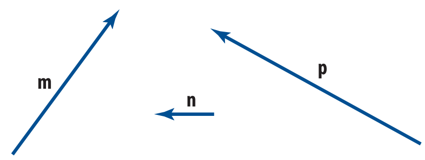
20. **BOATING** A charter boat leaves port on a heading of N60°W for 12 nautical miles. The captain changes course to a bearing of N25°E for the next 15 nautical miles. Determine the ship's distance and direction from port to its current location. (Example 3)

21. **HIKING** Mazen and Ayoub hiked 3.75 kilometers to a lake 55° east of south from their campsite. Then they hiked 33° west of north to the nature center 5.6 kilometers from the lake. Where is the nature center in relation to their campsite? (Example 3)

Determine the magnitude and direction of the resultant of each vector sum. (Example 3)

22. 18 newtons directly forward and then 20 newtons directly backward
23. 100 meters due north and then 350 meters due south
24. 10 kilograms of force at a bearing of 025° and then 15 kilograms of force at a bearing of 045°
25. 17 kilometers east and then 16 kilometers south
26. 15 meters per second squared at a 60° angle to the horizontal and then 9.8 meters per second squared downward

Use the set of vectors to draw a vector diagram of each expression. (Example 4)



27. $\mathbf{m} - 2\mathbf{n}$
28. $\mathbf{n} - \frac{3}{4}\mathbf{m}$
29. $\frac{1}{2}\mathbf{p} + 3\mathbf{n}$
30. $4\mathbf{n} + \frac{4}{5}\mathbf{p}$
31. $\mathbf{p} + 2\mathbf{n} - \mathbf{m}$
32. $-\frac{1}{3}\mathbf{m} + \mathbf{p} - 2\mathbf{n}$
33. $3\mathbf{n} - \frac{1}{2}\mathbf{p} + \mathbf{m}$
34. $\mathbf{m} - 3\mathbf{n} + \frac{1}{4}\mathbf{p}$

35. **RUNNING** A runner's resultant velocity is 8 miles per hour due west running with a wind of 3 miles per hour N28°W. What is the runner's speed, to the nearest mile per hour, without the effect of the wind? (Example 5)

36. **GLIDING** A glider is traveling at an air speed of 15 kilometers per hour due west. If the wind is blowing at 5 kilometers per hour in the direction N60°E, what is the resulting ground speed of the glider? (Example 5)

37. **CURRENT** Sally is swimming due west at a rate of 1.5 meters per second. A strong current is flowing S20°E at a rate of 1 meter per second. Find Sally's resulting speed and direction. (Example 5)

Draw a diagram that shows the resolution of each vector into its rectangular components. Then find the magnitudes of the vector's horizontal and vertical components. (Example 6)

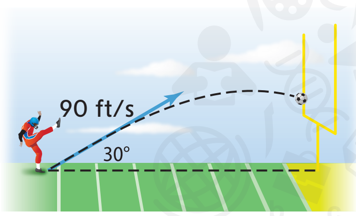
38. $2\frac{1}{8}$ centimeters at 310° to the horizontal

39. 1.5 centimeters at a bearing of N49°E

40. 3.2 centimeters per hour at a bearing of S78°W

41. $\frac{3}{4}$ centimeter per minute at a bearing of 255°

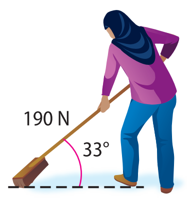
42. **FOOTBALL** For a goal attempt, a ball is kicked with the velocity shown in the diagram below.



a. Draw a diagram that shows the resolution of this force into its rectangular components.

b. Find the magnitudes of the horizontal and vertical components. (Example 6)

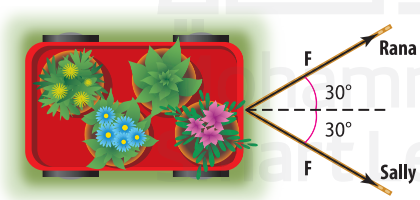
43. **CLEANING** A push broom is pushed with a force of 190 newtons at an angle of 33° with the ground. (Example 6)



a. Draw a diagram that shows the resolution of this force into its rectangular components.

b. Find the magnitudes of the horizontal and vertical components.

44. **GARDENING** Rana and Sally are pulling a wagon full of plants. Each person pulls on the wagon with equal force at an angle of 30° with the axis of the wagon. The resultant force is 120 newtons.



a. How much force is each person exerting?

b. If each person exerts a force of 75 newtons, what is the resultant force?

c. How will the resultant force be affected if Rana and Sally move closer together?

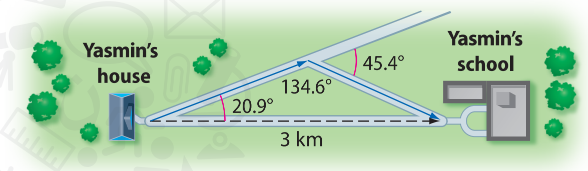
The magnitude and true bearings of three forces acting on an object are given. Find the magnitude and direction of the resultant of these forces.

45. 50 kg at 30°, 80 kg at 125°, and 100 kg at 220°

46. 8 newtons at 300°, 12 newtons at 45°, and 6 newtons at 120°

47. 18 kg at 190°, 3 kg at 20°, and 7 kg at 320°

48. **DRIVING** Yasmin's school is on a direct path three kilometers from her house. She drives on two different streets on her way to school. She travels at an angle of 20.9° with the path on the first street and then turns 45.4° onto the second street.



a. How far does Yasmin drive on the first street?

b. How far does she drive on the second street?

c. If it takes her 10 minutes to get to school and she averages 25 kilometers per hour on the first street, what speed does Yasmin average after she turns onto the second street?

49. **SLEDDING** Hamad is pulling his sister on a sled. The direction of his resultant force is 31°, and the horizontal component of the force is 86 newtons.

a. What is the vertical component of the force?

b. What is the magnitude of the resultant force?

50. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate multiplication of a vector by a scalar.

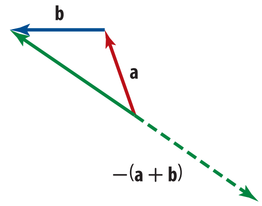
a. **GRAPHICAL** On a coordinate plane, draw a vector **a** so that the tail is located at the origin. Choose a value for a scalar k. Then draw the vector that results if you multiply the original vector by k on the same coordinate plane. Repeat the process for four additional vectors **b**, **c**, **d**, and **e**. Use the same value for k each time.

b. **TABULAR** Copy and complete the table below for each vector that you drew in part **a**.

Vector	Terminal Point of Vector	Terminal Point of Vector × k
a		
b		
c		
d		
e		

c. **ANALYTICAL** If the terminal point of a vector **a** is located at the point (a, b), what is the location of the terminal point of the vector k**a**?

An *equilibrant* vector is the opposite of a resultant vector. It balances a combination of vectors such that the sum of the vectors and the equilibrant is the zero vector. The equilibrant vector of a + b is −(a + b).



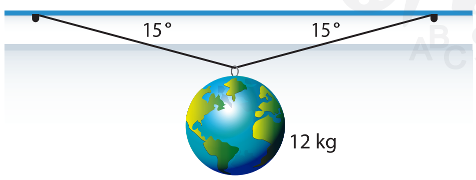
Find the magnitude and direction of the equilibrant vector for each set of vectors.

51. **a** = 15 kilometers per hour at a bearing of 125°
b = 12 kilometers per hour at a bearing of 045°

52. **a** = 4 meters at a bearing of N30W°
b = 6 meters at a bearing of N20E°

53. **a** = 23 meters per second at a bearing of 205°
b = 16 meters per second at a bearing of 345°

54. **MAGNITUDE** A round object is suspended from a ceiling by two wires of equal length as shown.

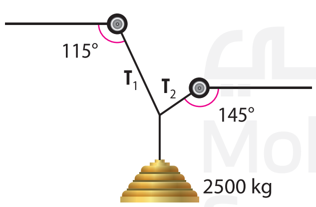


a. Draw a vector diagram of the situation that indicates that two tension vectors $\mathbf{T_1}$ and $\mathbf{T_2}$ with equal magnitude are keeping the object stationary or at equilibrium.

b. Redraw the diagram using the triangle method to find $\mathbf{T_1} + \mathbf{T_2}$.

c. Use your diagram from part **b** and the fact that the equilibrant of the resultant $\mathbf{T_1} + \mathbf{T_2}$ and the vector representing the weight of the object are equivalent vectors to calculate the magnitudes of $\mathbf{T_1}$ and $\mathbf{T_2}$.

55. **CABLE SUPPORT** Two cables with tensions $\mathbf{T_1}$ and $\mathbf{T_2}$ are tied together to support a 2500-kilogram load at equilibrium.



a. Write expressions to represent the horizontal and vertical components of $\mathbf{T_1}$ and $\mathbf{T_2}$.

b. Given that the equilibrant of the resultant $\mathbf{T_1} + \mathbf{T_2}$ and the vector representing the weight of the load are equivalent vectors, calculate the magnitudes of $\mathbf{T_1}$ and $\mathbf{T_2}$ to the nearest tenth of a kilogram.

c. Use your answers from parts **a** and **b** to find the magnitudes of the horizontal and vertical components of $\mathbf{T_1}$ and $\mathbf{T_2}$ to the nearest tenth of a kilogram.

Find the magnitude and direction of each vector given its vertical and horizontal components and the range of values for the angle of direction θ to the horizontal.

56. horizontal: 0.32 cm, vertical: 2.28 cm, $90° < \theta < 180°$

57. horizontal: 3.1 m, vertical: 4.2 m, $0° < \theta < 90°$

58. horizontal: 2.6 cm, vertical: 9.7 cm, $270° < \theta < 360°$

59. horizontal: 2.9 m, vertical: 1.8 m, $180° < \theta < 270°$

Draw any three vectors a, b, and c. Show geometrically that each of the following vector properties holds using these vectors.

60. Commutative Property: **a** + **b** = **b** + **a**

61. Associative Property: (**a** + **b**) + **c** = **a** + (**b** + **c**)

62. Distributive Property: $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$, for $k = 2, 0.5$, and -2

H.O.T. Problems Use Higher-Order Thinking Skills

63. **OPEN ENDED** Consider a vector of 5 units directed along the positive *x*-axis. Resolve the vector into two perpendicular components in which no component is horizontal or vertical.

64. **REASONING** Is it *sometimes, always*, or *never* possible to find the sum of two parallel vectors using the parallelogram method? Explain your reasoning.

65. **REASONING** Why is it important to establish a common reference for measuring the direction of a vector, for example, from the positive *x*-axis?

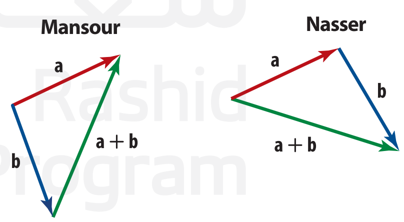
66. **CHALLENGE** The resultant of **a** + **b** is equal to the resultant of **a** − **b**. If the magnitude of **a** is $4x$, what is the magnitude of **b**?

67. **REASONING** Consider the statement $|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} + \mathbf{b}|$.

a. Express this statement using words.

b. Is this statement true or false? Justify your answer.

68. **ERROR ANALYSIS** Mansour and Nasser are finding the resultant of vectors **a** and **b**. Is either of them correct? Explain your reasoning.



69. **REASONING** Is it possible for the sum of two vectors to equal one of the vectors? Explain.

70. **WRITING IN MATH** Compare and contrast the parallelogram and triangle methods of finding the resultant of two or more vectors.

Spiral Review

71. **KICKBALL** Suppose a kickball player kicks a ball at a 32° angle to the horizontal with an initial speed of 20 meters per second. How far away will the ball land?

72. Graph $(x')^2 + y' - 5 = 1$ if it has been rotated 45° from its position in the xy-plane.

Write an equation for a circle that satisfies each set of conditions. Then graph the circle.

73. center at (4, 5), radius 4

74. center at (1, −4), diameter 7

Determine the equation of and graph the parabola with the given focus *F* and vertex *V*.

75. $F(2, 4)$, $V(2, 3)$

76. $F(1, 5)$, $V(-7, 5)$

77. **CRAFTS** Majed is selling wood carvings. He sells large statues for AED 60, clocks for AED 40, dollhouse furniture for AED 25, and chess pieces for AED 5. He takes the following number of items to the fair: 12 large statues, 25 clocks, 45 pieces of dollhouse furniture, and 50 chess pieces.

a. Write an inventory matrix for the number of each item and a cost matrix for the price of each item.

b. Find Majed's total income if he sells all of the items.

Solve each equation for all values of *x*.

78. $4 \sin x \cos x - 2 \sin x = 0$

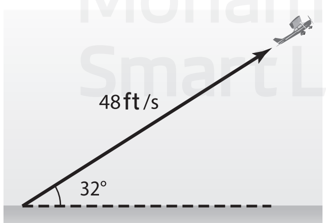
79. $\sin x - 2 \cos^2 x = -1$

Skills Review for Standardized Tests

80. **SAT/ACT** If town *A* is 12 kilometers from town *B* and town *C* is 18 kilometers from town *A*, then which of the following *cannot* be the distance from town *B* to town *C*?

A 5 kilometers
B 7 kilometers
C 10 kilometers
D 12 kilometers
E 18 kilometers

81. A remote control airplane flew along an initial path of 32° to the horizontal at a velocity of 48 feet per second as shown. Which of the following represent the magnitudes of the horizontal and vertical components of the velocity?



F 25.4 ft/s, 40.7 ft/s
G 40.7 ft/s, 25.4 ft/s
H 56.6 ft/s, 90.6 ft/s
J 90.6 ft/s, 56.6 ft/s

82. **REVIEW** Triangle *ABC* has vertices $A(-4, 2)$, $B(-4, -3)$, and $C(3, -3)$. After a dilation, triangle $A'B'C'$ has vertices $A'(-12, 6)$, $B'(-12, -9)$, and $C'(9, -9)$. How many times as great is the area of $\triangle A'B'C'$ than the area of $\triangle ABC$?

A $\frac{1}{9}$
B $\frac{1}{3}$
C 3
D 9

83. **REVIEW** Halima is drawing a map of her neighborhood. Her house is represented by quadrilateral *ABCD* with vertices $A(2, 2)$, $B(6, 2)$, $C(6, 6)$, and $D(2, 6)$. She wants to use the same coordinate system to make another map that is one half the size of the original map. What could be the new vertices of Halima's house?

F $A'(0, 0)$, $B'(2, 1)$, $C'(3, 3)$, $D'(0, 3)$
G $A'(0, 0)$, $B'(3, 1)$, $C'(2, 3)$, $D'(0, 2)$
H $A'(1, 1)$, $B'(3, 1)$, $C'(3, 3)$, $D'(1, 3)$
J $A'(1, 2)$, $B'(3, 0)$, $C'(2, 2)$, $D'(2, 3)$

LESSON 7-2 Vectors in the Coordinate Plane

Then

- You performed vector operations using scale drawings.

Now

1. Represent and operate with vectors in the coordinate plane.
2. Write a vector as a linear combination of unit vectors.

Why?

- Wind can impact the ground speed and direction of an airplane. While pilots can use scale drawings to determine the heading a plane should take to correct for wind, these calculations are more commonly calculated using vectors in the coordinate plane.

NewVocabulary
component form
unit vector
linear combination

1 Vectors in the Coordinate Plane

In Lesson 7-1, you found the magnitude and direction of the resultant of two or more forces geometrically by using a scale drawing. Since drawings can be inaccurate, an algebraic approach using a rectangular coordinate system is needed for situations where more accuracy is required or where the system of vectors is complex.

A vector $\overrightarrow{OP}$ in standard position on a rectangular coordinate system (as in Figure 7.2.1) can be uniquely described by the coordinates of its terminal point $P(x, y)$. We denote $\overrightarrow{OP}$ on the coordinate plane by $\langle x, y \rangle$. Notice that x and y are the rectangular components of $\overrightarrow{OP}$. For this reason, $\langle x, y \rangle$ is called the **component form** of a vector.

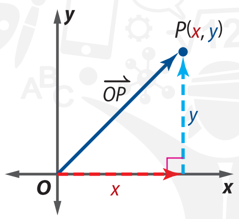


Figure 7.2.1

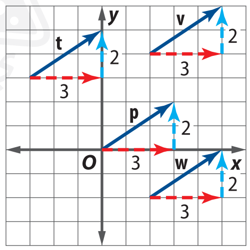


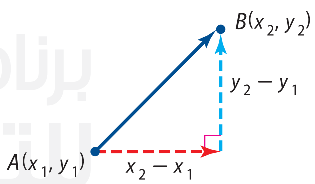
Figure 7.2.2

Since vectors with the same magnitude and direction are equivalent, many vectors can be represented by the same coordinates. For example, vectors **p**, **t**, **v**, and **w** in Figure 7.2.2 are *equivalent* because each can be denoted as $\langle 3, 2 \rangle$. To find the component form of a vector that is not in standard position, you can use the coordinates of its initial and terminal points.

KeyConcept Component Form of a Vector

The component form of a vector $\overrightarrow{AB}$ with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$ is given by

$$\langle x_2 - x_1, y_2 - y_1 \rangle.$$



Example 1 Express a Vector in Component Form

Find the component form of $\overrightarrow{AB}$ with initial point $A(-4, 2)$ and terminal point $B(3, -5)$.

$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$ — Component form

$= \langle 3 - (-4), -5 - 2 \rangle$ — $(x_1, y_1) = (-4, 2)$ and $(x_2, y_2) = (3, -5)$

$= \langle 7, -7 \rangle$ — Subtract.

GuidedPractice

Find the component form of $\overrightarrow{AB}$ with the given initial and terminal points.

1A. $A(-2, -7)$, $B(6, 1)$

1B. $A(0, 8)$, $B(-9, -3)$

The magnitude of a vector in the coordinate plane is found by using the Distance Formula.

ReadingMath

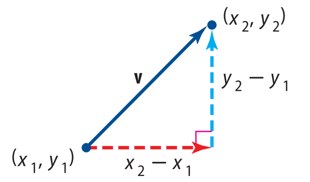
Norm The magnitude of a vector is sometimes called the *norm* of the vector.

KeyConcept Magnitude of a Vector in the Coordinate Plane

If **v** is a vector with initial point (x_1, y_1) and terminal point (x_2, y_2), then the magnitude of **v** is given by

$$|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

If **v** has a component form of $\langle a, b \rangle$, then $|\mathbf{v}| = \sqrt{a^2 + b^2}$.



Example 2 Find the Magnitude of a Vector

Find the magnitude of $\overrightarrow{AB}$ with initial point $A(-4, 2)$ and terminal point $B(3, -5)$.

$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ — Distance Formula

$= \sqrt{[3 - (-4)]^2 + (-5 - 2)^2}$ — $(x_1, y_1) = (-4, 2)$ and $(x_2, y_2) = (3, -5)$

$= \sqrt{98}$ or about 9.9 — Simplify.

CHECK From Example 1, you know that $\overrightarrow{AB} = \langle 7, -7 \rangle$. $|\overrightarrow{AB}| = \sqrt{7^2 + (-7)^2}$ or $\sqrt{98}$. ✓

GuidedPractice

Find the magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

2A. $A(-2, -7)$, $B(6, 1)$

2B. $A(0, 8)$, $B(-9, -3)$

Addition, subtraction, and scalar multiplication of vectors in the coordinate plane is similar to the same operations with matrices.

KeyConcept Vector Operations

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ are vectors and k is a scalar, then the following are true.

Vector Addition	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$
Vector Subtraction	$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$
Scalar Multiplication	$k\mathbf{a} = \langle ka_1, ka_2 \rangle$

Example 3 Operations with Vectors

Find each of the following for $\mathbf{w} = \langle -4, 1 \rangle$, $\mathbf{y} = \langle 2, 5 \rangle$, and $\mathbf{z} = \langle -3, 0 \rangle$.

a. $\mathbf{w} + \mathbf{y}$

$\mathbf{w} + \mathbf{y} = \langle -4, 1 \rangle + \langle 2, 5 \rangle$ — Substitute.

$= \langle -4 + 2, 1 + 5 \rangle$ or $\langle -2, 6 \rangle$ — Vector addition

b. $\mathbf{z} - 2\mathbf{y}$

$\mathbf{z} - 2\mathbf{y} = \mathbf{z} + (-2)\mathbf{y}$ — Rewrite subtraction as addition.

$= \langle -3, 0 \rangle + (-2)\langle 2, 5 \rangle$ — Substitute.

$= \langle -3, 0 \rangle + \langle -4, -10 \rangle$ or $\langle -7, -10 \rangle$ — Scalar multiplication and vector addition

GuidedPractice

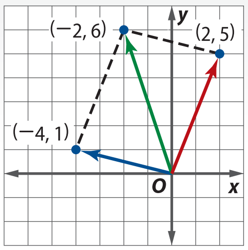
3A. $4\mathbf{w} + \mathbf{z}$

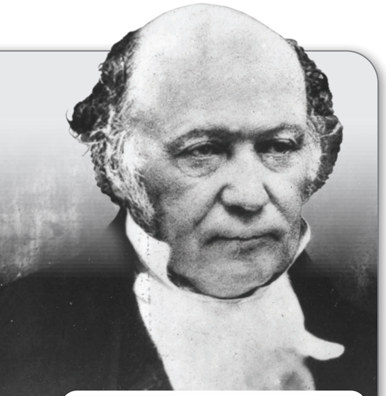
3B. $-3\mathbf{w}$

3C. $2\mathbf{w} + 4\mathbf{y} - \mathbf{z}$

StudyTip

Check Graphically A graphical check of Example 3a using the parallelogram method is shown below.





2 Unit Vectors

A vector that has a magnitude of 1 unit is called a **unit vector**. It is sometimes useful to describe a nonzero vector **v** as a scalar multiple of a unit vector **u** with the same direction as **v**. To find **u**, divide **v** by its magnitude $|\mathbf{v}|$.

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \text{or} \quad \frac{1}{|\mathbf{v}|}\mathbf{v}$$

Math HistoryLink

William Rowan Hamilton (1805–1865)
An Irish mathematician, Hamilton developed the theory of quaternions and published *Lectures on Quaternions.* Many basic concepts of vector analysis have their basis in this theory.

Example 4 Find a Unit Vector with the Same Direction as a Given Vector

Find a unit vector u with the same direction as v = ⟨−2, 3⟩.

$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v}$ — Unit vector with the same direction as v

$= \frac{1}{|\langle -2, 3\rangle|}\langle -2, 3\rangle$ — Substitute.

$= \frac{1}{\sqrt{(-2)^2 + 3^2}}\langle -2, 3\rangle$ — $|\langle a, b\rangle| = \sqrt{a^2 + b^2}$

$= \frac{1}{\sqrt{13}}\langle -2, 3\rangle$ — Simplify.

$= \left\langle -\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$ — Scalar multiplication

$= \left\langle -\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13} \right\rangle$ — Rationalize denominators.

CHECK Since **u** is a scalar multiple of **v**, it has the same direction as **v**. Verify that the magnitude of **u** is 1.

$|\mathbf{u}| = \sqrt{\left(-\frac{2\sqrt{13}}{13}\right)^2 + \left(\frac{3\sqrt{13}}{13}\right)^2}$ — Distance Formula

$= \sqrt{\frac{52}{169} + \frac{117}{169}}$ — Simplify.

$= \sqrt{1}$ or 1 ✓ — Simplify.

GuidedPractice

Find a unit vector with the same direction as the given vector.

4A. $\mathbf{w} = \langle 6, -2\rangle$

4B. $\mathbf{x} = \langle -4, -8\rangle$

WatchOut!

Unit Vector i Do not confuse the unit vector **i** with the imaginary number *i*. The unit vector is denoted by a bold, nonitalic letter **i**. The imaginary number is denoted by a bold italic letter ***i***.

The unit vectors in the direction of the positive *x*-axis and positive *y*-axis are denoted by $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$, respectively (Figure 7.2.3). Vectors **i** and **j** are called *standard unit vectors*.

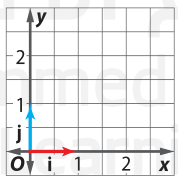


Figure 7.2.3

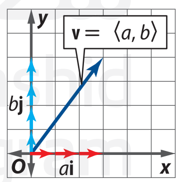


Figure 7.2.4

These vectors can be used to express any vector $\mathbf{v} = \langle a, b\rangle$ as $a\mathbf{i} + b\mathbf{j}$ as shown in Figure 7.2.4.

$\mathbf{v} = \langle a, b\rangle$ — Component form of v

$= \langle a, 0\rangle + \langle 0, b\rangle$ — Rewrite as the sum of two vectors.

$= a\langle 1, 0\rangle + b\langle 0, 1\rangle$ — Scalar multiplication

$= a\mathbf{i} + b\mathbf{j}$ — $\langle 1, 0\rangle = \mathbf{i}$ and $\langle 0, 1\rangle = \mathbf{j}$

The vector sum $a\mathbf{i} + b\mathbf{j}$ is called a **linear combination** of the vectors **i** and **j**.

Example 5 Write a Vector as a Linear Combination of Unit Vectors

Let $\overrightarrow{DE}$ be the vector with initial point $D(-2, 3)$ and terminal point $E(4, 5)$. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j.

First, find the component form of $\overrightarrow{DE}$.

$\overrightarrow{DE} = \langle x_2 - x_1, y_2 - y_1 \rangle$ Component form

$= \langle 4 - (-2), 5 - 3 \rangle$ $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (4, 5)$

$= \langle 6, 2 \rangle$ Simplify.

Then rewrite the vector as a linear combination of the standard unit vectors.

$\overrightarrow{DE} = \langle 6, 2 \rangle$ Component form

$= 6\mathbf{i} + 2\mathbf{j}$ $\langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$

GuidedPractice

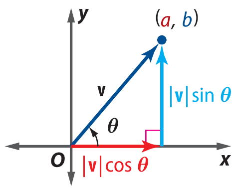
Let $\overrightarrow{DE}$ be the vector with the given initial and terminal points. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j.

5A. $D(-6, 0), E(2, 5)$ **5B.** $D(-3, -8), E(-7, 1)$

StudyTip

Unit Vector From the statement that $\mathbf{v} = \langle |\mathbf{v}| \cos \theta, |\mathbf{v}| \sin \theta \rangle$, it follows that the unit vector in the direction of **v** has the form $\mathbf{v} = |1 \cos \theta, 1 \sin \theta|$
$= \langle \cos \theta, \sin \theta \rangle$.

A way to specify the direction of a vector $\mathbf{v} = \langle a, b \rangle$ is to state the direction angle θ that **v** makes with the positive x-axis. From Figure 7.2.5, it follows that **v** can be written in component form or as a linear combination of **i** and **j** using the magnitude and direction angle of the vector.



$\mathbf{v} = \langle a, b \rangle$ Component form

$= \langle |\mathbf{v}| \cos \theta, |\mathbf{v}| \sin \theta \rangle$ Substitution

$= |\mathbf{v}| (\cos \theta)\mathbf{i} + |\mathbf{v}| (\sin \theta)\mathbf{j}$ Linear combination of i and j

Example 6 Find Component Form

Find the component form of the vector v with magnitude 10 and direction angle 120°.

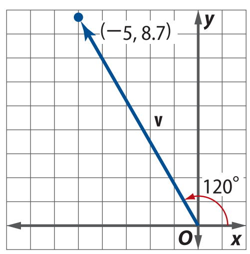
$\mathbf{v} = \langle |\mathbf{v}| \cos \theta, |\mathbf{v}| \sin \theta \rangle$ Component form of v in terms of $|\mathbf{v}|$ and θ

$= \langle 10 \cos 120°, 10 \sin 120° \rangle$ $|\mathbf{v}| = 10$ and $\theta = 120°$

$= \left\langle 10\left(-\frac{1}{2}\right), 10\left(\frac{\sqrt{3}}{2}\right) \right\rangle$ $\cos 120° = -\frac{1}{2}$ and $\sin 120° = \frac{\sqrt{3}}{2}$

$= \langle -5, 5\sqrt{3} \rangle$ Simplify.

CHECK Graph $\mathbf{v} = \langle -5, 5\sqrt{3} \rangle \approx \langle -5, 8.7 \rangle$. The measure of the angle **v** makes with the positive x-axis is about 120° as shown, and $|\mathbf{v}| = \sqrt{(-5)^2 + (5\sqrt{3})^2}$ or 10. ✓



GuidedPractice

Find the component form of v with the given magnitude and direction angle.

6A. $|\mathbf{v}| = 8, \theta = 45°$ **6B.** $|\mathbf{v}| = 24, \theta = 210°$

It also follows from Figure 7.2.5 on the previous page that the direction angle θ of vector $\mathbf{v} = \langle a, b \rangle$ can be found by solving the trigonometric equation $\tan\theta = \frac{|\mathbf{v}|\sin\theta}{|\mathbf{v}|\cos\theta}$ or $\tan\theta = \frac{b}{a}$.

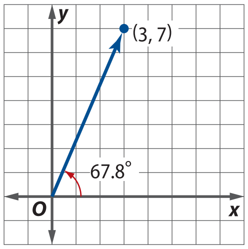


Figure 7.2.6

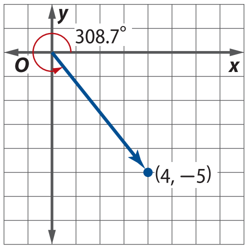


Figure 7.2.7

Example 7 Direction Angles of Vectors

Find the direction angle of each vector to the nearest tenth of a degree.

a. $\mathbf{p} = 3\mathbf{i} + 7\mathbf{j}$

$\tan\theta = \frac{b}{a}$ — Direction angle equation

$\tan\theta = \frac{7}{3}$ — $a = 3$ and $b = 7$

$\theta = \tan^{-1}\frac{7}{3}$ — Solve for θ.

$\theta \approx 66.8°$ — Use a calculator.

So, the direction angle of vector **p** is about 67.8° as shown in Figure 7.2.6.

b. $\mathbf{r} = \langle 4, -5 \rangle$

$\tan\theta = \frac{b}{a}$ — Direction angle equation

$\tan\theta = \frac{-5}{4}$ — $a = 4$ and $b = -5$

$\theta = \tan^{-1}\left(-\frac{5}{4}\right)$ — Solve for θ.

$\theta \approx -51.3°$ — Use a calculator.

Since **r** lies in Quadrant IV as shown in Figure 7.2.7, $\theta = 360 + (-51.3)$ or 308.7°.

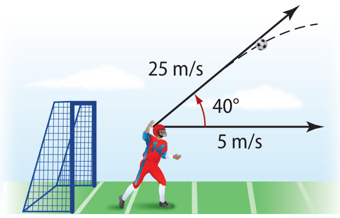
GuidedPractice

7A. $-6\mathbf{i} + 2\mathbf{j}$

7B. $\langle -3, -8 \rangle$

Real-World Example 8 Applied Vector Operations

SOCCER **A goalkeeper running forward at 5 meters per second throws a ball with a velocity of 25 meters per second at an angle of 40° with the horizontal. What is the resultant speed and direction of the pass?**



Since the goalkeeper moves straight forward, the component form of his velocity $\mathbf{v}_1$ is $\langle 5, 0 \rangle$. Use the magnitude and direction of the ball's velocity $\mathbf{v}_2$ to write this vector in component form.

$\mathbf{v}_2 = \langle |\mathbf{v}_2|\cos\theta, |\mathbf{v}_2|\sin\theta \rangle$ — Component form of v_2

$= \langle 25\cos 40°, 25\sin 40° \rangle$ — $|v_2| = 25$ and $\theta = 40°$

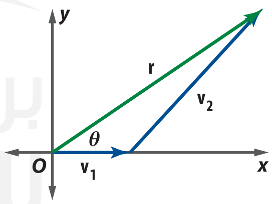
$\approx \langle 19.2, 16.1 \rangle$ — Simplify.

Add the algebraic vectors representing $\mathbf{v}_1$ and $\mathbf{v}_2$ to find the resultant velocity, vector **r**.

$\mathbf{r} = \mathbf{v}_1 + \mathbf{v}_2$ — Resultant vector

$= \langle 5, 0 \rangle + \langle 19.2, 16.1 \rangle$ — Substitution

$= \langle 24.2, 16.1 \rangle$ — Vector Addition



Note: Not drawn to scale.

The magnitude of this resultant is $|\mathbf{r}| = \sqrt{24.2^2 + 16.1^2}$ or about 29.1. Next find the resultant direction angle θ.

$\tan\theta = \frac{16.1}{24.2}$ — $\tan\theta = \frac{b}{a}$ where $\langle a, b \rangle = \langle 24.2, 16.1 \rangle$

$\theta = \tan^{-1}\frac{16.1}{24.2}$ or about 33.6° — Solve for θ.

Therefore, the resultant velocity of the pass is about 29.1 meters per second at an angle of about 33.6° with the horizontal.

GuidedPractice

8. **SOCCER** What would the resultant velocity of the ball be if the goalkeeper made the same pass running 5 meters per second backward?

Exercises

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points. (Examples 1 and 2)

1. $A(-3, 1), B(4, 5)$

2. $A(2, -7), B(-6, 9)$

3. $A(10, -2), B(3, -5)$

4. $A(-2, 7), B(-9, -1)$

5. $A(-5, -4), B(8, -2)$

6. $A(-2, 6), B(1, 10)$

7. $A(2.5, -3), B(-4, 1.5)$

8. $A(-4.3, 1.8), B(9.4, -6.2)$

9. $A\left(\frac{1}{2}, -9\right), B\left(6, \frac{5}{2}\right)$

10. $A\left(\frac{3}{5}, -\frac{2}{5}\right), B(-1, 7)$

Find each of the following for $\mathbf{f} = \langle 8, 0\rangle$, $\mathbf{g} = \langle -3, -5\rangle$, and $\mathbf{h} = \langle -6, 2\rangle$. (Example 3)

11. $4\mathbf{h} - \mathbf{g}$

12. $\mathbf{f} + 2\mathbf{h}$

13. $3\mathbf{g} - 5\mathbf{f} + \mathbf{h}$

14. $2\mathbf{f} + \mathbf{g} - 3\mathbf{h}$

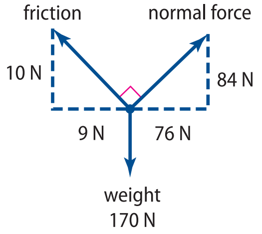
15. $\mathbf{f} - 2\mathbf{g} - 2\mathbf{h}$

16. $\mathbf{h} - 4\mathbf{f} + 5\mathbf{g}$

17. $4\mathbf{g} - 3\mathbf{f} + \mathbf{h}$

18. $6\mathbf{h} + 5\mathbf{f} - 10\mathbf{g}$

19. **PHYSICS** In physics, force diagrams are used to show the effects of all the different forces acting upon an object. The following force diagram could represent the forces acting upon a child sliding down a slide. (Example 3)



a. Using the blue dot representing the child as the origin, express each force as a vector in component form.

b. Find the component form of the resultant vector representing the force that causes the child to move down the slide.

Find a unit vector u with the same direction as v. (Example 4)

20. $\mathbf{v} = \langle -2, 7\rangle$

21. $\mathbf{v} = \langle 9, -3\rangle$

22. $\mathbf{v} = \langle -8, -5\rangle$

23. $\mathbf{v} = \langle 6, 3\rangle$

24. $\mathbf{v} = \langle -2, 9\rangle$

25. $\mathbf{v} = \langle -1, -5\rangle$

26. $\mathbf{v} = \langle 1, 7\rangle$

27. $\mathbf{v} = \langle 3, -4\rangle$

Let $\overrightarrow{DE}$ be the vector with the given initial and terminal points. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j. (Example 5)

28. $D(4, -1), E(5, -7)$

29. $D(9, -6), E(-7, 2)$

30. $D(3, 11), E(-2, -8)$

31. $D(9.5, 1), E(0, -7.3)$

32. $D(-3, -5.7), E(6, -8.1)$

33. $D(-4, -6), E(9, 5)$

34. $D\left(\frac{1}{8}, 3\right), E\left(-4, \frac{2}{7}\right)$

35. $D(-3, 1.5), E(-3, 1.5)$

36. **COMMUTE** To commute to school, Lamis leaves her house and drives north on Al Nasr Street for 2.4 kilometers. She turns left on Freedom Street for 3.1 kilometers and then turns right on Hope Street for 5.8 kilometers. Express Lamis' commute as a linear combination of unit vectors **i** and **j**. (Example 5)

37. **ROWING** Najat is rowing across a river at a speed of 5 kilometers per hour perpendicular to the shore. The river has a current of 3 kilometers per hour heading downstream. (Example 5)

a. At what speed is she traveling?

b. At what angle is she traveling with respect to the shore?

Find the component form of v with the given magnitude and direction angle. (Example 6)

38. $|\mathbf{v}| = 12, \theta = 60°$

39. $|\mathbf{v}| = 4, \theta = 135°$

40. $|\mathbf{v}| = 6, \theta = 240°$

41. $|\mathbf{v}| = 16, \theta = 330°$

42. $|\mathbf{v}| = 28, \theta = 273°$

43. $|\mathbf{v}| = 15, \theta = 125°$

Find the direction angle of each vector to the nearest tenth of a degree. (Example 7)

44. $3\mathbf{i} + 6\mathbf{j}$

45. $-2\mathbf{i} + 5\mathbf{j}$

46. $8\mathbf{i} - 2\mathbf{j}$

47. $-4\mathbf{i} - 3\mathbf{j}$

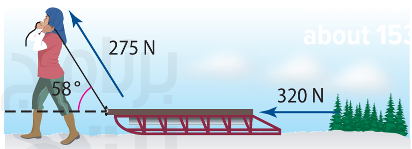
48. $\langle -5, 9\rangle$

49. $\langle 7, 7\rangle$

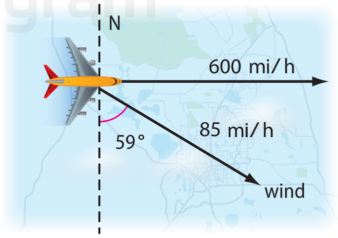
50. $\langle -6, -4\rangle$

51. $\langle 3, -8\rangle$

52. **SLEDDING** Hiyam is pulling a sled with a force of 275 newtons by holding its rope at a 58° angle. Her brother is going to help by pushing the sled with a force of 320 newtons. Determine the magnitude and direction of the total resultant force on the sled. (Example 8)



53. **NAVIGATION** An airplane is traveling due east with a speed of 600 miles per hour. The wind blows at 85 miles per hour at an angle of S59°E. (Example 8)



a. Determine the speed of the airplane's flight.

b. Determine the angle of the airplane's flight.

54. **HEADING** A pilot needs to plot a course that will result in a velocity of 500 miles per hour in a direction of due west. If the wind is blowing 100 miles per hour from the directed angle of 192°, find the direction and the speed the pilot should set to achieve this resultant.

Determine whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ with the initial and terminal points given are equivalent. If so, prove that $\overrightarrow{AB} = \overrightarrow{CD}$. If not, explain why not.

55. $A(3, 5)$, $B(6, 9)$, $C(-4, -4)$, $D(-2, 0)$

56. $A(-4, -5)$, $B(-8, 1)$, $C(3, -3)$, $D(1, 0)$

57. $A(1, -3)$, $B(0, -10)$, $C(11, 8)$, $D(10, 1)$

58. **RAFTING** Hana's family is rafting across a river. Suppose that they are on a stretch of the river that is 150 meters wide, flowing south at a rate of 1.0 meter per second. In still water, their raft travels 5.0 meters per second.

a. What is the speed of the raft?

b. How far downriver will the raft land?

c. How long does it take them to travel from one bank to the other if they head directly across the river?

59. **NAVIGATION** A jet is flying with an air speed of 480 miles per hour at a bearing of N82°E. Because of the wind, the ground speed of the plane is 518 miles per hour at a bearing of N79°E.

a. Draw a diagram to represent the situation.

b. What are the speed and direction of the wind?

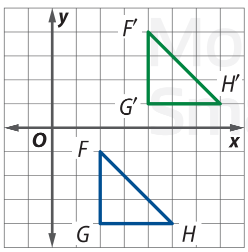
c. If the pilot increased the air speed of the plane to 500 miles per hour, what would be the resulting ground speed and direction of the plane?

60. **TRANSLATIONS** You can translate a figure along a translation vector $\langle a, b\rangle$ by adding a to each x-coordinate and b to each y-coordinate. Consider the triangles shown below.

a. Describe the translation from $\triangle FGH$ to $\triangle F'G'H'$ using a translation vector.

b. Graph $\triangle F'G'H'$ and its translated image $\triangle F''G''H''$ along $\langle -3, -6\rangle$.

c. Describe the translation from $\triangle FGH$ to $\triangle F''G''H''$ using a translation vector.

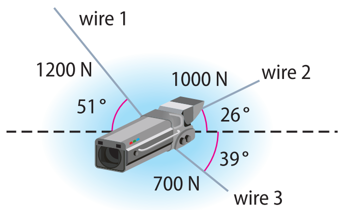


Given the initial point and magnitude of each vector, determine a possible terminal point of the vector.

61. $(-1, 4)$; $\sqrt{37}$

62. $(-3, -7)$; 10

63. **CAMERA** A video camera that follows the action at a sporting event is supported by three wires. The tension in each wire can be modeled by a vector.

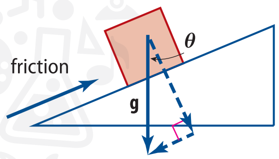


a. Find the component form of each vector.

b. Find the component form of the resultant vector acting on the camera.

c. Find the magnitude and direction of the resulting force.

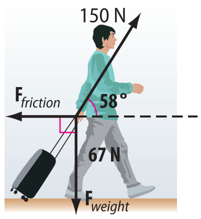
64. **FORCE** A box is stationary on a ramp. Both gravity **g** and friction are exerted on the box. The components of gravity are shown in the diagram. What must be true of the force of friction for this scenario to be possible?



H.O.T. Problems Use Higher-Order Thinking Skills

65. **REASONING** If vectors **a** and **b** are parallel, write a vector equation relating **a** and **b**.

66. **CHALLENGE** To pull luggage, Ahmed exerts a force of 150 newtons at an angle of 58° with the horizontal. If the resultant force on the luggage is 72 newtons at an angle of 56.7° with the horizontal, what is the magnitude of the resultant of $\mathbf{F}_{friction}$ and $\mathbf{F}_{weight}$?



67. **REASONING** If given the initial point of a vector and its magnitude, describe the locus of points that represent possible locations for the terminal point.

68. **WRITING IN MATH** Explain how to find the direction angle of a vector in the fourth quadrant.

69. **CHALLENGE** The direction angle of $\langle x, y\rangle$ is $(4y)°$. Find x in terms of y.

PROOF **Prove each vector property. Let $\mathbf{a} = \langle x_1, y_1\rangle$, $\mathbf{b} = \langle x_2, y_2\rangle$, and $\mathbf{c} = \langle x_3, y_3\rangle$.**

70. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

71. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

72. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$, where k is a scalar

73. $|k\mathbf{a}| = |k|\,|\mathbf{a}|$, where k is a scalar

Spiral Review

74. TOYS Fahd is pulling a toy by exerting a force of 1.5 newtons on a string attached to the toy.

a. The string makes an angle of 52° with the floor. Find the horizontal and vertical components of the force.

b. If Fahd raises the string so that it makes a 78° angle with the floor, what are the magnitudes of the horizontal and vertical components of the force?

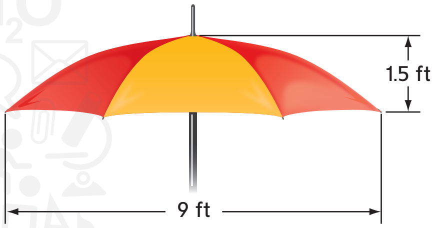
Write each pair of parametric equations in rectangular form.

75. $y = t^2 + 2,\ x = 3t - 6$

76. $y = t^2 - 5,\ x = 2t + 8$

77. $y = 7t,\ x = t^2 - 1$

78. UMBRELLAS A beach umbrella has an arch in the shape of a parabola. Write an equation to model the arch, assuming that the origin and the vertex are at the point where the pole and the canopy of the umbrella meet. Express y in terms of x.



Decompose each expression into partial fractions.

79. $\dfrac{5z - 11}{2z^2 + z - 6}$

80. $\dfrac{7x^2 + 18x - 1}{(x^2 - 1)(x + 2)}$

81. $\dfrac{9 - 9x}{x^2 - 9}$

Verify each identity.

82. $\sin(\theta + 180°) = -\sin\theta$

83. $\sin(60° + \theta) + \sin(60° - \theta) = \sqrt{3}\cos\theta$

Express each logarithm in terms of ln 3 and ln 7.

84. $\ln 189$

85. $\ln 5.\overline{4}$

86. $\ln 441$

87. $\ln \dfrac{9}{343}$

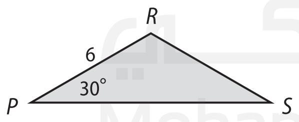
Find each $f(c)$ using synthetic substitution.

88. $f(x) = 6x^6 - 9x^4 + 12x^3 - 16x^2 + 8x + 24;\ c = 6$

89. $f(x) = 8x^5 - 12x^4 + 18x^3 - 24x^2 + 36x - 48,\ c = 4$

Skills Review for Standardized Tests

90. SAT/ACT If $PR = RS$, what is the area of triangle PRS?



A $9\sqrt{2}$
B $9\sqrt{3}$
C $18\sqrt{2}$
D $18\sqrt{3}$
E $36\sqrt{3}$

91. REVIEW Faleh has made a game for his younger sister's graduation celebration. The playing board is a circle divided evenly into 8 sectors. If the circle has a radius of 18 centimeters, what is the approximate area of one of the sectors?

F 4 cm^2
G 32 cm^2
H 127 cm^2
J 254 cm^2

92. Paramedics Ibrahim and Ismail are moving a person on a stretcher. Ibrahim is pushing the stretcher from behind with a force of 135 newtons at 58° with the horizontal, while Ismail is pulling the stretcher from the front with a force of 214 newtons at 43° with the horizontal. What is the magnitude of the horizontal force exerted on the stretcher?

A 228 newtons
B 260 newtons
C 299 newtons
D 346 newtons

93. REVIEW Find the center and radius of the circle with equation $(x - 4)^2 + y^2 - 16 = 0$.

F $C(-4, 0);\ r = 4$ units
G $C(-4, 0);\ r = 16$ units
H $C(4, 0);\ r = 4$ units
J $C(4, 0);\ r = 16$ units

LESSON 7-3

Dot Products and Vector Projections

Then

- You found the magnitudes of and operated with algebraic vectors.

Now

1. Find the dot product of two vectors, and use the dot product to find the angle between them.
2. Find the projection of one vector onto another.

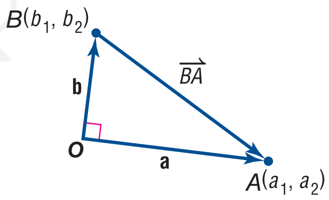
Why?

- The word *work* can have different meanings in everyday life; but in physics, its definition is very specific. Work is the magnitude of a force applied to an object multiplied by the distance through which the object moves parallel to this applied force. Work, such as that done to push a car a specific distance, can also be calculated using a vector operation called a *dot product*.

NewVocabulary
dot product
orthogonal
vector projection
work

1 Dot Product

In Lesson 7-2, you studied the vector operations of vector addition and scalar multiplication. In this lesson, you will use a third vector operation. Consider two perpendicular vectors in standard position **a** and **b**. Let $\overrightarrow{BA}$ be the vector between their terminal points as shown in the figure. By the Pythagorean Theorem, we know that



$$|\overrightarrow{BA}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2.$$

Using the definition of the magnitude of a vector, we can find $|\overrightarrow{BA}|^2$.

$|\overrightarrow{BA}| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$ — Definition of vector magnitude

$|\overrightarrow{BA}|^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2$ — Square each side.

$|\overrightarrow{BA}|^2 = a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2$ — Expand each binomial square.

$|\overrightarrow{BA}|^2 = (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2(a_1b_1 + a_2b_2)$ — Group the squared terms.

$|\overrightarrow{BA}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(a_1b_1 + a_2b_2)$ — $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$ so $|\mathbf{a}|^2 = a_1^2 + a_2^2$ and $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2}$, so $|\mathbf{b}|^2 = b_1^2 + b_2^2$.

Notice that the expressions $|\mathbf{a}|^2 + |\mathbf{b}|^2$ and $|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(a_1b_1 + a_2b_2)$ are equivalent if and only if $a_1b_1 + a_2b_2 = 0$. The expression $a_1b_1 + a_2b_2$ is called the **dot product** of **a** and **b**, denoted $\mathbf{a} \cdot \mathbf{b}$ and read as *a dot b*.

KeyConcept Dot Product of Vectors in a Plane

The dot product of $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$.

Notice that unlike vector addition and scalar multiplication, the dot product of two vectors yields a scalar, not a vector. As demonstrated above, two nonzero vectors are perpendicular if and only if their dot product is 0. Two vectors with a dot product of 0 are said to be **orthogonal**.

KeyConcept Orthogonal Vectors

The vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

The terms *perpendicular* and *orthogonal* have essentially the same meaning, except when **a** or **b** is the zero vector. The zero vector is orthogonal to any vector **a**, since $\langle 0, 0 \rangle \cdot \langle a_1, a_2 \rangle = 0a_1 + 0a_2$ or 0. However, since the zero vector has no magnitude or direction, it cannot be perpendicular to **a**.

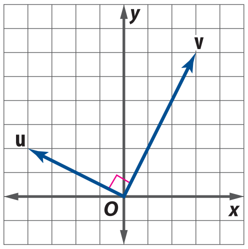


Figure 7.3.1

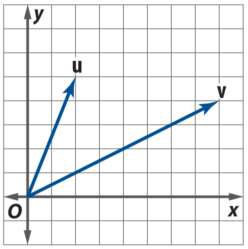


Figure 7.3.2

Example 1 Find the Dot Product to Determine Orthogonal Vectors

Find the dot product of u and v. Then determine if u and v are orthogonal.

a. $\mathbf{u} = \langle 3, 6 \rangle, \mathbf{v} = \langle -4, 2 \rangle$

$$\mathbf{u} \cdot \mathbf{v} = 3(-4) + 6(2) = 0$$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, **u** and **v** are orthogonal, as illustrated in Figure 7.3.1.

b. $\mathbf{u} = \langle 2, 5 \rangle, \mathbf{v} = \langle 8, 4 \rangle$

$$\mathbf{u} \cdot \mathbf{v} = 2(8) + 5(4) = 36$$

Since $\mathbf{u} \cdot \mathbf{v} \neq 0$, **u** and **v** are not orthogonal, as illustrated in Figure 7.3.2.

GuidedPractice

1A. $\mathbf{u} = \langle 3, -2 \rangle, \mathbf{v} = \langle -5, 1 \rangle$

1B. $\mathbf{u} = \langle -2, -3 \rangle, \mathbf{v} = \langle 9, -6 \rangle$

Dot products have the following properties.

KeyConcept Properties of the Dot Product

If **u**, **v**, and **w** are vectors and k is a scalar, then the following properties hold.

Commutative Property	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
Distributive Property	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
Scalar Multiplication Property	$k(\mathbf{u} \cdot \mathbf{v}) = k\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot k\mathbf{v}$
Zero Vector Dot Product Property	$\mathbf{0} \cdot \mathbf{u} = 0$
Dot Product and Vector Magnitude Relationship	$\mathbf{u} \cdot \mathbf{u} = \lvert\mathbf{u}\rvert^2$

Proof

Proof $\mathbf{u} \cdot \mathbf{u} = \lvert\mathbf{u}\rvert^2$

Let $\mathbf{u} = \langle u_1, u_2 \rangle$.

$\mathbf{u} \cdot \mathbf{u} = {u_1}^2 + {u_2}^2$ — Dot product

$= \left(\sqrt{({u_1}^2 + {u_2}^2)}\right)^2$ — Write as the square of the square root of $u_1^2 + u_2^2$.

$= \lvert\mathbf{u}\rvert^2$ — $\sqrt{u_1^2 + u_2^2} = \lvert\mathbf{u}\rvert$

You will prove the first three properties in Exercises 70–72.

ReadingMath

Inner and Scalar Products The dot product is also called the *inner product* or the *scalar product*.

Example 2 Use the Dot Product to Find Magnitude

Use the dot product to find the magnitude of $\mathbf{a} = \langle -5, 12 \rangle$.

Since $\lvert\mathbf{a}\rvert^2 = \mathbf{a} \cdot \mathbf{a}$, then $\lvert\mathbf{a}\rvert = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

$\lvert\langle -5, 12 \rangle\rvert = \sqrt{\langle -5, 12 \rangle \cdot \langle -5, 12 \rangle}$ — $\mathbf{a} = \langle -5, 12 \rangle$

$= \sqrt{(-5)^2 + 12^2}$ or 13 — Simplify.

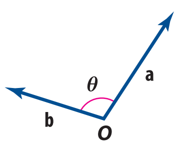
GuidedPractice

Use the dot product to find the magnitude of the given vector.

2A. $\mathbf{b} = \langle 12, 16 \rangle$

2B. $\mathbf{c} = \langle -1, -7 \rangle$

The angle θ between any two nonzero vectors **a** and **b** is the corresponding angle between these vectors when placed in standard position, as shown. This angle is always measured such that $0 \leq \theta \leq \pi$ or $0° \leq \theta \leq 180°$. The dot product can be used to find the angle between two nonzero vectors.



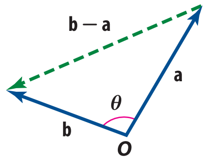
StudyTip

Parallel and Perpendicular Vectors Two vectors are perpendicular if the angle between them is 90°. Two vectors are parallel if the angle between them is 0° or 180°.

KeyConcept Angle Between Two Vectors

If θ is the angle between nonzero vectors **a** and **b**, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|}.$$



Proof

Consider the triangle determined by **a**, **b**, and **b** − **a** in the figure above.

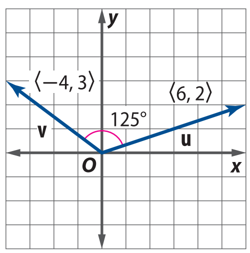
$\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\ \lvert\mathbf{b}\rvert \cos \theta = \lvert\mathbf{b} - \mathbf{a}\rvert^2$	Law of Cosines
$\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\ \lvert\mathbf{b}\rvert \cos \theta = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$	$\lvert\mathbf{u}\rvert^2 = \mathbf{u} \cdot \mathbf{u}$
$\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\ \lvert\mathbf{b}\rvert \cos \theta = \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a}$	Distributive Property for Dot Products
$\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\ \lvert\mathbf{b}\rvert \cos \theta = \lvert\mathbf{b}\rvert^2 - 2\mathbf{a} \cdot \mathbf{b} + \lvert\mathbf{a}\rvert^2$	$\mathbf{u} \cdot \mathbf{u} = \lvert\mathbf{u}\rvert^2$
$-2\lvert\mathbf{a}\rvert\ \lvert\mathbf{b}\rvert \cos \theta = -2\mathbf{a} \cdot \mathbf{b}$	Subtract $\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2$ from each side.
$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert\ \lvert\mathbf{b}\rvert}$	Divide each side by $-2\lvert\mathbf{a}\rvert\ \lvert\mathbf{b}\rvert$.

Example 3 Find the Angle Between Two Vectors

Find the angle θ between vectors u and v to the nearest tenth of a degree.

a. u = ⟨6, 2⟩ and v = ⟨−4, 3⟩

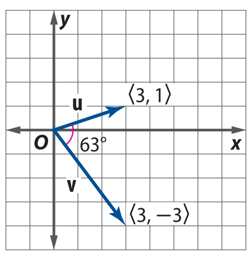
$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{u}\rvert\ \lvert\mathbf{v}\rvert}$	Angle between two vectors
$\cos \theta = \frac{\langle 6, 2\rangle \cdot \langle -4, 3\rangle}{\lvert\langle 6, 2\rangle\rvert\ \lvert\langle -4, 3\rangle\rvert}$	$\mathbf{u} = \langle 6, 2\rangle$ and $\mathbf{v} = \langle -4, 3\rangle$
$\cos \theta = \frac{-24 + 6}{\sqrt{40}\sqrt{25}}$	Evaluate.
$\cos \theta = \frac{-9}{5\sqrt{10}}$	Simplify.
$\theta = \cos^{-1} \frac{-9}{5\sqrt{10}}$ or about 124.7°	Solve for θ.



The measure of the angle between **u** and **v** is about 124.7°.

b. u = ⟨3, 1⟩ and v = ⟨3, −3⟩

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{u}\rvert\ \lvert\mathbf{v}\rvert}$	Angle between two vectors
$\cos \theta = \frac{\langle 3, 1\rangle \cdot \langle 3, -3\rangle}{\lvert\langle 3, 1\rangle\rvert\ \lvert\langle 3, -3\rangle\rvert}$	$\mathbf{u} = \langle 3, 1\rangle$ and $\mathbf{v} = \langle 3, -3\rangle$
$\cos \theta = \frac{9 + (-3)}{\sqrt{10}\sqrt{18}}$	Evaluate.
$\cos \theta = \frac{1}{\sqrt{5}}$	Simplify.
$\theta = \cos^{-1} \frac{1}{\sqrt{5}}$ or about 63.4°	Solve for θ.



The measure of the angle between **u** and **v** is about 63.4°.

GuidedPractice

3A. **u** = ⟨−5, −2⟩ and **v** = ⟨4, 4⟩

3B. **u** = ⟨9, 5⟩ and **v** = ⟨−6, 7⟩

2 Vector Projection

In Lesson 7-1, you learned that a vector can be resolved or decomposed into two perpendicular components. While these components are often horizontal and vertical, it is sometimes useful instead for one component to be parallel to another vector.

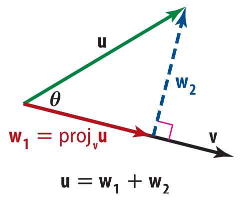
StudyTip

Perpendicular Component The vector $\mathbf{w}_2$ is called the *component of* **u** *perpendicular to* **v**.

KeyConcept Projection of u onto v

Let **u** and **v** be nonzero vectors, and let $\mathbf{w}_1$ and $\mathbf{w}_2$ be vector components of **u** such that $\mathbf{w}_1$ is parallel to **v** as shown. Then vector $\mathbf{w}_1$ is called the **vector projection** of **u** onto **v**, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, and

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.$$



Proof

Since $\text{proj}_{\mathbf{v}}\mathbf{u}$ is parallel to **v**, it can be written as a scalar multiple of **v**. As a scalar multiple of a unit vector $\mathbf{v}_x$ with the same direction as **v**, $\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{w}_1|\ \mathbf{v}_x$. Use the right triangle formed by $\mathbf{w}_1$, $\mathbf{w}_2$, and **u** and the cosine ratio to find an expression for $|\mathbf{w}_1|$.

$\cos\theta = \frac{|\mathbf{w}_1|}{|\mathbf{u}|}$ — Cosine ratio

$|\mathbf{u}||\mathbf{v}|\cos\theta = \cancel{|\mathbf{u}|}|\mathbf{v}|\frac{|\mathbf{w}_1|}{\cancel{|\mathbf{u}|}}$ — Multiply each side by the scalar quantity $|\mathbf{u}|\ |\mathbf{v}|$.

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{v}|\ |\mathbf{w}_1|$ — $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\ |\mathbf{v}|}$, so $|\mathbf{u}|\ |\mathbf{v}|\cos\theta = \mathbf{u} \cdot \mathbf{v}$.

$|\mathbf{w}_1| = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$ — Solve for $|\mathbf{w}_1|$.

Now use $\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{w}_1|\ \mathbf{v}_x$ to find $\text{proj}_{\mathbf{v}}\mathbf{u}$ as a scalar multiple of **v**.

$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{w}_1|\ \mathbf{v}_x$

$= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$ — $|\mathbf{w}_1| = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$ and $\mathbf{v}_x = \frac{\mathbf{v}}{|\mathbf{v}|}$

$= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$ — Multiply magnitudes.

Example 4 Find the Projection of u onto v

Find the projection of u = ⟨3, 2⟩ onto v = ⟨5, −5⟩. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v.

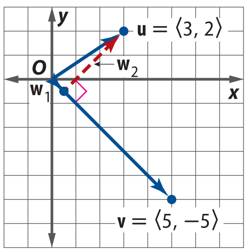


Figure 7.3.5

Step 1 Find the projection of **u** onto **v**.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

$$= \frac{\langle 3, 2\rangle \cdot \langle 5, -5\rangle}{|\langle 5, -5\rangle|^2}\langle 5, -5\rangle$$

$$= \frac{5}{50}\langle 5, -5\rangle$$

$$= \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$$

Step 2 Find $\mathbf{w}_2$.

Since $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$$

$$= \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$$

$$= \langle 3, 2\rangle - \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$= \left\langle \frac{5}{2}, \frac{5}{2}\right\rangle$$

Therefore, $\text{proj}_{\mathbf{v}}\mathbf{u}$ is $\mathbf{w}_1 = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$ as shown in Figure 7.3.5, and $\mathbf{u} = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle + \left\langle \frac{5}{2}, \frac{5}{2}\right\rangle$

GuidedPractice

4. Find the projection of **u** = ⟨1, 2⟩ onto **v** = ⟨8, 5⟩. Then write **u** as the sum of two orthogonal vectors, one of which is the projection of **u** onto **v**.

While the projection of **u** onto **v** is a vector parallel to **v**, this vector will not necessarily have the same direction as **v**, as illustrated in the next example.

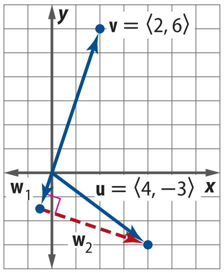


Figure 7.3.6

Example 5 Projection with Direction Opposite v

Find the projection of u = ⟨4, −3⟩ onto v = ⟨2, 6⟩. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v.

Notice that the angle between **u** and **v** is obtuse, so the projection of **u** onto **v** lies on the vector opposite **v** or **−v**, as shown in Figure 7.3.6.

Step 1 Find the projection of **u** onto **v**.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

$$= \frac{\langle 4, -3\rangle \cdot \langle 2, 6\rangle}{|\langle 2, 6\rangle|^2}\langle 2, 6\rangle$$

$$= \frac{-10}{40}\langle 2, 6\rangle \text{ or } \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$$

Step 2 Find $\mathbf{w}_2$.

Since $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ or $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 4, -3\rangle - \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$$

$$= \left\langle \frac{9}{2}, -\frac{3}{2}\right\rangle$$

Therefore, $\text{proj}_{\mathbf{v}}\mathbf{u}$, $\mathbf{w}_1 = \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$ and $\mathbf{u} = \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle + \left\langle \frac{9}{2}, -\frac{3}{2}\right\rangle$.

GuidedPractice

5. Find the projection of u = ⟨−3, 4⟩ onto **v** = ⟨6, 1⟩. Then write **u** as the sum of two orthogonal vectors, one of which is the projection of **u** onto **v**.

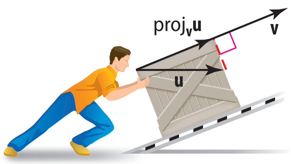
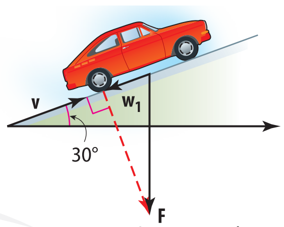


Figure 7.3.7

If the vector **u** represents a force, then $\text{proj}_{\mathbf{v}}\mathbf{u}$ represents the effect of that force acting in the direction of **v**. For example, if you push a box uphill (in the direction **v**) with a force **u** (Figure 7.3.7), the effective force is a component push in the direction of **v**, $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Real-World Example 6 Use a Vector Projection to Find a Force

CARS **A 3000- pounds car sits on a hill inclined at 30° as shown. Ignoring the force of friction, what force is required to keep the car from rolling down the hill?**



The weight of the car is the force exerted due to gravity, **F** = ⟨0, −3000⟩. To find the force $-\mathbf{w}_1$ required to keep the car from rolling down the hill, project **F** onto a unit vector **v** in the direction of the side of the hill.

Step 1 Find a unit vector **v** in the direction of the hill.

$$\mathbf{v} = \langle |\mathbf{v}|(\cos\theta), |\mathbf{v}|(\sin\theta)\rangle \quad \text{Component form of v in terms of } |\mathbf{v}| \text{ and } \theta$$

$$= \langle 1(\cos 30°), 1(\sin 30°)\rangle \text{ or } \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle \quad |\mathbf{v}| = 1 \text{ and } \theta = 30°$$

Step 2 Find $\mathbf{w}_1$, the projection of **F** onto unit vector **v**, $\text{proj}_{\mathbf{v}}\mathbf{F}$.

$$\text{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \quad \text{Projection of F onto v}$$

$$= (\mathbf{F} \cdot \mathbf{v})\mathbf{v} \quad \text{Since v is a unit vector, } |\mathbf{v}| = 1. \text{ Simplify.}$$

$$= \left(\langle 0, -3000\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\right)\mathbf{v} \quad \mathbf{F} = \langle 0, -3000\rangle \text{ and } \mathbf{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$$

$$= -1500\mathbf{v} \quad \text{Find the dot product.}$$

The force required is $-\mathbf{w}_1 = -(-1500\ \mathbf{v})$ or $1500\mathbf{v}$. Since **v** is a unit vector, this means that this force has a magnitude of 1500 pounds and is in the direction of the side of the hill.

GuidedPractice

6. **SLEDDING** Nisreen sits on a sled on the side of a hill inclined at 60°. What force is required to keep the sled from sliding down the hill if the weight of Nisreen and the sled is 125 kilograms?

Another application of vector projection is the calculation of the work done by a force. Consider a constant force **F** acting on an object to move it from point A to point B as shown in Figure 7.3.8. If **F** is parallel to $\overrightarrow{AB}$, then the **work** W done by **F** is the magnitude of the force times the distance from A to B or $W = |\mathbf{F}||\overrightarrow{AB}|$.

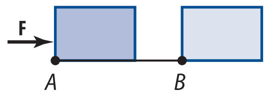


Figure 7.3.8

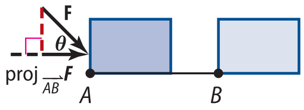


Figure 7.3.9

To calculate the work done by a constant force **F** in *any* direction to move an object from point A to B, as shown in Figure 7.3.9 you can use the vector projection of **F** onto $\overrightarrow{AB}$.

$W = |\text{proj}_{\overrightarrow{AB}}\mathbf{F}|\,|\overrightarrow{AB}|$ — Projection formula for work

$= |\mathbf{F}|\,(\cos\theta)\,|\overrightarrow{AB}|$ — $\cos\theta = \dfrac{|\text{proj}_{\overrightarrow{AB}}\mathbf{F}|}{|\mathbf{F}|}$, so $|\text{proj}_{\overrightarrow{AB}}\mathbf{F}| = |\mathbf{F}|\cos\theta$.

$= \mathbf{F} \cdot \overrightarrow{AB}$ — $\cos\theta = \dfrac{\mathbf{F}\cdot\overrightarrow{AB}}{|\mathbf{F}|\,|\overrightarrow{AB}|}$, so $|\mathbf{F}|\,|\overrightarrow{AB}|\cos\theta = \mathbf{F}\cdot\overrightarrow{AB}$.

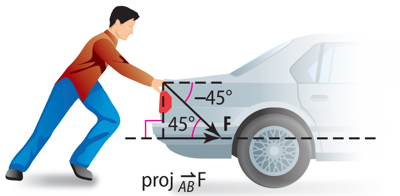
Therefore, this work can be calculated by finding the dot product of the constant force **F** and the directed distance $\overrightarrow{AB}$.

StudyTip

Units for Work Work is measured in foot-pounds in the customary system of measurement and in newton-meters (N·m) or joules (J) in the metric system.

Real-World Example 7 Calculate Work

ROADSIDE ASSISTANCE A person pushes a car with a constant force of 120 newtons at a constant angle of 45° as shown. Find the work done in joules moving the car 10 meters.



Method 1 Use the projection formula for work.

The magnitude of the projection of **F** onto $\overrightarrow{AB}$ is $|\mathbf{F}|\cos\theta = 120\cos 45°$. The magnitude of the directed distance $\overrightarrow{AB}$ is 10.

$W = |\text{proj}_{\overrightarrow{AB}}\mathbf{F}||\overrightarrow{AB}|$ — Projection formula for work

$= (120\cos 45°)(10)$ or about 848.5 — Substitution

Method 2 Use the dot product formula for work.

The component form of the force vector **F** in terms of magnitude and direction angle given is $\langle 120\cos(-45°), 120\sin(-45°)\rangle$. The component form of the directed distance the car is moved is $\langle 10, 0\rangle$.

$W = \mathbf{F}\cdot\overrightarrow{AB}$ — Dot product formula for work

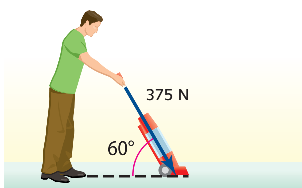
$= \langle 120\cos(-45°), 120\sin(-45°)\rangle \cdot \langle 10, 0\rangle$ — Substitution

$= [120\cos(-45°)](10)$ or about 848.5 — Dot product

Therefore, the person does about 848.5 joules of work pushing the car.

GuidedPractice

7. **CLEANING** Faris is pushing a vacuum cleaner with a force of 375 Newtons. The handle of the vacuum cleaner makes a 60° angle with the floor. How much work in Newton-meters does he do if he pushes the vacuum cleaner 2 meters?



Exercises

Find the dot product of u and v. Then determine if u and v are orthogonal. (Example 1)

1. $\mathbf{u} = \langle 3, -5 \rangle, \mathbf{v} = \langle 6, 2 \rangle$
2. $\mathbf{u} = \langle -10, -16 \rangle, \mathbf{v} = \langle -8, 5 \rangle$
3. $\mathbf{u} = \langle 9, -3 \rangle, \mathbf{v} = \langle 1, 3 \rangle$
4. $\mathbf{u} = \langle 4, -4 \rangle, \mathbf{v} = \langle 7, 5 \rangle$
5. $\mathbf{u} = \langle 1, -4 \rangle, \mathbf{v} = \langle 2, 8 \rangle$
6. $\mathbf{u} = 11\mathbf{i} + 7\mathbf{j}; \mathbf{v} = -7\mathbf{i} + 11\mathbf{j}$
7. $\mathbf{u} = \langle -4, 6 \rangle, \mathbf{v} = \langle -5, -2 \rangle$
8. $\mathbf{u} = 8\mathbf{i} + 6\mathbf{j}; \mathbf{v} = -\mathbf{i} + 2\mathbf{j}$

9. **SPORTING GOODS** The vector $\mathbf{u} = \langle 406, 297 \rangle$ gives the numbers of men's basketballs and women's basketballs, respectively, in stock at a sporting goods store. The vector $\mathbf{v} = \langle 27.5, 15 \rangle$ gives the prices in dirhams of the two types of basketballs, respectively. (Example 1)
   a. Find the dot product $\mathbf{u} \cdot \mathbf{v}$.
   b. Interpret the result in the context of the problem.

Use the dot product to find the magnitude of the given vector. (Example 2)

10. $\mathbf{m} = \langle -3, 11 \rangle$
11. $\mathbf{r} = \langle -9, -4 \rangle$
12. $\mathbf{n} = \langle 6, 12 \rangle$
13. $\mathbf{v} = \langle 1, -18 \rangle$
14. $\mathbf{p} = \langle -7, -2 \rangle$
15. $\mathbf{t} = \langle 23, -16 \rangle$

Find the angle θ between u and v to the nearest tenth of a degree. (Example 3)

16. $\mathbf{u} = \langle 0, -5 \rangle, \mathbf{v} = \langle 1, -4 \rangle$
17. $\mathbf{u} = \langle 7, 10 \rangle, \mathbf{v} = \langle 4, -4 \rangle$
18. $\mathbf{u} = \langle -2, 4 \rangle, \mathbf{v} = \langle 2, -10 \rangle$
19. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}, \mathbf{v} = -4\mathbf{i} - 2\mathbf{j}$
20. $\mathbf{u} = \langle -9, 0 \rangle, \mathbf{v} = \langle -1, -1 \rangle$
21. $\mathbf{u} = -\mathbf{i} - 3\mathbf{j}, \mathbf{v} = -7\mathbf{i} - 3\mathbf{j}$
22. $\mathbf{u} = \langle 6, 0 \rangle, \mathbf{v} = \langle -10, 8 \rangle$
23. $\mathbf{u} = -10\mathbf{i} + \mathbf{j}, \mathbf{v} = 10\mathbf{i} - 5\mathbf{j}$

24. **CAMPING** Omar and Ali set off from their campsite to search for firewood. The path that Omar takes can be represented by $\mathbf{u} = \langle 3, -5 \rangle$. The path that Ali takes can be represented by $\mathbf{v} = \langle -7, 6 \rangle$. Find the angle between the pair of vectors. (Example 3)

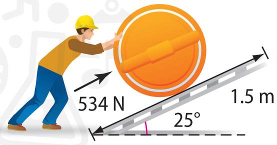
Find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v. (Examples 4 and 5)

25. $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j}, \mathbf{v} = -5\mathbf{i} + 2\mathbf{j}$
26. $\mathbf{u} = \langle 5, 7 \rangle, \mathbf{v} = \langle -4, 4 \rangle$
27. $\mathbf{u} = \langle 8, 2 \rangle, \mathbf{v} = \langle -4, 1 \rangle$
28. $\mathbf{u} = 6\mathbf{i} + \mathbf{j}, \mathbf{v} = -3\mathbf{i} + 9\mathbf{j}$
29. $\mathbf{u} = \langle 2, 4 \rangle, \mathbf{v} = \langle -3, 8 \rangle$
30. $\mathbf{u} = \langle -5, 9 \rangle, \mathbf{v} = \langle 6, 4 \rangle$
31. $\mathbf{u} = 5\mathbf{i} - 8\mathbf{j}, \mathbf{v} = 6\mathbf{i} - 4\mathbf{j}$
32. $\mathbf{u} = -2\mathbf{i} - 5\mathbf{j}, \mathbf{v} = 9\mathbf{i} + 7\mathbf{j}$

33. **WAGON** Eissa is pulling his sister in a wagon up a small slope at an incline of 15°. If the combined weight of Eissa's sister and the wagon is 344 Newtons, what force is required to keep her from rolling down the slope? (Example 6)

34. **SLIDE** Najla is going down a slide but stops herself when she notices that another student is lying hurt at the bottom of the slide. What force is required to keep her from sliding down the slide if the incline is 53° and she weighs 273 N? (Example 6)

35. **PHYSICS** Ali is pushing a construction barrel up a ramp 1.5 meters long into the back of a truck. She is using a force of 534 newtons and the ramp is 25° from the horizontal. How much work in joules is Ali doing? (Example 7)

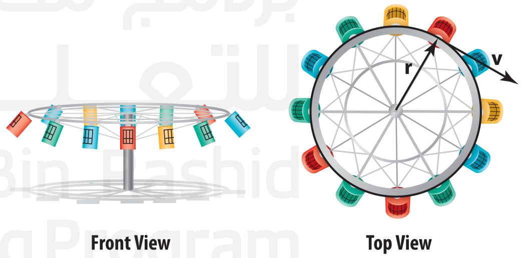


36. **SHOPPING** Suha is pushing a shopping cart with a force of 125 newtons at a downward angle, or angle of depression, of 52°. How much work in joules would Suha do if she pushed the shopping cart 200 meters? (Example 7)

Find a vector orthogonal to each vector.

37. $\langle -2, -8 \rangle$
38. $\langle 3, 5 \rangle$
39. $\langle 7, -4 \rangle$
40. $\langle -1, 6 \rangle$

41. **RIDES** For a circular amusement park ride, the position vector **r** is perpendicular to the tangent velocity vector **v** at any point on the circle, as shown below.



   a. If the radius of the ride is 20 feet and the speed of the ride is constant at 40 feet per second, write the component forms of the position vector **r** and the tangent velocity vector **v** when **r** is at a directed angle of 35°.
   b. What method can be used to prove that the position vector and the velocity vector that you developed in part **a** are perpendicular? Show that the two vectors are perpendicular.

Given v and u • v, find u.

42. $\mathbf{v} = \langle 3, -6 \rangle$, $\mathbf{u} \cdot \mathbf{v} = 33$

43. $\mathbf{v} = \langle 4, 6 \rangle$, $\mathbf{u} \cdot \mathbf{v} = 38$

44. $\mathbf{v} = \langle -5, -1 \rangle$, $\mathbf{u} \cdot \mathbf{v} = -8$

45. $\mathbf{v} = \langle -2, 7 \rangle$, $\mathbf{u} \cdot \mathbf{v} = -43$

46. **SCHOOL** A student rolls her backpack from her Chemistry classroom to her English classroom using a force of 175 newtons.

a. If she exerts 3060 joules to pull her backpack 31 meters, what is the angle of her force?

b. If she exerts 1315 joules at an angle of 60°, how far did she pull her backpack?

Determine whether each pair of vectors are *parallel*, *perpendicular*, or *neither*. Explain your reasoning.

47. $\mathbf{u} = \left\langle -\frac{2}{3}, \frac{3}{4} \right\rangle$, $\mathbf{v} = \langle 9, 8 \rangle$

48. $\mathbf{u} = \langle -1, -4 \rangle$, $\mathbf{v} = \langle 3, 6 \rangle$

49. $\mathbf{u} = \langle 5, 7 \rangle$, $\mathbf{v} = \langle -15, -21 \rangle$

50. $\mathbf{u} = \langle \sec \theta, \csc \theta \rangle$, $\mathbf{v} = \langle \csc \theta, -\sec \theta \rangle$

Find the angle between the two vectors in radians.

51. $\mathbf{u} = \cos\left(\frac{\pi}{3}\right)\mathbf{i} + \sin\left(\frac{\pi}{3}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j}$

52. $\mathbf{u} = \cos\left(\frac{7\pi}{6}\right)\mathbf{i} + \sin\left(\frac{7\pi}{6}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{5\pi}{4}\right)\mathbf{i} + \sin\left(\frac{5\pi}{4}\right)\mathbf{j}$

53. $\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{2\pi}{3}\right)\mathbf{i} + \sin\left(\frac{2\pi}{3}\right)\mathbf{j}$

54. $\mathbf{u} = \cos\left(\frac{\pi}{4}\right)\mathbf{i} + \sin\left(\frac{\pi}{4}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{5\pi}{6}\right)\mathbf{i} + \sin\left(\frac{5\pi}{6}\right)\mathbf{j}$

55. **WORK** Adnan lifts his nephew, who weighs 16 kilograms, a distance of 0.9 meter. The force of weight in newtons can be calculated using $F = mg$, where m is the mass in kilograms and g is 9.8 meters per second squared. How much work did Adnan do to lift his nephew?

The vertices of a triangle on the coordinate plane are given. Find the measures of the angles of each triangle using vectors. Round to the nearest tenth of a degree.

56. (2, 3), (4, 7), (8, 1)

57. (−3, −2), (−3, −7), (3, −7)

58. (−4, −3), (−8, −2), (2, 1)

59. (1, 5), (4, −3), (−4, 0)

Given u, |v|, and θ, the angle between u and v, find possible values of v. Round to the nearest hundredth.

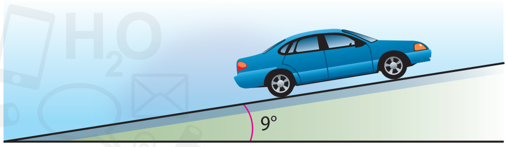
60. $\mathbf{u} = \langle 4, -2 \rangle$, $|\mathbf{v}| = 10$, 45°

61. $\mathbf{u} = \langle 3, 4 \rangle$, $|\mathbf{v}| = \sqrt{29}$, 121°

62. $\mathbf{u} = \langle -1, -6 \rangle$, $|\mathbf{v}| = 7$, 96°

63. $\mathbf{u} = \langle -2, 5 \rangle$, $|\mathbf{v}| = 12$, 27°

64. **CARS** A car is stationary on a 9° incline. Assuming that the only forces acting on the car are gravity and the 275 newton force applied by the brakes, about how much does the car weigh?



H.O.T. Problems Use Higher-Order Thinking Skills

65. **REASONING** Determine whether the statement below is *true* or *false*. Explain.

If $|\mathbf{d}|$, $|\mathbf{e}|$, and $|\mathbf{f}|$ form a Pythagorean triple, and the angles between **d** and **e** and between **e** and **f** are acute, then the angle between **d** and **f** must be a right angle. Explain your reasoning.

66. **ERROR ANALYSIS** Mahmoud and Mohammad are studying the properties of the dot product. Mahmoud concludes that the dot product is associative because it is commutative; that is, $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$. Mohammad disagrees. Is either of them correct? Explain your reasoning.

67. **REASONING** Determine whether the statement below is *true* or *false*.
If **a** and **b** are both orthogonal to **v** in the plane, then **a** and **b** are parallel. Explain your reasoning.

68. **CHALLENGE** If **u** and **v** are perpendicular, what is the projection of **u** onto **v**?

69. **PROOF** Show that if the angle between vectors **u** and **v** is 90°, $\mathbf{u} \cdot \mathbf{v} = 0$ using the formula for the angle between two nonzero vectors.

PROOF Prove each dot product property. Let $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, and $\mathbf{w} = \langle w_1, w_2 \rangle$.

70. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

71. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

72. $k(\mathbf{u} \cdot \mathbf{v}) = k\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot k\mathbf{v}$

73. **WRITING IN MATH** Explain how to find the dot product of two nonzero vectors.

Spiral Review

Find each of the following for a = ⟨10, 1⟩, b = ⟨−5, 2.8⟩, and c = $\left\langle \frac{3}{4}, -9 \right\rangle$.

74. b − a + 4c

75. c − 3a + b

76. 2a − 4b + c

77. **GOLF** Yousif drives a golf ball with a velocity of 62.5 meters per second at an angle of 32° with the ground. On the same hole, Saeed drives a golf ball with a velocity of 57.9 meters per second at an angle of 41°. Find the magnitudes of the horizontal and vertical components for each force.

Graph the hyperbola given by each equation.

78. $\frac{(x-5)^2}{48} - \frac{y^2}{5} = 1$

79. $\frac{x^2}{81} - \frac{y^2}{49} = 1$

80. $\frac{y^2}{36} - \frac{x^2}{4} = 1$

Find the exact value of each expression, if it exists.

81. $\arcsin\left(\sin \frac{\pi}{6}\right)$

82. $\arctan\left(\tan \frac{1}{2}\right)$

83. $\sin\left(\cos^{-1} \frac{3}{4}\right)$

Solve each equation.

84. $\log_{12}(x^3 + 2) = \log_{12} 127$

85. $\log_2 x = \log_2 6 + \log_2 (x - 5)$

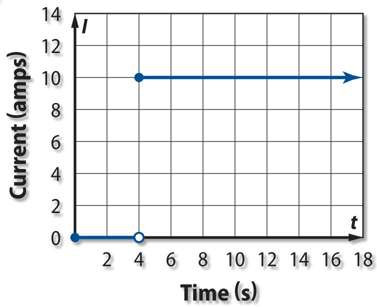
86. $e^{5x-4} = 70$

87. **ELECTRICITY** A simple electric circuit contains only a power supply and a resistor. When the power supply is off, there is no current in the circuit. When the power supply is turned on, the current almost instantly becomes a constant value. This situation can be modeled by a graph like the one shown at the right. I represents current in amps, and t represents time in seconds.

a. At what t-value is this function discontinuous?

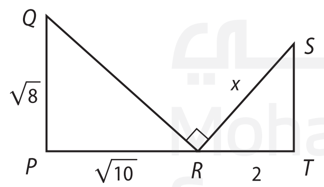
b. When was the power supply turned on?

c. If the person who turned on the power supply left and came back hours later, what would he or she measure the current in the circuit to be?



Skills Review for Standardized Tests

88. **SAT/ACT** In the figure below, △PQR ~ △TRS. What is the value of x?



A $\sqrt{2}$ **C** 3 **E** 6

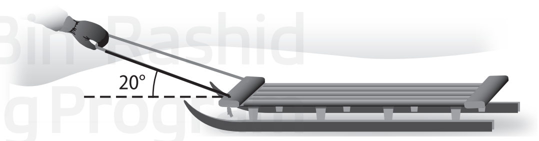
B $\sqrt{5}$ **D** $3\sqrt{2}$

89. **REVIEW** Consider $C(-9, 2)$ and $D(-4, -3)$. Which of the following is the component form and magnitude of $\overrightarrow{CD}$?

F ⟨5, −5⟩, $5\sqrt{2}$ **H** ⟨6, −5⟩, $5\sqrt{2}$

G ⟨5, −5⟩, $6\sqrt{2}$ **J** ⟨6, −6⟩, $6\sqrt{2}$

90. A snow sled is pulled by exerting a force of 25 Newtons on a rope that makes a 20° angle with the horizontal, as shown in the figure. What is the approximate work done in pulling the sled 50 meters?



A 428 N-m **C** 1175 N-m

B 1093 N-m **D** 1250 N-m

91. **REVIEW** If **s** = ⟨4, −3⟩ **t** = ⟨−6, 2⟩, which of the following represents **t** − 2**s**?

F ⟨14, 8⟩ **H** ⟨−14, 8⟩

G ⟨14, 6⟩ **J** ⟨−14, −8⟩

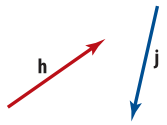
CHAPTER 7

Mid-Chapter Quiz

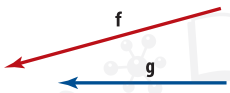
Lessons 7-1 through 7-3

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant in centimeters and its direction relative to the horizontal. (Lesson 7-1)

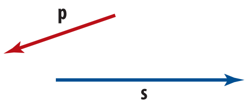
1.



2.



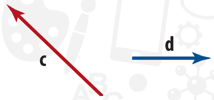
3.



4.

5. SLEDDING Ali pulls a sled through the snow with a force of 50 newtons at an angle of 35° with the horizontal. Find the magnitude of the horizontal and vertical components of the force. (Lesson 7-1)

6. Draw a vector diagram of $\frac{1}{2}\mathbf{c} - 3\mathbf{d}$. (Lesson 7-1)



Let $\overrightarrow{BC}$ be the vector with the given initial and terminal points. Write $\overrightarrow{BC}$ as a linear combination of the vectors i and j. (Lesson 7-2)

7. $B(3, -1)$, $C(4, -7)$

8. $B(10, -6)$, $C(-8, 2)$

9. $B(1, 12)$, $C(-2, -9)$

10. $B(4, -10)$, $C(4, -10)$

11. MULTIPLE CHOICE Which of the following is the component form of $\overrightarrow{AB}$ with initial point $A(-5, 3)$ and terminal point $B(2, -1)$? (Lesson 7-2)

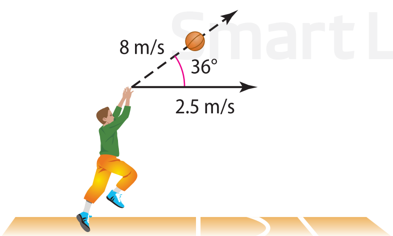
A $\langle 4, -1 \rangle$

B $\langle 7, -4 \rangle$

C $\langle 7, 4 \rangle$

D $\langle -6, 4 \rangle$

12. BASKETBALL With time running out in a game, Maysoun runs towards the basket at a speed of 2.5 meters per second and from half-court, launches a shot at a speed of 8 meters per second at an angle of 36° to the horizontal. (Lesson 7-2)



a. Write the component form of the vectors representing Maysoun's velocity and the path of the ball.

b. What is the resultant speed and direction of the shot?

