Teacher Edition

Volume 2

Reveal MATH™

Integrated I

McGraw Hill

Teacher Edition

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Reveal MATH®

Integrated I

McGraw Hill

mheducation.com/prek-12

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Contents in Brief

Reveal Math Guiding Principles

Academic research and the science of learning provide the foundation for this powerful K–12 math program designed to help reveal the mathematician in every student.

Reveal Math is built on a solid foundation of **RESEARCH** that shaped the **PEDAGOGY** of the program.

Reveal Math Integrated I, Integrated II, and Integrated III (Reveal Math Integrated) used findings from research on teaching and learning mathematics to develop its instructional model. Based on analyses of research findings, these areas form the foundational structure of the program:

- Rigor
- Productive Struggle
- Formative Assessment
- Rich Tasks
- Mathematical Discourse
- Collaborative Learning

Instructional Model

1 Launch

WARM UP

During the **Warm Up,** students complete exercises to activate prior knowledge and review prerequisite concepts and skills.

LAUNCH THE LESSON

In **Launch the Lesson**, students view a real-world scenario and image to pique their interest in the lesson content. They are introduced to questions that they will be able to answer at the end of the lesson.

EXPLORE

During the **Explore** activity, students work in partners or small groups to explore a rich mathematical problem related to the lesson content.

INDIVIDUAL ACTIVITY
GROUP ACTIVITY
CLASS ACTIVITY

2 Explore and Develop

LEARN

In the **Learn** section, students gain the foundational knowledge needed to actively work through upcoming Examples.

EXAMPLES & CHECK

Students work through **Examples** related to the key concepts and engage in mathematical discourse.

Students complete a **Check** after several Examples as a quick formative assessment to help teachers adjust instruction as needed.

3 Reflect and Practice

EXIT TICKET

The **Exit Ticket** gives students an opportunity to convey their understanding of the lesson concepts.

PRACTICE

Students complete **Practice** exercises individually or collaboratively to solidify their understanding of lesson concepts and build proficiency with lesson skills.

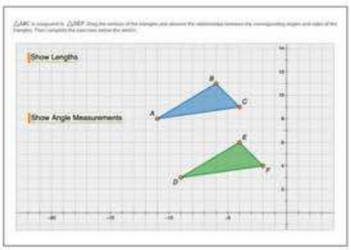
Klaus Vedfelt/DigitalVision/Getty Images

Reveal Math Key Areas of Focus

Reveal Math Integrated I, II, III (Reveal Math Integrated) have a strong focus on rigor—especially the development of conceptual understanding—an emphasis on student mindset, and ongoing formative assessment feedback loops.

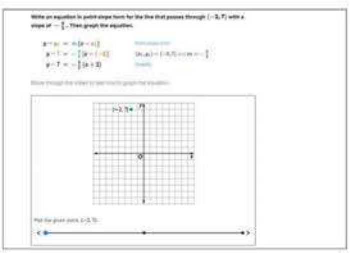
Rigor

Reveal Math Integrated has been thoughtfully designed to incorporate a balance of the three elements of rigor: conceptual understanding, procedural skills and fluency, and application.



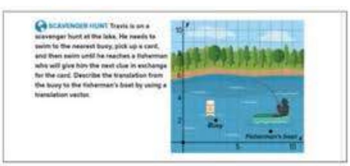
Conceptual Understanding

Explore activities give all students an opportunity to work collaboratively and discuss their thinking as they build conceptual understanding of new concepts. In the **Explore** activity to the left, students use **Web Sketchpad**® to build understanding of the relationships between corresponding sides and angles in congruent triangles.



Procedural Skills and Fluency

Students use different strategies and tools to build procedural fluency. In the **Example** shown, students build proficiency with writing equations in point-slope form.



Application

Real-world examples and practice problems are opportunities for students to apply their learning to new situations. In the real-world example shown, students apply their understanding by solving a multi-step problem with translations.

Student Mindset

Mindset Matters tips located in each module provide specific examples of how Reveal Math Integrated content can be used to promote a growth mindset in all students. Another feature focused on promoting a growth mindset is **Ignite! Activities** developed by Dr. Raj Shah to spark student curiosity about why the math works. An **Ignite!** delivers problem sets that are flexible enough so that students with varying background knowledge can engage with the content and motivates them to ask questions, solve complex problems, and develop a can-do attitude toward math.

Mindset Matters

Growth vs.Fixed Mindset

Everyone has a core belief or mindset about how they learn. People with a *growth mindset* believe that hard work will make them smarter. Those with a *fixed mindset* believe that they can learn new things, but can't become smarter. When a student changes their mindset they are more likely to work through challenging problems, learn from their mistakes, and ultimately learn more deeply.

How Can I Apply It?

Assign students tasks, such as the **Explore** activities, that can help them to develop their intelligence. Let them know that each time they learn a new idea an electric current fires that connects different parts of the brain!

Teacher Edition Mindset Tip

IGNITE!

Student Ignite! Activity

Formative Assessment

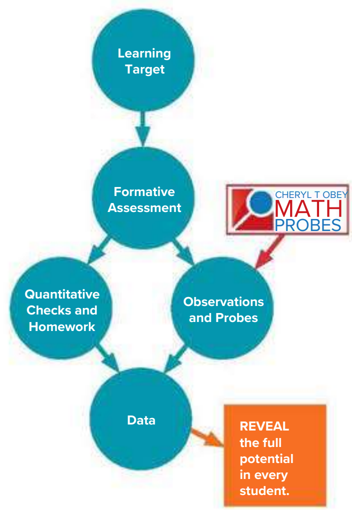
The key to reaching all learners is to adjust instruction based on each student's understanding. Reveal Math Integrated offers powerful formative assessment tools that help teachers to efficiently and effectively differentiate instruction for all students.

Math Probes

Each module includes a **Cheryl Tobey Formative Assessment Math Probe** that is focused on addressing student misconceptions about key math topics. Students can complete these probes at the beginning, middle, or end of a module. The teacher support includes a list of recommended differentiated resources that teachers assign based on students' responses.

Example Checks

After multiple examples, a formative assessment **Check** that students complete on their own allows teachers to gauge students' understanding of the concept or skill presented. When students complete the Check online, the teacher receives resource recommendations which can be assigned to students.



A Powerful Blended Learning Experience

The *Reveal Math Integrated Course I, Course II, Course III* (Reveal Math Integrated) blended learning experience was designed to include purposeful print and digital components focused on sparking student curiosity and providing teachers with flexible implementation options.

Reveal Math Integrated has been thoughtfully developed to provide a rich learning experience no matter where a district, school, or classroom falls on the digital spectrum. All of the instructional content can be projected or can be accessed via desktop, laptop, or tablet.

Lesson

1 Launch

WARM UP

The **Warm Up** exercise can be projected on an interactive whiteboard.

LAUNCH THE LESSON

Launch the Lesson can be projected or assigned to students to access on their own devices.

EXPLORE

The **Explore** activity can be projected while students record their observations in a notebook or can be assigned for students to complete on individual devices.

Launch the Lesson

Explore

INDIVIDUAL ACTIVITY
GROUP ACTIVITY
CLASS ACTIVITY
INTERACTIVE PRESENTATION
PRINT STUDENT EDITION

(l)Andrey_Kuzmin/iStock/Getty Images, (r)McGraw-Hill Education

2 Explore and Develop

3 Reflect and Practice

LEARN

As students are introduced to the key lesson concepts, they can progress through the **Learn** by recording notes in a notebook or on their own devices.

EXAMPLES & CHECK

Either in a notebook or on an individual device, students work through one or more **Examples** related to key lesson concepts.

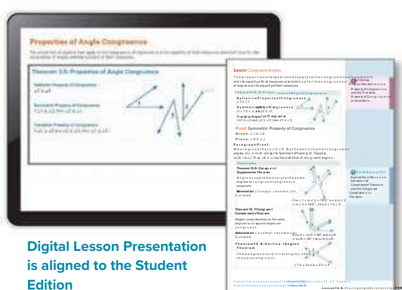
A **Check** follows several Examples in either the Student Edition or on each student device.

EXIT TICKET

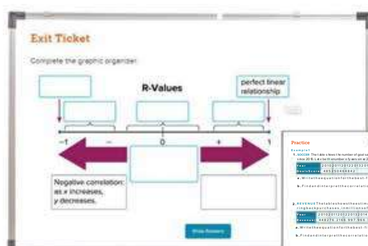
The **Exit Ticket** is projected or accessed via student devices to provide students with lesson closure and an opportunity to revisit the lesson concepts.

PRACTICE

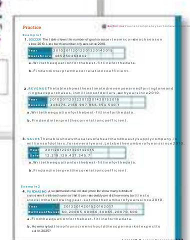
Assign students **Practice** problems from their Student Edition or create a digital assignment for them to work on their device in class or at home to solidify lesson concepts.



Digital Lesson Presentation is aligned to the Student Edition



Exit Ticket



Practice

Supporting All Learners

The *Reveal Math Integrated I, II,* and *III* (Reveal Math Integrated) programs were designed so that all students have access to:

- rich tasks that promote productive struggle,
- opportunities to develop proficiency with the habits of mind and thinking strategies of mathematicians, and
- prompts to promote mathematical discourse and build academic language.

Resources for Differentiating Instruction

When needed, resources are available to differentiate math instruction for students who may need to see a concept in a different way, practice prerequisite skills, or are ready to extend their learning.

AL

Approaching Level Resources

- Remediation Activities
- Extra Examples

BL

Beyond Level Resources

- Beyond Level Differentiated Activities
- Extension Activities

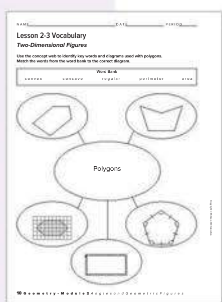
Resources for English Language Learners

Reveal Math Integrated also includes student and teacher resources to support students who are simultaneously learning grade-level math and building their English proficiency. Appropriate, research-based language scaffolds are also provided to support students as they engage in rigorous mathematical tasks and discussions.

ELL

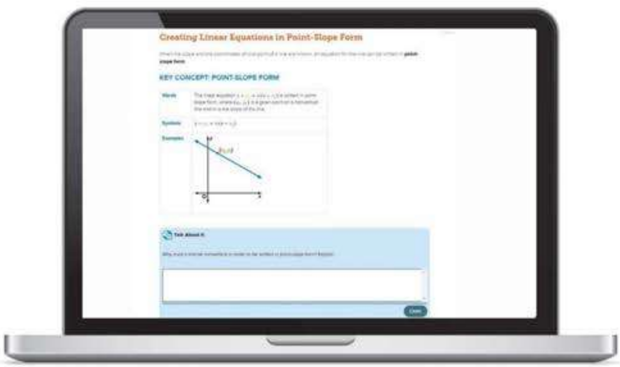
English Language Learners

- Spanish Personal Tutors
- Math Language-Building Activities
- Language Scaffolds
- *Think About It!* and *Talk About It!* Prompts
- Multilingual eGlossary
- Audio
- Graphic Organizers
- Web Sketchpad, Desmos, and eTools



Developing Mathematical Thinking and Strategic Questioning

Reveal Math Integrated I, II, and *III* (Reveal Math Integrated) are comprised of high-quality math content designed to be accessible and relevant to each student. Throughout the program, students are presented with a variety of thoughtfully designed questioning strategies related to the content. Using these questions provides you with an additional, built-in type of formative assessment that can be used to modify instruction. They also strengthen students' ownership of mathematical content knowledge and daily use of the Standards for Mathematical Practice.



Key Concept Introduction followed by a Talk About It question to discuss with a classmate.

You will find these types of questioning strategies throughout Reveal Math Integrated. The related Standard for Mathematical Practice for each is also indicated.

- **Talk About It** questions encourage students to engage in mathematical discourse with classmates (MP3)
- **Alternate Method** shows students another way to solve a problem and asks them to compare and contrast the methods and solutions (MP1)
- **Avoid a Common Error** shows students a problem similar to an example but with a flaw in reasoning, and students have to find and explain the error (MP3)
- **State Your Assumptions** requires that student state the assumptions they made to solve a problem (MP4)
- **Use a Source** asks students to find information using an external source, such as the Internet, and use it to pose or solve a problem (MP5)
- **Think About It** questions help students make sense of mathematical problems (MP1)
- **Concept Checks** prompt students to analyze how the Key Concepts of the lesson apply to various use cases (MP3)

Reveal Student Readiness with Individualized Learning Tools

Reveal Math Integrated I, II, and *III* (Reveal Math Integrated) incorporate innovative, technology-based tools that are designed to extend the teacher's reach in the classroom to help address a wide range of knowledge gaps, set and align academic goals, and meet student individualized learning needs.

LEARNSMART®

Topic-Mastery

With embedded **LearnSmart,**® students have a built-in study partner for topic practice and review to prepare for multi-module or mid-year tests.

LearnSmart's revolutionary adaptive technology measures students' awareness of their own learning, time on topic, answer accuracy, and suggests alternative resources to support student learning, confidence, and topic mastery.

ALEKS®

Individualized Learning Pathways

Learners of all levels benefit from the use of ***ALEKS'*** adaptive, online math technology designed to pinpoint what each student knows, does not know, and most importantly, what each student is ready to learn.

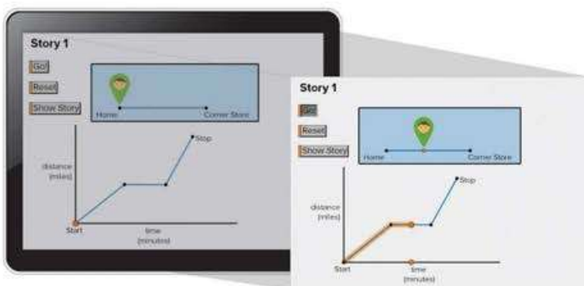
When paired with Reveal Math Integrated, ***ALEKS*** is a powerful tool designed to provide integrated instructionally actionable data enabling teachers to utilize Reveal Math Integrated resources for individual students, groups, or the entire classroom.

Activity Report

Scott Dunlap/iStockvectors/Getty Images

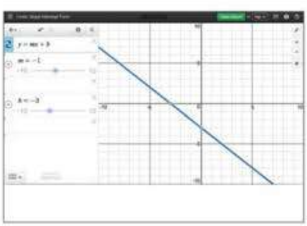
Powerful Tools for Modeling Mathematics

Reveal Math Integrated I, II, and *III* (Reveal Math Integrated) have been designed with purposeful, embedded digital tools to increase student engagement and provide unique modeling opportunities.



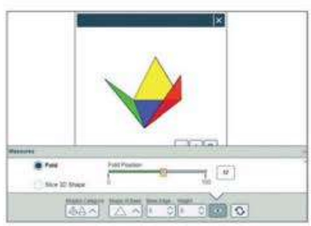
Web Sketchpad® Activities

The leading dynamic mathematics visualization software has now been integrated with **Web Sketchpad Activities** at point of use within Reveal Math Integrated. Student exploration (and practice) using **Web Sketchpad** encourages problem solving and visualization of abstract math concepts.



desmos

The powerful **Desmos** graphing calculator is available in Reveal Math Integrated for students to explore, model, and apply math to the real-world.



eTools

By using a wide variety of digital **eTools** embedded within Reveal Math Integrated, students gain additional hands-on experience while they learn and teachers have the option to create problem-based learning opportunities.

Technology-Enhanced Items

Embedded within the digital lesson, technology-enhanced items—such as drag-and-drop, flashcard flips, or diagram completion—are strategically placed to give students the practice with common computer functions needed to master computer-based testing.

TYPE

SWIPE

DRAG & DROP

FLASHCARDS

eTOOLS

MULTI-SELECT

WATCH

EXPAND

Assessment Tools to Reveal Student Progress and Success

Reveal Math Integrated I, II, and *III* (Reveal Math Integrated) provide a comprehensive array of assessment tools, with both print and digital administration options, to measure student understanding and progress. The digital assessment tools include next-generation assessment items, such as multiple-response, selected-response, and technology-enhanced items.

Assessment Solutions

Reveal Math Integrated provides embedded, regular formative checkpoints to monitor student learning and provide feedback that can be used to modify instruction and help direct student learning using reports and recommendations based on resulting scores.

Summative assessments built in Reveal Math Integrated evaluate student learning at the module conclusion by comparing it against the state standards covered.

Formative Assessment Resources

- Cheryl Tobey Formative Assessment Math Probes
- Checks
- Exit Tickets
- Put It All Together

Summative Assessment Resources

- Module Tests
- Performance Tasks
- End-of-Course Tests
- LearnSmart

Or **Build Your Own** assessments focused on standards or objectives. Access to banks of questions, including those with tech-enhanced capabilities, enable a wide range of options to mirror high-stakes assessment formats.

Reporting

Clear, instructionally actionable data is a click away with the Reveal Math Integrated Reporting Dashboard.

Activity Report Real-time class and student reporting of activities completed by the class. Includes average score, submission rate, and skills covered for the class and each student.

- **Item Analysis Report** A detailed analysis of response rates and patterns, answers, and question types in a class snapshot or by student.

Standards Report Performance data by class or individual student are aggregated by standards, skills, or objectives linked to the related activities completed.

Activity Report

Scott Dunlap/iStockvectors/Getty Images

Professional Development Support for Continuous Learning

McGraw-Hill Education supports lifelong learning and demonstrates commitment to teachers with a built-in professional learning environment designed for support during planning or extended learning opportunities.

What You Will Find

- Best-practice resources
- Implementation support
- Teaching Strategies
- Classroom Videos
- Math Misconception Videos
- Content and Pedagogy Videos
- Content Progression Information

Why Professional Development Is so Important

- Research-based understanding of student learning
- Improved student performance
- Evidence-based instructional best practices
- Collaborative content strategy planning
- Extended knowledge of program how-to's

Kevin Dodge/Blend Images/Getty Images

Reveal Math Expert Advisors

Cathy Seeley, Ed.D.

Austin, Texas

Mathematics educator, speaker, and writer, former Senior Fellow at the Charles A. Dana Center at The University of Texas at Austin, past President of NCTM, former Director of K-12 Mathematics for the State of Texas

Areas of expertise:
Mathematics Teaching, Equity, Assessment, STEM Learning, Informal Learning, Upside-Down Teaching, Productive Struggling, Mathematical Practices, Mathematical Habits of Mind, Family and Community Outreach, Mathematics Education Policy, Advocacy

"We want students to believe deeply that mathematics makes sense–in generating answers to problems, discussing their thinking and other students' thinking, and learning new material."

—Seeley, 2016, Making Sense of Math

Cheryl R. Tobey, M.Ed.

Gardiner, Maine

Senior Mathematics Associate at Education Development Center (EDC)

Areas of expertise:
Formative assessment and professional development for mathematics teachers; tools and strategies to uncovering misconceptions

"Misunderstandings and partial understandings develop as a normal part of learning mathematics. Our job as educators is to minimize the chances of students' harboring misconceptions by knowing the potential difficulties students are likely to encounter using assessments to elicit misconceptions and implementing instruction designed to build new and accurate mathematical ideas.

— Tobey, et al 2007, 2009, 2010, 2013, 2104, Uncovering Student Thinking Series

Nevels Nevels, Ph.D.

Saint Louis, Missouri

PK-12 Mathematics Curriculum Coordinator for Hazelwood School District

Areas of expertise:
Mathematics Teacher Education; Student Agency & Identity; Socio-Cultural Perspective in Mathematics Learning

"A school building is one setting for learning mathematics. It is understood that all children should be expected to learn meaningful mathematics within its walls. Additionally, teachers should be expected to learn within the walls of this same building. More poignantly, I posit that if teachers are not learning mathematics in their school building, then it is not a school."

—Nevels, 2018

Raj Shah, Ph.D.

Columbus, Ohio

Founder of Math Plus Academy, a STEM enrichment program and founding member of The Global Math Project

Areas of expertise:
Sparking student curiosity, promoting productive struggle, and creating math experiences that kids love

"As teachers, it's imperative that we start every lesson by getting students to ask more questions because curiosity is the fuel that drives engagement, deeper learning and perseverance."

—Shah, 2017

(l to r, t to b) Photo by Damon Leo, Shannon Larsen, Matthew McClellan, Raj Shah

Walter Secada, Ph.D.

Coral Gables, Florida

Professor of Teaching and Learning at the University of Miami

Areas of expertise:
Improving education for English language learners, equity in education, mathematics education, bilingual education, school restructuring, professional development of teachers, student engagement, Hispanic dropout and prevention, and reform

"The best lessons take place when teachers have thought about how their individual English language learners will respond not just to the mathematical content of that lesson, but also to its language demands and mathematical practices."

—Secada, 2018

Ryan Baker, Ph.D.

Philadelphia, Pennsylvania

Associate Professor and Director of Penn Center for Learning Analytics at the University of Pennsylvania

Areas of expertise:
Interactions between students and educational software; data mining and learning analytics to understand student learning

"The ultimate goal of the field of Artificial Intelligence in Education is not to promote artificial intelligence, but to promote education... systems that are designed intelligently, and that leverage teachers' intelligence. Modern online learning systems used at scale are leveraging human intelligence to improve their design, and they're bringing human beings into the decision-making loop and trying to inform them."

—Baker, 2016

Chris Dede, Ph.D.

Cambridge, Massachusetts

Timothy E. Wirth Professor in Learning Technologies at Harvard Graduate School of Education

Areas of expertise:
Provides leadership in educational innovation; educational improvements using technology

"People are very diverse in how they prefer to learn. Good instruction is like an ecosystem that has many niches for alternative types of learning: lectures, games, engaging video-based animations, readings, etc. Learners then can navigate to the niche that best fulfills their current needs."

—Dede, 2017

Dinah Zike, M.Ed.

Comfort, Texas

President of Dinah.com in San Antonio, Texas and Dinah Zike Academy

Areas of expertise:
Developing educational materials that include three-dimensional graphic organizers; interactive notebook activities for differentiation; and kinesthetic, cross-curricular manipulatives

"It is education's responsibility to meet the unique needs of students, and not the students' responsibility to meet education's need for uniformity."

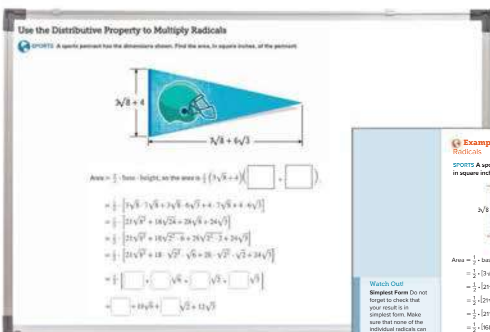
—Zike, 2017, InRIGORating Math Notebooks

Reveal Everything Needed for Effective Instruction

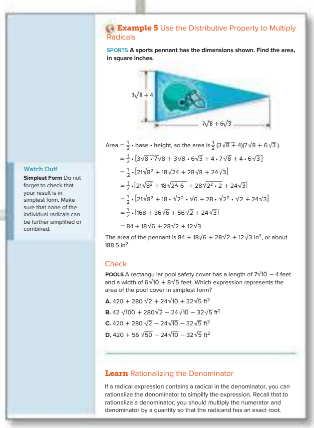
Reveal Math Integrated I, II, and *III* (Reveal Math Integrated) provide both print and innovative, technology-based tools designed to address a wide range of classrooms. No matter whether you're in a 1:1 district, or have a classroom projector, Reveal Math Integrated provides you with the resources you need for a rich learning experience.

Blended Classrooms

Focused on projection of the **Interactive Presentation,** students follow along, taking notes and working through problems in a notebook during class time. Also included in the Interactive Student Edition is a glossary, selected answers, and a reference sheet.