Find the component form and magnitude of the vector with each initial and terminal point. (Lesson 7-2)

13. $A(-4, 2)$, $B(3, 6)$

14. $Q(1, -5)$, $R(-7, 8)$

15. $X(-3, -5)$, $Y(2, 5)$

16. $P(9, -2)$, $S(2, -5)$

Find the angle θ between u and v to the nearest tenth of a degree. (Lesson 7-3)

17. $\mathbf{u} = \langle 9, -4 \rangle$, $\mathbf{v} = \langle -1, -2 \rangle$

18. $\mathbf{u} = \langle 5, 2 \rangle$, $\mathbf{v} = \langle -4, 10 \rangle$

19. $\mathbf{u} = \langle 8, 4 \rangle$, $\mathbf{v} = \langle -2, 4 \rangle$

20. $\mathbf{u} = \langle 2, -2 \rangle$, $\mathbf{v} = \langle 3, 8 \rangle$

21. MULTIPLE CHOICE If $\mathbf{u} = \langle 2, 3 \rangle$, $\mathbf{v} = \langle -1, 4 \rangle$, and $\mathbf{w} = \langle 8, -5 \rangle$, find $(\mathbf{u} \cdot \mathbf{v}) + (\mathbf{w} \cdot \mathbf{v})$. (Lesson 7-3)

F −18

G −2

H 15

J 38

Find the dot product of u and v. Then determine if u and v are orthogonal. (Lesson 7-3)

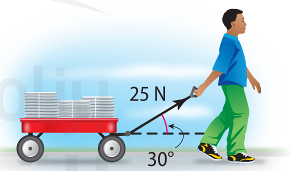
22. $\langle 2, -5 \rangle \cdot \langle 4, 2 \rangle$

23. $\langle 4, -3 \rangle \cdot \langle 7, 4 \rangle$

24. $\langle 1, -6 \rangle \cdot \langle 5, 8 \rangle$

25. $\langle 3, -6 \rangle \cdot \langle 10, 5 \rangle$

26. WAGON Hamad uses a wagon to carry newspapers for his paper route. He is pulling the wagon with a force of 25 newtons at an angle of 30° with the horizontal. (Lesson 7-3)



a. How much work in joules is Hamad doing when he pulls the wagon 150 meters?

b. If the handle makes an angle of 40° with the ground and he pulls the wagon with the same distance and force, is Hamad doing more or less work? Explain your answer.

Find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v. (Lesson 7-3)

27. $\mathbf{u} = \langle 7, -3 \rangle$, $\mathbf{v} = \langle 2, 5 \rangle$

28. $\mathbf{u} = \langle 2, 4 \rangle$, $\mathbf{v} = \langle 1, 3 \rangle$

29. $\mathbf{u} = \langle 3, 8 \rangle$, $\mathbf{v} = \langle -9, 2 \rangle$

30. $\mathbf{u} = \langle -1, 4 \rangle$, $\mathbf{v} = \langle -6, 1 \rangle$

LESSON 7-4 Vectors in Three-Dimensional Space

Then

- You represented vectors both geometrically and algebraically in two-dimensions.

Now

1. Plot points and vectors in the three-dimensional coordinate system.
2. Express algebraically and operate with vectors in space.

Why?

- The direction of a rocket after takeoff is given in terms of both a two-dimensional bearing and a third-dimensional angle relative to the horizontal. Since directed distance, velocities, and forces are not restricted to the plane, the concept of vectors must extend from two- to three-dimensional space.

NewVocabulary
three-dimensional coordinate system
z-axis
octant
ordered triple

1 Coordinates in Three Dimensions

The Cartesian plane is a two-dimensional coordinate system made up of the *x*- and *y*-axes that allows you to identify and locate points in a plane. We need a **three-dimensional coordinate system** to represent a point in space.

Start with the *xy*-plane and position it so that it gives the appearance of depth (Figure 7.4.1). Then add a third axis called the **z-axis** that passes through the origin and is perpendicular to both the *x*- and *y*-axes (Figure 7.4.2). The additional axis divides space into eight regions called **octants**. To help visualize the first octant, look at the corner of a room (Figure 7.4.3). The floor represents the *xy*-plane, and the walls represent the *xz*- and *yz*-planes.

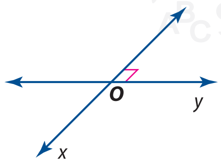


Figure 7.4.1

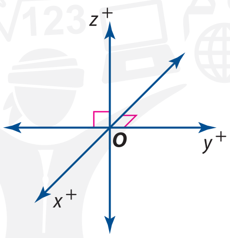


Figure 7.4.2

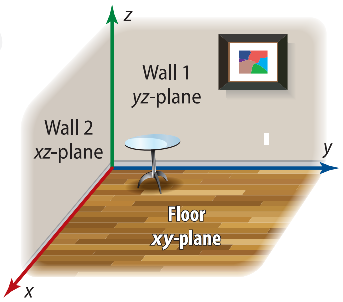


Figure 7.4.3

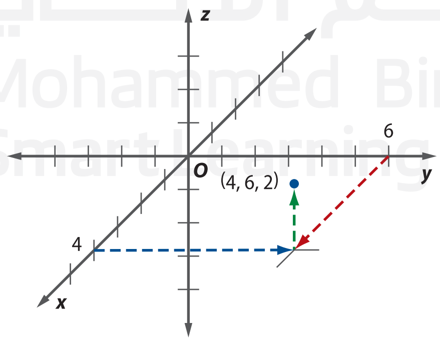
A point in space is represented by an **ordered triple** of real numbers (x, y, z). To plot such a point, first locate the point (x, y) in the *xy*-plane and move up or down parallel to the *z*-axis according to the directed distance given by *z*.

Example 1 Locate a Point in Space

Plot each point in a three-dimensional coordinate system.

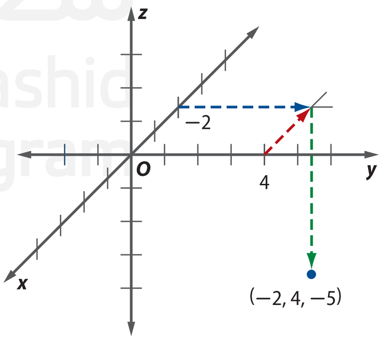
a. **(4, 6, 2)**

Locate (4, 6) in the *xy*-plane and mark it with a cross. Then plot a point **2** units up from this location parallel to the *z*-axis.



b. **(−2, 4, −5)**

Locate (−2, 4) in the *xy*-plane and mark it with a cross. Then plot a point **5** units down from this location parallel to the *z*-axis.



GuidedPractice

1A. (−3, −4, 2) **1B.** (3, 2, −3) **1C.** (5, −4, −1)

Finding the distance between points and the midpoint of a segment in space is similar to finding distance and a midpoint in the coordinate plane.

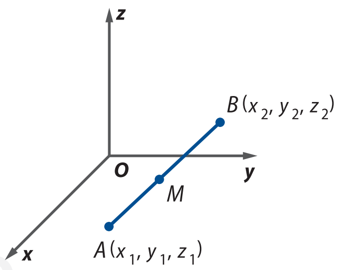
KeyConcept Distance and Midpoint Formulas in Space

The distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The midpoint M of $\overline{AB}$ is given by

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$



You will prove these formulas in Exercise 66.

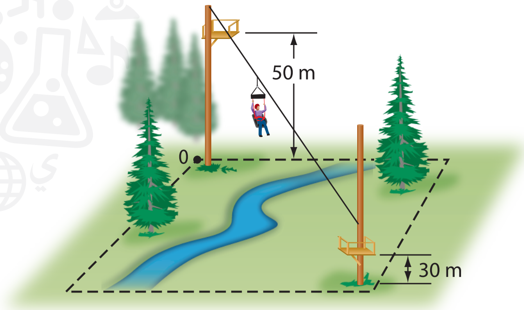
Real-WorldLink

A tour at Monteverde, Costa Rica, allows visitors to view nature from a system of trails, suspension bridges, and zip-lines. The zip-lines allow the guests to view the surroundings from as much as 456 feet above the ground.

Source: Monteverde Info

Real-World Example 2 Distance and Midpoint of Points in Space

ZIP-LINE **A tour of the Sierra Madre Mountains lets guests experience nature by zip-lining from one platform to another over the scenic surroundings. Two platforms that are connected by a zip-line are represented by the coordinates (10, 12, 50) and (70, 92, 30), where the coordinates are given in meters.**



a. Find the length of the zip-line needed to connect the two platforms.

Use the Distance Formula for points in space.

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ — Distance Formula

$= \sqrt{(70 - 10)^2 + (92 - 12)^2 + (30 - 50)^2}$ — $(x_1, y_1, z_1) = (10, 12, 50)$ and $(x_2, y_2, z_2) = (70, 92, 30)$

≈ 101.98 — Simplify.

The zip-line needs to be about 102 meters long to connect the two towers.

b. An additional platform is to be built halfway between the existing platforms. Find the coordinates of the new platform.

Use the Midpoint Formula for points in space.

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$ — Midpoint Formula

$= \left(\frac{10 + 70}{2}, \frac{12 + 92}{2}, \frac{50 + 30}{2}\right)$ or (40, 52, 40) — $(x_1, y_1, z_1) = (10, 12, 50)$ and $(x_2, y_2, z_2) = (70, 92, 30)$

The coordinates of the new platform will be (40, 52, 40).

GuidedPractice

2. **AIRPLANES** Safety regulations require airplanes to be at least a half a kilometer apart when in the sky. Two planes are flying above Cleveland with the coordinates (300, 150, 30000) and (450, −250, 28000), where the coordinates are given in meters.
   - **A.** Are the two planes in violation of the safety regulations? Explain.
   - **B.** If a firework was launched and exploded directly in between the two planes, what are the coordinates of the firework explosion?

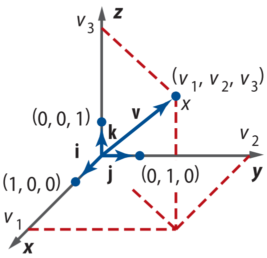


Figure 7.4.4

2 Vectors in Space

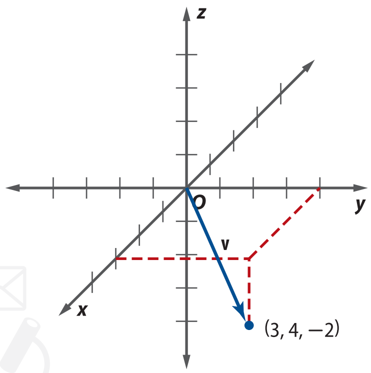
In space, a vector **v** in standard position with a terminal point located at (v_1, v_2, v_3) is denoted by $\langle v_1, v_2, v_3 \rangle$. The zero vector is $\mathbf{0} = \langle 0, 0, 0 \rangle$, and the standard unit vectors are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ as shown in Figure 7.4.4. The component form of **v** can be expressed as a linear combination of these unit vectors, $\langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

Example 3 Locate a Vector in Space

Locate and graph $\mathbf{v} = \langle 3, 4, -2 \rangle$.

Plot the point $(3, 4, -2)$.

Draw **v** with terminal point at $(3, 4, -2)$.



GuidedPractice

Locate and graph each vector in space.

3A. $\mathbf{u} = \langle -4, 2, -3 \rangle$

3B. $\mathbf{w} = -\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

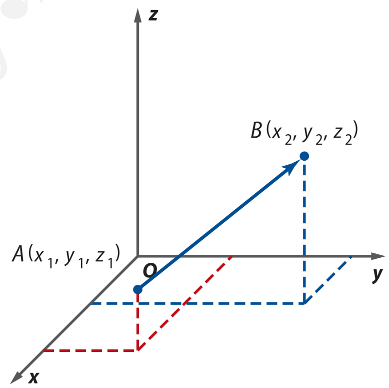
As with two-dimensional vectors, to find the component form of the directed line segment from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$, you subtract the coordinates of its initial point from its terminal point.

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Then $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ or

if $\overrightarrow{AB} = \langle a_1, a_2, a_3 \rangle$, then $|\overrightarrow{AB}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

A unit vector **u** in the direction of $\overrightarrow{AB}$ is still $\mathbf{u} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$.



Example 4 Express Vectors in Space Algebraically

Find the component form and magnitude of $\overrightarrow{AB}$ with initial point $A(-4, -2, 1)$ and terminal point $B(3, 6, -6)$. Then find a unit vector in the direction of $\overrightarrow{AB}$.

$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ — Component form of vector

$= \langle 3 - (-4), 6 - (-2), -6 - 1 \rangle$ or $\langle 7, 8, -7 \rangle$ — $(x_1, y_1, z_1) = (-4, -2, 1)$ and $(x_2, y_2, z_2) = (3, 6, -6)$

Using the component form, the magnitude of $\overrightarrow{AB}$ is

$|\overrightarrow{AB}| = \sqrt{7^2 + 8^2 + (-7)^2}$ or $9\sqrt{2}$. — $\overrightarrow{AB} = \langle 7, 8, -7 \rangle$

Using this magnitude and component form, find a unit vector **u** in the direction of $\overrightarrow{AB}$.

$\mathbf{u} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ — Unit vector in the direction of $\overrightarrow{AB}$

$= \dfrac{\langle 7, 8, -7 \rangle}{9\sqrt{2}}$ or $\left\langle \dfrac{7\sqrt{2}}{18}, \dfrac{4\sqrt{2}}{9}, -\dfrac{7\sqrt{2}}{18} \right\rangle$ — $\overrightarrow{AB} = \langle 7, 8, -7 \rangle$ and $|\overrightarrow{AB}| = 9\sqrt{2}$

GuidedPractice

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points. Then find a unit vector in the direction of $\overrightarrow{AB}$.

4A. $A(-2, -5, -5)$, $B(-1, 4, -2)$

4B. $A(-1, 4, 6)$, $B(3, 3, 8)$

As with vectors in the plane, when vectors in space are in component form or expressed as a linear combination of unit vectors, they can be added, subtracted, or multiplied by a scalar.

KeyConcept Vector Operations in Space

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and any scalar k, then

Vector Addition $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

Vector Subtraction $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

Scalar Multiplication $k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$

StudyTip

Vector Operations The properties for vector operations in space are the same as those for operations in the plane.

Example 5 Operations with Vectors in Space

Find each of the following for $\mathbf{y} = \langle 3, -6, 2 \rangle$, $\mathbf{w} = \langle -1, 4, -4 \rangle$, and $\mathbf{z} = \langle -2, 0, 5 \rangle$.

a. $4\mathbf{y} + 2\mathbf{z}$

$4\mathbf{y} + 2\mathbf{z} = 4\langle 3, -6, 2 \rangle + 2\langle -2, 0, 5 \rangle$ Substitute.

$= \langle 12, -24, 8 \rangle + \langle -4, 0, 10 \rangle$ or $\langle 8, -24, 18 \rangle$ Scalar multiplication and vector addition

b. $2\mathbf{w} - \mathbf{z} + 3\mathbf{y}$

$2\mathbf{w} - \mathbf{z} + 3\mathbf{y} = 2\langle -1, 4, -4 \rangle - \langle -2, 0, 5 \rangle + 3\langle 3, -6, 2 \rangle$ Substitute.

$= \langle -2, 8, -8 \rangle + \langle 2, 0, -5 \rangle + \langle 9, -18, 6 \rangle$ Scalar multiplication

$= \langle 9, -10, -7 \rangle$ Vector addition

GuidedPractice

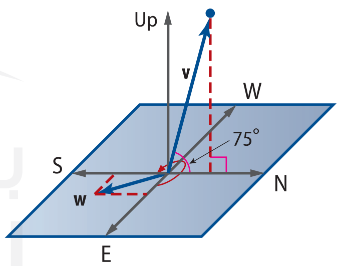
5A. $4\mathbf{w} - 8\mathbf{z}$

5B. $3\mathbf{y} + 3\mathbf{z} - 6\mathbf{w}$

Real-World Example 6 Use Vectors in Space

ROCKETS After liftoff, a model rocket is headed due north and climbing at an angle of 75° relative to the horizontal at 200 kilometers per hour. If the wind blows from the northwest at 5 kilometers per hour, find a vector for the resultant velocity of the rocket relative to the point of liftoff.

Let **i** point east, **j** point north, and **k** point up. Vector **v** representing the rocket's velocity and vector **w** representing the wind's velocity are shown. Notice that **w** points toward the southeast, since the wind is blowing *from* the northwest.



Since **v** has a magnitude of 200 and a direction angle of 75°, we can find the component form of **v**, as shown in Figure 7.4.5.

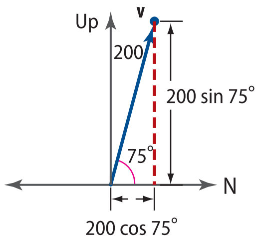


Figure 7.4.5

$\mathbf{v} = \langle 0, 200 \cos 75°, 200 \sin 75° \rangle$ or about $\langle 0, 51.8, 193.2 \rangle$

With east as the positive x-axis, **w** has direction angle of 315°. Since $|\mathbf{w}| = 5$, the component form of this vector is $\mathbf{w} = \langle 5 \cos 315°, 5 \sin 315°, 0 \rangle$ or about $\langle 3.5, -3.5, 0 \rangle$, as shown in Figure 7.4.6.

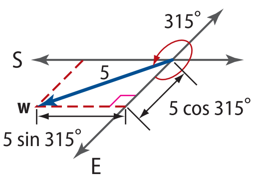


Figure 7.4.6

The resultant velocity of the rocket is $\mathbf{v} + \mathbf{w}$.

$\mathbf{v} + \mathbf{w} = \langle 0, 51.8, 193.2 \rangle + \langle 3.5, -3.5, 0 \rangle$

$= \langle 3.5, 48.3, 193.2 \rangle$ or $3.5\mathbf{i} + 48.3\mathbf{j} + 193.2\mathbf{k}$

GuidedPractice

6. AVIATION After takeoff, an airplane is headed east and is climbing at an angle of 18° relative to the horizontal. Its air speed is 250 kilometers per hour. If the wind blows from the northeast at 10 kilometers per hour, find a vector that represents the resultant velocity of the plane relative to the point of takeoff. Let **i** point east, **j** point north, and **k** point up.

Exercises

Plot each point in a three-dimensional coordinate system. (Example 1)

1. $(1, -2, -4)$
2. $(3, 2, 1)$
3. $(-5, -4, -2)$
4. $(-2, -5, 3)$
5. $(-5, 3, 1)$
6. $(2, -2, 3)$
7. $(4, -10, -2)$
8. $(-16, 12, -13)$

Find the length and midpoint of the segment with the given endpoints. (Example 2)

9. $(-4, 10, 4), (1, 0, 9)$
10. $(-6, 6, 3), (-9, -2, -2)$
11. $(6, 1, 10), (-9, -10, -4)$
12. $(8, 3, 4), (-4, -7, 5)$
13. $(-3, 2, 8), (9, 6, 0)$
14. $(-7, 2, -5), (-2, -5, -8)$

15. **VACATION** A family from Wichita, Kansas, is using a GPS device to plan a vacation to Castle Rock, Colorado. According to the device, the coordinates for the family's home are (37.7°, 97.2°, 433 m), and the coordinates to Castle Rock are (39.4°, 104.8°, 1981 m). Determine the longitude, latitude, and altitude of the halfway point between Wichita and Castle Rock. (Example 2)

16. **FIGHTER PILOTS** During a training session, the location of two F-18 fighter jets are represented by the coordinates (675, −121, 19,300) and (−289, 715, 16,100), where the coordinates are given in feet. (Example 2)
    a. Determine the distance between the two jets.
    b. To what location would one of the fighter pilots have to fly the F-18 in order to reduce the distance between the two jets by half?

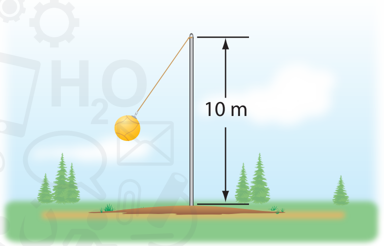
Locate and graph each vector in space. (Example 3)

17. $\mathbf{a} = \langle 0, -4, 4 \rangle$
18. $\mathbf{b} = \langle -3, -3, -2 \rangle$
19. $\mathbf{c} = \langle -1, 3, -4 \rangle$
20. $\mathbf{d} = \langle 4, -2, -3 \rangle$
21. $\mathbf{v} = 6\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}$
22. $\mathbf{w} = -10\mathbf{i} + 5\mathbf{k}$
23. $\mathbf{m} = 7\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$
24. $\mathbf{n} = \mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points. Then find a unit vector in the direction of $\overrightarrow{AB}$. (Example 4)

25. $A(-5, -5, -9), B(11, -3, -1)$
26. $A(-4, 0, -3), B(-4, -8, 9)$
27. $A(3, 5, 1), B(0, 0, -9)$
28. $A(-3, -7, -12), B(-7, 1, 8)$
29. $A(2, -5, 4), B(1, 3, -6)$
30. $A(8, 12, 7), B(2, -3, 11)$
31. $A(3, 14, -5), B(7, -1, 0)$
32. $A(1, -18, -13), B(21, 14, 29)$
33. $A(-5, 12, 17), B(6, -11, 4)$
34. $A(9, 3, 7), B(-5, -7, 2)$

35. **TETHERBALL** In the game of tetherball, a ball is attached to a 3-meter pole by a length of rope. Two players hit the ball in opposing directions in an attempt to wind the entire length of rope around the pole. To serve, a certain player holds the ball so that its coordinates are (5, 3.6, 4.7) and the coordinates of the end of the rope connected to the pole are (0, 0, 9.8), where the coordinates are given in feet. Determine the magnitude of the vector representing the length of the rope. (Example 4)



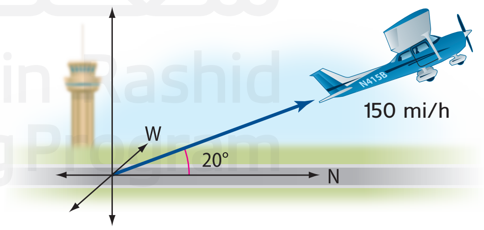
Find each of the following for $\mathbf{a} = \langle -5, -4, 3 \rangle$, $\mathbf{b} = \langle 6, -2, -7 \rangle$, and $\mathbf{c} = \langle -2, 2, 4 \rangle$. (Example 5)

36. $6\mathbf{a} - 7\mathbf{b} + 8\mathbf{c}$
37. $7\mathbf{a} - 5\mathbf{b}$
38. $2\mathbf{a} + 5\mathbf{b} - 9\mathbf{c}$
39. $6\mathbf{b} + 4\mathbf{c} - 4\mathbf{a}$
40. $8\mathbf{a} - 5\mathbf{b} - \mathbf{c}$
41. $-6\mathbf{a} + \mathbf{b} + 7\mathbf{c}$

Find each of the following for $\mathbf{x} = -9\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, $\mathbf{y} = 6\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$, and $\mathbf{z} = -2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. (Example 5)

42. $7\mathbf{x} + 6\mathbf{y}$
43. $3\mathbf{x} - 5\mathbf{y} + 3\mathbf{z}$
44. $4\mathbf{x} + 3\mathbf{y} + 2\mathbf{z}$
45. $-8\mathbf{x} - 2\mathbf{y} + 5\mathbf{z}$
46. $-6\mathbf{y} - 9\mathbf{z}$
47. $-\mathbf{x} - 4\mathbf{y} - \mathbf{z}$

48. **AIRPLANES** An airplane is taking off headed due north with an air speed of 150 miles per hour at an angle of 20° relative to the horizontal. The wind is blowing with a velocity of 8 miles per hour from the southwest. Find a vector that represents the resultant velocity of the plane relative to the point of takeoff. Let **i** point east, **j** point north, and **k** point up. (Example 6)



49. **TRACK AND FIELD** Maysa throws a javelin due south at a speed of 18 miles per hour and at an angle of 48° relative to the horizontal. If the wind is blowing with a velocity of 12 miles per hour at an angle of S15°E, find a vector that represents the resultant velocity of the javelin. Let **i** point east, **j** point north, and **k** point up. (Example 6)

50. **SUBMARINE** A submarine heading due west dives at a speed of 25 knots and an angle of decline of 55°. The current is moving with a velocity of 4 knots at an angle of S20°W. Find a vector that represents the resultant velocity of the submarine relative to the initial point of the dive. Let **i** point east, **j** point north, and **k** point up. (Example 6)

If N is the midpoint of $\overline{MP}$, find P.

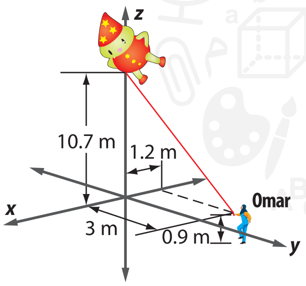
51. $M(3, 4, 5)$; $N\left(\frac{7}{2}, 1, 2\right)$

52. $M(-1, -4, -9)$; $N(-2, 1, -5)$

53. $M(7, 1, 5)$; $N\left(5, -\frac{1}{2}, 6\right)$

54. $M\left(\frac{3}{2}, -5, 9\right)$; $N\left(-2, -\frac{13}{2}, \frac{11}{2}\right)$

55. **VOLUNTEERING** Omar is volunteering to help guide a balloon in a parade. If the balloon is 10.7 meters high and he is holding the tether 0.9 meters above the ground as shown, how long is the tether to the nearest foot?



Determine whether the triangle with the given vertices is *isosceles* or *scalene*.

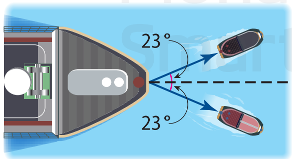
56. $A(3, 1, 2)$, $B(5, -1, 1)$, $C(1, 3, 1)$

57. $A(4, 3, 4)$, $B(4, 6, 4)$, $C(4, 3, 6)$

58. $A(-1, 4, 3)$, $B(2, 5, 1)$, $C(0, -6, 6)$

59. $A(-2.2, 4.3, 5.6)$, $B(0.7, 9.3, 15.6)$, $C(3.6, 14.3, 5.6)$

60. **TUGBOATS** Two tugboats are pulling a disabled supertanker. One of the tow lines makes an angle 23° west of north and the other makes an angle 23° east of north. Each tug exerts a constant force of 2.5×10^6 newtons depressed 15° below the point where the lines attach to the supertanker. They pull the supertanker two miles due north.



a. Write a three-dimensional vector to describe the force from each tugboat.

b. Find the vector that describes the total force on the supertanker.

c. If each tow line is 300 feet long, about how far apart are the tugboats?

61. **SPHERES** Use the distance formula for two points in space to prove that the standard form of the equation of a sphere with center (h, k, ℓ) and radius r is $r^2 = (x - h)^2 + (y - k)^2 + (z - \ell)^2$.

Use the formula from Exercise 61 to write an equation for the sphere with the given center and radius.

62. center $= (-4, -2, 3)$; radius $= 4$

63. center $= (6, 0, -1)$; radius $= \frac{1}{2}$

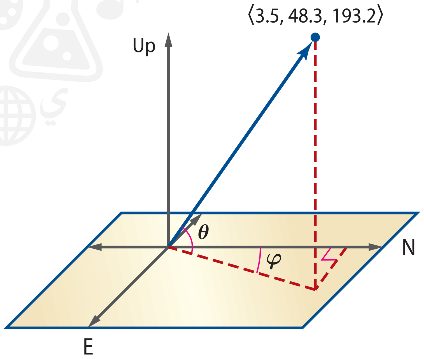
64. center $= (5, -3, 4)$; radius $= \sqrt{3}$

65. center $= (0, 7, -1)$; radius $= 12$

H.O.T. Problems Use Higher-Order Thinking Skills

66. **REASONING** Prove the Distance Formula in Space. (*Hint*: Use the Pythagorean Theorem twice.)

67. **CHALLENGE** Refer to Example 6.

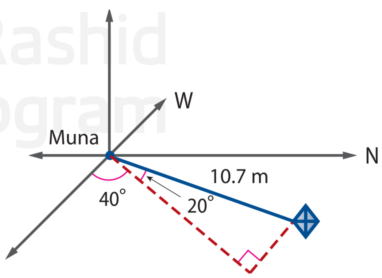


a. Calculate the resultant speed of the rocket.

b. Find the quadrant bearing φ of the rocket.

c. Calculate the resultant angle of incline θ of the rocket relative to the horizontal.

68. **CHALLENGE** Muna is standing in an open field facing N50°E. She is holding a kite on a 10.7-meter string that is flying at a 20° angle with the field. Find the components of the vector from Muna to the kite. (*Hint*: Use trigonometric ratios and two right triangles to find **x**, **y**, and **z**.)



69. **WRITING IN MATH** Describe a situation where it is more reasonable to use a two-dimensional coordinate system and one where it is more reasonable to use a three-dimensional coordinate system.

Spiral Review

Find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v.

70. $\mathbf{u} = \langle 6, 8\rangle, \mathbf{v} = \langle 2, -1\rangle$

71. $\mathbf{u} = \langle -1, 4\rangle, \mathbf{v} = \langle 5, 1\rangle$

72. $\mathbf{u} = \langle 5, 4\rangle, \mathbf{v} = \langle 4, -2\rangle$

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

73. $A(6, -4), B(-7, -7)$

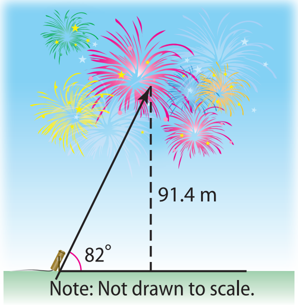
74. $A(-4, -8), B(1, 6)$

75. $A(-5, -12), B(1, 6)$

76. **ENTERTAINMENT** The UAE National Day fireworks at Burj Khalifa are fired at an angle of 82° with the horizontal. The technician firing the shells expects them to explode about 91.4 meters in the air 4.8 seconds after they are fired.

a. Find the initial speed of a shell fired from ground level.

b. Safety barriers will be placed around the launch area to protect spectators. If the barriers are placed 91.4 meters from the point directly below the explosion of the shells, how far should the barriers be from the point where the fireworks are launched?



77. **CONSTRUCTION** A stone door that was designed as an arch in the shape of a semi-ellipse will have an opening with a height of 3 meters at the center and a width of 8 meters along the base. To sketch an outline of the door, a contractor uses a string tied to two thumbtacks.

a. At what locations should the thumbtacks be placed?

b. What length of string needs to be used? Explain your reasoning.

Solve each equation for all values of θ.

78. $\csc \theta + 2 \cot \theta = 0$

79. $\sec^2 \theta - 9 = 0$

80. $2 \csc \theta - 3 = 5 \sin \theta$

Sketch the graph of each function.

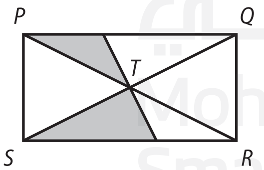
81. $y = \cos^{-1}(x - 2)$

82. $y = \arccos x + 3$

83. $y = \sin^{-1} 3x$

Skills Review for Standardized Tests

84. **SAT/ACT** What percent of the area of rectangle $PQRS$ is shaded?



A 22%
B 25%
C 30%
D $33\frac{1}{3}$%
E 35%

85. **REVIEW** A ship leaving port sails for 75 kilometers in a direction of 35° north of east. At that point, how far north of its starting point is the ship?

F 43 kilometers
G 55 kilometers
H 61 kilometers
J 72 kilometers

86. During a storm, the force of the wind blowing against a skyscraper can be expressed by the vector $\langle 132, 3454, -76\rangle$, where the force of the wind is measured in newtons. What is the approximate magnitude of this force?

A 3457 N
B 3568 N
C 3692 N
D 3717 N

87. **REVIEW** An airplane is flying due west at a velocity of 100 meters a second. The wind is blowing from the south at 30 meters a second. What is the approximate magnitude of the airplane's resultant velocity?

F 4 m/s
G 95.4 m/s
H 100 m/s
J 104.4 m/s

EXTEND 7-4 Graphing Technology Lab
Vector Transformations with Matrices

Objective

- Use a graphing calculator to transform vectors using matrices.

In Lesson 7-4, you learned that a vector in space can be transformed when written in component form or when expressed as a linear combination. A vector in space can also be transformed when written as a 3×1 or 1×3 matrix.

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ or } [x \;\; y \;\; z]$$

Once in matrix form, a vector can be transformed using matrix-vector multiplication.

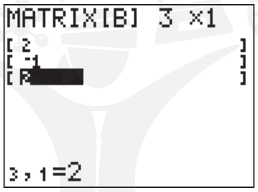
Activity Matrix-Vector Multiplication

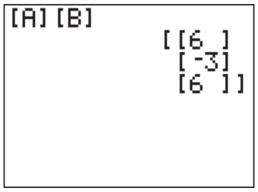
Multiply the vector $\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ by the transformation matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and graph both vectors.

Step 1 Write **B** as a matrix.

$$\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Step 2 Enter **B** and A in a graphing calculator and find $A\mathbf{B}$. Convert to vector form.

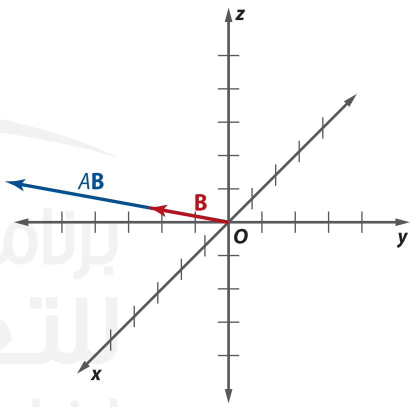




$A\mathbf{B} = 6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

Step 3 Graph **B** and $A\mathbf{B}$ on a coordinate plane.

$A\mathbf{B}$ is a dilation of **B** by a factor of 3.



Exercises

Multiply each vector by the transformation matrix. Graph both vectors.

1. $\mathbf{h} = 4\mathbf{i} + \mathbf{j} + 8\mathbf{k}$

$$B = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

2. $\mathbf{e} = 5\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}$

$$V = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

3. $\mathbf{f} = \mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$

$$W = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4. **REASONING** Multiply $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ by the transformation matrix $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, and graph both vectors. Explain the type of transformation that was performed.

4. $4\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$; $B\mathbf{v}$ is a rotation of $\mathbf{v}$ about the y-axis.

LESSON 7-5 Dot and Cross Products of Vectors in Space

Then

- You found the dot product of two vectors in the plane.

Now

1. Find dot products of and angles between vectors in space.
2. Find cross products of vectors in space, and use cross products to find area and volume.

Why?

- The tendency of a hinged door to rotate when pushed is affected by the distance between the location of the push and the hinge, the magnitude of the push, and the direction of the push.

A quantity called *torque* measures how effectively a force applied to a lever causes rotation about an axis.

NewVocabulary
cross product
torque
parallelepiped
triple scalar product

1 Dot Products in Space

Calculating the dot product of two vectors in space is similar to calculating the dot product of two vectors in a plane. As with vectors in a plane, nonzero vectors in space are perpendicular if and only if their dot product equals zero.

KeyConcept Dot Product and Orthogonal Vectors in Space

The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. The vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 1 Find the Dot Product to Determine Orthogonal Vectors in Space

Find the dot product of u and v. Then determine if u and v are orthogonal.

a. $\mathbf{u} = \langle -7, 3, -3 \rangle, \mathbf{v} = \langle 5, 17, 5 \rangle$

$$\mathbf{u} \cdot \mathbf{v} = -7(5) + 3(17) + (-3)(5)$$
$$= -35 + 51 + (-15) \text{ or } 1$$

Since $\mathbf{u} \cdot \mathbf{v} \neq 0$, **u** and **v** are not orthogonal.

b. $\mathbf{u} = \langle 3, -3, 3 \rangle, \mathbf{v} = \langle 4, 7, 3 \rangle$

$$\mathbf{u} \cdot \mathbf{v} = 3(4) + (-3)(7) + 3(3)$$
$$= 12 + (-21) + 9 \text{ or } 0$$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, **u** and **v** are orthogonal.

GuidedPractice

1A. $\mathbf{u} = \langle 3, -5, 4 \rangle, \mathbf{v} = \langle 5, 7, 5 \rangle$

1B. $\mathbf{u} = \langle 4, -2, -3 \rangle, \mathbf{v} = \langle 1, 3, -2 \rangle$

As with vectors in a plane, if θ is the angle between nonzero vectors **a** and **b**, then $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.

Example 2 Angle Between Two Vectors in Space

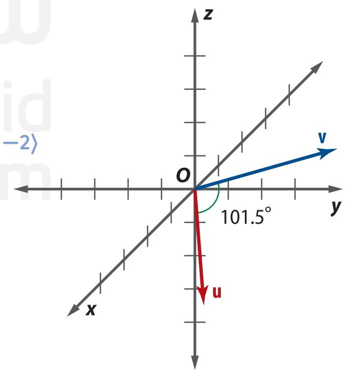
Find the angle θ between u and v to the nearest tenth of a degree if $\mathbf{u} = \langle 3, 2, -1 \rangle$ and $\mathbf{v} = \langle -4, 3, -2 \rangle$.

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$ Angle between two vectors

$$\cos \theta = \frac{\langle 3, 2, -1 \rangle \cdot \langle -4, 3, -2 \rangle}{|\langle 3, 2, -1 \rangle|\,|\langle -4, 3, -2 \rangle|}$$ $u = \langle 3, 2, -1 \rangle$ and $v = \langle -4, 3, -2 \rangle$

$$\cos \theta = \frac{-4}{\sqrt{14}\sqrt{29}}$$ Evaluate the dot product and magnitudes.

$$\theta = \cos^{-1} \frac{-4}{\sqrt{406}} \text{ or about } 101.5°$$ Simplify and solve for θ.



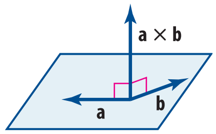
The measure of the angle between **u** and **v** is about 101.5°.

GuidedPractice

2. Find the angle between $\mathbf{u} = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{k}$ to the nearest tenth of a degree.

2 Cross Products

Another important product involving vectors in space is the cross product. Unlike the dot product, the **cross product** of two vectors **a** and **b** in space, denoted **a** × **b** and read *a cross b*, is a vector, not a scalar. The vector **a** × **b** is perpendicular to the plane containing vectors **a** and **b**.



KeyConcept Cross Product of Vectors in Space

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the cross product of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

ReviewVocabulary

2 × 2 Determinant The determinant of the 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$.

If we apply the formula for calculating the determinant of a 3 × 3 matrix to the following *determinant form* involving **i**, **j**, **k**, and the components of **a** and **b**, we arrive at the same formula for **a** × **b**.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Put the unit vectors i, j, and k in Row 1.
Put the components of a in Row 2.
Put the components of b in Row 3.

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$ Apply the formula for a 3 × 3 determinant.

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$ Compute each 2 × 2 determinant.

WatchOut!

Cross Product The cross product definition applies only to vectors in three-dimensional space. The cross product is not defined for vectors in the two-dimensional coordinate system.

Example 3 Find the Cross Product of Two Vectors

Find the cross product of u = ⟨3, −2, 1⟩ and v = ⟨−3, 3, 1⟩. Then show that u × v is orthogonal to both u and v.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -3 & 3 & 1 \end{vmatrix}$$ $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

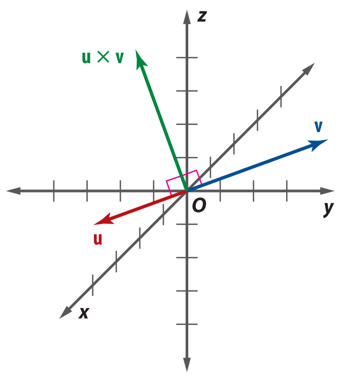
$$= \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ -3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ -3 & 3 \end{vmatrix}\mathbf{k}$$ Determinant of a 3 × 3 matrix

$$= (-2 - 3)\mathbf{i} - [3 - (-3)]\mathbf{j} + (9 - 6)\mathbf{k}$$ Determinants of 2 × 2 matrices

$$= -5\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$$ Simplify.

$$= \langle -5, -6, 3 \rangle$$ Component form

In the graph of **u**, **v**, and **u** × **v**, **u** × **v** is orthogonal to **u** and **u** × **v** is orthogonal to **v**.



To show that **u** × **v** is orthogonal to both **u** and **v**, find the dot product of **u** × **v** with **u** and **u** × **v** with **v**.

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$
$= \langle -5, -6, 3 \rangle \cdot \langle 3, -2, 1 \rangle$
$= -5(3) + (-6)(-2) + 3(1)$
$= -15 + 12 + 3$ or 0 ✓

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$
$= \langle -5, -6, 3 \rangle \cdot \langle -3, 3, 1 \rangle$
$= -5(-3) + (-6)(3) + 3(1)$
$= 15 + (-18) + 3$ or 0 ✓

Because both dot products are zero, the vectors are orthogonal.

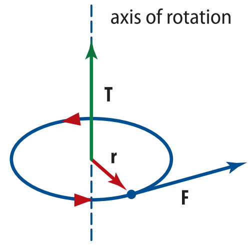
GuidedPractice

Find the cross product of u and v. Then show that u × v is orthogonal to both u and v.

3A. $\mathbf{u} = \langle 4, 2, -1 \rangle$, $\mathbf{v} = \langle 5, 1, 4 \rangle$

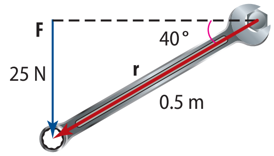
3B. $\mathbf{u} = \langle -2, -1, -3 \rangle$, $\mathbf{v} = \langle 5, 1, 4 \rangle$

You can use the cross product to find a vector quantity called **torque**. Torque measures how effectively a force applied to a lever causes rotation along the axis of rotation. The torque vector **T** is perpendicular to the plane containing the directed distance **r** from the axis of rotation to the point of the applied force and the applied force **F** as shown. Therefore, the torque vector is $\mathbf{T} = \mathbf{r} \times \mathbf{F}$ and is measured in newton-meters (N • m).



Real-World Example 4 Torque Using Cross Product

AUTO REPAIR **Abdulkarim uses a lug wrench to tighten a lug nut. The wrench he uses is 50 centimeters or 0.5 meter long. Find the magnitude and direction of the torque about the lug nut if he applies a force of 25 newtons straight down to the end of the handle when it is 40° below the positive *x*-axis as shown.**



Step 1 Graph each vector in standard position (Figure 7.5.1).

Step 2 Determine the component form of each vector.

The component form of the vector representing the directed distance from the axis of rotation to the end of the handle can be found using the triangle in Figure 7.5.2 and trigonometry. Vector **r** is therefore $\langle 0.5 \cos 40°, 0, -0.5 \sin 40°\rangle$ or about $\langle 0.38, 0, -0.32\rangle$. The vector representing the force applied to the end of the handle is 25 newtons straight down, so $\mathbf{F} = \langle 0, 0, -25\rangle$.

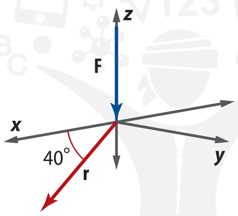


Figure 7.5.1

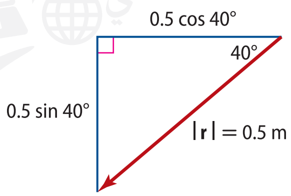


Figure 7.5.2

Step 3 Use the cross product of these vectors to find the vector representing the torque about the lug nut.

$\mathbf{T} = \mathbf{r} \times \mathbf{F}$ — Torque Cross Product Formula

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.38 & 0 & -0.32 \\ 0 & 0 & -25 \end{vmatrix}$$ Cross product of r and F

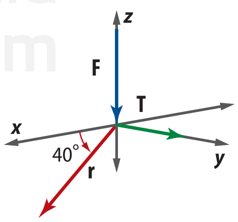
$$= \begin{vmatrix} 0 & -0.32 \\ 0 & -25 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0.38 & -0.32 \\ 0 & -25 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0.38 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{k}$$ Determinant of a 3 × 3 matrix

$= 0\mathbf{i} - (-9.5)\mathbf{j} + 0\mathbf{k}$ — Determinants of 2 × 2 matrices

$= \langle 0, 9.5, 0\rangle$ — Component form

Step 4 Find the magnitude and direction of the torque vector.

The component form of the torque vector $\langle 0, 9.5, 0\rangle$ tells us that the magnitude of the vector is about 9.5 newton-meters parallel to the positive *y*-axis as shown.



Guided Practice

4. **AUTO REPAIR** Find the magnitude of the torque if Abdulkarim applied the same amount of force to the end of the handle straight down when the handle makes a 40° angle above the positive *x*-axis as shown in Figure 7.5.3.

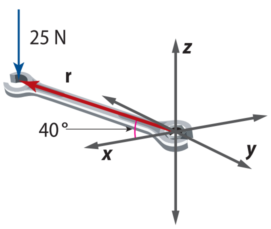


Figure 7.5.3

Real-World Career

Automotive Mechanic

Automotive mechanics perform repairs ranging from simple mechanical problems to high-level, technology–related repairs. They should have good problem-solving skills, mechanical aptitude, and knowledge of electronics and mathematics. Most mechanics complete a vocational training program in automotive service technology.

The cross product of two vectors has several geometric applications. One is that the magnitude of $\mathbf{u} \times \mathbf{v}$ represents the area of the parallelogram that has **u** and **v** as its adjacent sides (Figure 7.5.4).

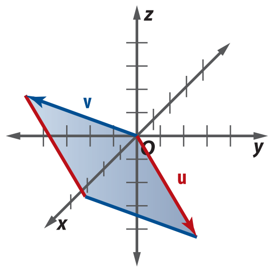


Figure 7.5.4

Example 5 Area of a Parallelogram in Space

Find the area of the parallelogram with adjacent sides $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

Step 1 Find $\mathbf{u} \times \mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -3 \\ 1 & -5 & 3 \end{vmatrix}$$ $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$

$$= \begin{vmatrix} 4 & -3 \\ -5 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 4 \\ 1 & -5 \end{vmatrix}\mathbf{k}$$ Determinant of a 3×3 matrix

$$= -3\mathbf{i} - 9\mathbf{j} - 14\mathbf{k}$$ Determinants of 2×2 matrices

Step 2 Find the magnitude of $\mathbf{u} \times \mathbf{v}$.

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{(-3)^2 + (-9)^2 + (-14)^2}$$ Magnitude of a vector in space

$$= \sqrt{286} \text{ or about } 16.9$$ Simplify.

The area of the parallelogram shown in Figure 7.5.4 is about 16.9 square units.

GuidedPractice

5. Find the area of the parallelogram with adjacent sides $\mathbf{u} = -6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Three vectors that lie in different planes but share the same initial point determine the adjacent edges of a **parallelepiped**, a polyhedron with faces that are all parallelograms (Figure 7.5.5). The absolute value of the **triple scalar product** of these vectors represents the volume of the parallelepiped.

StudyTip

Triple Scalar Product Notice that to find the triple scalar product of **t**, **u**, and **v**, you write the determinant representing $\mathbf{u} \times \mathbf{v}$ and replace the top row with the values for the vector **t**.

KeyConcept Triple Scalar Product

If $\mathbf{t} = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, the triple scalar product is given by $\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.

Example 6 Volume of a Parallelepiped

Find the volume of the parallelepiped with adjacent edges $\mathbf{t} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

$$\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} 4 & -2 & -2 \\ 2 & 4 & -3 \\ 1 & -5 & 3 \end{vmatrix}$$ $\mathbf{t} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$

$$= \begin{vmatrix} 4 & -3 \\ -5 & 3 \end{vmatrix}(4) - \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix}(-2) + \begin{vmatrix} 2 & 4 \\ 1 & -5 \end{vmatrix}(-2)$$ Determinant of a 3×3 matrix

$$= -12 + 18 + 28 \text{ or } 34$$ Simplify.

The volume of the parallelepiped shown in Figure 7.5.5 is $|\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v})|$ or 34 cubic units.

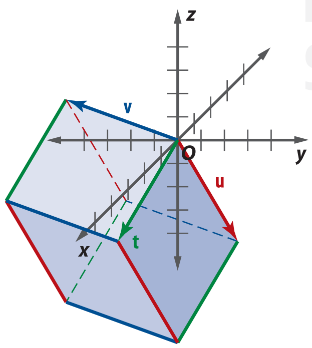


Figure 7.5.5

GuidedPractice

6. Find the volume of the parallelepiped with adjacent edges $\mathbf{t} = 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{u} = -6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Exercises

Find the dot product of u and v. Then determine if u and v are orthogonal. (Example 1)

1. $\mathbf{u} = \langle 3, -9, 6 \rangle$, $\mathbf{v} = \langle -8, 2, 7 \rangle$
2. $\mathbf{u} = \langle 5, 0, -4 \rangle$, $\mathbf{v} = \langle 6, -1, 4 \rangle$
3. $\mathbf{u} = \langle 2, -8, -7 \rangle$, $\mathbf{v} = \langle 5, 9, -7 \rangle$
4. $\mathbf{u} = \langle -7, -3, 1 \rangle$, $\mathbf{v} = \langle -4, 5, -13 \rangle$
5. $\mathbf{u} = \langle 11, 4, -2 \rangle$, $\mathbf{v} = \langle -1, 3, 8 \rangle$
6. $\mathbf{u} = 6\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
7. $\mathbf{u} = 3\mathbf{i} - 10\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
8. $\mathbf{u} = 9\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$, $\mathbf{v} = 6\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

9. **CHEMISTRY** A water molecule, in which the oxygen atom is centered at the origin, has one hydrogen atom centered at $\langle 55.5, 55.5, -55.5 \rangle$ and the second hydrogen atom centered at $\langle -55.5, -55.5, -55.5 \rangle$. Determine the bond angle between the vectors formed by the oxygen-hydrogen bonds. (Example 2)

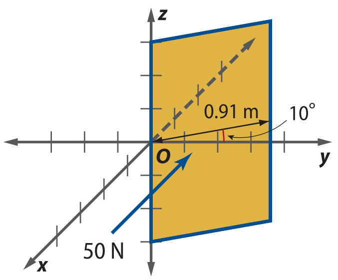
Find the angle θ between vectors u and v to the nearest tenth of a degree. (Example 2)

10. $\mathbf{u} = \langle 3, -2, 2 \rangle$, $\mathbf{v} = \langle 1, 4, -7 \rangle$
11. $\mathbf{u} = \langle 6, -5, 1 \rangle$, $\mathbf{v} = \langle -8, -9, 5 \rangle$
12. $\mathbf{u} = \langle -8, 1, 12 \rangle$, $\mathbf{v} = \langle -6, 4, 2 \rangle$
13. $\mathbf{u} = \langle 10, 0, -8 \rangle$, $\mathbf{v} = \langle 3, -1, -12 \rangle$
14. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + 9\mathbf{k}$, $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}$
15. $\mathbf{u} = -6\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{v} = -4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

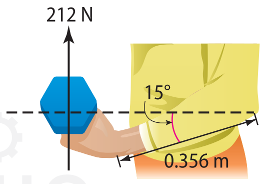
Find the cross product of u and v. Then show that u × v is orthogonal to both u and v. (Example 3)

16. $\mathbf{u} = \langle -1, 3, 5 \rangle$, $\mathbf{v} = \langle 2, -6, -3 \rangle$
17. $\mathbf{u} = \langle 4, 7, -2 \rangle$, $\mathbf{v} = \langle -5, 9, 1 \rangle$
18. $\mathbf{u} = \langle 3, -6, 2 \rangle$, $\mathbf{v} = \langle 1, 5, -8 \rangle$
19. $\mathbf{u} = \langle 5, -8, 0 \rangle$, $\mathbf{v} = \langle -4, -2, 7 \rangle$
20. $\mathbf{u} = -2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{v} = 7\mathbf{i} + \mathbf{j} - 6\mathbf{k}$
21. $\mathbf{u} = -4\mathbf{i} + \mathbf{j} + 8\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$

22. **RESTAURANTS** A restaurant server applies 50 newtons of force to open a door. Find the magnitude and direction of the torque about the door's hinge. (Example 4)



23. **WEIGHTLIFTING** A weightlifter doing bicep curls applies 212 newtons of force to lift the dumbbell. The weightlifter's forearm is 0.356 meters long and she begins the bicep curl with her elbow bent at a 15° angle below the horizontal in the direction of the positive *x*-axis. (Example 4)



   a. Find the vector representing the torque about the weightlifter's elbow in component form.
   b. Find the magnitude and direction of the torque.

Find the area of the parallelogram with adjacent sides u and v. (Example 5)

24. $\mathbf{u} = \langle 2, -5, 3 \rangle$, $\mathbf{v} = \langle 4, 6, -1 \rangle$
25. $\mathbf{u} = \langle -9, 1, 2 \rangle$, $\mathbf{v} = \langle 6, -5, 3 \rangle$
26. $\mathbf{u} = \langle 4, 3, -1 \rangle$, $\mathbf{v} = \langle 7, 2, -2 \rangle$
27. $\mathbf{u} = 6\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{v} = 5\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$
28. $\mathbf{u} = \mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$
29. $\mathbf{u} = -3\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + 6\mathbf{k}$

Find the volume of the parallelepiped having t, u, and v as adjacent edges. (Example 6)

30. $\mathbf{t} = \langle -1, -9, 2 \rangle$, $\mathbf{u} = \langle 4, -7, -5 \rangle$, $\mathbf{v} = \langle 3, -2, 6 \rangle$
31. $\mathbf{t} = \langle -6, 4, -8 \rangle$, $\mathbf{u} = \langle -3, -1, 6 \rangle$, $\mathbf{v} = \langle 2, 5, -7 \rangle$
32. $\mathbf{t} = \langle 2, -3, -1 \rangle$, $\mathbf{u} = \langle 4, -6, 3 \rangle$, $\mathbf{v} = \langle -9, 5, -4 \rangle$
33. $\mathbf{t} = -4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{u} = 5\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$
34. $\mathbf{t} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - 6\mathbf{j} + 8\mathbf{k}$
35. $\mathbf{t} = 5\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$, $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$, $\mathbf{v} = 8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

Find a vector that is orthogonal to each vector.

36. $\langle 3, -8, 4 \rangle$
37. $\langle -1, -2, 5 \rangle$
38. $\left\langle 6, -\frac{1}{3}, -3 \right\rangle$
39. $\langle 7, 0, 8 \rangle$

Given v and u · v, find u.

40. $\mathbf{v} = \langle 2, -4, -6 \rangle$, $\mathbf{u} \cdot \mathbf{v} = -22$
41. $\mathbf{v} = \left\langle \frac{1}{2}, 0, 4 \right\rangle$, $\mathbf{u} \cdot \mathbf{v} = \frac{31}{2}$
42. $\mathbf{v} = \langle -2, -6, -5 \rangle$, $\mathbf{u} \cdot \mathbf{v} = 35$

Determine whether the points are collinear.

43. $(-1, 7, 7)$, $(-3, 9, 11)$, $(-5, 11, 13)$
44. $(11, 8, -1)$, $(17, 5, -7)$, $(8, 11, 5)$

Determine whether each pair of vectors are parallel.

45. $\mathbf{m} = \langle 2, -10, 6 \rangle$, $\mathbf{n} = \langle 3, -15, 9 \rangle$

46. $\mathbf{a} = \langle 6, 3, -7 \rangle$, $\mathbf{b} = \langle -4, -2, 3 \rangle$

47. $\mathbf{w} = \left\langle -\frac{3}{2}, \frac{3}{4}, -\frac{9}{8} \right\rangle$, $\mathbf{z} = \langle -4, 2, -3 \rangle$

Write the component form of each vector.

48. **u** lies in the *yz*-plane, has a magnitude of 8, and makes a 60° angle above the positive *y*-axis.

49. **v** lies in the *xy*-plane, has a magnitude of 11, and makes a 30° angle to the left of the negative *x*-axis.

Given the four vertices, determine whether quadrilateral *ABCD* is a parallelogram. If it is, find its area, and determine whether it is a rectangle.

50. $A(3, 0, -2)$, $B(0, 4, -1)$, $C(0, 2, 5)$, $D(3, 2, 4)$

51. $A(7, 5, 5)$, $B(4, 4, 4)$, $C(4, 6, 2)$, $D(7, 7, 3)$

52. **AIR SHOWS** In an air show, two airplanes take off simultaneously. The first plane starts at the position $(0, -2, 0)$ and is at the position $(6, -10, 15)$ after three seconds. The second plane starts at the position $(0, 2, 0)$ and is at the position $(6, 10, 15)$ after three seconds. Are the paths of the two planes parallel? Explain.

For $\mathbf{u} = \langle 3, 2, -2 \rangle$ and $\mathbf{v} = \langle -4, 4, 5 \rangle$, find each of the following, if possible.

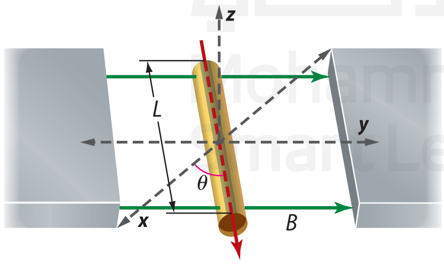
53. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$

54. $\mathbf{v} \times (\mathbf{u} \cdot \mathbf{v})$

55. $\mathbf{u} \times \mathbf{u} \times \mathbf{v}$

56. $\mathbf{v} \cdot \mathbf{v} \cdot \mathbf{u}$

57. **ELECTRICITY** When a wire carrying an electric current is placed in a magnetic field, the force on the wire in newtons is given by $\vec{F} = I\vec{L} \times \vec{B}$, where I represents the current flowing through the wire in amps, $\vec{L}$ represents the vector length of the wire pointing in the direction of the current in meters, and $\vec{B}$ is the force of the magnetic field in teslas. In the figure below, the wire is rotated through an angle θ in the *xy*-plane.

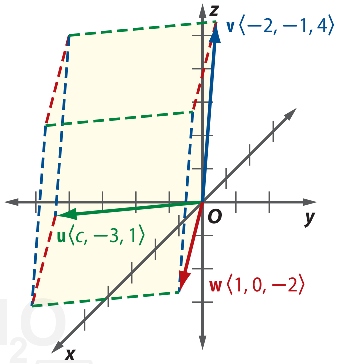


a. If the force of a magnetic field is 1.1 teslas, find the magnitude of the force on a wire in the *xy*-plane that is 0.15 meter in length carrying a current of 25 amps at an angle of 60°.

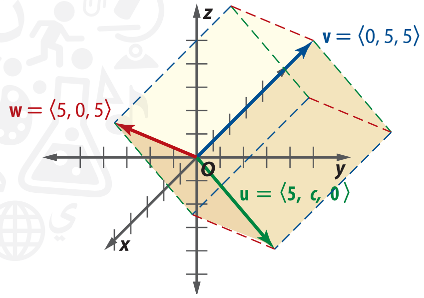
b. If the force on the wire is $\vec{F} = \langle 0, 0, -0.63 \rangle$, what is the angle of the wire?

Given v, w, and the volume of the parallelepiped having adjacent edges u, v, and w, find *c*.

58. $\mathbf{v} = \langle -2, -1, 4 \rangle$, $\mathbf{w} = \langle 1, 0, -2 \rangle$, $\mathbf{u} = \langle c, -3, 1 \rangle$, and $V = 7$ cubic units

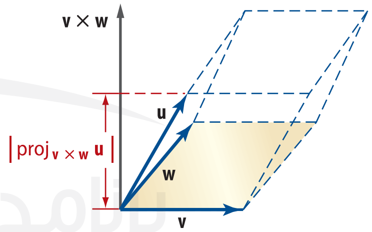


59. $\mathbf{v} = \langle 0, 5, 5 \rangle$, $\mathbf{w} = \langle 5, 0, 5 \rangle$, $\mathbf{u} = \langle 5, c, 0 \rangle$, and $V = 250$ cubic units



H.O.T. Problems Use Higher-Order Thinking Skills

60. **PROOF** Verify the formula for the volume of a parallelepiped. (*Hint*: Use the projection of **u** onto $\mathbf{v} \times \mathbf{w}$.)



61. **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain.

For any two nonzero, nonparallel vectors in space, there is a vector that is perpendicular to both.

62. **REASONING** If **u** and **v** are parallel in space, then how many vectors are perpendicular to both? Explain.

63. **CHALLENGE** Given $\mathbf{u} = \langle 4, 6, c \rangle$ and $\mathbf{v} = \langle -3, -2, 5 \rangle$, find the value of c for which $\mathbf{u} \times \mathbf{v} = 34\mathbf{i} - 26\mathbf{j} + 10\mathbf{k}$.

64. **REASONING** Explain why the cross product is not defined for vectors in the two-dimensional coordinate system.

65. **WRITING IN MATH** Compare and contrast the methods for determining whether vectors in space are parallel or perpendicular.

Spiral Review

Find the length and midpoint of the segment with the given endpoints.

66. (1, 10, 13), (−2, 22, −6)

67. (12, −1, −14), (21, 19, −23)

68. (−22, 24, −9), (10, 10, 2)

Find the dot product of u and v. Then determine if u and v are orthogonal.

69. $\langle -8, -7 \rangle \cdot \langle 1, 2 \rangle$

70. $\langle -4, -6 \rangle \cdot \langle 7, 5 \rangle$

71. $\langle 6, -3 \rangle \cdot \langle -3, 5 \rangle$

72. **BAKERY** Abdulaziz's bakery has racks that can hold up to 900 bagels and muffins. Due to costs, the number of bagels produced must be less than or equal to 300 plus twice the number of muffins produced. The demand for bagels is at least three times that of muffins. Abdulaziz makes a profit of AED 3 per muffin sold and AED 1.25 per bagel sold. How many of each item should he make to maximize profit?

73. Decompose $\frac{2m + 16}{m^2 - 16}$ into partial fractions.

Verify each identity.

74. $\tan^2 \theta + \cos^2 \theta + \sin^2 \theta = \sec^2 \theta$

75. $\sec^2 \theta \cot^2 \theta - \cot^2 \theta = 1$

76. $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$

Solve each triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree.

77. $a = 20$, $c = 24$, $B = 47°$

78. $A = 25°$, $B = 78°$, $a = 13.7$

79. $a = 21.5$, $b = 16.7$, $c = 10.3$

Write each decimal degree measure in DMS form and each DMS measure in decimal degree form to the nearest thousandth.

80. −72.775°

81. 29° 6′ 6″

82. 132° 18′ 31″

Skills Review for Standardized Tests

83. **SAT/ACT** The graph represents the set of all possible solutions to which of the following statements?

A $|x - 1| > 1$

B $|x - 1| < 1$

C $|x + 1| < 1$

D $|x + 1| > 1$

84. What is the cross product of $\mathbf{u} = \langle 3, 8, 0 \rangle$ and $\mathbf{v} = \langle -4, 2, 6 \rangle$?

F $48\mathbf{i} - 18\mathbf{j} + 38\mathbf{k}$

G $48\mathbf{i} - 22\mathbf{j} + 38\mathbf{k}$

H $46\mathbf{i} - 22\mathbf{j} + 38\mathbf{k}$

J $46\mathbf{i} - 18\mathbf{j} + 38\mathbf{k}$

85. **FREE RESPONSE** A batter hits a ball at a 30° angle with the ground at an initial speed of 90 feet per second.

a. Find the magnitude of the horizontal and vertical components of the velocity.

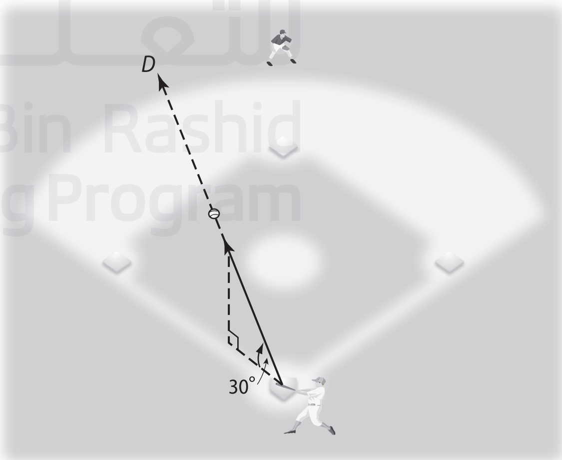
b. Are the values in part **a** vectors or scalars?

c. Assume that the ball is not caught and the player hit it one yard off the ground. How far will it travel in the air?

d. Assume that home plate is at the origin and second base lies due north. If the ball is hit at a bearing of N20°W and lands at point D, find the component form of $\overrightarrow{CD}$.

e. Determine the unit vector of $\overrightarrow{CD}$.

f. The fielder is standing at (0, 150) when the ball is hit. At what quadrant bearing should the fielder run in order to meet the ball where it will hit the ground?



LESSON 7-6

Properties of the Dot and Cross Products of Vectors

Before

- Getting to know dot products.

Now

1. Identify multiplication properties of dot products.
2. Learning characteristic of vectors.

Why?

- Compare the measurement of vectors in space and the relationship between especially the possibility that they could be parallel or perpendicular.

In sections 7-1 and 7-2, we defined vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ and examined many of the properties of vectors, including how to add and subtract two vectors. It turns out that two different kinds of products involving vectors have proved to be useful: the dot product (or scalar product) and the cross product (or vector product). We introduce the first of these two products in this section.

Definition 7-1

The **dot product** of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in V_3 is defined by

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1b_1 + a_2b_2 + a_3b_3. \qquad (7\text{-}1)$$

Likewise, the dot product of two vectors in V_2 is defined by

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1b_1 + a_2b_2.$$

Be sure to notice that the dot product of two vectors is a *scalar* (i.e., a number, not a vector). For this reason, the dot product is also called the **scalar product**.

Example 7-1 Computing a Dot Product in $\mathbb{R}^3$

Compute the dot product $\mathbf{a} \cdot \mathbf{b}$ for $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle 5, -3, 4 \rangle$.

Solution We have

$$\mathbf{a} \cdot \mathbf{b} = \langle 1, 2, 3 \rangle \cdot \langle 5, -3, 4 \rangle = (1)(5) + (2)(-3) + (3)(4) = 11.$$

Certainly, dot products are very simple to compute, whether a vector is written in component form or written in terms of the standard basis vectors, as in example 7-2.

Example 7-2 Computing a Dot Product in $\mathbb{R}^2$

Find the dot product of the two vectors $\mathbf{a} = 2\mathrm{i} - 5\mathrm{j}$ and $\mathbf{b} = 3\mathrm{i} + 6\mathrm{j}$.

Solution We have

$$\mathbf{a} \cdot \mathbf{b} = (2)(3) + (-5)(6) = 6 - 30 = -24.$$

The dot product in V_2 or V_3 satisfies the following simple properties.

Remark 7-1

Since vectors in V_2 can be thought of as special cases of vectors in V_3 (where the third component is zero), all of the results we prove for vectors in V_3 hold equally for vectors in V_2.

Theorem 7-1

For vectors **a**, **b** and **c** and any scalar d, the following hold:

(i) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutativity)
(ii) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributive law)
(iii) $(d\mathbf{a}) \cdot \mathbf{b} = d(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (d\mathbf{b})$
(iv) $\mathbf{0} \cdot \mathbf{a} = 0$ and
(v) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

Proof

We prove (i) and (v) for $\mathbf{a},\mathbf{b} \in V_3$. The remaining parts are left as exercises.

(i) For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, we have from (7-1) that

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1b_1 + a_2b_2 + a_3b_3 \\ &= b_1a_1 + b_2a_2 + b_3a_3 = \mathbf{b} \cdot \mathbf{a},\end{aligned}$$

Since multiplication of real numbers is commutative.

(v) For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, we have

$$\mathbf{a} \cdot \mathbf{a} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2.$$

Notice that properties (i)–(iv) of Theorem 7-1 are also properties of multiplication of real numbers. This is why we use the word *product* in dot product. However, there are some properties of multiplication of real numbers not shared by the dot product. For instance, we will see that $\mathbf{a} \cdot \mathbf{b} = 0$ does not imply that either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

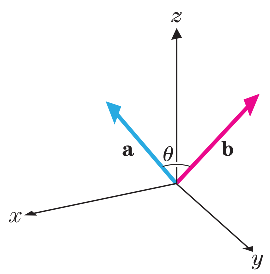


Figure 7.2.3a
The angle between two vectors

For two *nonzero* vectors **a** and **b** in V_3, we define the **angle** θ $(0 \le \theta \le \pi)$ **between the vectors** to be the smaller angle between **a** and **b**, formed by placing their initial points at the same point, as illustrated in Figure 7.2.3a.

Notice that if **a** and **b** have the *same* direction, then $\theta = 0$. If **a** and **b** have *opposite* directions, then $\theta = \pi$. We say that **a** and **b** are **orthogonal** (or **perpendicular**) if $\theta = \frac{\pi}{2}$. We consider the zero vector **0** to be orthogonal to every vector. The general case is stated in Theorem 7-2.

Theorem 7-2

Let θ be the angle between nonzero vectors **a** and **b**. Then,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta. \qquad (7\text{-}2)$$

Proof

We must prove the theorem for three separate cases.

(i) If **a** and **b** have the *same direction*, then $\mathbf{b} = c\mathbf{a}$, for some scalar $c > 0$ and the angle between **a** and **b** is $\theta = 0$. This says that

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (c\mathbf{a}) = c\mathbf{a} \cdot \mathbf{a} = c|\mathbf{a}|^2.$$

Further,

$$|\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}||c||\mathbf{a}|\cos 0 = c|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{b},$$

since for $c > 0$, we have $|c| = c$.

(ii) If **a** and **b** have the opposite direction, the proof is nearly identical to case (i) above and we leave the details as an exercise.

(iii) If **a** and **b** are not parallel, then we have that $0 < \theta < \pi$, as shown in Figure 7.23b. Recall that the Law of Cosines allows us to relate the lengths of the sides of triangles like the one in Figure 7.23b. We have

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta. \qquad (7\text{-}3)$$

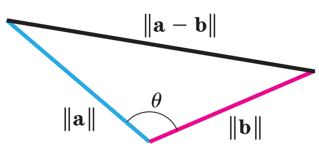


Figure 7.23b
The angle between two vectors

Now, observe that

$$\begin{aligned}|\mathbf{a} - \mathbf{b}|^2 &= |\langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle|^2 \\ &= (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \\ &= (a_1^2 - 2a_1b_1 + b_1^2) + (a_2^2 - 2a_2b_2 + b_2^2) + (a_3^2 - 2a_3b_3 + b_3^2) \\ &= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2(a_1b_1 + a_2b_2 + a_3b_3) \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} \qquad (7\text{-}4)\end{aligned}$$

Equating the right-hand sides of (7-3) and (7-4), we get (7-2), as desired.

We can use (7-2) to find the angle between two vectors, as in example 7-3.

Example 7-3 Finding the Angle between Two Vectors

Find the angle between the vectors $\mathbf{a} = \langle 2, 1, -3\rangle$ and $\mathbf{b} = \langle 1, 5, 6\rangle$.

Solution From (7-2), we have

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{-11}{\sqrt{14}\sqrt{62}}.$$

It follows that $\theta = \cos^{-1}\left(\frac{-11}{\sqrt{14}\sqrt{62}}\right) \approx 1.953$ (radians) (or about 112°), since $0 \le \theta \le \pi$ and the inverse cosine function returns an angle in this range.

The following result is an immediate and important consequence of Theorem 7-2.

Corollary 7-1

Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.

Proof

First, observe that if either **a** or **b** is the zero vector, then $\mathbf{a}\cdot\mathbf{b} = 0$ and **a** and **b** are orthogonal, as the zero vector is considered orthogonal to every vector. If **a** and **b** are nonzero vectors and if θ is the angle between **a** and **b**, we have from Theorem 7-2 that

$$|\mathbf{a}||\mathbf{b}|\cos\theta = \mathbf{a}\cdot\mathbf{b} = 0$$

if and only if $\cos\theta = 0$ (since neither **a** nor **b** is the zero vector). This occurs if and only if $\theta = \frac{\pi}{2}$, which is equivalent to having **a** and **b** orthogonal and so, the result follows.

Example 7-4 Determining Whether Two Vectors Are Orthogonal

Determine whether the following pairs of vectors are orthogonal: (a) $\mathbf{a} = \langle 1, 3, -5\rangle$ and $\mathbf{b} = \langle 2, 3, 10\rangle$ and (b) $\mathbf{a} = \langle 4, 2, -1\rangle$ and $\mathbf{b} = \langle 2, 3, 14\rangle$.

Solution For (a), we have:

$$\mathbf{a}\cdot\mathbf{b} = 2 + 9 - 50 = -39 \neq 0,$$

so that **a** and **b** are *not* orthogonal.

For (b), we have

$$\mathbf{a}\cdot\mathbf{b} = 8 + 6 - 14 = 0,$$

so that **a** and **b** are orthogonal, in this case.

The following two results provide us with some powerful tools for comparing the magnitudes of vectors.

Theorem 7-3 (Cauchy-Schwartz Inequality)

For any vectors **a** and **b**,

$$|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}||\mathbf{b}|.$$

Proof

If either **a** or **b** is the zero vector, notice that (7-5) simply says that $0 \le 0$, which is certainly true. On the other hand, if neither **a** nor **b** is the zero vector, we have from (7-2) that

$$|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos\theta| \le |\mathbf{a}||\mathbf{b}|,$$

since $|\cos\theta| \le 1$ for all values of θ.

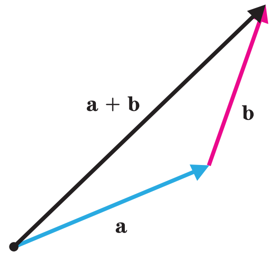


FIGURE 7.24
The Triangle Inequality

One benefit of the Cauchy—Schwartz Inequality is that it allows us to prove the following very useful result. If you were going to learn only one inequality in your lifetime, this is probably the one you would want to learn.

Theorem 7-4 (The Triangle Inequality)

For any vectors **a** and **b**,

$$|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|. \qquad (7\text{-}6)$$

Before we prove the theorem, consider the triangle formed by the vectors **a**, **b** and **a** + **b**, shown in Figure 7.24. Notice that the Triangle Inequality says that the length of the vector **a** + **b** never exceeds the sum of the individual lengths of **a** and **b**.

Proof

From Theorem 7-1 (i), (ii) and (v), we have

$$|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}$$
$$= |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2.$$

From the Cauchy-Schwartz Inequality (7-5), we have $\mathbf{a} \cdot \mathbf{b} \le |\mathbf{a} \cdot \mathbf{b}| \le |\mathbf{a}|\,|\mathbf{b}|$ and so, we have

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$$
$$\le |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 = (|\mathbf{a}| + |\mathbf{b}|)^2.$$

Taking square roots gives us (7-6).

Components and Projections Think about the case where a vector represents a force. Often, it's impractical to exert a force in the direction you'd like. For instance, in pulling a child'swagon,we exert a force in the direction determined by the position of the handle, instead of in the direction of motion. (See Figure 7.25) An important question is whether there is a force of smaller magnitude that can be exerted in a different direction and still produce the same effect on the wagon. Notice that it is the horizontal portion of the force that most directly contributes to the motion of the wagon. (The vertical portion of the force only acts to reduce friction.) We now consider how to compute such a component of a force.

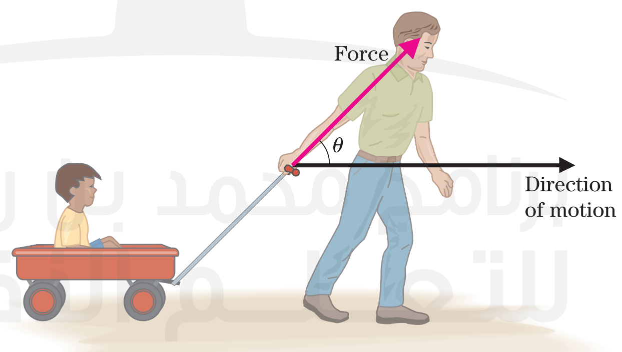


FIGURE 7.25
Pulling a wagon

For any two nonzero position vectors **a** and **b**, let θ be the angle between the vectors. If we drop a perpendicular line segment from the terminal point of **a** to the line containing the vector **b**, then from elementary trigonometry, the base of the triangle (in the case where $0 < \theta < \frac{\pi}{2}$) has length given by $|\mathbf{a}| \cos \theta$. (See Figure 7.26a.)

Today in mathematics

Lene Hau (1959–)
A Danish mathematician and physicist known for her experiments to slow down and stop light. Although neither of her parents had a background in science or mathematics, she says that as a student, "I loved mathematics. I would rather do mathematics than go to the movies in those days. But after awhile, I discovered quantum mechanics and I've been hooked ever since." Hau credits a culture of scientific achievement with her success. "I was lucky to be a Dane. Denmark has a long scientific tradition that included the great Niels Bohr . . . . In Denmark, physics is widely respected by laymen as well as scientists and laymen contribute to physics."

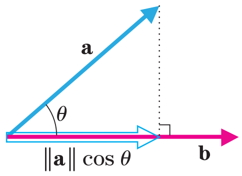


FIGURE 7.26a
$\text{comp}_{\mathbf{b}}\,\mathbf{a}$, for $0 < \theta < \frac{\pi}{2}$

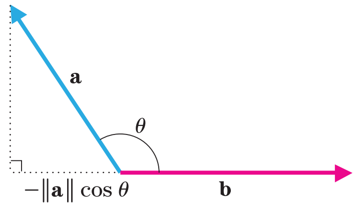


FIGURE 7.26b
$\text{comp}_{\mathbf{b}}\,\mathbf{a}$, for $\frac{\pi}{2} < \theta < \pi$

On the other hand, notice that if $\frac{\pi}{2} < \theta < \pi$, the length of the base is given by $-|\mathbf{a}|\cos\theta$. (See Figure 7.26b) In either case, we refer to $|\mathbf{a}|\cos\theta$ as the **component** of **a** along **b**, denoted $\text{comp}_{\mathbf{b}}\,\mathbf{a}$. Using (7-2), observe that we can rewrite this as

$$\text{comp}_{\mathbf{b}}\,\mathbf{a} = |\mathbf{a}|\cos\theta = \frac{|\mathbf{a}|\,|\mathbf{b}|}{|\mathbf{b}|}\cos\theta$$
$$= \frac{1}{|\mathbf{b}|}|\mathbf{a}|\,|\mathbf{b}|\cos\theta = \frac{1}{|\mathbf{b}|}\mathbf{a}\cdot\mathbf{b}$$

or Component of a along b

$$\boxed{\text{comp}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}.} \tag{7-7}$$

Notice that $\text{comp}_{\mathbf{b}}\,\mathbf{a}$ is a scalar and that we divide the dot product in (7-7) by $|\mathbf{b}|$ and not by $|\mathbf{a}|$. One way to keep this straight is to recognize that the components in Figures 7.26a and 7.26b depend on how long **a** is but not on how long **b** is. We can view (7-7) as the dot product of the vector **a** and a unit vector in the direction of **b**, given by $\frac{\mathbf{b}}{|\mathbf{b}|}$.

Once again, consider the case where the vector **a** represents a force. Rather than the component of **a** along **b**, we are often interested in finding a force vector parallel to **b** having the same component along **b** as **a**.We call this vector the **projection** of **a** onto **b**, denoted $\textbf{proj}_{\mathbf{b}}\,\mathbf{a}$, as indicated in Figures 7.27a and 7.27b. Since the projection has magnitude $|\text{comp}_{\mathbf{b}}\,\mathbf{a}|$ and points in the direction of **b**, for $0 < \theta < \frac{\pi}{2}$ and opposite **b**, for $\frac{\pi}{2} < \theta < \pi$, we have from (6.7) that

$$\textbf{proj}_{\mathbf{b}}\,\mathbf{a} = (\text{comp}_{\mathbf{b}}\,\mathbf{a})\frac{\mathbf{b}}{|\mathbf{b}|} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|},$$

or Projection of a onto b

$$\boxed{\textbf{proj}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b},} \tag{7-8}$$

where $\frac{\mathbf{b}}{|\mathbf{b}|}$ represents a unit vector in the direction of **b**.

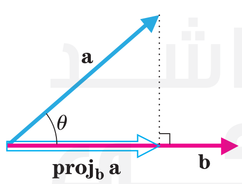


FIGURE 7.27a
$\textbf{proj}_{\mathbf{b}}\,\mathbf{a}$, for $0 < \theta < \frac{\pi}{2}$

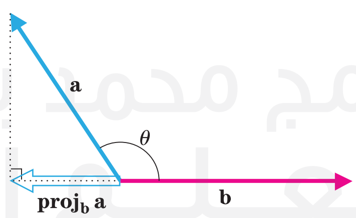


FIGURE 7.27b
$\textbf{proj}_{\mathbf{b}}\,\mathbf{a}$, for $\frac{\pi}{2} < \theta < \pi$

Caution

Be careful to distinguish between the *projection* of **a** onto **b** (a vector) and the *component* of **a** along **b** (a scalar). It is very common to confuse the two.

In example 7-5, we illustrate the process of finding components and projections.

Example 7-5 Finding Components and Projections

For $\mathbf{a} = \langle 2, 3\rangle$ and $\mathbf{b} = \langle -1, 5\rangle$, find the component of **a** along **b** and the projection of **a** onto **b**.

Solution From (7-7), we have

$$\text{comp}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \frac{\langle 2,3\rangle\cdot\langle -1,5\rangle}{|\langle -1,5\rangle|} = \frac{-2+15}{\sqrt{1+5^2}} = \frac{13}{\sqrt{26}}.$$

Similarly, from (7-8), we have

$$\begin{aligned}\textbf{proj}_{\mathbf{b}}\,\mathbf{a} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|} = \left(\frac{13}{\sqrt{26}}\right)\frac{\langle -1,5\rangle}{\sqrt{26}}\\ &= \frac{13}{\sqrt{26}}\langle -1,5\rangle = \frac{1}{2}\langle -1,5\rangle = \left\langle -\frac{1}{2},\frac{5}{2}\right\rangle.\end{aligned}$$

We leave it as an exercise to show that, in general, $\text{comp}_{\mathbf{b}}\,\mathbf{a} \neq \text{comp}_{\mathbf{a}}\,\mathbf{b}$ and $\textbf{proj}_{\mathbf{b}}\,\mathbf{a} \neq \textbf{proj}_{\mathbf{a}}\,\mathbf{b}$. One reason for needing to consider components of a vector in a given direction is to compute work, as we see in example 7-6.

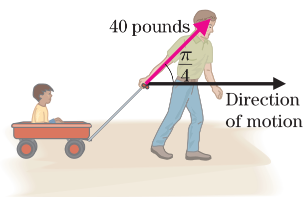


FIGURE 7.28
Pulling a wagon

Example 7-6 Calculating Work

You exert a constant force of 180 Newtons in the direction of the handle of the wagon pictured in Figure 7.28. If the handle makes an angle of $\frac{\pi}{4}$ with the horizontal and you pull the wagon along a flat surface for 1.6 kilometers 1,609 meters), find the work done.

Solution First, recall from our discussion in Chapter 5 that if we apply a constant force F for a distance d, the work done is given by $W = Fd$. In this case, the force exerted in the direction of motion is not given. However, since the magnitude of the force is, the force vector must be

$$F = 180\left\langle \cos\frac{\pi}{4},\ \sin\frac{\pi}{4}\right\rangle = 180\left\langle \frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right\rangle = \langle 90\sqrt{2}, 90\sqrt{2}\rangle.$$

The force exerted in the direction of motion is simply the component of the force along the vector **i** (that is, the horizontal component of **F**) or $90\sqrt{2}$. The work done is then

$$W = Fd = 90\sqrt{2}(1609) \approx 203{,}632 \text{ Newton-meter.}$$

More generally, if a constant force **F** moves an object from point P to point Q, we refer to the vector $\mathbf{d} = \overrightarrow{PQ}$ as the **displacement vector**. The work done is the product of the component of **F** along **d** and the distance:

$$\begin{aligned}W &= \text{comp}_{\mathbf{d}}\,\mathbf{F}\,|\mathbf{d}|\\ &= \frac{\mathbf{F}\cdot\mathbf{d}}{|\mathbf{d}|}\cdot|\mathbf{d}| = \mathbf{F}\cdot\mathbf{d}.\end{aligned}$$

Here, this gives us

$$W = \langle 90\sqrt{2}, 90\sqrt{2}\rangle\cdot\langle 1609, 0\rangle = 90\sqrt{2}(1609), \text{ as before.}$$

BEYOND FORMULAS The dot product gives us a shortcut for computing components and projections. The dot product test for perpendicular vectors follows directly from this interpretation. In general, components and projections are used to isolate a particular portion of a larger problem for detailed analysis. This sort of reductionism is central to much of modern science.

Definition 7-2

The **determinant** of a 2×2 matrix of real numbers is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

Example 7-7 Computing a 2 × 2 Determinant

Evaluate the determinant $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

Solution From (7-2), we have

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2.$$

Definition 7-3

The **determinant** of a 3×3 matrix of real numbers is defined as a combination of three 2×2 determinants, as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \quad (7\text{-}3)$$

Equation (7-3) is referred to as an **expansion** of the determinant **along the first row.** Notice that the multipliers of each of the 2×2 determinants are the entries of the first row of the 3×3 matrix. Each 2×2 determinant is the determinant you get if you eliminate the row and column in which the corresponding multiplier lies. That is, for the *first* term, the multiplier is a_1 and the 2×2 determinant is found by eliminating the first row and *first* column from the 3×3 matrix:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}.$$

Likewise, the *second* 2 × 2 determinant is found by eliminating the first row and the *second* column from the 3 × 3 determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}.$$

Be certain to notice the minus sign in front of this term. Finally, the *third* determinant is found by eliminating the first row and the *third* column from the 3 × 3 determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Example 7-8 Evaluating a 3 × 3 Determinant

Evaluate the determinant $\begin{vmatrix} 1 & 2 & 4 \\ -3 & 3 & 1 \\ 3 & -2 & 5 \end{vmatrix}$.