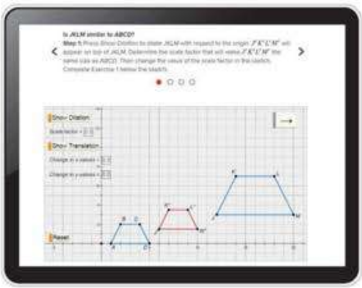
Digital Lesson Presentation is aligned to Student Edition



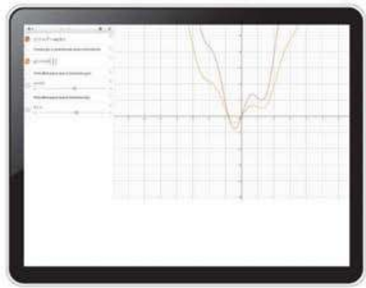
Andrey_Kuzmin/iStock/Getty Images

Digital Classrooms

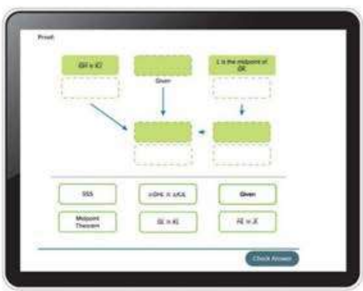
Projection is a focal point for key areas of the course with students interacting with the lesson using their own devices. Each student can access teacher-assigned sections of the lessons for **Explore** activities, **Learn** sections, and **Examples**. Point of use videos, animations, as well as interactive content enable students to experience math in interesting and impactful ways.



Web Sketchpad

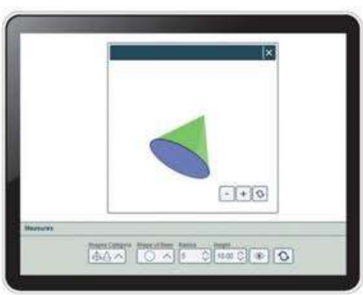


Desmos



Drag-and-Drop

Video



eTools

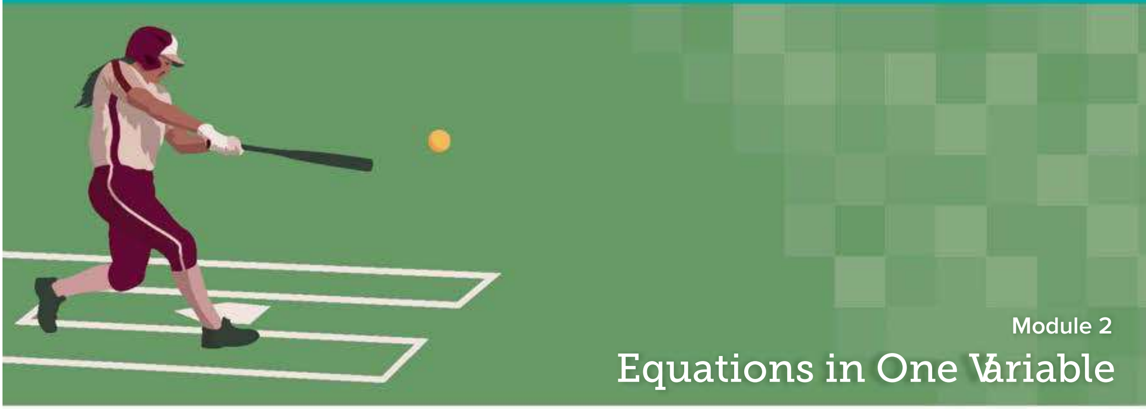
In Reveal Math Integrated, R is for—

- Research
- Rigor
- Relevant Connections

Are you... READY to start?

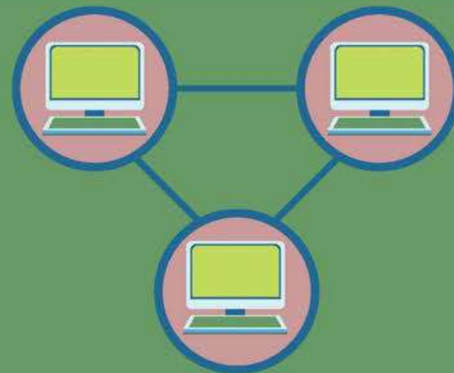
Module 1

Expressions



Module 2

Equations in One Variable



Module 3

Relations and Functions

Module 4

Linear and Nonlinear Functions

Module 5

Creating Linear Equations

Module 6

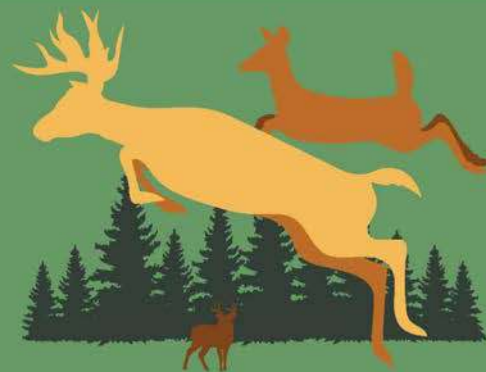
Linear Inequalties

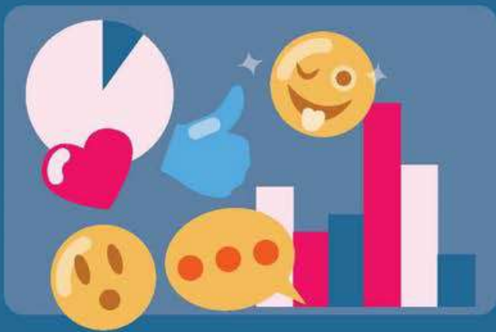
Module 7

Systems of Linear Equations and Inequalities

Module 8

Exponential Functions





Module 9

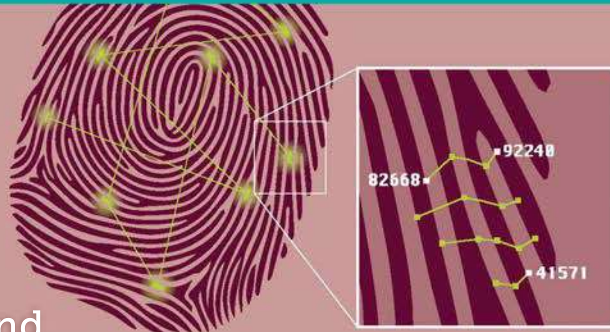
Statistics

Module 10

Tools of Geometry

Module 11

Angles and Geometric Figures



Module 12

Logical Arguments and Line Relationships

Module 13
Transformations and Symmetry

Module 14

Triangles and Congruence

Standards for Mathematical Content, Reveal Math Integrated I

This correlation shows the alignment of *Reveal Math Integrated I* to the Standards for Mathematical Content from the Common Core State Standards for Mathematics.

Standard		Lesson(s)
Number and Quantity		
Quantities ★ N.Q		
N.Q.1	Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.	**2-7, 3-1, 9-2, 9-4**
N.Q.2	Define appropriate quantities for the purpose of descriptive modeling.	**1-6**
N.Q.3	Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.	**1-6**
Algebra		
Seeing Structure in Expressions A.SSE		
A.SSE.1	Interpret expressions that represent a quantity in terms of its context.★ **a.** Interpret parts of an expression, such as terms, factors, and coefficients. **b.** Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of P and a factor not depending on P.*	**1-1, 1-2, 1-4, 4-6, 4-7**
Creating Equations ★ A.CED		
A.CED.1	Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear quadratic functions, and simple rational and exponential functions.	**2-1, 2-2, 2-3, 2-4, 2-5, 2-6, 6-1,6-2, 6-3, 6-4**
A.CED.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.	**4-1, 4-3, 4-4, 4-5, 4-6, 4-7, 5-1, 5-2, 5-3, 5-5, 5-6, 8-1, 8-2, 8-3, 8-5, 8-6**
A.CED.3	Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context.	**2-1, 2-7, 3-6, 5-1, 5-2, 6-1, 6-2, 6-3, 6-4, 6-5, 7-1, 7-2, 7-3, 7-4, 7-5**
A.CED.4	Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.	**2-7**
Reasoning with Equations and Inequalities A.REI		
A.REI.1	Explain each step in solving a linear equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.	**2-2, 2-3, 2-4, 2-5, 2-6, 4-3, 5-1, 5-2**
A.REI.3	Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.	**2-2, 2-3, 2-4, 2-5, 2-6, 2-7, 6-1, 6-2**
A.REI.5	Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.	**7-4**
A.REI.6	Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.	**7-1, 7-2, 7-3, 7-4**
A.REI.10	Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).	**3-4, 4-1, 4-3, 8-1**

★ Mathematical Modeling Standards

Standard		Lesson(s)
A.REI.11	Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★	**7-1**
A.REI.12	Graph the solutions to a linear inequality in two variables as a halfplane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.	**6-5, 7-5**
Functions		
Interpreting Functions F.IF		
F.IF.1	Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If *f* is a function and *x* is an element of its domain, then $f(x)$ denotes the output of *f* corresponding to the input *x*. The graph of *f* is the graph of the equation $y = f(x)$.	**3-1, 3-2**
F.IF.2	Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.	**3-2**
F.IF.3	Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.	**4-5, 8-5, 8-6**
F.IF.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★	**3-3, 3-4, 3-5, 3-6, 4-4, 4-6, 4-7, 8-1**
F.IF.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.	**3-2, 3-3, 3-6, 8-1**
F.IF.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★	**4-2, 5-1**
F.IF.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ **a.** Graph linear and quadratic functions and show intercepts, maxima, and minima. **e.** Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.	**4-1, 4-3, 4-4, 8-1, 8-2**
F.IF.9	Compare properties of two functions, each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).	**3-6, 4-3, 4-6, 8-1**
Building Linear or Exponential Functions F.BF		
F.BF.1	Write a function that describes a relationship between two quantities.★ **a.** Determine an explicit expression, a recursive process, or steps for calculation from a context. **b.** Combine standard function types using arithmetic operations.	**4-5**
F.BF.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★	**4-5, 8-5, 8-6**

★ Mathematical Modeling Standards

Standard		Lesson(s)
F.BF.3	Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.	4-4, 4-7, 8-2
Linear and Exponential ★ F.LE		
F.LE.1	Distinguish between situations that can be modeled with linear functions and with exponential functions. **a.** Prove that linear functions grow by equal differences over equal intervals; exponential functions grow by equal factors over equal intervals.	Expand 4-3, 8-1, Expand 8-5
	b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.	
	c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.	
F.LE.2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).	4-5, 8-3, 8-5
F.LE.3	Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly.	Standard F.LE.3 is taught in Integrated Math Course II, 12-8 Modeling and Curve Fitting
F.LE.5	Interpret the parameters in a linear or exponential function in terms of a context.	4-1, 4-2, 4-3, 8-1, 8-3, 8-5
Geometry		
Congruence G.CO		
G.CO.1	Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.	10-2, 10-3, 10-4, 11-1, 11-2, 12-7
G.CO.2	Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).	11-4
G.CO.3	Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.	13-6
G.CO.4	Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.	13-1, 13-2, 13-3, 13-5, 13-6
G.CO.5	Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.	13-1, 13-2, 13-3, 13-4, 13-5, 13-6
G.CO.6	Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide whether they are congruent.	13-1, 13-2, 13-3, 13-4, 13-6
G.CO.7	Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.	14-2

★ Mathematical Modeling Standards

Standard		Lesson(s)
G.CO.8	Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.	**14-3, 14-4**
G.CO.12	Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.	**10-3, 10-7, 11-1, 11-2, 12-5, 12-9, 12-10, 13-1, 14-3, 14-4, 14-6**
G.CO.13	Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle. Emphasize the ability to formalize and defend how these constructions result in the desired objects.	**Standard G.CO.13 is taught in Integrated Math Course II, 5-5 Tangents**
Expressing Geometric Properties With Equations G.GPE		
G.GPE.4	Use coordinates to prove simple geometric theorems algebraically.	**14-7**
G.GPE.5	Prove the slope criteria for parallel and perpendicular lines; use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).	**Expand 4-2, 12-8**
G.GPE.7	Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.★	**11-3**
Statistics and Probability ★		
Interpreting Categorical and Quantitative Data S.ID		
S.ID.1	Represent data with plots on the real number line (dot plots, histograms, and box plots).	**9-2, 9-4**
S.ID.2	Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.	**9-4, 9-6**
S.ID.3	Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).	**9-5, 9-6**
S.ID.5	Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.	**9-7**
S.ID.6	Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. **a.** Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions, or choose a function suggested by the context. Emphasize linear and exponential models.	**5-3, 5-5**
	b. Informally assess the fit of a function by plotting and analyzing residuals. Focus on situations for which linear models are appropriate.	
	c. Fit a linear function for scatter plots that suggest a linear association.	
S.ID.7	Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.	**5-1, 5-3**
S.ID.8	Compute (using technology) and interpret the correlation coefficient of a linear fit.	**5-5**
S.ID.9	Distinguish between correlation and causation.	**5-4**

★ Mathematical Modeling Standards

Standards for Mathematical Practice

This correlation shows the alignment of *Reveal Math Integrated I* to the Standards for Mathematical Practice, from the Common Core State Standards.

Standard	Lesson(s)
1 Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.	*Reveal Math Integrated I* requires students to make sense of problems and persevere in solving them in Examples and Practice throughout the program. Some specific lessons for review are: Lessons 1-1, 1-4, 2-5, 3-1, 3-3, 3-4, 3-5, 4-1, 4-3, 4-4, 4-7, 5-1, 5-4, 5-5, 5-6, 6-1, 6-2, 6-4, 7-2, 8-2, 9-2, 9-7, 10-4, 10-7, 11-3, 12-1, 12-7, 12-8, 12-10, 13-2, 14-3, 14-5, 14-7
2 Reason abstractly and quantitatively. Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.	*Reveal Math Integrated I* requires students to reason abstractly and quantitatively in Think About It features and Higher Order Thinking Skills throughout the program. Some specific lessons for review are: Lessons 1-2, 1-6, 2-1, 2-2, 2-3, 2-4, 2-6, 2-7 3-3, 3-4, 3-5, 4-2, 5-1, 5-2, 6-1, 6-2, 7-3, 7-4, 8-4, 8-5, 9-4, 10-3, 10-4, 11-3, 11-6, 12-2, 12-9, 13-4, 14-3, 14-7
3 Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.	*Reveal Math Integrated I* requires students to construct viable arguments and critique the reasoning of others in Talk About It features and Practice throughout the program. Some specific lessons for review are: Lessons 1-3, 2-4, 3-2, 3-3, 4-5, 5-4, 6-4, 7-5, 8-1, 8-5, 9-1, 9-3, 10-1, 10-2, 10-5, 11-2, 11-8, 12-1, 12-5, 12-6, 12-8, 12-9, 12-10, 13-1, 13-4, 13-5, 14-1, 14-3, 14-5, 14-7
4 Model with mathematics. Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.	*Reveal Math Integrated I* requires students to model with mathematics, collaborate, and discuss mathematics in Examples and Practice throughout the program. Some specific lessons for review are: Lessons 1-1, 1-2, 1-3, 1-4, 1-5, 1-6, 2-1, 2-5, 2-6, 3-2, 3-5, 4-2, 4-3, 4-6, 5-1, 5-2, 5-6, 6-3, 6-5, 7-3, 7-4, 8-1, 8-4, 8-6, 9-1, 9-2, 9-4, 9-5, 9-6, 9-7, 10-2, 10-6, 10-7, 11-1, 11-4, 11-5, 11-6, 12-3, 12-4, 12-6, 12-9, 12-10, 13-1, 13-4, 14-1, 14-4, 14-5

Standard	Lesson(s)
5 Use appropriate tools strategically. Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.	*Reveal Math Integrated I* requires students to use appropriate tools strategically in Explore activities throughout the program. Some specific lessons for review are: Lessons 1-4, 2-2, 2-3, 3-4, 4-1, 4-3, 4-4, 4-7, 5-3, 5-4, 5-5, 5-6, 6-1, 6-5, 7-1, 8-2, 8-5, 9-5, 9-6, 10-2, 10-6, 11-4, 11-8, 12-1, 12-7, 12-8, 13-1, 13-3, 13-5, 13-6, 14-2, 14-4, 14-6
6 Attend to precision. Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.	*Reveal Math Integrated I* requires students to attend to precision in Examples and Practice throughout the program. Some specific lessons for review are: Lessons 1-4, 1-6, 2-7, 3-1, 3-6, 4-6, 5-3, 5-5, 6-4, 7-2, 7-5, 8-3, 8-4, 9-3, 10-1, 11-1, 11-6, 11-7, 11-8, 12-2, 12-3, 12-4, 12-5, 12-6, 12-7, 13-2, 13-3, 13-6, 14-2, 14-4, 14-6
7 Look for and make use of structure. Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.	*Reveal Math Integrated I* requires students to look for and make use of structure in Explore activities and Higher Order Thinking Skills throughout the program. Some specific lessons for review are: Lessons 1-2, 1-3, 1-5, 2-2, 2-3, 2-4, 2-5, 3-6, 4-4, 4-7, 5-3, 6-3, 7-5, 8-1, 8-2, 8-6, 9-1, 9-7, 10-5, 11-5, 12-2, 13-3, 13-6, 14-2, 14-6
8 Look for and express regularity in repeated reasoning. Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.	*Reveal Math Integrated I* requires students to look for and express regularity in repeated reasoning in Concept Check and Think About It features and Higher Order Thinking Skills throughout the program. Some specific lessons for review are: Lessons 1-5, 2-7, 3-1, 4-5, 5-2, 6-3, 7-1, 8-3, 8-6, 9-6, 10-3, 11-2, 12-3, 12-4, 13-2, 14-1

Module 8

Exponential Functions

Module Goals

- Students write and solve exponential functions.
- Students graph and transform exponential functions.
- Students understand geometric sequences.

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (including reading these from a table).

Also addresses A.SSE.3c, F.LE.1c, F.LE.5, F.BF.2, F.BF.3, F.IF.3, and F.IF.8b

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Coherence

Vertical Alignment

Previous

Students understood that linear functions have a constant rate of change.

8.F.4

Now

Students graph exponential functions, showing intercepts and end behavior, and interpret the parameters of the function in terms of a context.

F.IF.7e, F.LE.2

Next

Students will relate the inverses of exponential functions to logarithmic functions.

F.LE.2 (Course 3)

Rigor

The Three Pillars of Rigor

Students will use the three pillars of rigor to help them meet standards. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Practice.

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION
EXPLORE	LEARN	EXAMPLE & PRACTICE

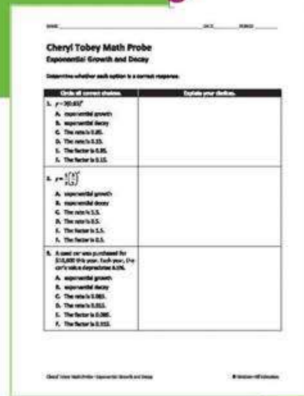
Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
8-1 Exponential Functions	F.IF.7e, F.LE.1c, F.LE.5	1	0.5
8-2 Transformations of Exponential Functions	F.IF.7e, F.BF.3	3	1.5
8-3 Writing Exponential Functions	F.LE.2, F.LE.5	2	1
Put It All Together: Lessons 8-1 through 8-3		1	0.5
8-4 Transforming Exponential Expressions	A.SSE.3c, F.IF.8b	1	0.5
8-5 Geometric Sequences	F.BF.2, F.LE.2	1	0.5
8-6 Recursive Formulas	F.IF.3, F.BF.2	2	1
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	14	7

CHERYL TOBEY MATH PROBES

Formative Assessment Math Probe
Exponential Growth and Decay

Module Resource


Cheryl Tobey Math Probe
Exponential Growth and Decay

Correct Answers: 1. B, D, E 2. A, D, E 3. B, C, F

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will determine whether various equations or descriptions describe a given exponential expression and explain their choices.

Targeted Concepts Exponential functions of the form $y = ab^x$ can be analyzed in terms of growth or decay, growth/decay factors and growth/decay rates.

Targeted Misconceptions

- The student confuses exponential growth and decay.
- The student does not understand how to determine the growth/decay factor. They either confuse it with the initial value or the growth/decay rate.
- The student does not understand how to determine the percent rate of change or confuses it with the growth/decay factor.

Use the Probe after Lesson 8-3.

Collect and Assess Student Answers

If the student selects these responses...	**Then** the student likely...
1. A **2.** B **3.** A	either does not know the difference between growth and decay or is confusing the initial value (a) with the growth/decay factor (b) in the exponential equation $y = ab^x$. **Example:** For Item 2, the initial value is $\frac{1}{2}$ and the factor is $\frac{3}{2}$. Because the factor is greater than 1, it indicates exponential growth.
1. C **2.** C **3.** D	does not understand how to calculate the growth/decay rate from the growth/decay factor (1 − factor) in an equation or does not recognize the decimal form of the rate. **Example:** For Item 1, the rate is 1 − 0.85, which is 0.15. Students will often forget to subtract the factor from 1.
1. F **2.** F **3.** E	is confusing the factor with the rate. **Example:** For Item 3, the decay rate is 8.5%. To find the factor, subtract the rate from 1: 1 − 0.085 = 0.915.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS® Exponential Functions
- Lesson 8-3, all Learns, all Examples

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

![IGNITE!]

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

When and how can exponential functions represent real-world situations? Sample answer: Exponential functions can be used in real life to represent situations that grow or decay. One example is representing compound interest.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

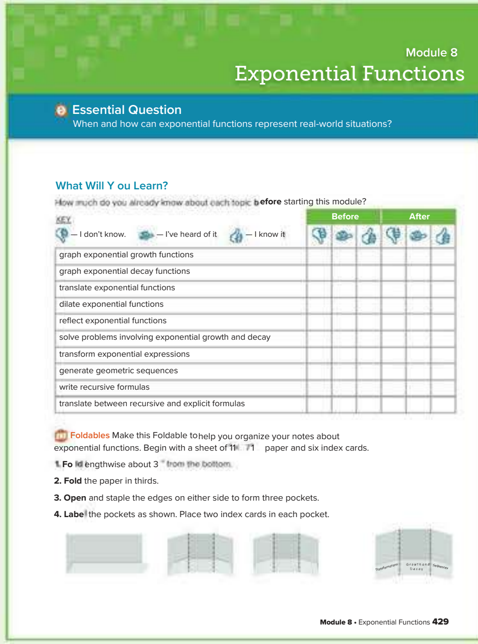
Focus Students create a tabbed book on which they organize information about exponential functions and geometric sequences.

Teach Have students make and label their Foldables as illustrated. Before beginning each lesson, ask students to think of one question that comes to mind as they skim through the lesson. Have them write the questions on an index card of the appropriate lesson. As they read and work through the lesson, ask them to record the answers to their questions on the index cards.

When to Use It Encourage students to add to their Foldables as they work through the Module and to use them to review for the Module Assessment.

Launch the Module

For this module, the Launch the Module video uses exponential growth in savings as a way of introducing the concept of exponential growth. Students are exposed to how exponential functions can model growth and decay in appropriate situations.



Module 8

Exponential Functions

Essential Question

When and how can exponential functions represent real-world situations?

What Will You Learn?

How much do you already know about each topic **before** starting this module?

KEY

— I don't know. — I've heard of it. — I know it!

	Before			After		
graph exponential growth functions						
graph exponential decay functions						
translate exponential functions						
dilate exponential functions						
reflect exponential functions						
solve problems involving exponential growth and decay						
transform exponential expressions						
generate geometric sequences						
write recursive formulas						
translate between recursive and explicit formulas						

Foldables Make this Foldable to help you organize your notes about exponential functions. Begin with a sheet of 11" × 17" paper and six index cards.

1. **Fold** lengthwise about 3" from the bottom.
2. **Fold** the paper in thirds.
3. **Open** and staple the edges on either side to form three pockets.
4. **Label** the pockets as shown. Place two index cards in each pocket.

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Interactive Presentation

What Vocabulary Will Y ou Learn?