Solution Expanding along the first row, we have:

$$\begin{vmatrix} 1 & 2 & 4 \\ -3 & 3 & 1 \\ 3 & -2 & 5 \end{vmatrix} = (1)\begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix} - (2)\begin{vmatrix} -3 & 1 \\ 3 & 5 \end{vmatrix} + (4)\begin{vmatrix} -3 & 3 \\ 3 & -2 \end{vmatrix}$$

$$= (1)[(3)(5) - (1)(-2)] - (2)[(-3)(5) - (1)(3)]$$

$$+ (4)[(-3)(-2) - (3)(3)]$$

$$= 41.$$

We use determinant notation as a convenient device for defining the cross product, as follows.

Definition 7-4

For two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in V_3, we define the **cross product** (or **vector product**) of **a** and **b** to be

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_2 & a_2 \\ b_2 & b_2 \end{vmatrix}\mathbf{k}. \qquad (7\text{-}4)$$

Notice that $\mathbf{a} \times \mathbf{b}$ is also a vector in V_3. To compute $\mathbf{a} \times \mathbf{b}$, you must write the components of **a** in the second row and the components of **b** in the third row; *the order is important!* Also note that while we've used the determinant notation, the 3 × 3 determinant indicated in (7-4) is not really a determinant, in the sense in which we defined them, since the entries in the first row are vectors instead of scalars. Nonetheless, we find this slight abuse of notation convenient for computing cross products and we use it routinely.

Example 7-9 Computing a Cross Product

Compute $\langle 1, 2, 3\rangle \times \langle 4, 5, 6\rangle$.

Solution From (7-10), we have

$$\langle 1,2,3\rangle \times \langle 4,5,6\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}\mathbf{k}$$

$$= -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} = \langle -3, 6, -3\rangle.$$

Theorem 7-5

For any vector $\mathbf{a} \in V_3$, $\mathbf{a} \times \mathbf{a} = 0$ and $\mathbf{a} \times 0 = 0$.

Remark 7-2

The cross product is defined only for vectors in V_3. There is no corresponding operation for vectors in V_2.

Proof

We prove the first of these two results. The second, we leave as an exercise. For $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, we have from (7-3) that

$$\mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix}\mathbf{k}$$

$$= (a_2a_3 - a_3a_2)\mathbf{i} - (a_1a_3 - a_3a_1)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k} = 0.$$

Let's take a brief look back at the result of example 7-3. There, we saw that

$$\langle 1,2,3\rangle \times \langle 4,5,6\rangle = \langle -3, 6, -3\rangle.$$

There is something rather interesting to observe here. Note that

$$\langle 1,2,3\rangle \cdot \langle -3,6,-3\rangle = 0$$

and

$$\langle 4,5,6\rangle \cdot \langle -3,6,-3\rangle = 0.$$

That is, both $\langle 1, 2, 3\rangle$ and $\langle 4, 5, 6\rangle$ are orthogonal to their cross product. As it turns out, this is true in general, as we see in Theorem 7-6.

Theorem 7-6

For any vectors **a** and **b** in V_3, $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**.

Proof

Recall that two vectors are orthogonal if and only if their dot product is zero. Now, using (7-3), we have

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \langle a_1, a_2, a_3\rangle \cdot \left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}\right]$$

$$= a_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= a_1[a_2b_3 - a_3b_2] - a_2[a_1b_3 - a_3b_1] + a_3[a_1b_2 - a_2b_1]$$

$$= a_1a_2b_3 - a_1a_3b_2 - a_1a_2b_3 + a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_1$$

$$= 0,$$

so that **a** and $(\mathbf{a} \times \mathbf{b})$ are orthogonal. We leave it as an exercise to show that $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$, also.

Historical Notes

Josiah Willard Gibbs (1839–1903) American physicist and mathematician who introduced and named the dot product and the cross product. A graduate of Yale, Gibbs published important papers in thermodynamics, statistical mechanics and the electromagnetic theory of light. Gibbs used vectors to determine the orbit of a comet from only three observations. Originally produced as printed notes for his students, Gibbs' vector system greatly simplified the original system developed by Hamilton. Gibbs was well liked but not famous in his lifetime. One biographer wrote of Gibbs that, "The greatness of his intellectual achievements will never overshadow the beauty and dignity of his life."

Notice that for nonzero and nonparallel vectors **a** and **b**, since $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**, it is also orthogonal to every vector lying in the plane containing **a** and **b**. (We also say that $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane, in this case.) But, given a plane, out of which side of the plane does $\mathbf{a} \times \mathbf{b}$ point? We can get an idea by computing some simple cross products.

Notice that

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k} = \mathbf{k}.$$

Likewise,

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}.$$

These are illustrations of the **right-hand rule**: If you align the fingers of your *right* hand along the vector **a** and bend your fingers around in the direction of rotation from **a** toward **b** (through an angle of less than 180°), your thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$, as in Figure 7.29a. Now, following the right-hand rule, $\mathbf{b} \times \mathbf{a}$ will point in the direction opposite $\mathbf{a} \times \mathbf{b}$. (See Figure 7.29b.) In particular, notice that

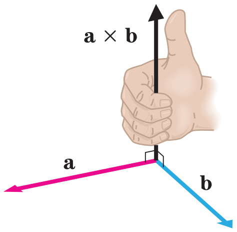


FIGURE 7.29a
$\mathbf{a} \times \mathbf{b}$

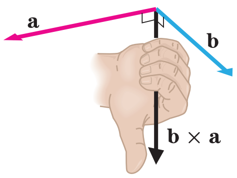


FIGURE 7.29b
$\mathbf{b} \times \mathbf{a}$

$$\mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}.$$

We leave it as an exercise to show that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i},$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j}, \quad \text{and} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

Take the time to think through the right-hand rule for each of these cross products.

There are several other unusual things to observe here. Notice that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \neq -\mathbf{k} = \mathbf{j} \times \mathbf{i},$$

which says that the cross product is *not* commutative. Further, notice that

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i},$$

while

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0},$$

so that the cross product is also *not* associative. That is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$$

Since the cross product does not follow several of the rules you might expect a product to satisfy, you might ask what rules the cross product *does* satisfy. We summarize these in Theorem 7-7.

Theorem 7-7

For any vectors **a**, **b** and **c** in V_3 and any scalar d, the following hold:

(i)	$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$	(anticommutativity)
(ii)	$(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$	
(iii)	$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$	(distributive law)
(iv)	$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$	(distributive law)
(v)	$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$	(scalar triple product) and
(vi)	$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$	(vector triple product).

Proof

We prove parts (i) and (iii) only. The remaining parts are left as exercises.

(i) For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, we have from (7-10) that

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_2 & a_2 \\ b_2 & b_2 \end{vmatrix} \mathbf{k}$$

$$= -\begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} b_1 & b_3 \\ a_1 & a_3 \end{vmatrix} \mathbf{j} - \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} \mathbf{k} = -(\mathbf{b} \times \mathbf{a}),$$

since swapping two rows in a 2 × 2 matrix (or in a 3 × 3 matrix, for that matter) changes the sign of its determinant.

(iii) For $c = \langle c_1, c_2, c_3 \rangle$, we have

$$\mathbf{b} + \mathbf{c} = \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$

and so,
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}.$$

Looking only at the **i** component of this, we have

$$\begin{vmatrix} a_2 & a_3 \\ b_2 + c_2 & b_3 + c_3 \end{vmatrix} = a_2(b_3 + c_3) - a_3(b_2 + c_2)$$

$$= (a_2 b_3 - a_3 b_2) + (a_2 c_3 - a_3 c_2)$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix},$$

which you should note is also the i component of $\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$. Similarly, you can show that the **j** and **k** components also match, which establishes the result.

Always keep in mind that vectors are specified by two things: magnitude and direction. We have already shown that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**. In Theorem 7-8, we make a general (and quite useful) statement about $|\mathbf{a} \times \mathbf{b}|$.

Theorem 7-8

For nonzero vectors **a** and **b** in V_3, if θ is the angle between **a** and **b** ($0 \le \theta \le \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta. \tag{7-8}$$

Proof

From (7-3), we get

$$|\mathbf{a} \times \mathbf{b}|^2 = [a_2 b_3 - a_3 b_2]^2 + [a_1 b_3 - a_3 b_1]^2 + [a_1 b_2 - a_2 b_1]^2$$

$$= a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + a_1^2 b_3^2 - 2a_1 a_3 b_1 b_3 + a_3^2 b_1^2 + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2$$

$$= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2\theta$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2\theta)$$

$$= |a|^2 |b|^2 \sin^2\theta.$$

Taking square roots, we get

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta,$$

since $\sin\theta \ge 0$, for $0 \le \theta \le \pi$.

The following characterization of parallel vectors is an immediate consequence of Theorem 7-8.

Corollary 7-2

Two nonzero vectors $\mathbf{a}, \mathbf{b} \in V_3$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Proof

Recall that $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if the angle θ between them is either 0 or π. In either case, $\sin\theta = 0$ and so, by Theorem 7-8,

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta = |\mathbf{a}| \times |\mathbf{b}|\,(0) = 0.$$

The result then follows from the fact that the only vector with zero magnitude is the zero vector.

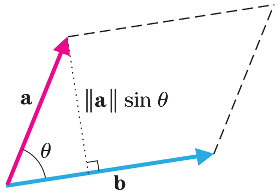


FIGURE 7.30 Parallelogram

Theorem 7-8 also provides us with the following interesting geometric interpretation of the cross product. For any two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, as long as $\mathbf{a}$ and $\mathbf{b}$ are not parallel, they form two adjacent sides of a parallelogram, as seen in Figure 7.30. Notice that the area of the parallelogram is given by the product of the base and the altitude. We have

$$\text{Area} = (\text{base})(\text{altitude})$$

$$= |\mathbf{b}|\ |\mathbf{a}| \sin\theta = |\mathbf{a} \times \mathbf{b}|, \qquad (7\text{-}5)$$

from Theorem 7-8. That is, the magnitude of the cross product of two vectors gives the area of the parallelogram with two adjacent sides formed by the vectors.

Example 7-10 Finding the Area of a Parallelogram Using the Cross Product

Find the area of the parallelogram with two adjacent sides formed by the vectors $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 4, 5, 6\rangle$.

Solution First notice that

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = \langle -3, 6, -3\rangle$$

From (7-5), the area of the parallelogram is given by

$$|\mathrm{a} \times \mathrm{b}| = |\langle -3, 6, -3\rangle| = \sqrt{54} \approx 7.348.$$

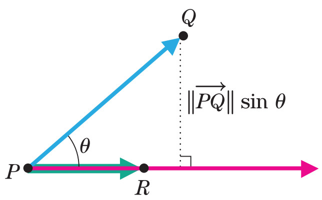


FIGURE 7.31 Distance from a point to a line

We can also use Theorem 7-8 to find the distance from a point to a line in $\mathbb{R}^3$, as follows. Let d represent the distance from the point Q to the line through the points P and R. From elementary trigonometry, we have that

$$d = \left|\overrightarrow{PQ}\right| \sin\theta,$$

where θ is the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. (See Figure 7.6.9.) From (7.12), we have

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \left|\overrightarrow{PQ}\right| \left|\overrightarrow{PR}\right| \sin\theta = \left|\overrightarrow{PR}\right| (d).$$

Solving this for d, we get

$$d = \frac{\left|\overrightarrow{PQ}\right| \times \left|\overrightarrow{PR}\right|}{\left|\overrightarrow{PR}\right|}. \qquad (7\text{-}6)$$

Example 7-11 Finding the Distance from a Point to a Line

Find the distance from the point $Q(1, 2, 1)$ to the line through the points $P(2, 1, -3)$ and $R(2, -1, 3)$.

Solution First, the position vectors corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \langle -1, 1, 4 \rangle \quad \text{and} \quad \overrightarrow{PR} = \langle 0, -2, 6 \rangle,$$

and
$$\langle -1, 1, 4 \rangle \times \langle 0, -2, 6 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 4 \\ 0 & -2 & 6 \end{vmatrix} = \langle 14, 6, 2 \rangle.$$

We then have from (7-14) that

$$d = \frac{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}{\overrightarrow{PR}} = \frac{|\langle 14, 6, 2 \rangle|}{|\langle 0, -2, 6 \rangle|} = \frac{\sqrt{236}}{\sqrt{40}} \approx 2.429.$$

For any three nonzero and noncoplanar vectors **a**, **b** and **c** (i.e., three vectors that do not lie in a single plane), consider the parallelepiped formed using the vectors as three adjacent edges. (See Figure 7.32.) Recall that the volume of such a solid is given by

$$\text{Volume} = (\text{Area of base})(\text{altitude}).$$

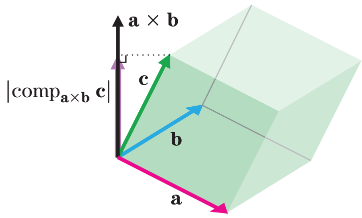


FIGURE 7.32
Parallelepiped formed by the vectors a, b and c

Further, since two adjacent sides of the base are formed by the vectors **a** and **b**, we know that the area of the base is given by $|\mathbf{a} \times \mathbf{b}|$. Referring to Figure 7.32, notice that the altitude is given by

$$|\text{comp}_{\mathbf{a}\times\mathbf{b}}\,\mathbf{c}| = \frac{|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|}{|\mathbf{a} \times \mathbf{b}|},$$

from (7-3). The volume of the parallelepiped is then

$$\text{Volume} = |\mathbf{a} \times \mathbf{b}| \frac{|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|}{|\mathbf{a} \times \mathbf{b}|} = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|.$$

The scalar $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ is called the **scalar triple product** of the vectors **a**, **b** and **c**. It turns out that we can evaluate the scalar triple product by computing a single determinant, as follows. Note that for $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, we have

$$\begin{aligned} \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{c} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \langle c_1, c_2, c_3 \rangle \cdot \left(\mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \\ &= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \end{aligned} \qquad (7\text{-}7)$$

Example 7-12 Finding the Volume of a Parallelepiped Using the Cross Product

Find the volume of the parallelepiped with three adjacent edges formed by the vectors $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle 4, 5, 6 \rangle$ and $\mathbf{c} = \langle 7, 8, 0 \rangle$.

Solution First, note that Volume $= |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$. From (7-4), we have that

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} 7 & 8 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 7(-3) - 8(-6) = 27.$$

So, the volume of the parallelepiped is Volume $= |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| = |27| = 27$.

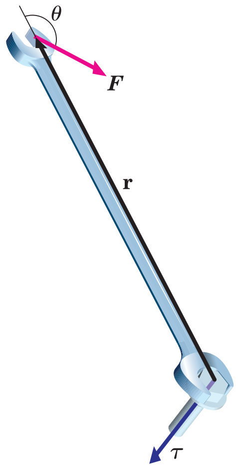


FIGURE 7.33
Torque, τ

Consider the action of a wrench on a bolt, as shown in Figure 7.33. In order to tighten the bolt, we apply a force **F** at the end of the handle, in the direction indicated in the figure. This force creates a **torque $\boldsymbol{\tau}$** acting along the axis of the bolt, drawing it in tight. Notice that the torque acts in the direction perpendicular to both **F** and the position vector **r** for the handle as indicated in Figure 7.6.11. In fact, using the right-hand rule, the torque acts in the same direction as $\mathbf{r} \times \mathbf{F}$ and physicists define the torque vector to be

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

In particular, this says that

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}|\sin\theta, \tag{7-8}$$

from (7-4). There are several observations we can make from this. First, this says that the farther away from the axis of the bolt we apply the force (i.e., the larger $|\mathbf{r}|$ is), the greater the magnitude of the torque. So, a longer wrench produces a greater torque, for a given amount of force applied. Second, notice that $\sin\theta$ is maximized when $\theta = \frac{\pi}{2}$, so that from (7-8) the magnitude of the torque is maximized when $\theta = \frac{\pi}{2}$ (when the force vector **F** is orthogonal to the position vector **r**). If you've ever spent any time using a wrench, this should fit well with your experience.

Example 7-13 Finding the Torque Applied by a Wrench

If you apply a force of magnitude 110 Newtons at the end of a 40 centimeter-long wrench, at an angle of $\frac{\pi}{3}$ to the wrench, find the magnitude of the torque applied to the bolt. What is the maximum torque that a force of 110 Newtons applied at that point can produce?

Solution From (7-8), we have

$$|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}|\,\sin\theta = \sin\frac{\pi}{3}$$

$$= (40)\frac{\sqrt{3}}{2} \approx 3810.5 \text{ centimeter-Newton.}$$

Further, the maximum torque is obtained when the angle between the wrench and the force vector is $\frac{\pi}{2}$. This would give us a maximum torque of

$$|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}|\,\sin\theta = (40)\,110\,(1) = 4,400 \text{ centimeter-Newton.}$$

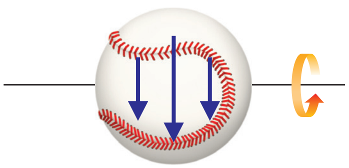


FIGURE 7.34
Spinning ball

In many sports, the action is at least partially influenced by the motion of a spinning ball. For instance, in baseball, batters must contend with pitchers' curveballs and in golf, players try to control their slice. In tennis, players hit shots with topspin, while in basketball, players improve their shooting by using backspin. The list goes on and on. These are all examples of the **Magnus force**, which we describe below.

Suppose that a ball is spinning with angular velocity ω, measured in radians per second (i.e., ω is the rate of change of the rotational angle). The ball spins about an axis, as shown in Figure 7.34. We define the spin vector **s** to have magnitude ω and direction parallel to the spin axis. We use a right-hand rule to distinguish between the two directions parallel to the spin axis: curl the fingers of your right hand around the ball in the direction of the spin, and your thumb will point in the correct direction. Two examples are shown in Figures 7.35a and 7.35b. The motion of the ball disturbs the air through which it travels, creating a Magnus force $\mathbf{F}_m$ acting on the ball. For a ball moving with velocity **v** and spin vector **s**, $\mathbf{F}_m$ is given by

$$\mathbf{F}_m = c(\mathbf{s} \times \mathbf{v}),$$

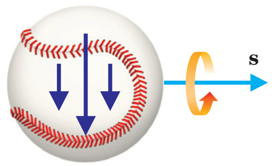


FIGURE 7.35a
Backspin

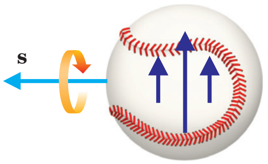


FIGURE 7.35b
Topspin

for some positive constant c. Suppose the balls in Figure 7.35a and Figure 7.35b are moving into the page and away from you. Using the usual sports terminology, the first ball has backspin and the second ball has topspin. Using the right-hand rule, we see that the Magnus force acting on the first ball acts in the upward direction, as indicated in Figure 7.36a. This says that backspin (for example, on a basketball or golf shot) produces an upward force that helps the ball land more softly than a ball with no spin. Similarly, the Magnus force acting on the second ball acts in the downward direction (see Figure 7.36b), so that topspin (for example, on a tennis shot or baseball hit) produces a downward force that causes the ball to drop to the ground more quickly than a ball with no spin.

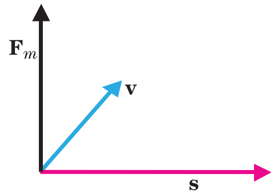


FIGURE 7.36a
Magnus force for a ball with backspin

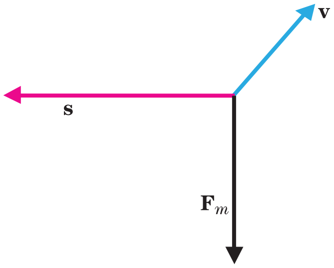


FIGURE 7.36b
Magnus force for a ball with topspin

Example 7-14 Finding the Direction of a Magnus Force

The balls shown in Figures 7.37a and 7.37b are moving into the page and away from you with spin as indicated. The first ball represents a right-handed baseball pitcher's curveball, while the second ball represents a right-handed golfer's shot. Determine the direction of the Magnus force and discuss the effects on the ball.

Solution For the first ball, notice that the spin vector points up and to the left, so that $\mathbf{s} \times \mathbf{v}$ points down and to the left as shown in Figure 7.38a. Such a ball will curve to the left and drop faster than a ball that is not spinning, making it more difficult to hit. For the second ball, the spin vector points down and to the right, so $\mathbf{s} \times \mathbf{v}$ points up and to the right. Such a ball will move to the right (a "slice") and stay in the air longer than a ball that is not spinning. (See Figure 7.38b.)

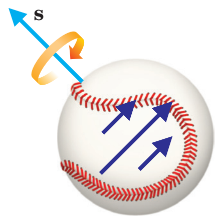


FIGURE 7.37a
Right-hand curveball

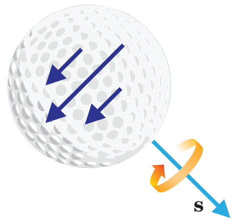


FIGURE 7.37b
Right-hand golf shot

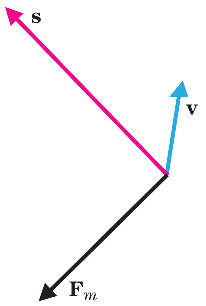


FIGURE 7.38a
Magnus force for a right-handed curveball

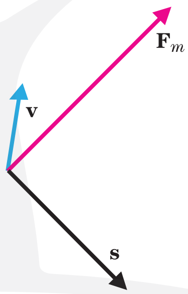


FIGURE 7.38b
Magnus force for a right-handed golf shot

Exercises 7-6-1

WRITING EXERCISES

1. Explain in words why the Triangle Inequality is true.
2. The dot product is called a "product" because the properties listed in Theorem 7-1 are true for multiplication of real numbers. Two other properties of multiplication of real numbers involve factoring: (1) if $ab = ac$ $(a \neq 0)$ then $b = c$ and (2) if $ab = 0$ then $a = 0$ or $b = 0$. Discuss the extent to which these properties are true for the dot product.
3. To understand the importance of unit vectors, identify the simplification in formulas for finding the angle between vectors and for finding the component of a vector, if the vectors are unit vectors. There is also a theoretical benefit to using unit vectors. Compare the number of vectors in a particular direction to the number of unit vectors in that direction. (For this reason, unit vectors are sometimes called **direction vectors**.)
4. It is important to understand why work is computed using only the component of force in the direction of motion. Suppose you push on a door to close it. If you are pushing on the edge of the door straight at the door hinges, are you accomplishing anything useful? In this case, the work done would be zero. If you change the angle at which you push very slightly, what happens? As the angle increases, discuss how the component of force in the direction of motion changes and how the work done changes.

In exercises 1–6, compute a · b.

1. $\mathbf{a} = \langle 3, 1\rangle$, $\mathbf{b} = \langle 2, 4\rangle$
2. $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j}$
3. $\mathbf{a} = \langle 2, -1, 3\rangle$, $\mathbf{b} = \langle 0, 2, -4\rangle$
4. $\mathbf{a} = \langle 3, 2, 0\rangle$, $\mathbf{b} = \langle -2, 4, 3\rangle$
5. $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{j} - \mathbf{k}$
6. $\mathbf{a} = 3\mathbf{i} + 3\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j}$

In exercises 7–10, compute the angle between the vectors.

7. $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$
8. $\mathbf{a} = \langle 2, 0, -2\rangle$, $\mathbf{b} = \langle 0, -2, 4\rangle$
9. $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
10. $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{k}$

In exercises 11–14, determine whether the vectors are orthogonal.

11. $\mathbf{a} = \langle 2, -1\rangle$, $\mathbf{b} = \langle 2, 4\rangle$
12. $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$
13. $\mathbf{a} = 3\mathbf{i}$, $\mathbf{b} = 6\mathbf{j} - 2\mathbf{k}$
14. $\mathbf{a} = \langle 4, -1, 1\rangle$, $\mathbf{b} = \langle 2, 4, 4\rangle$

In exercises 15–18, (a) find a 3-dimensional vector perpendicular to the given vector and (b) find a vector of the form $\langle a, 2, -3\rangle$ that is perpendicular to the given vector.

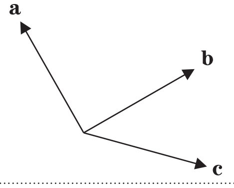
15. $\langle 2, -1, 0\rangle$
16. $\langle 4, -1, 1\rangle$
17. $6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
18. $2\mathbf{i} - 3\mathbf{k}$

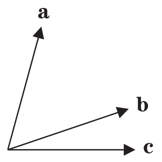
In exercises 19–24, find $\text{comp}_{\mathbf{b}}\,\mathbf{a}$ and $\text{proj}_{\mathbf{b}}\,\mathbf{a}$.

19. $\mathbf{a} = \langle 2, 1\rangle$, $\mathbf{b} = \langle 3, 4\rangle$
20. $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 3\mathbf{j}$
21. $\mathbf{a} = \langle 2, -1, 3\rangle$, $\mathbf{b} = \langle 1, 2, 2\rangle$
22. $\mathbf{a} = \langle 1, 4, 5\rangle$, $\mathbf{b} = \langle -2, 1, 2\rangle$
23. $\mathbf{a} = \langle 2, 0, -2\rangle$, $\mathbf{b} = \langle 0, -3, 4\rangle$
24. $\mathbf{a} = \langle 3, 2, 0\rangle$, $\mathbf{b} = \langle -2, 2, 1\rangle$

25. Repeat example 7-6 with an angle of $\frac{\pi}{3}$ with the horizontal.
26. Repeat example 7-6 with an angle of $\frac{\pi}{6}$ with the horizontal.
27. Explain why the answers to exercises 25 and 26 aren't the same, even though the force exerted is the same. In this setting, explain why a larger amount of work corresponds to a more efficient use of the force.
28. Find the force needed in exercise 25 to produce the same amount of work as in example 7-6.
29. A constant force of $\langle 30, 20\rangle$ kilograms moves an object in a straight line from the point (0, 0) to the point (24, 10). Compute the work done.
30. A constant force of $\langle 60, -30\rangle$ kilograms moves an object in a straight line from the point (0, 0) to the point (10, −10). Compute the work done.
31. Label each statement as true or false. If it is true, briefly explain why; if it is false, give a counterexample.
    - **a.** If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.
    - **b.** If $\mathbf{b} = \mathbf{c}$, then $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$.
    - **c.** $|\mathbf{a} \cdot \mathbf{a}| = |\mathbf{a}|^2$.
    - **d.** If $|\mathbf{a}| > |\mathbf{b}|$ then $\mathbf{a} \cdot \mathbf{c} > \mathbf{b} \cdot \mathbf{c}$.
    - **e.** If $|\mathbf{a}| = |\mathbf{b}|$ then $\mathbf{a} = \mathbf{b}$.
32. To compute $\mathbf{a} \cdot \mathbf{b}$, where $\mathbf{a} = \langle 2, 5\rangle$ and $\mathbf{b} = \frac{\langle 4, 1\rangle}{\sqrt{17}}$, You can first compute $\langle 2, 5\rangle \cdot \langle 4, 1\rangle$ and then divide the result (13) by $\sqrt{17}$. Which property stated in Theorem 6.1 is being used?

In exercises 33 and 34, use the figure to sequence a · b, a · c and b · c in increasing order.

33.


34.


35. If $\mathbf{a} = \langle 2, 1\rangle$, find a vector $\mathbf{b}$ such that (a) $\text{comp}_{\mathbf{b}}\, \mathbf{a} = 1$; (b) $\text{comp}_{\mathbf{a}}\, \mathbf{b} = -1$.

36. If $\mathbf{a} = \langle 4, -2\rangle$, find a vector $\mathbf{b}$ such that (a) $\text{proj}_{\mathbf{b}}\, \mathbf{a} = \langle 4, 0\rangle$; (b) $\text{proj}_{\mathbf{a}}\, \mathbf{b} = \langle 4, -2\rangle$.

37. Find the angles in the triangle with vertices (1, 2, 0), (3, 0, −1) and (1, 1, 1).

38. Find the angles in the quadrilateral $ABCD$ with vertices $A = (2, 0, 1)$, $B = (2, 1, 4)$, $C = (4, -2, 5)$ and $D = (4, 0, 2)$.

39. The distance from a point P to a line L is the length of the line segment connecting P to L at a right angle. Show that the distance from (x_1, y_1) to the line $ax + by + c = 0$ equals $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

40. Prove that the distance between lines $ax + by + c = 0$ and $ax + by + d = 0$ equals $\dfrac{|d - c|}{\sqrt{a^2 + b^2}}$.

41. (a) Find the angle between the diagonal of a square and an adjacent side. (b) Find the angle between the diagonal of a cube and an adjacent side. (c) Extend the results of parts (a) and (b) to a hypercube of dimension $n \geq 4$.

42. Prove that $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$. State this result in terms of properties of the parallelogram formed by vectors $\mathbf{a}$ and $\mathbf{b}$.

Exercises 43–53 involve the Cauchy-Schwartz and Triangle Inequalities.

43. By the Cauchy-Schwartz Inequality, $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. What relationship must exist between $\mathbf{a}$ and $\mathbf{b}$ to have $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$?

44. By the Triangle Inequality, $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. What relationship must exist between $\mathbf{a}$ and $\mathbf{b}$ to have $|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$?

45. Use the Triangle Inequality to prove that $|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$.

46. For vectors $\mathbf{a}$ and $\mathbf{b}$, use the Cauchy-Schwartz Inequality to find the maximum value of $\mathbf{a} \cdot \mathbf{b}$ if $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 5$.

47. Find a formula for $\mathbf{a}$ in terms of $\mathbf{b}$ if $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and $\mathbf{a} \cdot \mathbf{b}$ is maximum.

48. Use the Cauchy-Schwartz Inequality in n dimensions to show that $\left(\sum_{k=1}^{n} |a_k b_k|\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$. If both $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ converge, what can be concluded? Apply the result to $a_k = \frac{1}{k}$ and $b_k = \frac{1}{k^2}$.

49. Show that $\sum_{k=1}^{n} |a_k b_k| \leq \frac{1}{2}\sum_{k=1}^{n} a_k^2 + \frac{1}{2}\sum_{k=1}^{n} b_k^2$. If both $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ converge, what can be concluded? Apply the result to $a_k = \frac{1}{k}$ and $b_k = \frac{1}{k^2}$. Is this bound better or worse than the bound found in exercise 48?

50. (a) Use the Cauchy-Schwartz Inequality in n dimensions to show that $\sum_{k=1}^{n} |a_k| \leq \sqrt{n}\left(\sum_{k=1}^{n} a_k^2\right)^{1/2}$. (b) If $p_1, p_2, \ldots, p_n$ are nonnegative numbers that sum to 1, show that $\sum_{k=1}^{n} p_k^2 \geq \frac{1}{n}$.
(c) Among all sets of nonnegative numbers $p_1, p_2, \ldots, p_n$ that sum to 1, find the choice of $p_1, p_2, \ldots, p_n$ that minimizes $\sum_{k=1}^{n} p_k^2$.

51. Use the Cauchy-Schwartz Inequality in n dimensions to show that $\sum_{k=1}^{n} |a_k| \leq \left(\sum_{k=1}^{n} |a_k|^{2/3}\right)^{1/2}\left(\sum_{k=1}^{n} |a_k|^{4/3}\right)^{1/2}$.

52. Show that $\sum_{k=1}^{n} a_k^2 b_k^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$ and then $\left(\sum_{k=1}^{n} a_k b_k c_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)\left(\sum_{k=1}^{n} c_k^2\right)$.

53. Show that $\sqrt{\dfrac{x+y}{x+y+z}} + \sqrt{\dfrac{y+z}{x+y+z}} + \sqrt{\dfrac{x+z}{x+y+z}} \leq \sqrt{6}$.

54. Prove that $\text{comp}_{\mathbf{c}}(\mathbf{a} + \mathbf{b}) = \text{comp}_{\mathbf{c}}\, \mathbf{a} + \text{comp}_{\mathbf{c}}\, \mathbf{b}$ for any nonzero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

55. The **orthogonal projection** of vector $\mathbf{a}$ along vector $\mathbf{b}$ is defined as $\mathbf{orth_b\, a} = \mathbf{a} - \mathbf{proj_b\, a}$. Sketch a picture showing vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{proj_b\, a}$ and $\mathbf{orth_b\, a}$, and explain what is orthogonal about $\mathbf{orth_b\, a}$.

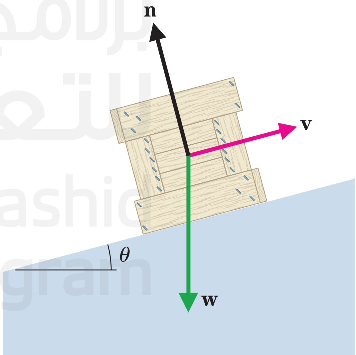
56. Write the given vector as $\mathbf{a} + \mathbf{b}$, where $\mathbf{a}$ is parallel to $\langle 1, 2, 3\rangle$ and $\mathbf{b}$ is perpendicular to $\langle 1, 2, 3\rangle$, for (a) $\langle 3, -1, 2\rangle$ and (b) $\langle 0, 4, 2\rangle$.

Applications

57. In a methane molecule (CH_4), a carbon atom is surrounded by four hydrogen atoms. Assume that the hydrogen atoms are at (0, 0, 0), (1, 1, 0), (1, 0, 1) and (0, 1, 1) and the carbon atom is at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Compute the **bond angle,** the angle from hydrogen atom to carbon atom to hydrogen atom.

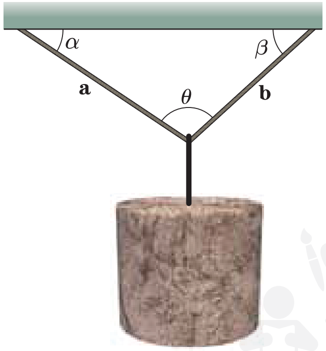
58. Suppose that a beam of an oil rig is installed in a direction parallel to $\langle 10, 1, 5\rangle$. (a) If a wave exerts a force of $\langle 0, -200, 0\rangle$ newtons, find the component of this force along the beam. (b) Repeat with a force of $\langle 13, -190, -61\rangle$ newtons. The forces in parts (a) and (b) have nearly identical magnitudes. Explain why the force components are different.

59. In the diagram, a crate of weight w kilograms is placed on a ramp inclined at angle θ above the horizontal. The vector $\mathbf{v}$ along the ramp is given by $\mathbf{v} = \langle \cos\theta, \sin\theta\rangle$ and the **normal** vector by $\mathbf{n} = \langle -\sin\theta, \cos\theta\rangle$. (a) Show that $\mathbf{v}$ and $\mathbf{n}$ are perpendicular. Find the component of $\mathbf{w} = \langle 0, -w\rangle$ along $\mathbf{v}$ and the component of $\mathbf{w}$ along $\mathbf{n}$.



(b) If the coefficient of static friction between the crate and ramp equals μ_s, the crate will slide down the ramp if the component of $\mathbf{w}$ along $\mathbf{v}$ is greater than the product of μ_s and the component of $\mathbf{w}$ along $\mathbf{n}$. Show that this occurs if the angle θ is steep enough that $\theta > \tan^{-1}\mu_s$.

60. A weight of 500 kilograms is supported by two ropes that exert forces of $\mathbf{a} = \langle -100, 200\rangle$ kilograms and $\mathbf{b} = \langle 100, 300\rangle$ kilograms. Find the angles α, β and θ between the ropes.



61. A car makes a turn on a banked road. If the road is banked at 10°, show that a vector parallel to the road is $\langle \cos 10°, \sin 10°\rangle$.
(a) If the car has weight 2000 kilograms, find the component of the weight vector along the road vector. This component of weight provides a force that helps the car turn. Compute the ratio of the component of weight along the road to the component of weight into the road. Discuss why it might be dangerous if this ratio is very small or very large.

(b) Repeat part (a) for a 2500-kilogram car on a 15° bank.

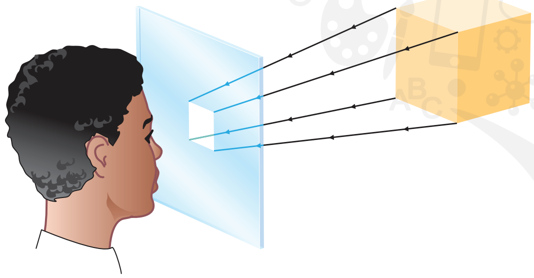
62. The racetrack at Bristol, Tennessee, is famous for its short length and its steeply banked curves. The track is an oval of length 857 meters and the corners are banked at 36°. Circular motion at a constant speed v requires a centripetal force of $F = \frac{mv^2}{r}$, where r is the radius of the circle and m is the mass of the car. For a track banked at angle A, the weight of the car provides a centripetal force of $mg \sin A$, where g is the gravitational constant. Setting the two equal gives $\frac{v^2}{r} = g \sin A$. Assuming that the Bristol track is circular (it's not really) and using $g = 9.8\,\text{m/s}^2$, find the speed supported by the Bristol bank. Cars actually complete laps at over 190 km/h. Discuss where the additional force for this higher speed might come from.

63. Suppose a small business sells three products. In a given month, if 3000 units of product A are sold, 2000 units of product B are sold and 4000 units of product C are sold, then the **sales vector** for that month is defined by $\mathbf{s} = \langle 3000, 2000, 4000\rangle$. If the prices of products A, B and C are AED 20, AED 15 and AED 25, respectively, then the **price vector** is defined by $\mathbf{p} = \langle 20, 15, 25\rangle$. Compute $\mathbf{s} \cdot \mathbf{p}$ and discuss how it relates to monthly revenue.

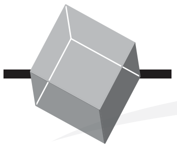
64. Suppose that in a particular county, ice cream sales (in thousands of liters) for a year is given by the vector $\mathbf{s} = \langle 3, 5, 12, 40, 60, 100, 120, 160, 110, 50, 10, 2\rangle$. That is, 3000 liters were sold in January, 5000 liters were sold in February, and so on. In the same county, suppose that murders for the year are given by the vector $\mathbf{m} = \langle 2, 0, 1, 6, 4, 8, 10, 13, 8, 2, 0, 6\rangle$. Show that the average monthly ice cream sales is $\overline{s} = 56000$ liters and that the average monthly number of murders is $\overline{m} = 5$. Compute the vectors **a** and **b**, where the components of **a** equal the components of **s** with the mean 56 subtracted (so that $\mathbf{a} = \langle -53, -51, -44, \ldots\rangle$) and the components of **b** equal the components of **m** with the mean 5 subtracted. The correlation between ice cream sales and murders is defined as $\rho = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$. Often, a positive correlation is incorrectly interpreted as meaning that **a** "causes" **b**. (In fact, correlation should *never* be used to infer a cause-and-effect relationship.) Explain why such a conclusion would be invalid in this case.

Exploratory Exercises

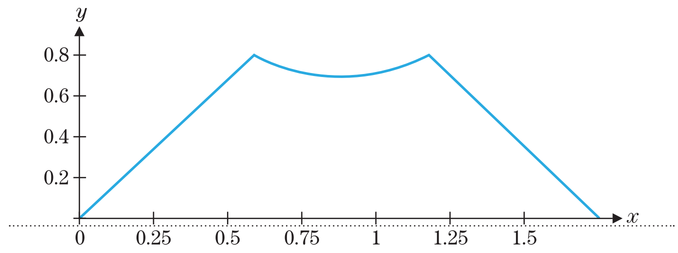
1. This exercise develops a basic principle used in computer graphics. In the drawing, an artist traces the image of an object onto a pane of glass. Explain why the trace will be distorted unless the artist keeps the pane of glass perpendicular to the line of sight. The trace is thus a projection of the object onto the pane of glass. To make this precise, suppose that the artist is at the point (100, 0, 0) and the point $P_1 = (2, 1, 3)$is part of the object being traced. Find the projection $\mathbf{p_1}$ of the position vector $\langle 2, 1, 3\rangle$ along the artist's position vector $\langle 100, 0, 0\rangle$. Then find the vector $\mathbf{q_1}$ such that $\langle 2, 1, 3\rangle = \mathbf{p_1} + \mathbf{q_1}$. Which of the vectors $\mathbf{p_1}$ and $\mathbf{q_1}$ does the artist actually see and which one is hidden? Repeat this with the point $P_2 = (-2, 1, 3)$ and find vectors $\mathbf{p_2}$ and $\mathbf{q_2}$ such that $\langle -2, 1, 3\rangle = \mathbf{p_2} + \mathbf{q_2}$. The artist would plot both points P_1 and P_2 at the same point on the pane of glass. Identify which of the vectors $\mathbf{p_1}, \mathbf{q_1}, \mathbf{p_2}$, and $\mathbf{q_2}$ correspond to this point. From the artist's perspective, one of the points P_1 or P_2 is hidden behind the other. Identify which point is hidden and explain how the information in the vectors $\mathbf{p_1}, \mathbf{q_1}, \mathbf{p_2}$, and $\mathbf{q_2}$ can be used to determine which point is hidden.

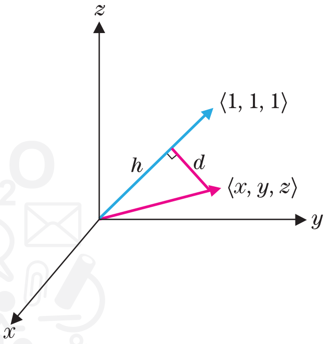


2. Take a cube and spin it around a diagonal.



If you spin it rapidly, you will see a curved outline appear in the middle. (See the figure below.) How does a cube become curved? This exercise answers that question. Suppose that the cube is a unit cube with $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$, and we rotate about the diagonal from (0, 0, 0) to (1, 1, 1). Spinning the cube, we see the combination of points on the cube at their maximum distance from the diagonal. The points on the edge of the cube have the maximum distance. If (x, y, z) is a point on an edge of the cube, define h to be the component of the vector $\langle x, y, z\rangle$ along the diagonal $\langle 1, 1, 1\rangle$. The distance d from (x, y, z) to the diagonal is then $d = \sqrt{|\langle x, y, z\rangle|^2 - h^2}$, as in the diagram below. The curve is produced by the edge from (0, 0, 1) to (0, 1, 1). Parametric equations for this segment are $x = 0$, $y = t$ and $z = 1$, for $0 \le t \le 1$. For the vector $\langle 0, t, 1\rangle$, compute h and then d. Graph $d(t)$. You should see a curve similar to the middle of the outline shown below. Show that this curve is actually part of a hyperbola. Then find the outline created by other sides of the cube. Which ones produce curves and which produce straight lines?





Exercises 7-6-2

WRITING EXERCISES

1. In this chapter, we have developed several tests for geometric relationships. Briefly describe how to test whether two vectors are (a) parallel; (b) perpendicular. Briefly describe how to test whether (c) three points are colinear; (d) four points are coplanar.

2. The flip side of the problems in exercise 1 is to construct vectors with desired properties. Briefly describe how to construct a vector (a) parallel to a given vector; (b) perpendicular to a given vector. (c) Given a vector, describe how to construct two other vectors such that the three vectors are mutually perpendicular.

3. In example 6.13, how would the torque change if the force **F** were replaced with the force −**F**? Answer both in mathematical terms and in physical terms.

4. Sketch a picture and explain in geometric terms why $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$.

In exercises 1–4, compute the given determinant.

1. $\begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & 1 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 2 & 3 & -1 \\ 0 & 1 & 0 \\ -2 & -1 & 3 \end{vmatrix}$

4. $\begin{vmatrix} -2 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 2 \end{vmatrix}$

In exercises 5–10, compute the cross product a × b.

5. $\mathbf{a} = \langle 1, 2, -1 \rangle$, $\mathbf{b} = \langle 1, 0, 2 \rangle$

6. $\mathbf{a} = \langle 3, 0, -1 \rangle$, $\mathbf{b} = \langle 1, 2, 2 \rangle$

7. $\mathbf{a} = \langle 0, 1, 4 \rangle$, $\mathbf{b} = \langle -1, 2, -1 \rangle$

8. $\mathbf{a} = \langle 2, -2, 0 \rangle$, $\mathbf{b} = \langle 3, 0, 1 \rangle$

9. $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{j} + \mathbf{k}$

10. $\mathbf{a} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{j} - \mathbf{k}$

In exercises 11–16, find two unit vectors orthogonal to the two given vectors.

11. $\mathbf{a} = \langle 1, 0, 4 \rangle$, $\mathbf{b} = \langle 1, -4, 2 \rangle$

12. $\mathbf{a} = \langle 2, -2, 1 \rangle$, $\mathbf{b} = \langle 0, 0, -2 \rangle$

13. $\mathbf{a} = \langle 2, -1, 0 \rangle$, $\mathbf{b} = \langle 1, 0, 3 \rangle$

14. $\mathbf{a} = \langle 0, 2, 1 \rangle$, $\mathbf{b} = \langle 1, 0, -1 \rangle$

15. $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{b} = 4\mathbf{j} + \mathbf{k}$

16. $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$

In exercises 17–20, find the distance from the point Q to the given line.

17. $Q = (1, 2, 0)$, line through $(0, 1, 2)$ and $(3, 1, 1)$

18. $Q = (2, 0, 1)$, line through $(1, -2, 2)$ and $(3, 0, 2)$

19. $Q = (3, -2, 1)$, line through $(2, 1, -1)$ and $(1, 1, 1)$

20. $Q = (1, 3, 1)$, line through $(1, 3, -2)$ and $(1, 0, -2)$

In exercises 21–26, find the indicated area or volume.

21. Area of the parallelogram with two adjacent sides formed by $\langle 2, 3 \rangle$ and $\langle 1, 4 \rangle$

22. Area of the parallelogram with two adjacent sides formed by $\langle -2, 1 \rangle$ and $\langle 1, 3 \rangle$

23. Area of the triangle with vertices $(0, 0, 0)$, $(2, 3, -1)$ and $(3, -1, 4)$

24. Area of the triangle with vertices $(1, 1, 0)$, $(0, -2, 1)$ and $(1, -3, 0)$

25. Volume of the parallelepiped with three adjacent edges formed by $\langle 2, 1, 0 \rangle$, $\langle -1, 2, 0 \rangle$ and $\langle 1, 1, 2 \rangle$

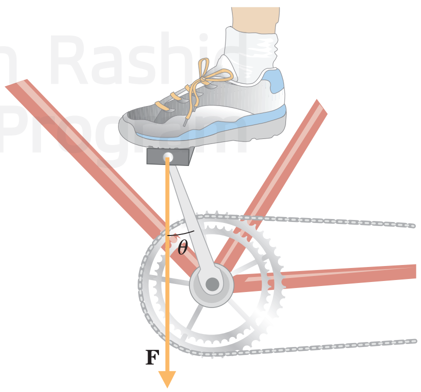
26. Volume of the parallelepiped with three adjacent edges formed by $\langle 0, -1, 0 \rangle$, $\langle 0, 2, -1 \rangle$ and $\langle 1, 0, 2 \rangle$

27. If you apply a force of magnitude 90 Newton's at the end of an 20 centimeter-long wrench at an angle of $\frac{\pi}{4}$ to the wrench, find the magnitude of the torque applied to the bolt.

28. If you apply a force of magnitude 180 Newton's at the end of an 45-centimeter-long wrench at an angle of $\frac{\pi}{3}$ to the wrench, find the magnitude of the torque applied to the bolt.

29. Use the torque formula $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ to explain the positioning of doorknobs. In particular, explain why the knob is placed as far as possible from the hinges and at a height that makes it possible for most people to push or pull on the door at a right angle to the door.

30. In the diagram, a foot applies a force **F** vertically to a bicycle pedal. Compute the torque on the sprocket in terms of θ and **F**. Determine the angle θ at which the torque is maximized. When helping a young person to learn to ride a bicycle, most people rotate the sprocket so that the pedal sticks straight out to the front. Explain why this is helpful.



In exercises 31–34, assume that the balls are moving into the page (and away from you) with the indicated spin. Determine the direction of the spin vector and of the Magnus force.

31. **a.**  **b.** 

32. **a.**  **b.** 

33. **a.**  **b.** 

34. **a.**  **b.** 

In exercises 35–40, label each statement as true or false. If it is true, briefly explain why. If it is false, give a counterexample.

35. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

36. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

37. $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|^2$

38. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$

39. If the force is doubled, the torque doubles.

40. If the spin rate is doubled, the Magnus force is doubled.

In exercises 41–44, use the cross product to determine the angle θ between the vectors, assuming that $0 < \theta \le \frac{\pi}{2}$.

41. $\mathbf{a} = \langle 1, 0, 4 \rangle$, $\mathbf{b} = \langle 2, 0, 1 \rangle$

42. $\mathbf{a} = \langle 2, 2, 1 \rangle$, $\mathbf{b} = \langle 0, 0, 2 \rangle$

43. $\mathbf{a} = 3\mathbf{i} + \mathbf{k}$, $\mathbf{b} = 4\mathbf{j} + \mathbf{k}$

44. $\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$

In exercises 45–50, draw pictures to identify the cross product (do not compute!).

45. $\mathbf{i} \times (3\mathbf{k})$

46. $\mathbf{k} \times (2\mathbf{i})$

47. $\mathbf{i} \times (\mathbf{j} \times \mathbf{k})$

48. $\mathbf{j} \times (\mathbf{j} \times \mathbf{k})$

49. $\mathbf{j} \times (\mathbf{j} \times \mathbf{i})$

50. $(\mathbf{j} \times \mathbf{i}) \times \mathbf{k}$

In exercises 51–54, use the parallelepiped volume formula to determine whether the vectors are coplanar.

51. $\langle 2, 3, 1 \rangle$, $\langle 1, 0, 2 \rangle$ and $\langle 0, 3, -3 \rangle$

52. $\langle 1, -3, 1 \rangle$, $\langle 2, -1, 0 \rangle$ and $\langle 0, -5, 2 \rangle$

53. $\langle 1, 0, -2 \rangle$, $\langle 3, 0, 1 \rangle$ and $\langle 2, 1, 0 \rangle$

54. $\langle 1, 1, 2 \rangle$, $\langle 0, -1, 0 \rangle$ and $\langle 3, 2, 4 \rangle$

55. Show that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

56. Show that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

57. Show that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$.

58. Prove parts (ii), (iv), (v) and (vi) of Theorem 6-7.

59. In each of the situations shown here, $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 4$. In which case is $|\mathbf{a} \times \mathbf{b}|$ larger? What is the maximum possible value for $|\mathbf{a} \times \mathbf{b}|$?

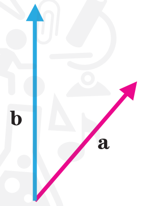


FIGURE A

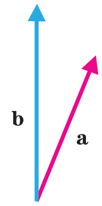


FIGURE B

60. In Figures A and B, if the angles between **a** and **b** are 50° and 20°, respectively, find the exact values for $|\mathbf{a} \times \mathbf{b}|$. Also, state whether $\mathbf{a} \times \mathbf{b}$ points into or out of the page.

61. Identify the expressions that are undefined.

a. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ **b.** $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$

c. $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$ **d.** $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

62. Explain why each equation is true.

a. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ **b.** $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{a}) = 0$

Applications

In exercises 63–70, a sports situation is described, with the typical ball spin shown in the indicated exercise. Discuss the effects on the ball and how the game is affected.

63. Baseball overhand fastball, spin in exercise 31(a)

64. Baseball right-handed curveball, spin in exercise 33(a)

65. Tennis topspin groundstroke, spin in exercise 34(a)

66. Tennis left-handed slice serve, spin in exercise 32(b)

67. Soccer spiral pass, spin in exercise 34(b)

68. Soccer left-footed "curl" kick, spin in exercise 31(b)

69. Golf "pure" hit, spin in exercise 31(a)

70. Golf right-handed "hook" shot, spin in exercise 33(b)

Exploratory Exercises

1. Devise a test that quickly determines whether $|\mathbf{a} \times \mathbf{b}| < |\mathbf{a} \cdot \mathbf{b}|$, $|\mathbf{a} \times \mathbf{b}| > |\mathbf{a} \cdot \mathbf{b}|$ or $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a} \cdot \mathbf{b}|$. Apply your test to the following vectors: (a) $\langle 2, 1, 1\rangle$ and $\langle 3, 1, 2\rangle$, (b) $\langle 2, 1, -1\rangle$ and $\langle -1, -2, 1\rangle$ and (c) $\langle 2, 1, 1\rangle$ and $\langle -1, 2, 2\rangle$. For randomly chosen vectors, which of the three cases is the most likely?

2. In this exercise, we explore the equation of motion for a general projectile in three dimensions. Newton's second law is $\mathbf{F} = m\mathbf{a}$. Three forces that could affect the motion of the projectile are gravity, air drag and the Magnus force. Orient the axes such that positive z is up, positive x is right and positive y is straight ahead. The force due to gravity is weight, given by $\mathbf{F}_g = \langle 0 . 0, -mg\rangle$. Air drag has magnitude proportional to the square of speed and direction opposite that of velocity. Show that if $\mathbf{v}$ is the velocity vector, then $\mathbf{F}_g = |\mathbf{v}|\mathbf{v}$ satisfies both properties. The Magnus force is proportional to $\mathbf{s} \times \mathbf{v}$, where s is the spin vector. The full model is then

$$\frac{d\mathbf{v}}{dt} = \langle 0, 0, -g\rangle - c_d|\mathbf{v}|\mathbf{v} + c_m(\mathbf{s} \times \mathbf{v}),$$

for positive constants c_d and c_m . With $\mathbf{v} = \langle v_x, v_y, v_z\rangle$ and $\mathbf{s} = \langle s_x, s_y, s_z\rangle$, expand this equation into separate differential equations for v_x, v_y and v_z, We can't solve these equations, but we can get some information by considering signs. For a golf drive, the spin produced could be pure backspin, in which case the spin vector is $\mathbf{s} = \langle \omega, 0, 0\rangle$ for some large $\omega > 0$. (A golf shot can have spins of 4000 rpm.) The initial velocity of a good shot would be straight ahead with some loft, $\mathbf{v}(0) = \langle 0, b, c\rangle$ for positive constants b and c. At the beginning of the flight, show that $v_y' < 0$ and thus, v_y decreases. If the ball spends approximately the same amount of time going up as coming down, conclude that the ball will travel farther downrange while going up than coming down. Next, consider the case of a ball with some sidespin, so that $s_x > 0$ and $s_y > 0$. By examining the sign of v_x', determine whether this ball will curve to the right or left. Examine the other equations and determine what other effects this sidespin may have.

Study Guide and Review

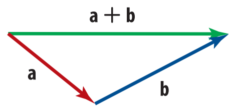
Chapter Summary

Key Concepts

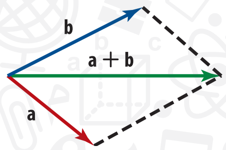
Introduction to Vectors (Lesson 7-1)

- The direction of a vector is the directed angle between the vector and a horizontal line. The magnitude of a vector is its length.
- When two or more vectors are combined, their sum is a single vector called the resultant, which can be found using the triangle or parallelogram method.

Triangle Method



Parallelogram Method



Vectors in the Coordinate Plane (Lesson 7-2)

- The component form of a vector with rectangular components x and y is $\langle x, y \rangle$.
- The component form of a vector that is not in standard position, with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, is given by $\langle x_2 - x_1, y_2 - y_1 \rangle$.
- The magnitude of a vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is given by $|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ are vectors and k is a scalar, then $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$, and $k\mathbf{a} = \langle ka_1, ka_2 \rangle$.
- The standard unit vectors $\mathbf{i}$ and $\mathbf{j}$ can be used to express any vector $\mathbf{v} = \langle a, b \rangle$ as $a\mathbf{i} + b\mathbf{j}$.

Dot Products (Lesson 7-3)

- The dot product of $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$.
- If θ is the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.

Vectors in Three-Dimensional Space (Lesson 7-4)

- The distance between $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- The midpoint of $\overline{AB}$ is given by $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Dot and Cross Products of Vectors in Space (Lesson 7-5)

- The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$.

Key Vocabulary

component form
components
cross product
direction
dot product
equivalent vectors
initial point
linear combination
magnitude
octants
opposite vectors
ordered triple
orthogonal
parallelepiped
parallelogram method
parallel vectors
quadrant bearing
rectangular components
resultant
standard position
terminal point
three-dimensional coordinate system
torque
triangle method
triple scalar product
true bearing
unit vector
vector
vector projection
work
z-axis
zero vector

Vocabulary Check

Determine whether each statement is *true* or *false*. If false, replace the underlined term or expression to make the statement true.

1. The terminal point of a vector is where the vector <u>begins</u>.
2. If $\mathbf{a} = \langle -4, 1 \rangle$ and $\mathbf{b} = \langle 3, 2 \rangle$, the dot product is calculated by <u>$-4(1) + 3(2)$</u>.
3. The midpoint of $\overline{AB}$ with $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.
4. The <u>magnitude</u> of $\mathbf{r}$ if the initial point is $A(-1, 2)$ and the terminal point is $B(2, -4)$ is $\langle 3, -6 \rangle$.
5. Two vectors are <u>equal</u> only if they have the same direction and magnitude.
6. When two nonzero vectors are orthogonal, the angle between them is <u>180°</u>.
7. The <u>component</u> of $\mathbf{u}$ onto $\mathbf{v}$ is the vector with direction that is parallel to $\mathbf{v}$ *and* with length that is the component of $\mathbf{u}$ along $\mathbf{v}$.
8. To find at least one vector orthogonal to any two vectors in space, calculate the <u>cross product</u> of the two original vectors.
9. When a vector is subtracted, it is equivalent to adding the opposite vector.
10. If $\mathbf{v}$ is a unit vector in the same direction as $\mathbf{u}$, then $\mathbf{v} = \underline{\frac{|\mathbf{u}|}{\mathbf{u}}}$.

Study Guide and Review *Continued*

Lesson-by-Lesson Review

7-1 Introduction to Vectors

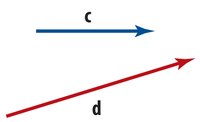
State whether each quantity described is a *vector* quantity or a *scalar* quantity.

11. a car driving 50 kilometers an hour due east

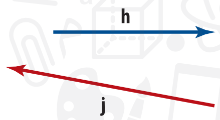
12. a gust of wind blowing 5 meters per second

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest tenth of a centimeter and its direction relative to the horizontal.

13.



14.



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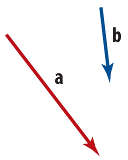
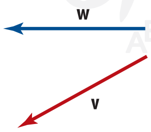


16.



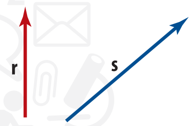
Determine the magnitude and direction of the resultant of each vector sum.

17. 70 meters due west and then 150 meters due east

18. 8 newtons directly backward and then 12 newtons directly backward

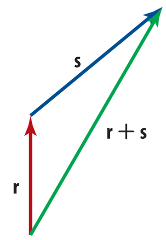
Example 1

Find the resultant of r and s using either the triangle or parallelogram method. State the magnitude of the resultant in centimeters and its direction relative to the horizontal.



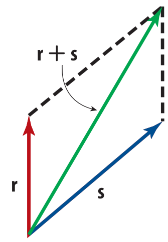
Triangle Method

Translate **r** so that the tip of **r** touches the tail of **s**. The resultant is the vector from the tail of **r** to the tip of **s**.



Parallelogram Method

Translate **s** so that the tail of **s** touches the tail of **r**. Complete the parallelogram that has **r** and **s** as two of its sides. The resultant is the vector that forms the indicated diagonal of the parallelogram.



The magnitude of the resultant is 3.4 cm and the direction is 59°.

7-2 Vectors in the Coordinate Plane

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

19. $A(-1, 3)$, $B(5, 4)$

20. $A(7, -2)$, $B(-9, 6)$

21. $A(-8, -4)$, $B(6, 1)$

22. $A(2, -10)$, $B(3, -5)$

Find each of the following for $\mathbf{p} = \langle 4, 0\rangle$, $\mathbf{q} = \langle -2, -3\rangle$, and $\mathbf{t} = \langle -4, 2\rangle$.

23. $2\mathbf{q} - \mathbf{p}$

24. $\mathbf{p} + 2\mathbf{t}$

25. $\mathbf{t} - 3\mathbf{p} + \mathbf{q}$

26. $2\mathbf{p} + \mathbf{t} - 3\mathbf{q}$

Find a unit vector u with the same direction as v.

27. $\mathbf{v} = \langle -7, 2\rangle$

28. $\mathbf{v} = \langle 3, -3\rangle$

29. $\mathbf{v} = \langle -5, -8\rangle$

30. $\mathbf{v} = \langle 9, 3\rangle$

Example 2

Find the component form and magnitude of $\overrightarrow{AB}$ with initial point $A(3, -2)$ and terminal point $B(4, -1)$.

$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1\rangle$	Component form
$= \langle 4 - 3, -1 - (-2)\rangle$	Substitute.
$= \langle 1, 1\rangle$	Subtract.

Find the magnitude using the Distance Formula.

$\lvert\overrightarrow{AB}\rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{[(4 - 3)]^2 + [-1 - (-2)]^2}$	Substitute.
$= \sqrt{2}$ or about 1.4	Simplify.

7-3 Dot Products and Vector Projections

Find the dot product of **u** and **v**. Then determine if **u** and **v** are orthogonal.

31. $\mathbf{u} = \langle -3, 5\rangle, \mathbf{v} = \langle 2, 1\rangle$

32. $\mathbf{u} = \langle 4, 4\rangle, \mathbf{v} = \langle 5, 7\rangle$

33. $\mathbf{u} = \langle -1, 4\rangle, \mathbf{v} = \langle 8, 2\rangle$

34. $\mathbf{u} = \langle -2, 3\rangle, \mathbf{v} = \langle 1, 3\rangle$

Find the angle θ between **u** and **v** to the nearest tenth of a degree.

35. $\mathbf{u} = \langle 5, -1\rangle, \mathbf{v} = \langle -2, 3\rangle$

36. $\mathbf{u} = \langle -1, 8\rangle, \mathbf{v} = \langle 4, 2\rangle$

Example 3

Find the dot product of $x = \langle 2, -5\rangle$ and $\mathbf{y} = \langle -4, 7\rangle$. Then determine if x and y are orthogonal.

$x \cdot \mathbf{y} = x_1y_1 + x_2y_2$ — Dot product

$= 2(-4) + -5(7)$ — Substitute.

$= -8 + (-35)$ or -43 — Simplify.

Since $\mathbf{x} \cdot \mathbf{y} \neq 0$, **x** and **y** are not orthogonal.

7-4 Vectors in Three-Dimensional Space

Plot each point in a three-dimensional coordinate system.

37. (1, 2, −4)

38. (3, 5, 3)

39. (5, −3, −2)

40. (−2, −3, −2)

Find the length and midpoint of the segment with the given endpoints.

41. (−4, 10, 4), (2, 0, 8)

42. (−5, 6, 4), (−9, −2, −2)

43. (3, 2, 0), (−9, −10, 4)

44. (8, 3, 2), (−4, −6, 6)

Locate and graph each vector in space.

45. $\mathbf{a} = \langle 0, -3, 4\rangle$

46. $\mathbf{b} = -3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

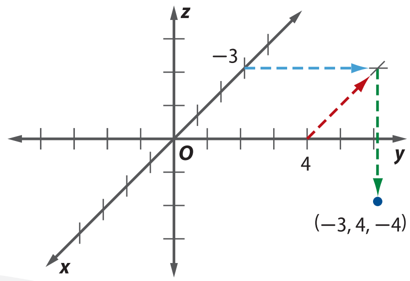
47. $\mathbf{c} = -2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

48. $\mathbf{d} = \langle -4, -5, -3\rangle$

Example 4

Plot (−3, 4, −4) in a three-dimensional coordinate system.

Locate the point (−3, 4) in the *xy*-plane and mark it with a cross. Then plot a point 4 units down from this location parallel to the *z*-axis.



7-5 Vectors in Three-Dimensional Space

Find the dot product of **u** and **v**. Then determine if **u** and **v** are orthogonal.

49. $\mathbf{u} = \langle 2, 5, 2\rangle, \mathbf{v} = \langle 8, 2, -13\rangle$

50. $\mathbf{u} = \langle 5, 0, -6\rangle, \mathbf{v} = \langle -6, 1, 3\rangle$

Find the cross product of **u** and **v**. Then show that **u** × **v** is orthogonal to both **u** and **v**.

51. $\mathbf{u} = \langle 1, -3, -2\rangle, \mathbf{v} = \langle 2, 4, -3\rangle$

52. $\mathbf{u} = \langle 4, 1, -2\rangle, \mathbf{v} = \langle 5, -4, -1\rangle$

Example 5

Find the cross product of $\mathbf{u} = \langle -4, 2, -3\rangle$ and $\mathbf{v} = \langle 7, 11, 2\rangle$. Then show that u × v is orthogonal to both u and v.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} 2 & -3 \\ 11 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & -3 \\ 7 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 2 \\ 7 & 11 \end{vmatrix}\mathbf{k}$$

$$= \langle 37, -13, -58\rangle$$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle 37, -13, -58\rangle \cdot \langle -4, 2, -3\rangle$

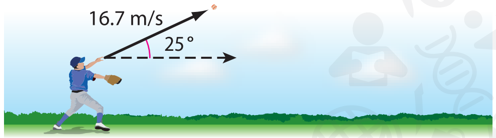
$= -148 - 26 + 174$ or 0 ✔

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \langle 37, -13, -58\rangle \cdot \langle 7, 11, 2\rangle$

$= 259 - 143 - 116$ or 0 ✔

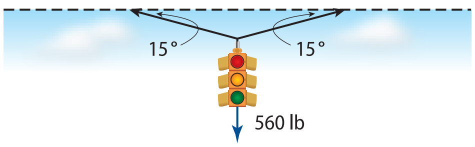
Applications and Problem Solving

53. **BASEBALL** A player throws a baseball with an initial velocity of 16.7 meters per second at an angle of 25° above the horizontal, as shown below. Find the magnitude of the horizontal and vertical components. (Lesson 7-1)

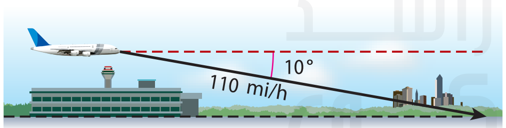


54. **STROLLER** Laila is pushing a stroller with a force of 200 newtons at an angle of 20° below the horizontal. Find the magnitude of the horizontal and vertical components of the force. (Lesson 7-1)

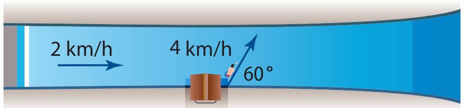
55. **LIGHTS** A traffic light at an intersection is hanging from two wires of equal length at 15° below the horizontal as shown. If the traffic light weighs 560 pounds, what is the tension in each wire keeping the light at equilibrium? (Lesson 7-1)



56. **AIRPLANE** An airplane is descending at a speed of 110 miles per hour at an angle of 10° below the horizontal. Find the component form of the vector that represents the velocity of the airplane. (Lesson 7-2)



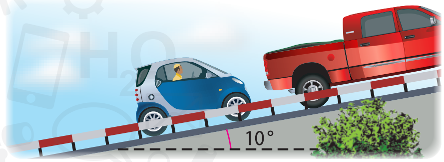
57. **LIFEGUARD** A lifeguard at a wave pool swims at a speed of 4 kilometers per hour at a 60° angle to the side of the pool as shown. (Lesson 7-2)



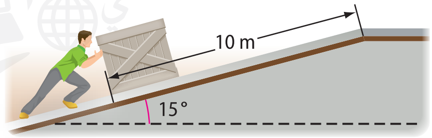
a. At what speed is the lifeguard traveling if the current in the pool is 2 kilometers per hour parallel to the side of the pool as shown?

b. At what angle is the lifeguard traveling with respect to the starting side of the pool?

58. **TRAFFIC** A 680.4-kilogram car is stopped in traffic on a hill that is at an incline of 10°. Determine the force that is required to keep the car from rolling down the hill. (Lesson 7-3)



59. **WORK** At a warehouse, Jassim pushes a box on sliders with a constant force of 80 newtons up a ramp that has an incline of 15° with the horizontal. Determine the amount of work in joules that Jassim does if he pushes the dolly 10 meters. (Lesson 7-3)



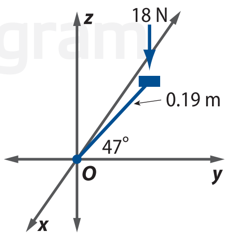
60. **SATELLITES** The positions of two satellites that are in orbit can be represented by the coordinates (28,625, 32,461, −38,426) and (−31,613, −29,218, 43,015), where (0, 0, 0) represents the center of Earth and the coordinates are given in miles. The radius of Earth is about 3963 miles. (Lesson 7-4)

a. Determine the distance between the two satellites.

b. If a third satellite were to be placed directly between the two satellites, what would the coordinates be?

c. Can a third satellite be placed at the coordinates found in part **b**? Explain your reasoning.

61. **BICYCLES** A bicyclist applies 18 newtons of force down on the pedal to put the bicycle in motion. The pedal has an initial position of 47° above the y-axis, and a length of 0.19 meters to the pedal's axle, as shown. (Lesson 7-5)



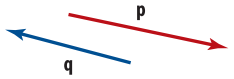
a. Find the vector representing the torque about the axle of the bicycle pedal in component form.

b. Find the magnitude and direction of the torque.

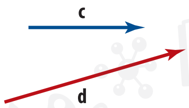
Practice Test

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest tenth of a centimeter and its direction relative to the horizontal.

1.



2.

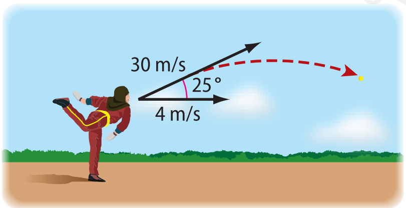


Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

3. $A(1, -3), B(-5, 1)$

4. $A\left(\frac{1}{2}, \frac{3}{2}\right), B(-1, 7)$

5. SOFTBALL A batter on the opposing softball team hits a ground ball that rolls out to Lamya in left field. She runs toward the ball at a velocity of 4 meters per second, scoops it, and proceeds to throw it to the catcher at a speed of 30 meters per second and at an angle of 25° with the horizontal in an attempt to throw out a runner. What is the resultant speed and direction of the throw?



Find a unit vector in the same direction as u.

6. $\mathbf{u} = \langle -1, 4\rangle$

7. $\mathbf{u} = \langle 6, -3\rangle$

Find the dot product of u and v. Then determine if u and v are orthogonal.

8. $\mathbf{u} = \langle 2, -5\rangle, \mathbf{v} = \langle -3, 2\rangle$

9. $\mathbf{u} = \langle 4, -3\rangle, \mathbf{v} = \langle 6, 8\rangle$

10. $\mathbf{u} = 10\mathbf{i} - 3\mathbf{j}, \mathbf{v} = \mathbf{i} + 8\mathbf{j}$

11. MULTIPLE CHOICE Write **u** as the sum of two orthogonal vectors, one of which being the projection of **u** onto **v** if $\mathbf{u} = \langle 1, 3\rangle$ and $\mathbf{v} = \langle -4, 2\rangle$.

A $\mathbf{u} = \left\langle \frac{2}{5}, -\frac{3}{5}\right\rangle + \left\langle \frac{3}{5}, \frac{18}{5}\right\rangle$

B $\mathbf{u} = \left\langle \frac{2}{5}, \frac{3}{5}\right\rangle + \left\langle \frac{3}{5}, \frac{12}{5}\right\rangle$

C $\mathbf{u} = \left\langle -\frac{4}{5}, \frac{2}{5}\right\rangle + \left\langle \frac{9}{5}, \frac{13}{5}\right\rangle$

D $\mathbf{u} = \left\langle -\frac{2}{5}, \frac{1}{5}\right\rangle + \left\langle \frac{7}{5}, \frac{14}{5}\right\rangle$

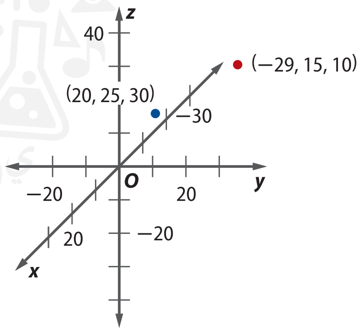
12. MOVING Lamis is pushing a box along a level floor with a force of 54.4 kilograms at an angle of depression of 20°. Determine how much work is done if the box is moved 25 meters.

Find each of the following for $\mathbf{a} = \langle 2, 4, -3\rangle$, $\mathbf{b} = \langle -5, -7, 1\rangle$, and $\mathbf{c} = \langle 8, 5, -9\rangle$.

13. $2\mathbf{a} + 5\mathbf{b} - 3\mathbf{c}$

14. $\mathbf{b} - 6\mathbf{a} + 2\mathbf{c}$

15. HOT AIR BALLOONS During a festival, twelve hot air balloons take off. A few minutes later, the coordinates of the first two balloons are (20, 25, 30) and (−29, 15, 10) as shown, where the coordinates are given in feet.



a. Determine the distance between the first two balloons that took off.

b. A third balloon is halfway between the first two balloons. Determine the coordinates of the third balloon.

c. Find a unit vector in the direction of the first balloon if it took off at (0, 0, 0).

Find the angle θ between vectors u and v to the nearest tenth of a degree.

16. $\mathbf{u} = \langle -2, 4, 6\rangle, \mathbf{v} = \langle 3, 7, 12\rangle$

17. $\mathbf{u} = -9\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}, \mathbf{v} = -5\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$

Find the cross product of u and v. Then show that u × v is orthogonal to both u and v.

18. $\mathbf{u} = \langle 1, 7, 3\rangle, \mathbf{v} = \langle 9, 4, 11\rangle$

19. $\mathbf{u} = -6\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{v} = 5\mathbf{i} - 3\mathbf{j} - 2k$

20. BOATING The tiller is a lever that controls the position of the rudder on a boat. When force is applied to the tiller, the boat will turn. Suppose the tiller on a certain boat is 0.75 meter in length and is currently resting in the *xy*-plane at a 15° angle from the positive *x*-axis. Find the magnitude of the torque that is developed about the axle of the tiller if 50 newtons of force is applied in a direction parallel to the positive *y*-axis.

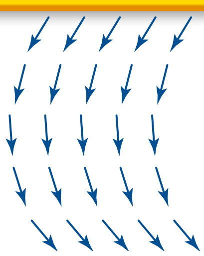
CHAPTER 7

Connect to AP Calculus

Vector Fields

Objective

- Graph vectors in and identify graphs of vector fields.



In Chapter 7, you examined the effects that wind and water currents have on a moving object. The force produced by the wind and current was represented by a single vector. However, we know that the current in a body of water or the force produced by wind is not necessarily constant; instead it differs from one region to the next. If we want to represent the entire current or air flow in an area, we would need to assign a vector to each point in space, thus creating a *vector field*.

Vector fields are commonly used in engineering and physics to model air resistance, magnetic and gravitational forces, and the motion of liquids. While these applications of vector fields require multiple dimensions, we will analyze vector fields in only two dimensions.

A vector field $\mathbf{F}(x, y)$ is a function that converts two-dimensional coordinates into sets of two-dimensional vectors.

$$\mathbf{F}(x, y) = \langle f_1(x, y), f_2(x, y) \rangle, \text{ where } f_1(x, y) \text{ and } f_2(x, y) \text{ are scalar functions.}$$

To graph a vector field, evaluate $\mathbf{F}(x, y)$ at (x, y) and graph the vector using (x, y) as the initial point. This is done for several points.

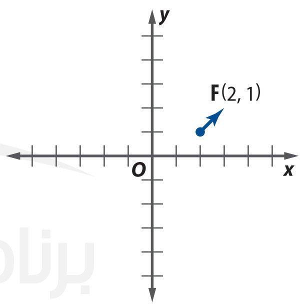
Activity 1 Vector Fields

Evaluate $\mathbf{F}(2, 1)$, $\mathbf{F}(-1, -1)$, $\mathbf{F}(1.5, -2)$, and $\mathbf{F}(-3, 2)$ for the vector field $\mathbf{F}(x, y) = \langle y^2, x - 1 \rangle$. Graph each vector using (x, y) as the initial point.

Step 1 To evaluate $\mathbf{F}(2, 1)$, let $x = 2$ and $y = 1$.

$$\langle y^2, x - 1 \rangle = \langle 1^2, 2 - 1 \rangle$$
$$= \langle 1, 1 \rangle$$

Step 2 To graph, let $(2, 1)$ represent the initial point of the vector. This makes $(2 + 1, 1 + 1)$ or $(3, 2)$ the terminal point.



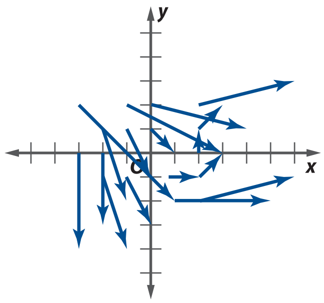
Step 3 Repeat Steps 1–2 for $\mathbf{F}(-1, -1)$, $\mathbf{F}(1.5, -2)$ and $\mathbf{F}(-3, 2)$.

Analyze the Results

1. Are the magnitudes and directions of the vectors the *same* or *different*?
2. Make a conjecture as to why a vector field can be defined as a function.
3. Is it possible to graph every vector in a vector field? Explain your reasoning.

2. Sample answer: There is exactly one vector for each point (x, y) in a vector field.
3. No; sample answer: Every point (x, y) in a plane has a vector associated to it, and there are infinitely many points in a given plane.

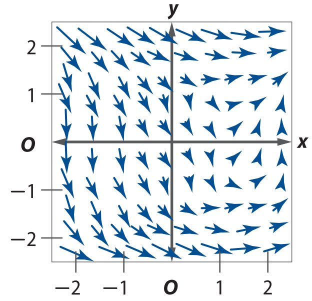
A graph of a vector field $\mathbf{F}(x, y)$ should include a variety of vectors all with initial points at (x, y). Graphing devices are typically used to graph vector fields because sketching vector fields by hand is often too difficult.



StudyTip

Graphs of Vector Fields Every point in a plane has a corresponding vector. The graphs of vector fields only show a selection of points.

To keep vectors from overlapping each other and to prevent the graph from looking too jumbled, the graphing devices proportionally reduce the lengths of the vectors and only construct vectors at certain intervals. For example, if we continue to graph more vectors for the vector field from Activity 1, the result would be the graph on the right.



Analyze the component functions of a vector field to identify the type of graph it will produce.

Activity 2 Vector Fields

Match each vector field to its graph.

$\mathbf{F}(x, y) = \langle 2, 1 + 2xy \rangle$ $\mathbf{G}(x, y) = \langle e^y, x \rangle$ $\mathbf{H}(x, y) = \langle e^y, y \rangle$

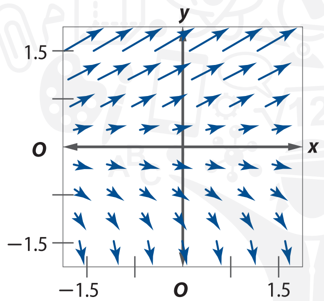


Figure 1

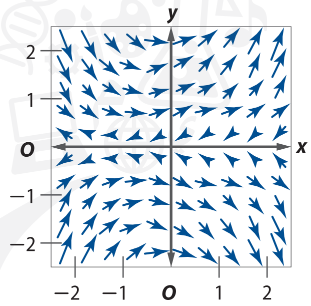


Figure 2

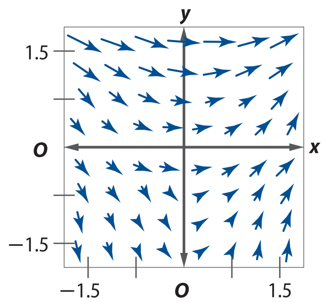


Figure 3

Step 1 Start by analyzing the components that make up $\mathbf{F}(x, y)$. The second component $(1 + 2xy)$ will produce a positive outcome when x and y have the same sign. The vertical component of the vectors in Quadrants I and III is positive, which makes the vectors in these quadrants point upward.

Step 2 The graph that has vectors pointing upward in Quadrants I and III is Figure 2.

Step 3 Repeat Steps 1–2 for the remaining vector fields.

Analyze the Results

4. Suppose the vectors in a vector field represent a force. What is the relationship between the force, the magnitude, and the length of a vector?

5. Representing wind by a single vector assumed that the force created remained constant for an entire area. If the force created by wind is represented by multiple vectors in a vector field, what assumption would have to be made about the third dimension?

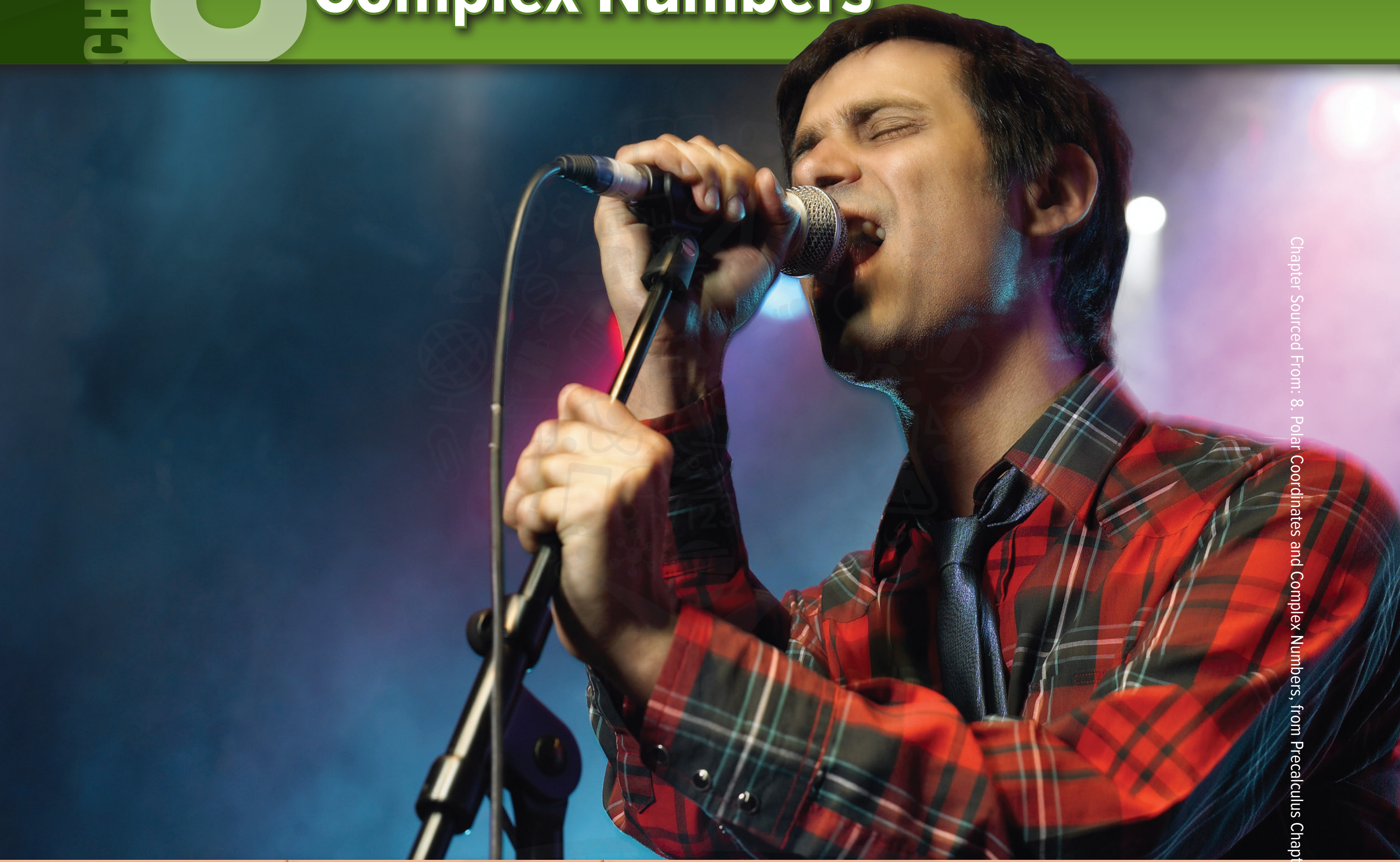
Model and Apply

6. Complete the table for the vector field $\mathbf{F}(x, y) = \langle -y, x \rangle$. Then graph each vector.

(x, y)	$\langle -y, x \rangle$	(x, y)	$\langle -y, x \rangle$
(2, 0)		(−2, 1)	
(1, 2)		(−2, 0)	
(2, 1)		(−1, −2)	
(0, 2)		(0, −2)	
(−1, 2)		(1, −2)	
(−2, −1)		(2, −1)	

CHAPTER 8

Polar Coordinates and Complex Numbers



Then

In **a previous chapter**, you identified and graphed rectangular equations of conic sections.

Now

In **Chapter 8**, you will:

- Graph polar coordinates and equations.
- Convert between polar and rectangular coordinates and equations.
- Identify polar equations of conic sections.
- Convert complex numbers between polar and rectangular form.

Why? ▲

CONCERTS Polar equations can be used to model sound patterns to help determine stage arrangement, speaker and microphone placement, and volume and recording levels. Polar equations can also be used with lighting and camera angles when concerts are filmed.

PREREAD Use the Lesson Openers in Chapter 8 to make two or three predictions about what you will learn in this chapter.

Get Ready for the Chapter

1 Textbook Option Take the Quick Check below.

QuickCheck

Graph each function using a graphing calculator. Analyze the graph to determine whether each function is *even*, *odd*, or *neither*. Confirm your answer algebraically. If odd or even, describe the symmetry of the graph of the function.

1. $f(x) = x^2 + 10$

2. $f(x) = -2x^3 + 5x$

3. $g(x) = \sqrt{x + 9}$

4. $h(x) = \sqrt{x^2 - 3}$

5. $g(x) = 3x^5 - 7x$

6. $h(x) = \sqrt{x^2} - 5$

7. BALLOON The distance in meters between a balloon and a person can be represented by $d(t) = \sqrt{t^2 + 3000}$, where t represents time in seconds. Analyze the graph to determine whether the function is *even*, *odd*, or *neither*.

Approximate to the nearest hundredth the relative or absolute extrema of each function. State the *x*-values where they occur.

8. $f(x) = 4x^2 - 20x + 24$

9. $g(x) = -2x^2 + 9x - 1$

10. $f(x) = -x^3 + 3x - 2$

11. $f(x) = x^3 + x^2 - 5x$

12. ROCKET A rocket is fired into the air. The function $h(t) = -16t^2 + 35t + 15$ represents the height h of the rocket in feet after t seconds. Find the extrema of this function.

Identify all angles that are coterminal with the given angle. Then find and draw one positive and one negative angle coterminal with the given angle.

13. 165°

14. 205°

15. −10°

16. $\frac{\pi}{6}$

17. $\frac{4\pi}{3}$

18. $-\frac{\pi}{4}$

NewVocabulary

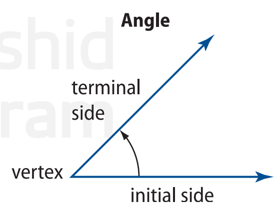
English

polar coordinate system
pole
polar axis
polar coordinates
polar equation
polar graph
limaçon
cardioid
rose
lemniscate
spiral of Archimedes
complex plane
real axis
imaginary axis
Argand plane
absolute value of a complex number
polar form
trigonometric form
modulus
argument

ReviewVocabulary

initial side of an angle the starting position of the ray

terminal side of an angle the ray's position after rotation



measure of an angle the amount and direction of rotation necessary to move from the initial side to the terminal side of the angle

---

The term interest refers to an amount of money that is paid or received when borrowing or lending money. If a customer borrows money from a bank, the customer pays the bank interest for the use of its money. If a customer saves money in a bank account, the bank pays the customer interest for the use of his or her money.

The amount of money that is initially borrowed or saved is called the principal. The interest rate is a percentage earned or charged during a certain time period. Simple interest is the amount of interest charged or earned after the interest rate is applied to the principal.

Simple interest (I) is the product of three values: the principal (P), the interest rate written as a decimal number (r), and time (*t*): $I = P \times r \times t$.

LESSON 8-1 Polar Coordinates

Then

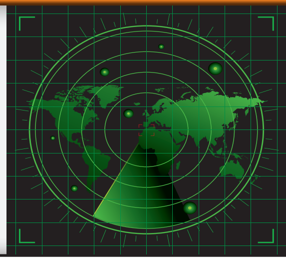
- You drew positive and negative angles given in degrees and radians in standard position.

Now

1. Graph points with polar coordinates.
2. Graph simple polar equations.

Why?

- To provide safe routes and travel, air traffic controllers use advanced radar systems to direct the flow of airplane traffic. This ensures that airplanes keep a sufficient distance from other aircraft and landmarks. The radar uses angle measure and directional distance to plot the positions of aircraft. Controllers can then relay this information to the pilots.



NewVocabulary

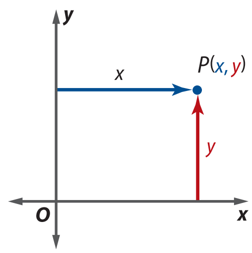
polar coordinate system
pole
polar axis
polar coordinates
polar equation
polar graph

1 Graph Polar Coordinates

To this point, you have graphed equations in a rectangular coordinate system. When air traffic controllers record the locations of airplanes using distances and angles, they are using a **polar coordinate system** or polar plane.

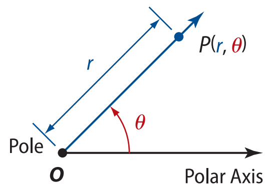
In a rectangular coordinate system, the *x*- and *y*-axes are horizontal and vertical lines, respectively, and their point of intersection *O* is called the origin. The location of a point *P* is identified by rectangular coordinates of the form (x, y), where *x* and *y* are the horizontal and vertical *directed distances*, respectively, to the point. For example, the point $(3, -4)$ is 3 units to the right of the *y*-axis and 4 units below the *x*-axis.

Rectangular Coordinate System



In a polar coordinate system, the origin is a fixed point *O* called the **pole**. The **polar axis** is an initial ray from the pole, usually horizontal and directed toward the right. The location of a point *P* in the polar coordinate system can be identified by **polar coordinates** of the form (r, θ), where *r* is the directed distance from the pole to the point and θ is the *directed angle* from the polar axis to $\overrightarrow{OP}$.

Polar Coordinate System



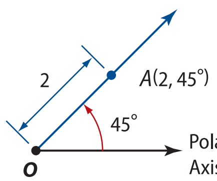
To graph a point given in polar coordinates, remember that a positive value of θ indicates a counterclockwise rotation from the polar axis, while a negative value indicates a clockwise rotation. If *r* is positive, then *P* lies on the terminal side of θ. If *r* is negative, *P* lies on the ray opposite the terminal side of θ.

Example 1 Graph Polar Coordinates

Graph each point.

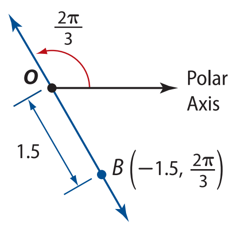
a. $A(2, 45°)$

Because $\theta = 45°$, sketch the terminal side of a 45° angle with the polar axis as its initial side. Because $r = 2$, plot a point 2 units from the pole along the terminal side of this angle.



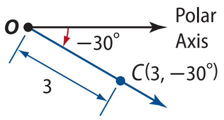
b. $B\left(-1.5, \frac{2\pi}{3}\right)$

Because $\theta = \frac{2\pi}{3}$, sketch the terminal side of a $\frac{2\pi}{3}$ angle with the polar axis as its initial side. Because *r* is negative, extend the terminal side of the angle in the *opposite* direction and plot a point 1.5 units from the pole along this extended ray.



c. $C(3, -30°)$

Because $\theta = -30°$, sketch the terminal side of a $-30°$ angle with the polar axis as its initial side. Because $r = 3$, plot a point 3 units from the pole along the terminal side of this angle.



GuidedPractice

Graph each point.

1A. $D\left(-1, \frac{\pi}{2}\right)$

1B. $E(2.5, 240°)$

1C. $F\left(4, -\frac{5\pi}{6}\right)$

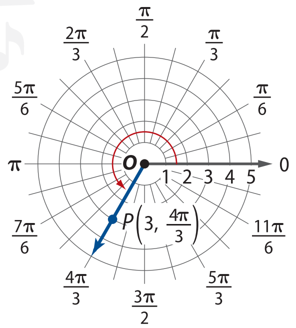
Just as rectangular coordinates are graphed on a rectangular grid, polar coordinates are graphed on a circular or *polar* grid representing the polar plane.

Example 2 Graph Points on a Polar Grid

Graph each point on a polar grid.

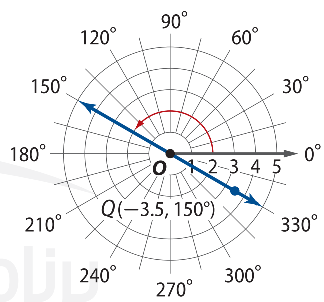
a. $P\left(3, \frac{4\pi}{3}\right)$

Because $\theta = \frac{4\pi}{3}$, sketch the terminal side of a $\frac{4\pi}{3}$ angle with the polar axis as its initial side. Because $r = 3$, plot a point 3 units from the pole along the terminal side of the angle.



b. $Q(-3.5, 150°)$

Because $\theta = 150°$, sketch the terminal side of a $150°$ angle with the polar axis as its initial side. Because r is negative, extend the terminal side of the angle in the *opposite* direction and plot a point 3.5 units from the pole along this extended ray.



GuidedPractice

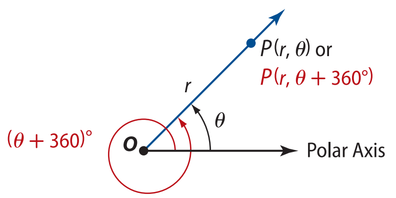
2A. $R\left(1.5, -\frac{7\pi}{6}\right)$

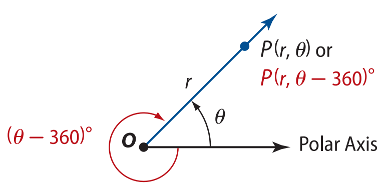
2B. $S(-2, -135°)$

StudyTip

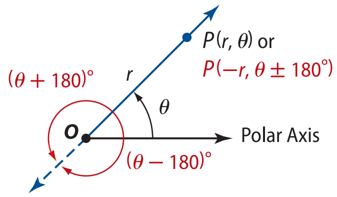
Pole The pole can be represented by $(0, \theta)$, where θ is any angle.

In a rectangular coordinate system, each point has a unique set of coordinates. This is *not* true in a polar coordinate system. In Lesson 4-2, you learned that a given angle has infinitely many coterminal angles. As a result, if a point has polar coordinates (r, θ), then it also has polar coordinates $(r, \theta \pm 360°)$ or $(r, \theta \pm 2\pi)$ as shown.





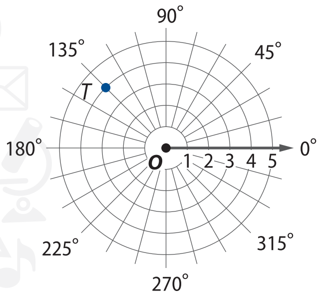
Additionally, because r is a directed distance, (r, θ) and $(-r, \theta \pm 180°)$ or $(-r, \theta \pm \pi)$ represent the same point as shown.



In general, if n is any integer, the point with polar coordinates (r, θ) can also be represented by polar coordinates of the form $(r, \theta + 360n°)$ or $(-r, \theta + (2n + 1)180°)$. Likewise, if θ is given in radians and n is any integer, the other representations of (r, θ) are of the form $(r, \theta + 2n\pi)$ or $(-r, \theta + (2n + 1)\pi)$.

Example 3 Multiple Representations of Polar Coordinates

Find four different pairs of polar coordinates that name point T if $-360° \leq \theta \leq 360°$.



One pair of polar coordinates that name point T is $(4, 135°)$. The other three representations are as follows.

$(4, 135°) = (4, 135° - 360°)$ Subtract 360° from θ.
$= (4, -225°)$

$(4, 135°) = (-4, 135° + 180°)$ Replace r with $-r$ and
$= (-4, 315°)$ add 180° to θ.

$(4, 135°) = (-4, 135° - 180°)$ Replace r with $-r$ and
$= (-4, -45°)$ subtract 180° from θ.

GuidedPractice

Find three additional pairs of polar coordinates that name the given point if $-360° \leq \theta \leq 360°$ or $-2\pi \leq \theta \leq \pi$.

3A. $(5, 240°)$

3B. $\left(2, \frac{\pi}{6}\right)$

2 Graphs of Polar Equations

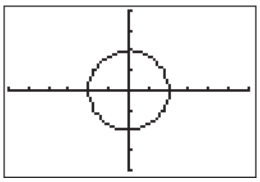
An equation expressed in terms of polar coordinates is called a **polar equation**. For example, $r = 2 \sin \theta$ is a polar equation. A **polar graph** is the set of all points with coordinates (r, θ) that satisfy a given polar equation.

You already know how to graph equations in the Cartesian, or *rectangular*, coordinate system. Graphs of equations involving constants like $x = 2$ and $y = -3$ are considered basic in the Cartesian coordinate system. Similarly, the graphs of the polar equations $r = k$ and $\theta = k$, where k is a constant, are considered basic in the polar coordinate system.

TechnologyTip

Graphing Polar Equations

To graph the polar equation $r = 2$ on a graphing calculator, first press MODE and change the graphing setting from FUNC to POL. When you press Y=, notice that the dependent variable has changed from Y to r and the independent variable from x to θ. Graph $r = 2$.



$[0, 2\pi]$ scl: $\frac{\pi}{16}$ by $[-6, 6]$ scl: 1 by $[-4, 4]$ scl: 1

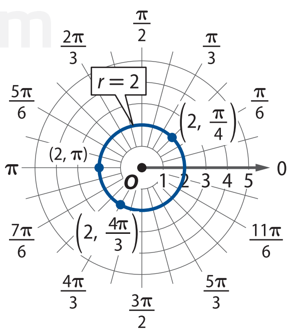
Example 4 Graph Polar Equations

Graph each polar equation.

a. $r = 2$

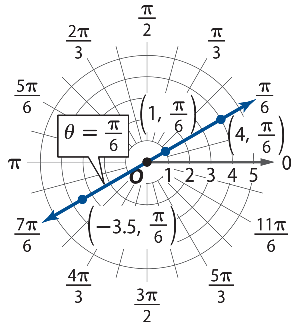
The solutions of $r = 2$ are ordered pairs of the form $(2, \theta)$, where θ is any real number.

The graph consists of all points that are 2 units from the pole, so the graph is a circle centered at the origin with radius 2.



b. $\theta = \frac{\pi}{6}$

The solutions of $\theta = \frac{\pi}{6}$ are ordered pairs of the form $\left(r, \frac{\pi}{6}\right)$, where r is any real number. The graph consists of all points on the line that makes an angle of $\frac{\pi}{6}$ with the positive polar axis.



GuidedPractice

Graph each polar equation.

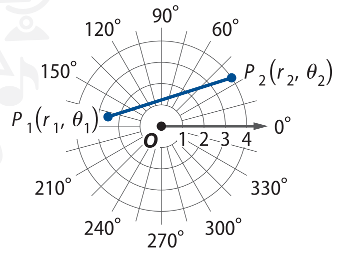
4A. $r = 3$ **4B.** $\theta = \frac{2\pi}{3}$

The distance between two points in the polar plane can be found using the following formula.

KeyConcept Polar Distance Formula

If $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ are two points in the polar plane, then the distance P_1P_2 is given by

$$\sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}.$$



You will prove this formula in Exercise 63.

WatchOut!

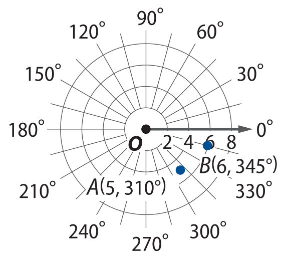
Mode When using the Polar Distance Formula, if θ is given in degrees, make sure your graphing calculator is set in degree mode.

Real-World Example 5 Find the Distance Between Polar Coordinates

AIR TRAFFIC **An air traffic controller is tracking two airplanes that are flying at the same altitude. The coordinates of the planes are $A(5, 310°)$ and $B(6, 345°)$, where the directed distance is measured in kilometers.**

a. Sketch a graph of this situation.

Airplane A is located 5 kilometers from the pole on the terminal side of the angle 310°, and airplane B is located 6 kilometers from the pole on the terminal side of the angle 345°, as shown.



b. If regulations prohibit airplanes from passing within three kilometers of each other, are these airplanes in violation? Explain.

Use the Polar Distance Formula.

$AB = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$ Polar Distance Formula

$= \sqrt{5^2 + 6^2 - 2(5)(6)\cos(345° - 310°)}$ or about 3.44 $(r_2, \theta_2) = (6, 345°)$ and $(r_1, \theta_1) = (5, 310°)$

The planes are about 3.44 kilometers apart, so they are not in violation of this regulation.

GuidedPractice

5. **BOATS** A naval radar is tracking two aircraft carriers. The coordinates of the two carriers are (8, 150°) and (3, 65°), with r measured in kilometers.

A. Sketch a graph of this situation.

B. What is the distance between the two aircraft carriers?

Real-WorldLink

Germany developed a radar device in 1936 that could detect planes in an 128-kilometer radius. The following year, Germany was credited with supplying a battleship, the *Graf Spee*, with the first radar system.

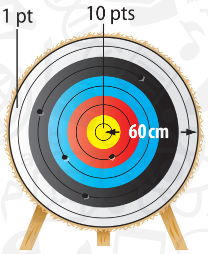
Source: A History of the World Semiconductor Industry

Exercises

Graph each point on a polar grid. (Examples 1 and 2)

1. $R(1, 120°)$
2. $T(-2.5, 330°)$
3. $F\left(-2, \frac{2\pi}{3}\right)$
4. $A\left(3, \frac{\pi}{6}\right)$
5. $Q\left(4, -\frac{5\pi}{6}\right)$
6. $B(5, -60°)$
7. $D\left(-1, -\frac{5\pi}{3}\right)$
8. $G\left(3.5, -\frac{11\pi}{6}\right)$
9. $C(-4, \pi)$
10. $M(0.5, 270°)$
11. $P(4.5, -210°)$
12. $W(-1.5, 150°)$

13. **ARCHERY** The target in competitive target archery consists of 10 evenly spaced concentric circles with score values from 1 to 10 points from the outer circle to the center. Suppose an archer using a target with a 60-centimeter radius shoots arrows at (57, 45°), (41, 315°), and (15, 240°). (Examples 1 and 2)



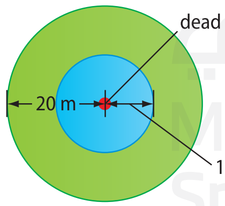
a. Plot the points where the archer's arrows hit the target on a polar grid.

b. How many points did the archer earn?

Find three different pairs of polar coordinates that name the given point if $-360° \le \theta \le 360°$ or $-2\pi \le \theta \le 2\pi$. (Example 3)

14. $(1, 150°)$
15. $(-2, 300°)$
16. $\left(4, -\frac{7\pi}{6}\right)$
17. $\left(-3, \frac{2\pi}{3}\right)$
18. $\left(5, \frac{11\pi}{6}\right)$
19. $\left(-5, -\frac{4\pi}{3}\right)$
20. $(2, -30°)$
21. $(-1, -240°)$

22. **SKYDIVING** In competitive accuracy landing, skydivers attempt to land as near as possible to "dead center," the center of a target marked by a disk 2 meters in diameter. (Example 4)



a. Write polar equations representing the three target boundaries.

b. Graph the equations on a polar grid.

Graph each polar equation. (Example 4)

23. $r = 4$
24. $\theta = 225°$
25. $r = 1.5$
26. $\theta = -\frac{7\pi}{6}$
27. $\theta = -15°$
28. $r = -3.5$

29. **DARTBOARD** A certain dartboard has a radius of 225 millimeters. The bull's-eye has two sections. The 50-point section has a radius of 6.3 millimeters. The 25-point section surrounds the 50-point section for an additional 9.7 millimeters. (Example 4)

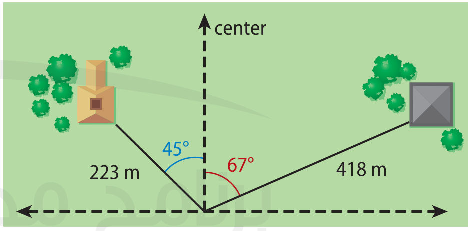
a. Write and graph polar equations representing the boundaries of the dartboard and these sections.

b. What percentage of the dartboard's area does the bull's-eye comprise?

Find the distance between each pair of points. (Example 5)

30. $(2, 30°), (5, 120°)$
31. $\left(3, \frac{\pi}{2}\right), \left(8, \frac{4\pi}{3}\right)$
32. $(6, 45°), (-3, 300°)$
33. $\left(7, -\frac{\pi}{3}\right), \left(1, \frac{2\pi}{3}\right)$
34. $\left(-5, \frac{7\pi}{6}\right), \left(4, \frac{\pi}{6}\right)$
35. $(4, -315°), (1, 60°)$
36. $(-2, -30°), (8, 210°)$
37. $\left(-3, \frac{11\pi}{6}\right), \left(-2, \frac{5\pi}{6}\right)$
38. $\left(1, -\frac{\pi}{4}\right), \left(-5, \frac{7\pi}{6}\right)$
39. $(7, -90°), (-4, -330°)$
40. $\left(8, -\frac{2\pi}{3}\right), \left(4, -\frac{3\pi}{4}\right)$
41. $(-5, 135°), (-1, 240°)$

42. **SURVEYING** A surveyor mapping out the land where a new housing development will be built identifies a landmark 223 meters away and 45° left of center. A second landmark is 418 meters away and 67° right of center. Determine the distance between the two landmarks. (Example 5)



43. **SURVEILLANCE** A mounted surveillance camera oscillates and views part of a circular region determined by $-60° \le \theta \le 150°$ and $0 \le r \le 40$, where r is in meters.

a. Sketch a graph of the region that the security camera can view on a polar grid.

b. Find the area of the region.

Find a different pair of polar coordinates for each point such that $0 \le \theta \le 180°$ or $0 \le \theta \le \pi$.

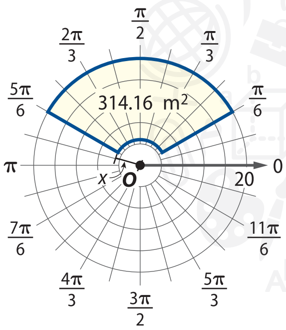
44. $(5, 960°)$
45. $\left(-2.5, \frac{5\pi}{2}\right)$
46. $\left(4, \frac{11\pi}{4}\right)$
47. $(1.25, -920°)$
48. $\left(-1, -\frac{21\pi}{8}\right)$
49. $(-6, -1460°)$

50. **AMPHITHEATER** Suppose a singer is performing at an amphitheater. We can model this situation with polar coordinates by assuming that the singer is standing at the pole and is facing the direction of the polar axis. The seats can then be described as occupying the area defined by $-45° \leq \theta \leq 45°$ and $30 \leq r \leq 240$, where r is measured in meters.

a. Sketch a graph of this region on a polar grid.

b. If each person needs 5 square meters of space, how many seats can fit in the amphitheater?

51. **SECURITY** A security light mounted above a house illuminates part of a circular region defined by $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$ and $x \leq r \leq 20$, where r is measured in meters. If the total area of the region is approximately 314.16 square meters, find x.



Find a value for the missing coordinate that satisfies the following condition.

52. $P_1 = (3, 35°)$; $P_2 = (r, 75°)$; $P_1P_2 = 4.174$

53. $P_1 = (5, 125°)$; $P_2 = (2, \theta)$; $P_1P_2 = 4$; $0 \leq \theta \leq 180°$

54. $P_1 = (3, \theta)$; $P_2 = \left(4, \frac{7\pi}{9}\right)$; $P_1P_2 = 5$; $0 \leq \theta \leq \pi$

55. $P_1 = (r, 120°)$; $P_2 = (4, 160°)$; $P_1P_2 = 3.297$

56. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the relationship between polar coordinates and rectangular coordinates.

a. **GRAPHICAL** Plot points $A\left(2, \frac{\pi}{3}\right)$ and $B\left(4, \frac{5\pi}{6}\right)$ on a polar grid. Sketch a rectangular coordinate system on top of the polar grid so that the origins coincide and the x-axis aligns with the polar axis. The y-axis should align with the line $\theta = \frac{\pi}{2}$. Form one right triangle by connecting point A to the origin and perpendicular to the x-axis. Form another right triangle by connecting point B to the origin and perpendicular to the x-axis.

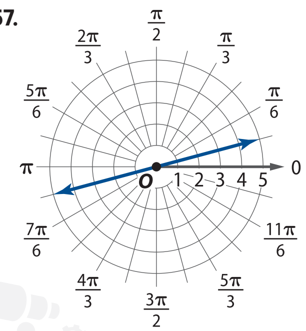
b. **NUMERICAL** Calculate the lengths of the legs of each triangle.

c. **ANALYTICAL** How do the lengths of the legs relate to rectangular coordinates for each point?

d. **ANALYTICAL** Explain the relationship between the polar coordinates (r, θ) and the rectangular coordinates (x, y).

Write an equation for each polar graph.

57.



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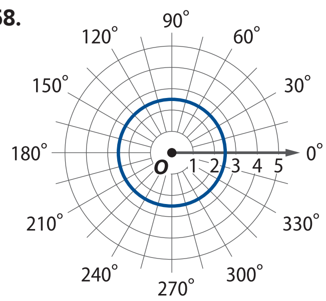

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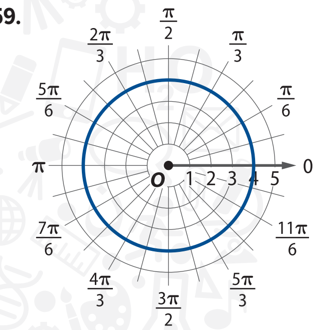
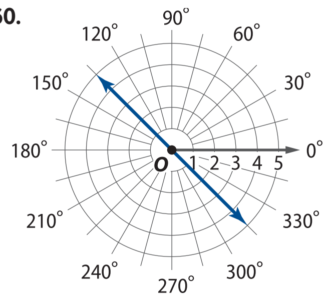


60.



H.O.T. Problems Use Higher-Order Thinking Skills

61. **REASONING** Explain why the order of the points used in the Polar Distance Formula is not important. That is, why can you choose either point to be P_1 and the other to be P_2?

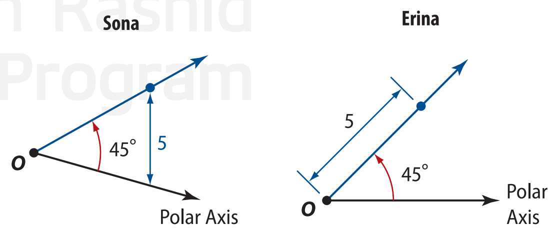
62. **CHALLENGE** Find an ordered pair of polar coordinates to represent the point with rectangular coordinates $(-3, -4)$. Round the angle measure to the nearest degree.

63. **PROOF** Prove that the distance between two points with polar coordinates $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ is

$$P_1P_2 = \sqrt{r_1{}^2 + r_2{}^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}.$$

64. **REASONING** Describe what happens to the Polar Distance Formula when $\theta_2 - \theta_1 = \frac{\pi}{2}$. Explain this change.

65. **ERROR ANALYSIS** Sona and Suhaila both graphed the polar coordinates (5, 45°). Is either of them correct? Explain your reasoning.



66. **WRITING IN MATH** Make a conjecture as to why having the polar coordinates for an aircraft is not enough to determine its exact position.

Spiral Review

Find the dot product of u and v. Then determine if u and v are orthogonal.

67. $\mathbf{u} = \langle 4, 10, 1 \rangle$, $\mathbf{v} = \langle -5, 1, 7 \rangle$

68. $\mathbf{u} = \langle -5, 4, 2 \rangle$, $\mathbf{v} = \langle -4, -9, 8 \rangle$

69. $\mathbf{u} = \langle -8, -3, 12 \rangle$, $\mathbf{v} = \langle 4, -6, 0 \rangle$

Find each of the following for $\mathbf{a} = \langle -4, 3, -2 \rangle$, $\mathbf{b} = \langle 2, 5, 1 \rangle$, and $\mathbf{c} = \langle 3, -6, 5 \rangle$.

70. $3\mathbf{a} + 2\mathbf{b} + 8\mathbf{c}$

71. $-2\mathbf{a} + 4\mathbf{b} - 5\mathbf{c}$

72. $5\mathbf{a} - 9\mathbf{b} + \mathbf{c}$

For each equation, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola.

73. $-14(x - 2) = (y - 7)^2$

74. $(x - 7)^2 = -32(y - 12)$

75. $y = \frac{1}{2}x^2 - 3x + \frac{19}{2}$

76. **STATE FAIR** If Hareb and Zayed each purchased the number of game and ride tickets shown below, what was the price for each type of ticket?

Person	Ticket Type	Tickets	Total (AED)
Hareb	game ride	6 15	93
Zayed	game ride	7 12	81

Write the augmented matrix for the system of linear equations.

77.
$12w + 14x - 10y = 23$
$4w - 5y + 6z = 33$
$11w - 13x + 2z = -19$
$19x - 6y + 7z = -25$

78.
$-6x + 2y + 5z = 18$
$5x - 7y + 3z = -8$
$y - 12z = -22$
$8x - 3y + 2z = 9$

79.
$x + 8y - 3z = 25$
$2x - 5y + 11z = 13$
$-5x + 8z = 26$
$y - 4z = 17$

Solve each equation for all values of *x*.

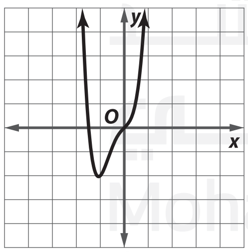
80. $2\cos^2 x + 5\sin x - 5 = 0$

81. $\tan^2 x + 2\tan x + 1 = 0$

82. $\cos^2 x + 3\cos x = -2$

Skills Review for Standardized Tests

83. **SAT/ACT** If the figure shows part of the graph of $f(x)$, then which of the following could be the range of $f(x)$?



A $\{y \mid y \geq -2\}$

B $\{y \mid y \leq -2\}$

C $\{y \mid -2 < y < 1\}$

D $\{y \mid -2 \leq y \leq 1\}$

E $\{y \mid y > -2\}$

84. **REVIEW** Which of the following is the component form of $\overrightarrow{RS}$ with initial point $R(-5, 3)$ and terminal point $S(2, -7)$?

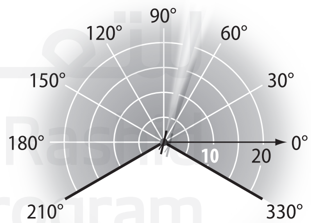
F $\langle 7, -10 \rangle$

G $\langle -3, 10 \rangle$

H $\langle -7, 10 \rangle$

J $\langle -3, -10 \rangle$

85. The lawn sprinkler shown can cover the part of a circular region determined by the polar inequalities $-30^\circ \leq \theta \leq 210^\circ$ and $0 \leq r \leq 20$, where r is measured in meters. What is the approximate area of this region?



A 821 square meters

B 838 square meters

C 852 square meters

D 866 square meters

86. **REVIEW** What type of conic is represented by $25y^2 = 400 + 16x^2$?

F circle

G ellipse

H hyperbola

J parabola

EXPLORE

8-2 Graphing Technology Lab
Investigate Graphs of Polar Equations

Objective

- Use a graphing calculator to explore the shape and symmetry of graphs of polar equations.

StudyTip

Square the Window To view the graphs in this activity without any distortion, square the window by selecting ZSquare under the ZOOM menu.

In Lesson 8-1, you graphed polar coordinates and simple polar equations on the polar coordinate system. Now you will explore the shape and symmetry of more complex graphs of polar equations by using a graphing calculator.

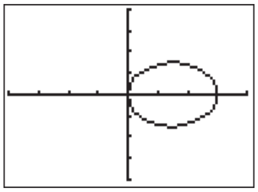
Activity Graph Polar Equations

Graph each equation. Then describe the shape and symmetry of the graph.

a. $r = 3 \cos \theta$

First, change the graph mode to polar and the angle mode to radians. Next, enter $r = 3 \cos \theta$ for r_1 in the Y= list. Use the viewing window shown.

The graph of $r = 3 \cos \theta$ is a circle with a center at (1.5, 0) and radius 1.5 units. The graph is symmetric with respect to the polar axis.

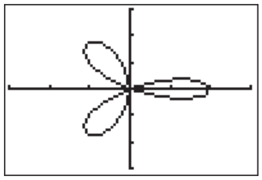


$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-4, 4]$ scl: 1 by $[-4, 4]$ scl: 1

b. $r = 2 \cos 3\theta$

Clear the equation from part **a** in the Y= list and insert $r = 2 \cos 3\theta$. Use the window shown.

The graph of $r = 2 \cos 3\theta$ is a classic polar curve called a rose, which will be covered in Lesson 8-2. The graph has 3 petals and is symmetric with respect to the polar axis.

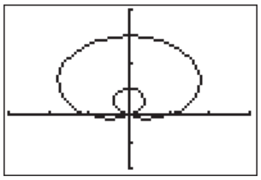


$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-3, 3]$ scl: 1 by $[-3, 3]$ scl: 1

c. $r = 1 + 2 \sin \theta$

Clear the equation from part **b** in the **Y=** list, and enter $r = 1 + 2 \sin \theta$. Adjust the window to display the entire graph.

The graph of $r = 1 + 2 \sin \theta$ is a classic polar curve called a *limaçon,* which will be covered in Lesson 8-2. The graph has a curve with an inner loop and is symmetric with respect to the line $\theta = \frac{\pi}{2}$.



$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-3, 3]$ scl: 1 by $[-2, 4]$ scl: 1

Exercises

Graph each equation. Then describe the shape and symmetry of the graph.

1. $r = -3 \cos \theta$
2. $r = 3 \sin \theta$
3. $r = -3 \sin \theta$
4. $r = 2 \cos 4\theta$
5. $r = 2 \cos 5\theta$
6. $r = 2 \cos 6\theta$
7. $r = 2 + 4 \sin \theta$
8. $r = 1 - 3 \sin \theta$
9. $r = 1 + 2 \sin (-\theta)$

Analyze the Results

10. ANALYTICAL Explain how each value affects the graph of the given equation.

a. the value of n in $r = a \cos n\theta$

b. the value of $|a|$ in $r = b \pm a \sin n\theta$

11. MAKE A CONJECTURE Without graphing $r = 10 \cos 24\theta$, describe the shape and symmetry of the graph. Explain your reasoning.

10a. Sample answer: If n is odd, the number of petals will be equal to n; if n is even, the number of petals will be equal to $2n$.

11. Sample answer: Since the equation is similar to the graph of $r = 2 \cos 4\theta$, which is a rose, $r = 10 \cos 24\theta$ will also be a rose. Since n is even, the rose will have 2(24) or 48 petals and will be symmetric to the polar axis and the line $\theta = \frac{\pi}{2}$.

LESSON 8-2 Graphs of Polar Equations

Then

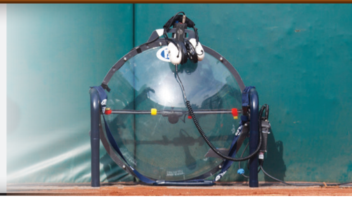
- You graphed functions in the rectangular coordinate system.

Now

1. Graph polar equations.
2. Identify and graph classical curves.

Why?

- To reduce background noise, networks that broadcast sporting events use directional microphones to capture the sounds of the game. Directional microphones have the ability to pick up sound primarily from one direction or region. The sounds that these microphones can detect can be expressed as polar functions.



NewVocabulary
limaçon
cardioid
rose
lemniscate
spiral of Archimedes

1 Graphs of Polar Equations

When you graphed equations on a rectangular coordinate system, you began by using an equation to obtain a set of ordered pairs. You then plotted these coordinates as points and connected them with a smooth curve. In this lesson, you will approach the graphing of polar equations in a similar manner.

Example 1 Graph Polar Equations by Plotting Points

Graph each equation.

a. $r = \cos \theta$

Make a table of values to find the r-values corresponding to various values of θ on the interval $[0, 2\pi]$. Round each r-value to the nearest tenth.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \cos \theta$	1	0.9	0.5	0	−0.5	−0.9	−1	−0.9	−0.5	0	0.5	0.9	1

Graph the ordered pairs (r, θ) and connect them with a smooth curve. It appears that the graph shown in Figure 8.2.1 is a circle with center at (0.5, 0) and radius 0.5 unit.

b. $r = \sin \theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \sin \theta$	0	0.5	0.9	1	0.9	0.5	0	−0.5	−0.9	−1	−0.9	−0.5	0

Graph the ordered pairs and connect them with a smooth curve. It appears that the graph shown in Figure 8.2.2 is a circle with center at $\left(0.5, \frac{\pi}{2}\right)$ and radius 0.5 unit.

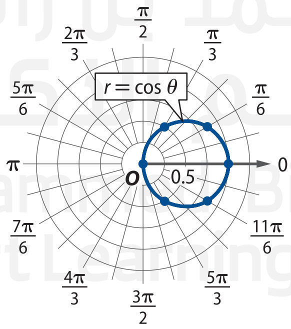


Figure 8.2.1

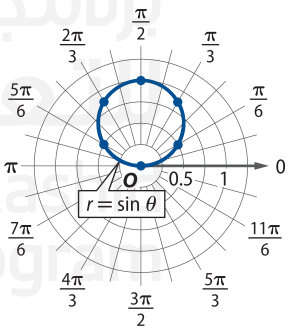


Figure 8.2.2

GuidedPractice

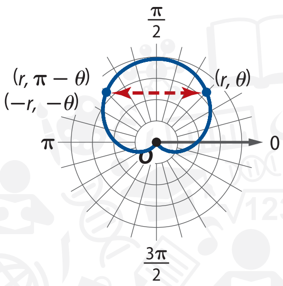
1A. $r = -\sin \theta$ **1B.** $r = 2 \cos \theta$ **1C.** $r = \sec \theta$

Notice that as θ increases on $[0, 2\pi]$, each graph above is traced twice. This is because the polar coordinates obtained on $[0, \pi]$ represent the same points as those obtained on $[\pi, 2\pi]$.

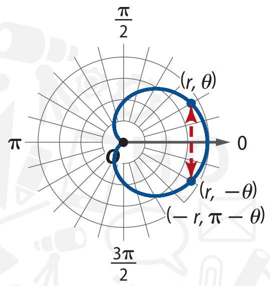
Like knowing whether a graph in the rectangular coordinate system has symmetry with respect to the x-axis, y-axis, or origin, knowing whether the graph of a polar equation is symmetric can help reduce the number of points needed to sketch its graph. Graphs of polar equations can be symmetric with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole, as shown below.

KeyConcept Symmetry of Polar Graphs

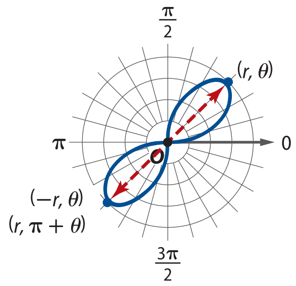
Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$



Symmetry with Respect to Polar Axis



Symmetry with Respect to the Pole



The graphical definitions above provide a way of testing a polar equation for symmetry. For example, if replacing (r, θ) in a polar equation with $(r, -\theta)$ or $(-r, \pi - \theta)$ produces an equivalent equation, then its graph is symmetric with respect to the polar axis. If an equation passes one of the symmetry tests, this is sufficient to guarantee that the equation has that type of symmetry. The converse, however, is *not* true. If a polar equation fails all of these tests, the graph may still have symmetry.

Example 2 Polar Axis Symmetry

Use symmetry to graph $r = 1 - 2 \cos \theta$.

Replacing (r, θ) with $(r, -\theta)$ yields $r = 1 - 2 \cos(-\theta)$. Because cosine is an even function, $\cos(-\theta) = \cos \theta$, so this equation simplifies to $r = 1 - 2 \cos \theta$. Because the replacement produced an equation equivalent to the original equation, the graph of this equation is symmetric with respect to the polar axis.

Because of this symmetry, you need only make a table of values to find the r-values corresponding to θ on the interval $[0, \pi]$.

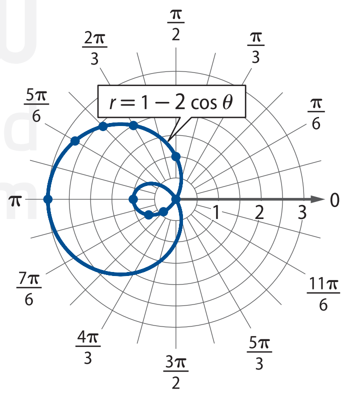
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 1 - 2\cos\theta$	−1	−0.7	−0.4	0	1	2	2.4	2.7	3

StudyTip

Graphing Polar Equations It is customary to graph polar functions in radians, rather than in degrees.

Plotting these points and using polar axis symmetry, you obtain the graph shown.

The type of curve is called a **limaçon**. Some limaçons have inner loops like this one. Other limaçons come to a point, have a dimple, or just curve outward.



GuidedPractice

Use symmetry to graph each equation.

2A. $r = 1 - \cos \theta$

2B. $r = 2 + \cos \theta$

In Examples 1 and 2, notice that the graphs of $r = \cos\theta$ and $r = 1 - 2\cos\theta$ are symmetric with respect to the polar axis, while the graph of $r = \sin\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$. These observations can be generalized as follows.

KeyConcept Quick Tests for Symmetry in Polar Graphs

Words The graph of a polar equation is symmetric with respect to

- the polar axis if it is a function of $\cos\theta$, and
- the line $\theta = \frac{\pi}{2}$ if it is a function of $\sin\theta$.

Example The graph of $r = 3 + \sin\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

You will justify these tests for specific cases in Exercises 65–66.

Symmetry can be used to graph polar functions that model real-world situations.

Real-World Example 3 Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$

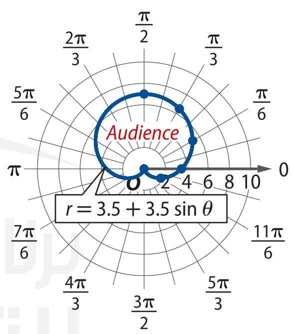
AUDIO TECHNOLOGY During a concert, a directional microphone was placed facing the audience from the center of stage to capture the crowd noise for a live recording. The area of sound the microphone captures can be represented by $r = 3.5 + 3.5\sin\theta$. Suppose the front of the stage faces due north.

a. Graph the polar pattern of the microphone.

Because this polar equation is a function of the sine function, it is symmetric with respect to the line $\theta = \frac{\pi}{2}$. Therefore, make a table and calculate the values of r on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$r = 3.5 + 3.5\sin\theta$	0	0.5	1.0	1.8	3.5	5.25	6.0	6.5	7

Plotting these points and using symmetry with respect to the line $\theta = \frac{\pi}{2}$, you obtain the graph shown. This type of curve is called a **cardioid** (CAR-dee-oid). A cardioid is a special limaçon that has a heart shape.



b. Describe what the polar pattern tells you about the microphone.

The polar pattern indicates that the microphone will pick up sounds up to 7 units away directly in front of the microphone and up to 3.5 units away directly to the left or right of the microphone.

GuidedPractice

3. **VIDEOTAPING** A high school teacher is videotaping presentations performed by her students using a stationary video camera positioned in the back of the room. The area of sound captured by the camera's microphone can be represented by $r = 5 + 2\sin\theta$. Suppose the front of the classroom is due north of the camera.

   A. Graph the polar pattern of the microphone.

   B. Describe what the polar pattern tells you about the microphone.

Real-WorldLink

Live Aid was a 1985 rock concert held in an effort to raise AED 3.6 million for Ethiopian aid. Concerts in London, Philadelphia, and other cities were televised and viewed by 1.9 billion people in 150 countries. The event raised AED 514 million.

Source: CNN

WatchOut!

Graphing over the Period
Usually the period of the trigonometric function used in a polar equation is sufficient to trace the entire graph, but sometimes it is not. The best way to know if you have graphed enough to discern a pattern is to plot more points.

Previously, you used maximum and minimum points along with zeros to aid in graphing trigonometric functions. On the graph of a polar function, r is at its maximum for a value of θ when the distance between that point (r, θ) and the pole is maximized. To find the maximum point(s) on the graph of a polar equation, find the θ-values for which $|r|$ is maximized. Additionally, if $r = 0$ for some value of θ, you know that the graph intersects the pole.

Example 4 Symmetry, Zeros, and Maximum r-Values

Use symmetry, zeros, and maximum r-values to graph $r = 2 \cos 3\theta$.

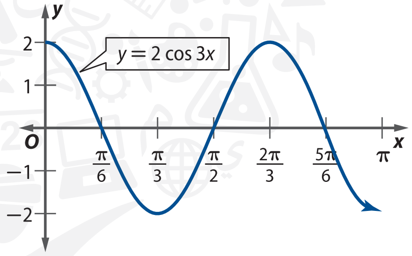
Determine the symmetry of the graph.

This function is symmetric with respect to the polar axis, so you can find points on the interval $[0, \pi]$ and then use polar axis symmetry to complete the graph.

Find the zeros and the maximum r-value.

Sketch the graph of the rectangular function $y = 2 \cos 3x$ on the interval $[0, \pi]$.

From the graph, you can see that $|y| = 2$ when $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, and π and $y = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}$, and $\frac{5\pi}{6}$.



Interpreting these results in terms of the polar equation $r = 2 \cos 3\theta$, we can say that $|r|$ has a maximum value of 2 when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or π and $r = 0$ when $\theta = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$.

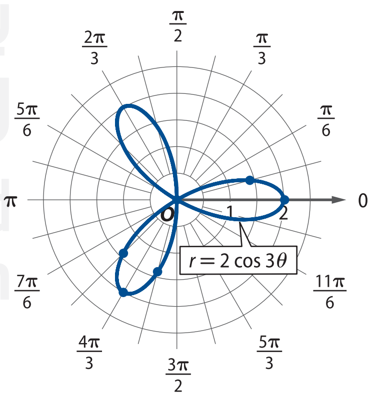
Graph the function.

Use these and a few additional points to sketch the graph of the function.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$r = 2 \cos 3\theta$	2	1.4	0	−1	−2	−1.4	0	1.4	2	1.4	0	−1.4	−2

Notice that polar axis symmetry can be used to complete the graph after plotting points on $\left[0, \frac{\pi}{2}\right]$.

This type of curve is called a **rose**. Roses can have three or more equal loops.



StudyTip

Alternative Method Solving the rectangular function $y = 2 \cos 3x$, we find that the function has extrema when $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or π. Similarly, the function has zeros when $x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$.

GuidedPractice

Use symmetry, zeros, and maximum r-values to graph each function.

4A. $r = 3 \sin 2\theta$

4B. $r = \cos 5\theta$

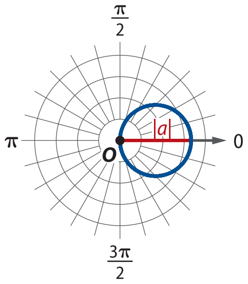
2 Classic Polar Curves

Circles, limaçons, cardioids, and roses are examples of classic curves. The forms and model graphs of these and other classic curves are summarized below.

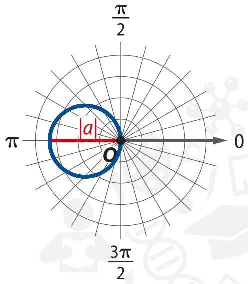
ConceptSummary Special Types of Polar Graphs

Circles

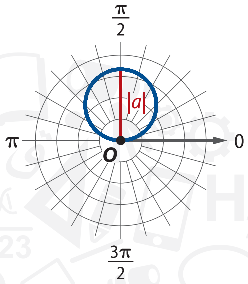
$r = a\cos\theta$ or $r = a\sin\theta$



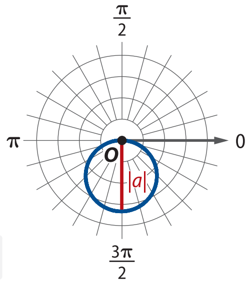
$a > 0$
$r = a\cos\theta$



$a < 0$
$r = a\cos\theta$



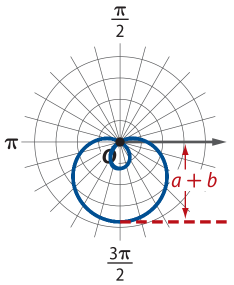
$a > 0$
$r = a\sin\theta$



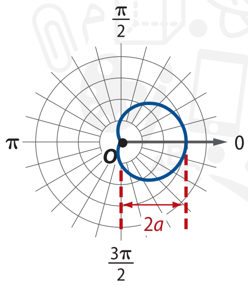
$a < 0$
$r = a\sin\theta$

Limaçons

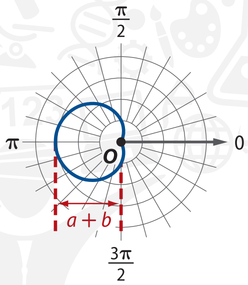
$r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$, where a and b are both positive



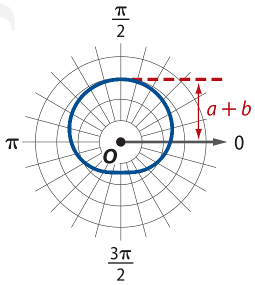
Limaçon with inner loop
$a < b$
$r = a - b\sin\theta$



Cardioid
$a = b$
$r = a + b\cos\theta$



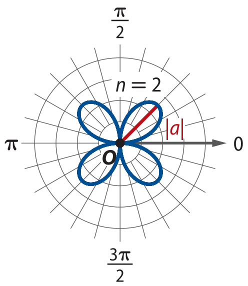
Dimpled limaçon
$b < a < 2b$
$r = a - b\cos\theta$



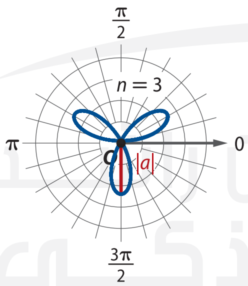
Convex limaçon
$a \geq 2b$
$r = a + b\sin\theta$

Roses

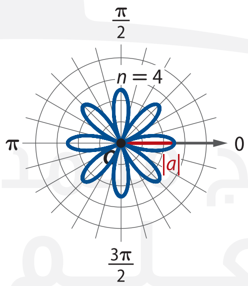
$r = a\cos n\theta$ or $r = a\sin n\theta$, where $n \geq 2$ is an integer
The rose has n petals if n is odd and $2n$ petals if n is even.



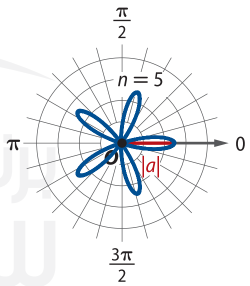
$r = a\sin n\theta$



$r = a\sin n\theta$



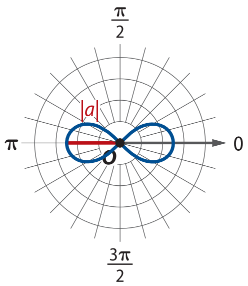
$r = a\cos n\theta$



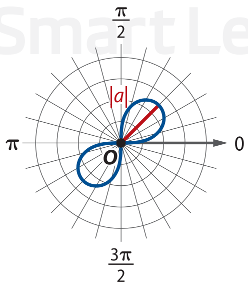
$r = a\cos n\theta$

Lemniscates (LEM-nis-keyts)

$r^2 = a^2\cos 2\theta$ or $r^2 = a^2\sin 2\theta$



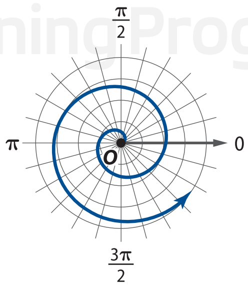
$r^2 = a^2\cos 2\theta$



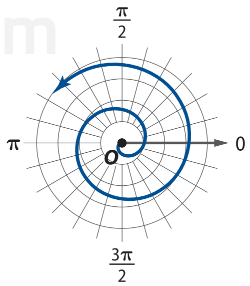
$r^2 = a^2\sin 2\theta$

Spirals of Archimedes (ahr-kuh-MEE-deez)

$r = a\theta + b$



$\theta > 0, r = a\theta + b$



$\theta < 0, r = a\theta + b$

Example 5 Identify and Graph Classic Curves

Identify the type of curve given by each equation. Then use symmetry, zeros, and maximum *r*-values to graph the function.

a. $r^2 = 16 \sin 2\theta$

Type of Curve and Symmetry

The equation is of the form $r^2 = a^2 \sin 2\theta$, so its graph is a lemniscate. Replacing (r, θ) with $(-r, \theta)$ yields $(-r)^2 = 16 \sin 2\theta$ or $r^2 = 16 \sin 2\theta$. Therefore, the function has symmetry with respect to the pole.

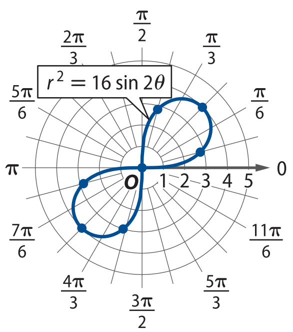
Maximum *r*-Value and Zeros

The equation $r^2 = 16 \sin 2\theta$ is equivalent to $r = \pm 4\sqrt{\sin 2\theta}$, which is undefined when $\sin 2\theta < 0$. Therefore, the domain of the function is restricted to the intervals $\left[0, \frac{\pi}{2}\right]$ or $\left[\pi, \frac{3\pi}{2}\right]$. Because you can use pole symmetry, you need only graph points in the interval $\left[0, \frac{\pi}{2}\right]$. The function attains a maximum *r*-value of $|a|$ or 4 when $\theta = \frac{\pi}{4}$ and zero *r*-value when $x = 0$ and $\frac{\pi}{2}$.

Graph

Use these points and the indicated symmetry to sketch the graph of the function.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	0	±2.8	±3.7	±4	±3.7	±2.8	0



b. $r = 3\theta$

Type of Curve and Symmetry

The equation is of the form $r = a\theta + b$, so its graph is a spiral of Archimedes. Replacing (r, θ) with $(-r, -\theta)$ yields $(-r) = 3(-\theta)$ or $r = 3\theta$. Therefore, the function has symmetry with respect to the line $\theta = \frac{\pi}{2}$.

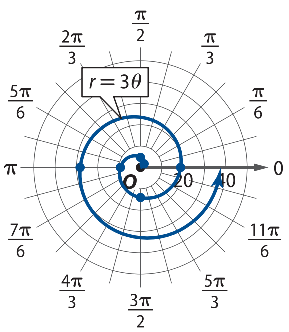
Maximum *r*-Value and Zeros

Spirals are unbounded. Therefore, the function has no maximum *r*-values and only one zero when $\theta = 0$.

Graph

Use points on the interval $[0, 4\pi]$ to sketch the graph of the function. To show symmetry, points on the interval $[-4\pi, 0]$ should also be graphed.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	3π	4π
r	0	2.4	4.7	9.4	14.1	18.8	28.3	37.7



TechnologyTip

Window Settings θmin and θmax determine the values of θ that will be graphed. Normal settings for these are θmin=0 and θmax=2π, although it may be necessary to change these values to obtain a complete graph. θstep determines the interval for plotting points. The smaller this value is, the smoother the look of the graph.

GuidedPractice

5A. $r^2 = 9 \cos 2\theta$

5B. $r = 3 \sin 5\theta$

Exercises

Graph each equation by plotting points. (Example 1)

1. $r = -\cos\theta$
2. $r = \csc\theta$
3. $r = \frac{1}{2}\cos\theta$
4. $r = 3\sin\theta$
5. $r = -\sec\theta$
6. $r = \frac{1}{3}\sin\theta$
7. $r = -4\cos\theta$
8. $r = -\csc\theta$

Use symmetry to graph each equation. (Examples 2 and 3)

9. $r = 3 + 3\cos\theta$
10. $r = 1 + 2\sin\theta$
11. $r = 4 - 3\cos\theta$
12. $r = 2 + 4\cos\theta$
13. $r = 2 - 2\sin\theta$
14. $r = 3 - 5\cos\theta$
15. $r = 5 + 4\sin\theta$
16. $r = 6 - 2\sin\theta$

Use symmetry, zeros, and maximum r-values to graph each function. (Example 4)

17. $r = \sin 4\theta$
18. $r = 2\cos 2\theta$
19. $r = 5\cos 3\theta$
20. $r = 3\sin 2\theta$
21. $r = \frac{1}{2}\sin 3\theta$
22. $r = 4\cos 5\theta$
23. $r = 2\sin 5\theta$
24. $r = 3\cos 4\theta$

25. **MARINE BIOLOGY** Rose curves can be observed in marine wildlife. Determine the symmetry, zeros, and maximum r-values of each function modeling a marine species for $0 \le \theta \le \pi$. Then use the information to graph the function. (Example 4)
    a. The pores forming the petal pattern of a sand dollar (Figure 8.2.3) can be modeled by $r = 3\cos 5\theta$.
    b. The outline of the body of a crown-of-thorns sea star (Figure 8.2.4) can be modeled by $r = 20\cos 8\theta$.

Figure 8.2.3

Figure 8.2.4

Identify the type of curve given by each equation. Then use symmetry, zeros, and maximum r-values to graph the function. (Example 5)

26. $r = \frac{1}{3}\cos\theta$
27. $r = 4\theta + 1;\ \theta > 0$
28. $r = 2\sin 4\theta$
29. $r = 6 + 6\cos\theta$
30. $r^2 = 4\cos 2\theta$
31. $r = 5\theta + 2;\ \theta > 0$
32. $r = 3 - 2\sin\theta$
33. $r^2 = 9\sin 2\theta$

34. **FIGURE SKATING** The original focus of figure skating was to carve figures, known as *compulsory figures,* into the ice. The shape of one of these figures can be modeled by $r^2 = 25\cos 2\theta$. (Example 5)
    a. Which classic curve does the figure model?
    b. Graph the model.

Write an equation for each graph.

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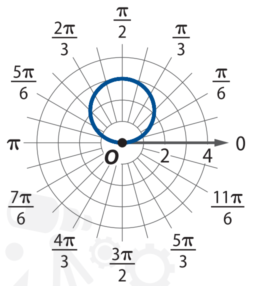

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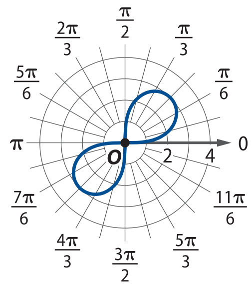

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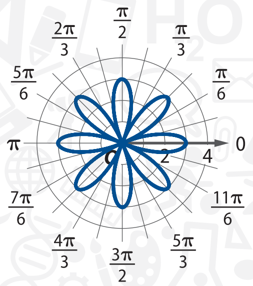

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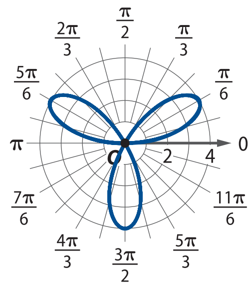

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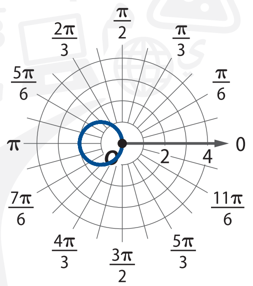

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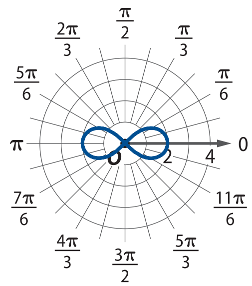


41. **FAN** A ceiling fan has a central motor with five blades that each extend 4 units from the center. The shape of the fan can be represented by a rose curve.
    a. Write two polar equations that can be used to represent the fan.
    b. Sketch two graphs of the fan using the equations that you wrote.

Use one of the three tests to prove the specified symmetry.

42. $r = 3 + \sin\theta$, symmetric about the line $\theta = \frac{\pi}{2}$
43. $r^2 = 4\sin 2\theta$, symmetric about the pole
44. $r = 3\sin 2\theta$, symmetric about the polar axis
45. $r = 5\cos 8\theta$, symmetric about the line $\theta = \frac{\pi}{2}$
46. $r = 2\sin 4\theta$, symmetric about the pole

47. **FOUR-LEAF CLOVER** The shape of a certain type of clover can be represented using a rose curve. Write a polar equation for the clover if it has:
    a. 5 petals with a length of 2 units each.
    b. 4 petals with a length of 7 units each.
    c. 8 petals with a length of 6 units each.

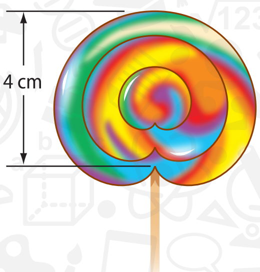
48. **CONCERT** For a concert, a circular stage is constructed and placed in the center so fans can completely surround the musicians. To record the sound of the crowd, two directional microphones are placed next to each other on the stage, one facing due east and the other facing due west. The patterns of the microphones can be represented by the polar equations $r = 2.5 + 2.5 \cos \theta$ and $r = -2.5 - 2.5 \cos \theta$.

   a. Identify the type of curve given by each polar equation.

   b. Sketch the graph of each microphone pattern on the same polar grid.

   c. Describe what the graph tells you about the area covered by the microphones.

49. **CANDY** Write an equation that can model this lollipop in the shape of a limaçon if it is symmetric with respect to the line $\theta = \frac{\pi}{2}$ and measures 4 centimeters from the top of the lollipop to where the candy meets the stick.



Match each equation with its graph.

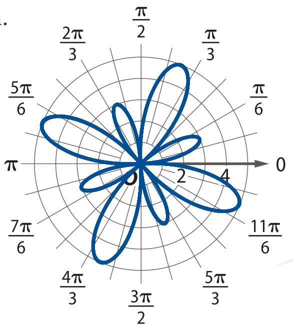
50. $r = 1 + 4 \cos 3\theta$

51. $r = 1 - 4 \sin 4\theta$

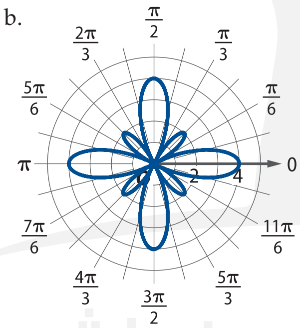
52. $r = 1 - 3 \sin 3\theta$

53. $r = 1 + 3 \cos 4\theta$

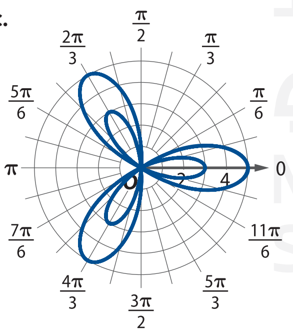
a.



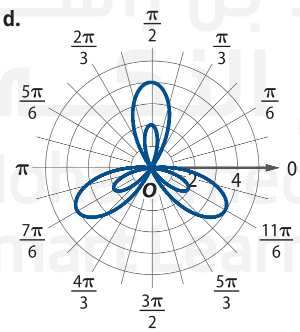
b.



c.



d.



Find x for the interval $0 \leq \theta \leq x$ so that x is a minimum and the graph is complete.

54. $r = 3 + 2 \cos \theta$

55. $r = 2 - \sin 2\theta$

56. $r = 1 + \cos \frac{\theta}{3}$

Match each equation with an equation that produces an equivalent graph.

57. $r = 5 + 4 \cos \theta$

58. $r = -5 + 4 \sin \theta$

59. $r = 5 - 4 \sin \theta$

60. $r = -5 - 4 \cos \theta$

a. $r = 5 + 4 \sin \theta$

b. $r = -5 + 4 \cos \theta$

c. $r = 5 - 4 \cos \theta$

d. $r = -5 - 4 \sin \theta$

61. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate a spiral of Archimedes.

   a. **GRAPHICAL** Sketch separate graphs of $r = \theta$ for the intervals $0 \leq \theta \leq 3\pi$, $-3\pi \leq \theta \leq 0$, and $-3\pi \leq \theta \leq 3\pi$.

   b. **VERBAL** Make a conjecture as to the symmetry of $r = \theta$. Explain your reasoning.

   c. **ANALYTICAL** Prove your conjecture from part **b** by using one of the symmetry tests discussed in this lesson.

   d. **VERBAL** How does changing the interval for θ affect the other classic curves? How does this differ from how the interval affects a spiral of Archimedes? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

62. **ERROR ANALYSIS** Halima and Eiman are graphing polar equations. Eiman says that $r = 7 \sin 2\theta$ is not a function because it does not pass the vertical line test. Halima says the vertical line test does not apply in a polar grid. Is either of them correct? Explain your reasoning.

63. **REASONING** Sketch the graphs of $r_1 = \cos \theta$, $r_2 = \cos\left(\theta - \frac{\pi}{2}\right)$, and $r_3 = \cos(\theta - \pi)$ on the same polar grid. Describe the relationship between the three graphs. Make a conjecture as to the change in a graph when a value d is subtracted from θ.

64. **CHALLENGE** Solve the following system of polar equations algebraically on $[0, 2\pi]$. Graph the system and compare the points of intersection with the solutions that you found. Explain any discrepancies.

   $r = 1 + 2 \sin \theta$

   $r = 4 \sin \theta$

65. **PROOF** Prove that the graph of $r = a + b \cos 2\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

66. **PROOF** Prove that the graph of $r = a \sin 2\theta$ is symmetric with respect to the polar axis.

67. **WRITING IN MATH** Describe the effect of a in the graph of $r = a \cos \theta$.

68. **OPEN ENDED** Sketch the graph of a rose with 8 petals. Then write the equation for your graph.

Spiral Review

Graph each polar equation. (Lesson 8-1)

69. $r = 3.5$

70. $\theta = -\frac{\pi}{3}$

71. $\theta = 225°$

Find the angle θ between vectors u and v to the nearest tenth of a degree.

72. $\mathbf{u} = \langle 4, -3, 5\rangle$, $\mathbf{v} = \langle 2, 6, -8\rangle$

73. $\mathbf{u} = 2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$, $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j} - 11\mathbf{k}$

74. $\mathbf{u} = \langle -1, 1, 5\rangle$, $\mathbf{v} = \langle 7, -6, 9\rangle$

Let $\overrightarrow{DE}$ be the vector with the given initial and terminal points. Write $\overrightarrow{DE}$ as a linear combinatioån of the vectors i and j.

75. $D\left(-5, \frac{2}{3}\right)$, $E\left(-\frac{4}{5}, 0\right)$

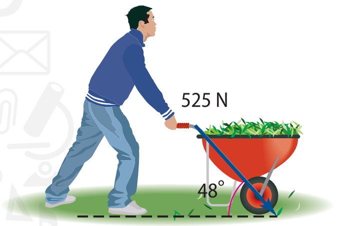
76. $D\left(-\frac{1}{2}, \frac{4}{7}\right)$, $E\left(-\frac{3}{4}, \frac{5}{7}\right)$

77. $D(9.7, -2.4)$, $E(-6.1, -8.5)$

78. YARDWORK Ahmed is pushing a wheelbarrow full of leaves with a force of 525 newtons at a 48° angle with the ground.

a. Draw a diagram that shows the resolution of the force that Kyle is exerting into its rectangular components.

b. Find the magnitudes of the horizontal and vertical components of the force.



Graph the hyperbola given by each equation.

79. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

80. $\frac{(y-4)^2}{16} - \frac{(x+2)^2}{9} = 1$

81. $\frac{(x+1)^2}{4} - \frac{(y+3)^2}{9} = 1$

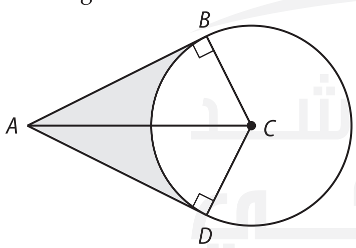
Write an equation for and graph each parabola with focus F and the given characteristics.

82. $F(-5, 8)$; opens right; contains $(-5, 12)$

83. $F(-1, -5)$; opens left; contains $(-1, 5)$

Skills Review for Standardized Tests

84. SAT/ACT In the figure, C is the center of the circle, $AC = 12$, and $m\angle BAD = 60°$. What is the perimeter of the shaded region?



A $12 + 3\pi$

B $6\sqrt{3} + 4\pi$

C $6\sqrt{3} + 3\pi$

D $12\sqrt{3} + 3\pi$

E $12\sqrt{3} + 4\pi$

85. REVIEW While mapping a level site, a surveyor identifies a landmark 450 meters away and 30° left of center and another landmark 600 meters away and 50° right of center. What is the approximate distance between the two landmarks?

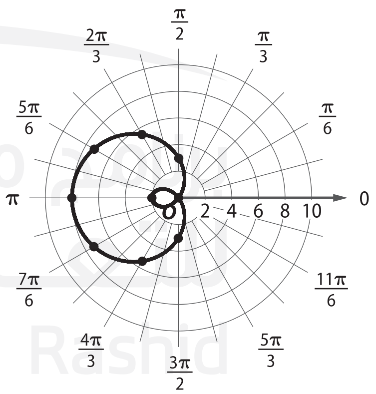
F 672 meters

G 685 meters

H 691 meters

J 703 meters

86. Which type of curve does the figure represent?



A lemniscate

B limaçon

C rose

D cardioid

87. REVIEW An air traffic controller is tracking two jets at the same altitude. The coordinates of the jets are (5, 310°) and (6, 345°), with r measured in kilometers. What is the approximate distance between the jets?

F 2.97 kilometers

G 3.25 kilometers

H 3.44 kilometers

J 3.71 kilometers

LESSON

8-3 Polar and Rectangular Forms of Equations

Then

- You used a polar coordinate system to graph points and equations. (Lessons 9-1 and 9-2)

Now

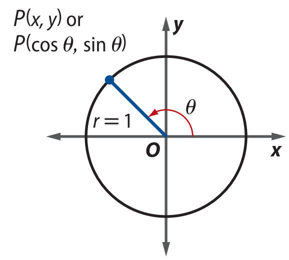
1. Convert between polar and rectangular coordinates.
2. Convert between polar and rectangular equations.

Why?

- An ultrasonic sensor attached to a robot emits an outward beam that rotates through a full circle. The sensor receives a return signal when the beam intercepts an object, and it calculates the position of the object in terms of its distance r and the angle measure θ relative to the front of the robot. The sensor relays these polar coordinates to the robot, which converts them to rectangular coordinates so it can plot the object on an internal map.

1 Polar and Rectangular Coordinates

Recall from Chapter 4 that the coordinates of a point $P(x, y)$ corresponding to an angle θ on a unit circle with radius 1 can be written in terms of θ as $P(\cos\theta, \sin\theta)$ because



$$\cos\theta = \frac{x}{r} = \frac{x}{1} \text{ or } x \qquad \text{and} \qquad \sin\theta = \frac{y}{r} = \frac{y}{1} \text{ or } y.$$

If we let r take on any real value, we can write a point $P(x, y)$ in terms of both r and θ.

$$\cos\theta = \frac{x}{r} \qquad \text{and} \qquad \sin\theta = \frac{y}{r}$$

$$r\cos\theta = x \qquad\qquad r\sin\theta = y \qquad \text{Multiply each side by } r.$$

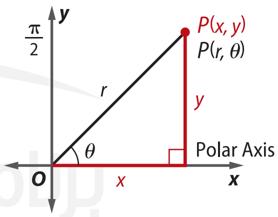
If we let the polar axis and pole in the polar coordinate system coincide with the positive x-axis and origin in the rectangular coordinate system, respectively, we now have a means of converting polar coordinates to rectangular coordinates.

KeyConcept Convert Polar to Rectangular Coordinates

If a point P has polar coordinates (r, θ), then the rectangular coordinates (x, y) of P are given by

$x = r\cos\theta$ and $y = r\sin\theta$.

That is, $(x, y) = (r\cos\theta, r\sin\theta)$.



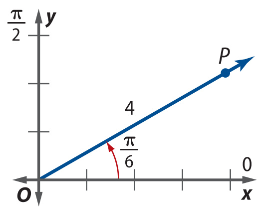
Example 1 Polar Coordinates to Rectangular Coordinates

Find the rectangular coordinates for each point with the given polar coordinates.

a. $P\left(4, \frac{\pi}{6}\right)$

For $P\left(4, \frac{\pi}{6}\right)$, $r = 4$ and $\theta = \frac{\pi}{6}$.

$x = r\cos\theta$	Conversion formula	$y = r\sin\theta$
$= 4\cos\frac{\pi}{6}$	$r = 4$ and $\theta = \frac{\pi}{6}$	$= 4\sin\frac{\pi}{6}$
$= 4\left(\frac{\sqrt{3}}{2}\right)$	Simplify.	$= 4\left(\frac{1}{2}\right)$
$= 2\sqrt{3}$		$= 2$

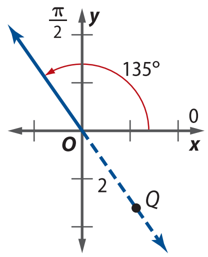


The rectangular coordinates of P are $(2\sqrt{3}, 2)$ or approximately $(3.46, 2)$ as shown.

b. $Q(-2, 135°)$

For $Q(-2, 135°)$, $r = -2$ and $\theta = 135°$.

$x = r \cos \theta$	Conversion formula	$y = r \sin \theta$
$= -2 \cos 135°$	$r = -2$ and $\theta = 135°$	$= -2 \sin 135°$
$= -2\left(-\frac{\sqrt{2}}{2}\right)$	Simplify.	$= -2\left(\frac{\sqrt{2}}{2}\right)$
$= \sqrt{2}$		$= -\sqrt{2}$

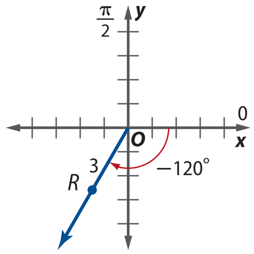


The rectangular coordinates of Q are $(\sqrt{2}, -\sqrt{2})$ or approximately $(1.41, -1.41)$ as shown.

c. $V(3, -120°)$

For $V(3, -120°)$, $r = 3$ and $\theta = -120°$.

$x = r \cos \theta$	Conversion formula	$y = r \sin \theta$
$= 3 \cos -120°$	$r = 3$ and $\theta = -120°$	$= 3 \sin -120°$
$= 3\left(-\frac{1}{2}\right)$	Simplify.	$= 3\left(-\frac{\sqrt{3}}{2}\right)$
$= -\frac{3}{2}$		$= -\frac{3\sqrt{3}}{2}$



The rectangular coordinates of V are $\left(-\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$ or approximately $(-1.5, -2.6)$ as shown.

GuidedPractice

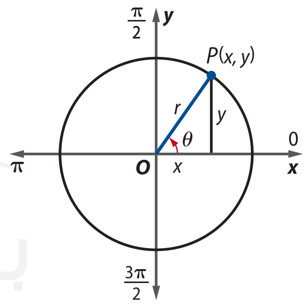
1A. $R(-6, -120°)$ **1B.** $S\left(5, \frac{\pi}{3}\right)$ **1C.** $T(-3, 45°)$

StudyTip

Coordinate Conversions The process for converting rectangular coordinates to polar coordinates is the same as the process used to determine the magnitude and direction of vectors.

To write a pair of rectangular coordinates in polar form, you need to find the distance r a point (x, y) is from the origin or pole and the angle measure θ that point is from the x- or polar axis.

To find the distance r from the point (x, y) to the origin, use the Pythagorean Theorem.



$r^2 = x^2 + y^2$ Pythagorean Theorem

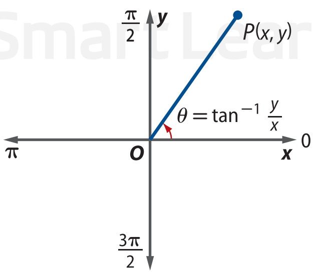
$r = \sqrt{x^2 + y^2}$ Take the positive square root of each side.

The angle θ is related to x and y by the tangent function.

$\tan \theta = \frac{y}{x}$ Tangent Ratio

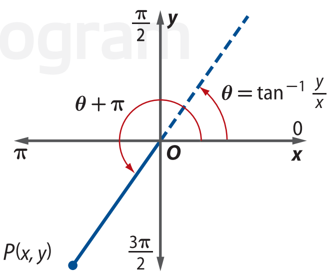
$\theta = \tan^{-1} \frac{y}{x}$ Definition of inverse tangent function

Recall that the inverse tangent function is only defined on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $[-90°, 90°]$. In the rectangular coordinate system, this refers to θ-values in Quadrants I and IV or when $x > 0$, as shown in Figure 8.3.1. If a point is located in Quadrant II or III, which is when $x < 0$, you must add π or 180° to the angle measure given by the inverse tangent function, as shown in Figure 8.3.2.



When $x > 0$, $\theta = \tan^{-1} \frac{y}{x}$.

Figure 8.3.1



When $x < 0$, $\theta = \tan^{-1} \frac{y}{x} + \pi$ or $\theta = \tan^{-1} \frac{y}{x} + 180°$.

Figure 8.3.2

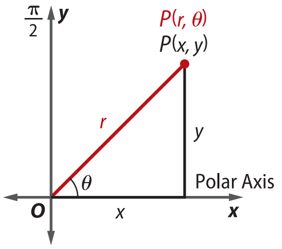
KeyConcept Convert Rectangular to Polar Coordinates

If a point P has rectangular coordinates (x, y) then the polar coordinates (r, θ) of P are given by

$r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$, when $x > 0$

$\theta = \tan^{-1}\frac{y}{x} + \pi$ or

$\theta = \tan^{-1}\frac{y}{x} + 180°$, when $x < 0$.



Recall that polar coordinates are not unique. The conversion from rectangular coordinates to polar coordinates results in just *one* representation of the polar coordinates. There are, however, infinitely many polar representations for a point given in rectangular form.

TechnologyTip

Coordinate Conversions

To convert rectangular coordinates to polar coordinates using a calculator, press [2nd] [APPS] to view the ANGLE menu. Select R▸Pr(and enter the coordinates. This will calculate the value of r. To calculate θ, repeat this process but select R▸Pθ).

Example 2 Rectangular Coordinates to Polar Coordinates

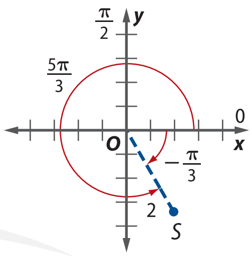
Find two pairs of polar coordinates for each point with the given rectangular coordinates.

a. $S(1, -\sqrt{3})$

For $S(x, y) = (1, -\sqrt{3})$, $x = 1$ and $y = -\sqrt{3}$. Because $x > 0$, use $\tan^{-1}\frac{y}{x}$ to find θ.

$r = \sqrt{x^2 + y^2}$	Conversion formula	$\theta = \tan^{-1}\frac{y}{x}$
$= \sqrt{1^2 + (-\sqrt{3})^2}$	$x = 1$ and $y = -\sqrt{3}$	$= \tan^{-1}\frac{-\sqrt{3}}{1}$
$= \sqrt{4}$ or 2	Simplify.	$= -\frac{\pi}{3}, \frac{5\pi}{3}$

One set of polar coordinate for S is $\left(2, -\frac{\pi}{3}\right)$. Another representation that uses a positive θ-value is $\left(2, -\frac{\pi}{3} + 2\pi\right)$ or $\left(2, \frac{5\pi}{3}\right)$, as shown.



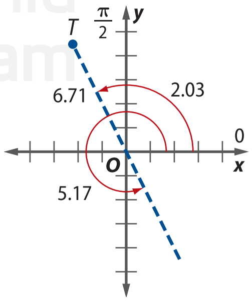
b. $T(-3, 6)$

For $T(x, y) = (-3, 6)$, $x = -3$ and $y = 6$.

Because $x < 0$, use $\tan^{-1}\frac{y}{x} + \pi$ to find θ.

$r = \sqrt{x^2 + y^2}$	Conversion formula	$\theta = \tan^{-1}\frac{y}{x} + \pi$
$= \sqrt{(-3)^2 + 6^2}$	$x = -3$ and $y = 6$	$= \tan^{-1}\left(-\frac{6}{3}\right) + \pi$
$= \sqrt{45}$ or about 6.71	Simplify.	$= \tan^{-1}(-2) + \pi$ or about 2.03

One set of polar coordinates for T is approximately (6.71, 2.03). Another representation that uses a negative r-value is $(-6.71, 2.03 + \pi)$ or $(-6.71, 5.17)$, as shown.



GuidedPractice

Find two pairs of polar coordinates for each point with the given rectangular coordinates. Round to the nearest hundredth, if necessary.

2A. $V(8, 10)$

2B. $W(-9, -4)$

For some real-world phenomena, it is useful to be able to convert between polar coordinates and rectangular coordinates.

Real-World Example 3 Conversion of Coordinates

ROBOTICS **Refer to the beginning of the lesson. Suppose the robot is facing due east and its sensor detects an object at (5, 295°).**

a. What are the rectangular coordinates that the robot will need to calculate?

$x = r \cos \theta$	Conversion formula	$y = r \sin \theta$
$= 5 \cos 295°$	$r = 5$ and $\theta = 295°$	$= 5 \sin 295°$
≈ 2.11	Simplify.	≈ -4.53

The object is located at the rectangular coordinates (2.11, −4.53).

b. If a previously detected object has rectangular coordinates of (3, 7), what are the distance and angle measure of the object relative to the front of the robot?

$r = \sqrt{x^2 + y^2}$	Conversion formula	$\theta = \tan^{-1} \frac{y}{x}$
$= \sqrt{3^2 + 7^2}$	$x = 3$ and $y = 7$	$= \tan^{-1} \frac{7}{3}$
≈ 7.62	Simplify.	$\approx 66.8°$

The object is located at the polar coordinates (7.62, 66.8°).

Guided Practice

3. **FISHING** A fish finder is a type of radar that is used to locate fish under water. Suppose a boat is facing due east, and a fish finder gives the polar coordinates of a school of fish as (6, 125°).

   A. What are the rectangular coordinates for the school of fish?

   B. If a previously detected school of fish had rectangular coordinates of (−2, 6), what are the distance and angle measure of the school relative to the front of the boat?

Real-World Link

NASA's Special Purpose Dexterous Manipulator, or Dextre, is a 1542-kilogram robot that stands 3.7 meters tall with an arm span of 3.4 meter. Dextre is responsible for performing jobs in space that previously required astronauts.

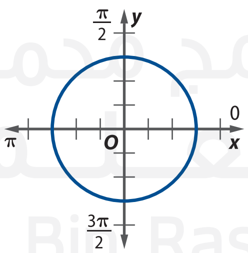
Source: *The New York Times*

2 Polar and Rectangular Equations

In calculus, you will sometimes need to convert from the rectangular form of an equation to its polar form and vice versa to facilitate some calculations. Some complicated rectangular equations have much simpler polar equations. Consider the rectangular and polar equations of the circle graphed below.

Rectangular Equation

$x^2 + y^2 = 9$



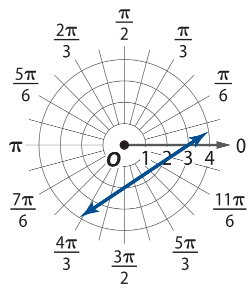
Polar Equation

$r = 3$

Likewise, some polar equations have much simpler rectangular equations, such as the line graphed below.

Polar Equation

$r = \dfrac{6}{2 \cos \theta - 3 \sin \theta}$



Rectangular Equation

$2x - 3y = 6$

The conversion of a rectangular equation to a polar equation is fairly straightforward. Replace x with $r\cos\theta$ and y with $r\sin\theta$, and then simplify the resulting equation using algebraic manipulations and trigonometric identities.

Example 4 Rectangular Equations to Polar Equations

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation.

a. $(x-4)^2 + y^2 = 16$

The graph of $(x-4)^2 + y^2 = 16$ is a circle with radius 4 centered at (4, 0). To find the polar form of this equation, replace x with $r\cos\theta$ and y with $r\sin\theta$. Then simplify.

$(x-4)^2 + y^2 = 16$	Original equation
$(r\cos\theta - 4)^2 + (r\sin\theta)^2 = 16$	$x = r\cos\theta$ and $y = r\sin\theta$
$r^2\cos^2\theta - 8r\cos\theta + 16 + r^2\sin^2\theta = 16$	Multiply.
$r^2\cos^2\theta - 8r\cos\theta + r^2\sin^2\theta = 0$	Subtract 16 from each side.
$r^2\cos^2\theta + r^2\sin^2\theta = 8r\cos\theta$	Isolate the squared terms.
$r^2(\cos^2\theta + \sin^2\theta) = 8r\cos\theta$	Factor.
$r^2(1) = 8r\cos\theta$	Pythagorean Identity
$r = 8\cos\theta$	Divide each side by r.

The graph of this polar equation (Figure 8.3.3) is a circle with radius 4 centered at (4, 0).

StudyTip

Trigonometric Identities You will find it helpful to review trigonometric identities you to help you simplify the polar forms of rectangular equations. A summary of these identities is found inside the back cover of this text.

b. $y = x^2$

The graph of $y = x^2$ is a parabola with vertex at the origin that opens up.

$y = x^2$	Original equation
$r\sin\theta = (r\cos\theta)^2$	$x = r\cos\theta$ and $y = r\sin\theta$
$r\sin\theta = r^2\cos^2\theta$	Multiply.
$\dfrac{\sin\theta}{\cos^2\theta} = r$	Divide each side by $r\cos^2\theta$.
$\dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{1}{\cos\theta} = r$	Rewrite.
$\tan\theta\sec\theta = r$	Quotient and Reciprocal Identities

The graph of the polar equation $r = \tan\theta\sec\theta$ (Figure 8.3.4) is a parabola with vertex at the pole that opens up.

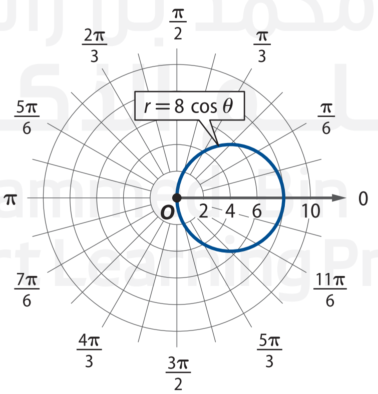


Figure 8.3.3

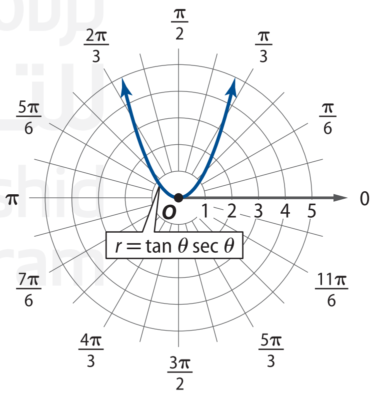


Figure 8.3.4

GuidedPractice

4A. $x^2 + (y-3)^2 = 9$

4B. $x^2 - y^2 = 1$

To write a polar equation in rectangular form, you also make use of the relationships $r^2 = x^2 + y^2$, $x = r \cos \theta$, and $y = r \sin \theta$, as well as the relationship $\tan \theta = \frac{y}{x}$. The process, however, is not as straightforward as converting from rectangular to polar form.

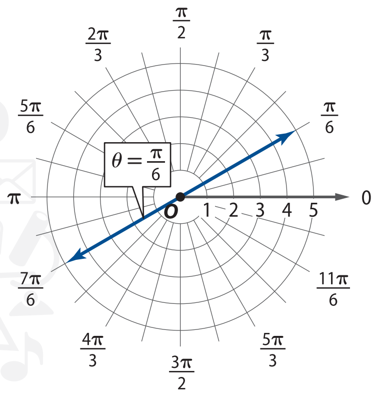
Example 5 Polar Equations to Rectangular Equations

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

a. $\theta = \frac{\pi}{6}$

$\theta = \frac{\pi}{6}$	Original equation
$\tan \theta = \frac{\sqrt{3}}{3}$	Find the tangent of each side.
$\frac{y}{x} = \frac{\sqrt{3}}{3}$	$\tan \theta = \frac{y}{x}$
$y = \frac{\sqrt{3}}{3}x$	Multiply each side by x.

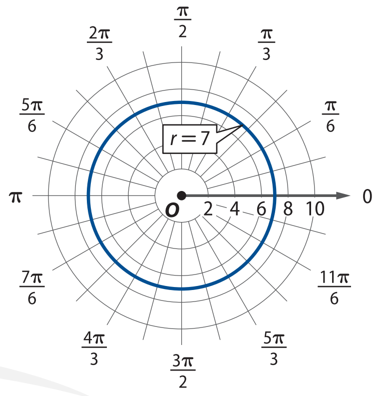
The graph of this equation is a line through the origin with slope $\frac{\sqrt{3}}{3}$ or about $\frac{2}{3}$, as supported by the graph of $\theta = \frac{\pi}{6}$ shown.



b. $r = 7$

$r = 7$	Original equation
$r^2 = 49$	Square each side.
$x^2 + y^2 = 49$	$r^2 = x^2 + y^2$

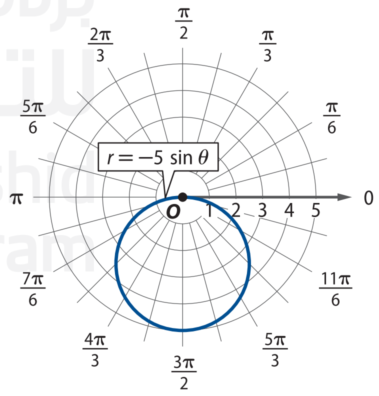
The graph of this rectangular equation is a circle with center at the origin and radius 7, supported by the graph of $r = 7$ shown.



c. $r = -5 \sin \theta$

$r = -5 \sin \theta$	Original equation
$r^2 = -5r \sin \theta$	Multiply each side by r.
$x^2 + y^2 = -5y$	$r^2 = x^2 + y^2$ and $y = r \sin \theta$
$x^2 + y^2 + 5y = 0$	Add $5y$ to each side.