- asymptote
- common ratio
- compound interest
- explicit formula
- exponential decay functions
- exponential function
- exponential growth function
- geometric sequence
- recursive formula

Are Y ou Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review

Example 1		Example 2	
Evaluate $2x^3$ for $x = 5$.		**Divide $\frac{5}{6} \div \frac{1}{3}$.**	
$2x^3$	Original expression	$\frac{5}{6} \div \frac{1}{3} = \frac{5}{6} \cdot \frac{3}{1}$	Multiply by the reciprocal.
$= 2(5)^3$	Substitute 5 for *x*.	$= \frac{15}{6}$	Multiply the numerators and multiply the denominators.
$= 2(125)$	Evaluate the exponent.		
$= 250$	Multiply.	$= \frac{5}{2}$	Find the simplest form.

Quick Check

Evaluate each expression for the given value.	Divide.
1. $-4x^2$ for $x = 7$ -196	**5.** $128 \div 4$ 32
2. $3x^2$ for $x = 3$ 27	**6.** $\frac{1}{3} \div 2$ $\frac{1}{6}$
3. $0.25x^4$ for $x = 1$ 0.25	**7.** $-9 \div 3$ -3
4. x^5 for $x = 3$ 243	**8.** $\frac{11}{8} \div \frac{}{2}$ $\frac{1}{4}$

How did you do?
Which exercises did you answer correctly in the Quick Check?

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What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define An exponential function is a function of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$.

Example $f(x) = 2(3)^x$

Ask Can you identify another exponential function? Possible answers: $g(x) = 0.5(4)^x$, $h(t) = 2(0.3)^t$

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- finding function values
- transforming linear functions
- writing linear functions
- evaluating expressions with exponents
- writing explicit formulas to represent arithmetic sequences
- writing explicit formulas to represent geometric sequences

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the **Exponents** section to ensure student success in this module.

Mindset Matters

Promote Process Over Results

The process that a student takes as he or she encounters a new problem is just as important as—if not more important than—the result.

How Can I Apply It?

Encourage students to consider the **Think About It!** prompts in their *Student Edition*. Have students discuss their problem-solving strategies with a partner. Be sure to support the process and reward effort as students explore and work through problems.

Lesson 8-1

Exponential Functions

F.IF.7e, F.LE.1c, F.LE.5

LESSON GOAL

Students graph exponential functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore:

- Exponential Behavior

Develop:

Identifying Exponential Behavior
- Identify Exponential Behavior

Explore:

- Restrictions on Exponential Functions

Develop:

Graphing Exponential Functions
- Exponential Growth Function
- Exponential Decay Function

You may want your students to complete the Checks online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL	
Remediation: Functions	●	●			●
Extension: Logarithmic Functions			●	●	●

Language Development Handbook

Assign page 43 of the *Language Development Handbook* to help your students build mathematical language related to graphing exponential functions.

ELL You can use the tips and suggestions on page T43 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F.LE.1c Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

F.LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students understood that linear functions have a constant rate of change.
8.F.4

Now
Students graph exponential functions.
F.IF.7e, F.LE.1c, F.LE.5

Next
Students will identify the effects of transformations on the graphs of exponential functions.
F.IF.7e, F.BF.3

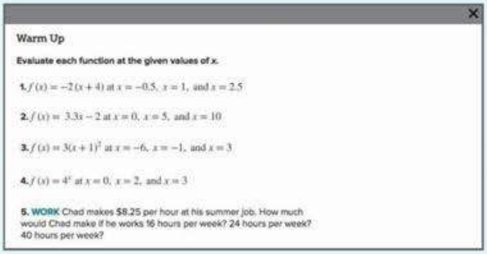
Rigor

The Three Pillars of Rigor

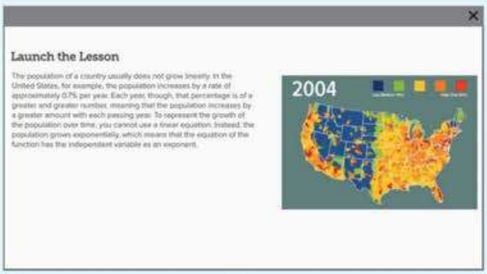
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop understanding of exponential functions and use it to build fluency by graphing exponential functions. They apply their understanding of exponential functions by solving real-world problems.

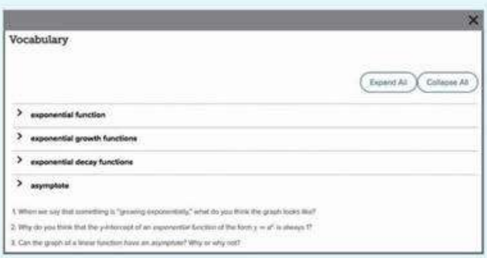
Interactive Presentation



Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding function values

Answers:

1. −7, −10, −13
2. −2, 14.5, 31
3. 75, 0, 48
4. 1, 16, 64
5. $132; $198; $330

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of exponential functions.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Mathematical Background

This lesson introduces the graphing of exponential functions in the form $y = ab^x$. Students explore the differences between the exponential growth function and the exponential decay function and the contexts in which each apply.

Explore Exponential Behavior

Objective

Students explore the differences between exponential and linear behavior.

Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in the rates of change in this Explore.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

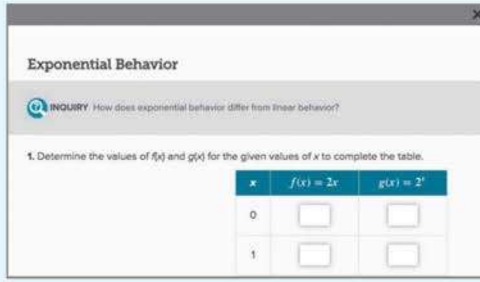
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will be presented with a linear function and an exponential function. They will create a table of values for the two functions and then will identify the differences in the rates of change. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



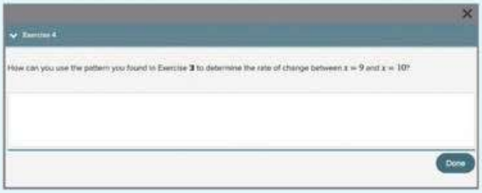
Explore

Explore

TYPE

Students complete the calculations to find the rates of change for exponential functions.

Interactive Presentation



Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Exponential Behavior (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How does looking at the rate of change help you see the relationships over an interval? Sample answer: If the change is constant, the data represent a linear function. If the rate of change varies, the data represent a nonlinear function.
- Why do you think you compared the exponential function with a linear function? Sample answer: A linear function has a constant rate of change, so it is easy to recognize and compare to other functions.

Inquiry

How does exponential behavior differ from linear behavior?
Sample answer: Functions with linear behavior have a constant rate of change, whereas functions with exponential behavior have a rate of change that increases by a constant factor for each equal-sized interval.

Go Online to find additional teaching notes and answers for the guiding exercises.

Explore Restrictions on Exponential Functions

Objective

Students use a sketch to explore the restrictions on exponential functions.

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this Explore, students will need to use the sketch. Work with students to explore and deepen their understanding of exponential functions.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

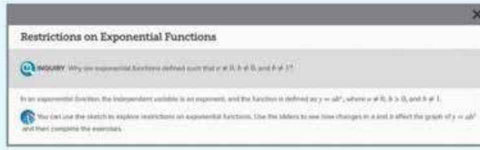
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

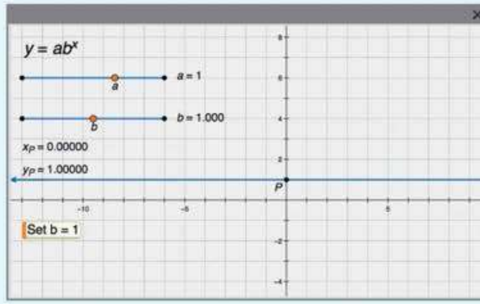
Students will complete guiding exercises throughout the Explore activity. Students will be presented with an exponential function. They will use a sketch to change the parameters of an exponential function, identifying special cases for the various parameters. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

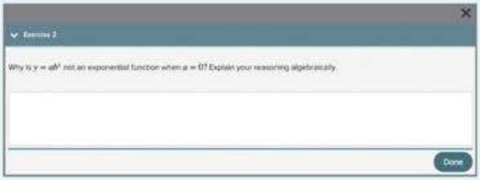


Explore

WEB SKETCHPAD

Students use a sketch to explore restrictions on exponential functions.

Interactive Presentation



Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Restrictions on Exponential Functions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How does the variable a affect the exponential function $y = ab$? Sample answer: a will multiply the exponential function b^x.
- Does the value of x have any restrictions? Explain. No; sample answer: Changing the value of x does not determine whether the function is exponential or not, it just evaluates the exponential function at a certain value.

Inquiry

Why are exponential functions defined such that $a \neq 0$, $b > 0$, and $b \neq 1$? Sample answer: In all three cases, the function becomes a horizontal line or ray, which means that all three cases result in a function that is linear rather than exponential.

Go Online to find additional teaching notes and answers for the guiding exercises.

Learn Identifying Exponential Behavior

Objective

Students recognize situations modeled by linear or exponential functions by examining rates of change.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the equations, graphs, and rates of change for linear and exponential functions.

Common Misconception

Be sure that students do not confuse quadratic functions and exponential functions. While $y = x^2$ and $y = 2^x$ each have an exponent, $y = x^2$ is a quadratic function, and $y = 2^x$ is an exponential function.

Essential Question Follow-Up

Students have begun to explore exponential behavior in real-world situations.

Ask:

Why is it important to identify whether a relationship is represented by a straight line or a curve? Sample answer: A relationship that is represented by a straight line is modeled with a linear equation. If a relationship is not modeled by a line, then the model will need to be based on a nonlinear function.

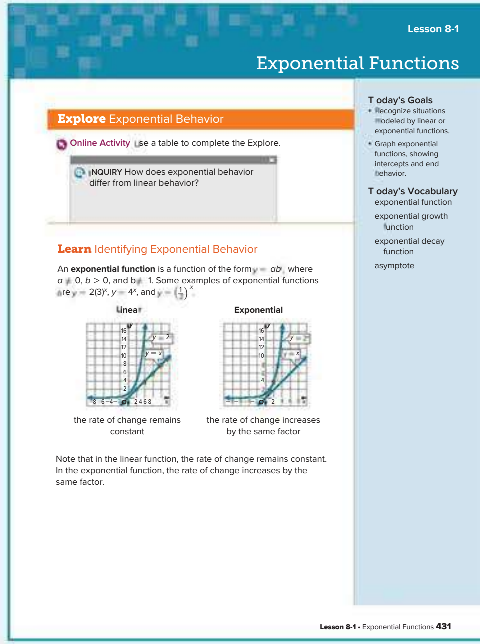
DIFFERENTIATE

Enrichment Activity BL

Ask students to write a comparison of an exponential function and a linear function.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.


Lesson 8-1

Exponential Functions

Explore Exponential Behavior

Online Activity Use a table to complete the Explore.

INQUIRY How does exponential behavior differ from linear behavior?

Learn Identifying Exponential Behavior

An **exponential function** is a function of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$. Some examples of exponential functions are $y = 2(3)^x$, $y = 4^x$, and $y = \left(\frac{1}{3}\right)^x$.

Linear | Exponential

the rate of change remains constant | the rate of change increases by the same factor

Note that in the linear function, the rate of change remains constant. In the exponential function, the rate of change increases by the same factor.

Today's Goals
- Recognize situations modeled by linear or exponential functions.
- Graph exponential functions, showing intercepts and end behavior.

Today's Vocabulary
exponential function
exponential growth function
exponential decay function
asymptote

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Interactive Presentation

Learn

TAP

Students tap on each button to see the rate of change for linear and exponential functions.

Think About It!
Is a function that relates an independent variable to a change in order of magnitude always exponential? Explain your reasoning.

Yes; sample answer: Because an order of magnitude is a number rounded to the nearest power of 10, the rate of change between magnitudes increases by a factor of 10 each time, making it exponential.

Think About It!
Can you confirm that the entire data set displays exponential behavior by looking at the first two intervals?