Because in standard form, $x^2 + (y + 2.5)^2 = 6.25$, you can identify the graph of this equation as a circle centered at $(0, -2.5)$ with radius 2.5, as supported by the graph of $r = -5 \sin \theta$.



GuidedPractice

5A. $r = -3$ **5B.** $\theta = \frac{\pi}{3}$ **5C.** $r = 3 \cos \theta$

StudyTip

Alternative Method Two points on the line $\theta = \frac{\pi}{6}$ are $\left(2, \frac{\pi}{6}\right)$ and $\left(4, \frac{\pi}{6}\right)$. In rectangular form, these points are $(\sqrt{3}, 1)$ and $(2\sqrt{3}, 2)$. The equation of the line through these points is $y = \frac{\sqrt{3}}{3}x$.

StudyTip

Converting to Rectangular Form Other useful substitutions are variations of the equations $x = r \cos \theta$ and $y = r \sin \theta$, such as $r = \frac{x}{\cos \theta}$ and $r = \frac{y}{\sin \theta}$.

Exercises

Find the rectangular coordinates for each point with the given polar coordinates. Round to the nearest hundredth, if necessary. (Example 1)

1. $\left(2, \frac{\pi}{4}\right)$

2. $\left(\frac{1}{4}, \frac{\pi}{2}\right)$

3. $(5, 240°)$

4. $(2.5, 250°)$

5. $\left(-2, \frac{4\pi}{3}\right)$

6. $(-13, -70°)$

7. $\left(3, \frac{\pi}{2}\right)$

8. $\left(\frac{1}{2}, \frac{3\pi}{4}\right)$

9. $(-2, 270°)$

10. $(4, 210°)$

11. $\left(-1, -\frac{\pi}{6}\right)$

12. $\left(5, \frac{\pi}{3}\right)$

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \le \theta \le 2\pi$. Round to the nearest hundredth, if necessary. (Example 2)

13. $(7, 10)$

14. $(-13, 4)$

15. $(-6, -12)$

16. $(4, -12)$

17. $(2, -3)$

18. $(0, -173)$

19. $(a, 3a), a > 0$

20. $(-14, 14)$

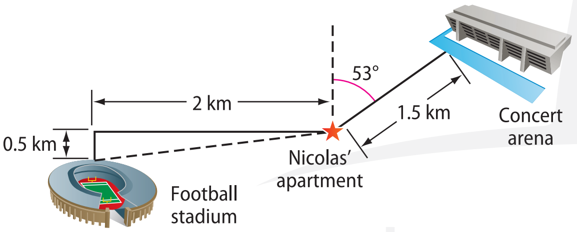
21. $(52, -31)$

22. $(3b, -4b), b > 0$

23. $(1, -1)$

24. $(2, \sqrt{2})$

25. DISTANCE Standing on top of his apartment building, Nicolas determines that a concert arena is 53° east of north. Suppose the arena is exactly 1.5 kilometers from Nicolas' apartment. (Example 3)



a. How many kilometers north and east will Nicolas have to travel to reach the arena?

b. If a football stadium is 2 kilometers west and 0.5 kilometers south of Nicolas' apartment, what are the polar coordinates of the stadium if Nicolas' apartment is at the pole?

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation. (Example 4)

26. $x = -2$

27. $(x + 5)^2 + y^2 = 25$

28. $y = -3$

29. $x = y^2$

30. $(x - 2)^2 + y^2 = 4$

31. $(x - 1)^2 - y^2 = 1$

32. $x^2 + (y + 3)^2 = 9$

33. $y = \sqrt{3}x$

34. $x^2 + (y + 1)^2 = 1$

35. $x^2 + (y - 8)^2 = 64$

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation. (Example 5)

36. $r = 3 \sin \theta$

37. $\theta = -\frac{\pi}{3}$

38. $r = 10$

39. $r = 4 \cos \theta$

40. $\tan \theta = 4$

41. $r = 8 \csc \theta$

42. $r = -4$

43. $\cot \theta = -7$

44. $\theta = \frac{3\pi}{4}$

45. $r = \sec \theta$

46. EARTHQUAKE An equation to model the seismic waves of an earthquake is $r = 12.6 \sin \theta$, where r is measured in kilometers. (Example 5)

a. Graph the polar pattern of the earthquake.

b. Write an equation in rectangular form to model the seismic waves.

c. Find the rectangular coordinates of the epicenter of the earthquake, and describe the area that is affected by the earthquake.

47. MICROPHONE The polar pattern for a directional microphone at a football game is given by $r = 2 + 2 \cos \theta$. (Example 5)

a. Graph the polar pattern.

b. Will the microphone detect a sound that originates from the point with rectangular coordinates $(-2, 0)$? Explain.

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

48. $r = \frac{1}{\cos \theta + \sin \theta}$

49. $r = 10 \csc \left(\theta + \frac{7\pi}{4}\right)$

50. $r = 3 \csc \left(\theta - \frac{\pi}{2}\right)$

51. $r = -2 \sec \left(\theta - \frac{11\pi}{6}\right)$

52. $r = 4 \sec \left(\theta - \frac{4\pi}{3}\right)$

53. $r = \frac{5 \cos \theta + 5 \sin \theta}{\cos^2 \theta - \sin^2 \theta}$

54. $r = 2 \sin \left(\theta + \frac{\pi}{3}\right)$

55. $r = 4 \cos \left(\theta + \frac{\pi}{2}\right)$

56. ASTRONOMY Polar equations are used to model the paths of satellites or other orbiting bodies in space. Suppose the path of a satellite is modeled by $r = \frac{4}{4 + 3 \sin \theta}$, where r is measured in tens of thousands of kilometers, with Earth at the pole.

a. Sketch a graph of the path of the satellite.

b. Determine the minimum and maximum distances the satellite is from Earth at any time.

c. Suppose a second satellite passes through a point with rectangular coordinates $(1.5, -3)$. Are the two satellites at risk of ever colliding at this point? Explain.

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation.

57. $6x - 3y = 4$

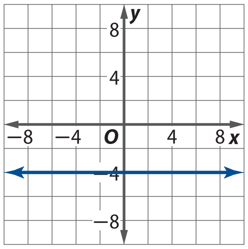
58. $2x + 5y = 12$

59. $(x - 6)^2 + (y - 8)^2 = 100$

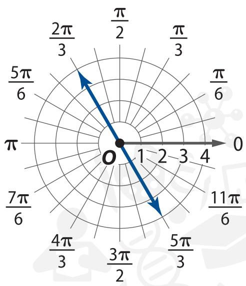
60. $(x + 3)^2 + (y - 2)^2 = 13$

Write rectangular and polar equations for each graph.

61.



62.



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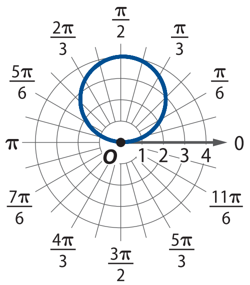

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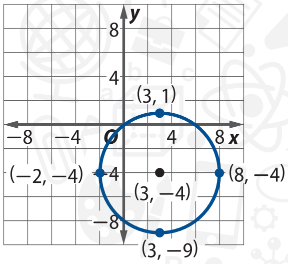
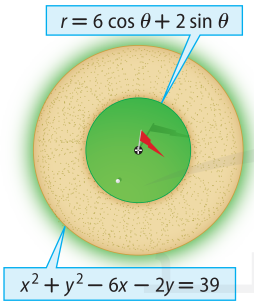
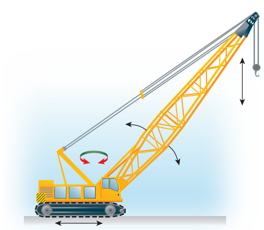
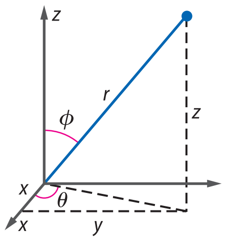


65. **GOLF** On the 18th hole at Hilly Pines Golf Course, the circular green is surrounded by a ring of sand as shown in the figure. Find the area of the region covered by sand assuming the hole acts as the pole for both equations and units are given in meters.



66. **CONSTRUCTION** Boom cranes operate on three-dimensional counterparts of polar coordinates called *spherical coordinates*. A point in space has spherical coordinates (r, θ, ϕ), where r represents the distance from the pole, θ represents the angle of rotation about the vertical axis, and ϕ represents the polar angle from the positive vertical axis. Given a point in spherical coordinates (r, θ, ϕ) find the rectangular coordinates (x, y, z) in terms of r, θ, and ϕ.





67. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the relationship between complex numbers and polar coordinates.

a. **GRAPHICAL** The complex number $a + bi$ can be plotted on a complex plane using the ordered pair (a, b), where the x-axis is the real axis R and the y-axis is the imaginary axis $\boldsymbol{i}$. Graph the complex number $6 + 8i$.

b. **NUMERICAL** Find polar coordinates for the complex number using the rectangular coordinates plotted in part **a** if $0 < \theta < 360°$. Graph the coordinates on a polar grid.

c. **GRAPHICAL** Graph the complex number $-3 + 3i$ on a rectangular coordinate system.

d. **GRAPHICAL** Find polar coordinates for the complex number using the rectangular coordinates plotted in part **c** if $0 < \theta < 360°$. Graph the coordinates on a polar grid.

e. **ANALYTICAL** For a complex number $a + bi$, find an expression for converting to polar coordinates.

H.O.T. Problems Use Higher-Order Thinking Skills

68. **ERROR ANALYSIS** Usama and Saleh are writing the polar equation $r = \sin\theta$ in rectangular form. Saleh believes that the answer is $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. Usama believes that the answer is simply $y = \sin x$. Is either of them correct? Explain your reasoning.

69. **CHALLENGE** The equation for a circle is $r = 2a\cos\theta$. Write this equation in rectangular form. Find the center and radius of the circle.

70. **REASONING** Given a set of rectangular coordinates (x, y) and a value for r, write expressions for finding θ in terms of sine and in terms of cosine. (*Hint:* You may have to write multiple expressions for each function, similar to the expressions given in this lesson using tangent.)

71. **WRITING IN MATH** Make a conjecture about when graphing an equation is made easier by representing the equation in polar form rather than rectangular form and vice versa.

72. **PROOF** Use $x = r\cos\theta$ and $y = r\sin\theta$ to prove that $r = x\sec\theta$ and $r = y\csc\theta$.

73 **CHALLENGE** Write $r^2(4\cos^2\theta + 3\sin^2\theta) + r(-8a\cos\theta + 6b\sin\theta) = 12 - 4a^2 - 3b^2$ in rectangular form. (*Hint:* Distribute before substituting values for r^2 and r. The rectangular equation should be a conic.)

74. **WRITING IN MATH** Use the definition of a polar axis given in Lesson 8-1 to explain why it was necessary to state that the robot in Example 3 was facing due east. How can the use of quadrant bearings help to eliminate this?

Spiral Review

Use symmetry to graph each equation. (Lesson 8-2)

75. $r = 1 - 2 \sin \theta$

76. $r = -2 - 2 \sin \theta$

77. $r = 2 \sin 3\theta$

Find three different pairs of polar coordinates that name the given point if $-360° < \theta \leq 360°$ or $-2\pi < \theta \leq 2\pi$. (Lesson 8-1)

78. $T(1.5, 180°)$

79. $U\left(-1, \frac{\pi}{3}\right)$

80. $V(4, 315°)$

Find the angle θ between u and v to the nearest tenth of a degree.

81. $\mathbf{u} = \langle 6, -4 \rangle, \mathbf{v} = \langle -5, -7 \rangle$

82. $\mathbf{u} = \langle 2, 3 \rangle, \mathbf{v} = \langle -9, 6 \rangle$

83. $\mathbf{u} = \langle 1, 10 \rangle, \mathbf{v} = \langle 8, -2 \rangle$

Write each pair of parametric equations in rectangular form. Then graph and state any restrictions on the domain.

84. $y = t + 6$ and $x = \sqrt{t}$

85. $y = \frac{t}{2} + 1$ and $x = \frac{t^2}{4}$

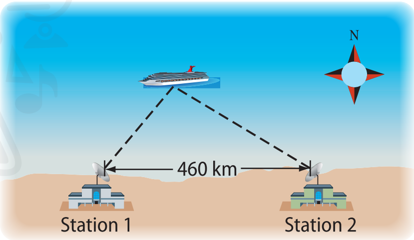
86. $y = -3 \sin t$ and $x = 3 \cos t$

87. **NAVIGATION** Two LORAN broadcasting stations are located 460 kilometers apart. A ship receives signals from both stations and determines that it is 108 kilometers farther from Station 2 than Station 1.

a. Determine the equation of the hyperbola centered at the origin on which the ship is located.

b. Graph the equation, indicating on which branch of the hyperbola the ship is located.

c. Find the coordinates of the location of the ship on the coordinate grid if it is 110 kilometers from the x-axis.



88. **BICYCLES** Woodland Bicycles makes two models of off-road bicycles: the Adventure, which sells for AED 250, and the Grande Venture, which sells for AED 350. Both models use the same frame. The painting and assembly time required for the Adventure is 2 hours, while the time is 3 hours for the Grande Venture. If there are 175 frames and 450 hours of labor available for production, how many of each model should be produced to maximize revenue? What is the maximum revenue?

Solve each system of equations using Gauss-Jordan elimination.

89. $3x + 9y + 6z = 21$
$4x - 10y + 3z = 15$
$-5x + 12y - 2z = -6$

90. $x + 5y - 3z = -14$
$2x - 4y + 5z = 18$
$-7x - 6y - 2z = 1$

91. $2x - 4y + z = 20$
$5x + 2y - 2z = -4$
$6x + 3y + 5z = 23$

Skills Review for Standardized Tests

92. **SAT/ACT** A square is inscribed in circle B. If the circumference of the circle is 50π, what is the length of the diagonal of the square?

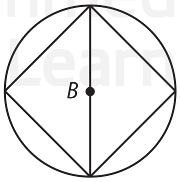
A $10\sqrt{2}$

B 25

C $25\sqrt{2}$

D 50

E $50\sqrt{2}$



93. **REVIEW** Which of the following could be an equation for a rose with three petals?

F $r = 3 \sin \theta$

G $r = \sin 3\theta$

H $r = 6 \sin \theta$

J $r = \sin 6\theta$

94. What is the polar form of $x^2 + (y - 2)^2 = 4$?

A $r = \sin \theta$

B $r = 2 \sin \theta$

C $r = 4 \sin \theta$

D $r = 8 \sin \theta$

95. **REVIEW** Which of the following could be an equation for a spiral of Archimedes that passes through $A\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$?

F $r = \frac{\sqrt{2}\pi}{2} \cos \theta$

G $r = \theta$

H $r = \frac{3}{4}$

J $r = \frac{\theta}{2}$

CHAPTER 8

Mid-Chapter Quiz

Lessons 8-1 through 8-3

Graph each point on a polar grid. (Lesson 8-1)

1. $A(-2, 45°)$

2. $D(1, 315°)$

3. $C\left(-1.5, -\frac{4\pi}{3}\right)$

4. $B\left(3, -\frac{5\pi}{6}\right)$

Graph each polar equation. (Lesson 8-1)

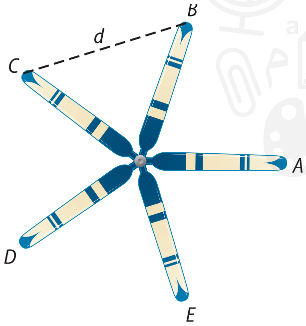
5. $r = 3$

6. $\theta = -\frac{3\pi}{4}$

7. $\theta = 60°$

8. $r = -1.5$

9. HELICOPTERS A toy helicopter rotor consists of five equally spaced blades. Each blade is 11.5 centimeters long. (Lesson 8-1)



a. If the angle blade A makes with the polar axis is 3°, write an ordered pair to represent the tip of each blade on a polar grid. Assume that the rotor is centered at the pole.

b. What is the distance d between the tips of the helicopter blades to the nearest centimeter?

Graph each equation. (Lesson 8-2)

10. $r = \frac{1}{4}\sec\theta$

11. $r = \frac{1}{3}\cos\theta$

12. $r = 3\csc\theta$

13. $r = 4\sin\theta$

14. STAINED GLASS A rose window is a circular window seen in gothic architecture. The pattern of the window radiates from the center. The window shown can be approximated by the equation $r = 3\sin 6\theta$. Use symmetry, zeros, and maximum r-values of the function to graph the function. (Lesson 8-2)

Identify and graph each classic curve. (Lesson 8-2)

15. $r = \frac{1}{2}\sin\theta$

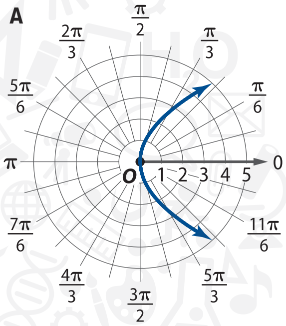
16. $r = \frac{1}{3}\theta + 3, \theta \geq 0$

17. $r = 1 + 2\cos\theta$

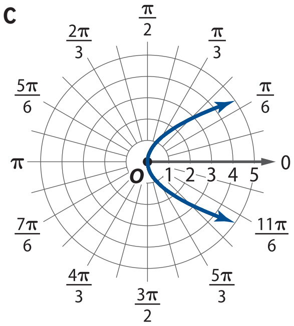
18. $r = 5\sin 3\theta$

19. MULTIPLE CHOICE Identify the polar graph of $y^2 = \frac{1}{2}x$. (Lesson 8-3)

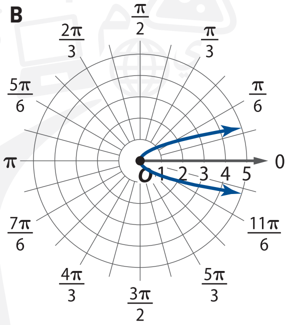
A



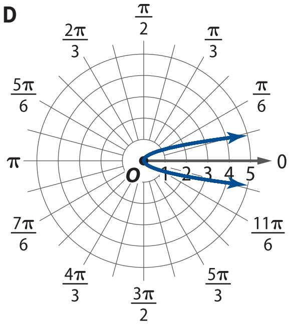
C



B



D



Find the rectangular coordinates for each point with the given polar coordinates. (Lesson 8-3)

20. $\left(4, \frac{2\pi}{3}\right)$

21. $\left(-2, -\frac{\pi}{4}\right)$

22. $(-1, 210°)$

23. $(3, 30°)$

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \leq \theta \leq 2\pi$. Round to the nearest hundredth. (Lesson 8-3)

24. $(-3, 5)$

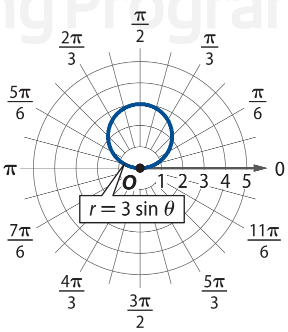
25. $(8, 1)$

26. $(7, -6)$

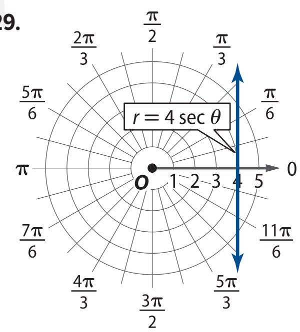
27. $(-4, -10)$

Write a rectangular equation for each graph. (Lesson 8-3)

28.



29.



LESSON 8-4 Polar Forms of Conic Sections

Then

- You defined conic sections.

Now

1. Identify polar equations of conics.
2. Write and graph the polar equation of a conic given its eccentricity and the equation of its directrix.

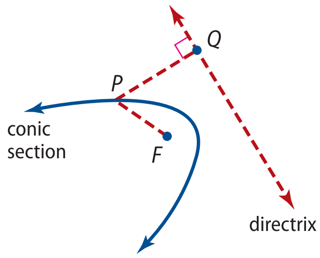
Why?

- Polar equations of conic sections can be used to model orbital motion, such as the orbit of a planet around the Sun or the orbit of a satellite around a planet.

1 Use Polar Equations of Conics

Previously, you defined conic sections in terms of the distance between a focus and directrix (parabola) or between two foci (ellipse and hyperbola). Alternatively, we can define all of these curves using the focus-directrix definition of a parabola.

In general, a conic section can be defined as the locus of points such that the distance from a point P to the focus and the distance from the point to a fixed line not containing P (the directrix) is a constant ratio. This constant ratio $\frac{PF}{PQ}$ represents the eccentricity of a conic and is denoted e.



***e* as Constant Ratio** $\quad$ $e = \frac{PF}{PQ}$

or

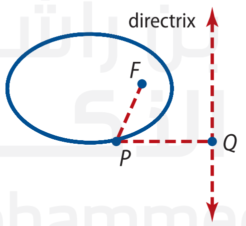
***e* as Constant Multiplier** $\quad$ $PF = e \cdot PQ$

Recall that for a parabola, $PF = PQ$. Therefore, a parabola has eccentricity $\frac{PQ}{PQ}$ or 1. Other values of e give us other conics. These eccentricities are summarized below.

ConceptSummary Eccentricities of Conics

Ellipse

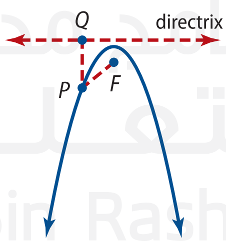
$0 < e < 1$



$0 < \frac{PF}{PQ} < 1$

Parabola

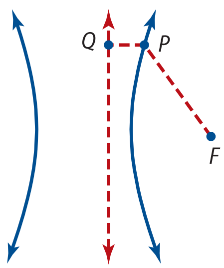
$e = 1$



$\frac{PF}{PQ} = 1$

Hyperbola

$e > 1$



$\frac{PF}{PQ} > 1$

Recall too that when the center of a conic section lies at the origin, the rectangular equations of conics take on a simpler form.

Ellipses

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

Parabolas

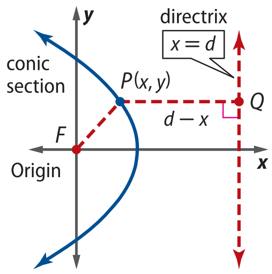
$x^2 = 4py$ or $y^2 = 4px$

Hyperbolas

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Using the focus-directrix definition, the equation of a conic in polar form is simplified if a *focus* of the conic lies at the origin.

Consider a conic with its focus located at the origin and its directrix to the right at $x = d$. For any point $P(x, y)$ on the curve, the distance PF is given by $\sqrt{x^2 + y^2}$, and the distance PQ is given by $d - x$. We can substitute these expressions in the definition of a conic section.



StudyTip

Other Conics When defining conics in terms of their eccentricity, *e* is a strictly positive constant. There are no circles, lines, or other degenerate conics.

$PF = e \cdot PQ$	Definition of a conic section
$\sqrt{x^2 + y^2} = e(d - x)$	$PF = \sqrt{x^2 + y^2}$ and $PQ = d - x$

The expression $\sqrt{x^2 + y^2}$ should make you think of polar coordinates. In fact, the equation above has a simpler form in the polar coordinate system.

$\sqrt{x^2 + y^2} = e(d - x)$	Rectangular form of conic defined in terms of its eccentricity *e*
$r = e(d - r\cos\theta)$	$r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$
$r = ed - er\cos\theta$	Distributive Property
$r + er\cos\theta = ed$	Isolate *r*-terms.
$r(1 + e\cos\theta) = ed$	Factor.
$r = \dfrac{ed}{1 + e\cos\theta}$	Solve for *r*.

This last equation is the polar form of an equation for the conic sections with focus at the pole and vertical directrix and center or vertex to the right of the pole. Different orientations of the focus and directrix can produce different forms of this polar equation as summarized below.

ReadingMath

Eccentricity In each of these polar equations, the letter *e* is a variable that represents the eccentricity of the conic. It should *not* be confused with the transcendental number *e*, which is a constant.

KeyConcept Polar Equations of Conics

The conic section with eccentricity $e > 0$, $d > 0$, and focus at the pole has the polar equation:

- $r = \dfrac{ed}{1 + e\cos\theta}$ if the directrix is the vertical line $x = d$ (Figure 8.4.1),
- $r = \dfrac{ed}{1 - e\cos\theta}$ if the directrix is the vertical line $x = -d$ (Figure 8.4.2),
- $r = \dfrac{ed}{1 + e\sin\theta}$ if the directrix is the horizontal line $y = d$ (Figure 8.4.3), and
- $r = \dfrac{ed}{1 - e\sin\theta}$ if the directrix is the horizontal line $y = -d$ (Figure 8.4.4).

In each of the examples below, $e = 1$, so the conic takes the form of a parabola.

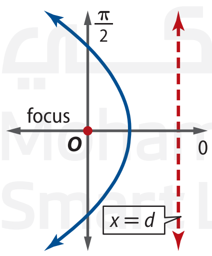


Figure 8.4.1

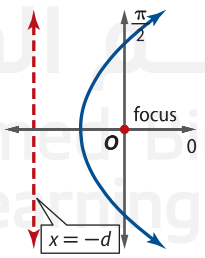


Figure 8.4.2

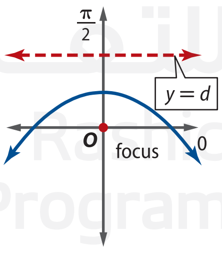


Figure 8.4.3

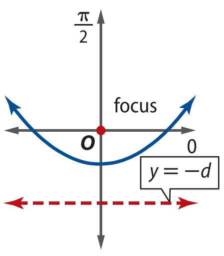


Figure 8.4.4

You will derive the last three of these equations in Exercises 50–52.

Notice that for $r = \dfrac{ed}{1 - e\cos\theta}$, the directrix of the conic is to the left of the pole. For $r = \dfrac{ed}{1 + e\sin\theta}$, the directrix is above the pole. For $r = \dfrac{ed}{1 - e\sin\theta}$, the directrix is below the pole.

To analyze the polar equation of a conic, begin by writing the equation in standard form, $r = \frac{ed}{1 \pm e \cos \theta}$ or $r = \frac{ed}{1 \pm e \sin \theta}$. In this form, determine the eccentricity and use this value to identify the type of conic the equation represents. Then determine the equation of the directrix, and use it to describe the orientation of the conic.

Example 1 Identify Conics from Polar Equations

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation.

a. $r = \frac{9}{3 + 2.25 \cos \theta}$

Write the equation in standard form, $r = \frac{ed}{1 + e \cos \theta}$.

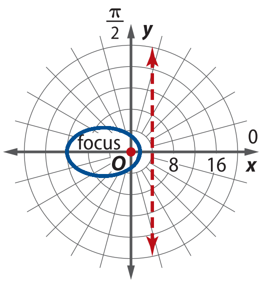
$r = \frac{9}{3 + 2.25 \cos \theta}$ Original equation

$r = \frac{3(3)}{3(1 + 0.75 \cos \theta)}$ Factor the numerator and denominator.

$r = \frac{3}{1 + 0.75 \cos \theta}$ Divide the numerator and denominator by 3.

In this form, you can see from the denominator that $e = 0.75$. Therefore, the conic is an ellipse. For polar equations of this form, the equation of the directrix is $x = d$. From the numerator, we know that $ed = 3$, so $d = 3 \div 0.75$ or 4. Therefore, the equation of the directrix is $x = 4$.

CHECK Sketch the graph of $r = \frac{9}{3 + 2.25 \cos \theta}$ and its directrix $x = 4$ using either the techniques shown in Lesson 8-2 or a graphing calculator. The graph is an ellipse with its directrix to the right of the pole. ✔



StudyTip

Focus-Directrix Pairs While a parabola has one focus and one directrix, ellipses and hyperbolas have two foci-directrix pairs. Either focus-directrix pair can be used to generate the conic.

b. $r = \frac{-16}{4 \sin \theta - 2}$

Write the equation in standard form.

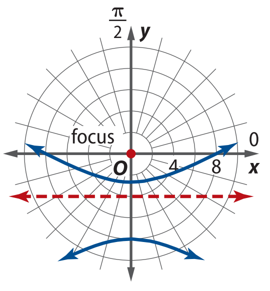
$r = \frac{-16}{4 \sin \theta - 2}$ Original equation

$r = \frac{-2(8)}{-2(1 - 2 \sin \theta)}$ Factor the numerator and denominator.

$r = \frac{8}{1 - 2 \sin \theta}$ Divide the numerator and denominator by −2.

The equation is of the form $r = \frac{ed}{1 - e \sin \theta}$, so $e = 2$. Therefore, the conic is a hyperbola. For polar equations of this form, the equation of the directrix is $y = -d$. Because $ed = 8$, $d = 8 \div 2$ or 4. Therefore, the equation of the directrix is $y = -4$.

CHECK Sketch the graph of $r = \frac{-16}{4 \sin \theta - 2}$ and its directrix $y = -4$. The graph is a hyperbola with one focus at the origin, above the directrix. ✔



GuidedPractice

1A. $r = \frac{-6}{3 \cos \theta - 1}$

1B. $r = \frac{9}{3 + 3 \sin \theta}$

1C. $r = \frac{1}{6 + 1.2 \cos \theta}$

2 Write Polar Equations of Conics

You can write the polar equation of a conic given its eccentricity and the equation of the directrix or its eccentricity and some other characteristics.

Example 2 Write Polar Equations of Conics

Write and graph a polar equation and directrix for the conic with the given characteristics.

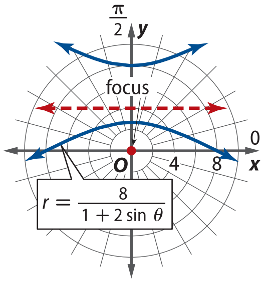
a. $e = 2$; **directrix:** $y = 4$

Because $e = 2$, the conic is a hyperbola. The directrix $y = 4$ is above the pole, so the equation is of the form $r = \frac{ed}{1 + e \sin \theta}$. Use the values for e and d to write the equation.

$r = \frac{ed}{1 + e \sin \theta}$ — Polar form of conic with directrix $y = d$

$r = \frac{2(4)}{1 + 2 \sin \theta}$ or $\frac{8}{1 + 2 \sin \theta}$ — $e = 2$ and $d = 4$

Sketch the graph of this polar equation and its directrix. The graph is a hyperbola with its directrix above the pole.



b. $e = 0.5$; **vertices at** $(-4, 0)$ **and** $(12, 0)$

Because $e = 0.5$, the conic is an ellipse. The center of the ellipse is at $(4, 0)$, the midpoint of the segment between the given vertices. This point is to the right of the pole. Therefore, the directrix will be to the left of the pole at $x = -d$. The polar equation of a conic with this d irectrix is $r = \frac{ed}{1 - e \cos \theta}$.

Use the value of e and the polar form of a point on the conic to find the value of d. The vertex point $(12, 0)$ has polar coordinates $(r, \theta) = \left(\sqrt{12^2 + 0^2}, \tan^{-1} \frac{0}{12}\right)$ or $(12, 0)$.

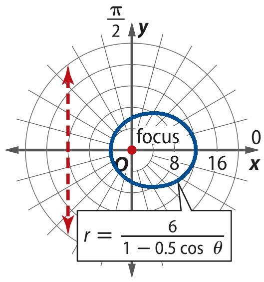
$r = \frac{ed}{1 - e \cos \theta}$ — Polar form of conic with directrix $x = -d$

$12 = \frac{0.5d}{1 - 0.5 \cos 0}$ — $e = 0.5$, $r = 12$, and $\theta = 0$

$12 = \frac{0.5d}{0.5}$ — $\cos 0 = 1$

$12 = d$ — Simplify.

Therefore, the equation for the ellipse is $r = \frac{0.5 \cdot 12}{1 - 0.5 \cos \theta}$ or $r = \frac{6}{1 - 0.5 \cos \theta}$. Because $d = 12$, the equation of the directrix is $x = -12$. The graph is an ellipse with vertices at $(-4, 0)$ and $(12, 0)$.



Guided Practice

2A. $e = 1$; directrix: $x = 2$

2B. $e = 2.5$; vertices at $(0, -3)$ and $(0, -7)$

> **Study Tip**
> **Effects of Various Eccentricities** You will investigate the effects of various eccentricities for a fixed directrix and various directrixes for a fixed eccentricity in Exercise 49.

Previously, you analyzed the rectangular equations of conics in standard form to describe the geometric properties of parabolas, ellipses, and hyperbolas. You can use the geometric analysis of the graph of a conic given in polar form to write the equation in rectangular form.

Example 3 Write the Polar Form of Conics in Rectangular Form

Write each polar equation in rectangular form.

a. $r = \dfrac{4}{1 - \sin\theta}$

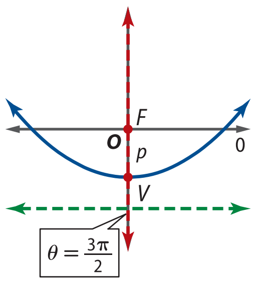


Figure 8.4.5

Step 1 Analyze the polar equation.

For this equation, $e = 1$ and $d = 4$. The eccentricity and form of the equation determine that this is a parabola that opens vertically with focus at the pole and a directrix $y = -4$. The general equation of such a parabola in rectangular form is $(x - h)^2 = 4p(y - k)$.

Step 2 Determine values for h, k, and p.

The vertex lies between the focus F and directrix of the parabola, occurring when $\theta = \frac{3\pi}{2}$, as shown in Figure 8.4.5. Evaluating the function at this value, we find that the vertex lies at polar coordinates $\left(2, \frac{3\pi}{2}\right)$, which correspond to rectangular coordinates $(0, -2)$. So, $(h, k) = (0, -2)$. The distance p from the vertex at $(0, -2)$ to the focus at $(0, 0)$ is 2.

Step 3 Substitute the values for h, k, and p into the standard form of an equation for a parabola.

$(x - h)^2 = 4p(y - k)$ — Standard form of a parabola

$(x - 0)^2 = 4(2)[y - (-2)]$ — $h = 0$, $k = -2$, and $p = 2$

$x^2 = 8y + 16$ — Simplify.

b. $r = \dfrac{3.2}{1 - 0.6\cos\theta}$

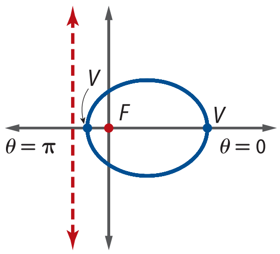


Figure 8.4.6

Step 1 Analyze the polar equation.

For this equation, $e = 0.6$ and $d \approx 5.3$. The eccentricity and form of the equation determine that this is an ellipse with directrix $x = -5.3$. Therefore, the major axis of the ellipse lies along the polar or x-axis. The general equation of such an ellipse in rectangular form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

Step 2 Determine values for h, k, a, and b.

The vertices are the endpoints of the major axis and occur when $\theta = 0$ and π as shown in Figure 8.4.6. Evaluating the function at these values, we find that the vertices have polar coordinates $(8, 0)$ and $(2, \pi)$, which correspond to rectangular coordinates $(8, 0)$ and $(-2, 0)$. The ellipse's center is the midpoint of the segment between the vertices, so $(h, k) = (3, 0)$.

The distance a between the center and each vertex is 5. The distance c from the center to the focus at $(0, 0)$ is 3. By the Pythagorean relation $b = \sqrt{a^2 - c^2}$, $b = \sqrt{5^2 - 3^2}$ or 4.

Step 3 Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ — Standard form of an ellipse

$\frac{(x-3)^2}{5^2} + \frac{(y-0)^2}{4^2} = 1$ — $h = 3$, $k = 0$, $a = 5$, and $b = 4$

$\frac{(x-3)^2}{25} + \frac{y^2}{16} = 1$ — Simplify.

GuidedPractice

3A. $r = \dfrac{2.5}{1 - 1.5\cos\theta}$

3B. $r = \dfrac{5}{1 + \sin\theta}$

Exercises

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation. (Example 1)

1. $r = \dfrac{20}{4 + 4 \sin \theta}$

2. $r = \dfrac{18}{2 - 6 \cos \theta}$

3. $r = \dfrac{21}{3 \cos \theta + 1}$

4. $r = \dfrac{24}{4 \sin \theta + 8}$

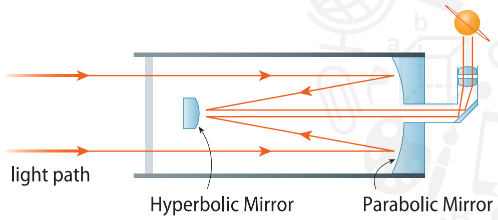
5. $r = \dfrac{-12}{6 \cos \theta - 6}$

6. $r = \dfrac{9}{4 - 3 \sin \theta}$

7. $r = \dfrac{-8}{\sin \theta - 0.25}$

8. $r = \dfrac{10}{2.5 + 2.5 \cos \theta}$

9. **TELESCOPES** The Cassegrain Telescope, invented in 1692, produces an image by reflecting light off of parabolic and hyperbolic mirrors. Determine the eccentricity, type of conic, and the equation of the directrix for each equation modeling a mirror in the telescope. (Example 1)



a. $r = \dfrac{7}{2 \sin \theta + 2}$

b. $r = \dfrac{28}{12.5 \cos \theta + 5}$

Write and graph a polar equation and directrix for the conic with the given characteristics. (Example 2)

10. $e = 1$: directrix: $y = 6$

11. $e = 0.75$; directrix: $x = -8$

12. $e = 5$; directrix: $x = 2$

13. $e = 0.1$; directrix: $y = 8$

14. $e = 6$; directrix: $y = -7$

15. $e = 1$; directrix: $x = -1.5$

16. $e = 0.8$; vertices at $(-36, 0)$ and $(4, 0)$

17. $e = 1.5$; vertices at $(-3, 0)$ and $(-15, 0)$

Write each polar equation in rectangular form. (Example 3)

18. $r = \dfrac{4.8}{1 + \sin \theta}$

19. $r = \dfrac{30}{4 + \cos \theta}$

20. $r = \dfrac{5}{1 - 1.5 \cos \theta}$

21. $r = \dfrac{5.1}{1 + 0.7 \sin \theta}$

22. $r = \dfrac{12}{1 - \cos \theta}$

23. $r = \dfrac{6}{0.25 - 0.75 \sin \theta}$

24. $r = \dfrac{4.5}{1 + 1.25 \sin \theta}$

25. $r = \dfrac{8.4}{1 - 0.4 \cos \theta}$

GRAPHING CALCULATOR **Determine the type of conic for each polar equation. Then graph each equation.**

26. $r = \dfrac{2}{2 + \sin\left(\theta + \frac{\pi}{3}\right)}$

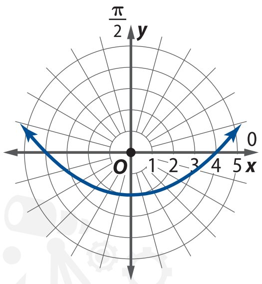
27. $r = \dfrac{3}{1 + \cos\left(\theta - \frac{\pi}{4}\right)}$

28. $r = \dfrac{2}{1 - \cos\left(\theta + \frac{\pi}{6}\right)}$

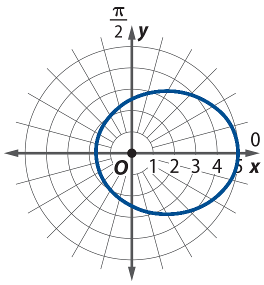
29. $r = \dfrac{4}{1 + 2 \sin\left(\theta + \frac{3\pi}{4}\right)}$

Match each polar equation with its graph.

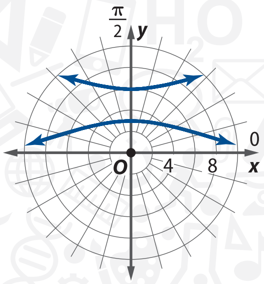
a.



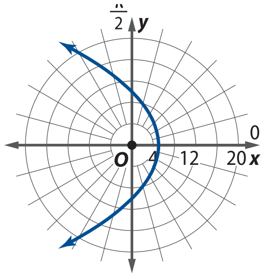
b.



c.



d.



30. $r = \dfrac{10}{1 + \cos \theta}$

31. $r = \dfrac{4}{1 - \sin \theta}$

32. $r = \dfrac{5}{2 - \cos \theta}$

33. $r = \dfrac{12}{1 + 3 \sin \theta}$

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation. Then sketch the graph of the equation, and label the directrix.

34. $r = \dfrac{12}{2 - 0.75 \cos \theta}$

35. $r = \dfrac{1}{0.2 - 0.2 \sin \theta}$

36. $r = \dfrac{6}{1.2 \sin \theta + 0.3}$

37. $r = \dfrac{8}{\cos \theta + 5}$

38. **ASTRONOMY** The comet Borrelly travels in an elliptical orbit around the Sun with eccentricity $e = 0.624$. The point in a comet's orbit nearest to the Sun is defined as the *perihelion,* while the farthest point from the Sun is defined as the *aphelion.* The aphelion occurs at a distance of 5.83 AU (astronomical units, based on the distance between Earth and the Sun) from the Sun and the perihelion occurs at a distance of 1.35 AU. The diameter of the Sun is about 0.0093 AU.

a. Write a polar equation for the elliptical orbit of the comet Borrelly, and graph the equation.

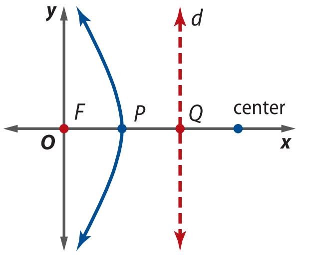
b. Determine the distance in kilometers between the comet Borrelly and the Sun at the aphelion and perihelion if 1 AU ≈ 149.7 million kilometers.

PROOF **Prove each of the following.**

39. $b = a\sqrt{1 - e^2}$ for an ellipse

40. $b = a\sqrt{e^2 - 1}$ for a hyperbola

41. **PROOF** Use the definition for the eccentricity of a conic, $PF = ePQ$, and the drawing of the hyperbola shown below, to verify that $d = \frac{a(e^2 - 1)}{e}$ for any hyperbola.



Write each rectangular equation in polar form.

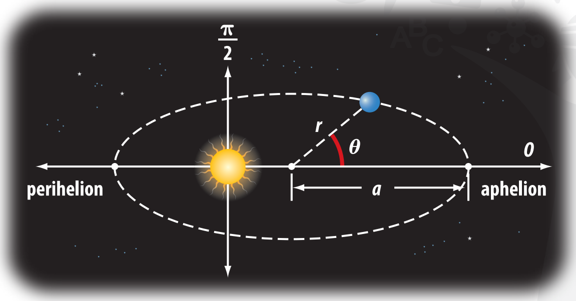
42. $x^2 = 4y + 4$

43. $-10y + 25 = x^2$

44. $\frac{(x-2)^2}{16} + \frac{y^2}{12} = 1$

45. $\frac{(x+4)^2}{64} + \frac{y^2}{48} = 1$

46. **ASTRONOMY** The planets travel around the Sun in approximately elliptical orbits with the Sun at one focus, as shown below.



a. Show that the polar equation of the planets' orbit can be written as $r = \frac{a(1 - e^2)}{(1 - e \cos \theta)}$.

b. Prove that the perihelion distance of any planet is $a(1 - e)$, and the aphelion distance is $a(1 + e)$.

c. Use the formulas from part **a** to find the perihelion and aphelion distances for each of the planets.

Planet	a	e	Planet	a	e
Earth	1.000	0.017	Neptune	30.06	0.009
Jupiter	5.203	0.048	Saturn	9.539	0.056
Mars	1.524	0.093	Uranus	19.18	0.047
Mercury	0.387	0.206	Venus	0.723	0.007

d. For which planet is the distance between the perihelion and aphelion the smallest? the greatest?

Write each equation in polar form. (*Hint*: Translate each conic so that a focus lies on the pole.)

47. $\frac{(x-2)^2}{64} - \frac{y^2}{36} = 1$

48. $3(x + 5)^2 + 4y^2 = 192$

49. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the effects of varying the eccentricity and the directrix on graphs of conic sections.

a. **NUMERICAL** Write an equation for a conic section with focus (0, 0) and directrix $x = 3$ for $e = 0.4$, 0.6, 1, 1.6, and 2. Then identify the type of conic that each equation represents.

b. **GRAPHICAL** Graph and label the eccentricity for each of the equations that you found in part **a** on the same coordinate plane.

c. **VERBAL** Describe the changes in the graphs from part **b** as e approaches 2.

d. **NUMERICAL** Write an equation for a conic section with focus (0, 0) and eccentricity $e = 0.5$ for $d = 0.25$, 1, and 4.

e. **GRAPHICAL** Graph each of the equations on the same coordinate plane.

f. **VERBAL** Describe the relationship between the value of d and the distances between the vertices and the foci of the graphs from part **e**.

Derive each of the following polar equations of conics for the equation $r = \frac{ed}{1 + e \cos \theta}$. Include a diagram with each derivation.

50. $r = \frac{ed}{1 - e \cos \theta}$

51. $r = \frac{ed}{1 + e \sin \theta}$

52. $r = \frac{ed}{1 - e \sin \theta}$

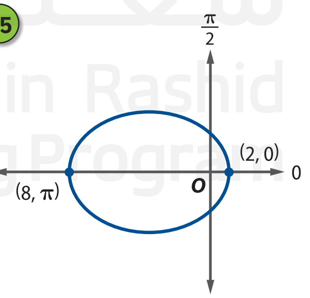
H.O.T. Problems Use Higher-Order Thinking Skills

53. **WRITING IN MATH** Describe two definitions that can be used to define a conic section.

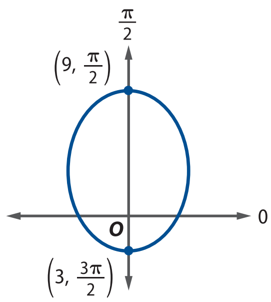
54. **REASONING** Explain why $r = \frac{ed}{1 + e \sin \theta}$ does not produce a true circle for any value of e.

CHALLENGE Determine a polar equation for the ellipse with the given vertices if one focus is at the pole.

55.



56.



57. **WRITING IN MATH** Explain how you can find the distance from the focus at (0, 0) to any point on the conic when the rectangular coordinates, polar coordinates, or θ is provided.

Spiral Review

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \le \theta \le 2\pi$. If necessary, round to the nearest hundredth. (Lesson 8-3)

58. $\left(-\sqrt{2}, \sqrt{2}\right)$ **59.** $(-2, -5)$ **60.** $(8, -12)$

Identify and graph each classic curve. (Lesson 8-2)

61. $r = 3 + 3\cos\theta$ **62.** $r = -2\sin 3\theta$ **63.** $r = \frac{5}{2}\theta,\ \theta \ge 0$

Determine an equation of an ellipse with each set of characteristics.

64. co-vertices $(5, 8)$, $(5, 0)$; foci $(8, 4)$, $(2, 4)$

65. major axis $(-2, 4)$ to $(8, 4)$; minor axis $(3, 1)$ to $(3, 7)$

66. foci $(1, -1)$, $(9, -1)$; length of minor axis equals 6

67. OLYMPICS In the Olympic Games, team standings are determined according to each team's total points. Each type of Olympic medal earns a team a given number of points. Use the information to determine the Olympics in which the United States earned the most points.

Olympics	Gold	Silver	Bronze
1996	44	32	25
2000	37	24	31
2004	35	39	29
2008	36	38	36

Medal	Points
gold	3
silver	2
bronze	1

Find the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ for the given value and interval.

68. $\sin\theta = \frac{2}{3}$, $(0°, 90°)$ **69.** $\tan\theta = -\frac{24}{7}$, $\left(\frac{\pi}{2}, \pi\right)$ **70.** $\sin\theta = -\frac{4}{5}$, $\left(\pi, \frac{3\pi}{2}\right)$

Locate the vertical asymptotes, and sketch the graph of each function.

71. $y = \sec\left(x + \frac{\pi}{3}\right)$ **72.** $y = 4\cot\frac{x}{2}$ **73.** $y = 2\cot\left[\frac{2}{3}\left(x - \frac{\pi}{2}\right)\right] + 0.75$

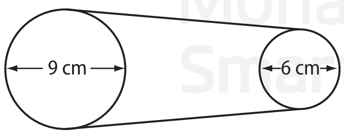
Find the exact values of the five remaining trigonometric functions of θ.

74. $\sec\theta = 2$, where $\sin\theta > 0$ and $\cos\theta > 0$

75. $\csc\theta = \sqrt{5}$, where $\sin\theta > 0$ and $\cos\theta > 0$

Skills Review for Standardized Tests

76. SAT/ACT A pulley with a 9-centimeter diameter is belted to a pulley with a 6-centimeter diameter, as shown in the figure. If the larger pulley runs at 120 rpm (revolutions per minute), how fast does the smaller pulley run?



A 80 rpm
B 120 rpm
C 160 rpm
D 180 rpm
E 200 rpm

77. What type of conic is given by $r = \frac{3}{2 - 0.5\cos\theta}$?

F circle
G ellipse
H parabola
J hyperbola

78. REVIEW Which of the following includes the component form and magnitude of $\overrightarrow{AB}$ with initial point $A(3, 4, -2)$ and terminal point $B(-5, 2, 1)$?

A $\langle -8, -2, 3\rangle$, $\sqrt{77}$
B $\langle 8, -2, 3\rangle$, $\sqrt{77}$
C $\langle -8, -2, 3\rangle$, $\sqrt{109}$
D $\langle 8, -2, 3\rangle$, $\sqrt{109}$

79. REVIEW What is the eccentricity of the ellipse described by $\frac{y^2}{47} + \frac{(x - 12)^2}{34} = 1$?

F 0.38
G 0.41
H 0.53
J 0.62

LESSON 8-5

Complex Numbers and DeMoivre's Theorem

Then

- You performed operations with complex numbers written in rectangular form.

Now

1. Convert complex numbers from rectangular to polar form and vice versa.
2. Find products, quotients, powers, and roots of complex numbers in polar form.

Why?

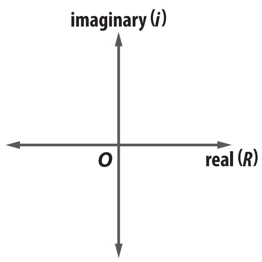
- Electrical engineers use complex numbers to describe certain relationships of electricity. Voltage E, impedance Z, and current I are the three quantities related by the equation $E = I \cdot Z$ used to describe alternating current. Each variable can be written as a complex number in the form $a + bj$, where j is an imaginary number (engineers use j to not be confused with current I). For impedance, the real part a represents the opposition to current flow due to resistors, and the imaginary part b is related to the opposition due to inductors and capacitors.

New Vocabulary

complex plane
real axis
imaginary axis
Argand plane
absolute value of a complex number
polar form
trigonometric form
modulus
argument
*p*th roots of unity

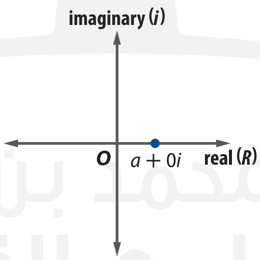
1 Polar Forms of Complex Numbers

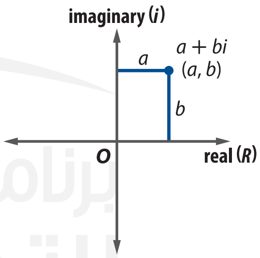
A complex number given in rectangular form, $a + b\mathbf{i}$, has a real component a and an imaginary component $b\mathbf{i}$. You can graph a complex number on the **complex plane** by representing it with the point (a, b). Similar to a coordinate plane, we need two axes to graph a complex number. The real component is plotted on the horizontal axis called the **real axis**, and the imaginary component is plotted on the vertical axis called the **imaginary axis**. The complex plane may also be referred to as the **Argand Plane** (ar GON).



Complex Plane

Consider a complex number where $b = 0$, $a + 0\mathbf{i}$. The result is a real number a that can be graphed using just a real number line or the real axis. When $b \neq 0$, the imaginary axis is needed to represent the imaginary component.



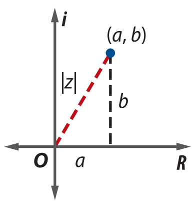


Recall that the absolute value of a real number is its distance from zero on the number line. Similarly, the **absolute value of a complex number** is its distance from zero in the complex plane. When $a + b\mathbf{i}$ is graphed in the complex plane, the distance from zero can be calculated using the Pythagorean Theorem.

Key Concept Absolute Value of a Complex Number

The absolute value of the complex number $z = a + b\mathbf{i}$ is

$$|z| = |a + b\mathbf{i}| = \sqrt{a^2 + b^2}.$$

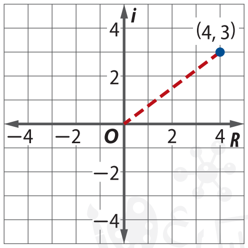


Example 1 Graphs and Absolute Values of Complex Numbers

Graph each number in the complex plane, and find its absolute value.

a. $z = 4 + 3i$

$(a, b) = (4, 3)$



$|z| = \sqrt{a^2 + b^2}$ Absolute value formula

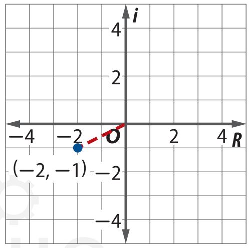
$= \sqrt{4^2 + 3^2}$ $a = 4$ and $b = 3$

$= \sqrt{25}$ or 5 Simplify.

The absolute value of $4 + 3i = 5$.

b. $z = -2 - i$

$(a, b) = (-2, -1)$



$|z| = \sqrt{a^2 + b^2}$ Absolute value formula

$= \sqrt{(-2)^2 + (-1)^2}$ $a = -2$ and $b = -1$

$= \sqrt{5}$ or 2.24 Simplify.

The absolute value of $-2 - i \approx 2.24$.

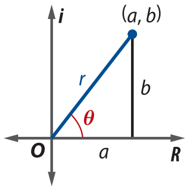
GuidedPractice

1A. $5 + 2i$

1B. $-3 + 4i$

> **WatchOut!**
>
> **Polar Form** The *polar form of a complex number* should not be confused with *polar coordinates of a complex number*. The polar form of a complex number is another way to represent a complex number. Polar coordinates of a complex number will be discussed later in this lesson.

Just as rectangular coordinates (x, y) can be written in polar form, so can the coordinates that represent the graph of a complex number in the complex plane. The same trigonometric ratios that were used to find values of x and y can be applied to represent values for a and b.



$\cos \theta = \frac{a}{r}$ and $\sin \theta = \frac{b}{r}$

$r \cos \theta = a$ $r \sin \theta = b$ Multiply each side by r.

Substituting the polar representations for a and b, we can calculate the **polar form** or **trigonometric form** of a complex number.

$z = a + bi$ Original complex number

$= r \cos \theta + (r \sin \theta)i$ $a = r \cos \theta$ and $b = r \sin \theta$

$= r(\cos \theta + i \sin \theta)$ Factor.

In the case of a complex number, r represents the absolute value, or **modulus**, of the complex number and can be found using the same process you used when finding the absolute value, $r = |z| = \sqrt{a^2 + b^2}$. The angle θ is called the **argument** of the complex number. Similar to finding θ with rectangular coordinates (x, y), when using a complex number, $\theta = \tan^{-1} \frac{b}{a}$ or $\theta = \tan^{-1} \frac{b}{a} + \pi$ if $a < 0$.

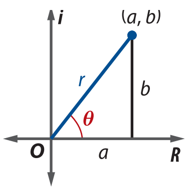
> **StudyTip**
>
> **Argument** The argument of a complex number is also called the *amplitude*. Just as in polar coordinates, θ is not unique, although it is normally given in the interval $-2\pi < \theta < 2\pi$.

KeyConcept Polar Form of a Complex Number

The polar or trigonometric form of the complex number $z = a + bi$ is

$z = r(\cos \theta + i \sin \theta)$, where

$r = |z| = \sqrt{a^2 + b^2}$, $a = r \cos \theta$, $b = r \sin \theta$, and $\theta = \tan^{-1} \frac{b}{a}$ for $a > 0$ or $\theta = \tan^{-1} \frac{b}{a} + \pi$ for $a < 0$.



Example 2 Complex Numbers in Polar Form

Express each complex number in polar form.

a. $-6 + 8i$

Find the modulus r and argument θ.

$r = \sqrt{a^2 + b^2}$ Conversion formula $\theta = \tan^{-1}\frac{b}{a} + \pi$

$= \sqrt{(-6)^2 + 8^2}$ or 10 $a = -6$ and $b = 8$ $= \tan^{-1} -\frac{8}{6} + \pi$ or about 2.21

The polar form of $-6 + 8i$ is about $10(\cos 2.21 + i \sin 2.21)$.

b. $4 + \sqrt{3}i$

Find the modulus r and argument θ.

$r = \sqrt{a^2 + b^2}$ Conversion formula $\theta = \tan^{-1}\frac{b}{a}$

$= \sqrt{4^2 + (\sqrt{3})^2}$ $a = 4$ and $b = \sqrt{3}$ $= \tan^{-1}\frac{\sqrt{3}}{4}$

$= \sqrt{19}$ or about 4.36   Simplify. ≈ 0.41

The polar form of $4 + \sqrt{3}i$ is about $4.36(\cos 0.41 + i \sin 0.41)$.

GuidedPractice

2A. $9 + 7i$ **2B.** $-2 - 2i$

ReadingMath

Polar Form $r(\cos\theta + i\sin\theta)$ is often abbreviated as $r \text{ cis } \theta$. In Example 2a, $-6 + 8i$ can also be expressed as 10 cis 2.21, where $10 = \sqrt{(-6)^2 + 8^2}$ and $2.21 = \tan^{-1} -\frac{8}{6}$.

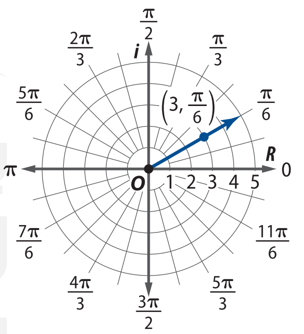
You can use the polar form of a complex number to graph the number on a polar grid by using the r and θ values as your polar coordinates (r, θ). You can also take a complex number written in polar form and convert it to rectangular form by evaluating.

Example 3 Graph and Convert the Polar Form of a Complex Number

Graph $z = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ on a polar grid. Then express it in rectangular form.

The value of r is 3, and the value of θ is $\frac{\pi}{6}$.

Plot the polar coordinates $\left(3, \frac{\pi}{6}\right)$.



To express the number in rectangular form, evaluate the trigonometric values and simplify.

$3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ Polar form

$= 3\left[\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]$ Evaluate for cosine and sine.

$= \frac{3\sqrt{3}}{2} + \frac{3}{2}i$ Distributive Property

The rectangular form of $z = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ is $z = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$.

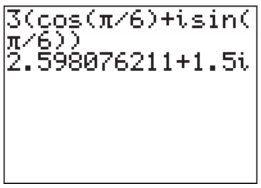
GuidedPractice

Graph each complex number on a polar grid. Then express it in rectangular form.

3A. $5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ **3B.** $4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

TechnologyTip

Complex Number Conversions You can convert a complex number in polar form to rectangular form by entering the expression in polar form, then selecting ENTER. To be in polar mode, select MODE then $a + bi$.



2 Products, Quotients, Powers, and Roots of Complex Numbers

The polar form of complex numbers, along with the sum and difference formulas for cosine and sine, greatly aid in the multiplication and division of complex numbers. The formula for the product of two complex numbers in polar form can be derived by performing the multiplication.

$z_1z_2 = r_1(\cos\theta_1 + \boldsymbol{i}\sin\theta_1)\cdot r_2(\cos\theta_2 + \boldsymbol{i}\sin\theta_2)$	Original equation
$= r_1r_2(\cos\theta_1\cos\theta_2 + \boldsymbol{i}\cos\theta_1\sin\theta_2 + \boldsymbol{i}\sin\theta_1\cos\theta_2 + \boldsymbol{i}^2\sin\theta_1\sin\theta_2)$	FOIL
$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\boldsymbol{i}\cos\theta_1\sin\theta_2 + \boldsymbol{i}\sin\theta_1\cos\theta_2)]$	$i^2 = -1$ and group imaginary terms.
$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \boldsymbol{i}(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$	Factor out i.
$= r_1r_2[\cos(\theta_1+\theta_2) + \boldsymbol{i}\sin(\theta_1+\theta_2)]$	Sum identities for cosine and sine

KeyConcept Product and Quotient of Complex Numbers in Polar Form

Given the complex numbers $z_1 = r_1(\cos\theta_1 + \boldsymbol{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \boldsymbol{i}\sin\theta_2)$:

Product Formula $z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + \boldsymbol{i}\sin(\theta_1+\theta_2)]$

Quotient Formula $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + \boldsymbol{i}\sin(\theta_1-\theta_2)]$, where z_2 and $r_2 \neq 0$

You will prove the Quotient Formula in Exercise 77.

ReadingMath

Plural Forms *Moduli* is the plural of *modulus*.

Notice that when multiplying complex numbers, you multiply the moduli and add the arguments. When dividing, you divide the moduli and subtract the arguments.

Example 4 Product of Complex Numbers in Polar Form

Find $2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\cdot 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ in polar form. Then express the product in rectangular form.

$2\left(\cos\frac{5\pi}{3} + \boldsymbol{i}\sin\frac{5\pi}{3}\right)\cdot 4\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right)$	Original expression
$= 2(4)\left[\cos\left(\frac{5\pi}{3}+\frac{\pi}{6}\right) + \boldsymbol{i}\sin\left(\frac{5\pi}{3}+\frac{\pi}{6}\right)\right]$	Product Formula
$= 8\left(\cos\frac{11\pi}{6} + \boldsymbol{i}\sin\frac{11\pi}{6}\right)$	Simplify.

Now find the rectangular form of the product.

$8\left(\cos\frac{11\pi}{6} + \boldsymbol{i}\sin\frac{11\pi}{6}\right)$	Polar form
$= 8\left(\frac{\sqrt{3}}{2} - \boldsymbol{i}\frac{1}{2}\right)$	Evaluate.
$= 4\sqrt{3} - 4\boldsymbol{i}$	Distributive Property

The polar form of the product is $8\left(\cos\frac{11\pi}{6} + \boldsymbol{i}\sin\frac{11\pi}{6}\right)$. The rectangular form of the product is $4\sqrt{3} - 4\boldsymbol{i}$.

GuidedPractice

Find each product in polar form. Then express the product in rectangular form.

4A. $3\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)\cdot 5\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)$

4B. $-6\left(\cos\frac{3\pi}{4} + \boldsymbol{i}\sin\frac{3\pi}{4}\right)\cdot 2\left(\cos\frac{2\pi}{3} + \boldsymbol{i}\sin\frac{2\pi}{3}\right)$

As mentioned at the beginning of this lesson, quotients of complex numbers can be used to show relationships in electricity.

Real-WorldCareer

Electrical Engineers Electrical engineers design and create new technology used to manufacture global positioning systems, giant generators that power entire cities, turbine engines used in jet aircrafts, and radar and navigation systems. They also work on improving various products such as cell phones, cars, and robots.

Real-World Example 5 Quotient of Complex Numbers in Polar Form

ELECTRICITY **If a circuit has a voltage *E* of 150 volts and an impedance *Z* of 6 − 3*j* ohms, find the current *I* amps in the circuit in rectangular form. Use *E* = *I* • *Z*.**

Express each number in polar form.

$150 = 150(\cos 0 + j \sin 0)$ — $r = \sqrt{150^2 + 0^2}$ or 150; $\theta = \tan^{-1}\frac{0}{150}$ or 0

$6 - 3j = 3\sqrt{5}[\cos(-0.46) + j \sin(-0.46)]$ — $r = \sqrt{6^2 + (-3)^2}$ or $3\sqrt{5}$; $\theta = \tan^{-1} -\frac{3}{6}$ or about -0.46

Solve for the current I in $I \cdot Z = E$.

$I \cdot Z = E$ — Original equation

$I = \frac{E}{Z}$ — Divide each side by Z.

$I = \frac{150(\cos 0 + j \sin 0)}{3\sqrt{5}[\cos(-0.46) + j \sin(-0.46)]}$ — $E = 150(\cos 0 + j \sin 0)$ and $Z = 3\sqrt{5}[\cos(-0.46) + j \sin(-0.46)]$

$I = \frac{150}{3\sqrt{5}}\{\cos[0 - (-0.46)] + j \sin[0 - (-0.46)]\}$ — Quotient Formula

$I = 10\sqrt{5}(\cos 0.46 + j \sin 0.46)$ — Simplify.

Now convert the current to rectangular form.

$I = 10\sqrt{5}(\cos 0.46 + j \sin 0.46)$ — Original equation

$= 10\sqrt{5}(0.90 + 0.44j)$ — Evaluate.

$= 20.12 + 9.84j$ — Distributive Property

The current is about $20.12 + 9.84j$ amps.

GuidedPractice

5. **ELECTRICITY** If a circuit has a voltage of 120 volts and a current of $8 + 6j$ amps, find the impedance of the circuit in rectangular form.

Before calculating the powers and roots of complex numbers, it may be helpful to express the complex numbers in polar form. Abraham DeMoivre is credited with discovering a useful pattern for evaluating powers of complex numbers.

We can use the formula for the product of complex numbers to help visualize the pattern that DeMoivre discovered.

First, find z^2 by taking the product of $z \cdot z$.

$z \cdot z = r(\cos\theta + i \sin\theta) \cdot r(\cos\theta + i \sin\theta)$ — Multiply.

$z^2 = r^2[\cos(\theta + \theta) + i \sin(\theta + \theta)]$ — Product Formula

$z^2 = r^2(\cos 2\theta + i \sin 2\theta)$ — Simplify.

Now find z^3 by calculating $z^2 \cdot z$.

$z^2 \cdot z = r^2(\cos 2\theta + i \sin 2\theta) \cdot r(\cos\theta + i \sin\theta)$ — Multiply.

$z^3 = r^3[\cos(2\theta + \theta) + i \sin(2\theta + \theta)]$ — Product Formula

$z^3 = r^3(\cos 3\theta + i \sin 3\theta)$ — Simplify.

Notice that when calculating these powers of a complex number, you take the nth power of the modulus and multiply the argument by n.

This pattern is summarized below.

Math HistoryLink

Abraham DeMoivre

(1667–1754)

A French mathematician, DeMoivre is known for the theorem named for him and his book on probability theory, *The Doctrine of Chances*. He is credited with being one of the pioneers of analytic geometry and probability.

KeyConcept DeMoivre's Theorem

If the polar form of a complex number is $z = r(\cos\theta + \boldsymbol{i}\sin\theta)$, then for positive integers n

$$z^n = [r(\cos\theta + \boldsymbol{i}\sin\theta)]^n \text{ or } r^n(\cos n\theta + \boldsymbol{i}\sin n\theta).$$

You will prove DeMoivre's Theorem in Lesson 10-4.

Example 6 DeMoivre's Theorem

Find $\left(4 + 4\sqrt{3}\boldsymbol{i}\right)^6$, and express it in rectangular form.

First, write $4 + 4\sqrt{3}\boldsymbol{i}$ in polar form.

$r = \sqrt{a^2 + b^2}$	Conversion formula	$\theta = \tan^{-1}\frac{b}{a}$	
$= \sqrt{4^2 + (4\sqrt{3})^2}$	$a = 4$ and $b = 4\sqrt{3}$	$= \tan^{-1}\frac{4\sqrt{3}}{4}$	
$= \sqrt{16 + 48}$	Simplify.	$= \tan^{-1}\sqrt{3}$	
$= 8$	Simplify.	$= \frac{\pi}{3}$	

The polar form of $4 + 4\sqrt{3}\boldsymbol{i}$ is $8\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$.

Now use DeMoivre's Theorem to find the sixth power.

$(4 + 4\sqrt{3}\boldsymbol{i})^6 = \left[8\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)\right]^6$	Original equation
$= 8^6\left[\cos 6\left(\frac{\pi}{3}\right) + \boldsymbol{i}\sin 6\left(\frac{\pi}{3}\right)\right]$	DeMoivre's Theorem
$= 262{,}144(\cos 2\pi + \boldsymbol{i}\sin 2\pi)$	Simplify.
$= 262{,}144(1 + 0\boldsymbol{i})$	Evaluate.
$= 262{,}144$	Simplify.

Therefore, $(4 + 4\sqrt{3}\boldsymbol{i})^6 = 262{,}144$.

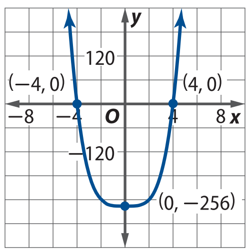
GuidedPractice

Find each power, and express it in rectangular form.

6A. $(1 + \sqrt{3}\boldsymbol{i})^4$

6B. $(2\sqrt{3} - 2\boldsymbol{i})^8$

In the real number system, $x^4 = 256$ has two solutions, 4 and -4. The graph of $y = x^4 - 256$ shows that there are two real zeros at $x = 4$ and -4. In the complex number system, however, there are two real solutions and two complex solutions.



Through the Fundamental Theorem of Algebra polynomials of degree n have exactly n zeros in the complex number system. Therefore, the equation $x^4 = 256$, rewritten as $x^4 - 256 = 0$, has exactly four solutions, or roots: 4, -4, $4\boldsymbol{i}$, and $-4\boldsymbol{i}$. In general, all nonzero complex numbers have p distinct pth roots. That is, they each have two square roots, three cube roots, four fourth roots, and so on.

ReviewVocabulary

Fundamental Theorem of Algebra A polynomial function of degree n, where $n > 0$, has at least one zero (real or imaginary) in the complex number system.

To find all of the roots of a polynomial, we can use DeMoivre's Theorem to arrive at the following expression.

KeyConcept Distinct Roots

For a positive integer p, the complex number $r(\cos\theta + i\sin\theta)$ has p distinct pth roots. They are found by

$$r^{\frac{1}{p}}\left(\cos\frac{\theta + 2n\pi}{p} + i\sin\frac{\theta + 2n\pi}{p}\right),$$

where $n = 0, 1, 2, \ldots, p - 1$.

We can use this formula for the different values of n, but we can stop when $n = p - 1$. When n equals or exceeds p, the roots repeat as the following shows.

$$\frac{\theta + 2\pi p}{p} = \frac{\theta}{p} + 2\pi$$ Coterminal with $\frac{\theta}{p}$, when $n = 0$

Example 7 *p*th Roots of a Complex Number

Find the fourth roots of $-4 - 4i$.

First, write $-4 - 4i$ in polar form.

$$-4 - 4i = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$ $r = \sqrt{(-4)^2 + (-4)^2}$ or $4\sqrt{2}$; $\theta = \tan^{-1}\frac{-4}{-4} + \pi$ or $\frac{5\pi}{4}$

Now write an expression for the fourth roots.

$$(4\sqrt{2})^{\frac{1}{4}}\left(\cos\frac{\frac{5\pi}{4} + 2n\pi}{4} + i\sin\frac{\frac{5\pi}{4} + 2n\pi}{4}\right)$$ $\theta = \frac{5\pi}{4}$, $p = 4$, and $r^{\frac{1}{p}} = (4\sqrt{2})^{\frac{1}{4}}$

$$= \sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{n\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{n\pi}{2}\right)\right]$$ Simplify.

Let $n = 0, 1, 2$, and 3 successively to find the fourth roots.

Let $n = 0$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(0)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(0)\pi}{2}\right)\right]$ Distinct Roots

$= \sqrt[8]{32}\left(\cos\frac{5\pi}{16} + i\sin\frac{5\pi}{16}\right)$ or $0.86 + 1.28i$ First fourth root

Let $n = 1$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(1)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(1)\pi}{2}\right)\right]$

$= \sqrt[8]{32}\left(\cos\frac{13\pi}{16} + i\sin\frac{13\pi}{16}\right)$ or $-1.28 + 0.86i$ Second fourth root

Let $n = 2$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(2)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(2)\pi}{2}\right)\right]$

$= \sqrt[8]{32}\left(\cos\frac{21\pi}{16} + i\sin\frac{21\pi}{16}\right)$ or $-0.86 - 1.28i$ Third fourth root

Let $n = 3$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(3)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(3)\pi}{2}\right)\right]$

$= \sqrt[8]{32}\left(\cos\frac{29\pi}{16} + i\sin\frac{29\pi}{16}\right)$ or $1.28 - 0.86i$ Fourth fourth root

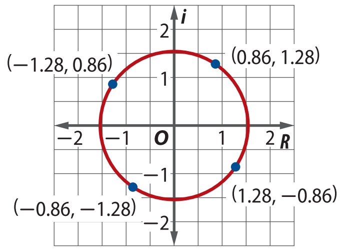
The fourth roots of $-4 - 4i$ are approximately $0.86 + 1.28i$, $-1.28 + 0.86i$, $-0.86 - 1.28i$, and $1.28 - 0.86i$.

GuidedPractice

7A. Find the cube roots of $2 + 2i$.

7B. Find the fifth roots of $4\sqrt{3} - 4i$.

We can make observations about the distinct roots of a number by graphing the roots on a coordinate plane. As shown at the right, the four fourth roots found in Example 7 lie on a circle. If we look at the polar form of each complex number, each has the same modulus of $\sqrt[8]{32}$, which acts as the radius of the circle. The roots are also equally spaced around the circle as a result of the arguments differing by $\frac{\pi}{2}$.



A special case of finding roots occurs when finding the pth roots of 1. When 1 is written in polar form, $r = 1$. As mentioned in the previous paragraph, the modulus of our roots is the radius of the circle that is formed from plotting the roots on a coordinate plane. Thus, the pth roots of 1 lie on the unit circle. Finding the pth roots of 1 is referred to as finding the **pth roots of unity**.

StudyTip

The pth Roots of a Complex Number Each root will have the same modulus of $r^{\frac{1}{p}}$. The argument of the first root is $\frac{\theta}{p}$, and each successive root is found by repeatedly adding $\frac{2\pi}{p}$ to the argument.

Example 8 The pth Roots of Unity

Find the eighth roots of unity.

First, write 1 in polar form.

$1 = 1 \cdot (\cos 0 + \boldsymbol{i} \sin 0)$ — $r = \sqrt{1^2 + 0^2}$ or 1 and $\theta = \tan^{-1} \frac{0}{1}$ or 0

Now write an expression for the eighth roots.

$1\left(\cos \frac{0 + 2n\pi}{8} + \boldsymbol{i} \sin \frac{0 + 2n\pi}{8}\right)$ — $\theta = 0$, $p = 8$, and $r^{\frac{1}{p}} = 1^{\frac{1}{8}}$ or 1

$= \cos \frac{n\pi}{4} + \boldsymbol{i} \sin \frac{n\pi}{4}$ — Simplify.

Let $n = 0$ to find the first root of 1.

$\boldsymbol{n = 0}$ $\quad \cos \frac{(0)\pi}{4} + \boldsymbol{i} \sin \frac{(0)\pi}{4}$ — Distinct Roots

$= \cos 0 + \boldsymbol{i} \sin 0$ or 1 — First root

Notice that the modulus of each complex number is 1. The arguments are found by $\frac{n\pi}{4}$, resulting in θ increasing by $\frac{\pi}{4}$ for each successive root. Therefore, we can calculate the remaining roots by adding $\frac{\pi}{4}$ to each previous θ.

$\cos 0 + \boldsymbol{i} \sin 0$ or 1 — 1st root

$\cos \frac{\pi}{4} + \boldsymbol{i} \sin \frac{\pi}{4}$ or $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\boldsymbol{i}$ — 2nd root

$\cos \frac{\pi}{2} + \boldsymbol{i} \sin \frac{\pi}{2}$ or $\boldsymbol{i}$ — 3rd root

$\cos \frac{3\pi}{4} + \boldsymbol{i} \sin \frac{3\pi}{4}$ or $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\boldsymbol{i}$ — 4th root

$\cos \pi + \boldsymbol{i} \sin \pi$ or -1 — 5th root

$\cos \frac{5\pi}{4} + \boldsymbol{i} \sin \frac{5\pi}{4}$ or $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\boldsymbol{i}$ — 6th root

$\cos \frac{3\pi}{2} + \boldsymbol{i} \sin \frac{3\pi}{2}$ or $-\boldsymbol{i}$ — 7th root

$\cos \frac{7\pi}{4} + \boldsymbol{i} \sin \frac{7\pi}{4}$ or $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\boldsymbol{i}$ — 8th root

The eighth roots of 1 are 1, $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\boldsymbol{i}$, $\boldsymbol{i}$, $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\boldsymbol{i}$, -1, $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\boldsymbol{i}$, $-\boldsymbol{i}$, and $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\boldsymbol{i}$ as shown in Figure 8.5.1.

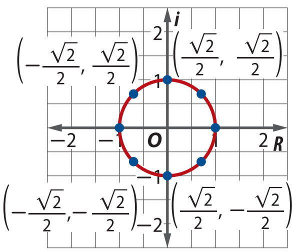


Figure 8.5.1

GuidedPractice

8A. Find the cube roots of unity.

8B. Find the seventh roots of unity.

Exercises

Graph each number in the complex plane, and find its absolute value. (Example 1)

1. $z = 4 + 4i$
2. $z = -3 + i$
3. $z = -4 - 6i$
4. $z = 2 - 5i$
5. $z = 3 + 4i$
6. $z = -7 + 5i$
7. $z = -3 - 7i$
8. $z = 8 - 2i$

9. **VECTORS** The force on an object is given by $z = 10 + 15i$, where the components are measured in newtons (N). (Example 1)
   a. Represent z as a vector in the complex plane.
   b. Find the magnitude and direction angle of the vector.

Express each complex number in polar form. (Example 2)

10. $4 + 4i$
11. $-2 + i$
12. $4 - \sqrt{2}i$
13. $2 - 2i$
14. $4 + 5i$
15. $-2 + 4i$
16. $-1 - \sqrt{3}i$
17. $3 + 3i$

Graph each complex number on a polar grid. Then express it in rectangular form. (Example 3)

18. $10(\cos 6 + i \sin 6)$
19. $2(\cos 3 + i \sin 3)$
20. $4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$
21. $3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$
22. $\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$
23. $2\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$
24. $-3(\cos 180° + i \sin 180°)$
25. $\frac{3}{2}(\cos 360° + i \sin 360°)$

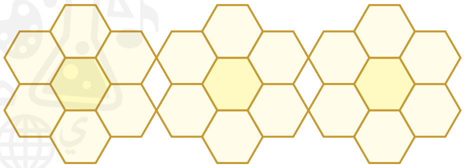
Find each product or quotient, and express it in rectangular form. (Examples 4 and 5)

26. $6\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \cdot 4\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$
27. $5(\cos 135° + i \sin 135°) \cdot 2\,(\cos 45° + i \sin 45°)$
28. $3\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) \div \frac{1}{2}(\cos \pi + i \sin \pi)$
29. $2(\cos 90° + i \sin 90°) \cdot 2(\cos 270° + i \sin 270°)$
30. $3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) \div 4\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$
31. $4\left(\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4}\right) \div 2\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$
32. $\frac{1}{2}(\cos 60° + i \sin 60°) \cdot 6(\cos 150° + i \sin 150°)$
33. $6\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) \div 2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$
34. $5(\cos 180° + i \sin 180°) \cdot 2(\cos 135° + i \sin 135°)$
35. $\frac{1}{2}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \div 3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$

Find each power, and express it in rectangular form. (Example 6)

36. $(2 + 2\sqrt{3}i)^6$
37. $(12i - 5)^3$
38. $\left[4\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)\right]^4$
39. $(\sqrt{3} - i)^3$
40. $(3 - 5i)^4$
41. $(2 + 4i)^4$
42. $(3 - 6i)^4$
43. $(2 + 3i)^2$
44. $\left[3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)\right]^3$
45. $\left[2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)\right]^4$

46. **DESIGN** Suha works for an advertising agency. She wants to incorporate a design comprised of regular hexagons as the artwork for one of her proposals. Suha can locate the vertices of one of the central regular hexagons by graphing the solutions to $x^6 - 1 = 0$ in the complex plane. Find the vertices of this hexagon. (Example 7)



Find all of the distinct *p*th roots of the complex number. (Examples 7 and 8)

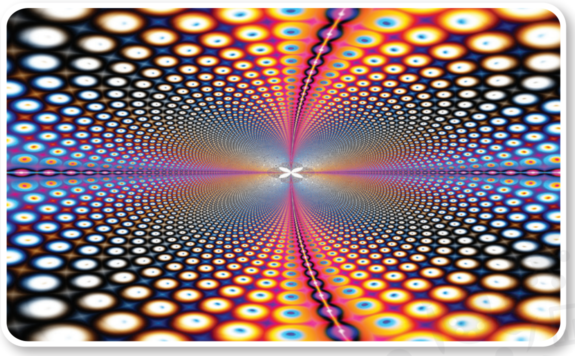
47. sixth roots of i
48. fifth roots of $-i$
49. fourth roots of $4\sqrt{3} - 4i$
50. cube roots of $-117 + 44i$
51. fifth roots of $-1 + 11\sqrt{2}i$
52. square root of $-3 - 4i$
53. find the square roots of unity
54. find the fourth roots of unity

55. **ELECTRICITY** The impedance in one part of a series circuit is $5(\cos 0.9 + j \sin 0.9)$ ohms. In the second part of the circuit, it is $8(\cos 0.4 + j \sin 0.4)$ ohms.
   a. Convert each expression to rectangular form.
   b. Add your answers from part **a** to find the total impedance in the circuit.
   c. Convert the total impedance back to polar form.

Find each product. Then repeat the process by multiplying the polar forms of each pair of complex numbers using the Product Formula.

56. $(1 - i)(4 + 4i)$
57. $(3 + i)(3 - i)$
58. $(4 + i)(3 - i)$
59. $(-6 + 5i)(2 - 3i)$
60. $(\sqrt{2} + 2i)(1 + i)$
61. $(3 - 2i)(1 + \sqrt{3}i)$

62. FRACTALS A *fractal* is a geometric figure that is made up of a pattern that is repeated indefinitely on successively smaller scales, as shown below.



In this problem, you will generate a fractal through iterations of $f(z) = z^2$. Consider $z_0 = 0.8 + 0.5i$.

a. Calculate $z_1, z_2, z_3, z_4, z_5, z_6$, and z_7 where $z_1 = f(z_0)$, $z_2 = f(z_1)$, and so on.

b. Graph each of the numbers on the complex plane.

c. Predict the location of z_{100}. Explain.

63. TRANSFORMATIONS There are certain operations with complex numbers that correspond to geometric transformations in the complex plane. Describe the transformation applied to point z to obtain point w in the complex plane for each of the following operations.

a. $w = z + (3 - 4i)$

b. w is the complex conjugate of z.

c. $w = i \cdot z$

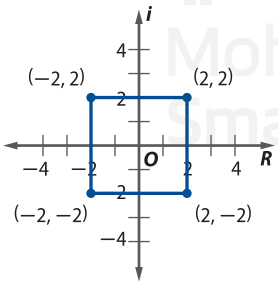
d. $w = 0.25z$

Find z and the pth roots of z given each of the following.

64. $p = 3$, one cube root is $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

65. $p = 4$, one fourth root is $-1 - i$

66. GRAPHICS By representing each vertex by a complex number in polar form, a programmer dilates and then rotates the square below 45° counterclockwise so that the new vertices lie at the midpoints of the sides of the original square.



a. By what complex number should the programmer multiply each number to produce this transformation?

b. What happens if the numbers representing the original vertices are multiplied by the square of your answer to part **a**?

Use the Distinct Roots Formula to find all of the solutions of each equation. Express the solutions in rectangular form.

67. $x^3 = i$

68. $x^3 + 3 = 128$

69. $x^4 = 81i$

70. $x^5 - 1 = 1023$

71. $x^3 + 1 = i$

72. $x^4 - 2 + i = -1$

H.O.T. Problems Use Higher-Order Thinking Skills

73. ERROR ANALYSIS Alma and Bilal are evaluating $\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^5$. Alma uses DeMoivre's Theorem and gets an answer of $\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}$. Bilal tells her that she has only completed part of the problem. Is either of them correct? Explain your reasoning.

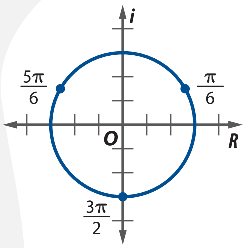
74. REASONING Suppose $z = a + bi$ is one of the 29th roots of 1.

a. What is the maximum value of a?

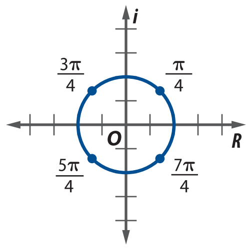
b. What is the maximum value of b?

CHALLENGE Find the roots shown on each graph and write them in polar form. Then identify the complex number with the given roots.

75.



76.



77. PROOF Given $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, where $r_2 \neq 0$, prove that $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$.

REASONING Determine whether each statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

78. The pth roots of a complex number z are equally spaced around the circle centered at the origin with radius $r^{\frac{1}{p}}$.

79. The complex conjugate of $z = a + bi$ is $\overline{z} = a - bi$. For any z, $z + \overline{z}$ and $z \cdot \overline{z}$ are real numbers.

80. OPEN ENDED Find two complex numbers $a + bi$ in which $a \neq 0$ and $b \neq 0$ with an absolute value of $\sqrt{17}$.

81. WRITING IN MATH Explain why the sum of the imaginary parts of the p distinct pth roots of any positive real number must be zero. (*Hint*: The roots are the vertices of a regular polygon.)

Spiral Review

Write each polar equation in rectangular form. (Lesson 8-4)

82. $r = \dfrac{15}{1 + 4\cos\theta}$

83. $r = \dfrac{14}{2\cos\theta + 2}$

84. $r = \dfrac{-6}{\sin\theta - 2}$

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation. (Lesson 8-3)

85. $(x - 3)^2 + y^2 = 9$

86. $x^2 - y^2 = 1$

87. $x^2 + y^2 = 2y$

Graph the conic given by each equation.

88. $y = x^2 + 3x + 1$

89. $y^2 - 2x^2 - 16 = 0$

90. $x^2 + 4y^2 + 2x - 24y + 33 = 0$

Find the center, foci, and vertices of each ellipse.

91. $\dfrac{(x + 8)^2}{9} + \dfrac{(y - 7)^2}{81} = 1$

92. $25x^2 + 4y^2 + 150x + 24y = -161$

93. $4x^2 + 9y^2 - 56x + 108y = -484$

Solve each system of equations using Gauss-Jordan elimination.

94. $x + y + z = 12$
$6x - 2y - z = 16$
$3x + 4y + 2z = 28$

95. $9g + 7h = -30$
$8h + 5j = 11$
$-3g + 10j = 73$

96. $2k - n = 2$
$3p = 21$
$4k + p = 19$

97. POPULATION In the beginning of 2008, the world's population was about 6.7 billion. If the world's population grows continuously at a rate of 2%, the future population P, in billions, can be predicted by $P = 6.5e^{0.02t}$, where t is the time in years since 2008.

a. According to this model, what will be the world's population in 2018?

b. Some experts have estimated that the world's food supply can support a population of at most 18 billion people. According to this model, for how many more years will the food supply be able to support the trend in world population growth?

Skills Review for Standardized Tests

98. SAT/ACT The graph on the xy-plane of the quadratic function g is a parabola with vertex at (3, −2). If $g(0) = 0$, then which of the following must also equal 0?

A $g(2)$
B $g(3)$
C $g(4)$
D $g(6)$
E $g(7)$

99. Which of the following expresses the complex number $20 - 21\boldsymbol{i}$ in polar form?

F $29(\cos 5.47 + \boldsymbol{i}\sin 5.47)$
G $29(\cos 5.52 + \boldsymbol{i}\sin 5.52)$
H $32(\cos 5.47 + \boldsymbol{i}\sin 5.47)$
J $32(\cos 5.52 + \boldsymbol{i}\sin 5.52)$

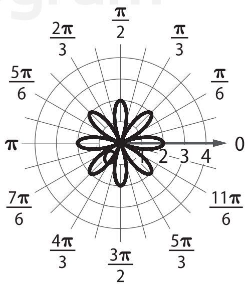
100. FREE RESPONSE Consider the graph at the right.

a. Give a possible equation for the function.

b. Describe the symmetries of the graph.

c. Give the zeroes of the function over the domain $0 \le \theta \le 2\pi$.

d. What is the minimum value of r over the domain $0 \le \theta \le 2\pi$?



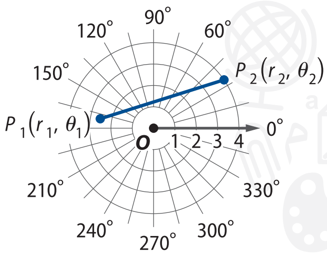
CHAPTER 8 Study Guide and Review

Chapter Summary

KeyConcepts

Polar Coordinates (Lesson 8-1)

- In the polar coordinate system, a point (r, θ) is located using its directed distance r and directed angle θ.
- The distance between $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ in the polar plane is $P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$.



Graphs of Polar Equations (Lesson 8-2)

- Circle: $r = a \cos \theta$ or $r = a \sin \theta$
- Limaçon: $r = a \pm b \cos \theta$ or $r = a \pm b \sin \theta$, $a > 0$, $b > 0$
- Rose: $r = a \cos n\theta$ or $r = a \sin n\theta$, $n \geq 2$, $n \in \mathbb{Z}$
- Lemniscate: $r^2 = a^2 \cos 2\theta$ or $r^2 = a^2 \sin 2\theta$
- Spirals of Archimedes: $r = a\theta + b$, $\theta \geq 0$

Polar and Rectangular Forms of Equations (Lesson 8-3)

- A point $P(r, \theta)$ has rectangular coordinates $(r \cos \theta, r \sin \theta)$.
- To convert a point $P(x, y)$ from rectangular to polar coordinates, use the equations $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$, when $x > 0$ or $\theta = \tan^{-1}\frac{y}{x} + \pi$, when $x < 0$.

Polar Forms of Conic Sections (Lesson 8-4)

- The polar equation of a conic section is of the form $r = \frac{ed}{1 \pm e \cos \theta}$ or $r = \frac{ed}{1 \pm e \sin \theta}$, depending on the location and orientation of the directrix.

Complex Numbers and DeMoivre's Theorem (Lesson 8-5)

- The polar or trigonometric form of the complex number $a + bi$ is $r(\cos \theta + \boldsymbol{i} \sin \theta)$.
- The product formula for two complex numbers z_1 and z_2 is $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + \boldsymbol{i} \sin(\theta_1 + \theta_2)]$.
- The quotient formula for two complex numbers z_1 and z_2 is $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \boldsymbol{i} \sin(\theta_1 - \theta_2)]$, where z_2 and $r_2 \neq 0$.
- DeMoivre's Theorem states that if the polar form of a complex number is $z = r(\cos \theta + \boldsymbol{i} \sin \theta)$, then $z^n = r^n(\cos n\theta + \boldsymbol{i} \sin n\theta)$ for positive integers n.

KeyVocabulary

absolute value of a complex number
Argand plane
argument
cardioid
complex plane
imaginary axis
lemniscate
limaçon
modulus
polar axis
polar coordinate system
polar coordinates
polar equation
polar form
polar graph
pole
*p*th roots of unity
real axis
rose
spiral of Archimedes
trigonometric form

VocabularyCheck

Choose the correct term from the list above to complete each sentence.

1. A(n) ____________ is the set of all points with coordinates (r, θ) that satisfy a given polar equation.
2. A plane that has an axis for the real component and an axis for the imaginary component is a(n) ____________.
3. The location of a point in the ____________ is identified using the directed distance from a fixed point and the angle from a fixed axis.
4. A special type of limaçon with equation of the form $r = a + b \cos \theta$ where $a = b$ is called a(n) ____________.
5. The ____________ is the angle θ of a complex number written in the form $r(\cos \theta + \boldsymbol{i} \sin \theta)$.
6. The origin of a polar coordinate system is called the ____________.
7. The absolute value of a complex number is also called the ____________.
8. The ____________ is another name for the complex plane.
9. The graph of a polar equation of the form $r^2 = a^2 \cos 2\theta$ or $r^2 = a^2 \sin 2\theta$ is called a(n) ____________.
10. The ____________ is an initial ray from the pole, usually horizontal and directed toward the right.

Lesson-by-Lesson Review

8-1 Polar Coordinates

Graph each point on a polar grid.

11. $W(-0.5, 210°)$

12. $X\left(1.5, \frac{7\pi}{4}\right)$

13. $Y(4, -120°)$

14. $Z\left(-3, \frac{5\pi}{6}\right)$

Graph each polar equation.

15. $\theta = -60°$

16. $r = \frac{9}{2}$

17. $r = 7$

18. $\theta = \frac{11\pi}{6}$

Find the distance between each pair of points.

19. $\left(5, \frac{\pi}{2}\right), \left(2, -\frac{7\pi}{6}\right)$

20. $(-3, 60°), (4, 240°)$

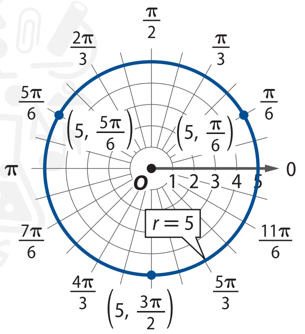
21. $(-1, -45°), (6, 270°)$

22. $\left(7, \frac{5\pi}{6}\right), \left(2, \frac{4\pi}{3}\right)$

Example 1

Graph $r = 5$.

The solutions of $r = 5$ are ordered pairs of the form $(5, \theta)$ where θ is any real number. The graph consists of all points that are 5 units from the pole, so the graph is a circle centered at the pole with radius 5.



8-2 Graphs of Polar Equations

Use symmetry, zeros, and maximum r-values to graph each function.

23. $r = \sin 3\theta$

24. $r = 2 \cos \theta$

25. $r = 5 \cos 2\theta$

26. $r = 4 \sin 4\theta$

27. $r = 2 + 2 \cos \theta$

28. $r = 1.5\theta, \theta \geq 0$

Use symmetry to graph each equation.

29. $r = 2 - \sin \theta$

30. $r = 1 + 5 \cos \theta$

31. $r = 3 - 2 \cos \theta$

32. $r = 4 + 4 \sin \theta$

33. $r = -3 \sin \theta$

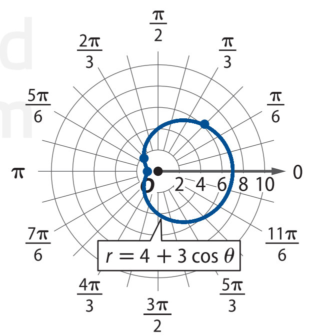
34. $r = -5 + 3 \cos \theta$

Example 2

Use symmetry to graph $r = 4 + 3 \cos \theta$.

Replacing (r, θ) with $(r, -\theta)$ yields $r = 4 + 3 \cos(-\theta)$, which simplifies to $r = 4 + 3 \cos \theta$ because cosine is even. The equations are equivalent, so the graph of this equation is symmetric with respect to the polar axis. Therefore, you can make a table of values to find the r-values corresponding to θ on the interval $[0, \pi]$.

θ	r
0	7
$\frac{\pi}{4}$	$\frac{8 + 3\sqrt{2}}{2}$
$\frac{\pi}{2}$	4
$\frac{3\pi}{4}$	$\frac{8 - 3\sqrt{2}}{2}$
π	1



By plotting these points and using polar axis symmetry, you obtain the graph shown.

8-3 Polar and Rectangular Forms

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \leq \theta \leq 2\pi$. Round to the nearest hundredth.

35. $(-1, 5)$

36. $(3, 7)$

37. $(2a, 0)$, $a > 0$

38. $(4b, -6b)$, $b > 0$

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

39. $r = 5$

40. $r = -4 \sin \theta$

41. $r = 6 \sec \theta$

42. $r = \frac{1}{3} \csc \theta$

Example 3

Write $r = 2 \cos \theta$ in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

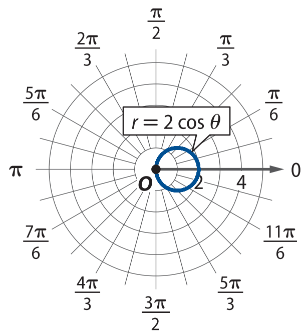
$r = 2 \cos \theta$ — Original equation

$r^2 = 2r \cos \theta$ — Multiply each side by r.

$x^2 + y^2 = 2x$ — $r^2 = x^2 + y^2$ and $x = r \cos \theta$

$x^2 + y^2 - 2x = 0$ — Subtract $2x$ from each side.

In standard form, $(x - 1)^2 + y^2 = 1$, you can identify the graph of this equation as a circle centered at $(1, 0)$ with radius 1, as supported by the graph of $r = 2 \cos \theta$.



8-4 Polar Forms of Conic Sections

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation.

43. $r = \frac{3.5}{1 + \sin \theta}$

44. $r = \frac{1.2}{1 + 0.3 \cos \theta}$

45. $r = \frac{14}{1 - 2 \sin \theta}$

46. $r = \frac{6}{1 - \cos \theta}$

Write and graph a polar equation and directrix for the conic with the given characteristics.

47. $e = 0.5$; vertices at $(0, -2)$ and $(0, 6)$

48. $e = 1.5$; directrix: $x = 5$

Write each polar equation in rectangular form.

49. $r = \frac{1.6}{1 - 0.2 \sin \theta}$

50. $r = \frac{5}{1 + \cos \theta}$

Example 4

Determine the eccentricity, type of conic, and equation of the directrix for $r = \frac{7}{3.5 - 3.5 \cos \theta}$.

Write the equation in standard form, $r = \frac{ed}{1 + e \cos \theta}$.

$r = \frac{7}{3.5 - 3.5 \cos \theta}$ — Original equation

$r = \frac{3.5(2)}{3.5(1 - \cos \theta)}$ — Factor the numerator and denominator.

$r = \frac{2}{1 - \cos \theta}$ — Divide the numerator and denominator by 3.5.

In this form, you can see from the denominator that $e = 1$; therefore, the conic is a parabola. For polar equations of this form, the equation of the directrix is $x = -d$. From the numerator, we know that $ed = 2$, so $d = 2 \div 1$ or 2. Therefore, the equation of the directrix is $x = -2$.

8-5 Complex Numbers and DeMoivre's Theorem

Graph each number in the complex plane, and find its absolute value.

51. $z = 3 - i$

52. $z = 4i$

53. $z = -4 + 2i$

54. $z = 6 - 3i$

Express each complex number in polar form.

55. $3 + \sqrt{2}i$

56. $-5 + 8i$

57. $-4 - \sqrt{3}i$

58. $\sqrt{2} + \sqrt{2}i$

Graph each complex number on a polar grid. Then express it in rectangular form.

59. $z = 3\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$

60. $z = 5\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

61. $z = -2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

62. $z = 4\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$

Find each product or quotient, and express it in rectangular form.

63. $-2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) \cdot -4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

64. $8(\cos 225° + i \sin 225°) \cdot \frac{1}{2}(\cos 120° + i \sin 120°)$

65. $5\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \div \frac{1}{3}\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$

66. $6(\cos 210° + i \sin 210°) \div 3(\cos 150° + i \sin 150°)$

Find each power, and express it in rectangular form.

67. $(4 - i)^5$

68. $(\sqrt{2} + 3i)^4$

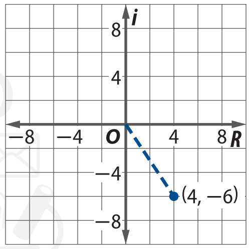
Find all of the distinct *p*th roots of the complex number.

69. cube roots of $6 - 4i$

70. fourth roots of $1 + i$

Example 5

Graph $4 - 6i$ in the complex plane and express in polar form.



Find the modulus.

$r = \sqrt{a^2 + b^2}$ — Conversion formula

$= \sqrt{4^2 + (-6)^2}$ or $2\sqrt{13}$ — $a = 4$ and $b = -6$

Find the argument.

$\theta = \tan^{-1}\frac{b}{a}$ — Conversion formula

$= \tan^{-1}\left(-\frac{6}{4}\right)$ — $a = 4$ and $b = -6$

$= -0.98$ — Simplify.

The polar form of $4 - 6i$ is approximately $2\sqrt{13}\,[\cos(-0.98) + i \sin(-0.98)]$.

Example 6

Find $-3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \cdot 5\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right)$ in polar form. Then express the product in rectangular form.

$-3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \cdot 5\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right)$ — Original expression

$= (-3 \cdot 5)\left[\cos\left(\frac{\pi}{4} + \frac{7\pi}{6}\right) + i \sin\left(\frac{\pi}{4} + \frac{7\pi}{6}\right)\right]$ — Product Formula

$= -15\left[\cos\left(\frac{17\pi}{12}\right) + i \sin\left(\frac{17\pi}{12}\right)\right]$ — Simplify.

Now find the rectangular form of the product.

$-15\left[\cos\left(\frac{17\pi}{12}\right) + i \sin\left(\frac{17\pi}{12}\right)\right]$ — Polar form

$= -15[-0.26 + i(-0.97)$ — Evaluate.

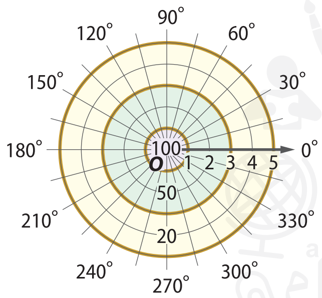
$= 3.9 + 14.5i$ — Distributive Property

The polar form of the product is $-15\left[\cos\left(\frac{17\pi}{12}\right) + i \sin\left(\frac{17\pi}{12}\right)\right]$

The rectangular form of the product is $3.9 + 14.5i$.

Applications and Problem Solving

71. GAMES An arcade game consists of rolling a ball up an incline at a target. The region in which the ball lands determines the number of points earned. The model shows the point value for each region. (Lesson 8-1)



a. If, on a turn, a player rolls the ball to the point (3.5, 165°), how many points does he get?

b. Give two possible locations that a player will receive 50 points.

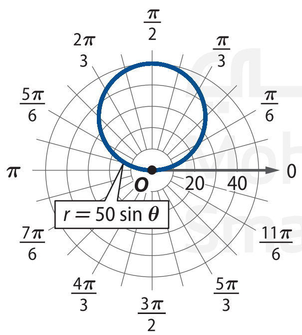
72. LANDSCAPING A landscaping company uses an adjustable lawn sprinkler that can rotate 360° and can cover a circular region with radius of 20 meters. (Lesson 8-1)

a. Graph the dimensions of the region that the sprinkler can cover on a polar grid if it is set to rotate 360°.

b. Find the area of the region that the sprinkler covers if the rotation is adjusted to $-30° \leq \theta \leq 210°$.

73. BIOLOGY The pattern on the shell of a snail can be modeled using $r = \frac{1}{3}\theta + \frac{1}{2}$, $\theta \geq 0$. Identify and graph the classic curve that models this pattern. (Lesson 8-2)

74. RIDES The path of a Ferris wheel can be modeled by $r = 50 \sin \theta$, where r is given in meters. (Lesson 8-3)



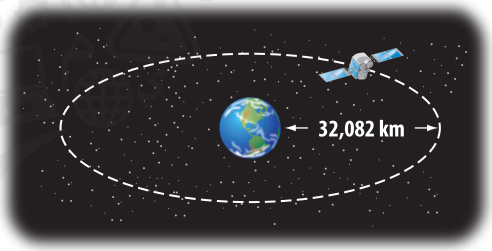
a. What are the polar coordinates of a rider located at $\theta = \frac{\pi}{12}$? Round to the nearest tenth, if necessary.

b. What are the rectangular coordinates of the rider's location? Round to the nearest tenth, if necessary.

c. What is the rider's distance above the ground if the polar axis represents the ground?

75. ORIENTEERING Orienteering requires participants to make their way through an area using a topographic map. One orienteer starts at Checkpoint A and walks 5000 meters at an angle of 35° measured clockwise from due east. A second orienteer starts at Checkpoint A and walks 3000 meters due west and then 2000 meters due north. How far, to the nearest meter, are the two orienteers from each other? (Lesson 8-3)

76. SATELLITE The orbit of a satellite around Earth has eccentricity of 0.05, and the distance from a vertex of the path to the center of Earth is 32,082 kilometers. Write a polar equation that can be used to model the path of the satellite if Earth is located at the focus closest to the given vertex. (Lesson 8-4)

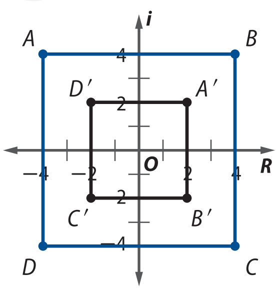


77. ELECTRICITY Most circuits in Europe are designed to accommodate 220 volts. For parts **a** and **b**, use $E = I \cdot Z$, where voltage E is measured in volts, impedance Z is measured in ohms, and current I is measured in amps. Round to the nearest tenth. (Lesson 8-5)

a. If the circuit has a current of $2 + 5j$ amps, what is the impedance?

b. If a circuit has an impedance of $1 - 3j$ ohms, what is the current?

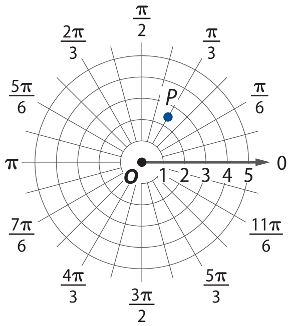
78. COMPUTER GRAPHICS Geometric transformation of figures can be performed using complex numbers. If a programmer starts with square $ABCD$, as shown below, each of the vertices can be represented by a complex number in polar form. Multiplication can then be used to rotate and dilate the square, producing the square $A'B'C'D'$. By what complex number should the programmer multiply each number to produce this transformation? (Lesson 8-5)



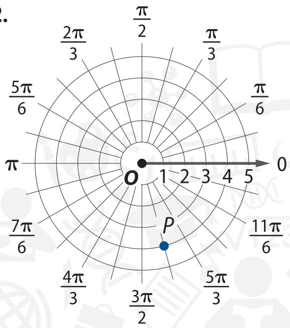
CHAPTER 8 Practice Test

Find four different pairs of polar coordinates that name point P if $-2\pi \leq \theta \leq 2\pi$.

1.



2.



Graph each polar equation.

3. $\theta = 30°$

4. $r = 1$

5. $r = 2.5$

6. $\theta = \frac{5\pi}{3}$

7. $r = \frac{2}{3} \sin \theta$

8. $r = -\frac{1}{2} \sec \theta$

9. $r = -4 \csc \theta$

10. $r = 2 \cos \theta$

Identify and graph each classic curve.

11. $r = 1.5 + 1.5 \cos \theta$

12. $r^2 = 6.25 \sin 2\theta$

13. RADAR An air traffic controller is tracking an airplane with a current location of (66, 115°). The value of r is given in kilometers.

a. What are the rectangular coordinates of the airplane? Round to the nearest tenth kilometers.

b. If a second plane is located at the point (50, −75), what are the polar coordinates of the plane if $r > 0$ and $0 \leq \theta \leq 360°$? Round to the nearest kilometers and the nearest tenth of a degree, if necessary.

c. What is the distance between the two planes? Round to the nearest kilometers.

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation.

14. $(x - 7)^2 + y^2 = 49$

15. $y = 3x^2$

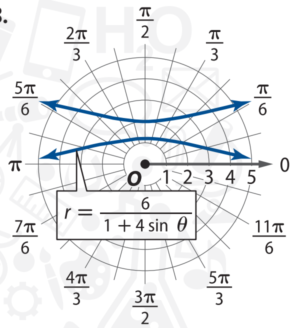
Determine the eccentricity, type of conic, and equation of the directrix for each polar equation.

16. $r = \frac{2}{1 - 0.4 \sin \theta}$

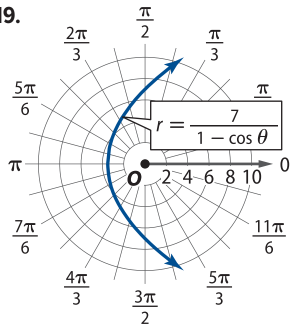
17. $r = \frac{6}{2 \cos \theta + 1}$

Write the equation for each polar graph in rectangular form.

18.



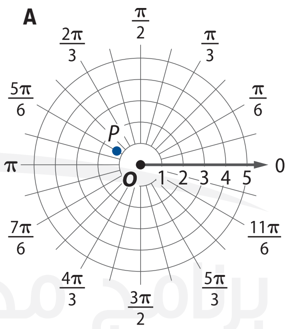
19.



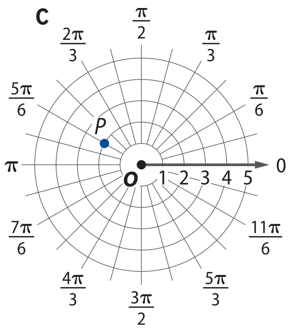
20. ELECTRICITY If a circuit has a voltage E of 135 volts and a current I of $3 - 4j$ amps, find the impedance Z of the circuit in ohms in rectangular form. Use the equation $E = I \cdot Z$.

21. MULTIPLE CHOICE Identify the graph of point P with complex coordinates $(-\sqrt{3}, -1)$ on the polar coordinate plane.

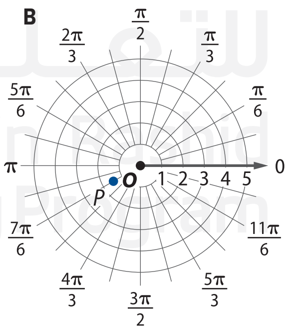
A



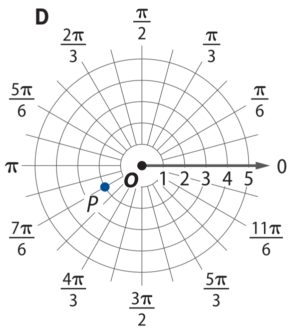
C



B



D



Find each power, and express it in rectangular form. Round to the nearest tenth.

22. $(-1 + 4i)^3$

23. $(-7 - 3i)^5$

24. $(6 + i)^4$

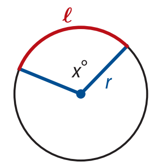
25. $(2 - 5i)^6$

CHAPTER 8 Connect to AP Calculus
Arc Length

:: Objective

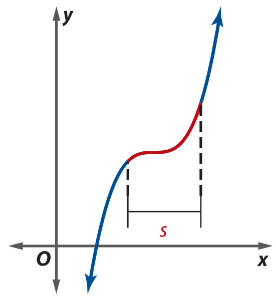
- Approximate the arc length of a curve.

You can find the length of a line segment by using the Distance Formula. You can find the length of an arc by using proportions. In calculus, you will need to calculate many lengths that are not represented by line segments or sections of a circle.

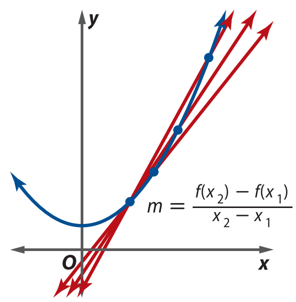


$$\frac{\ell}{2\pi r} = \frac{x}{360}$$

Integral calculus focuses on areas, volumes, and lengths. It can be used to find the length of a curve for which we do not have a standard equation, such as a curve defined by a quadratic, cubic, or polar function. *Riemann sums* and *definite integrals*, two concepts that you will learn more about in the following chapters, are needed to calculate the exact length of a curve, or *arc length*, denoted *s*.



In this lesson, we will approximate the arc length of a curve using a process similar to the method that you applied to approximate the rate of change at a point. Recall that in Chapter 1, you calculated the slopes of secant lines to approximate the rates of change for graphs at specific points. Decreasing the distance between the two points on the secant lines increased the accuracy of the approximations as shown in the graph at the right.



Activity 1 Approximate Arc Length

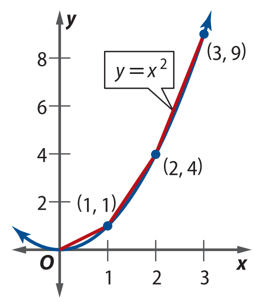
Approximate the arc length of the graph of $y = x^2$ for $0 \le x \le 3$.

Step 1 Graph $y = x^2$ for $0 \le x \le 3$ as shown.

Step 2 Graph points on the curve at $x = 1$, 2, and 3. Connect the points using line segments as shown.

Step 3 Use the Distance Formula to find the length of each line segment.

Step 4 Approximate the length of the arc by finding the sum of the lengths of the line segments.

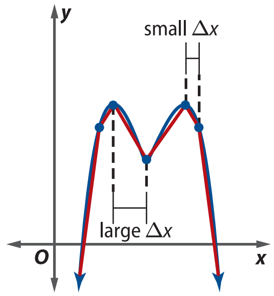


Analyze the Results

1. Is your approximation *greater* or *less* than the actual length? Explain your reasoning.
2. Approximate the arc length a second time using 6 line segments that are formed by the points $x = 0$, 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0. Include a sketch of the graph with your approximation.
3. Describe what happens to the approximation for the arc length as shorter line segments are used.
4. For the two approximations, the endpoints of the line segments were equally spaced along the *x*-axis. Do you think this will always produce the most accurate approximation? Explain your reasoning.

Notice that for the first activity, the endpoints of the line segments were equally spaced 0.5 units apart along the x-axis. When using advanced methods of calculus to find *exact* arc length, a constant difference between a pair of endpoints along the x-axis is essential. This difference is denoted Δx.

Accurately approximating arc length by using a constant Δx to create the line segments may not always be the most efficient method. The shape of the arc will dictate the spacing of the endpoints, thus creating different values for Δx. For example, if a graph shows an increase or decrease over a large interval for x, a large line segment may be used for the approximation. If a graph includes a turning point, it is better to use small line segments to account for the curve in the graph.



Previously, you learned how to calculate the distance between polar coordinates. This formula can be used to approximate the arc length of a curve represented by a polar equation.

Activity 2 Approximate Arc Length

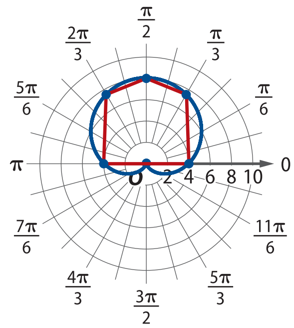
Approximate the arc length of the graph of $r = 4 + 4 \sin \theta$ for $0 \leq x \leq 2\pi$.

Step 1 Graph $r = 4 + 4 \sin \theta$ for $0 \leq x \leq 2\pi$ as shown.

Step 2 Draw 6 points on the curve at $\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$, and $\frac{3\pi}{2}$. Connect the points using line segments as shown.

Step 3 Use the Polar Distance Formula to find the length of each line segment.

Step 4 Approximate the length of the arc by finding the sum of the lengths of the line segments.



StudyTip

Polar Graphs Create a table of values for r and θ when calculating the arc length for a polar graph. This will help to reduce errors created by functions that produce negative values for r.

Analyze the Results

5. Explain how symmetry can be used to reduce the number of calculations in Step 3.

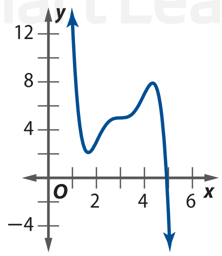
6. Approximate the arc length using at least 10 segments. Include a sketch of the graph.

7. Let n be the number of line segments used in an approximation and $\Delta\theta$ be a constant difference in θ between the endpoints of a line segment. Make a conjecture regarding the relationship between n, θ, and the approximation for an arc length.

Model and Apply

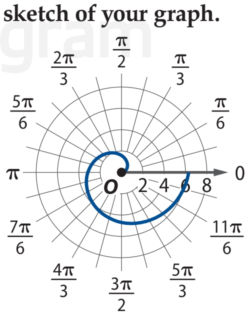
Approximate the arc length for each graph. Include a sketch of your graph.

8.



$y = -(x - 3)^5 + 3(x - 3)^3 + 5$ for $1 \leq x \leq 5$

9.



$r = \theta$ for $0 \leq \theta \leq 2\pi$

CHAPTER 9

Sequences and Series

Then

You simplified and evaluated algebraic expressions.

Now

You will:

- Use arithmetic and geometric sequences and series.
- Use special sequences and iterate functions.
- Expand powers by using the Binomial Theorem.
- Prove statements by using mathematical induction.

Why?

CONSERVATION AND NATURE Mathematics occurs in aspects of nature in astonishing ways. The Fibonacci sequence manifests itself in seeds, flowers, pine cones, fruits, and vegetables. Sequences and series can further help us conserve our natural resources by making water filtration systems more efficient.

Get Ready for the Chapter

QuickCheck

Solve each equation.

1. $-6 = 7x + 78$
2. $768 = 3x^4$
3. $23 - 5x = 8$
4. $2x^3 + 4 = -50$
5. **PLANTS** Laila has 48 plants for her two gardens. She plants 12 in the small garden. In the other garden she wants 4 plants in each row. How many rows will she have?

Graph each function.

6. $\{(1, 3), (2, 5), (3, 7), (4, 9), (5, 11)\}$
7. $\{(1, -15), (2, -12), (3, -9), (4, -6), (5, -3)\}$
8. $\left\{(1, 27), (2, 9), (3, 3), (4, 1), \left(5, \frac{1}{3}\right)\right\}$
9. $\left\{(1, 1), (2, 2), \left(3, \frac{5}{2}\right), \left(4, \frac{11}{4}\right), \left(5, \frac{23}{8}\right)\right\}$
10. **DAY CARE** A child care center has expenses of AED 450 per day. They charge AED 150 per child per day. The function $P(c) = 150c - 450$ gives the amount of money the center makes when there are c children there. How much will they make if there are 8 children?

Evaluate each expression for the given value(s) of the variable(s).

11. $\frac{a}{3}(b + c)$ if $a = 9$, $b = -2$, and $c = -8$
12. $r + (n - 2)t$ if $r = 15$, $n = 5$, and $t = -1$
13. $x \cdot y^{z+1}$ if $x = -2$, $y = \frac{1}{3}$, and $z = 5$
14. $\frac{a(1 - bc)^2}{1 - b}$ if $a = -3$, $b = -4$, and $c = 1$

QuickReview

Example 1

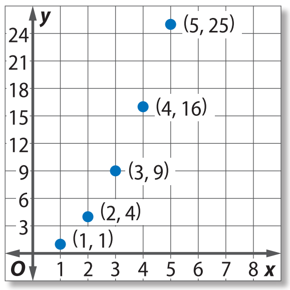
Solve $25 = 3x^3 + 400$.

$25 = 3x^3 + 400$	Original equation
$-375 = 3x^3$	Subtract 400 from each side.
$-125 = x^3$	Divide each side by 3.
$\sqrt[3]{-125} = \sqrt[3]{x^3}$	Take the cube root of each side.
$-5 = x$	Simplify.

Example 2

Graph the function $\{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$. State the domain and range.

The domain of a function is the set of all possible x-values. So, the domain of the function is $\{1, 2, 3, 4, 5\}$. The range of a function is the set of all possible y-values. So, the range of this function is $\{1, 4, 9, 16, 25\}$.



Example 3

Evaluate $2 \cdot 3^{x+y}$ if $x = -2$ and $y = -3$.

$2 \cdot 3^{x+y} = 2 \cdot 3^{-2+(-3)}$	Substitute.
$= 2 \cdot 3^{-5}$	Simplify.
$= \frac{2}{3^5}$	Rewrite with positive exponent.
$= \frac{2}{243}$	Evaluate the power.

Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 9. To get ready, identify important terms and organize your resources.

FOLDABLES® StudyOrganizer

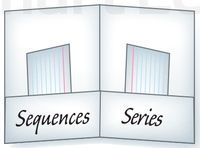
Sequences and Series Make this Foldable to help you organize your Chapter 9 notes about sequences and series. Begin with one $8\frac{1}{2}''$ by $11''$ sheet of paper.

1 **Fold** in half, matching the short sides.

2 **Unfold** and fold the long side up to form a pocket.

3 **Staple** or glue the outer edges to complete the pocket.

4 **Label** each side as shown. Use index cards to record notes and examples.



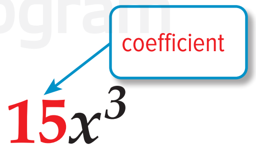
NewVocabulary

English

sequence
finite sequence
infinite sequence
arithmetic sequence
common difference
geometric sequence
common ratio
arithmetic means
series
arithmetic series
partial sum
geometric means
geometric series
convergent series
divergent series
recursive sequence
iteration
mathematical induction
induction hypothesis

ReviewVocabulary

coefficient the numerical factor of a monomial



formula a mathematical sentence that expresses the relationship between certain quantities

function a relation in which each element of the domain is paired with exactly one element in the range

LESSON 9-1 Sequences as Functions

Then

- You analyzed linear and exponential functions.

Now

1. Relate arithmetic sequences to linear functions.
2. Relate geometric sequences to exponential functions.

Why?

- During their routine, a high school marching band marches in rows. There is one performer in the first row, three performers in the next row, and five in the third row. This pattern continues for the rest of the rows.

NewVocabulary
sequence
term
finite sequence
infinite sequence
arithmetic sequence
common difference
geometric sequence
common ratio

Mathematical Practices
Reason abstractly and quantitatively.
Look for and make use of structure.

1 Arithmetic Sequences

A **sequence** is a set of numbers in a particular order or pattern. Each number in a sequence is called a **term**. A sequence may be a **finite sequence** containing a limited number of terms, such as {−2, 0, 2, 4, 6}, or an **infinite sequence** that continues without end, such as {0, 1, 2, 3, ...}. The first term of a sequence is denoted a_1, the second term is denoted a_2, and so on.

KeyConcept Sequences as Functions

Words A sequence is a function in which the domain consists of natural numbers, and the range consists of real numbers.

Symbols

Domain:	1	2	3	...	n	the position of a term
Range:	a_1	a_2	a_3	...	a_n	the terms of the sequence

Examples

Finite Sequence	Infinite Sequence
{3, 6, 9, 12, 15}	{3, 6, 9, 12, 15, ...}
Domain: {1, 2, 3, 4, 5}	Domain: {all natural numbers}
Range: {3, 6, 9, 12, 15}	Range: {y | y is a multiple of 3, $y \geq 3$}

In an **arithmetic sequence**, each term is determined by adding a constant value to the previous term. This constant value is called the **common difference**.

Consider the sequence 3, 6, 9, 12, 15. This sequence is arithmetic because the terms share a common difference. Each term is 3 more than the previous term.

3 6 9 12 15
+3 +3 +3 +3

Example 1 Identify Arithmetic Sequences

Determine whether each sequence is arithmetic.

a. 5, −6, −17, −28, ...

5 −6 −17 −28
−11 −11 −11

The common difference is −11.
The sequence is arithmetic.

b. −4, 12, 28, 42, ...

−4 12 28 42
+16 +16 +14

There is no common difference.
This is not an arithmetic sequence.

GuidedPractice

1A. 7, 12, 16, 20, ...

1B. −6, 3, 12, 21, ...

You can use the common difference to find terms of an arithmetic sequence.

Example 2 Graph an Arithmetic Sequence

Consider the arithmetic sequence 18, 14, 10, … .

a. Find the next four terms of the sequence.

Step 1 To determine the common difference, subtract any term from the term directly after it. The common difference is 10 − 14 or −4.

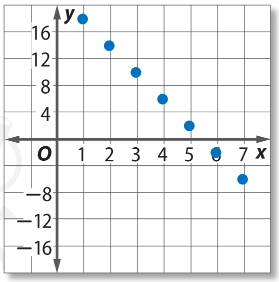
Step 2 To find the next term, add −4 to the last term.

Continue to add −4 to find the following terms.

10 →(+(−4)) 6 →(+(−4)) 2 →(+(−4)) −2 →(+(−4)) −6

The next four terms are 6, 2, −2, and −6.

b. Graph the first seven terms of the sequence.
The domain contains the terms {1, 2, 3, 4, 5, 6, 7} and the range contains the terms {18, 14, 10, 6, 2, −2, −6}. So, graph the corresponding ordered pairs.



GuidedPractice

2. Find the next four terms of the arithmetic sequence 18, 11, 4, … . Then graph the first seven terms.

Notice that the graph of the terms of the arithmetic sequence lie on a line. An arithmetic sequence is a linear function in which the term number n is the independent variable, the term a_n is the dependent variable, and the common difference is the slope.

Real-WorldLink

Each year, about 100 bands compete in the Bands of America Grand National Championships.

Source: Bands of America

Real-World Example 3 Find a Term

MARCHING BANDS Refer to the beginning of the lesson. Suppose the director wants to determine how many performers will be in the 14th row during the routine.

Understand Because the difference between any two consecutive rows is 2, the common difference for the sequence is 2.

Plan Use point-slope form to write an equation for the sequence. Let $m = 2$ and $(x_1, y_1) = (3, 5)$. Then solve for $x = 14$.

Solve

$(y - y_1) = m(x - x_1)$	Point-slope form
$(y - 5) = 2(x - 3)$	$m = 2$ and $(x_1, y_1) = (3, 5)$
$y - 5 = 2x - 6$	Multiply.
$y = 2x - 1$	Add 5 to each side.
$y = 2(14) - 1$	Replace x with 14.
$y = 28 - 1$ or 27	Simplify.

Check You can find the terms of the sequence by adding 2, starting with row 1, until you reach row 14.

GuidedPractice

3. MONEY Usama's employer offers him a pay rate of AED 33 per hour with a AED 0.50 raise every three months. How much will Usama earn per hour after 3 years?

2 Geometric Sequences

Another type of sequence is a geometric sequence. In a **geometric sequence**, each term is determined by multiplying a nonzero constant by the previous term. This constant value is called the **common ratio**.

Consider the sequence $\frac{1}{16}, \frac{1}{4}, 1, 4, 16$. This sequence is geometric because the terms share a common ratio. Each term is 4 times as much as the previous term.

$$\frac{1}{16} \xrightarrow{\times 4} \frac{1}{4} \xrightarrow{\times 4} 1 \xrightarrow{\times 4} 4 \xrightarrow{\times 4} 16$$

Example 4 Identify Geometric Sequences

Determine whether each sequence is geometric.

a. −2, 6, −18, 54, ...

Find the ratios of the consecutive terms.

$\frac{6}{-2} = -3$ $\quad \frac{-18}{6} = -3$ $\quad \frac{54}{-18} = -3$

The ratios are the same, so the sequence is geometric.

b. 8, 16, 24, 32, ...

$\frac{16}{8} = 2$ $\quad \frac{24}{16} = 1.5$ $\quad \frac{32}{24} = 1.\overline{3}$

The ratios are not the same, so the sequence is not geometric.

WatchOut!

Ratios If you find the ratio of a term to the previous term, set up the remaining ratios the same way.

GuidedPractice

4A. −8, 2, −0.5, 0.125, ...

4B. 1, 3, 7, 15, ...

When given a set of information, you can create a problem that relates a story.

Example 5 Graph a Geometric Sequence

Consider the geometric sequence 32, 8, 2, ...

a. Find the next three terms of the sequence.

Step 1 Find the value of the common ratio: $\frac{2}{8}$ or $\frac{1}{4}$.

Step 2 To find the next term, multiply the previous term by $\frac{1}{4}$.

Continue multiplying by $\frac{1}{4}$ to find the following terms.

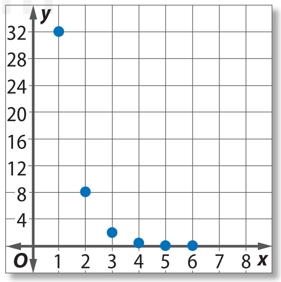
$$2 \xrightarrow{\times \frac{1}{4}} \frac{1}{2} \xrightarrow{\times \frac{1}{4}} \frac{1}{8} \xrightarrow{\times \frac{1}{4}} \frac{1}{32}$$

The next three terms are $\frac{1}{2}$, $\frac{1}{8}$, and $\frac{1}{32}$.

b. Graph the first six terms of the sequence.

Domain: {1, 2, 3, 4, 5, 6}

Range: $\left\{32, 8, 2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}\right\}$



GuidedPractice

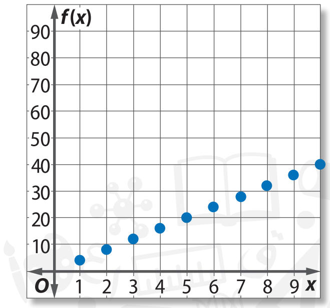
5. Find the next two terms of 7, 21, 63, ... Then graph the first five terms.

Review Vocabulary

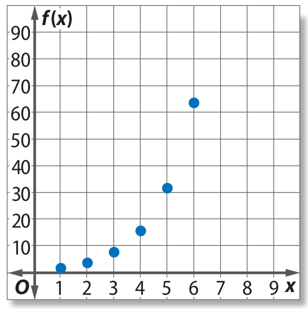
exponential function a function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$

Examine the graph in Example 5. While the graph of an arithmetic sequence is linear, the graph of a geometric sequence is exponential and can be represented by $f(x) = r^x$, where r is the common ratio, $r > 0$, and $r \neq 1$.

Arithmetic



Geometric



x	1	2	3	4	5	6	7	8	9	10
$f(x)$	4	8	12	16	20	24	28	32	36	40

x	1	2	3	4	5	6
$f(x)$	2	4	8	16	32	64

Arithmetic and geometric sequences are functions in which the domain, defined by the term number n, contains the set of or subset of positive integers. The characteristics of arithmetic and geometric sequences can be used to classify sequences.

Example 6 Classify Sequences

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

a. 16, 24, 36, 54, …

Check for a common difference.

$54 - 36 = 18$ $\qquad 36 - 24 = 12$ ✗

Check for a common ratio.

$\frac{54}{36} = \frac{3}{2}$ $\qquad \frac{36}{24} = \frac{3}{2}$ $\qquad \frac{24}{16} = \frac{3}{2}$ ✓

Because there is a common ratio, the sequence is geometric.

b. 1, 4, 9, 16, …

Check for a common difference.

$16 - 9 = 7$ $\qquad 9 - 4 = 5$ ✗

Check for a common ratio.

$\frac{16}{9} = 1.\overline{7}$ $\qquad \frac{9}{4} = 2.25$ ✗

Because there is no common difference or ratio, the sequence is neither arithmetic nor geometric.

c. 23, 17, 11, 5, …

Check for a common difference.

$5 - 11 = -6$ $\qquad 11 - 17 = -6$ $\qquad 17 - 23 = -6$ ✓

Because there is a common difference, the sequence is arithmetic.

Guided Practice

6A. $\frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, \ldots$

6B. $2, -\frac{3}{2}, \frac{9}{8}, -\frac{27}{32}, \ldots$

6C. $-4, 4, 5, -5, \ldots$

Check Your Understanding

Example 1 **Determine whether each sequence is arithmetic. Write *yes* or *no*.**

1. $8, -2, -12, -22,$

2. $-19, -12, -5, 2, 9$

3. $1, 2, 4, 8, 16$

4. $0.6, 0.9, 1.2, 1.8, \ldots$

Example 2 **Find the next four terms of each arithmetic sequence. Then graph the sequence.**

5. $6, 18, 30, \ldots$

6. $15, 6, -3, \ldots$

7. $-19, -11, -3, \ldots$

8. $-26, -33, -40, \ldots$

Example 3

9. **FINANCIAL LITERACY** Yasmin is saving her money to buy a car. She has AED 950, and she plans to save AED 320 per week from her job as a babysitter.

a. How much will Yasmin have saved after 8 weeks?

b. If the car costs AED 7,350, how long will it take her to save enough money at this rate?

Example 4 **Determine whether each sequence is geometric. Write *yes* or *no*.**

10. $-8, -5, -1, 4, \ldots$

11. $4, 12, 36, 108, \ldots$

12. $27, 9, 3, 1, \ldots$

13. $7, 14, 21, 28, \ldots$

Example 5 **Find the next three terms of each geometric sequence. Then graph the sequence.**

14. $8, 12, 18, 27, \ldots$

15. $8, 16, 32, 64, \ldots$

16. $250, 50, 10, 2, \ldots$

17. $9, -3, 1, -\frac{1}{3}, \ldots$

Example 6 **Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.**

18. $5, 1, 7, 3, 9, \ldots$

19. $200, -100, 50, -25, \ldots$

20. $12, 16, 20, 24, \ldots$

Practice and Problem Solving

Example 1 **Determine whether each sequence is arithmetic. Write *yes* or *no*.**

21. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

22. $-9, -3, 0, 3, 9$

23. $14, -5, -19, \ldots$

24. $\frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \frac{11}{9}, \ldots$

Example 2 **Find the next four terms of each arithmetic sequence. Then graph the sequence.**

25. $-4, -1, 2, 5, \ldots$

26. $10, 2, -6, -14, \ldots$

27. $-5, -11, -17, -23, \ldots$

28. $-19, -2, 15, \ldots$

29. $\frac{1}{5}, \frac{4}{5}, \frac{7}{5}, \ldots$

30. $\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}$

Example 3

31. **THEATER** There are 28 seats in the front row of a theater. Each successive row contains two more seats than the previous row. If there are 24 rows, how many seats are in the last row of the theater?

32. **SENSE-MAKING** Ibrahim began an exercise program to get back in shape. He plans to row 5 minutes on his rowing machine the first day and increase his rowing time by one minute and thirty seconds each day.

a. How long will he row on the 18th day?

b. On what day will Ibrahim first row an hour or more?

c. Is it reasonable for this pattern to continue indefinitely? Explain.

Example 4 **Determine whether each sequence is geometric. Write *yes* or *no*.**

33. 21, 14, 7, ...
34. 124, 186, 248, ...
35. −27, 18, −12, ...
36. 162, 108, 72, ...
37. $\frac{1}{2}, -\frac{1}{4}, 1, -\frac{1}{2}, \ldots$
38. −4, −2, 0, 2, ...

Example 5 **Find the next three terms of the sequence. Then graph the sequence.**

39. 0.125, −0.5, 2, ...
40. 18, 12, 8, ...
41. 64, 48, 36, ...
42. 81, 108, 144, ...
43. $\frac{1}{3}, 1, 3, 9, \ldots$
44. 1, 0.1, 0.01, 0.001, ...

Example 6 **Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.**

45. 3, 12, 27, 48, ...
46. 1, −2, −5, −8, ...
47. 12, 36, 108, 324, ...
48. $-\frac{2}{5}, -\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, \ldots$
49. $\frac{5}{2}, 3, \frac{7}{2}, 4, \ldots$
50. 6, 9, 14, 21, ...

51. READING Asma took an 800-page book on vacation. If she was already on page 112 and is going to be on vacation for 8 days, what is the minimum number of pages she needs to read per day to finish the book by the end of her vacation?

52. DEPRECIATION Amna's car is expected to depreciate at a rate of 15% per year. If her car is currently valued at AED 88,200, to the nearest dirham, how much will it be worth in 6 years?

53. REGULARITY When a piece of paper is folded onto itself, it doubles in thickness. If a piece of paper that is 0.1 mm thick could be folded 37 times, how thick would it be?

H.O.T. Problems Use Higher-Order Thinking Skills

54. REASONING Explain why the sequence 8, 10, 13, 17, 22 is not arithmetic.

55. OPEN ENDED Describe a real-life situation that can be represented by an arithmetic sequence with a common difference of 8.

56. CHALLENGE The sum of three consecutive terms of an arithmetic sequence is 6. The product of the terms is −42. Find the terms.

57. ERROR ANALYSIS Badr and Salem are determining whether the sequence 8, 8, 8, ... is *arithmetic*, *geometric*, *neither*, or *both*. Is either of them correct? Explain your reasoning.

Badr	Salem
The sequence has a common difference of 0. The sequence is arithmetic.	The sequence has a common ratio of 1. The sequence is geometric.

58. OPEN ENDED Find a geometric sequence, an arithmetic sequence, and a sequence that is neither geometric nor arithmetic that begins 3, 9, ...

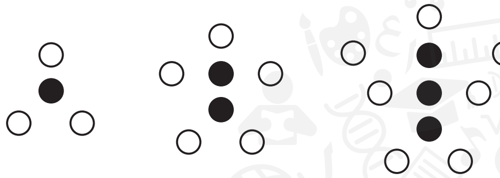
59. REASONING If a geometric sequence has a ratio r such that $|r| < 1$, what happens to the terms as n increases? What would happen to the terms if $|r| \geq 1$?

60. WRITING IN MATH Describe what happens to the terms of a geometric sequence when the common ratio is doubled. What happens when it is halved? Explain your reasoning.

Standardized Test Practice

61. SHORT RESPONSE Fawzia's rectangular bedroom measures 4.5 meters by 3.5 meters. She wants to purchase carpet for the bedroom that costs AED 108 per meter square, including tax. How much will it cost to carpet her bedroom?

62. The pattern of filled circles and white circles below can be described by a relationship between two variables.



Which rule relates w, the number of white circles, to f, the number of dark circles?

A $w = 3f$

B $f = \frac{1}{2}w - 1$

C $w = 2f + 1$

D $f = \frac{1}{3}w$

63. SAT/ACT Rana wanted to determine the average of her six test scores. She added the scores correctly to get T, but divided by 7 instead of 6. Her average was 12 less than the actual average. Which equation could be used to determine the value of T?

F $6T + 12 = 7T$

G $\frac{T}{7} = \frac{T - 12}{6}$

H $\frac{T}{7} + 12 = \frac{T}{6}$

J $\frac{T}{6} = \frac{T - 12}{7}$

K $\frac{T}{6} = 12 - \frac{T}{7}$

64. Find the next term in the geometric sequence $8, 6, \frac{9}{2}, \frac{27}{8}, \ldots$.

A $\frac{11}{8}$

B $\frac{27}{16}$

C $\frac{9}{4}$

D $\frac{81}{32}$

Spiral Review

Solve each system of equations.

65. $y = 5$
$y^2 = x^2 + 9$

66. $y - x = 1$
$x^2 + y^2 = 25$

67. $3x = 8y^2$
$8y^2 - 2x^2 = 16$

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

68. $6x^2 + 6y^2 = 162$

69. $4y^2 - x^2 + 4 = 0$

70. $x^2 + y^2 + 6y + 13 = 40$

Graph each function.

71. $f(x) = \frac{6}{(x - 2)(x + 3)}$

72. $f(x) = \frac{-3}{(x - 2)^2}$

73. $f(x) = \frac{x^2 - 36}{x + 6}$

74. HEALTH A certain medication is eliminated from the bloodstream at a steady rate. It decays according to the equation $y = ae^{-0.1625t}$, where t is in hours. Find the half-life of this substance.

Skills Review

Write an equation of each line.

75. passes through (6, 4), $m = 0.5$

76. passes through $\left(2, \frac{1}{2}\right)$, $m = -\frac{3}{4}$

77. passes through (0, −6), $m = 3$

78. passes through (0, 4), $m = \frac{1}{4}$

79. passes through (1, 3) and $\left(8, -\frac{1}{2}\right)$

80. passes through (−5, 1) and (5, 16)

LESSON 9-2

Sequences, Series, and Sigma Notation

Then

- You used functions to generate ordered pairs and used graphs to analyze end behavior.

Now

1. Investigate several different types of sequences.
2. Use sigma notation to represent and calculate sums of series.

Why?

- Wafa developed a Web site where students at her high school can post their own social networking Web pages. A student at the high school is given a free page if he or she refers the Web site to five friends. The site starts with one page created by Wafa, who in turn, refers five friends that each create a page. Those five friends refer five more people each, all of whom develop pages, and so on.

NewVocabulary

sequence
term
finite sequence
infinite sequence
recursive sequence
explicit sequence
Fibonacci sequence
converge
diverge
series
finite series
*n*th partial sum
infinite series
sigma notation

1 Sequences

In mathematics, a **sequence** is an ordered list of numbers. Each number in the sequence is known as a **term**. A **finite sequence**, such as 1, 3, 5, 7, 9, 11, contains a finite number of terms. An **infinite sequence**, such as 1, 3, 5, 7, ..., contains an infinite number of terms.

Each term of a sequence is a function of its position. Therefore, an infinite sequence is a function whose domain is the set of natural numbers and can be written as $f(1) = a_1$, $f(2) = a_2$, $f(3) = a_3$, ..., $f(n) = a_n$, ..., where a_n denotes the nth term. If the domain of the function is only the first n natural numbers, the sequence is finite.

Infinitely many sequences exist with the same first few terms. To sufficiently define a *unique* sequence, a formula for the nth term or other information *must* be given. When defined *explicitly*, an **explicit formula** gives the nth term a_n as a function of n.

Example 1 Find Terms of Sequences

a. Find the next four terms of the sequence 2, 7, 12, 17, ...

The nth term of this sequence is not given. One possible pattern is that each term is 5 greater than the previous term. Therefore, a sample answer for the next four terms is 22, 27, 32, and 37.

b. Find the next four terms of the sequence 2, 5, 10, 17, ...

The nth term of this sequence is not given. If we subtract each term from the term that follows, we start to see a possible pattern.

$a_2 - a_1 = 5 - 2$ or 3 $\qquad$ $a_3 - a_2 = 10 - 5$ or 5 $\qquad$ $a_4 - a_3 = 17 - 10$ or 7

It appears that each term is generated by adding the next successive odd number. However, looking at the pattern, it may also be determined that each term is 1 more than each perfect square, or $a_n = n^2 + 1$. Using either pattern, a sample answer for the next four terms is 26, 37, 50, and 65.

c. Find the first four terms of the sequence given by $a_n = 2n(-1)^n$.

Use the explicit formula given to find a_n for $n = 1, 2, 3,$ and 4.

$a_1 = 2 \cdot 1 \cdot (-1)^1$ or -2 $\quad$ $n = 1$ $\qquad$ $a_2 = 2 \cdot 2 \cdot (-1)^2$ or 4 $\quad$ $n = 2$

$a_3 = 2 \cdot 3 \cdot (-1)^3$ or -6 $\quad$ $n = 3$ $\qquad$ $a_4 = 2 \cdot 4 \cdot (-1)^4$ or 8 $\quad$ $n = 4$

The first four terms in the sequence are -2, 4, -6, and 8.

GuidedPractice

Find the next four terms of each sequence.

1A. 32, 16, 8, 4, ... $\qquad$ **1B.** 1, 2, 4, 7, 11, 16, 22, ...

1C. Find the first four terms of the sequenc e given by $a_n = n^3 - 10$.

Sequences can also be defined *recursively*. Recursively defined sequences give one or more of the first few terms and then define the terms that follow using those previous terms. The formula defining the nth term of the sequence is called a **recursive formula** or a *recurrence relation*.

StudyTip

Notation The term denoted a_n represents the nth term of a sequence. The term denoted a_{n-1} represents the term immediately before a_n. The term a_{n-2} represents the term two terms before a_n.

Example 2 Recursively Defined Sequences

Find the fifth term of the recursively defined sequence $a_1 = 1$, $a_n = a_{n-1} + 2n - 1$, where $n \geq 2$.

Since the sequence is defined recursively, all the terms before the fifth term must be found first. Use the given first term, $a_1 = 2$, and the recursive formula for a_n.

$a_2 = a_{2-1} + 2(2) - 1$ — $n = 2$

$= a_1 + 3$ — Simplify.

$= 2 + 3$ or 5 — $a_1 = 2$

$a_3 = a_{3-1} + 2(3) - 1$ — $n = 3$

$= a_2 + 5$ or 10 — $a_2 = 5$

$a_4 = a_{4-1} + 2(4) - 1$ — $n = 4$

$= a_3 + 7$ or 17 — $a_3 = 10$

$a_5 = a_{5-1} + 2(5) - 1$ — $n = 5$

$= a_4 + 9$ or 26 — $a_4 = 17$

GuidedPractice

Find the sixth term of each sequence.

2A. $a_1 = 3$, $a_n = (-2)a_{n-1}$, $n \geq 2$

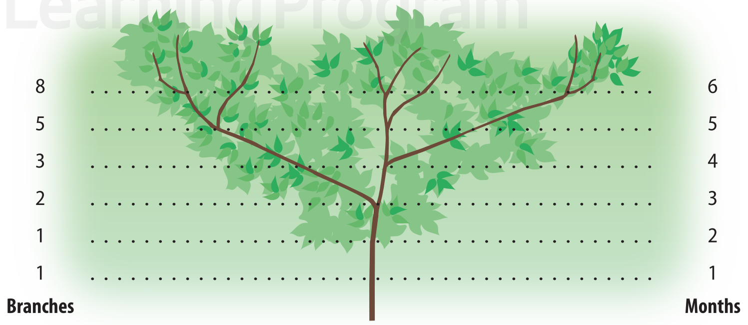
2B. $a_1 = 8$, $a_n = 2a_{n-1} - 7$, $n \geq 2$

The **Fibonacci sequence** describes many patterns found in nature. This sequence is often defined recursively.

Real-World Example 3 Fibonacci Sequence

NATURE **Suppose that when a plant first starts to grow, the stem has to grow for two months before it is strong enough to support branches. At the end of the second month, it sprouts a new branch and will continue to sprout one new branch each month. The new branches also each grow for two months and then start to sprout one new branch each month. If this pattern continues, how many branches will the plant have after 10 months?**

During the first two months, there will only be one branch, the stem. At the end of the second month, the stem will produce a new branch, making the total for the third month two branches. The new branch will grow and develop two months before producing a new branch of its own, but the original branch will now produce a new branch each month.



Real-WorldLink

Along with being found in flower petals, sea shells, and the bones in a human hand, Fibonacci sequences can also be found in pieces of art, music, poetry, and architecture.

Source: *Universal Principles of Design*

WatchOut!

Notation The first term of a sequence is occasionally denoted as a_0. When this occurs, the domain of the function describing the sequence is the set of whole numbers.

The following table shows the pattern.

Month	1	2	3	4	5	6	7	8	9	10
Branches	1	1	2	3	5	8	13	21	34	55

Each term is the sum of the previous two terms. This pattern can be written as the recursive formula $a_0 = 1$, $a_1 = 1$, $a_n = a_{n-2} + a_{n-1}$, where $n \geq 2$.

GuidedPractice

3. NATURE How many branches will a plant like the one described in Example 3 have after 15 months if no branches are removed?

Previously, you examined the end behavior of the graphs of functions. You learned that as the domains of some functions approach ∞, the ranges approach a unique number called a limit. As a function, an infinite sequence may also have a limit. If a sequence has a limit such that the terms approach a unique number, then it is said to **converge**. If not, the sequence is said to **diverge**.

TechnologyTip

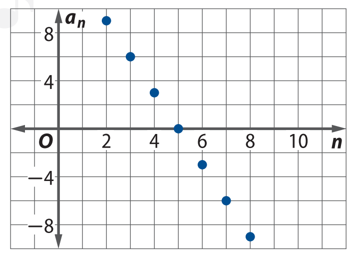
Convergent or Divergent Sequences If an explicit formula for a sequence is known, you can enter the formula in the Y= menu of a graphing calculator and graph the related function. Analyzing the end behavior of the graph can help you to determine whether the sequence is convergent or divergent.

Example 4 Convergent and Divergent Sequences

Determine whether each sequence is *convergent* or *divergent*.

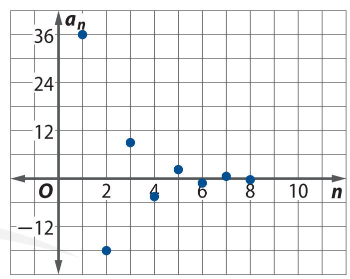
a. $a_n = -3n + 12$

The first eight terms of this sequence are 12, 9, 6, 3, 0, −3, −6, and −9. From the graph at the right, you can see that a_n does not approach a finite number. Therefore, this sequence is divergent.



b. $a_1 = 36$, $a_n = -\frac{1}{2}a_{n-1}$, $a \geq 2$

The first eight terms of this sequence are 36, −18, 9, −4.5, 2.25, −1.125, 0.5625, −0.28125, and 0.140625. From the graph at the right, you can see that a_n approaches 0 as n increases. This sequence has a limit and is therefore convergent.



c. $a_n = \frac{(-1)^n \cdot n}{4n + 1}$

The first twelve terms of this sequence are given or approximated below.

$a_1 = -0.2$ $a_2 \approx 0.222$

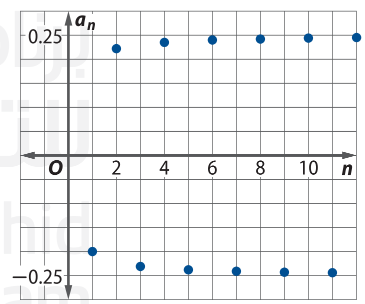
$a_3 \approx -0.231$ $a_4 \approx 0.235$

$a_5 \approx -0.238$ $a_6 = 0.24$

$a_7 \approx -0.241$ $a_8 \approx 0.242$

$a_9 \approx -0.243$ $a_{10} \approx 0.244$

$a_{11} \approx -0.244$ $a_{12} \approx 0.245$



It appears that when n is odd, a_n approaches $-\frac{1}{4}$, and when n is even, a_n approaches $\frac{1}{4}$. Since a_n does not approach one particular value, the sequence has no limit. Therefore, the sequence is divergent.

GuidedPractice

4A. $a_n = \frac{64}{2n}$ **4B.** $a_1 = 9$, $a_n = a_{n-1} + 4$ **4C.** $a_n = 3(-1)^n$

2 Series

A **series** is the indicated sum of all of the terms of a sequence. Like sequences, series can be finite or infinite. A **finite series** is the indicated sum of all the terms of a finite sequence, and an **infinite series** is the indicated sum of all the terms of an infinite sequence.

	Sequence	Series
Finite	1, 3, 5, 7, 9	1 + 3 + 5 + 7 + 9
Infinite	1, 3, 5, 7, 9, …	1 + 3 + 5 + 7 + 9 + …

The sum of the first n terms of a series is called the **nth partial sum** and is denoted S_n. The nth partial sum of any series can be found by calculating each term up to the nth term and then finding the sum of those terms.

Example 5 The nth Partial Sum

a. Find the fourth partial sum of $a_n = (-2)^n + 3$.

Find the first four terms.

$a_1 = (-2)^1 + 3$ or 1 $\quad n = 1$

$a_2 = (-2)^2 + 3$ or 7 $\quad n = 2$

$a_3 = (-2)^3 + 3$ or -5 $\quad n = 3$

$a_4 = (-2)^4 + 3$ or 19 $\quad n = 4$

The fourth partial sum is $S_4 = 1 + 7 + (-5) + 19$ or 22.

b. Find S_3 of $a_n = \frac{4}{10^n}$.

Find the first three terms.

$a_1 = \frac{4}{10^1}$ or 0.4 $\quad n = 1$

$a_2 = \frac{4}{10^2}$ or 0.04 $\quad n = 2$

$a_3 = \frac{4}{10^3}$ or 0.004 $\quad n = 3$

The third partial sum is $S_3 = 0.4 + 0.04 + 0.004$ or 0.444.

Guided Practice

5A. Find the sixth partial sum of $a_1 = 8$, $a_n = 0.5(a_{n-1})$, $n \geq 2$.

5B. Find the seventh partial sum of $a_n = 3\left(\frac{1}{10}\right)^n$.

Study Tip

Converging Infinite Sequences While it is necessary for an infinite sequence to converge to 0 in order for the corresponding infinite series to have a sum, it is not sufficient. Some infinite sequences converge to 0 and the corresponding infinite series still do not have sums.

Since an infinite series does not have a finite number of terms, you might assume that an infinite series has no sum S. This is true for the series below.

Infinite Sequence	Infinite Series	Sequence of First Four Partial Sums
1, 4, 7, 10, …	1 + 4 + 7 + 10 + …	1, 5, 12, 22, …

However, some infinite series do have sums. For an infinite series to have a fixed sum S, the infinite sequence associated with this series must converge to 0. Notice the sequence of partial sums in the infinite series below appears to approach a sum of $0.\overline{1}$ or $\frac{1}{9}$.

Infinite Sequence	Infinite Series	Sequence of First Three Partial Sums
0.1, 0.01, 0.001, …	0.1 + 0.01 + 0.001 + …	0.1, 0.11, 0.111, …

We will take a closer look at sums of infinite sequences in Lesson 9-3.

Series are often more conveniently notated using the uppercase Greek letter sigma Σ. A series written using this letter is said to be expressed using *summation notation* or **sigma notation**.

ReadingMath

Sigma Notation $\sum_{n=1}^{k} a_n$ is read *the summation from $n = 1$ to k of a sub n.*

KeyConcept Sigma Notation

For any sequence $a_1, a_2, a_3, a_4, \ldots$, the sum of the first k terms is denoted

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \ldots + a_k,$$

where n is the index of summation, k is the upper bound of summation, and 1 is the lower bound of summation.

In this notation, the lower bound indicates where to begin summing the terms of the sequence and the upper bound indicates where to end the sum. If the upper bound is given as ∞, the sigma notation represents an infinite series.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

WatchOut!

Variations in Sigma Notation The index of summation does not have to be the letter n. It can be represented by any variable. For example, the summation in Example 6a could also be written as $\sum_{i=1}^{5}(4i - 3)$.

Example 6 Sums in Sigma Notation

Find each sum.

a. $\sum_{n=1}^{5} (4n - 3)$

$$\sum_{n=1}^{5} (4n - 3) = [4(1) - 3] + [4(2) - 3] + [4(3) - 3] + [4(4) - 3] + [4(5) - 3]$$

$$= 1 + 5 + 9 + 13 + 17 \text{ or } 45$$

b. $\sum_{n=3}^{7} \frac{6n - 3}{2}$

$$\sum_{n=3}^{7} \frac{6n - 3}{2} = \frac{6(3) - 3}{2} + \frac{6(4) - 3}{2} + \frac{6(5) - 3}{2} + \frac{6(6) - 3}{2} + \frac{6(7) - 3}{2}$$

$$= 7.5 + 10.5 + 13.5 + 16.5 + 19.5 \text{ or } 67.5$$

c. $\sum_{n=1}^{\infty} \frac{7}{10^n}$

$$\sum_{n=1}^{\infty} \frac{7}{10^n} = \frac{7}{10^1} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \frac{7}{10^5} + \ldots$$

$$= 0.7 + 0.07 + 0.007 + 0.0007 + 0.00007 + \ldots$$

$$= 0.77777\ldots \text{ or } \frac{7}{9}$$

GuidedPractice

6A. $\sum_{n=1}^{5} \frac{n^2 - 1}{2}$

6B. $\sum_{n=7}^{13} (n^3 - n^2)$

6C. $\sum_{n=1}^{\infty} \frac{6}{10^n}$

Note that while the lower bound of a summation is often 1, a sum can start with any term p in a sequence as long as $p < k$. In Example 6b, the summation started with the 3rd term of the sequence and ended with the 7th term.

Exercises

Find the next four terms of each sequence. (Example 1)

1. 1, 8, 15, 22, …
2. 3, −6, 12, −24, …
3. 81, 27, 9, 3, …
4. 1, 3, 7, 13, …
5. −2, −15, −28, −41…
6. 1, 4, 10, 19, …

Find the first four terms of each sequence. (Example 1)

7. $a_n = n^2 - 1$
8. $a_n = -2^n + 7$
9. $a_n = \frac{n+7}{9-n}$
10. $a_n = (-1)^{n+1} + n$

11. **AUTOMOBILE LEASES** Lease agreements often contain clauses that limit the number of kilometers driven per year by charging a per-kilometer fee over that limit. For the car shown below, the lease requires that the number of kilometers driven each year must be no more than 15,000. (Example 2)

a. Write the sequence describing the maximum number of allowed kilometers on the car at the end of every 12 months of the lease if the car has 1350 kilometers at the beginning of the lease.

b. Write the first 4 terms of the sequence that gives the cumulative cost of the lease for a given month.

c. Write an explicit formula to represent the sequence in part **b**.

d. Determine the total amount of money paid by the end of the lease.

Find the specified term of each sequence. (Example 2)

12. 4th term, $a_1 = 5$, $a_n = -3a_{n-1} + 10$, $n \geq 2$
13. 7th term, $a_1 = 14$, $a_n = 0.5a_{n-1} + 3$, $n \geq 2$
14. 4th term, $a_1 = 0$, $a_n = 3^{a_{n-1}}$, $n \geq 2$
15. 3rd term, $a_1 = 3$, $a_n = (a_{n-1})^2 - 5a_{n-1} + 4$, $n \geq 2$

16. **WEB SITE** Wafa, the student from the beginning of the lesson, had great success expanding her Web site. Each student who received a referral developed a Web page and referred five more students to Wafa's site. (Example 3)

a. List the first five terms of a sequence modeling the number of new Web pages created through Wafa's site.

b. Suppose the school has 1576 students. After how many rounds of referrals did the entire student body have a Web page?

17. **BEES** Female honeybees come from fertilized eggs (male and female parent), while male honeybees come from unfertilized eggs (one female parent). (Example 3)

a. Draw a family tree showing the 3 previous generations of a male honeybee (parents only).

b. Determine the number of parent bees in the 11th previous generation of a male honeybee.

Determine whether each sequence is *convergent* or *divergent*. (Example 4)

18. $a_1 = 4$, $1.5a_{n-1}$, $n \geq 2$
19. $a_n = \frac{5}{10^n}$
20. $a_n = -n^2 - 8n + 106$
21. $a_1 = -64$, $\frac{3}{4}a_{n-1}$, $n \geq 2$
22. $a_1 = 1$, $a_n = 4 - a_{n-1}$, $n \geq 2$
23. $a_n = n^2 - 3n + 1$
24. $a_n = \frac{n^2 + 4}{3 + n}$
25. $a_1 = 9$, $a_n = \frac{a_{n-1} + 3}{2}$, $n \geq 2$
26. $a_n = \frac{5n + 6}{n}$
27. $a_n = \frac{5n}{5^n} + 1$

Find the indicated sum for each sequence. (Example 5)

28. 5th partial sum of $a_n = n(n-4)(n-3)$
29. 6th partial sum of $a_n = \frac{-5n + 3}{n}$
30. S_8 of $a_1 = 1$, $a_n = a_{n-1} + (18 - n)$, $n \geq 2$
31. S_4 of $a_1 = 64$, $a_n = -\frac{3}{4}a_{n-1}$, $n \geq 2$
32. 11th partial sum of $a_1 = 4$, $a_n = (-1)^{n-1}(|a_{n-1}| + 3)$, $n \geq 2$
33. S_9 of $a_1 = -35$, $a_n = a_{n-1} + 8$, $n \geq 2$
34. 4th partial sum of $a_1 = 3$, $a_n = (a_{n-1} - 2)^3$, $n \geq 2$
35. S_4 of $a_n = \frac{(-3)^n}{10}$

Find each sum. (Example 6)

36. $\sum_{n=1}^{8} (6n - 11)$
37. $\sum_{n=4}^{11} (30 - 4n)$
38. $\sum_{n=1}^{7} [n^2(n-5)]$
39. $\sum_{n=2}^{7} (n^2 - 6n + 1)$
40. $\sum_{n=8}^{15} \left(\frac{n}{4} - 7\right)$
41. $\sum_{n=1}^{10} [(n-4)^2(n-5)]$
42. $\sum_{n=0}^{6} [(-2)^n - 9]$
43. $\sum_{n=1}^{3} 7\left(\frac{1}{10}\right)^{2n}$
44. $\sum_{n=1}^{\infty} 5\left(\frac{1}{10^n}\right)$
45. $\sum_{n=1}^{\infty} \frac{8}{10^n}$

46. **FINANCIAL LITERACY** Mazen's bank account had an initial deposit of AED 380, earning 3.5% interest per year compounded annually.

a. Find the balance each year for the first five years.

b. Write a recursive and an explicit formula defining his account balance.

c. For very large values of n, which formula gives a more accurate balance? Explain.

47. **INVESTING** Hiyam invests AED 200 every 3 months. The investment pays an annual percentage rate of 8%, and the interest is compounded quarterly. If Hiyam makes each payment at the beginning of the quarter and the interest is posted at the end of the quarter, what will the total value of the investment be after 2 years?

48. **RIDES** The table shows the number of riders of the Mean Streak each year from 1998 to 2007. This ridership data can be approximated by $a_n = -\frac{1}{20}n + 1.3$, where $n = 1$ represents 1998, $n = 2$ represents 1999, and so on.

Mean Streak Roller Coaster

Year	Number of Riders (millions)	Year	Number of Riders (millions)
1998	1.31	2003	0.99
1999	1.15	2004	0.95
2000	1.14	2005	0.89
2001	1.09	2006	0.81
2002	1.05	2007	0.82

Source: *Cedar Fair Entertainment Company*

a. Sketch a graph of the number of riders from 1998 to 2007. Then determine whether the sequence appears to be *convergent* or *divergent*. Does this make sense in the context of the situation? Explain your reasoning.

b. Use the table to find the total number of riders from 1998 to 2005. Then use the explicit sequence to find the 8th partial sum of a_n. Compare the results.

c. If the sequence continues, find a_{14}. What does this number represent?

Copy and complete the table.

	Recursive Formula	Sequence	Explicit Formula
49.		6, 8, 10, 12, ...	
50.	$a_1 = 15, a_n = a_{n-1} - 1, n \geq 2$		
51.		7, 21, 63, 189, ...	
52.			$a_n = 10(-2)^n$
53.			$a_n = 8n - 3$
54.	$a_1 = 2, a_n = 4a_{n-1}, n \geq 2$		
55.	$a_1 = 3, a_n = a_{n-1} + 2n - 1, n \geq 2$		
56.			$a_n = n^2 + 1$

Write each series in sigma notation. The lower bound is given.

57. $-2 - 1 + 0 + 1 + 2 + 3 + 4 + 5$; $n = 1$

58. $\frac{1}{20} + \frac{1}{25} + \frac{1}{30} + \frac{1}{35} + \frac{1}{40} + \frac{1}{45}$; $n = 4$

59. $8 + 27 + 64 + \cdots + 1000$; $n = 2$

60. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{128}$; $n = 1$

61. $-8 + 16 - 32 + 64 - 128 + 256 - 512$; $n = 3$

62. $8\left(-\frac{1}{3}\right) + 8\left(\frac{1}{9}\right) + 8\left(-\frac{1}{27}\right) + \cdots + 8\left(-\frac{1}{243}\right)$; $n = 1$

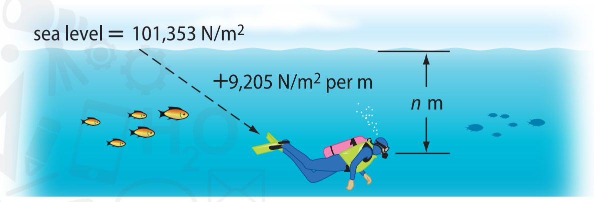
Determine whether each sequence is *convergent* or *divergent*. Then find the fifth partial sum of the sequence.

63. $a_n = \sin \frac{n\pi}{2}$

64. $a_n = n \cos \pi$

65. $a_n = e^{-\frac{n}{2}} \cos \pi n$

66. **WATER PRESSURE** The pressure exerted on the human body at sea level is 101,353 newton per square meter (N/m^2). For each additional meter below sea level, the pressure is about 9,205 N/m^2 greater, as shown.



a. Write a recursive formula to represent a_n, the pressure at n meters below sea level. (*Hint*: Let $a_0 = 14.7$.)

b. Write the first three terms of the sequence and describe what they represent.

c. Scuba divers cannot safely dive deeper than 100 meters. Write an explicit formula to represent a_n. Then use the formula to find the water pressure at 100 meters below sea level.

Match each sequence with its graph.

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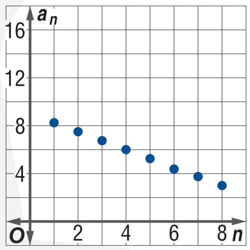

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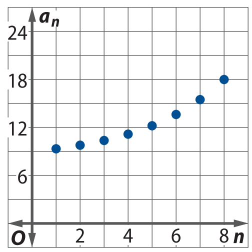

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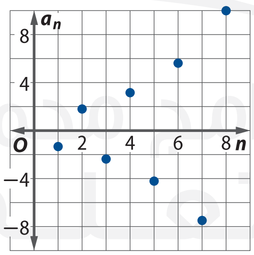

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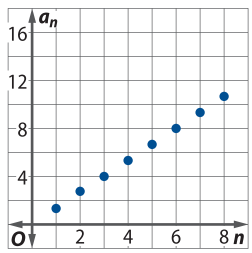

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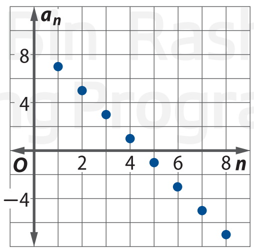

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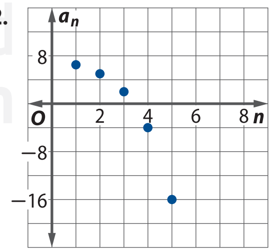


a. $a_n = \frac{4}{3}n$

b. $a_n = -\frac{3}{4}n + 9$

c. $a_n = \left(-\frac{4}{3}\right)^n$

d. $a_n = 8 - \frac{3}{4}(2^n)$

e. $a_n = 9 - 2n$

f. $a_n = \left(\frac{4}{3}\right)^n + 8$

73. **GOLDEN RATIO** Consider the Fibonacci sequence 1, 1, 2, 3, …, $a_{n-2} + a_{n-1}$.

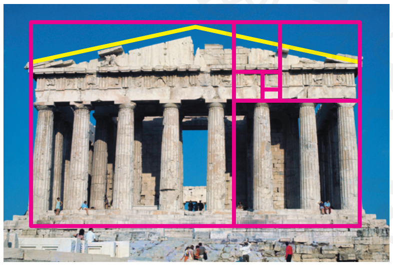
a. Find $\frac{a_n}{a_{n-1}}$ for the second through eleventh terms of the Fibonacci sequence.

b. Sketch a graph of the terms found in part **a.** Let $n - 1$ be the x-coordinate and $\frac{a_n}{a_{n-1}}$ be the y-coordinate.

c. Based on the graph found in part **b,** does this sequence appear to be convergent? If so, describe the limit to three decimal places. If not, explain why not.

d. In a *golden rectangle*, the ratio of the length to the width is about 1.61803399. This is called the *golden ratio*. How does the limit of the sequence $\frac{a_n}{a_{n-1}}$ compare to the golden ratio?

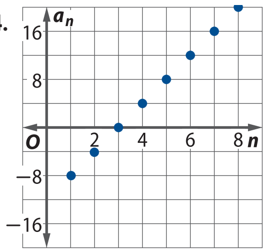
e. Golden rectangles are common in art and architecture. The Parthenon, in Greece, is an example of how golden rectangles are used in architecture.



Research golden rectangles and find two more examples of golden rectangles in art or architecture.

Determine whether each sequence is *convergent* or *divergent*.

74.



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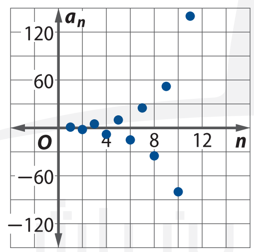

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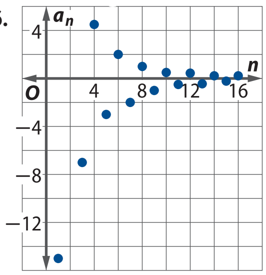

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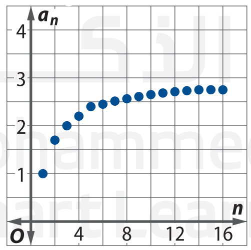


Write an explicit formula for each recursively defined sequence.

78. $a_1 = 10$; $a_n = a_{n-1} + 5$

79. $a_1 = 1.25$; $a_n = a_{n-1} - 0.5$

80. $a_1 = 128$; $a_n = 0.5a_{n-1}$

81. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate sums of infinite series.

a. NUMERICAL Calculate the first five terms of the infinite sequence $a_n = \frac{4}{10^n}$.

b. GRAPHICAL Use a graphing calculator to sketch $y = \frac{4}{10^x}$.

c. VERBAL Describe what is happening to the terms of the sequence as $n \to \infty$.

d. NUMERICAL Find the sum of the first 5 terms, 7 terms, and 9 terms of the series.

e. VERBAL Describe what is happening to the partial sums S_n as n increases.

f. VERBAL Predict the sum of the first n terms of the series. Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

82. **CHALLENGE** Consider the recursive sequence below.

$$a_n = a_{n - a_{n-1}} + a_{n - a_{n-2}} \text{ for } a_1 = 1, a_2 = 1, n \geq 3$$

a. Find the first eight terms of the sequence.

b. Describe the similarities and differences between this sequence and the other recursive sequences in this lesson.

83. **OPEN ENDED** Write a sequence either recursively or explicitly that has the following characteristics.

a. converges to 0

b. converges to 3

c. diverges

84. **WRITING IN MATH** Describe why an infinite sequence must not only converge, but converge to 0, in order for there to be a sum.

REASONING **Determine whether each statement is *true* or *false*. Explain your reasoning.**

85. $\sum_{n=1}^{5} (n^2 + 3n) = \sum_{n=1}^{5} n^2 + 3\sum_{n=1}^{5} n$

86. $\sum_{n=1}^{5} 3^n = \sum_{n=3}^{7} 3^{n-2}$

87. **CHALLENGE** Find the sum of the first 60 terms of the sequence below. Explain how you determined your answer.

15, 17, 2, −15, −17, …,
where $a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$

88. **WRITING IN MATH** Make an outline that could be used to describe the steps involved in finding the 300th partial sum of the infinite sequence $a_n = 2n - 3$. Then explain how to express the same sum using sigma notation.

Spiral Review

Graph each complex number on a polar grid. Then express it in rectangular form.

89. $2\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$

90. $2.5(\cos 1 + i \sin 1)$

91. $5(\cos 0 + i \sin 0)$

Determine the eccentricity, type of conic, and equation of the directrix given by each polar equation.

92. $r = \frac{3}{2 - 0.5 \cos \theta}$

93. $r = \frac{6}{1.2 \sin \theta + 0.3}$

94. $r = \frac{1}{0.2 - 0.2 \sin \theta}$

Determine whether the points are collinear. Write *yes* or *no*.

95. $(-3, -1, 4), (3, 8, 1), (5, 12, 0)$

96. $(4, 8, 6), (0, 6, 12), (8, 10, 0)$

97. $(0, -4, 3), (8, -10, 5), (12, -13, 2)$

98. $(-7, 2, -1), (-9, 3, -4), (-5, 1, 2)$

Find the length and the midpoint of the segment with the given endpoints.

99. $(2, -15, 12), (1, -11, 15)$

100. $(-4, 2, 8), (9, 6, 0)$

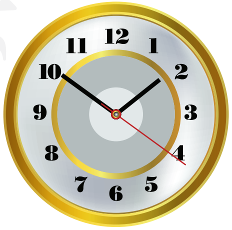
101. $(7, 1, 5), (-2, -5, -11)$

102. TIMING The path traced by the tip of the hour-hand of a clock can be modeled by a circle with parametric equations $x = 6 \sin t$ and $y = 6 \cos t$.

a. Find an interval for t in radians that can be used to describe the motion of the tip as it moves from 12 o'clock noon to 12 o'clock noon the next day.

b. Simulate the motion described in part **a** by graphing the equation in parametric mode on a graphing calculator.

c. Write an equation in rectangular form that models the motion of the hour-hand. Find the radius of the circle traced out by the hour-hand if x and y are given in centimeters.



Find the exact value of each expression.

103. $\tan \frac{\pi}{12}$

104. $\sin 75°$

105. $\cos 165°$

Find the partial fraction decomposition of each rational expression.

106. $\frac{10x^2 - 11x + 4}{2x^2 - 3x + 1}$

107. $\frac{1}{2x^2 + x}$

108. $\frac{x + 1}{x^3 + x}$

Skills Review for Standardized Tests

109. SAT/ACT The first term in a sequence is −5, and each subsequent term is 6 more than the term that immediately precedes it. What is the value of the 104th term?

A 607
B 613
C 618
D 619
E 615

110. REVIEW Find the exact value of $\cos 2\theta$ if $\sin \theta = -\frac{\sqrt{5}}{3}$ and $180° < \theta < 270°$.

F $-\frac{\sqrt{6}}{6}$
G $-\frac{4\sqrt{5}}{9}$
H $-\frac{\sqrt{30}}{6}$
J $-\frac{1}{9}$

111. The first four terms of a sequence are 144, 72, 36, and 18. What is the tenth term in the sequence?

A 0
B $\frac{9}{64}$
C $\frac{9}{32}$
D $\frac{9}{16}$

112. REVIEW How many 5-centimeter cubes can be stacked inside a box that is 10 centimeters long, 15 centimeters wide, and 5 centimeters tall?

F 5
G 6
H 15
J 20

LESSON 9-3 Arithmetic Sequences and Series

Then

- You determined whether a sequence was arithmetic.

Now

1. Find the nth term and arithmetic means for arithmetic sequences.
2. Find sums of arithmetic series.

Why?

- In the 18th century, a teacher asked his class of elementary students to find the sum of the counting numbers 1 through 100. A pupil named Karl Gauss correctly answered within seconds, astonishing the teacher. Gauss went on to become a great mathematician.

He solved this problem by using an arithmetic series.

NewVocabulary
arithmetic means
series
arithmetic series
partial sum
sigma notation

Mathematical Practices
Look for and express regularity in repeated reasoning.

1 Arithmetic Sequences

In Lesson 9-1, you used the point-slope form to find a specific term of an arithmetic sequence. It is possible to develop an equation for any term of an arithmetic sequence using the same process.

Consider the arithmetic sequence $a_1, a_2, a_3, \ldots, a_n$ in which the common difference is d.

$(y - y_1) = m(x - x_1)$	Point-slope form
$(a_n - a_1) = d(n - 1)$	$(x, y) = (n, a_n)$, $(x_1, y_1) = (1, a_1)$, and $m = d$
$a_n = a_1 + d(n - 1)$	Add a_1 to each side.

You can use this equation to find any term in an arithmetic sequence when you know the first term and the common difference.