No; sample answer: To determine whether the entire data set displays exponential behavior, you have to ensure that the ratio is equal for all equal-sized intervals.

Go Online
An alternate method is available for this example.

Example 1 Identify Exponential Behavior

EARTHQUAKES **The Richter Scale measures the energy that an earthquake releases and assigns a magnitude to it. These orders of magnitude can be approximated by comparing them to the explosive power of TNT. Determine whether the set of data displays exponential behavior.**

Magnitude	TNT (tons)
1	0.6
2	6
3	60
4	600
5	6000
6	60,000
7	600,000
8	6,000,000

Magnitudes 1 and 2

As the order of magnitude increases from 1 to 2, the amount of TNT that is approximately equal in magnitude increases from 0.6 tons to 6 tons. That is an increase by a factor of 10.

Magnitudes 2 and 3

As the order of magnitude increases from 2 to 3, the amount of TNT that is approximately equal in magnitude increases from 6 tons to 60 tons. That is an increase by a factor of 10.

Since the change in the amount of TNT increases by the same factor given an equal change in magnitude, the data set displays exponential behavior.

Check

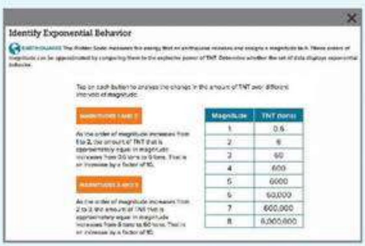
MEMORY In the 19th century, psychologist Hermann Ebbinghaus created a formula to approximate how quickly people forget information over time. The approximate percentage of the newly learned information a person retains over time is shown in the table. Determine whether the data displays exponential behavior.

Time (days)	% Retained
0	100
1	80
2	64
3	51.2
4	40.96

Source: Indiana University

The data set __?__ display exponential behavior. does

Interactive Presentation



Example 1

TYPE

Students answer a question to show they understand whether data exhibits exponential behavior.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Identify Exponential Behavior

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions to the questions in the Think About It! features.

Questions for Mathematical Discourse

AL How do you know that the change in this table is not linear? Sample answer: The change is not constant.

OL By what factor does the explosive power of TNT increase for each integer increase in magnitude? 10

BL How do you know that the behavior is exponential? Sample answer: The increase is by a factor of 10 each time.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Graphing Exponential Functions

Objective

Students graph exponential functions, showing intercepts and end behavior.

MP Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of graphs of exponential functions in this Learn.

Important to Know

To solidify the unique properties of exponential functions, students may need to review key features, including those for linear and other types of non-linear functions that have been previously studied. Compare the domain and range of exponential functions, both exponential growth and exponential decay, to other types of functions.

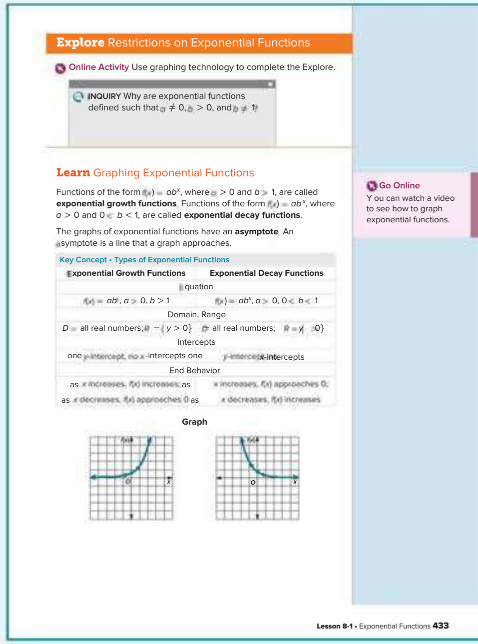
Common Misconception

Make sure students understand that the graphs of exponential functions never actually touch the asymptote. It is acceptable for hand-drawn graphs to show the graph nearly touching and parallel to an asymptote, as long as the students understand that the graph gets infinitely close to the asymptote without touching it.

DIFFERENTIATE

Language Development Activity AL ELL

Ask students where they have heard the term *exponential* before and what they think it means. Students may have heard terms like *exponential growth* on a television news program, and they might think that *exponential* means "enormous." Use students' answers to introduce the concept of exponential functions.



Explore Restrictions on Exponential Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY Why are exponential functions defined such that $a \neq 0$, $b > 0$, and $b \neq 1$?

Learn Graphing Exponential Functions

Functions of the form $f(x) = ab^x$, where $a > 0$ and $b > 1$, are called **exponential growth functions**. Functions of the form $f(x) = ab^x$, where $a > 0$ and $0 < b < 1$, are called **exponential decay functions**.

The graphs of exponential functions have an **asymptote**. An asymptote is a line that a graph approaches.

Go Online You can watch a video to see how to graph exponential functions.

Key Concept • Types of Exponential Functions

Exponential Growth Functions	Exponential Decay Functions
Equation	
$f(x) = ab^x$, $a > 0$, $b > 1$	$f(x) = ab^x$, $a > 0$, $0 < b < 1$
Domain, Range	
$D =$ all real numbers; $R = \{y \mid y > 0\}$	$D =$ all real numbers; $R = \{y \mid y > 0\}$
Intercepts	
one y-intercept, no x-intercepts	one y-intercept, no x-intercepts
End Behavior	
as x increases, $f(x)$ increases; as x decreases, $f(x)$ approaches 0	as x increases, $f(x)$ approaches 0; as x decreases, $f(x)$ increases
Graph	

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Interactive Presentation

Learn

SWIPE

Students move through the slides to see differences between exponential growth and decay functions.

WATCH

Students watch a video to see how to graph exponential functions.

Talk About It!

How can you determine the y-intercept without substituting into the original equation? (Hint: Consider what $x = 0$ and the y-intercept mean in the context of the situation.)

Sample answer: The y-intercept is the thickness of the paper when it has been folded 0 times, so it is the initial thickness of the paper, 0.05 millimeter.

Problem-Solving Tip

Make an Organized List Making an organized list of x-values and corresponding y-values is helpful in graphing the function. It can also help you identify patterns in the data.

Go Online You can watch a video to see how to use a graphing calculator with this example.

Example 2 Exponential Growth Function

FOLDING **Each time you fold a piece of paper in half, it doubles in thickness. If a piece of paper is 0.05 millimeter thick, then you can determine the thickness y of a piece of paper given the number of folds x with the function $y = 0.05(2)^x$. Identify the key features of the function, graph it, and then identify the relevant domain and range in the context of the situation.**

Part A Identify key features.

Because $a > 0$ and $b > 1$, $y = 0.05(2)^x$ is an exponential growth function. The domain is all real numbers and the range is $y > 0$.

The y-intercept is the value of y when $x = 0$.

$y = 0.05(2)^0$

$y = 0.05(1)$

$= 0.05$

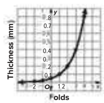
The y-intercept is 0.05.

Because $y = 0.05(2)^x$ is an exponential growth function, as x increases, y increases, and as x decreases, y approaches 0.

Part B Graph the function.

Make a table of values. Round to the nearest unit. Then, plot the points and draw a curve to approximate it.

x	$y = 0.05(2)^x$	y
−2	$y = 0.05(2)^{-2}$	0.0125
−1	$y = 0.05(2)^{-1}$	0.025
0	$y = 0.05(2)^0$	0.05
1	$y = 0.05(2)^1$	0.1
2	$y = 0.05(2)^2$	0.2
3	$y = 0.05(2)^3$	0.4
4	$y = 0.05(2)^4$	0.8



Part C Identify relevant domain and range.

Because the number of folds cannot be negative and folds must be counted in integers, the potential domain is the set of whole numbers and the potential range is the set of integers greater than or equal to 0.05. However, because the paper cannot be folded indefinitely, the thickness of the paper cannot continue to grow to infinity. So the domain will be restricted to the greatest possible number of folds, and the range will be restricted to the greatest thickness of the paper.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 2

EXPAND

Students tap to see the steps to identify key features and graph an exponential growth function.

Example 2 Exponential Growth Function

Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the quantities used in this problem.

Questions for Mathematical Discourse

AL Will the thickness of the paper increase or decrease as the paper is folded? increase

OL What is the domain of this function? Explain. Sample answer: The domain is the set of whole numbers. The folds must be an integer because there cannot be partial folds in this situation. The smallest number of folds is 0 because there cannot be negative folds.

BL How would the function change if you folded the paper into thirds instead of in half? The base of the exponent would be 3 instead of 2.

Check

Consider $y = 3^x$.

Part A

List the key features that apply to $y = 3^x$. Include the domain, range, y-intercept, and end behavior of the function.

D = {all real numbers}, R = {$y > 0$}, y-intercept = 1

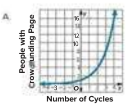
As x increases, y increases. As x decreases, y approaches 0.

FILM The function $y = 3^x$ can be used to model a real-world situation. Sarah wants to crowdfund a film project. T o spread the word, she shares the page with 3 friends, and requests that each friend share it with 3 more friends. The function that models the number of people with a link to the crowdfunding page y given the number of cycles x is defined by the function $y = 3^x$.

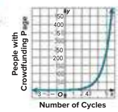
Part B

Select the correct graph of $y = 3^x$. **B**

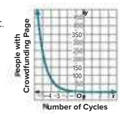
A.



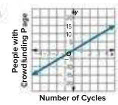
B.



C.



D.



Part C

Describe the relevant range in the context of the situation.

R = {y = 1, 3, 9, 27 , 81, ...}

Math History Minute

While a junior in high school, **Britney Gallivan (1985-)** showed that a single length of toilet paper 4000 feet long can be folded in half a maximum of twelve times. Britney's proof involved the exponential equation $L = \frac{\pi t}{6}(2^n + 4)(2^n - 1)$, where L is the length of the paper, t is the thickness of the material to be folded, and n represents the number of folds desired.

Interactive Presentation

Question 1

Consider $y = 3^x$.

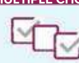
Part A

Select all of the key features that apply to $y = 3^x$.

- **A)** The domain is all real numbers.
- **B)** The range is all real numbers.
- **C)** The domain is $\{x \geq 0\}$.
- **D)** The range is $\{y > 0\}$.
- **E)** The y-intercept is 3.
- **F)** The y-intercept is 1.
- **G)** As x increases, y approaches 0. As x decreases, y increases.
- **H)** As x increases, y increases. As x decreases, y approaches 0.

Check

MULTIPLE CHOICE



Students select the graph of an exponential function.

Lesson 8-1 • Exponential Functions **435**

Use a Source

Research the amount of caffeine in another caffeinated drink. Write a function that models the amount of caffeine left in your system after x hours, and identify the key features of the function.

Sample answer: The amount of caffeine left in the body after drinking an 8-oz cup of black tea can be modeled by the function $y = 47\left(\frac{1}{2}\right)^{\frac{x}{5}}$, where x represents the number of hours and y represents the amount of caffeine in milligrams.

Example 3 Exponential Decay Function

CAFFEINE **The half-life of a substance describes how long it takes for the substance to deplete by half. The half-life of caffeine in the body of a healthy adult is approximately 5 hours, meaning that it takes 5 hours for the body to break down half of the caffeine. Suppose an energy drink contains 160 milligrams of caffeine. The amount of caffeine y left in your system after x hours is modeled by the function $y = 160\left(\frac{1}{2}\right)^{\frac{x}{5}}$. Identify the key features of the function, graph it, and then identify the relevant domain and range in the context of the situation.**

Part A Identify key features.

Because $a > 0$ and $0 < b < 1$, $y = 160\left(\frac{1}{2}\right)^{\frac{x}{5}}$ is an exponential decay function. The domain is all real numbers and the range is $y > 0$.

The y-intercept is the value of y when $x = 0$.

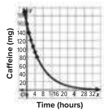
$y = 160\left(\frac{1}{2}\right)^{\frac{0}{5}}$

$= 160(1) = 160$

The y-intercept is 160.

Part B Graph the function.

x	$160\left(\frac{1}{2}\right)^{\frac{x}{5}}$	y
−1	$160\left(\frac{1}{2}\right)^{\frac{-1}{5}}$	184
0	$160\left(\frac{1}{2}\right)^{\frac{0}{5}}$	160
1	$160\left(\frac{1}{2}\right)^{\frac{1}{5}}$	139
2	$160\left(\frac{1}{2}\right)^{\frac{2}{5}}$	121
3	$160\left(\frac{1}{2}\right)^{\frac{3}{5}}$	106
4	$160\left(\frac{1}{2}\right)^{\frac{4}{5}}$	92
5	$160\left(\frac{1}{2}\right)^{\frac{5}{5}}$	80



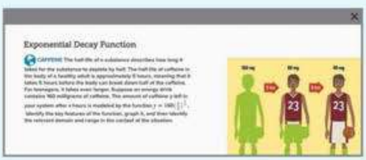
Part C Identify relevant domain and range.

Because time cannot be negative, the relevant domain is $\{x \geq 0\}$. Because the amount of caffeine cannot be negative and the amount of caffeine when $x = 0$ is 160 mg, the relevant range is $\{0 < y \leq 160\}$.

Go Online Y ou can complete an Extra Example online.

436 Module 8 • Exponential Functions

Interactive Presentation



Example 3

TYPE

Students research another caffeinated drink and write a function to model the situation.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Exponential Decay Functions

Teaching the Mathematical Practices

5 Use a Source Guide students to find external sources to answer the questions posed in the Use a Source feature.

Questions for Mathematical Discourse

AL Is the amount of caffeine increasing or decreasing in this situation? decreasing

OL In 10 hours, how many times will the caffeine break down by half? twice

BL How much caffeine will be left in the bloodstream 20 hours after drinking the caffeinated drink? 10 milligrams

Common Error

Because the half-life is 5 hours and Example 3 defines the independent variable as x hours, it is easy for students to become confused about where the exponent in the given equation comes from. Review the table in **Part B** to make it clear to the students that the half-life takes place at $x = 5$ hours. Emphasize that changes in x do not represent each half-life step.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–8
2	exercises that use a variety of skills from this lesson	9–15
3	exercises that emphasize higher-order and critical-thinking skills	16–19

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–15 odd, 16–19
- Extension: Logarithmic Functions
- ALEKS Exponential Functions

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–19 odd
- Remediation, Review Resources: Functions
- Personal Tutors
- Extra Examples 1–3
- ALEKS Sets, Relations, and Functions

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–7
- Remediation, Review Resources: Functions
- *Quick Review Math Handbook*: Exponential Functions
- Arrive**MATH** Take Another Look
- ALEKS Sets, Relations, and Functions

Answers

6c. Because time cannot be negative, the relevant domain is $\{x \mid x \geq 0\}$. Because the amount of nonrecycled paper, and cardboard cannot be negative and the amount when $x = 0$ is 1000 tons, the relevant range is $\{y \mid y \geq 1000\}$.

7c. Because time cannot be negative, the relevant domain is $\{x \mid x \geq 0\}$. Because the amount of Iodine 131 cannot be negative, and the amount when $x = 0$ is 50 mg, the relevant range is $\{y \mid 0 < y \leq 50\}$.

Practice

Go Online You can complete your homework online.

Example 1

Determine whether the set of data displays exponential behavior. Write *yes* or *no*. Explain why or why not.

1.

x	−3	−2	−1	0
y	9	12	15	18

No; the domain values are at regular intervals and the range values have a common difference 3.

2.

x	0	5	10	15
y	20	10	5	2.5

Yes; the domain values are at regular intervals and the range values have a common factor 0.5.

3.

x	4	8	12	16
y	20	40	80	160

Yes; the domain values are at regular intervals and the range values have a common factor 2.

4.

x	50	30	10	−10
y	90	70	50	30

No; the domain values are at regular intervals and the range values have a common difference 20.

5. PICTURE FRAMES Since a picture frame includes a border, the picture must be smaller in area than the entire frame. The table shows the relationship between picture area and frame length for a particular line of frames. Is this an exponential relationship? Explain. No; there is no common factor between the picture areas.

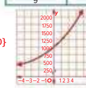
Frame Length (in.)	Picture Area (in^2)
5	6
6	12
7	20
8	30
9	42

Examples 2 and 3

6. WASTE Suppose the waste generated by nonrecycled paper and cardboard products in tons y after x days can be approximated by the function $y = 1000(2)^{0.3x}$.

a. Identify key features. y-intercept $= 1000$; D = {all real numbers}, R = $\{y \mid y > 0\}$

b. Graph the function.

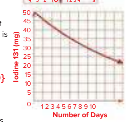


c. Identify relevant domain and range. See margin.

7. IODINE Iodine 131 is a radioisotope that is related to nuclear energy, medical diagnostic and treatment procedures, and natural gas production. A scientist is testing 50 milligrams of Iodine 131. The scientist knows that the half-life of Iodine 131 is about 8.02 days. The function $y = 50\left(\frac{1}{2}\right)^{\frac{x}{8.02}}$ represents the amount of Iodine 131 remaining in milligrams y after x days.

a. Identify key features. y-intercept $= 50$; D = {all real numbers}, R = $\{y \mid y > 0\}$

b. Graph the function.



c. Identify relevant domain and range. See margin.

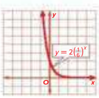
8. DEPRECIATION Suppose a company's computer equipment is decreasing in value according to the function $y = 4000(0.87)^x$. In the equation, x represents the number of years that have elapsed since the equipment was purchased and y represents the value in dollars. What was the value 5 years after the computer equipment was purchased? Round your answer to the nearest dollar. $1994

Lesson 8-1 • Exponential Functions 437

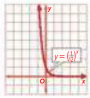
Mixed Exercises

MODELING **Graph each function. Find the y-intercept and state the domain, range, and the equation of the asymptote. See margin for y-intercept, domain, range, and asymptote.**

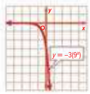
9. $y = 2\left(\frac{1}{6}\right)^x$



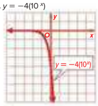
10. $y = \left(\frac{1}{12}\right)^x$



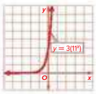
11. $y = -3(9^x)$



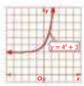
12. $y = -4(10^x)$



13. $y = 3(11^x)$



14. $y = 4^x + 3$



15. METEOROLOGY The atmospheric pressure in millibars at altitude x meters above sea level can be approximated by the function $f(x) = 1038(1.000134)^{-x}$ when x is between 0 and 10,000.

a. What is the atmospheric pressure at sea level? **1038 millibars**

b. The McDonald Observatory in Texas is at an altitude of 2000 meters. What is the approximate atmospheric pressure there? **about 794 millibars**

c. As altitude increases, what happens to atmospheric pressure? **It decreases.**

Higher-Order Thinking Skills

16. PERSEVERE Use tables and graphs to compare and contrast an exponential function $f(x) = ab^x + c$, where $a \neq 0$, $b > 0$, and $b \neq 1$, and a linear function $g(x) = ax + c$. Include intercepts, symmetry, end behavior, extrema, and intervals where the functions are increasing, decreasing, positive, or negative **See Mod. 8. Answer Appendix.**

17. CREATE Write an exponential function that passes through (0, 3) and (1, 6). **$f(x) = 3(2^x)$**

18. ANALYZE Determine whether the graph of $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$, *sometimes*, *always*, or *never* has an x-intercept. Justify your argument. **See margin.**

19. WRITE Find an exponential function that represents a real-world situation, and graph the function. Analyze the graph, and explain why the situation is modeled by an exponential function rather than a linear function. **See Mod. 8. Answer Appendix.**

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Answers

9. 2; D = {all real numbers}, R = $\{y \mid y > 0\}$; $y = 0$

10. 1; D = {all real numbers}, R = $\{y \mid y > 0\}$; $y = 0$

11. −3; D = {all real numbers}, R = $\{y \mid y < 0\}$; $y = 0$

12. −4; D = {all real numbers}, R = $\{y \mid y < 0\}$; $y = 0$

13. 3; D = {all real numbers}, R = $\{y \mid y > 0\}$; $y = 0$

14. 4; D = {all real numbers}, R = $\{y \mid y > 3\}$; $y = 3$

18. Never; the graph never intersects the x-axis because the powers of b are always positive and $a \neq 0$. Thus ab^x is never 0.

Lesson 8-2

F.IF.7e, F.BF.3

Transformations of Exponential Functions

LESSON GOAL

Students identify the effects of transformations of the graphs of exponential functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore:
- Translating Exponential Functions

Develop:

Translations of Exponential Functions
- Vertical Translations of Exponential Functions
- Horizontal Translations of Exponential Functions
- Multiple Translations of Exponential Functions
- Identify Exponential Functions from Graphs (Vertical Translations)
- Identify Exponential Functions from Graphs (Horizontal Translations)

Explore:
- Dilating Exponential Functions

Develop:

Dilations of Exponential Functions
- Vertical Dilations of Exponential Functions
- Horizontal Dilations of Exponential Functions
- Describe Dilations of Exponential Functions
- Identify Exponential Functions from Graphs (Dilations)

Explore:
- Reflecting Exponential Functions

Develop:

Reflections of Exponential Functions
- Vertical Reflections of Exponential Functions
- Horizontal Reflections of Exponential Functions

Transformations of Exponential Functions
- Multiple Transformations of Exponential Functions

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Suggested Pacing

90 min	1.5 days
45 min	3 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the values of k given the graphs.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
5 Use appropriate tools strategically.
7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students transformed linear functions and graphed exponential functions. **F.BF.3, F.IF.7e, F.LE.5**

Now
Students identify the effects of transformations on the graphs of exponential functions. **F.IF.7e, F.BF.3**

Next
Students will create exponential functions and solve problems involving exponential growth and decay. **F.LE.2, F.LE.5**

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

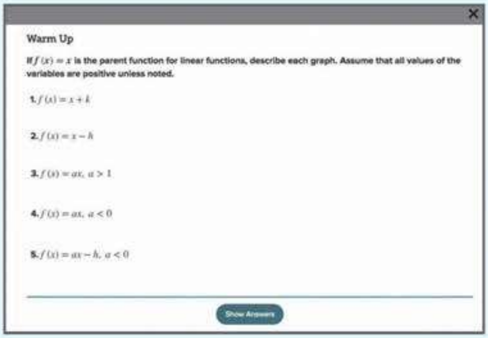
Resources	AL	OL	BL	ELL
Remediation: Transformations of Linear Functions	●	●		●
Extension: The Natural Base, *e*		●	●	●

Language Development Handbook

Assign page 44 of the *Language Development Handbook* to help your students build mathematical language related to transformations of the graphs of exponential functions.

ELL You can use the tips and suggestions on page T44 of the handbook to support students who are building English proficiency.

Interactive Presentation



Warm Up

Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

• transforming linear functions

Answers:

1. The graph of $f(x)$ moved up k units.
2. The graph of $f(x)$ moved down h units.
3. similar to the graph of $f(x)$, but steeper
4. similar to the graph of $f(x)$, but steeper and going down; a reflection of the graph in Exercise 3 over the y-axis
5. similar to the graph of $f(x)$, but steeper, going down, and moved down h units; a translation of the graph in Exercise 4 h units down

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of reflections of exponential functions.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Mathematical Background

This lesson focuses on transformations of exponential function graphs. Students will explore various methods of performing translations, dilations, and reflections of the graphs of exponential functions, as well as combinations of these transformations, by representing graphical functions in symbolic form.

Explore Translating Exponential Functions

Objective

Students use a graphing calculator to explore translations of exponential functions.

Teaching the Mathematical Practices

5 Analyze Graphs Help students analyze the graphs they have generated using graphing calculators. Point out that to see all the graphs, students may need to adjust the viewing window.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

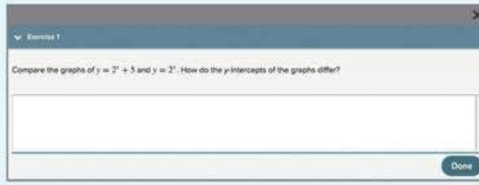
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use a graphing calculator to explore vertical and horizontal translations of exponential functions. They will identify how different function representations relate to the parent exponential function. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore

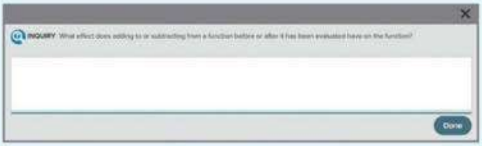


Explore

TAP

Students select a calculator to explore translations of exponential functions.