KeyConcept *n*th Term of an Arithmetic Sequence

The *n*th term a_n of an arithmetic sequence in which the first term is a_1 and the common difference is d is given by the following formula, where n is any natural number.

$$a_n = a_1 + (n - 1)d$$

You will prove this formula in Exercise 80.

Example 1 Find the *n*th Term

Find the 12th term of the arithmetic sequence 9, 16, 23, 30, … .

Step 1 Find the common difference.

$16 - 9 = 7 \qquad 23 - 16 = 7 \qquad 30 - 23 = 7$

So, $d = 7$.

Step 2 Find the 12th term.

$a_n = a_1 + (n - 1)d$	*n*th term of an arithmetic sequence
$a_{12} = 9 + (12 - 1)(7)$	$a_1 = 9$, $d = 7$, and $n = 12$
$= 9 + 77$ or 86	Simplify.

GuidedPractice

Find the indicated term of each arithmetic sequence.

1A. $a_1 = -4$, $d = 6$, $n = 9$

1B. a_{20} for $a_1 = 15$, $d = -8$

If you are given some terms of an arithmetic sequence, you can write an equation for the nth term of the sequence.

> **StudyTip**
>
> **Checking Solutions** Check your solution by using it to determine the first three terms of the sequence.

Example 2 Write Equations for the nth Term

Write an equation for the nth term of each arithmetic sequence.

a. $5, -13, -31, \ldots$

$d = -13 - 5$ or -18; 5 is the first term.

$a_n = a_1 + (n - 1)d$	nth term of an arithmetic sequence
$a_n = 5 + (n - 1)(-18)$	$a_1 = 5$ and $d = -18$
$a_n = 5 + (-18n + 18)$	Distributive Property
$a_n = -18n + 23$	Simplify.

b. $a_5 = 19, d = 6$

First, find a_1.

$a_n = a_1 + (n - 1)d$	nth term of an arithmetic sequence
$19 = a_1 + (5 - 1)(6)$	$a_5 = 19$, $n = 5$, and $d = 6$
$19 = a_1 + 24$	Multiply.
$-5 = a_1$	Subtract 24 from each side.

Then write the equation.

$a_n = a_1 + (n - 1)d$	nth term of an arithmetic sequence
$a_n = -5 + (n - 1)(6)$	$a_1 = -5$ and $d = 6$
$a_n = -5 + (6n - 6)$	Distributive Property
$a_n = 6n - 11$	Simplify.

GuidedPractice

2A. $12, 3, -6, \ldots$

2B. $a_6 = 12, d = 8$

Sometimes you are given two terms of a sequence, but they are not consecutive terms of the sequence. The terms between any two nonconsecutive terms of an arithmetic sequence, called **arithmetic means**, can be used to find missing terms of a sequence.

> **ReadingMath**
>
> **arithmetic mean** the average of two or more numbers
>
> **arithmetic means** the terms between any two nonconsecutive terms of an arithmetic sequence

Example 3 Find Arithmetic Means

Find the arithmetic means in the sequence -8, ? , ? , ? , ? , 22, … .

Step 1 Since there are four terms between the first and last terms given, there are $4 + 2$ or 6 total terms, so $n = 6$.

Step 2 Find d.

$a_n = a_1 + (n - 1)d$	nth term of an arithmetic sequence
$22 = -8 + (6 - 1)d$	$a_1 = -8$, $a_6 = 22$, and $n = 6$
$30 = 5d$	Distributive Property
$6 = d$	Divide each side by 5.

Step 3 Use d to find the four arithmetic means.

$-8 \quad -2 \quad 4 \quad 10 \quad 16 \quad 22$

$+6 \quad +6 \quad +6 \quad +6 \quad +6$

The arithmetic means are -2, 4, 10, and 16.

GuidedPractice

3. Find the five arithmetic means between -18 and 36.

2 Arithmetic Series

A **series** is formed when the terms of a sequence are added. An **arithmetic series** is the sum of the terms of an arithmetic sequence. The sum of the first n terms is called the **partial sum** and is denoted S_n.

KeyConcept Partial Sum of an Arithmetic Series

Formula	Given	The sum S_n of the first n terms is:
General	a_1 and a_n	$S_n = n\left(\frac{a_1 + a_n}{2}\right)$
Alternate	a_1 and d	$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

Sometimes a_1, a_n, or n must be determined before the sum of an arithmetic series can be found. When this occurs, use the formula for the nth term.

Example 4 Use the Sum Formulas

Find the sum of $12 + 19 + 26 + \ldots + 180$.

Step 1 $a_1 = 12$, $a_n = 180$, and $d = 19 - 12$ or 7.

We need to find n before we can use one of the formulas.

$a_n = a_1 + (n-1)d$ — nth term of an arithmetic sequence

$180 = 12 + (n-1)(7)$ — $a_n = 180$, $a_1 = 12$, and $d = 7$

$168 = 7n - 7$ — Simplify.

$25 = n$ — Solve for n.

Step 2 Use either formula to find S_n.

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$ — Sum formula

$S_{25} = \frac{25}{2}[2(12) + (25-1)(7)]$ — $n = 25$, $a_1 = 12$, and $d = 7$

$S_{25} = 12.5(192)$ or 2400 — Simplify.

GuidedPractice

Find the sum of each arithmetic series.

4A. $2 + 4 + 6 + \ldots + 100$

4B. $n = 16$, $a_n = 240$, and $d = 8$.

You can use a sum formula to find terms of a series.

WatchOut!

Common Difference Don't confuse the sign of the common difference in an arithmetic sequence. Check that the rule actually produces the terms of a sequence.

Example 5 Find the First Three Terms

Find the first three terms of the arithmetic series in which $a_1 = 7$, $a_n = 79$, and $S_n = 430$.

Step 1 Find n.

$S_n = n\left(\frac{a_1 + a_n}{2}\right)$ — Sum formula

$430 = n\left(\frac{7 + 79}{2}\right)$ — $S_n = 430$, $a_1 = 7$, and $a_n = 79$

$430 = n(43)$ — Simplify.

$10 = n$ — Divide each side by 43.

Step 2 Find d.

$a_n = a_1 + (n - 1)d$ — *nth term of an arithmetic sequence*
$79 = 7 + (10 - 1)d$ — $a_n = 79$, $a_1 = 7$, and $n = 10$
$72 = 9d$ — Subtract 7 from each side.
$8 = d$ — Divide each side by 9.

Step 3 Use d to determine a_2 and a_3.

$a_2 = 7 + 8$ or 15 $\quad a_3 = 15 + 8$ or 23

The first three terms are 7, 15, and 23.

GuidedPractice

Find the first three terms of each arithmetic series.

5A. $S_n = 120$, $n = 8$, $a_n = 36$ **5B.** $a_1 = -24$, $a_n = 288$, $S_n = 5280$

The sum of a series can be written in shorthand by using **sigma notation**.

ReadingMath

Sigma Notation The name comes from the Greek letter sigma, which is used in the notation.

KeyConcept Sigma Notation

Symbols

$$\sum_{k=1}^{n} f(k)$$

last value of k → n; first value of k → $k = 1$; formula for the terms of the series → $f(k)$

Example

$$\sum_{k=1}^{12}(4k + 2) = [4(1) + 2] + [4(2) + 2] + [4(3) + 2] + \cdots + [4(12) + 2]$$
$$= 6 + 10 + 14 + \cdots + 50$$

Standardized Test Example 6 Use Sigma Notation

Find $\sum_{k=4}^{18}(6k - 1)$.

A 846 **B** 910 **C** 975 **D** 1008

Read the Test Item

You need to find the sum of the series. Find a_1, a_n, and n.

Solve the Test Item

There are $18 - 4 + 1$ or 15 terms, so $n = 15$.

$a_1 = 6(4) - 1$ or 23 $\quad a_n = 6(18) - 1$ or 107

Find the sum.

$S_n = n\left(\frac{a_1 + a_n}{2}\right)$ — Sum formula

$S_{15} = 15\left(\frac{23 + 107}{2}\right)$ — $n = 15$, $a_1 = 23$, and $a_n = 107$

$S_{15} = 15(65)$ or 975 — The correct answer is C.

Test-TakingTip

Perseverance

Sometimes it is necessary to break a problem into parts, solve each part, then combine the solutions of the parts.

GuidedPractice

6. Find $\sum_{m=9}^{21}(5m + 6)$.

F 972 **G** 1053 **H** 1281 **J** 1701

Check Your Understanding

Example 1 **Find the indicated term of each arithmetic sequence.**

1. $a_1 = 14$, $d = 9$, $n = 11$

2. a_{18} for 12, 25, 38, ...

Example 2 **Write an equation for the *n*th term of each arithmetic sequence.**

3. 13, 19, 25, ...

4. $a_5 = -12$, $d = -4$

Example 3 **Find the arithmetic means in each sequence.**

5. 6, $\underline{?}$, $\underline{?}$, $\underline{?}$, 42

6. −4, $\underline{?}$, $\underline{?}$, $\underline{?}$, 8

Example 4 **Find the sum of each arithmetic series.**

7. the first 50 natural numbers

8. $4 + 8 + 12 + \cdots + 200$

9. $a_1 = 12$, $a_n = 188$, $d = 4$

10. $a_n = 145$, $d = 5$, $n = 21$

Example 5 **Find the first three terms of each arithmetic series.**

11. $a_1 = 8$, $a_n = 100$, $S_n = 1296$

12. $n = 18$, $a_n = 112$, $S_n = 1098$

Example 6

13. MULTIPLE CHOICE Find $\sum_{k=1}^{12} (3k + 9)$.

A 45

B 78

C 342

D 410

Practice and Problem Solving

Example 1 **Find the indicated term of each arithmetic sequence.**

14. $a_1 = -18$, $d = 12$, $n = 16$

15. $a_1 = -12$, $n = 66$, $d = 4$

16. $a_1 = 9$, $n = 24$, $d = -6$

17. a_{15} for −5, −12, −19, ...

18. a_{10} for −1, 1, 3, ...

19. a_{24} for 8.25, 8.5, 8.75, ...

Example 2 **Write an equation for the *n*th term of each arithmetic sequence.**

20. 24, 35, 46, ...

21. 31, 17, 3, ...

22. $a_9 = 45$, $d = -3$

23. $a_7 = 21$, $d = 5$

24. $a_4 = 12$, $d = 0.25$

25. $a_5 = 1.5$, $d = 4.5$

26. 9, 2, −5, ...

27. $a_6 = 22$, $d = 9$

28. $a_8 = -8$, $d = -2$

29. $a_{15} = 7$, $d = \frac{2}{3}$

30. −12, −17, −22, ...

31. $a_3 = -\frac{4}{5}$, $d = \frac{1}{2}$

32. STRUCTURE Jamal averaged 123 total pins per game in his bowling league this season. He is taking bowling lessons and hopes to bring his average up by 8 pins each new season.

a. Write an equation to represent the *n*th term of the sequence.

b. If the pattern continues, during what season will Jamal average 187 per game?

c. Is it reasonable for this pattern to continue indefinitely? Explain.

Example 3 **Find the arithmetic means in each sequence.**

33. 24, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, −1

34. −6, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, 49

35. −28, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, 7

36. 84, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, 39

37. −12, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, −66

38. 182, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, $\underline{?}$, 104

Example 4 **Find the sum of each arithmetic series.**

39. the first 100 even natural numbers
40. the first 200 odd natural numbers
41. the first 100 odd natural numbers
42. the first 300 even natural numbers
43. $-18 + (-15) + (-12) + \cdots + 66$
44. $-24 + (-18) + (-12) + \cdots + 72$
45. $a_1 = -16, d = 6, n = 24$
46. $n = 19, a_n = 154, d = 8$
47. **CONTESTS** The prizes in a weekly radio contest began at AED 150 and increased by AED 50 for each week that the contest lasted. If the contest lasted for eleven weeks, how much was awarded in total?

Example 5 **Find the first three terms of each arithmetic series.**

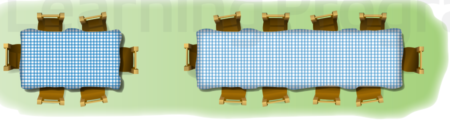
48. $n = 32, a_n = -86, S_n = 224$
49. $a_1 = 48, a_n = 180, S_n = 1368$
50. $a_1 = 3, a_n = 66, S_n = 759$
51. $n = 28, a_n = 228, S_n = 2982$
52. $a_1 = -72, a_n = 453, S_n = 6858$
53. $n = 30, a_n = 362, S_n = 4770$
54. $a_1 = 19, n = 44, S_n = 9350$
55. $a_1 = -33, n = 36, S_n = 6372$
56. **PRIZES** A radio station is offering a total of AED 8,500 in prizes over ten hours. Each hour, the prize will increase by AED 100. Find the amounts of the first and last prize.

Example 6 **Find the sum of each arithmetic series.**

57. $\sum_{k=1}^{16} (4k - 2)$
58. $\sum_{k=4}^{13} (4k + 1)$
59. $\sum_{k=5}^{16} (2k + 6)$
60. $\sum_{k=0}^{12} (-3k + 2)$
61. **FINANCIAL LITERACY** Najla borrowed some money from her parents. She agreed to pay AED 50 at the end of the first month and AED 25 more each additional month for 12 months. How much does she pay in total after the 12 months?
62. **GRAVITY** When an object is in free fall and air resistance is ignored, it falls 16 meters in the first second, an additional 48 meters during the next second, and 80 meters during the third second. How many total meters will the object fall in 10 seconds?

Use the given information to write an equation that represents the *n*th term in each arithmetic sequence

63. The 100th term of the sequence is 245. The common difference is 13.
64. The eleventh term of the sequence is 78. The common difference is -9.
65. The sixth term of the sequence is -34. The 23rd term is 119.
66. The 25th term of the sequence is 121. The 80th term is 506.
67. **MODELING** The rectangular tables in a reception hall are often placed end-to-end to form one long table. The diagrams below show the number of people who can sit at each of the table arrangements.



   a. Make drawings to find the next three numbers as tables are added one at a time to the arrangement.
   b. Write an equation representing the *n*th number in this pattern.
   c. Is it possible to have seating for exactly 100 people with such an arrangement? Explain.

68. **PERFORMANCE** A certain company pays its employees according to their performance. Badria is paid a flat rate of AED 800 per week plus AED 96 for every unit she completes. If she earned AED 2,048 in one week, how many units did she complete?

69. **SALARY** Tarek currently earns AED 112,000 per year. If Tarek expects a AED 16,000 increase in salary every year, after how many years will he have a salary of AED 400,000 per year?

70. **SPORTS** While training for cross country, Sindiyya plans to run 3 kilometers per day for the first week, and then increase the distance by a half kilometer each of the following weeks.

a. Write an equation to represent the nth term of the sequence.

b. If the pattern continues, during which week will she be running 10 kilometers per day?

c. Is it reasonable for this pattern to continue indefinitely? Explain.

71. **MULTIPLE REPRESENTATIONS** Consider $\sum_{k=1}^{x} (2k + 2)$.

a. Tabular Make a table of the partial sums of the series for $1 \leq k \leq 10$.

b. Graphical Graph (k, partial sum).

c. Graphical Graph $f(x) = x^2 + 3x$ on the same grid.

d. Verbal What do you notice about the two graphs?

e. Analytical What conclusions can you make about the relationship between quadratic functions and the sum of arithmetic series?

f. Algebraic Find the arithmetic series that relates to $g(x) = x^2 + 8x$.

Find the value of x.

72. $\sum_{k=3}^{x} (6k - 5) = 928$

73. $\sum_{k=5}^{x} (8k + 2) = 1032$

H.O.T. Problems Use Higher-Order Thinking Skills

74. **CRITIQUE** Eissa and Jassim are determining the formula for the nth term for the sequence $-11, -2, 7, 16, \ldots$. Is either of them correct? Explain your reasoning.

Eissa	Jassim
$d = 16 - 7$ or 9, $a_1 = -11$ $a_n = -11 + (n - 1)9$ $= 9n - 20$	$d = 16 - 7$ or 9, $a_1 = -11$ $a_n = 9n - 11$

75. **REASONING** If a is the third term in an arithmetic sequence, b is the fifth term, and c is the eleventh term, express c in terms of a and b.

76. **CHALLENGE** There are three arithmetic means between a and b in an arithmetic sequence. The average of the arithmetic means is 16. What is the average of a and b?

77. **CHALLENGE** Find S_n for $(x + y) + (x + 2y) + (x + 3y) + \ldots$.

78. **OPEN ENDED** Write an arithmetic series with 8 terms and a sum of 324.

79. **WRITING IN MATH** Compare and contrast arithmetic sequences and series.

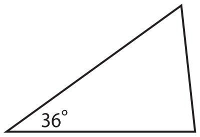
80. **PROOF** Prove the formula for the nth term of an arithmetic sequence.

81. **PROOF** Derive a sum formula that does not include a_1.

82. **PROOF** Derive the Alternate Sum Formula using the General Sum Formula.

Standardized Test Practice

83. SAT/ACT The measures of the angles of a triangle form an arithmetic sequence. If the measure of the smallest angle is 36°, what is the measure of the largest angle?



A 54°
B 75°
C 84°
D 90°
E 97°

84. The area of a triangle is $\frac{1}{2}q^2 - 8$ and the height is $q + 4$. Which expression best describes the triangle's base?

F $(q + 1)$
G $(q + 2)$
H $(q - 3)$
J $(q - 4)$.

85. The expression $1 + \sqrt{2} + \sqrt[3]{3}$ is equivalent to

A $\sum_{k=1}^{3} k^{\frac{1}{k}}$
B $\sum_{k=1}^{3} k^k$
C $\sum_{k=1}^{3} k^{-k}$
D $\sum_{k=1}^{3} \sqrt{k}$

86. SHORT RESPONSE Ahmed can type a 200-word essay in 6 hours. Husam can type the same essay in $4\frac{1}{2}$ hours. If they work together, how many hours will it take them to type the essay?

Spiral Review

Determine whether each sequence is arithmetic. Write *yes* or *no*. (Lesson 9-1)

87. $-6, 4, 14, 24, \ldots$

88. $2, \frac{7}{5}, \frac{4}{5}, \frac{1}{5}, \ldots$

89. $10, 8, 5, 1, \ldots$

Solve each system of inequalities by graphing.

90. $x + 2y > 1$
$x^2 + y^2 \leq 25$

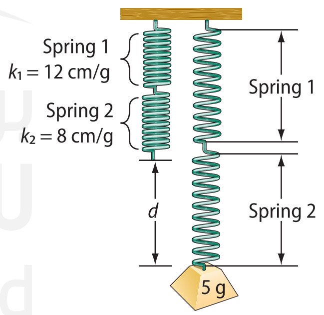
91. $x + y \leq 2$
$4x^2 - y^2 \geq 4$

92. $x^2 + y^2 \geq 4$
$4y^2 + 9x^2 \leq 36$

93. PHYSICS The distance a spring stretches is related to the mass attached to the spring. This is represented by $d = km$, where d is the distance, m is the mass, and k is the spring constant. When two springs with spring constants k_1 and k_2 are attached in a series, the resulting spring constant k is found by the equation $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$.

a. If one spring with constant of 12 centimeters per gram is attached in a series with another spring with constant of 8 centimeters per gram, find the resultant spring constant.

b. If a 5-gram object is hung from the series of springs, how far will the springs stretch? Is this answer reasonable in this context?



Graph each function. State the domain and range.

94. $f(x) = \frac{2}{3}(2^x)$

95. $f(x) = 4^x + 3$

96. $f(x) = 2\left(\frac{1}{3}\right)^x - 1$

Skills Review

Solve each equation. Round to the nearest ten-thousandth.

97. $5^x = 52$

98. $4^{3p} = 10$

99. $3^{n+2} = 14.5$

100. $16^{d-4} = 3^{3-d}$

LESSON 9-4

Geometric Sequences and Series

Then

- You determined whether a sequence was geometric.

Now

1. Find the nth term and geometric means for geometric sequences.
2. Find sums of geometric series.

Why?

- Hasan sees a new book in a bookshop. He e-mails a link for the author's Web site to five of his friends. They each forward the link to five of their friends. The link is forwarded again following the same pattern. How many people will receive the link on the eighth round of e-mails?

NewVocabulary
geometric means
geometric series

Mathematical Practices
Look for and express regularity in repeated reasoning.

1 Geometric Sequences

As with arithmetic sequences, there is a formula for the nth term of a geometric sequence. This formula can be used to determine any term of the sequence.

KeyConcept *n*th Term of a Geometric Sequence

The nth term a_n of a geometric sequence in which the first term is a_1 and the common ratio is r is given by the following formula, where n is any natural number.

$$a_n = a_1 r^{n-1}$$

You will prove this formula in Exercise 68.

Real-World Example 1 Find the *n*th Term

MUSIC **If the pattern continues, how many e-mails will be sent in the eighth round?**

Understand We need to determine the number of forwarded e-mails on the eighth round. Five e-mails were sent on the first round. Each of the five recipients sent five e-mails on the second round, and so on.

Plan This is a geometric sequence, and the common ratio is 5. Use the formula for the nth term of a geometric sequence.

Solve

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$a_8 = 5(5)^{8-1}$ — $a_1 = 5$, $r = 5$, and $n = 8$

$a_8 = 5(78{,}125)$ or $390{,}625$ — $5^7 = 78{,}125$

Check Write out the first eight terms by multiplying by the common ratio.

5, 25, 125, 625, 3125, 15,625, 78,125, 390,625

There will be 390,625 e-mails sent on the 8th round.

GuidedPractice

1. **E-MAILS** Sumayya receives a joke in an e-mail that asks her to forward it to four of her friends. She forwards it, then each of her friends forwards it to four of their friends, and so on. If the pattern continues, how many people will receive the e-mail on the ninth round of forwarding?

If you are given some of the terms of a geometric sequence, you can determine an equation for finding the *n*th term of the sequence.

Math History Link

Archytas (428–347 B.C.)
Geometric sequences, or geometric progressions, were first studied by the Greek mathematician Archytas. His studies of these sequences came from his interest in music and octaves.

Example 2 Write an Equation for the *n*th Term

Write an equation for the *n*th term of each geometric sequence.

a. $0.5, 2, 8, 32, \ldots$

$r = 8 \div 2$ or 4; 0.5 is the first term.

$a_n = a_1 r^{n-1}$ — *n*th term of a geometric sequence

$a_n = 0.5(4)^{n-1}$ — $a_1 = 0.5$ and $r = 4$

b. $a_4 = 5$ and $r = 6$

Step 1 Find a_1.

$a_n = a_1 r^{n-1}$ — *n*th term of a geometric sequence

$5 = a_1(6^{4-1})$ — $a_n = 5$, $r = 6$, and $n = 4$

$5 = a_1(216)$ — Evaluate the power.

$\frac{5}{216} = a_1$ — Divide each side by 216.

Step 2 Write the equation.

$a_n = a_1 r^{n-1}$ — *n*th term of a geometric sequence

$a_n = \frac{5}{216}(6)^{n-1}$ — $a_1 = \frac{5}{216}$ and $r = 6$

Guided Practice

Write an equation for the *n*th term of each geometric sequence.

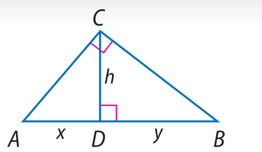
2A. $-0.25, 2, -16, 128, \ldots$

2B. $a_3 = 16$, $r = 4$

Like arithmetic means, **geometric means** are the terms between two nonconsecutive terms of a geometric sequence. The common ratio *r* can be used to find the geometric means.

Reading Math

Geometric Means
A geometric mean can also be represented geometrically. In the figure below, *h* is the geometric mean between *x* and *y*.



Example 3 Find Geometric Means

Find three geometric means between 2 and 1250.

Step 1 Since there are three terms between the first and last term, there are 3 + 2 or 5 total terms, so $n = 5$.

Step 2 Find *r*.

$a_n = a_1 r^{n-1}$ — *n*th term of a geometric sequence

$1250 = 2r^{5-1}$ — $a_n = 1250$, $a_1 = 2$, and $n = 5$

$625 = r^4$ — Divide each side by 2.

$\pm 5 = r$ — Take the 4th root of each side.

Step 3 Use *r* to find the four arithmetic means.

2 **10** **50** **250** 1250 or 2 **−10** **50** **−250** 1250

× 5 × 5 × 5 × 5 × −5 × −5 × −5 × −5

The geometric means are 10, 50, and 250 or −10, 50, and −250.

Guided Practice

3. Find four geometric means between 0.5 and 512.

2 Geometric Series

A **geometric series** is the sum of the terms of a geometric sequence. The sum of the first n terms of a series is denoted S_n. You can use either of the following formulas to find the partial sum S_n of the first n terms of a geometric series.

KeyConcept Partial Sum of a Geometric Series

Given	The sum S_n of the first n terms is:
a_1 and n	$S_n = \frac{a_1 - a_1 r^n}{1 - r}, r \neq 1$
a_1 and a_n	$S_n = \frac{a_1 - a_n r}{1 - r}, r \neq 1$

Real-World Example 4 Find the Sum of a Geometric Series

MUSIC Refer to the beginning of the lesson. If the pattern continues, what is the total number of e-mails sent in the eight rounds?

Five e-mails are sent in the first round and there are 8 rounds of e-mails. So, $a_1 = 5$, $r = 5$ and $n = 8$.

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ Sum formula

$S_8 = \frac{5 - 5 \cdot 5^8}{1 - 5}$ $a_1 = 5$, $r = 5$, and $n = 8$

$S_8 = \frac{-1{,}953{,}120}{-4}$ Simplify the numerator and denominator.

$S_8 = 488{,}280$ Divide.

There will be 488,280 e-mails sent after 8 rounds.

GuidedPractice

Find the sum of each geometric series.

4A. $a_1 = 2, n = 10, r = 3$

4B. $a_1 = 2000, a_n = 125, r = \frac{1}{2}$

As with arithmetic series, sigma notation can also be used to represent geometric series.

Example 5 Sum in Sigma Notation

Find $\sum_{k=3}^{10} 4(2)^{k-1}$.

Find a_1, r, and n. In the first term, $k = 3$ and $a_1 = 4 \cdot 2^{3-1}$ or 16. The base of the exponential function is r, so $r = 2$. There are $10 - 3 + 1$ or 8 terms, so $n = 8$.

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ Sum formula

$= \frac{16 - 16(2)^8}{1 - 2}$ $a_1 = 16$, $r = 2$, and $n = 8$

$= 4080$ Use a calculator.

GuidedPractice

Find each sum.

5A. $\sum_{k=4}^{12} \frac{1}{4} \cdot 3^{k-1}$

5B. $\sum_{k=2}^{9} \frac{2}{3} \cdot 4^{k-1}$

WatchOut!

Sigma Notation Notice in Example 5 that you are being asked to evaluate the sum from the 3rd term to the 10th term.

You can use the formula for the sum of a geometric series to help find a particular term of the series.

Example 6 Find the First Term of a Series

Find a_1 in a geometric series for which $S_n = 13{,}116$, $n = 7$, and $r = 3$.

$S_n = \dfrac{a_1 - a_1 r^n}{1 - r}$ — Sum formula

$13{,}116 = \dfrac{a_1 - a_1(3^7)}{1 - 3}$ — $S_n = 13{,}116$, $r = 3$, and $n = 7$

$13{,}116 = \dfrac{a_1(1 - 3^7)}{1 - 3}$ — Distributive Property

$13{,}116 = \dfrac{-2186a_1}{-2}$ — Subtract.

$13{,}116 = 1093a_1$ — Simplify.

$12 = a_1$ — Divide each side by 1093.

Guided Practice

6. Find a_1 in a geometric series for which $S_n = -26{,}240$, $n = 8$, and $r = -3$.

Check Your Understanding

Example 1

1. **REGULARITY** Ismail is making a family tree for his grandfather. He was able to trace many generations. If Ismail could trace his family back 10 generations, starting with his parents how many ancestors would there be?

Example 2

Write an equation for the nth term of each geometric sequence.

2. 2, 4, 8, ...
3. 18, 6, 2, ...
4. −4, 16, −64, ...
5. $a_2 = 4$, $r = 3$
6. $a_6 = \frac{1}{8}$, $r = \frac{3}{4}$
7. $a_2 = -96$, $r = -8$

Example 3

Find the geometric means of each sequence.

8. 0.25, ?, ?, ?, 64
9. 0.20, ?, ?, ?, 125

Example 4

10. **GAMES** Muna arranges some rows of dominoes so that after she knocks over the first one, each domino knocks over two more dominoes when it falls. If there are ten rows, how many dominoes does Muna use?

Example 5

Find the sum of each geometric series.

11. $\sum_{k=1}^{6} 3(4)^{k-1}$
12. $\sum_{k=1}^{8} 4\left(\frac{1}{2}\right)^{k-1}$

Example 6

Find a_1 for each geometric series described.

13. $S_n = 85\frac{5}{16}$, $r = 4$, $n = 6$
14. $S_n = 91\frac{1}{12}$, $r = 3$, $n = 7$
15. $S_n = 1020$, $a_n = 4$, $r = \frac{1}{2}$
16. $S_n = 121\frac{1}{3}$, $a_n = \frac{1}{3}$, $r = \frac{1}{3}$

Practice and Problem Solving

Example 1

17. **WEATHER** Heavy rain in Bilal's town caused the river to rise. The river rose three centimeters the first day, and each day after rose twice as much as the previous day. How much did the river rise in five days?

Find a_n for each geometric sequence.

18. $a_1 = 2400, r = \frac{1}{4}, n = 7$

19. $a_1 = 800, r = \frac{1}{2}, n = 6$

20. $a_1 = \frac{2}{9}, r = 3, n = 7$

21. $a_1 = -4, r = -2, n = 8$

22. **BIOLOGY** A certain bacteria grows at a rate of 3 cells every 2 minutes. If there were 260 cells initially, how many are there after 21 minutes?

Example 2

Write an equation for the *n*th term of each geometric sequence.

23. $-3, 6, -12, \ldots$

24. $288, -96, 32, \ldots$

25. $-1, 1, -1, \ldots$

26. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \ldots$

27. $8, 2, \frac{1}{2}, \ldots$

28. $12, -16, \frac{64}{3}, \ldots$

29. $a_3 = 28, r = 2$

30. $a_4 = -8, r = 0.5$

31. $a_6 = 0.5, r = 6$

32. $a_3 = 8, r = \frac{1}{2}$

33. $a_4 = 24, r = \frac{1}{3}$

34. $a_4 = 80, r = 4$

Example 3

Find the geometric means of each sequence.

35. 810, ?, ?, ?, 10

36. 640, ?, ?, ?, 2.5

37. $\frac{7}{2}$, ?, ?, ?, $\frac{56}{81}$

38. $\frac{729}{64}$, ?, ?, ?, $\frac{324}{9}$

39. Find two geometric means between 3 and 375.

40. Find two geometric means between 16 and -2.

Example 4

41. **PERSEVERANCE** A certain water filtration system can remove 70% of the contaminants each time a sample of water is passed through it. If the same water is passed through the system four times, what percent of the original contaminants will be removed from the water sample?

Find the sum of each geometric series.

42. $a_1 = 36, r = \frac{1}{3}, n = 8$

43. $a_1 = 16, r = \frac{1}{2}, n = 9$

44. $a_1 = 240, r = \frac{3}{4}, n = 7$

45. $a_1 = 360, r = \frac{4}{3}, n = 8$

46. **VACUUMS** A vacuum claims to pick up 80% of the dirt every time it is run over the carpet. Assuming this is true, what percent of the original amount of dirt is picked up after the seventh time the vacuum is run over the carpet?

Example 5

Find the sum of each geometric series.

47. $\sum_{k=1}^{7} 4(-3)^{k-1}$

48. $\sum_{k=1}^{8} (-3)(-2)^{k-1}$

49. $\sum_{k=1}^{9} (-1)(4)^{k-1}$

50. $\sum_{k=1}^{10} 5(-1)^{k-1}$

Example 6

Find a_1 for each geometric series described.

51. $S_n = -2912, r = 3, n = 6$

52. $S_n = -10{,}922, r = 4, n = 7$

53. $S_n = 1330, a_n = 486, r = \frac{3}{2}$

54. $S_n = 4118, a_n = 128, r = \frac{2}{3}$

55. $a_n = 1024, r = 8, n = 5$

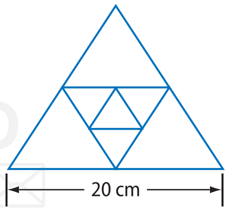
56. $a_n = 1875, r = 5, n = 7$

57. **SCIENCE** One minute after it is released, a gas-filled balloon has risen 100 meters. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will it rise in 5 minutes?

58. **CHEMISTRY** Radon has a half-life of about 4 days. This means that about every 4 days, half of the mass of radon decays into another element. How many grams of radon remain from an initial 60 grams after 4 weeks?

59. **REASONING** A virus goes through a computer, infecting the files. If one file was infected initially and the total number of files infected doubles every minute, how many files will be infected in 20 minutes?

60. **GEOMETRY** In the figure, the sides of each equilateral triangle are twice the size of the sides of its inscribed triangle. If the pattern continues, find the sum of the perimeters of the first eight triangles.



61. **PENDULUMS** The first swing of a pendulum travels 30 centimeters. If each subsequent swing travels 95% as far as the previous swing, find the total distance traveled by the pendulum after the 30th swing.

62. **PHONE CHAINS** A school established a phone chain in which every staff member calls two other staff members to notify them when the school closes due to weather. The first round of calls begins with the superintendent calling both principals. If there are 94 total staff members and employees at the school, how many rounds of calls are there?

63. **TELEVISIONS** High Tech Electronics advertises a weekly installment plan for the purchase of a popular brand of high definition television. The buyer pays AED 15 at the end of the first week, AED 16.50 at the end of the second week, AED 18.15 at the end of the third week, and so on for one year. (Assume that 1 year = 52 weeks.)

a. What will the payments be at the end of the 10th, 20th, and 40th weeks?

b. Find the total cost of the TV.

c. Why is the cost found in part **b** not entirely accurate?

H.O.T. Problems Use Higher-Order Thinking Skills

64. **PROOF** Derive the General Sum Formula using the Alternate Sum Formula.

65. **PROOF** Derive a sum formula that does not include a_1.

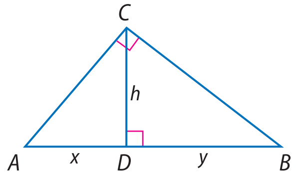
66. **OPEN ENDED** Write a geometric series for which $r = \frac{3}{4}$ and $n = 6$.

67. **REASONING** Explain how $\sum_{k=1}^{10} 3(2)^{k-1}$ needs to be altered to refer to the same series if $k = 1$ changes to $k = 0$. Explain your reasoning.

68. **PROOF** Prove the formula for the nth term of a geometric sequence.

69. **CHALLENGE** The fifth term of a geometric sequence is $\frac{1}{27}$th of the eighth term. If the ninth term is 702, what is the eighth term?

70. **CHALLENGE** Use the fact that h is the geometric mean between x and y in the figure at the right to find h^4 in terms of x and y.



71. **OPEN ENDED** Write a geometric series with 6 terms and a sum of 252.

72. **WRITING IN MATH** How can you classify a sequence? Explain your reasoning.

Standardized Test Practice

73. Which of the following is closest to $\sqrt[3]{7.32}$?

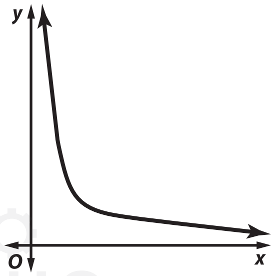
A 1.8
B 1.9
C 2.0
D 2.1

74. The first term of a geometric series is 5, and the common ratio is −2. How many terms are in the series if its sum is −6825?

F 5
G 9
H 10
J 12

75. SHORT RESPONSE Ayesha has a savings account. She withdraws half of the contents every year. After 4 years, she has AED 2,000 left. How much did she have in the savings account originally?

76. SAT/ ACT The curve below could be part of the graph of which function?



A $y = \sqrt{x}$
B $y = x^2 - 5x + 4$
C $y = -x + 20$
D $y = \log x$
E $xy = 4$

Spiral Review

77. MONEY Manal bought a high-definition LCD television at the electronics store. She paid AED 800 immediately and AED 300 each month for a year and a half. How much did Manal pay in total for the TV? (Lesson 9-2)

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. (Lesson 9-1)

78. $\frac{1}{10}, \frac{3}{5}, \frac{7}{20}, \frac{17}{20}, \ldots$

79. $-\frac{7}{25}, -\frac{13}{50}, -\frac{6}{25}, -\frac{11}{50}, \ldots$

80. $-\frac{22}{3}, -\frac{68}{9}, -\frac{208}{27}, -\frac{632}{81}, \ldots$

Find the center and radius of each circle. Then graph the circle.

81. $(x - 3)^2 + (y - 1)^2 = 25$

82. $(x + 3)^2 + (y + 7)^2 = 81$

83. $(x - 3)^2 + (y + 7)^2 = 50$

84. Suppose y varies jointly as x and z. Find y when $x = 9$ and $z = -5$, if $y = -90$ when $z = 15$ and $x = -6$.

85. SHOPPING A certain store found that the number of customers who will attend a sale can be modeled by $N = 125\sqrt[3]{100Pt}$, where N is the number of customers expected, P is the percent of the sale discount, and t is the number of hours the sale will last. Find the number of customers the store should expect for a sale that is 50% off and will last four hours.

Skills Review

Evaluate each expression if $a = -2$, $b = \frac{1}{3}$, and $c = -12$.

86. $\frac{3ab}{c}$

87. $\frac{a - c}{a + c}$

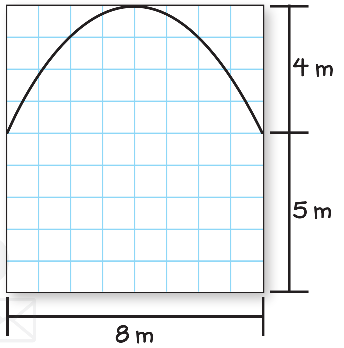
88. $\frac{a^3 - c}{b^2}$

89. $\frac{c + 3}{ab}$

EXPLORE

9-5 Algebra Lab
Area Under a Curve

A soccer stadium is being redesigned so that there is an archway above the main entrance. A scale drawing of the archway is created in which each line on the grid paper represents one meter of the actual archway. The designer modeled the shape of the top with the quadratic equation $y = -0.25x^2 + 3x$.



Activity

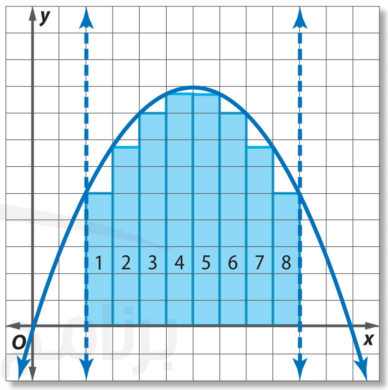
Find the area of the opening under the archway.

Method 1

Step 1 Make a table of values for $y = -0.25x^2 + 3x$. Then graph the equation.

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	2.75	5	6.75	8	8.75	9	8.75	8	6.75	5	2.75	0

Step 2 Divide the figure into regions.



To estimate the area inside the archway, you can divide the archway into rectangles as shown in blue.

Because the left and right sides of the archway are 5 meters high and $y = 5$ when $x = 2$ and when $x = 10$, the opening of the entrance extends from $x = 2$ to $x = 10$.

Step 3 Find the area of the regions.

Rectangle	1	2	3	4	5	6	7	8
Width (m)	1	1	1	1	1	1	1	1
Height (m)	5	6.75	8	8.75	8.75	8	6.75	5
Area (m^2)	5	6.75	8	8.75	8.75	8	6.75	5

The approximate area of the archway is the sum of the areas of the rectangles.

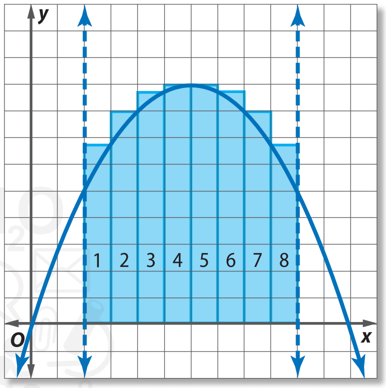
$5 + 6.75 + 8 + 8.75 + 8.75 + 8 + 6.75 + 5 = 57\ m^2$

(continued on the next page)

Area Under a Curve *Continued*

Method 2

Step 1 Draw a second graph of the equation and divide into regions. Divide the archway into rectangles as shown in blue.



Step 2 Find the area of the regions.

Rectangle	1	2	3	4	5	6	7	8
Width (m)	1	1	1	1	1	1	1	1
Height (m)	6.75	8	8.75	9	9	8.75	8	6.75
Area (m^2)	6.75	8	8.75	9	9	8.75	8	6.75

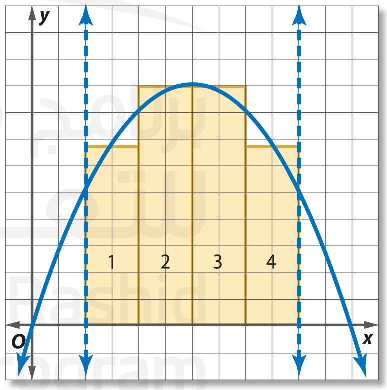
The approximate area of the archway is the sum of the areas of the rectangles.

$6.75 + 8 + 8.75 + 9 + 9 + 8.75 + 8 + 6.75 = 65 \text{ m}^2$

Both Method 1 and Method 2 illustrate how to approximate the area under a curve over a specified interval.

Analyze the Results

1. Is the area of the regions calculated using Method 1 greater than or less than the actual area of the archway? Explain your reasoning.
2. Is the area of the regions calculated using Method 2 greater than or less than the actual area of the archway? Explain your reasoning.
3. Compare the area estimates for both methods. How could you find the best estimate for the area inside the archway? Explain your reasoning.
4. The diagram shows a third method for finding an estimate of the area of the archway. Is this estimate for the area greater than or less than the actual area? How does this estimate compare to the other two estimates of the area?



Exercises

Estimate the area described by any method. Make a table of values, draw graphs with rectangles, and make a table for the areas of the rectangles. Compare each estimate to the actual area.

5. the area under the curve for $y = -x^2 + 4$, from $x = -2$ to $x = 2$, and above the x–axis
6. the area under the curve for $y = x^3$, from $x = 0$ to $x = 4$, and above the x–axis
7. the area under the curve for $y = x^2$, from $x = -3$ to $x = 3$, and above the x–axis

LESSON 9-5 Infinite Geometric Series

Then

- You found sums of finite geometric series.

Now

1. Find sums of infinite geometric series.
2. Write repeating decimals as fractions.

Why?

- In a game of American Football, with their opponent on the 10-yard line, the defense is penalized half the distance to the goal, placing the ball on the 5-yard line. If they continue to be penalized in this way, where will the ball eventually be placed? Will they ever reach the goal line? How many total penalty yards will the defense have incurred? These questions can be answered by looking at infinite geometric series.

NewVocabulary
infinite geometric series
convergent series
divergent series
infinity

Mathematical Practices
Attend to precision.
Look for and express regularity in repeated reasoning.

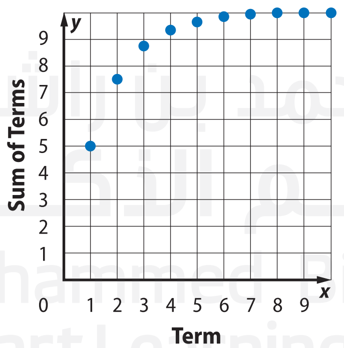
1 Infinite Geometric Series

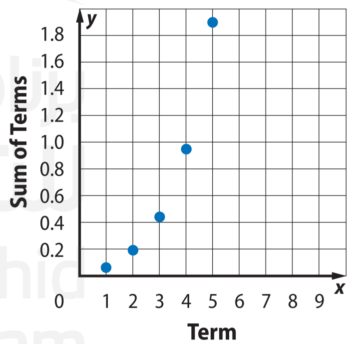
An **infinite geometric series** has an infinite number of terms. A series that has a sum is a **convergent series**, because its sum converges to a specific value. A series that does not have a sum is a **divergent series**.

When you evaluated the sum S_n of an infinite geometric series for the first n terms, you were finding the partial sum of the series. It is also possible to find the sum of an entire series. In the application above, it seems that the ball will eventually reach the goal line, and the defense will be penalized a total of 10 yards. This value is the actual sum of the infinite series $5 + 2.5 + 1.25 + \ldots$. The graph of S_n for $1 \le n \le 10$ is shown on the left below. As n increases, S_n approaches 10.

KeyConcept Convergent and Divergent Series

Convergent Series	Divergent Series
Words The sum approaches a finite value.	**Words** The sum does not approach a finite value.
Ratio $\lvert r \rvert < 1$	**Ratio** $\lvert r \rvert \ge 1$
Example $5 + 2.5 + 1.25 + \ldots$	**Example** $\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \ldots$





Example 1 Convergent and Divergent Series

Determine whether each infinite geometric series is *convergent* or *divergent*.

a. $\mathbf{54 + 36 + 24 + \ldots}$

Find the value of r.

$r = \frac{36}{54}$ or $\frac{2}{3}$; since $-1 < \frac{2}{3} < 1$, the series is convergent.

b. $8 + 12 + 18 + \dots$

$r = \frac{12}{8}$ or 1.5; since $1.5 > 1$, the series is divergent.

GuidedPractice

1A. $2 + 3 + 4.5 + \dots$

1B. $100 + 50 + 25 + \dots$

StudyTip

Absolute Value Recall that $|r| < 1$ means $-1 < r < 1$.

When $|r| < 1$, the value of r^n will approach 0 as n increases. Therefore, the partial sums of the infinite geometric series will approach $\frac{a_1 - a_1(0)}{1 - r}$ or $\frac{a_1}{1 - r}$.

KeyConcept Sum of an Infinite Geometric Series

The sum S of an infinite geometric series with $|r| < 1$ is given by

$$S = \frac{a_1}{1 - r}.$$

If $|r| \geq 1$, the series has no sum.

When an infinite geometric series is divergent, $|r| \geq 1$ and the series has no sum because the absolute value of r^n will increase infinitely as n increases.

The table at the right shows the partial sums for the divergent series $4 + 16 + 64 + \dots$. As n increases, S_n increases rapidly without limit.

n	S_n
5	1364
10	1,398,100
15	1,431,655,764

Example 2 Sum of an Infinite Series

Find the sum of each infinite series, if it exists.

Determine whether each infinite geometric series is *convergent* or *divergent*.

a. $\frac{2}{3} + \frac{6}{15} + \frac{18}{75} + \dots$

Step 1 Find the value of r to determine if the sum exists.

$r = \frac{6}{15} \div \frac{2}{3}$ or $\frac{3}{5}$ — Divide consecutive terms.

Since $\left|\frac{3}{5}\right| < 1$, the sum exists.

Step 2 Use the formula to find the sum.

$S = \frac{a_1}{1 - r}$ — Sum formula

$= \frac{\frac{2}{3}}{1 - \frac{3}{5}}$ — $a_1 = \frac{2}{3}$ and $r = \frac{3}{5}$

$= \frac{2}{3} \div \frac{2}{5}$ or $\frac{5}{3}$ — Simplify.

b. $6 + 9 + 13.5 + 20.25 + \dots$

$r = \frac{9}{6}$ or 1.5; since $|1.5| \geq 1$, the series diverges and the sum does not exist.

StudyTip

Convergence and Divergence A series converges when the absolute value of a term is smaller than the absolute value of the previous term. An infinite arithmetic series will always be divergent.

GuidedPractice

2A. $4 - 2 + 1 - 0.5 + \dots$

2B. $16 + 20 + 25 + \dots$

Sigma notation can be used to represent infinite series. If a sequence goes to **infinity**, it continues without end. The infinity symbol ∞ is placed above the $\sum$ to indicate that a series is infinite.

Example 3 Infinite Series in Sigma Notation

Find $\sum_{k=1}^{\infty} 18\left(\frac{4}{5}\right)^{k-1}$.

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{18}{1 - \frac{4}{5}}$ $a_1 = 18$ and $r = \frac{4}{5}$

$= \frac{18}{\frac{1}{5}}$ or 90 Simplify.

GuidedPractice

3. Find $\sum_{k=1}^{\infty} 12\left(\frac{3}{4}\right)^{k-1}$.

2 Repeating Decimals

A repeating decimal is the sum of an infinite geometric series. For instance, $0.\overline{45} = 0.454545...$ or $0.45 + 0.0045 + 0.000045 + ...$. The formula for the sum of an infinite series can be used to convert the decimal to a fraction.

Problem-SolvingTip

Sense-Making In many cases, it is possible to solve a problem in more than one way. Use the method with which you are most comfortable.

Example 4 Write a Repeating Decimal as a Fraction

Write $0.\overline{63}$ as a fraction.

Method 1 Use the sum of an infinite series.

$0.\overline{63} = 0.63 + 0.0063 + ...$

$= \frac{63}{100} + \frac{63}{10,000} + ...$

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{\frac{63}{100}}{1 - \frac{1}{100}}$ $a_1 = \frac{63}{100}$ and $r = \frac{1}{100}$

$= \frac{63}{99}$ or $\frac{7}{11}$ Simplify.

Method 2 Use algebraic properties.

$x = 0.\overline{63}$ Let $x = 0.\overline{63}$.

$x = 0.636363...$ Write as a repeating decimal.

$100x = 63.636363...$ Multiply each side by 100.

$99x = 63$ Subtract x from $100x$ and $0.\overline{63}$ from $63.\overline{63}$.

$x = \frac{63}{99}$ or $\frac{7}{11}$ Divide each side by 99.

StudyTip

Repeating Decimals Every repeating decimal is a rational number and can be written as a fraction.

GuidedPractice

4. Write $0.\overline{21}$ as a fraction.

Check Your Understanding

Example 1 Determine whether each infinite geometric series is *convergent* or *divergent*.

1. $16 - 8 + 4 - \ldots$

2. $32 - 48 + 72 - \ldots$

3. $0.5 + 0.7 + 0.98 + \ldots$

4. $1 + 1 + 1 + \ldots$

Example 2 Find the sum of each infinite series, if it exists.

5. $440 + 220 + 110 + \ldots$

6. $520 + 130 + 32.5 + \ldots$

7. $\frac{1}{4} + \frac{3}{8} + \frac{9}{16} + \ldots$

8. $\frac{32}{9} + \frac{16}{3} + 8 + \ldots$

9. **SENSE-MAKING** A certain medicine has a half-life of 8 hours after it is administered to a patient. What percent of the medicine is still in the patient's system after 24 hours?

Example 3 Find the sum of each infinite series, if it exists.

10. $\sum_{k=1}^{\infty} 5 \cdot 4^{k-1}$

11. $\sum_{k=1}^{\infty} (-2) \cdot (0.5)^{k-1}$

12. $\sum_{k=1}^{\infty} 3 \cdot \left(\frac{4}{5}\right)^{k-1}$

13. $\sum_{k=1}^{\infty} \frac{1}{2} \cdot \left(\frac{3}{4}\right)^{k-1}$

Example 4 Write each repeating decimal as a fraction.

14. $0.\overline{35}$

15. $0.\overline{642}$

Practice and Problem Solving

Example 1 Determine whether each infinite geometric series is *convergent* or *divergent*.

16. $21 + 63 + 189 + \ldots$

17. $480 + 360 + 270 + \ldots$

18. $\frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \ldots$

19. $\frac{5}{6} + \frac{10}{9} + \frac{40}{27} + \ldots$

20. $0.1 + 0.01 + 0.001 + \ldots$

21. $0.008 + 0.08 + 0.8 + \ldots$

Example 2 Find the sum of each infinite series, if it exists.

22. $18 + 21.6 + 25.92 + \ldots$

23. $-3 - 4.2 - 5.88 - \ldots$

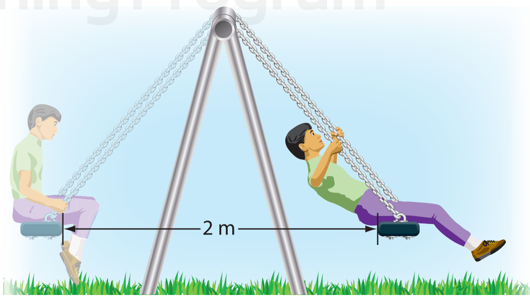
24. $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \ldots$

25. $\frac{12}{5} + \frac{6}{5} + \frac{3}{5} + \ldots$

26. $21 + 14 + \frac{28}{3} + \ldots$

27. $32 + 40 + 50 + \ldots$

28. **SWINGS** If Hassan does not push any harder after his initial swing, the distance traveled per swing will decrease by 10% with each swing. If his initial swing traveled 2 meters, find the total distance traveled when he comes to rest.



Example 3 **Find the sum of each infinite series, if it exists.**

29. $\sum_{k=1}^{\infty} \frac{4}{3} \cdot \left(\frac{5}{4}\right)^{k-1}$

30. $\sum_{k=1}^{\infty} \frac{1}{4} \cdot 3^{k-1}$

31. $\sum_{k=1}^{\infty} \frac{5}{3} \cdot \left(\frac{3}{7}\right)^{k-1}$

32. $\sum_{k=1}^{\infty} \frac{2}{3} \cdot \left(\frac{4}{3}\right)^{k-1}$

33. $\sum_{k=1}^{\infty} \frac{8}{3} \cdot \left(\frac{5}{6}\right)^{k-1}$

34. $\sum_{k=1}^{\infty} \frac{1}{8} \cdot \left(\frac{1}{12}\right)^{k-1}$

Example 4 **Write each repeating decimal as a fraction.**

35. $0.3\overline{21}$

36. $0.1\overline{45}$

37. $2.\overline{18}$

38. $4.\overline{96}$

39. $0.12\overline{14}$

40. $0.43\overline{36}$

41. FANS A fan is running at 10 revolutions per second. After it is turned off, its speed decreases at a rate of 75% per second. Determine the number of revolutions completed by the fan after it is turned off.

42. PRECISION Sally deposited AED 5,000 into an account at the beginning of the year. The account earns 8% interest (murabaha)* each year.

a. How much money will be in the account after 20 years? (*Hint*: Let $5000(1 + 0.08)^1$ represent the end of the first year.)

b. Is this series *convergent* or *divergent*? Explain.

43. RECHARGEABLE BATTERIES A certain rechargeable battery is advertised to recharge back to 99.9% of its previous capacity with every charge. If its initial capacity is 8 hours of life, how many total hours should the battery last?

Find the sum of each infinite series, if it exists.

44. $\frac{7}{5} + \frac{21}{20} + \frac{63}{80} + \ldots$

45. $\frac{15}{4} + \frac{5}{2} + \frac{5}{3} + \ldots$

46. $-\frac{16}{9} + \frac{4}{3} - 1 + \ldots$

47. $\frac{15}{8} + \frac{5}{2} + \frac{10}{3} + \ldots$

48. $\frac{21}{16} + \frac{7}{4} + \frac{7}{3} + \ldots$

49. $-\frac{18}{7} + \frac{12}{7} - \frac{8}{7} + \ldots$

50. MULTIPLE REPRESENTATIONS In this problem, you will use a square of paper that is at least 8 centimeters on a side.

a. Concrete Let the square be one unit. Cut away one half of the square. Call this piece Term 1. Next, cut away one half of the remaining sheet of paper. Call this piece Term 2. Continue cutting the remaining paper in half and labeling the pieces with a term number as long as possible. List the fractions represented by the pieces.

b. Numerical If you could cut the squares indefinitely, you would have an infinite series. Find the sum of the series.

c. Verbal How does the sum of the series relate to the original square of paper?

51. PHYSICS In a physics experiment, a steel ball on a flat track is accelerated, and then allowed to roll freely. After the first minute, the ball has rolled 120 meters. Each minute the ball travels only 40% as far as it did during the preceding minute. How far does the ball travel?

52. PENDULUMS A pendulum travels 12 centimeters on its first swing and 95% of that distance on each swing thereafter. Find the total distance traveled by the pendulum when it comes to rest.

53. TOYS If a rubber ball can bounce back to 95% of its original height, what is the total vertical distance that it will travel if it is dropped from an elevation of 30 meters?

54. CARS During a maintenance inspection, a tire is removed from a car and spun on a diagnostic machine. When the machine is turned off, the spinning tire completes 20 revolutions the first second and 98% of the revolutions each additional second. How many revolutions does the tire complete before it stops spinning?

*The term interest (murabaha) refers to an amount of money that is paid or received when borrowing or lending money. If a customer borrows money from a bank, the customer pays the bank interest (murabaha) for the use of its money. If a customer saves money in a bank account, the bank pays the customer interest (murabaha) for the use of his or her money.

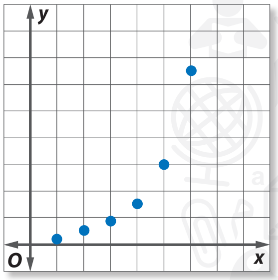
The amount of money that is initially borrowed or saved is called the principal. The interest rate (murabaha rate) is a percentage earned or charged during a certain time period. Simple interest (murabaha) is the amount of interest (murabaha) charged or earned after the interest rate (murabaha rate) is applied to the principal.

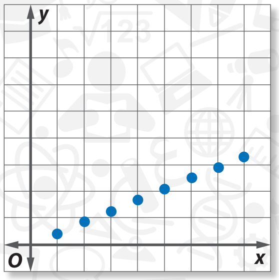
Simple interest (murabaha) (I) is the product of three values: the principal (P), the interest rate (murabaha rate) written as a decimal number (r), and time (t): $I = P \times r \times t$.

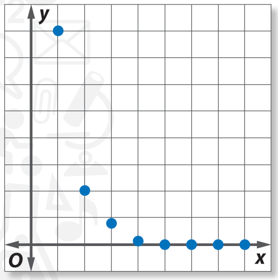
55. **ECONOMICS** The government decides to stimulate its economy by giving AED 500 to every adult. The government assumes that everyone who receives the money will spend 80% on consumer goods and that the producers of these goods will in turn spend 80% on consumer goods. How much money is generated for the economy for every AED 500 that the government provides?

56. **SCIENCE MUSEUM** An exhibit at a science museum offers visitors the opportunity to experiment with the motion of an object on a spring. One visitor pulls the object down and lets it go. The object travels 1.2 meters upward before heading back the other way. Each time the object changes direction, it decreases its distance by 20% when compared to the previous direction. Find the total distance traveled by the object.

Match each graph with its corresponding description.

57. 

58. 

59. 

a. converging geometric series

b. diverging geometric series

c. converging arithmetic series

d. diverging arithmetic series

H.O.T. Problems Use Higher-Order Thinking Skills

60. **ERROR ANALYSIS** Mahmoud and Faleh are asked to find the sum of $1 - 1 + 1 - \ldots$. Is either of them correct? Explain your reasoning.

Mahmoud	Faleh
The sum is 0 because the sum of each pair of terms in the sequence is 0.	There is no sum because $\lvert r \rvert \geq 1$, and the series diverges.

61. **PROOF** Derive the formula for the sum of an infinite geometric series.

62. **CHALLENGE** For what values of b does $3 + 9b + 27b^2 + 81b^3 + \ldots$ have a sum?

63. **REASONING** When does an infinite geometric series have a sum, and when does it not have a sum? Explain your reasoning.

64. **ARGUMENTS** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

If the absolute value of a term of any geometric series is greater than the absolute value of the previous term, then the series is divergent.

65. **OPEN ENDED** Write an infinite series with a sum that converges to 9.

66. **OPEN ENDED** Write $3 - 6 + 12 - \ldots$ using sigma notation in two different ways.

67. **WRITING IN MATH** Explain why an arithmetic series is always divergent.

Standardized Test Practice

68. SAT/ACT What is the sum of an infinite geometric series with a first term of 27 and a common ratio of $\frac{2}{3}$?

A 18
B 34
C 41
D 65
E 81

69. Hareb, Hamad, Humaid, and Hamdan each simplified the same expression at the board. Each student's work is shown below. The teacher said that while two of them had a correct answer, only one of them had arrived at the correct conclusion using correct steps.

Hareb's work

$$x^2x^{-5} = \frac{x^2}{x^{-5}}$$
$$= x^7, x \neq 0$$

Humaid's work

$$x^2x^{-5} = \frac{x^2}{x^5}$$
$$= \frac{1}{x^3}, x \neq 0$$

Hamad's work

$$x^2x^{-5} = \frac{x^2}{x^{-5}}$$
$$= x^{-3}, x \neq 0$$

Hamdan's work

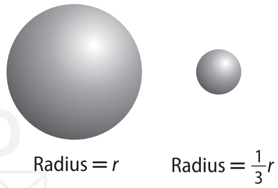
$$x^2x^{-5} = \frac{x^2}{x^5}$$
$$= x^3, x \neq 0$$

Which is a completely accurate simplification?

F Hareb's work
G Hamad's work
H Humaid's work
J Hamdan's work

70. GRIDDED RESPONSE Evaluate $\log_8 60$ to the nearest hundredth.

71. GEOMETRY The radius of a large sphere was multiplied by a factor of $\frac{1}{3}$ to produce a smaller sphere.



How does the volume of the smaller sphere compare to the volume of the larger sphere?

A The volume of the smaller sphere is $\frac{1}{9}$ as large.
B The volume of the smaller sphere is $\frac{1}{\pi^3}$ as large.
C The volume of the smaller sphere is $\frac{1}{27}$ as large.
D The volume of the smaller sphere is $\frac{1}{3}$ as large.

Spiral Review

72. CONTEST An audition is held for a TV contest. At the end of each round, one half of the prospective contestants are eliminated from the competition. On a particular day, 524 contestants begin the audition. (Lesson 9-3)

a. Write an equation for finding the number of contestants who are left after n rounds.

b. Using this method, will the number of contestants who are to be eliminated always be a whole number? Explain.

73. CLUBS A quilting club consists of 9 members. Every week, each member must bring one completed quilt square. (Lesson 9-2)

a. Find the first eight terms of the sequence that describes the total number of squares that have been made after each meeting.

b. One particular quilt measures 144 centimeters by 168 centimeters and is being designed with 8-centimeter squares. After how many meetings will the quilt be complete?

Skills Review

Find each function value.

74. $f(x) = 5x - 9, f(6)$

75. $g(x) = x^2 - x, g(4)$

76. $h(x) = x^2 - 2x - 1, h(3)$

Graphing Technology Lab
Limits

You may have noticed that in some geometric sequences, the later the term in the sequence, the closer the value is to 0. Another way to describe this is that as n increases, a_n approaches 0. The value that the terms of a sequence approach, in this case 0, is called the **limit** of the sequence. Other types of infinite sequences may also have limits. If the terms of a sequence do not approach a unique value, we say that the limit of the sequence does not exist.

You can use a TI-83/84 Plus graphing calculator to help find the limits of infinite sequences.

Activity

Find the limit of the geometric sequence $1, \frac{1}{4}, \frac{1}{16}, \ldots$.

Step 1 Enter the sequence.

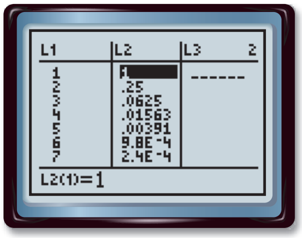
The formula for this sequence is $a_n = \left(\frac{1}{4}\right)^{n-1}$.

- Position the cursor on **L1** in the **STAT EDIT 1: Edit**... screen and enter the formula **seq(N,N,1,10,1)**. This generates the values 1, 2, ..., 10 of the index **N**.

KEYSTROKES: [STAT] [ENTER] [▲] [2nd] [STAT] [▶] 5 [X,T,θ,n] [,] [X,T,θ,n] [,] 1 [,] 10 [,] 1 [)] [ENTER]

- Position the cursor on **L2** and enter the formula **seq((1/4)^(N-1),N,1,10,1)**. This generates the first ten terms of the sequence.

KEYSTROKES: [▶] [▲] [2nd] [STAT] [▶] 5 [(] 1 [÷] 4 [)] [^] [(] [X,T,θ,n] [–] 1 [)] [,] [X,T,θ,n] [,] 1 [,] 10 [,] 1 [)] [ENTER]

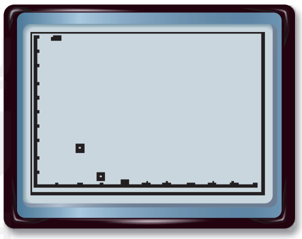


Notice that as n increases, the terms of the given sequence get closer and closer to 0. If you scroll down, you can see that for $n \geq 6$ the terms are so close to 0 that the calculator expresses them in scientific notation. This suggests that the limit of the sequence is 0.

Step 2 Graph the sequence.

Use **STAT PLOT** to graph the sequence. Use **L1** as the **Xlist** and **L2** as the **Ylist**.

The graph also shows that, as n increases, the terms approach 0. In fact, for $n \geq 3$, the marks appear to lie on the horizontal axis. This strongly suggests that the limit of the sequence is 0.



[0, 10] scl: 1 by [0, 1] scl: 0.1

Exercises

Find the limit of each sequence.

1. $a_n = \left(\frac{1}{3}\right)^n$

2. $a_n = \left(-\frac{1}{3}\right)^n$

3. $a_n = 5^n$

4. $a_n = \frac{1}{n^2}$

5. $a_n = \frac{3^n}{3^n + 1}$

6. $a_n = \frac{n^2}{n + 2}$

CHAPTER 2 Mid-Chapter Quiz

Lessons 1-1 through 1-4

Direction Line TK MCQ-DIR (Lesson Ref)

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. (Lesson 9-1)

1. Text TK
2. Text TK

1. $5, -3, -12, -22, -33...$

Direction Line TK MCQ-DIR (Lesson Ref)

2. $\frac{1}{5}, \frac{7}{10}, \frac{6}{5}, \frac{17}{10}, \frac{11}{5}, ...$

3. Text TK

4. **RUNIN HEAD** Text TK (Lesson Ref)

5. **MULTIPLE CHOICE** Text TK (Lesson Ref)
   - **A** Choice
   - **B** Choice
   - **C** Choice
   - **D** Choice

3. **HOUSING** Suha is a real estate agent. She needs to sell 15 houses in 6 months. (Lesson 9-1)
   - **a.** By the end of the first 2 months she has sold 4 houses. If she sells 2 houses each month for the rest of the 6 months, will she meet her goal? Explain.
   - **b.** If she has sold 5 houses by the end of the first month, how many will she have to sell on average each month in order to meet her goal?

4. **GEOMETRY** The figures below show a pattern of filled squares and white squares. (Lesson 9-1)

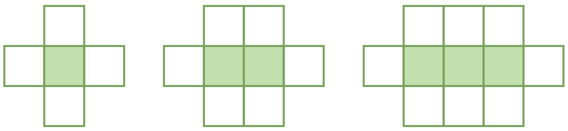


Figure 1 **Figure 2** **Figure 3**

   - **a.** Write an equation representing the nth number in this pattern where n is the number of white squares.
   - **b.** Is it possible to have exactly 84 white squares in an arrangement? Explain.

Find the indicated term of each arithmetic sequence. (Lesson 9-2)

5. $a_1 = 10, d = -5, n = 9$

6. $a_1 = -8, d = 4, n = 99$

Find the sum of each arithmetic series. (Lesson 9-2)

7. $-15 + (-11) + (-7) + \cdots + 53$

8. $a_1 = -12, d = 8, n = 22$

9. $\sum_{k=11}^{50} (-3k + 1)$

10. **MULTIPLE CHOICE** What is the sum of the first 50 odd numbers? (Lesson 9-2)
    - **A** 2550
    - **B** 2500
    - **C** 2499
    - **D** 2401

Find the indicated term for each geometric sequence. (Lesson 9-3)

11. $a_2 = 8, r = 2, a_8 = ?$

12. $a_3 = 0.5, r = 8, a_{10} = ?$

13. **MULTIPLE CHOICE** What are the geometric means of the sequence below? (Lesson 9-3)

    0.5, ______, ______, _____, 2048

    - **F** 512.375, 1024.25, 1536.125
    - **G** 683, 1365.5, 2048
    - **H** 2, 8, 32
    - **J** 4, 32, 256

14. **INCOME** Fahd works for a house building company for 4 months per year. He starts out making AED 9,000 per month. At the end of each month, his salary increases by 5%. How much money will he make in those 4 months? (Lesson 9-3)

Evaluate the sum of each geometric series. (Lesson 9-3)

15. $\sum_{k=1}^{8} 3 \cdot 2^{k-1}$

16. $\sum_{k=1}^{9} 4 \cdot (-1)^{k-1}$

17. $\sum_{k=1}^{20} -2\left(\frac{2}{3}\right)^{k-1}$

Find the sum of each infinite series, if it exists. (Lesson 9-4)

18. $\sum_{n=1}^{\infty} 9 \cdot 2^{n-1}$

19. $\sum_{n=1}^{\infty} (4) \cdot (0.5)^{n-1}$

20. $\sum_{n=1}^{\infty} 12 \cdot \left(\frac{2}{3}\right)^{n-1}$

CHAPTER

LESSON 9-6

Recursion and Iteration

Then

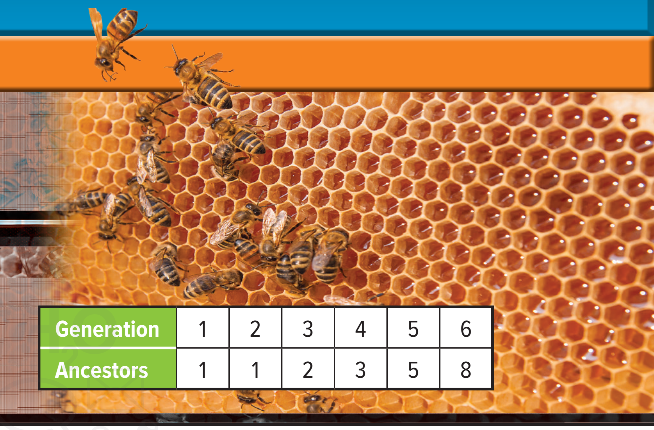
- You explored compositions of functions.

Now

- 1 Recognize and use special sequences.
- 2 Recognize recursive functions.

Why?

- The female honeybee is produced after the queen mates with a male, so the female has two parents, a male and a female. The male honeybee, however, is produced by the queen's unfertilized eggs and thus has only one parent, a female. The family tree for the honeybee follows a special sequence.



Generation	1	2	3	4	5	6
Ancestors	1	1	2	3	5	8

New Vocabulary

- Fibonacci sequence
- recursive sequence
- explicit formula
- recursive formula
- iteration

Mathematical Practices

Attend to precision.

Look for and express regularity in repeated reasoning.

1 Special Sequences

Notice that every term in the list of ancestors is the sum of the previous two terms. This special sequence is called the **Fibonacci sequence**, and it is found in many places in nature. The Fibonacci sequence is an example of a **recursive sequence**. In a recursive sequence, each term is determined by one or more of the previous terms.

The formulas you have used for sequences thus far have been explicit formulas. An **explicit formula** gives a_n as a function of n, such as $a_n = 3n + 1$. The formula that describes the Fibonacci sequence, $a_n = a_{n-2} + a_{n-1}$, is a **recursive formula**, which means that every term will be determined by one or more of the previous terms. An initial term must be given in a recursive formula.

Key Concept Recursive Formulas for Sequences

Arithmetic Sequence	$a_n = a_{n-1} + d$, where d is the common difference
Geometric Sequence	$a_n = r \cdot a_{n-1}$, where r is the common ratio

Example 1 Use a Recursive Formula

Find the first five terms of the sequence in which $a_1 = -3$ and $a_{n+1} = 4a_n - 2$, if $n \geq 1$.

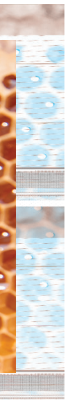
$a_{n+1} = 4a_n - 2$	Recursive formula
$a_{1+1} = 4a_1 - 2$	$n = 1$
$a_2 = 4(-3) - 2$ or -14	$a_1 = -3$
$a_3 = 4(-14) - 2$ or -58	$a_2 = -14$
$a_4 = 4(-58) - 2$ or -234	$a_3 = -58$
$a_5 = 4(-234) - 2$ or -938	$a_4 = -234$

The first five terms of the sequence are −3, −14, −58, −234, and −938.

Guided Practice

1. Find the first five terms of the sequence in which $a_1 = 8$ and $a_{n+1} = -3a_n + 6$, if $n \geq 1$.

In order to find a recursive formula, first determine the initial term. Then evaluate the pattern to generate the later terms. The recursive formula that generates a sequence does not include the value of the initial term.



StudyTip

Sequences and Recursive Formulas Like arithmetic and geometric sequences, recursive formulas define functions in which the domain is the set of positive integers, represented by the term number *n*.

Example 2 Write Recursive Formulas

Write a recursive formula for each sequence.

a. 2, 10, 18, 26, 34, ...

Step 1 Determine whether the sequence is arithmetic or geometric.
The sequence is arithmetic because each term after the first can be found by adding a common difference.

Step 2 Find the common difference.

$d = 10 - 2$ or 8

Step 3 Write the recursive formula.

$a_n = a_{n-1} + d$ Recursive formula for arithmetic sequence

$a_n = a_{n-1} + 8$ $d = 8$

A recursive formula for the sequence is $a_n = a_{n-1} + 8$, $a_1 = 2$.

b. 16, 56, 196, 686, 2401, ...

Step 1 Determine whether the sequence is arithmetic or geometric.
The sequence is geometric because each term after the first can be found after multiplying by a common ratio.

Step 2 Find the common ratio.

$r = \frac{56}{16}$ or 3.5

Step 3 Write the recursive formula.

$a_n = r \cdot a_{n-1}$ Recursive formula for geometric sequence

$a_n = 3.5a_{n-1}$ $r = 3.5$

A recursive formula for the sequence is $a_n = 3.5a_{n-1}$, $a_1 = 16$.

c. $a_4 = 108$ and $r = 3$

Step 1 Determine whether the sequence is arithmetic or geometric.
Because r is given, the sequence is geometric.

Step 2 Write the recursive formula.

$a_n = r \cdot a_{n-1}$ Recursive formula for geometric sequence

$a_n = 3a_{n-1}$ $r = 3$

A recursive formula for the sequence is $a_n = 3a_{n-1}$, $a_1 = 4$.

GuidedPractice

Write a recursive formula for each sequence.

2A. 8, 20, 50, 125, 312.5, ...   **2B.** 8, 17, 26, 35, 44, ...   **2C.** $a_3 = 16$ and $r = 4$

Real-WorldLink

In 2008, the average credit card debt for college students was about AED 3,173.

Source: *USA Today*

Real-World Example 3 Use a Recursive Formula

FINANCIAL LITERACY **Nasser had AED 15,000 in credit card debt when he graduated from college. The balance increased by 2% each month due to interest (murabaha), and Nasser could only make payments of AED 400 per month. Write a recursive formula for the balance of his account each month. Then determine the balance after five months.**

Step 1 Write the recursive formula.

Let a_n represent the balance of the account in the nth month. The initial balance a_1 is AED 15,000. After one month, **interest (murabaha) is added** and **a payment is made**.

	initial balance	+	balance times 2%	−	monthly payment
$a_2 =$	a_1	+	$(a_1 \times 0.02)$	−	400

$a_2 = 1.02a_1 - 400$

The formula is $a_n = 1.02a_{n-1} - 400$.

Step 2 Find the next five terms.

	Recursive formula
$a_n = 1.02a_{n-1} - 400$	
$a_2 = (15{,}000 \times 1.02) - 400$ or **14,900**	$a_1 = 15{,}000$
$a_3 = (\mathbf{14{,}900} \times 1.02) - 400$ or **14,798**	$a_2 = 14{,}900$
$a_4 = (\mathbf{14{,}798} \times 1.02) - 400$ or **14,693.96**	$a_3 = 14{,}798$
$a_5 = (\mathbf{14{,}693.96} \times 1.02) - 400$ or **14,587.84**	$a_4 = 14{,}693.96$
$a_6 = (\mathbf{14{,}587.84} \times 1.02) - 400$ or 14,479.60	$a_5 = 14{,}587.84$

After the fifth month, the balance will be AED 14,479.60.

GuidedPractice

3. Write a recursive formula for a AED 10,000 debt, at 2.5% interest (murabaha) per month, with a AED 600 monthly payment. Then find the first five balances.

ReviewVocabulary

Composition of functions A function is performed, and then a second function is performed on the result of the first function.

2 Iteration

Iteration is the process of repeatedly composing a function with itself. Consider x_0. The first iterate is $f(x_0)$, the second iterate is $f(f(x_0))$, the third iterate is $f(f(f(x_0)))$, and so on.

Iteration can be used to recursively generate a sequence. Start with the initial value x_0. Let $x_1 = f(x_0)$, $x_2 = f(f(x_0))$, and so on.

Example 4 Iterate a Function

Find the first three iterates x_1, x_2, and x_3 of $f(x) = 5x + 4$ for an initial value of $x_0 = 2$.

$x_1 = f(x_0)$	Iterate the function.
$= 5(2) + 4$ or **14**	$x_0 = 2$
$x_2 = f(x_1)$	Iterate the function.
$= 5(\mathbf{14}) + 4$ or **74**	$x_1 = 14$
$x_3 = f(x_2)$	Iterate the function.
$= 5(\mathbf{74}) + 4$ or 374	$x_2 = 74$

The first three iterates are 14, 74, and 374.

GuidedPractice

4. Find the first three iterates x_1, x_2, and x_3 of $f(x) = -3x + 8$ for an initial value of $x_0 = 6$.

Check Your Understanding

Example 1 **Find the first five terms of each sequence described.**

1. $a_1 = 16, a_{n+1} = a_n + 4$
2. $a_1 = -3, a_{n+1} = a_n + 8$
3. $a_1 = 5, a_{n+1} = 3a_n + 2$
4. $a_1 = -4, a_{n+1} = 2a_n - 6$

Example 2 **Write a recursive formula for each sequence.**

5. 3, 8, 18, 38, 78, ...
6. 5, 14, 41, 122, 365, ...

Example 3

7. **FINANCING** Faris financed a AED 1,500 rowing machine to help him train for the college rowing team. He could only make a AED 100 payment each month, and his bill increased by 1% due to interest (murabaha) at the end of each month.
   a. Write a recursive formula for the balance owed at the end of each month.
   b. Find the balance owed after the first four months.
   c. How much interest (murabaha) has accumulated after the first six months?

Example 4 **Find the first three iterates of each function for the given initial value.**

8. $f(x) = 5x + 2, x_0 = 8$
9. $f(x) = -4x + 2, x_0 = 5$
10. $f(x) = 6x + 3, x_0 = -4$
11. $f(x) = 8x - 4, x_0 = -6$

Practice and Problem Solving

Example 1 **PERSEVERANCE** **Find the first five terms of each sequence described.**

12. $a_1 = 10, a_{n+1} = 4a_n + 1$
13. $a_1 = -9, a_{n+1} = 2a_n + 8$
14. $a_1 = 12, a_{n+1} = a_n + n$
15. $a_1 = -4, a_{n+1} = 2a_n + n$
16. $a_1 = 6, a_{n+1} = 3a_n - n$
17. $a_1 = -2, a_{n+1} = 5a_n + 2n$
18. $a_1 = 7, a_2 = 10, a_{n+2} = 2a_n + a_{n+1}$
19. $a_1 = 4, a_2 = 5, a_{n+2} = 4a_n - 2a_{n+1}$
20. $a_1 = 4, a_2 = 3x, a_n = a_{n-1} + 4a_{n-2}$
21. $a_1 = 3, a_2 = 2x, a_n = 4a_{n-1} - 3a_{n-2}$
22. $a_1 = 2, a_2 = x + 3, a_n = a_{n-1} + 6a_{n-2}$
23. $a_1 = 1, a_2 = x, a_n = 3a_{n-1} + 6a_{n-2}$

Example 2 **Write a recursive formula for each sequence.**

24. 16, 10, 7, 5.5, 4.75, ...
25. 32, 12, 7, 5.75, ...
26. 4, 15, 224, 50,175, ...
27. 1, 2, 9, 730, ...
28. 9, 33, 129, 513, ...
29. 480, 128, 40, 18, ...
30. 393, 132, 45, 16, ...
31. 68, 104, 176, 320, ...

Example 3

32. **FINANCIAL LITERACY** Mr. Adnan and his company deposit AED 20,000 into his retirement account at the end of each year. The account earns 8% interest (murabaha) before each deposit.
   a. Write a recursive formula for the balance in the account at the end of each year.
   b. Determine how much is in the account at the end of each of the first 8 years.

Example 4 **Find the first three iterates of each function for the given initial value.**

33. $f(x) = 12x + 8, x_0 = 4$
34. $f(x) = -9x + 1, x_0 = -6$
35. $f(x) = -6x + 3, x_0 = 8$
36. $f(x) = 8x + 3, x_0 = -4$
37. $f(x) = -3x^2 + 9, x_0 = 2$
38. $f(x) = 4x^2 + 5, x_0 = -2$
39. $f(x) = 2x^2 - 5x + 1, x_0 = 6$
40. $f(x) = -0.25x^2 + x + 6, x_0 = 8$
41. $f(x) = x^2 + 2x + 3, x_0 = \frac{1}{2}$
42. $f(x) = 2x^2 + x + 1, x_0 = -\frac{1}{2}$

43. FRACTALS Consider the figures at the right. The number of blue triangles increases according to a specific pattern.

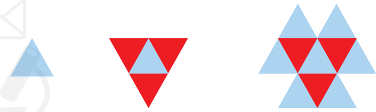
a. Write a recursive formula for the number of blue triangles in the sequence of figures.

b. How many blue triangles will be in the sixth figure?

44. FINANCIAL LITERACY Amer's monthly loan payment is AED 234.85. The recursive formula $b_n = 1.005b_{n-1} - 234.85$ describes the balance left on the loan after n payments. Find the balance of the AED 10,000 loan after each of the first eight payments.

45. CONSERVATION Suppose a lake is populated with 10,000 fish. A year later, 80% of the fish have died or been caught, and the lake is replenished with 10,000 new fish. If the pattern continues, will the lake eventually run out of fish? If not, will the population of the lake converge to any particular value? Explain.

46. GEOMETRY Consider the pattern at the right.



a. Write a sequence of the total number of triangles in the first six figures.

b. Write a recursive formula for the number of triangles.

c. How many triangles will be in the tenth figure?

47. SPREADSHEETS Consider the sequence with $x_0 = 20{,}000$ and $f(x) = 0.3x + 5000$.

a. Enter x_0 in cell A1 of your spreadsheet. Enter "= (0.3)*(A1) + 5000" in cell A2. What answer does it provide?

b. Copy cell A2, highlight cells A3 through A70, and paste. What do you notice about the sequence?

c. How do spreadsheets help analyze recursive sequences?

48. VIDEO GAMES The final monster in Hidaya's video game has 100 health points. During the final battle, the monster regains 10% of its health points after every 10 seconds. If Hidaya can inflict damage to the monster that takes away 10 health points every 10 seconds without getting hurt herself, will she ever kill the monster? If so, when?

H.O.T. Problems Use Higher-Order Thinking Skills

49. CRITIQUE Sultan and Saeed are finding the first three iterates of $f(x) = 5x - 3$ for an initial value of $x_0 = 4$. Is either of them correct? Explain.

Sultan	Saeed
f(4) = 5(4) − 3 or 17	f(4) = 5(4) − 3 or 17
f(17) = 5(17) − 3 or 82	f(17) = 5(17) − 3 or 82
The first three iterates are 4, 17, and 82.	f(82) = 5(82) − 3 or 407
	The first three iterates are 17, 82, and 407.

50. CHALLENGE Find a recursive formula for 5, 23, 98, 401, … .

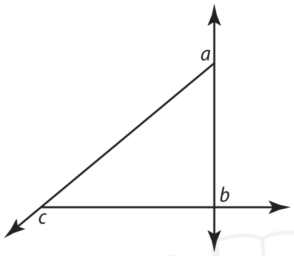
51. REASONING Is the statement *"If the first three terms of a sequence are identical, then the sequence is not recursive" sometimes, always,* or *never* true? Explain your reasoning.

52. OPEN ENDED Write a function for which the first three iterates are 9, 19, and 39.

53. WRITING IN MATH Why is it useful to represent a sequence with an explicit or recursive formula?

Standardized Test Practice

54. GEOMETRY In the figure shown, $a + b + c = ?$



A 180°
B 270°
C 360°
D 450°

55. EXTENDED RESPONSE Omar launches a model rocket from ground level. The rocket's height h in meters is given by the equation $h = -4.9t^2 + 56t$, where t is the time in seconds after the launch.

a. What is the maximum height the rocket will reach?

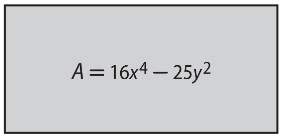
b. How long after it is launched will the rocket reach its maximum height? Round to the nearest tenth of a second.

c. How long after it is launched will the rocket land? Round to the nearest tenth of a second.

56. Which of the following is true about the graphs of $y = 3(x - 4)^2 + 5$ and $y = 3(x + 4)^2 + 5$?

F Their vertices are maximums.
G The graphs have the same shape with different vertices.
H The graphs have different shapes with different vertices.
J One graph has a vertex that is a maximum, while the other graph has a vertex that is a minimum.

57. Which factors could represent the length times the width?



A $(4x - 5y)(4x - 5y)$
B $(4x + 5y)(4x - 5y)$
C $(4x^2 - 5y)(4x^2 + 5y)$
D $(4x^2 + 5y)(4x^2 + 5y)$

Spiral Review

Write each repeating decimal as a fraction. (Lesson 9-4)

58. $0.\overline{7}$ **59.** $5.\overline{126}$ **60.** $6.\overline{259}$

61. SPORTS Obaid is training for a marathon, about 42 kilometers. He begins by running 2 kilometers. Then, when he runs every other day, he runs one and a half times the distance he ran the time before. (Lesson 9-3)

a. Write the first five terms of a sequence describing his training schedule.

b. When will he exceed 26 kilometers in one run?

c. When will he have run 100 total kilometers?

State whether the events are *independent* or *dependent*.

62. tossing a coin and rolling a number cube

63. choosing first and second place in an academic competition

Skills Review

Find each product.

64. $(y + 4)(y + 3)$ **65.** $(x - 2)(x + 6)$ **66.** $(a - 8)(a + 5)$

67. $(4h + 5)(h + 7)$ **68.** $(9p - 1)(3p - 2)$ **69.** $(2g + 7)(5g - 8)$

EXTEND

9-6 Spreadsheet Lab: Amortizing Loans

Mathematical Practices
Use appropriate tools strategically.

When a payment is made on a loan, part of the payment is used to cover the interest (murabaha) that has accumulated since the last payment. The rest is used to reduce the *principal*, or original amount of the loan. This process is called *amortization*. You can use a spreadsheet to analyze the payments, interest (murabaha), and balance on a loan. A table that shows this kind of information is called an *amortization schedule*.

Example

LOANS Najla just bought a new smartphone for AED 695. The store is letting her make monthly payments of AED 60.78 at an interest rate of 9% for one year. How much will she still owe after six months?

Every month, the interest (murabaha) on the remaining balance will be $\frac{9\%}{12}$ or 0.75%. You can find the balance after a payment by multiplying the balance after the previous payment by $1 + 0.0075$ or 1.0075 and then subtracting 60.78.

In a spreadsheet, the column of numbers represents the number of payments, and Column B shows the balance. Enter the interest (murabaha) rate and monthly payment in cells in Column A so that they can be easily updated if the information changes.

The spreadsheet at the right shows the formulas for the balances after each of the first six payments. After six months, Najla still owes AED 355.28.

Smartphone Loan

◇	A	B	C
1	Interest (murabaha) Rate	=695*(1+A2)−A5	
2	0.0075	=B1*(1+A2)−A5	
3		=B2*(1+A2)−A5	
4	Monthly payment	=B3*(1+A2)−A5	
5	60.78	=B4*(1+A2)−A5	
6		=B5*(1+A2)−A5	
7			

Sheet 1 | Sheet 2 | Sheet 3

Model and Analyze

1. Let b_n be the balance left on Najla's loan after n months. Write an equation relating b_n and b_{n+1}.
2. Payments at the beginning of a loan go more toward interest (murabaha) than payments at the end. What percent of Najla's loan remains to be paid after half a year?
3. Extend the spreadsheet to the whole year. What is the balance after 12 payments? Why is it not 0?
4. Suppose Najla decides to pay AED 70 every month. How long would it take her to pay off the loan?
5. Suppose that, based on how much she can afford, Najla will pay a variable amount each month in addition to the AED 60.78. Explain how the flexibility of a spreadsheet can be used to adapt to this situation.
6. Amer has a three-year, AED 12,000 moped loan. The annual interest (murabaha) rate is 6%, and his monthly payment is AED 365.06. After fifteen months, he receives an inheritance which he wants to use to pay off the loan. How much does he owe at that point?

LESSON 9-7

The Binomial Theorem

Then

- You worked with combinations.

Now

1. Use Pascal's Triangle to write binomial expansions.
2. Use the Binomial Theorem to expand powers of binomials.

Why?

- A manager plans to promote 8 employees. Not wanting to appear biased, the manager wants to promote a combination of senior and junior employees that has at least a 10% chance of occurring randomly. If there are an equal number of senior and junior employees applicants, is the probability of randomly promoting 6 senior and 2 junior employees less than 10%?

NewVocabulary
Pascal's triangle

Mathematical Practices
Model with mathematics.

1 Pascal's Triangle

In the 13th century, the Chinese discovered a pattern of numbers that would later be referred to as **Pascal's triangle**. This pattern can be used to determine the coefficients of an expanded binomial $(a + b)^n$.

$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	**1 5 10 10 5 1**

For example, the expanded form of
$(a + b)^5 = \mathbf{1}a^5 + \mathbf{5}a^4b + \mathbf{10}a^3b^2 + \mathbf{10}a^2b^3 + \mathbf{5}ab^4 + \mathbf{1}b^5$.

Real-World Example 1 Use Pascal's Triangle

Find the probability of promoting 6 senior and 2 junior employees by expanding $(m + f)^8$.

Write three more rows of Pascal's triangle and use the pattern to write the expansion.

5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1

$(m + f)^8 = m^8 + 8m^7f + 28m^6f^2 + 56m^5f^3 + 70m^4f^4 + 56m^3f^5 + 28m^2f^6 + 8mf^7 + f^8$

By adding the coefficients of the polynomial, we determine that there are 256 combinations of senior and junior employees that could be promoted.

$28m^6f^2$ represents the number of combinations with 6 senior and 2 junior employees. Therefore, there is a $\frac{28}{256}$ or about an 11% chance of randomly promoting 6 senior and 2 junior employees.

GuidedPractice

1. Expand $(c + d)^9$.

2 The Binomial Theorem

Instead of writing out row after row of Pascal's triangle, you can use the **Binomial Theorem** to expand a binomial. Recall that ${}_nC_r = \frac{n!}{r!(n-r)!}$.

StudyTip

Combinations Recall that both ${}_nC_0$ and ${}_nC_n$ equal 1.

KeyConcept Binomial Theorem

If n is a natural number, then $(a+b)^n =$

$${}_nC_0\, a^n b^0 + {}_nC_1\, a^{n-1} b^1 + {}_nC_2\, a^{n-2} b^2 + \cdots + {}_nC_n\, a^0 b^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k.$$

To use the theorem, replace n with the value of the exponent. Notice how the terms will follow the pattern of Pascal's triangle, and the coefficients will be symmetric.

Example 2 Use the Binomial Theorem

Expand $(a+b)^7$.

Method 1 Use combinations.

Replace n with 7 in the Binomial Theorem.

$$(a+b)^7 = {}_7C_0\, a^7 + {}_7C_1\, a^6 b + {}_7C_2\, a^5 b^2 + {}_7C_3\, a^4 b^3 + {}_7C_4\, a^3 b^4 + {}_7C_5\, a^2 b^5 + {}_7C_6\, ab^6 + {}_7C_7\, b^7$$

$$= a^7 + \frac{7!}{6!} a^6 b + \frac{7!}{2!5!} a^5 b^2 + \frac{7!}{3!4!} a^4 b^3 + \frac{7!}{4!3!} a^3 b^4 + \frac{7!}{5!2!} a^2 b^5 + \frac{7!}{6!} ab^6 + b^7$$

$$= a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7ab^6 + b^7$$

Method 2 Use Pascal's triangle.

Use the Binomial Theorem to determine exponents, but instead of finding the coefficients by using combinations, look at the seventh row of Pascal's triangle.

6			1		6		15		20		15		6		1		
7		1		7		21		35		35		21		7		1	
8	1		8		28		56		70		56		28		8		1

$$(a+b)^7 = a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7ab^6 + b^7$$

GuidedPractice

2. Expand $(x+y)^{10}$.

When the binomial to be expanded has coefficients other than 1, the coefficients will no longer be symmetric. In these cases, you may want to use the Binomial Theorem.

Example 3 Coefficients Other Than 1

StudyTip

Graphing Calculator You can calculate ${}_nC_r$ by using a graphing calculator.

Press MATH and choose PRB 3.

Expand $(5a-4b)^4$.

$$(5a-4b)^4$$

$$= {}_4C_0\,(5a)^4 + {}_4C_1\,(5a)^3(-4b) + {}_4C_2\,(5a)^2(-4b)^2 + {}_4C_3\,(5a)(-4b)^3 + {}_4C_4\,(-4b)^4$$

$$= 625a^4 + \frac{4!}{3!}(125a^3)(-4b) + \frac{4!}{2!2!}(25a^2)(16b^2) + \frac{4!}{3!}(5a)(-64b^3) + 256b^4$$

$$= 625a^4 - 2000a^3 b + 2400a^2 b^2 - 1280ab^3 + 256b^4$$

GuidedPractice

3. Expand $(3x+2y)^5$.

Sometimes you may need to find only one term in a binomial expansion. To do this, you can use the summation formula for the Binomial Theorem, $\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k$.

Example 4 Determine a Single Term

Find the fifth term of $(y + z)^{11}$.

Step 1 Use the Binomial Theorem to write the expansion in sigma notation.

$$(y + z)^{11} = \sum_{k=0}^{11} \frac{11!}{k!(11-k)!} y^{11-k} z^k$$

Step 2

$$\frac{11!}{k!(11-k)!} y^{11-k} z^k = \frac{11!}{4!(11-4)!} y^{11-4} z^4 \quad \text{For the fifth term, } k = 4.$$

$$= 330y^7z^4 \quad C(11, 4) = 330$$

GuidedPractice

4. Find the sixth term of $(c + d)^{10}$.

ConceptSummary Binomial Expansion

In a binomial expansion of $(a + b)^n$,

- there are $n + 1$ terms.
- n is the exponent of a in the first term and b in the last term.
- in successive terms, the exponent of a decreases by 1, and the exponent of b increases by 1.
- the sum of the exponents in each term is n.
- the coefficients are symmetric.

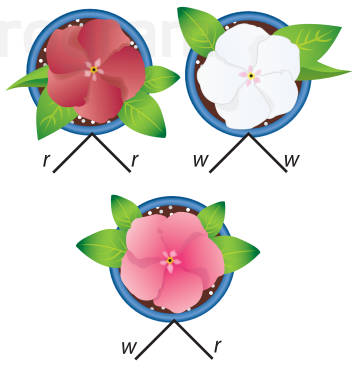
Check Your Understanding

Examples 1–3 **Expand each binomial.**

1. $(c + d)^5$
2. $(g + h)^7$
3. $(x - 4)^6$
4. $(2y - z)^5$
5. $(x + 3)^5$
6. $(y - 4z)^4$
7. **GENETICS** If a woman is equally as likely to have a baby boy or a baby girl, use binomial expansion to determine the probability that 5 of her 6 children are girls. Do not consider identical twins.

Example 4 **Find the indicated term of each expression.**

8. fourth term of $(b + c)^9$
9. fifth term of $(x + 3y)^8$
10. third term of $(a - 4b)^6$
11. sixth term of $(2c - 3d)^8$
12. last term of $(5x + y)^5$
13. first term of $(3a + 8b)^5$
14. **MODELING** The color of a particular flower is determined by the combination of two genes, also called *alleles*. If the flower has two red alleles r, the flower is red. If the flower has two white alleles w, the flower is white. If the flower has one allele of each color, the flower will be pink. In a lab, two pink flowers are mated and eventually produce 1000 offspring. How many of the 1000 offspring will be pink?



Practice and Problem Solving

Examples 1–3 **Expand each binomial.**

15. $(a - b)^6$

16. $(c - d)^7$

17. $(x + 6)^6$

18. $(y - 5)^7$

19. $(2a + 4b)^4$

20. $(3a - 4b)^5$

21. **COMMITTEES** If an equal number of seniors and juniors applied to be on a school sports committee and the committee needs a total of 10 people, find the probability that 7 of the members will be juniors. Assume that committee members will be chosen randomly.

22. **BASEBALL** If a pitcher is just as likely to throw a ball as a strike, find the probability that 11 of his first 12 pitches are balls.

Example 4 **Find the indicated term of each expression.**

23. third term of $(x + 2z)^7$

24. fourth term of $(y - 3x)^6$

25. seventh term of $(2a - 2b)^8$

26. sixth term of $(4x + 5y)^6$

27. fifth term of $(x - 4)^9$

28. fourth term of $(c + 6)^8$

Expand each binomial.

29. $\left(x + \frac{1}{2}\right)^5$

30. $\left(x - \frac{1}{3}\right)^4$

31. $\left(2b + \frac{1}{4}\right)^5$

32. $\left(3c + \frac{1}{3}\right)^5$

33. **SENSE-MAKING** In $\frac{n!}{k!(n-k)!}\, p^k q^{n-k}$, let p represent the likelihood of a success and q represent the likelihood of a failure.

a. If a penalty kick taker scores 70% of his penalties, find the likelihood that he scores 9 of his next 10 attempts.

b. If a midfielder completes 60% of his passes, find the likelihood that he completes 8 of his next 10 attempts.

c. If a team converts 30% of their free kicks, find the likelihood that they convert 2 of their next 5 free kicks.

H.O.T. Problems Use Higher-Order Thinking Skills

34. **CHALLENGE** Find the sixth term of the expansion of $\left(\sqrt{a} + \sqrt{b}\right)^{12}$. Explain your reasoning.

35. **REASONING** Explain how the terms of $(x + y)^n$ and $(x - y)^n$ are the same and how they are different.

36. **REASONING** Determine whether the following statement is *true* or *false*. Explain your reasoning.

The eighth and twelfth terms of $(x + y)^{20}$ have the same coefficients.

37. **OPEN ENDED** Write a power of a binomial for which the second term of the expansion is $6x^4y$.

38. **WRITING IN MATH** Explain how to write out the terms of Pascal's triangle.

Standardized Test Practice

39. PROBABILITY A desk drawer contains 7 sharpened red pencils, 5 sharpened yellow pencils, 3 unsharpened red pencils, and 5 unsharpened yellow pencils. If a pencil is taken from the drawer at random, what is the probability that it is yellow, given that it is one of the sharpened pencils?

A $\frac{5}{12}$

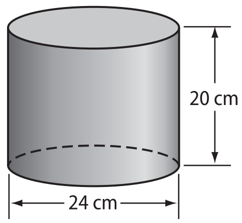
B $\frac{7}{20}$

C $\frac{5}{8}$

D $\frac{1}{5}$

40. GRIDDED RESPONSE Two people are 17.5 kilometers apart. They begin to walk toward each other along a straight line at the same time. One walks at the rate of 4 kilometers per hour, and the other walks at the rate of 3 kilometers per hour. In how many hours will they meet?

41. GEOMETRY Suha has a cylindrical block that she needs to paint for an art project.



What is the surface area of the cylinder in square centimeters rounded to the nearest square centimeter?

F 1960 **G** 2413 **H** 5127 **J** 6635

42. Which of the following is a linear function?

A $y = \frac{x+3}{x+2}$

B $y = (3x+2)^2$

C $y = \frac{x+3}{2}$

D $y = |3x| + 2$

Spiral Review

Find the first five terms of each sequence. (Lesson 9-5)

43. $a_1 = -2,\ a_{n+1} = a_n + 5$

44. $a_1 = 3,\ a_{n+1} = 4a_n - 10$

45. $a_1 = 4,\ a_{n+1} = 3a_n - 6$

Find the sum of each infinite geometric series, if it exists. (Lesson 9-4)

46. $-6 + 3 - \frac{3}{2} + \ldots$

47. $\frac{3}{4} + \frac{1}{4} + \frac{1}{12} + \ldots$

48. $\sqrt{3} + 3 + \sqrt{27} + \ldots$

49. TRAVEL A trip between two towns takes 4 hours under ideal conditions. The first 150 kilometers of the trip is on an interstate, and the last 130 kilometers is on a highway with a speed limit that is 10 kilometers per hour less than on the interstate.

a. If x represents the speed limit on the interstate, write expressions for the time spent at that speed and for the time spent on the other highway.

b. Write and solve an equation to find the speed limits on the two highways.

Skills Review

State whether each statement is *true* or *false* when $n = 1$. Explain.

50. $\frac{(n+1)(n+1)}{2} = 2$

51. $3n + 5$ is even.

52. $n^2 - 1$ is odd.

EXTEND 9-7

Algebra Lab
Combinations and Pascal's Triangle

Recall that an arrangement or selection of objects in which order is not important is called a *combination*. For example, selecting 2 snacks from a choice of 6 is a combination of 6 objects taken 2 at a time and can be written ${}_6C_2$ or $C(6, 2)$.

Activity

A contestant on a tv show has the opportunity to win up to five prizes, one for each of five rounds of the game. If the contestant wins a round, he or she may choose one prize. Determine the number of ways that prizes can be chosen.

Step 1 If a contestant does not win any rounds, he or she receives 0 prizes. This represents 5 items taken 0 at a time.

$${}_nC_r = \frac{n!}{(n-r)!\,r!} \quad \text{Definition of combination}$$

$${}_5C_0 = \frac{5!}{(5-0)!\,0!} \quad n = 5 \text{ and } r = 0$$

$$= \frac{120}{120(1)} \text{ or } 1 \quad 5! = 120 \text{ and } 0! = 1$$

There is 1 way to receive 0 prizes.

If a contestant wins one round, any one of the prizes can be selected. If a contestant wins two rounds, two prizes can be chosen. If three rounds are won, three prizes can be chosen, and so on. In how many ways can 1 prize be chosen? 2 prizes? 3, 4, and 5 prizes? We can determine these answers by examining Pascal's triangle.

Step 2 Examine Pascal's triangle.

List Rows 0 through 5 of Pascal's triangle.

Row 0						1					
Row 1					1		1				
Row 2				1		2		1			
Row 3			1		3		3		1		
Row 4		1		4		6		4		1	
Row 5	1		5		10		10		5		1

The number of ways one prize can be chosen from 5 can be determined by looking at Row 5. The first number in Row 5 represents the number of ways to choose 0 prizes, the second number represents the number of ways to choose 1 prize, and so on.

Analyze the Result

1. Make a conjecture about how the numbers in one of the rows can be used to find the number of ways that 0, 1, 2, 3, 4, ..., n objects can be selected from n objects.
2. Suppose the rules of the game are changed so that there are 6 rounds and 6 prizes from which to choose. Find the number of ways that 0, 1, 2, 3, 4, 5, or 6 prizes can be chosen. Which row of Pascal's triangle can be used to find the answers?
3. Use Pascal's triangle to find ${}_8C_0$, ${}_8C_1$, ${}_8C_2$, ${}_8C_3$, ${}_8C_4$, ${}_8C_5$, ${}_8C_6$, ${}_8C_7$, and ${}_8C_8$. State the row number that you used to find the answers.

LESSON 9-8 Proof by Mathematical Induction

:: Then

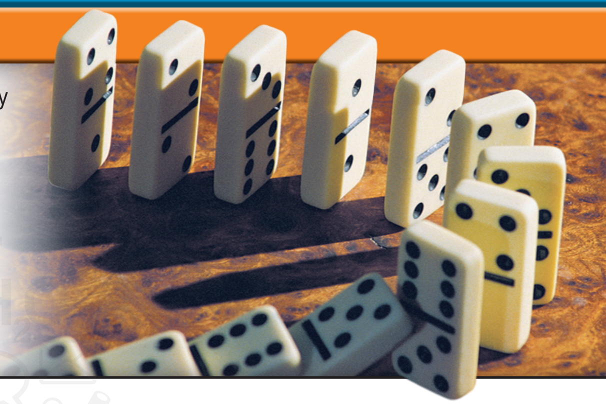
- You have proved the sum of an arithmetic series.

:: Now

1. Prove statements by using mathematical induction.
2. Disprove statements by finding a counterexample.

:: Why?

- When dominoes are set up closely and the first domino is knocked down, the rest of the dominoes come tumbling down. All that is needed with this setup is for the first domino to fall, and the rest will follow. The same is true with mathematical induction.



NewVocabulary
mathematical induction
induction hypothesis

Mathematical Practices
Make sense of problems and persevere in solving them.
Construct viable arguments and critique the reasoning of others.

1 Mathematical Induction

Mathematical induction is a method of proving statements involving natural numbers.

KeyConcept Mathematical Induction

To prove that a statement is true for all natural numbers n,

Step 1 Show that the statement is true for $n = 1$.

Step 2 Assume that the statement is true for some natural number k. This assumption is called the **induction hypothesis**.

Step 3 Show that the statement is true for the next natural number $k + 1$.

Example 1 Prove Summation

Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

Step 1 When $n = 1$, the left side of the equation is 1^3 or 1.
The right side is $\dfrac{1^2(1+1)^2}{4}$ or 1. Thus, the statement is true for $n = 1$.

Step 2 Assume that $1^3 + 2^3 + 3^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$ for a natural number k.

Step 3 Show that the given statement is true for $n = k + 1$.

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4} \quad \text{Inductive hypothesis}$$

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \text{Add } (k+1)^3 \text{ to each side.}$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \quad \text{The LCD is 4.}$$

$$= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \quad \text{Factor.}$$

$$= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \quad \text{Simplify.}$$

$$= \frac{(k+1)^2(k+2)^2}{4} \quad \text{Factor.}$$

The last expression is the statement to be proved, where n has been replaced by $k + 1$. This proves the conjecture.

GuidedPractice

1. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Along with summation, mathematical induction can be used to prove divisibility.

Example 2 Prove Divisibility

Prove that $8^n - 1$ is divisible by 7 for all natural numbers n.

Step 1 When $n = 1$, $8^n - 1 = 8^1 - 1$ or 7. Since 7 is divisible by 7, the statement is true for $n = 1$.

Step 2 Assume that $8^k - 1$ is divisible by 7 for some natural number k. This means that there is a natural number r such that $8^k - 1 = 7r$.

Step 3 Show that the statement is true for $n = k + 1$.

$8^k - 1 = 7r$ — Inductive hypothesis

$8^k = 7r + 1$ — Add 1 to each side.

$8(8^k) = 8(7r + 1)$ — Multiply each side by 8.

$8^{k+1} = 56r + 8$ — Simplify.

$8^{k+1} - 1 = 56r + 7$ — Subtract 1 from each side.

$8^{k+1} - 1 = 7(8r + 1)$ — Factor.

Since r is a natural number, $8r + 1$ is a natural number and $7(8r + 1)$ is divisible by 7. Therefore, $8^{k+1} - 1$ is divisible by 7.

This proves that 8^{n-1} is divisible by 7 for all natural numbers n.

StudyTip

Regularity r is a whole number used to show divisibility in proof. When a value equals $4r$, then it must be divisible by 4.

GuidedPractice

2. Prove that $7^n - 1$ is divisible by 6 for all natural numbers n.

2 Counterexamples

Statements can be proved false by using mathematical induction. An easier method is by finding a counterexample, which is a specific case in which the statement is false.

Example 3 Use a Counterexample to Disprove

Find a counterexample to disprove the statement that $2^n + 2n^2$ is divisible by 4 for any natural number n.

Test different values of n.

n	$2^n + 2n^2$	Divisible by 4?
1	$2^1 + 2(1)^2 = 2 + 2$ or 4	yes
2	$2^2 + 2(2)^2 = 4 + 8$ or 12	yes
3	$2^3 + 2(3)^2 = 8 + 18$ or 26	no

The value $n = 3$ is a counterexample for the statement.

ReviewVocabulary

counterexample One of the synonyms of *counter* is to *contradict*, so a counterexample is an example that contradicts a hypothesis.

GuidedPractice

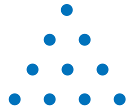
3. Find a counterexample to disprove $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(3n - 1)}{2}$.

Check Your Understanding

Example 1 **Prove that each statement is true for all natural numbers.**

1. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$
2. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

3. **NUMBER THEORY** A number is *triangular* if it can be represented visually by a triangular array.



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   a. The first triangular number is 1. Find the next 5 triangular numbers.

   b. Write a formula for the nth triangular number.

   c. Prove that the sum of the first n triangular numbers equals $\frac{n(n+1)(n+2)}{6}$.

Example 2 **Prove that each statement is true for all natural numbers.**

4. $10^n - 1$ is divisible by 9.
5. $4^n - 1$ is divisible by 3.

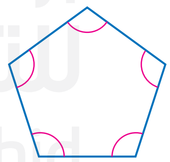
Example 3 **Find a counterexample to disprove each statement.**

6. $3^n + 1$ is divisible by 4.
7. $2^n + 3^n$ is divisible by 4.

Practice and Problem Solving

Example 1 **ARGUMENTS** **Prove that each statement is true for all natural numbers.**

8. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$
9. $2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n+1)}{2}$
10. $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$
11. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$
12. $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$
13. $3 + 7 + 11 + \cdots + (4n - 1) = 2n^2 + n$
14. $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
15. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{n(2n-1)(2n+1)}{3}$
16. **GEOMETRY** According to the Interior Angle Sum Formula, if a convex polygon has n sides, then the sum of the measures of the interior angles of a polygon equals $180(n - 2)$. Prove this formula for $n \geq 3$ using mathematical induction and geometry.



Example 2 **Prove that each statement is true for all natural numbers.**

17. $5^n + 3$ is divisible by 4.
18. $9^n - 1$ is divisible by 8.
19. $12^n + 10$ is divisible by 11.
20. $13^n + 11$ is divisible by 12.

Example 3 **Find a counterexample to disprove each statement.**

21. $1 + 2 + 3 + \cdots + n = n^2$
22. $1 + 8 + 27 + \cdots + n^3 = (2n + 2)^2$
23. $n^2 - n + 15$ is prime.
24. $n^2 + n + 23$ is prime.

25. **NATURE** The terms of the Fibonacci sequence are found in many places in nature. The number of spirals of seeds in sunflowers is a Fibonacci number, as is the number of spirals of scales on a pinecone. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, ... Each element after the first two is found by adding the previous two terms. If f_n stands for the nth Fibonacci number, prove that $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$.

Prove that each statement is true for all natural numbers or find a counterexample.

26. $7^n + 5$ is divisible by 6.

27. $18^n - 1$ is divisible by 17.

28. $n^2 + 21n + 7$ is a prime number.

29. $n^2 + 3n + 3$ is a prime number.

30. $500 + 100 + 20 + \cdots + 4 \cdot 5^{4-n} = 625\left(1 - \frac{1}{5^n}\right)$

31. $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

32. **PERSEVERANCE** Refer to the figures below.

Figure 1 Figure 2 Figure 3

a. There is a total of 5 squares in the second figure. How many squares are there in the third figure?

b. Write a sequence for the first five figures.

c. How many squares are there in a standard 8 × 8 checkerboard?

d. Write a formula to represent the number of squares in an $n \times n$ grid.

H.O.T. Problems Use Higher-Order Thinking Skills

33. **CHALLENGE** Suggest a formula to represent $2 + 4 + 6 + \cdots + 2n$, and prove your hypothesis using mathematical induction.

REASONING **Determine whether the following statements are *true* or *false*. Explain.**

34. If you cannot find a counterexample to a statement, then it is true.

35. If a statement is true for $n = k$ and $n = k + 1$, then it is also true for $n = 1$.

36. **CHALLENGE** Prove $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

37. **REASONING** Find a counterexample to $x^3 + 30 > x^2 + 20x$.

38. **OPEN ENDED** Write a sequence, the formula that produces it, and determine the formula for the sum of the terms of the sequence. Then prove the formula with mathematical induction.

39. **WRITING IN MATH** Explain how the concept of dominoes can help you understand the power of mathematical induction.

40. **WRITING IN MATH** Provide a real-world example other than dominoes that describes mathematical induction.

Standardized Test Practice

41. Which of the following is a counterexample to the statement below?

$$n^2 + n - 11 \text{ is prime.}$$

A $n = -6$
B $n = 4$
C $n = 5$
D $n = 6$

42. **PROBABILITY** Moza wants to create a 7-character password. She wants to use an arrangement of the first 3 letters of her first name (lat), followed by an arrangement of the 4 digits in 1986, the year she was born. How many possible passwords can she create in this way?

F 72
G 144
H 288
J 576

43. **GRIDDED RESPONSE** A gear that is 8 centimeters in diameter turns a smaller gear that is 3 centimeters in diameter. If the larger gear makes 36 revolutions, how many revolutions does the smaller gear make in that time?

44. **SHORT RESPONSE** Write an equation for the nth line. Show how it fits the pattern for each given line in the list.

Line 1: $1 \times 0 = 1 - 1$

Line 2: $2 \times 1 = 4 - 2$

Line 3: $3 \times 2 = 9 - 3$

Line 4: $4 \times 3 = 16 - 4$

Line 5: $5 \times 4 = 25 - 5$

Spiral Review

Find the indicated term of each expansion. (Lesson 9-6)

45. fourth term of $(x + 2y)^6$

46. fifth term of $(a + b)^6$

47. fourth term of $(x - y)^9$

48. **BIOLOGY** In a particular forest, scientists are interested in how the population of wolves will change over the next two years. One model for animal population is the Verhulst population model, $p_{n+1} = p_n + rp_n(1 - p_n)$, where n represents the number of time periods that have passed, p_n represents the percent of the maximum sustainable population that exists at time n, and r is the growth factor. (Lesson 9-5)

a. To find the population of the wolves after one year, evaluate $p_1 = 0.45 + 1.5(0.45)(1 - 0.45)$.

b. Explain what each number in the expression in part **a** represents.

c. The current population of wolves is 165. Find the new population by multiplying 165 by the value in part **a**.

Find the exact solution(s) of each system of equations.

49. $x^2 + y^2 - 18x + 24y + 200 = 0$
$4x + 3y = 0$

50. $4x^2 + y^2 = 16$
$x^2 + 2y^2 = 4$

Skills Review

Evaluate each expression.

51. $P(8, 2)$

52. $P(9, 1)$

53. $P(12, 6)$

54. $C(5, 2)$

55. $C(8, 4)$

56. $C(20, 17)$

57. $P(12, 2)$

58. $P(7, 2)$

59. $C(8, 6)$

60. $C(9, 4) \cdot C(5, 3)$

61. $C(6, 1) \cdot C(4, 1)$

62. $C(10, 5) \cdot C(8, 4)$

LESSON 9-9

Functions as Infinite Series

Then

- You found the nth term of an infinite series expressed using sigma notation.

Now

1. Use a power series to represent a rational function.
2. Use power series representations to approximate values of transcendental functions.

Why?

- The music to which you listen on a digital audio player is first performed by an artist. The waveform of each sound in that performance is then broken down into its component parts and stored digitally. These parts are then retrieved and combined to reproduce each original sound of the performance. The analysis of a special series is an essential ingredient in this process.

NewVocabulary
power series
exponential series
Euler's Formula

1 Power Series

Earlier in this chapter, you saw how some series of numbers can be expressed as functions. In this lesson, you will see that some functions can be broken down into infinite series of component functions.

You learned that the sum of an infinite geometric series,

$$1 + r + r^2 + \cdots + r^n + \cdots, \ a_1 = 1$$

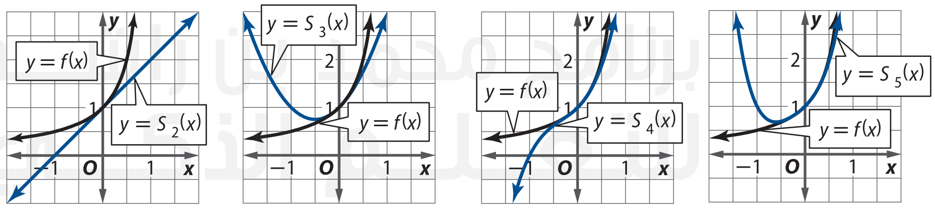
with common ratio r, converges to a sum of $\frac{a_1}{1 - r}$ if $|r| < 1$. Replacing r with x,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1 - x}, \text{ for } |x| < 1.$$

It follows that $f(x) = \frac{1}{1 - x}$ can be expressed as an infinite series. That is,

$$f(x) = \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \text{ or } 1 + x + x^2 + \cdots + x^n + \cdots \text{ for } |x| < 1.$$

The figures below show the graph of $f(x) = \frac{1}{1 - x}$ and the second through fifth partial sums $S_n(x)$ of the series: $S_2(x) = 1 + x$, $S_3(x) = 1 + x + x^2$, $S_4(x) = 1 + x + x^2 + x^3$, and $S_5(x) = 1 + x + x^2 + x^3 + x^4$.



Notice that as n increases, the graph of $S_n(x)$ appears to come closer and closer to the graph of $f(x)$ on the interval $(-1, 1)$ or $|x| < 1$. Notice too that each of the partial sums of the series is a polynomial function, so the series can be thought of as an "infinite" polynomial. An infinite series of this type is called a **power series**.

KeyConcept Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

where x and a_n can take on any values for $n = 0, 1, 2, \ldots$, is called a power series in x.

If you know the power series representation of one function, you can use it to find the power series representations of other related functions.

Example 1 Power Series Representation of a Rational Function

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x) = \frac{1}{3-x}$. Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.

To find the transformation that relates $f(x)$ to $g(x)$, use u-substitution. Substitute u for x in f, equate the two functions, and solve for u as shown.

$$g(x) = f(u)$$
$$\frac{1}{3-x} = \frac{1}{1-u}$$
$$1 - u = 3 - x$$
$$-u = 2 - x$$
$$u = x - 2$$

> **WatchOut!**
> When finding the kth partial sum of a series where the lower bound starts at 0 use the series $\sum_{n=0}^{k-1}$. For instance in Example 1, the sixth partial sum is called for, but since the lower bound is 0, the upper bound is 6 − 1 or 5, not 6.

Therefore, $g(x) = f(x-2)$. Replacing x with $x - 2$ in $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$ yields

$$f(x-2) = \sum_{n=0}^{\infty} (x-2)^n \text{ for } |x-2| < 1.$$

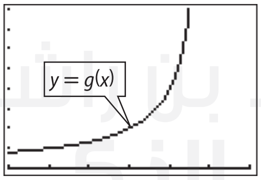
Therefore, $g(x) = \frac{1}{3-x}$ can be represented by the power series $\sum_{n=0}^{\infty} (x-2)^n$.

This series converges for $|x - 2| < 1$, which is equivalent to $-1 < x - 2 < 1$ or $1 < x < 3$.

The sixth partial sum $S_6(x)$ of this series is

$$\sum_{n=0}^{5} (x-2)^n \text{ or } 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + (x-2)^5.$$

The graphs of $g(x) = \frac{1}{3-x}$ and $S_6(x) = 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + (x-2)^5$ are shown. Notice that on the interval (1, 3), the graph of $S_6(x)$ comes close to the graph of $g(x)$.



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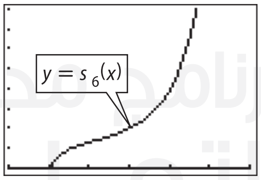

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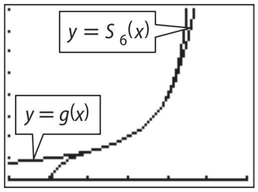

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> **StudyTip**
> **Graphs of Series** Notice that the graphs of $f(x)$ and $S_n(x)$ only converge on an interval. The graphs may differ greatly outside of that interval.

GuidedPractice

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$. Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.

1A. $g(x) = \frac{1}{1-2x}$

1B. $g(x) = \frac{2}{1-x}$

In calculus, power series representations are often easier to use in calculations than other representations of functions when determining functions called *derivatives* and *integrals*. A more immediate application can be seen by looking at the power series representations of transcendental functions such as $f(x) = e^x$, $f(x) = \sin x$, and $f(x) = \cos x$.

2 Transcendental Functions as Power Series

Previously, you learned that the transcendental number e is given by $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. Thus, $e^x = \lim_{x\to\infty}\left(1+\frac{1}{n}\right)^{nx}$. We can use this definition along with the Binomial Theorem to derive a power series representation for $f(x) = e^x$.

ReadingMath

Euler Number The Swiss mathematician Leonhard Euler (pronounced OY ler), published a work in which he developed this irrational number, called e, the Euler number.

If we let $u = \frac{1}{n}$ and $k = nx$, then $\left(1+\frac{1}{n}\right)^{nx}$ becomes $(1+u)^k$. Applying the Binomial Theorem,

$$(1+u)^k = {}_kC_0\,(1)^k u^0 + {}_kC_1\,(1)^{k-1}u + {}_kC_2\,(1)^{k-2}u^2 + {}_kC_3\,(1)^{k-3}u^3 + \cdots$$

$$= \frac{k!}{(k-0)!\,0!}(1) + \frac{k!}{(k-1)!\,1!}(1)u + \frac{k!}{(k-2)!\,2!}(1)u^2 + \frac{k!}{(k-3)!\,3!}(1)u^3 + \cdots$$

$$= 1 + \frac{k(k-1)!}{(k-1)!}u + \frac{k(k-1)(k-2)!}{(k-2)!\,2!}u^2 + \frac{k(k-1)(k-2)(k-3)!}{(k-3)!\,3!}u^3 + \cdots$$

$$= 1 + ku + \frac{k(k-1)}{2!}u^2 + \frac{k(k-1)(k-2)}{3!}u^3 + \cdots$$

Now replace u with $\frac{1}{n}$ and k with nx and find the limit as n approaches infinity. Use the fact that as n approaches infinity, the fraction $\frac{1}{n}$ gets increasingly smaller, so $\lim_{n\to\infty}\frac{1}{n} = 0$.

$$\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^{nx} = 1 + (nx)\frac{1}{n} + \frac{nx(nx-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{nx(nx-1)(nx-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{2!} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This series is often called the **exponential series**.

StudyTip

Defining *e* The exponential series provides yet another way to define e. When $x = 1$, $e^1 = \sum_{n=0}^{\infty}\frac{1^n}{n!}$ or $\sum_{n=0}^{\infty}\frac{1}{n!}$.

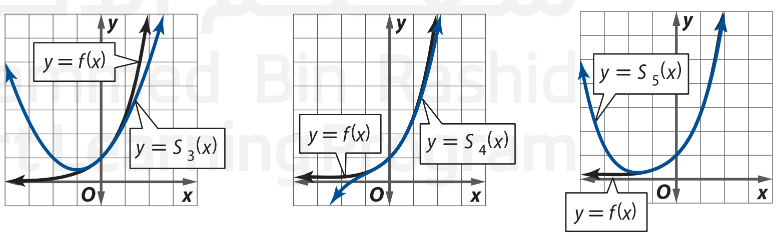
KeyConcept Exponential Series

The power series representing e^x is called the exponential series and is given by

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

which is convergent for all x.

The graph of $f(x) = e^x$ and the partial sums $S_3(x)$, $S_4(x)$, and $S_5(x)$ of the exponential series are shown below.



You can see from the graphs that the partial sums of the exponential series approximate the graph of $f(x) = e^x$ on increasingly wider intervals of the domain for increasingly greater values of n.

Notice that the calculations involved in the exponential series are relatively simple: multiplications (for powers and factorials), divisions, and additions. Because of this, calculators and computer programs use partial sums of the exponential series to evaluate e^x to desired degrees of accuracy.

WatchOut!

Evaluating e^x The fifth partial sum of the exponential series only gives reasonably good approximations of e^x for x on $[-1.5, 2.5]$. Subsequent partial sums, such as the sixth and seventh partial sums, are more accurate for wider intervals of x-values.

Example 2 Exponential Series

Use the fifth partial sum of the exponential series to approximate the value of $e^{1.5}$. Round to three decimal places.

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad e^x \approx \sum_{n=0}^{4} \frac{x^n}{n!}$$

$$e^{1.5} \approx 1 + 1.5 + \frac{1.5^2}{2!} + \frac{1.5^3}{3!} + \frac{1.5^4}{4!} \qquad x = 1.5$$

$$\approx 4.398 \qquad \text{Simplify.}$$

CHECK A calculator, using a partial sum of the exponential series with many more terms, returns an approximation of 4.48 for $e^{1.5}$. Therefore, an approximation of 4.398 is reasonable. ✔

GuidedPractice

Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places.

2A. $e^{-0.75}$

2B. $e^{0.25}$

Math HistoryLink

Mádhava of Sangamagramma (**1340–1425**)
An Indian mathematician born near Cochin, Mádhava discovered the series equivalent to the expansions of sin x, cos x, and arctan x around 1400, two hundred years before their discovery in Europe.

Other transcendental functions have power series representations as well. Calculators and computers use **power series** to approximate the values of cosine and sine functions.

KeyConcept Power Series for Cosine and Sine

The power series representations for cos x and sin x are given by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots, \text{ and}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots,$$

which are convergent for all x.

By replacing x with any angle measure expressed in radians and carrying out the computations, approximate values of the cosine and sine functions can be found to any desired degree of accuracy.

Example 3 Trigonometric Series

a. Use the fifth partial sum of the power series for cosine to approximate the value of cos $\frac{\pi}{7}$. Round to three decimal places.

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \qquad \cos x \approx \sum_{n=0}^{4} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos \frac{\pi}{7} \approx 1 - \frac{(0.449)^2}{2!} + \frac{(0.449)^4}{4!} - \frac{(0.449)^6}{6!} + \frac{(0.449)^8}{8!} \qquad x = \frac{\pi}{7} \text{ or about } 0.449$$

$$\approx 0.901 \qquad \text{Simplify.}$$

CHECK A calculator, using a partial sum of the power series for cosine with many more terms, returns an approximation of 0.901, to three decimal places, for cos $\frac{\pi}{7}$. Therefore, an approximation of 0.901 is reasonable. ✔

StudyTip

Fifth Partial Sum While additional partial sums provide a better approximation, the fifth partial sum typically is accurate to three decimal places.

b. Use the fifth partial sum of the power series for sine to approximate the value of $\sin \frac{\pi}{5}$. Round to three decimal places.

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \qquad \sin x \approx \sum_{n=0}^{5} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin \frac{\pi}{5} \approx 0.628 - \frac{(0.628)^3}{3!} + \frac{(0.628)^5}{5!} - \frac{(0.628)^7}{7!} + \frac{(0.628)^9}{9!} \qquad x = \frac{\pi}{5} \text{ or about } 0.628$$

$$\approx 0.588 \qquad \text{Simplify.}$$

CHECK Using a calculator, $\sin \frac{\pi}{5} \approx 0.588$. Therefore, an approximation of 0.588 is reasonable. ✓

GuidedPractice

Use the fifth partial sum of the power series for cosine or sine to approximate each value. Round to three decimal places.

3A. $\sin \frac{\pi}{11}$

3B. $\cos \frac{2\pi}{17}$

You may have noticed similarities in the power series representations of $f(x) = e^x$ and the power series representations of $f(x) = \sin x$ and $f(x) = \cos x$. A relationship is derived by replacing x by $i\theta$ in the exponential series, where i is the imaginary unit and θ is the measure of an angle in radians.

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \qquad e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \qquad i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ i^7 = -i$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - i\frac{\theta^7}{7!} + \cdots\right) \qquad \text{Group real and imaginary terms.}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \qquad \text{Distributive Property}$$

$$= \cos\theta + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \qquad \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$= \cos\theta + i\sin\theta \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots$$

This relationship is called **Euler's Formula**.

KeyConcept Euler's Formula

For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$.

From your work you should recognize the right-hand side of this equation as being part of the polar form of a complex number. Applying Euler's Formula to the polar form of a complex number yields the following result.

$$a + bi = r(\cos\theta + i\sin\theta) \qquad \text{Polar form of a complex number}$$

$$= re^{i\theta} \qquad \text{Euler's Formula}$$

Therefore, Euler's Formula gives us a way of expressing a complex number in exponential form.

KeyConcept Exponential Form of a Complex Number

The exponential form of a complex number $a + bi$ is given by

$$a + bi = re^{i\theta},$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$ for $a > 0$ and $\theta = \tan^{-1}\frac{b}{a} + \pi$ for $a < 0$.

Example 4 Write a Complex Number in Exponential Form

Write $-\sqrt{3} + i$ in exponential form.

Write the polar form of $-\sqrt{3} + i$. In this expression, $a = -\sqrt{3}$, $b = 1$, and $a < 0$. Find r.

$r = \sqrt{(-\sqrt{3})^2 + 1^2}$ — $r = \sqrt{a^2 + b^2}$

$= \sqrt{4}$ or 2 — Simplify.

Now find θ.

$\theta = \tan^{-1}\frac{1}{-\sqrt{3}} + \pi$ — $\theta = \tan^{-1}\frac{b}{a} + \pi$ for $a < 0$

$= -\frac{\pi}{6} + \pi$ or $\frac{5\pi}{6}$ — Simplify.

Therefore, because $a + bi = re^{i\theta}$, the exponential form of $-\sqrt{3} + i$ is $2e^{i\frac{5\pi}{6}}$.

GuidedPractice

Write each complex number in exponential form.

4A. $1 + \sqrt{3}i$

4B. $\sqrt{2} + \sqrt{2}i$

From your study of logarithms, you know that no *real* number can be the logarithm of a negative number. We can use Euler's Formula to show that the natural logarithm of a negative number does exist in the *complex* number system.

$e^{i\theta} = \cos\theta + i\sin\theta$ — Euler's Formula

$e^{i\pi} = \cos\pi + i\sin\pi$ — Let $\theta = \pi$.

$e^{i\pi} = -1 + i(0)$ — $\cos\pi = -1$ and $\sin\pi = 0$

$e^{i\pi} = -1$ — Simplify.

$\ln e^{i\pi} = \ln(-1)$ — Take the natural logarithm of each side.

$i\pi = \ln(-1)$ — Power Property of Logarithms

This result indicates that the natural logarithm of -1 exists and is the complex number $i\pi$. You can use this result to find the natural logarithm of any negative number $-k$, for $k > 0$.

$\ln(-k) = \ln[(-1)k]$ — $-k = (-1)k$

$= \ln(-1) + \ln k$ — Product Property of Logarithms

$= i\pi + \ln k$ — $\ln(-1) = i\pi$

$= \ln k + i\pi$ — Write in the form $a + bi$.

TechnologyTip

Complex Numbers You can use your calculator to evaluate the natural logarithm of a negative number by changing from REAL to $a + bi$ under MODE.

Example 5 Natural Logarithm of a Negative Number

Find the value of $\ln(-5)$ in the complex number system.

$\ln(-5) = \ln 5 + i\pi$ — $\ln(-k) = \ln k + i\pi$

$\approx 1.609 + i\pi$ — Use a calculator to compute ln 5.

GuidedPractice

Find the value of each natural logarithm in the complex number system.

5A. $\ln(-8)$

5B. $\ln(-6.24)$

Exercises

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$. **Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.** (Example 1)

1. $g(x) = \frac{4}{1-x}$

2. $g(x) = \frac{3}{1-2x}$

3. $g(x) = \frac{2}{1-x^2}$

4. $g(x) = \frac{3}{2-x}$

5. $g(x) = \frac{2}{5-3x}$

6. $g(x) = \frac{4}{3-2x^2}$

Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places. (Example 2)

7. $e^{0.5}$

8. $e^{-0.25}$

9. $e^{-2.5}$

10. $e^{0.8}$

11. $e^{-0.3}$

12. $e^{3.5}$

13. **ECOLOGY** The population density P per square meter of zebra mussels in the Upper Mississippi River can be modeled by $P = 3.5e^{0.08t}$, where t is measured in weeks. Use the fifth partial sum of the exponential series to estimate the zebra mussel population density after 4 weeks, 12 weeks, and 1 year. (Example 2)

Use the fifth partial sum of the power series for cosine or sine to approximate each value. Round to three decimal places. (Example 3)

14. $\sin \frac{\pi}{9}$

15. $\cos \frac{2\pi}{13}$

16. $\sin \frac{5\pi}{13}$

17. $\cos \frac{3\pi}{10}$

18. $\cos \frac{2\pi}{9}$

19. $\sin \frac{3\pi}{19}$

20. **AMUSEMENT PARK** A ride at an amusement park is in the shape of a giant pendulum that swings riders back and forth in a 240° arc to a maximum height of 41 meters. The pendulum is supported by a tower that is 26 meters tall and dips below ground-level into a pit when swinging below the tower. Use the fifth partial sum of the power series for cosine or sine to approximate the length of the pendulum. (Example 3)

Write each complex number in exponential form. (Example 4)

21. $\sqrt{3} + i$

22. $\sqrt{3} - i$

23. $\sqrt{2} - \sqrt{2}i$

24. $-\sqrt{3} - i$

25. $1 - \sqrt{3}i$

26. $-1 + \sqrt{3}i$

27. $-\sqrt{2} + \sqrt{2}i$

28. $-1 - \sqrt{3}i$

Find the value of each natural logarithm in the complex number system. (Example 5)

29. $\ln(-6)$

30. $\ln(-3.5)$

31. $\ln(-2.45)$

32. $\ln(-7)$

33. $\ln(-4.36)$

34. $\ln(-9.12)$

35. **POWER SERIES** Use the power series representations of $\sin x$ and $\cos x$ to answer each of the following questions.

a. Graph $f(x) = \sin x$ and the third partial sum of the power series representing $\sin x$. Repeat for the fourth and fifth partial sums. Describe the interval of convergence for each.

b. Repeat part **a** for $f(x) = \cos x$ and the third, fourth, and fifth partial sums of the power series representing $\cos x$. Describe the interval of convergence for each.

c. Describe how the interval of convergence changes as n increases. Then make a conjecture as to the relationship between each trigonometric function and its related power series as $n \to \infty$.

Solve for z over the complex numbers. Round to three decimal places.

36. $2e^z + 5 = 0$

37. $e^{2z} + 12 = 0$

38. $4e^{2z} + 7 = 6$

39. $3(e^z - 1) + 5 = -2$

40. $e^{2z} - e^z = 2$

41. $10e^{2z} + 17e^z = -3$

42. **ECONOMICS** The total value of an investment of P dirhams compounded continuously at an annual interest rate of r over t years is Pe^{rt}. Use the first five terms of the exponential series to approximate the value of an investment of AED 10,000 compounded continuously at 5.25% for 5 years.

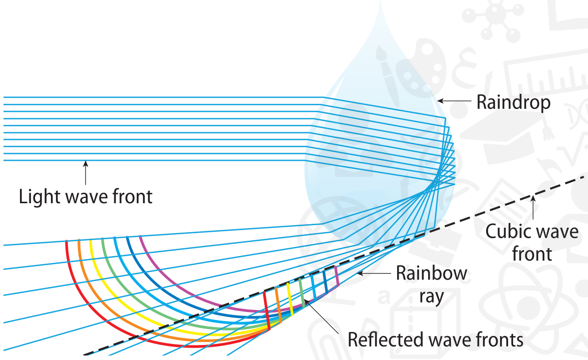
43. **RELATIVE ERROR** *Relative error* is the absolute error in estimating a quantity divided by its true value. The relative error of an approximation a of a quantity b is given by $\frac{|b-a|}{b}$. Find the relative error in approximating $e^{2.1}$ using two, three, and six terms of the exponential series.

Approximate the value of each expression using the first four terms of the power series for sine and cosine. Then find the expected value of each.

44. $\sin^2 \frac{1}{2} + \cos^2 \frac{1}{2}$

45. $\sec^2 1 - \tan^2 1$

46. **RAINBOWS** *Airy's equation*, which is used in physics to model the diffraction of light, can also be used to explain how a light wave front is converted into a curved wave front in forming rainbows.

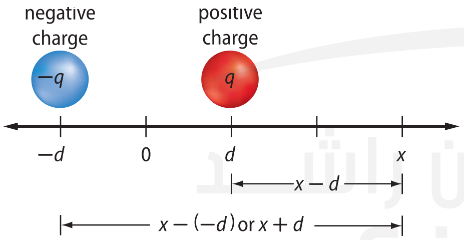


This equation can be represented by the power series shown below.

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2 \cdot 3)(5 \cdot 6) \cdots [(3k-1) \cdot (3k)]}$$

Use the fifth partial sum of the series to find $f(3)$. Round to the nearest hundredth.

47. **ELECTRICITY** When an electric charge is accompanied by an equal and opposite charge nearby, such an object is called an *electric dipole*. It consists of charge q at the point $x = d$ and charge $-q$ at $x = -d$, as shown below.



Along the x-axis, the electric field strength at x is the sum of the electric fields from each of the two charges. This is given by $E(x) = \frac{kq}{(x-d)^2} - \frac{kq}{(x+d)^2}$. Find a power series representing $E(x)$ if k is a constant and $d = 1$.

48. **SOUND** The *Fourier Series* represents a periodic function of time $f(t)$ as a summation of sine waves and cosine waves with frequencies that start at 0 and increase by integer multiples. The series below represents a sound wave from the digital data fed from a CD into a CD player.

$$f(t) = 0.7 + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n} \cos 270.6nt + \frac{1}{2n-1} \sin 270.6nt \right)$$

Graph the series for $n = 4$. Then analyze the graph.

IDENTITIES **Use power series representations from this lesson to verify each trigonometric identity.**

49. $\sin(-x) = -\sin x$

50. $\cos(-x) = \cos x$

51. **APPROXIMATIONS** The infinite series for the inverse tangent function $f(x) = \tan^{-1} x$, is given by $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$. However, this series is only valid for values of x on the interval $(-1, 1)$.

a. Write the first five terms of the infinite series representation for $f(x) = \tan^{-1} x$.

b. Use the first five terms of the series to approximate $\tan^{-1} 0.1$.

c. On the same coordinate plane, graph $f(x) = \tan^{-1} x$ and the third partial sum of the power series representing $f(x) = \tan^{-1} x$. On another coordinate plane, graph $f(x)$ and the fourth partial sum. Then graph $f(x)$ and the fifth partial sum.

d. Describe what happens on the interval $(-1, 1)$ and in the regions $x \geq 1$ or $x \leq -1$.

H.O.T. Problems Use Higher-Order Thinking Skills

52. **WRITING IN MATH** Describe how using additional terms in the approximating series for e^x affects the outcome.

53. **REASONING** Use the power series for sine to explain why, for x-values on the interval $[-0.1, 0.1]$, a close approximation of $\sin x$ is x.

54. **CHALLENGE** Prove that $2 \sin \theta \cos \theta = \frac{e^{2\theta i} - e^{-2\theta i}}{2i}$

55. **REASONING** For what values of α and β does $e^{i\alpha} = e^{i\beta}$? Explain.

PROOF **Show that for all real numbers x, the following are true.**

56. $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

57. $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

58. **CHALLENGE** The hyperbolic sine and hyperbolic cosine functions are analogs of the trigonometric functions. Just as the points $(\cos x, \sin x)$ form a unit circle, the points $(\cosh t, \sinh t)$ form the right half of an equilateral hyperbola. An equilateral hyperbola has perpendicular asymptotes. The hyperbolic sine (sinh) and hyperbolic cosine (cosh) functions are defined below. Find the power series representations for these functions.

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\qquad$ $\cosh x = \frac{e^x + e^{-x}}{2}$

Spiral Review

Use Pascal's triangle to expand each binomial.

59. $\left(3m + \sqrt{2}\right)^4$

60. $\left(\frac{1}{2}n + 2\right)^5$

61. $(p^2 + q)^8$

62. Prove that $4 + 7 + 10 + \cdots + (3n + 1) = \frac{n(3n + 5)}{2}$ for all positive integers n.

Find each power, and express it in rectangular form.

63. $(-2 + 2i)^3$

64. $\left(1 + \sqrt{3}i\right)^4$

65. $\left(\sqrt{2} + \sqrt{2}i\right)^{-2}$

66. Given $\mathbf{t} = \langle -9, -3, c\rangle$, $\mathbf{u} = \langle 8, -4, 3\rangle$, $\mathbf{v} = \langle 2, 5, -6\rangle$, and that the volume of the parallelepiped having adjacent edges **t**, **u**, and **v** is 93 cubic units, find c.

Use an inverse matrix to solve each system of equations, if possible.

67. $x - 8y = -7$
$2x + 5y = 28$

68. $4x + 7y = 22$
$-9x + 11y = 4$

69. $w + 2x + 3y = 18$
$4w - 8x + 7y = 41$
$-w + 9x - 2y = -4$

Determine whether A and B are inverse matrices.

70. $A = \begin{bmatrix} 1 & -2 \\ 7 & -6 \end{bmatrix}, B = \begin{bmatrix} -6 & 2 \\ -7 & 1 \end{bmatrix}$

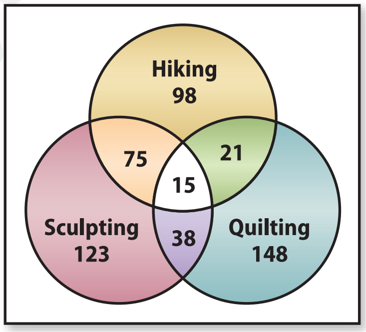
71. $A = \begin{bmatrix} -11 & -5 \\ 9 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ -9 & -11 \end{bmatrix}$

72. $A = \begin{bmatrix} 6 & 2 \\ -2 & 8 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ -3 & -5 \end{bmatrix}$

73. CONFERENCE A university sponsored a conference for 680 women. The Venn diagram shows the number of participants in three of the activities offered. Suppose women who attended the conference were randomly selected for a survey.

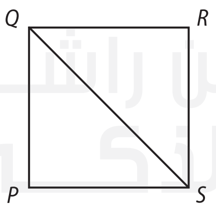
a. What is the probability that a woman selected participated in hiking or sculpting?

b. Describe a set of women such that the probability of being selected is about 0.39.



Skills Review for Standardized Tests

74. SAT/ACT $PQRS$ is a square. What is the ratio of the length of diagonal $\overline{QS}$ to the length of side $\overline{RS}$?



A 2

B $\sqrt{2}$

C 1

D $\frac{\sqrt{2}}{2}$

E $\frac{\sqrt{3}}{2}$

75. REVIEW What is the sum of the infinite geometric series $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \cdots$?

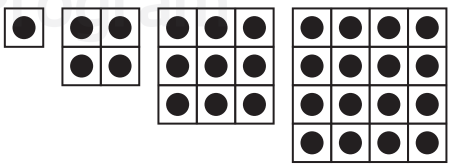
F $\frac{2}{3}$

G 1

H $1\frac{1}{3}$

J $1\frac{2}{3}$

76. FREE RESPONSE Consider the pattern of dots shown.



a. Draw the next figure in this sequence.

b. Write the sequence, starting with 1, that represents the number of dots that must be added to each figure in the sequence to get the number of dots in the next figure.

c. Find the expression for the nth term of the sequence found in part **b**.

d. Find the expression for the number of dots in the nth figure in the original sequence.

e. Prove, through mathematical induction, that the sum of the sequence found in part **b** is equal to the expression found in part **d**.

EXTEND

9-9 Spreadsheet Lab
Detecting Patterns in Data

Objective

- Organize and display data using spreadsheets to detect patterns and departures from patterns.

In Chapter 9, you learned how to detect patterns in a sequence and describe them by using functions.

Pattern in Data Sequence	Pattern in Graph of Data Sequence	Type of Sequence	Function Describing Sequence
common 1st differences	data in a linear pattern	arithmetic	linear
common ratio	data in an exponential pattern	geometric	exponential

In this lab, you will use a spreadsheet to organize and display paired data in order to look for such patterns.

Activity 1 Detect Patterns

DOGS **A certain sheep had a mass of 1.45 kilograms at birth. The table shows the sheep's mass in the first 70 days of its growth. Use a spreadsheet to find a pattern in the data.**

Days after Birth	10	20	30	40	50	60	70
Mass (kg)	1.91	2.46	3.17	4.10	5.22	6.81	8.63

Step 1 Enter the data into the spreadsheet.

Step 2 To determine if the sequence of masses is arithmetic, enter a formula in the next column to find the average daily rate of change in the sheep's mass.

Step 3 To determine if the sequence is geometric, enter the formula shown in the next column to find the average ratio of change in the sheep's mass each day.

◇	A	B	C	D
1	Days after Birth	Mass (kg)	Average Rate of Change	Average Ratio of Change
2	0	1.45		
3	10	1.91	=(B3-B2)/(A3-A2)	=(B3/B2)^(1/(A3-A2))-1
4	20	2.46		
5	30	3.17		
6	40	4.1		
7	50	5.22		
8	60	6.81		
9	70	8.63		

Sheet 1 | Sheet 2 | Sheet 3

2. There appears to be a pattern in the average ratio of change between consecutive pairs of data. These values cluster around a common average ratio of 0.026. This suggests that the sequence of mass values is geometric.

Analyze the Results

1. Explain the formulas used in the spreadsheet.
2. Describe any pattern you see in the data. What type of sequence approximates the data? Explain.
3. Use the chart tool to create a scatter plot of the data. Does this graph support your answer to Exercise 2? Explain.
4. Write an equation approximating the sheep's mass y after x days.
5. Use your equation to predict the sheep's mass 25 days after birth and 365 days after birth. Are these predictions reasonable? Explain.

You can also use a spreadsheet to detect and analyze departures from patterns.

Activity 2 Detect Departures from Patterns

HOMEWORK **Omar recorded the number of precalculus problems and how long he worked on them for eight nights. Look for a pattern in the data and any departures from that pattern.**

Number of Problems	0	3	5	6	8	9	10	15
Time (min)	0	27	44	70	72	82	95	140

Step 1 Enter the data into the spreadsheet.

Step 2 Enter formulas in the adjacent columns to detect whether the sequence of is arithmetic or geometric. Then copy these formulas into the cells below.

Step 3 Look for patterns. Notice that all but two of the rates of change cluster around 9.

	A	B	C
1	Number of Problems	Time (min)	Average Rate of Change
2	0	0	
3	3	27	9.00
4	5	44	8.50
5	6	70	26.00
6	8	72	1.00
7	9	81	9.00
8	10	89	8.00
9	15	136	9.40

Sheet 1 | Sheet 2 | Sheet 3

StudyTip

Series in Data To investigate series in data, you can use the Auto Sum tool. For Activity 2, enter =B2 in cell D2 and =SUM(B2,B3) in cell D3. Copy this second formula into the remaining cells in the column to create a sequence of partial sums.

Analyze the Results

6. Where does the departure in the pattern occur?
7. Write a spreadsheet formula that could model the data if this data value were removed.
8. Create a scatter plot that shows the actual data and the model of the data. Does this graph support your answer to Exercise 7? Explain.
9. Use your formula from Exercise 7 to predict how long it would take Omar to complete 12 problems and 20 problems. Are these predictions reasonable? Explain.

Exercises

Use a spreadsheet to organize and identify a pattern or departure from a pattern in each set of data. Then use a calculator to write an equation to model the data.

10. **INTERNET** The table shows the number of times the main page of a popular blog is read (hits) each month.

Month	2	4	6	8	10	12	15	20
Hits	83	171	266	368	479	732	1405	4017

11. **COLLEGE** The table shows the composite ACT scores and grade-point averages (GPA) of 20 students after their first semester in college. (*Hint:* First use the Sort Ascending tool to organize the data.)

ACT Score	27	16	15	22	20	21	25
College GPA	3.9	2.9	2.7	3.6	3.2	3.4	3.1
ACT Score	26	18	23	19	29	28	17
College GPA	4.0	3.1	3.6	2.6	4.0	3.9	3.0

Study Guide and Review

Study Guide

KeyConcepts

Arithmetic Sequences and Series (Lessons 9-1 and 9-2)

- The nth term a_n of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$, where n is any positive integer.
- The sum S_n of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$.

Geometric Sequences and Series (Lessons 9-3 and 9-4)

- The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1 \cdot r^{n-1}$, where n is any positive integer.
- The sum S_n of the first n terms of a geometric series is given by $S_n = \frac{a_1(1 - r^n)}{1 - r}$ or $S_n = \frac{a_1 - a_1 r^n}{1 - r}$, where $r \neq 1$.
- The sum S of an infinite geometric series with $-1 < r < 1$ is given by $S_n = \frac{a_1}{1 - r}$.

Recursion and Iteration (Lesson 9-5)

- In a recursive formula, each term is formulated from one or more previous terms.

The Binomial Theorem (Lesson 9-6)

- The Binomial Theorem:

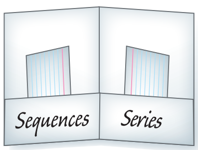
$$(a + b)^n = \sum_{k=0}^{n} \frac{n!}{(n - k)!\,k!} a^{n-k} b^k$$

Mathematical Induction (Lesson 9-7)

- Mathematical induction is a method of proof used to prove statements about the positive integers.

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.



KeyVocabulary

arithmetic means
arithmetic sequence
arithmetic series
common difference
common ratio
convergent series
divergent series
explicit formula
Fibonacci sequence
finite sequence
geometric means
geometric sequence
geometric series
induction hypothesis
infinite geometric series
infinite sequence
infinity
iteration
mathematical induction
partial sum
Pascal's triangle
recursive formula
recursive sequence
sequence
series
sigma notation
term

VocabularyCheck

State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.

1. An infinite geometric series that has a sum is called a convergent series.
2. Mathematical induction is the process of repeatedly composing a function with itself.
3. The arithmetic means of a sequence are the terms between any two non-successive terms of an arithmetic sequence.
4. A term is a list of numbers in a particular order.
5. The sum of the first n terms of a series is called the partial sum.
6. The formula $a_n = a_{n-2} + a_{n-1}$ is a recursive formula.
7. A geometric sequence is a sequence in which every term is determined by adding a constant value to the previous term.
8. An infinite geometric series that does not have a sum is called a partial sum.
9. Eleven and 17 are two geometric means between 5 and 23 in the sequence 5, 11, 17, 23.
10. Using the Binomial Theorem, $(x - 2)^4$ can be expanded to $x^4 - 8x^3 + 24x^2 - 32x + 16$.

Lesson-by-Lesson Review

9-1 Sequences as Functions

Find the indicated term of each arithmetic sequence.

11. $a_1 = 9, d = 3, n = 14$

12. $a_1 = -3, d = 6, n = 22$

13. $a_1 = 10, d = -4, n = 9$

14. $a_1 = -1, d = -5, n = 18$

Example 1

Find the 11th term of an arithmetic sequence if $a_1 = -15$ and $d = 6$.

$a_n = a_1 + (n - 1)d$	Formula for the nth term
$a_{11} = -15 + (11 - 1)6$	$n = 11, a_1 = -15, d = 6$
$a_{11} = 45$	Simplify.

Study Guide and Review *Continued*

9-2 Arithmetic Sequences and Series

Find the arithmetic means in each sequence.

15. −12, __, __, __, 8

16. 15, __, __, 29

17. 12, __, __, __, __, −8

18. 72, __, __, __, 24

19. **BANKING** Zayed saves AED 150 every 2 months. If he saves at this rate for two years, how much will he have at the end of two years?

Find S_n for each arithmetic series.

20. $a_1 = 16$, $a_n = 48$, $n = 6$

21. $a_1 = 8$, $a_n = 96$, $n = 20$

22. $9 + 14 + 19 + \cdots + 74$

23. $16 + 7 + -2 + \cdots + -65$

24. **DRAMA** Laila has a drama performance in 12 days. She plans to practice her lines each night. On the first night she rehearses her lines 2 times. The next night she rehearses her lines 4 times. The third night she rehearses her lines 6 times. On the eleventh night, how many times has she rehearsed her lines?

Find the sum of each arithmetic series.

25. $\sum_{k=5}^{21}(3k - 2)$

26. $\sum_{k=0}^{10}(6k - 1)$

27. $\sum_{k=4}^{12}(-2k + 5)$

Example 2

Find the two arithmetic means between 3 and 39.

$a_n = a_1 + (n - 1)d$ — Formula for the nth term

$a_4 = 3 + (4 - 1)d$ — $n = 4$, $a_1 = 3$

$39 = 3 + 3d$ — $a_4 = 39$

$12 = d$ — Simplify.

The arithmetic means are 3 + 12 or 15 and 15 + 12 or 27.

Example 3

Find S_n for the arithmetic series with $a_1 = 18$, $a_n = 56$, and $n = 8$.

$S_n = \frac{n}{2}(a_1 + a_n)$ — Sum formula

$S_8 = \frac{8}{2}(18 + 56)$ — $n = 8$, $a_1 = 18$, $a_n = 56$

$= 296$ — Simplify.

Example 4

Evaluate $\sum_{k=3}^{15} 5k + 1$.

Use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. There are 13 terms, $a_1 = 5(3) + 1$ or 16, and $a_{13} = 5(15) + 1$ or 76.

$S_{13} = \frac{13}{2}(16 + 76)$

$= 598$

9-3 Geometric Sequences and Series

Find the indicated term for each geometric sequence.

28. $a_1 = 5, r = 2, n = 7$

29. $a_1 = 11, r = 3, n = 3$

30. $a_1 = 128, r = -\frac{1}{2}, n = 5$

31. a_8 for $\frac{1}{8}, \frac{3}{8}, \frac{9}{8}, \dots$

Find the geometric means in each sequence.

32. 6, __, __, 162

33. 8, __, __, __, 648

34. −4, __, __, 108

35. **SAVINGS** Najat has a savings account with a current balance of AED 1,500. What would be Najat's account balance after 4 years if he receives 5% interest (murabaha) annually?

Find S_n for each geometric series.

36. $a_1 = 15, r = 2, n = 4$

37. $a_1 = 9, r = 4, n = 6$

38. $5 - 10 + 20 - \cdots$ to 7 terms

39. $243 + 81 + 27 + \cdots$ to 5 terms

Evaluate the sum of each geometric series.

40. $\sum_{k=1}^{7} 3 \cdot (-2)^{k-1}$

41. $\sum_{k=1}^{8} -1\left(\frac{2}{3}\right)^{k-1}$

42. **ADVERTISING** Nabila is handing out fliers to advertise the next student council meeting. She hands out fliers to 4 people. Then, each of those 4 people hand out 4 fliers to 4 other people. Those 4 then hand out 4 fliers to 4 new people. If Nabila is considered the first round, how many people will have been given fliers after 4 rounds?

Example 5

Find the sixth term of a geometric sequence for which $a_1 = 9$ and $r = 4$.

$a_n = a_1 \cdot r^{n-1}$ — Formula for the nth term

$a_6 = 9 \cdot 4^{6-1}$ — $n = 6, a_1 = 9, r = 4$

$a_6 = 9216$

The sixth term is 9216.

Example 6

Find two geometric means between 1 and 27.

$a_n = a_1 \cdot r^{n-1}$ — Formula for the nth term

$a_4 = 1 \cdot r^{4-1}$ — $n = 4$ and $a_1 = 1$

$27 = r^3$ — $a_4 = 27$

$3 = r$ — Simplify.

The geometric means are 1(3) or 3 and 3(3) or 9.

Example 7

Find the sum of a geometric series for which $a_1 = 3$, $r = 5$, and $n = 11$.

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ — Sum formula

$S_{11} = \frac{3 - 3 \cdot 5^{11}}{1 - 5}$ — $n = 11, a_1 = 3, r = 5$

$S_{11} = 36,621,093$ — Use a calculator.

Example 8

Evaluate $\sum_{k=1}^{6} 2 \cdot (4)^{k-1}$.

$S_6 = \frac{2 - 2 \cdot 4^6}{1 - 4}$ — $n = 6, a_1 = 2, r = 4$

$= \frac{-8190}{-3}$ — Simplify.

$= 2730$ — Simplify.

Study Guide and Review *Continued*

9-4 Infinite Geometric Series

Find the sum of each infinite series, if it exists.

43. $a_1 = 8, r = \frac{3}{4}$

44. $\frac{5}{6} - \frac{20}{18} + \frac{80}{54} - \frac{320}{162} + \ldots$

45. $\sum_{k=1}^{\infty} 3\left(\frac{1}{2}\right)^{k-1}$

46. PHYSICAL SCIENCE Maysoun drops a ball off of a building that is 20 meters high. Each time the ball bounces, it bounces back to $\frac{2}{3}$ its previous height. If the ball continues to follow this pattern, what will be the total distance that the ball travels?

Example 9

Find the sum of the infinite geometric series for which $a_1 = 15$ and $r = \frac{1}{3}$.

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{15}{1 - \frac{1}{3}}$ $a_1 = 15, r = \frac{1}{3}$

$= \frac{15}{\frac{2}{3}}$ or 22.5   Simplify.

9-5 Recursion and Iteration

Find the first five terms of each sequence.

47. $a_1 = -3, a_{n+1} = a_n + 4$

48. $a_1 = 5, a_{n+1} = 2a_n - 5$

49. $a_1 = 1, a_{n+1} = a_n + 5$

50. SAVINGS Shaikha has a savings account with a AED 12,000 balance. She has a 5% interest (murabaha) rate that is compounded monthly. Every month Shaikha adds AED 500 to the account. The recursive formula $b_n = 1.05b_{n-1} + 500$ describes the balance in Shaikha's savings account after n months. Find the balance of Shaikha's account after 3 months. Round your answer to the nearest fil.

Find the first three iterates of each function for the given initial value.

51. $f(x) = 2x + 1, x_0 = 3$

52. $f(x) = 5x - 4, x_0 = 1$

53. $f(x) = 6x - 1, x_0 = 2$

54. $f(x) = 3x + 1, x_0 = 4$

Example 10

Find the first five terms of the sequence in which $a_1 = 1$, $a_{n+1} = 3a_n + 2$.

$a_{n+1} = 3a_n + 2$	Recursive formula
$a_{1+1} = 3a_1 + 2$	$n = 1$
$a_2 = 3(1) + 2$ or 5	$a_1 = 1$
$a_{2+1} = 3a_2 + 2$	$n = 2$
$a_3 = 3(5) + 2$ or 17	$a_2 = 5$
$a_{3+1} = 3a_3 + 2$	$n = 3$
$a_4 = 3(17) + 2$ or 53	$a_3 = 17$
$a_{4+1} = 3a_4 + 2$	$n = 4$
$a_5 = 3(53) + 2$ or 161	$a_4 = 53$

The first five terms of the sequence are 1, 5, 17, 53, and 161.

Example 11

Find the first three iterates of the function $f(x) = 3x - 2$ for the initial value of $x_0 = 2$.

$x_1 = f(x_0)$	$x_2 = f(x_1)$	$x_3 = f(x_2)$
$= f(2)$	$= f(4)$	$= f(10)$
$= 3(2) - 2$	$= 3(4) - 2$	$= 3(10) - 2$
$= 4$	$= 10$	$= 28$

The first three iterates are 4, 10, and 28.

9-6 The Binomial Theorem

Expand each binomial.

55. $(a + b)^3$

56. $(y - 3)^7$

57. $(3 - 2z)^5$

58. $(4a - 3b)^4$

59. $\left(x - \frac{1}{4}\right)^5$

Find the indicated term of each expression.

60. third term of $(a + 2b)^8$

61. sixth term of $(3x + 4y)^7$

62. second term of $(4x - 5)^{10}$

Example 12

Expand $(x - 3y)^4$.

$$\begin{aligned}(x - 3y)^4 &= x^4 + {}_4C_1x^3(-3y) + {}_4C_2x^2(-3y)^2 + {}_4C_3(-3y)^4 + {}_4C_4(-3y)^4 \\ &= x^4 + \frac{4!}{3!}x^3(-3y) + \frac{4!}{2!2!}x^2(9y^2) + \frac{4!}{3!}x(-27y^3) + 81y^4 \\ &= x^4 + -12x^3y + 54x^2y^2 + -108xy^3 + 81y^4\end{aligned}$$

Example 13

Find the fourth term of $(x + y)^8$.

Use the Binomial Theorem to write the expansion in sigma notation.

$$(x + y)^8 = \sum_{k=0}^{8} \frac{8!}{k!(8-k)!}x^{8-k}y^k$$

For the fourth term, $k = 3$.

$$\frac{8!}{k!(8-k)!}x^{8-k}y^k = \frac{8!}{3!(8-3)!}x^{8-3}y^3$$

$$= 56x^5y^3$$

9-7 Proof by Mathematical Induction

Prove that each statement is true for all positive integers.

63. $2 + 6 + 12 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$

64. $7^n - 1$ is divisible by 6.

65. $5^n - 1$ is divisible by 4.

Find a counterexample for each statement.

66. $8^n + 3$ is divisible by 11.

67. $6^{n+1} - 2$ is divisible by 17.

68. $n^2 + 2n + 4$ is prime.

69. $n + 19$ is prime.

Example 14

Prove that $9^n + 3$ is divisible by 4.

Step 1 When $n = 1$, $9^n + 3 = 9^1 + 3$ or 12. Since 12 divided by 4 is 3, the statement is true for $n = 1$.

Step 2 Assume that $9^k + 3$ is divisible by 4 for some positive integer k. This means that $9^k + 3 = 4r$ for some whole number r.

Step 3

$$\begin{aligned}9^k + 3 &= 4r \\ 9^k &= 4r - 3 \\ 9^{k+1} &= 36r - 27 \\ 9^{k+1} + 3 &= 36r - 27 + 3 \\ 9^{k+1} + 3 &= 36r - 24 \\ 9^{k+1} + 3 &= 4(9r - 6)\end{aligned}$$

Since r is a whole number, $9r - 6$ is a whole number. Thus, $9^{k+1} + 3$ is divisible by 4, so the statement is true for $n = k + 1$.

Therefore, $9^n + 3$ is divisible by 4 for all positive integers n.

CHAPTER 9

Study Guide and Review

9-9 Functions as Infinite Series

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$.

Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the 6th partial sum of its power series.

42. $g(x) = \frac{1}{1 - 5x}$

43. $g(x) = \frac{3}{1 - 2x}$

Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places.

44. $e^{\frac{1}{4}}$

45. $e^{-1.5}$

Find the value of each natural logarithm in the complex number system.

46. $\ln(-4)$

47. $\ln(-7.15)$

Example 6

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x) = \frac{4}{1-x}$. Indicate the interval on which the series converges.

A geometric series converges to $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Replace x with $\frac{x+3}{4}$ since $g(x)$ is a transformation of $f(x)$ and: $g(x) = f\left(\frac{x+3}{4}\right)$. The result is

$$f\left(\frac{x+3}{4}\right) = \sum_{n=0}^{\infty}\left(\frac{x+3}{4}\right)^n \text{ for } \left|\frac{x+3}{4}\right| < 1.$$

Therefore, $g(x) = \frac{4}{1-x}$ can be represented by $\sum_{n=0}^{\infty}\left(\frac{x+3}{4}\right)^n$. This series converges for $\left|\frac{x+3}{4}\right| < 1$, which is equivalent to $-1 < \frac{x+3}{4} < 1$ or $-7 < x < 1$.

CHAPTER 9 Practice Test

1. Find the next 4 terms of the arithmetic sequence 81, 72, 63, ...

2. Find the 25th term of an arithmetic sequence for which $a_1 = 9$ and $d = 5$.

3. **MULTIPLE CHOICE** What is the eighth term in the arithmetic sequence that begins 18, 20.2, 22.4, 24.6, ...?

   A 26.8

   B 29

   C 31.2

   D 33.4

4. Find the four arithmetic means between −9 and 11.

5. Find the sum of the arithmetic series for which $a_1 = 11$, $n = 14$, and $a_n = 22$.

6. **MULTIPLE CHOICE** What is the next term in the geometric sequence below?

$$10, \frac{5}{2}, \frac{5}{8}, \frac{5}{32} \ldots$$

   F $\frac{5}{8}$

   G $\frac{5}{32}$

   H $\frac{5}{128}$

   J $\frac{5}{256}$

7. Find the three geometric means between 6 and 1536.

8. Find the sum of the geometric series for which $a_1 = 15$, $r = \frac{2}{3}$, and $n = 5$.

Find the sum of each series, if it exists.

9. $\sum_{k=2}^{12} (3k - 1)$

10. $\sum_{k=1}^{\infty} \frac{1}{2}(3^k)$

11. $45 + 37 + 29 + \cdots + -11$

12. $\frac{1}{8} + \frac{2}{24} + \frac{4}{72} + \cdots$

13. Write $0.\overline{65}$ as a fraction.

Find the first five terms of each sequence.

14. $a_1 = -1$, $a_{n+1} = 3a_n + 5$

15. $a_1 = 4$, $a_{n+1} = a_n + n$

16. **MULTIPLE CHOICE** What are the first 3 iterates of $f(x) = -5x + 4$ for an initial value of $x_0 = 3$?

   A 3, −11, 59

   B −11, 59, −291

   C −1, −6, −11

   D 59, −291, 1459

17. Expand $(2a - 3b)^4$.

18. What is the coefficient of the fifth term of $(m + 3n)^6$?

19. Find the fourth term of the expansion of $(c + d)^9$.

Prove that each statement is true for all positive integers.

20. $1 + 6 + 36 + \cdots + 6^{n-1} = \frac{1}{5}(6^n - 1)$.

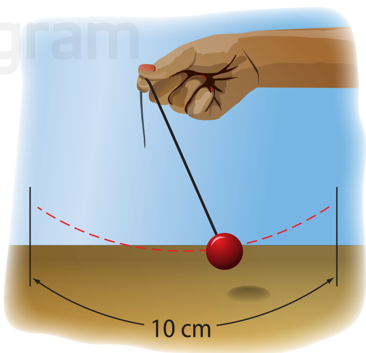
21. $11^n - 1$ is divisible by 10.

22. Find a counterexample for the following statement.

$$2^n + 4^n \text{ is divisible by } 4.$$

23. **SCHOOL** There are an equal number of 15 and 16 year old students in Mr. Khalid's science class. He needs to choose 8 students to represent his class at the science fair. What is the probability that 5 are 15 years old?

24. **PENDULUM** Laila swings a pendulum. The distance traveled per swing decreases by 15% with each swing. If the pendulum initially traveled 10 centimeters, find the total distance traveled when the pendulum comes to a rest.



CHAPTER 9

Preparing for Standardized Tests

Look For a Pattern

One of the most common problem-solving strategies is to look for a pattern. The ability to recognize patterns, model them algebraically, and extend them is a valuable problem-solving tool.

Strategies for Looking For a Pattern

Step 1

Identify the pattern.

- Compare the numbers, shapes, or graphs in the pattern.
- **Ask yourself:** How are the terms of the pattern related?
- **Ask yourself:** Are there any common operations that lead from one term to the next?

Step 2

Generalize the pattern.

- Write a rule using words to describe how the terms of the pattern are generated.
- Assign variables and write an algebraic expression to model the pattern if appropriate.

Step 3

Find missing terms, extend the pattern, and solve the problem.

- Use your pattern or your rule to finding missing terms and/or extend the pattern to solve the problem.
- Check your answer to make sure it makes sense.

Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

Use the sequence of squares shown. How many squares will be needed to make the ninth figure of the sequence?

A 55 C 74

B 65 D 82

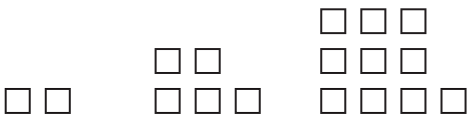
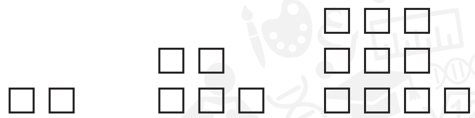


Figure 1 Figure 2 Figure 3

Read the problem statement carefully. You are given three figures of a sequence and asked to find how many squares will be needed to make the ninth figure.

Look for a pattern in the figures of squares. Count the number of squares in each figure.



Write an expression to model this pattern.

Words The number of squares is equal to the square of the figure number plus one.

Variable Let n represent the figure number.

Equation $a_n = n^2 + 1$

Use your expression to extend the pattern and find the number of squares in the ninth figure.

$a_9 = 9^2 + 1 = 82$

So, the ninth figure will have 82 squares. The correct answer is D.

Exercises

Read each problem. Use a pattern to solve the problem.

1. The numbers below form a famous mathematical sequence of numbers known as the Fibonacci sequence. What is the next Fibonacci number in the sequence?

1, 1, 2, 3, 5, 8, 13, 21, ...

A 36

B 34

C 31

D 29

2. What is the missing number in the table?

n	a_n
1	0
2	2
3	6
4	12
5	??
6	30

F 17

G 18

H 20

J 21

CHAPTER 9

Standardized Test Practice

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Find the next term of the arithmetic sequence.

7, 13, 19, 25, 31, ...

A 36
B 37
C 38
D 39

2. Maha gets an enlargement of a 10 centimeter by 15 centimeter picture so that the new print has dimensions that are 4 times the dimensions of her original. How does the area of the enlargement compare to the area of the original picture?

F The area is twice as large.
G The area is four times as large.
H The area is eight times as large.
J The area is sixteen times as large.

3. Evaluate $\sum_{k=1}^{15}(8k - 1)$.

A 119
B 826
C 945
D 1072

4. What is the effect on the graph of the equation $y = 3x^2$ when the equation is changed to $y = 2x^2$?

F The graph of $y = 2x^2$ is a reflection of the graph of $y = 3x^2$ across the y-axis.
G The graph is rotated 90 degrees about the origin.
H The graph is narrower.
J The graph is wider.

5. Write the formula for the nth term of the geometric sequence shown in the table.

n	a_n
1	5
2	10
3	20
4	40
5	80

A $a_n = (5)^n$
B $a_n = 5(2)^{n-1}$
C $a_n = 2(5)^{n-1}$
D $a_n = 5(2)^n$

6. The table shows a dimension of a square tent and the number of people that the tent can fit.

Length of Tent (meters)	Number of People
2	7
5	28
6	39
8	67
12	147

Let ℓ represent the length of the tent and n represent the number of people that can fit in the tent. Identify the equation that best represents the relationship between the length of the tent and the number of people that can fit in the tent.

F $\ell = n^2 + 3$
G $n = \ell^2 + 3$
H $\ell = 3n + 1$
J $n = 3\ell + 1$

7. An air filter claims to remove 90% of the contaminants in the air each time air is circulated through the filter. If the same volume of air is circulated through the filter three times, what percent of the original contaminants will be removed from the air?

A 0.01%
B 0.1%
C 99.0%
D 99.9%

8. At the movies, the cost of 2 boxes of popcorn and 1 soft drink is AED 34.50. The cost of 3 boxes of popcorn and 4 soft drinks is AED 81.75. Which pair of equations can be used to determine p, the cost of a box of popcorn, and s, the cost of a soft drink?

F $2p + s = 81.75$
$3p + 4s = 34.50$

G $2p - s = 34.50$
$3p - 4s = 81.75$

H $2p + s = 34.50$
$3p + 4s = 81.75$

J $p + s = 34.50$
$p + 4 = 81.75$

9. Which of the following geometric series does *not* converge to a sum?

A $\sum_{k=1}^{\infty} 4 \cdot \left(\frac{9}{10}\right)^{k-1}$

B $\sum_{k=1}^{\infty} \frac{1}{5} \cdot \left(\frac{3}{2}\right)^{k-1}$

C $\sum_{k=1}^{\infty} \frac{7}{6} \cdot \left(\frac{1}{3}\right)^{k-1}$

D $\sum_{k=1}^{\infty} (-2) \cdot \left(\frac{5}{6}\right)^{k-1}$

Test-Taking Tip

Question 9 Understand the terms used in Algebra and how to apply them. A geometric series converges to a sum if the common ratio r has an absolute value less than 1.

Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

10. What are the dimensions of the matrix that results from the multiplication shown?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$$

11. **GRIDDED RESPONSE** Consider the pattern below. Into how many pieces will the sixth figure of the pattern be divided?

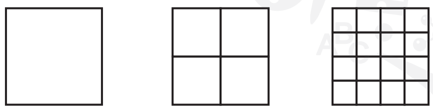
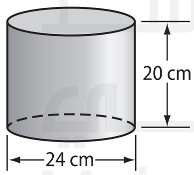


Figure 1 1 piece | Figure 2 4 pieces | Figure 3 16 pieces

12. Use the Binomial Theorem to expand the expression $(c + d)^6$.

13. **GRIDDED RESPONSE** Suhaila has a cylindrical container that she needs to fill with dirt so she can plant some flowers.



What is the volume of the cylinder in cubic centimeters rounded to the nearest cubic centimeter?

14. Bacteria in a culture are growing exponentially with time, as shown in the table.

Hours	Bacteria
0	1000
1	2000
2	4000

Write an equation to express the number of bacteria, y, with respect to time, t.

15. **GRIDDED RESPONSE** What is the value of $f[g(6)]$ if $f(x) = 2x + 4$ and $g(x) = x^2 + 5$?

Extended Response

Record your answers on a sheet of paper. Show your work.

16. Prove that the sum of any two odd integers is even.

17. The endpoints of a diameter of a circle are at $(-1, 0)$ and $(5, -8)$.

a. What are the coordinates of the center of the circle? Explain your method.

b. Find the radius of the circle. Explain your method.

c. Write an equation of the circle.

18. A cyclist travels from Dubai to Sharjah in 2.5 hours. If she increases her speed, she can make the trip in 2 hours.

a. Does this situation represent a direct or inverse variation? Explain your reasoning.

b. If the trip from Dubai to Sharjah takes 2.5 hours when traveling at 12 kilometers per hour, what must the speed be to make the trip in 2 hours?