Interactive Presentation



Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Translating Exponential Functions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How do the functions $y = 2^x + 5$ and $y = 2^{(x + 5)}$ differ? Sample answer: The first equation has a vertical translation, while the second has a horizontal translation.
- What does it mean to subtract a value "before it has been evaluated"? Sample answer: In this case, it means to subtract a value from *x* and then use the difference as the exponent.

Inquiry

What effect does adding to or subtracting from a function before or after it has been evaluated have on the function? Sample answer: Adding to or subtracting from a function after it has been evaluated results in a shift up or down, while adding to or subtracting from a function before it has been evaluated results in a shift right or left.

Go Online to find additional teaching notes and answers for the guiding exercises.

Explore Dilating Exponential Functions

Objective

Students use a graphing calculator to explore dilations of exponential functions.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

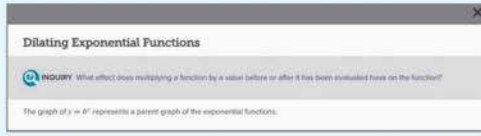
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

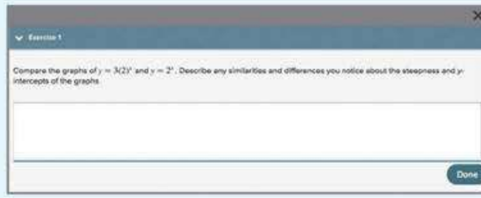
Students will complete guiding exercises throughout the Explore activity. Students are presented with a series of equations representing the dilation of exponential functions. They will graph these equations on graphing calculators and analyze the differences to identify the effects of dilating an exponential function. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

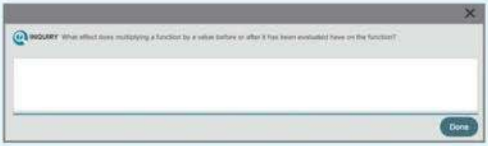


Explore

TAP

Students will select a calculator and tap through exercises to compare graphs.

Interactive Presentation



Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Dilating Exponential Functions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why does the y-intercept change between $y = 3(2)^x$ and $y = 2^x$?
  Sample answer: The y-intercept is 1 in the graph of $y = 2^x$ and this value is being multiplied by 3 in $y = 3(2)^x$. So the y-intercept becomes 3 in the graph of $y = 3(2)^x$.
- Why do the y-intercepts of $y = b^{ax}$ and $y = b^x$ stay the same?
  Sample answer: The value of a is multiplying x, which is zero at the y-intercept. Any number multiplied by zero is still zero, so the y-intercept will not change.

Inquiry

What effect does multiplying a function by a value before or after it has been evaluated have on the function? Sample answer: Multiplying by a after the function has been evaluated changes the steepness and y-intercept of the parent graph. Graphs in which a is greater are steeper. The y-intercept is multiplied by a. Multiplying by a before the function has been evaluated only changes the steepness of the parent graph.

Go Online to find additional teaching notes and answers for the guiding exercises.

Explore Reflecting Exponential Functions

Objective

Students use a graphing calculator to explore reflections of exponential functions.

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in the Explore activity, students will need to use a graphing calculator. Work with students to explore and deepen their understanding of reflections of exponential functions.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

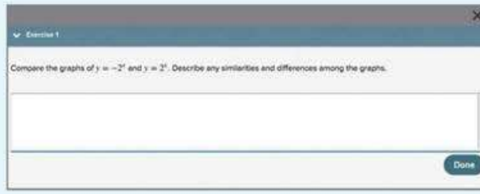
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will be presented with a series of exponential functions. They will graph these equations on graphing calculators and analyze the differences to identify the effects of reflecting an exponential functions. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore



Explore

TYPE

Students describe similarities and differences among the graphs.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Reflecting Exponential Functions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How would you describe the transformation of $y = -2$?
  Sample answer: vertical reflection across the x-axis
- How would you describe the transformation of $y = 2$?^{x}
  Sample answer: horizontal reflection across the y-axis

Inquiry

What effect does multiplying a function by −1 before or after it has been evaluated have on the function? Sample answer: Multiplying a function by −1 after it has been evaluated changes the direction in which the graph slopes, while multiplying a function by −1 before it has been evaluated reverses the end behavior of the graph.

Go Online to find additional teaching notes and answers for the guiding exercises.

Learn Translations of Exponential Functions

Objective

Students identify the effects on the graphs of exponential functions by replacing $f(x)$ with $f(x) + k$ and $f(x - h)$ for positive and negative values.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Example 1 Vertical Translations of Exponential Functions

Questions for Mathematical Discourse

AL What is the base in the function $g(x)$? 2

OL What is $f(0)$? What is $g(0)$? 1; 4

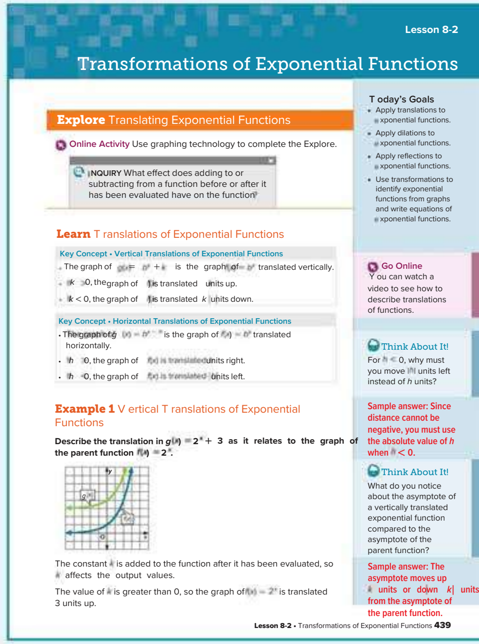
BL How does the range of $g(x)$ compare to the parent function? Sample answer: Because the function was translated up 3 units, all y-values will also increase by 3 units. The new range will be $y > 3$.

Common Error

Make sure that students connect to the equation $g(x) = f(x) + k$ to recognize that the translation happens to the output, or y-values, rather than to the input, or x-values.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 8-2

Transformations of Exponential Functions

Today's Goals
- Apply translations to exponential functions.
- Apply dilations to exponential functions.
- Apply reflections to exponential functions.
- Use transformations to identify exponential functions from graphs and write equations of exponential functions.

Explore Translating Exponential Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY What effect does adding to or subtracting from a function before or after it has been evaluated have on the function?

Learn Translations of Exponential Functions

Key Concept • Vertical Translations of Exponential Functions
The graph of $g(x) = b^x + k$ is the graph of $f(x) = b^x$ translated vertically.
- If $k > 0$, the graph of $f(x)$ is translated k units up.
- If $k < 0$, the graph of $f(x)$ is translated $|k|$ units down.

Key Concept • Horizontal Translations of Exponential Functions
The graph of $g(x) = b^{x-h}$ is the graph of $f(x) = b^x$ translated horizontally.
- If $h > 0$, the graph of $f(x)$ is translated h units right.
- If $h < 0$, the graph of $f(x)$ is translated $|h|$ units left.

Go Online You can watch a video to see how to describe translations of functions.

Think About It! For $h < 0$, why must you move $|h|$ units left instead of h units?

Sample answer: Since distance cannot be negative, you must use the absolute value of h when $h < 0$.

Example 1 Vertical Translations of Exponential Functions

Describe the translation in $g(x) = 2^x + 3$ as it relates to the graph of the parent function $f(x) = 2^x$.

The constant k is added to the function after it has been evaluated, so k affects the output values.

The value of k is greater than 0, so the graph of $f(x) = 2^x$ is translated 3 units up.

Think About It! What do you notice about the asymptote of a vertically translated exponential function compared to the asymptote of the parent function?

Sample answer: The asymptote moves up k units or down $|k|$ units from the asymptote of the parent function.

Lesson 8-2 • Transformations of Exponential Functions 439

Interactive Presentation

Learn

FLASHCARDS

Students tap on each card to compare translations of exponential functions to the parent function.

Think About It!
What do you notice about the asymptote of a horizontally translated exponential function compared to the asymptote of the parent function?

Sample answer: It is the same.

Example 2 Horizontal Translations of Exponential Functions

Describe the translation in $g(x) = 3^{x+1}$ as it relates to the graph of the parent function $f(x) = 3^x$.

The constant h is subtracted from x before the function is performed, so h affects the input values.

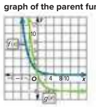
The value of h is less than 0, so the graph of $f(x) = 3^x$ is translated 1 unit left.

Think About It!
How can the placement of the constant tell you if the resulting transformation will be a vertical or horizontal translation?

Sample answer: If the constant is not in the exponent, it will be a vertical translation. If the constant is in the exponent, it will be a horizontal translation.

Example 3 Multiple Translations of Exponential Functions

Describe the translation in $g(x) = \left(\frac{1}{2}\right)^{x-2} - 4$ as it relates to the graph of the parent function $f(x) = \left(\frac{1}{2}\right)^x$.



The value of h is subtracted from x before the function is performed and is greater than 0, so the graph of $f(x) = \left(\frac{1}{2}\right)^x$ is translated 2 units right.

The value of k is added to the function after it has been evaluated and is less than 0, so the graph of $f(x) = \left(\frac{1}{2}\right)^x$ is also translated 4 units down.

Check

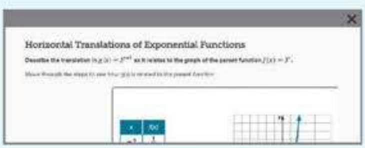
Describe the translation in $g(x) = 2^{x+1} - 8$ as it relates to the graph of the parent function $f(x) = 2^x$.

The graph of $g(x) = 2^{x+1} - 8$ is the translation of the graph of the parent function 1 unit ___?___ (left) and 8 units ___?___ (down).

Go Online You can complete an Extra Example online.

440 Module 8 • Exponential Functions

Interactive Presentation



Example 2

TAP

Students move through slides to see how to graph a horizontal translation of an exponential function.

Example 2 Horizontal Translations of Exponential Functions

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the parent function and the translated function used in this example.

Questions for Mathematical Discourse

AL What is the exponent in the function $g(x)$? $x + 1$

OL When is $f(x) = 1$? When is $g(x) = 1$? $x = 0$; $x = -1$

BL What would be the input of $g(x)$ that would result in $f(x)$? Why? $x - 1$; Sample answer: Substituting $x - 1$ for x, the exponent simplifies to $(x - 1) + 1 = x$.

Example 3 Multiple Translations of Exponential Functions

MP Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of equations of translations in this example to identify the vertical or horizontal translations in this example.

Questions for Mathematical Discourse

AL How can you identify a horizontal translation? Sample answer: Look for a value that is added to or subtracted from x. For exponential functions, that would be in the exponent.

OL What about $g(x)$ indicates that there will be a vertical translation? Explain. Minus four; sample answer: Subtracting 4 from the exponential function represents a vertical translation downward.

BL What about $g(x)$ indicates that there will be a horizontal translation? Explain. Minus 2 in the exponent; sample answer: Subtracting from the exponent represents a translation to the right.

Example 4 Identify Exponential Functions from Graphs (Vertical Translations)

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to use the relationships between the graphs of the parent function and translated function and their equations in this example.

Questions for Mathematical Discourse

AL What is the y-intercept of $g(x)$? −1

OL How is $g(x)$ vertically translated from the parent function $f(x)$? 2 units down

BL What y-intercept would indicate a vertical translation 2 units up from the parent graph? (0, 3)

Example 5 Identify Exponential Functions from Graphs (Horizontal Translations)

MP Teaching the Mathematical Practices

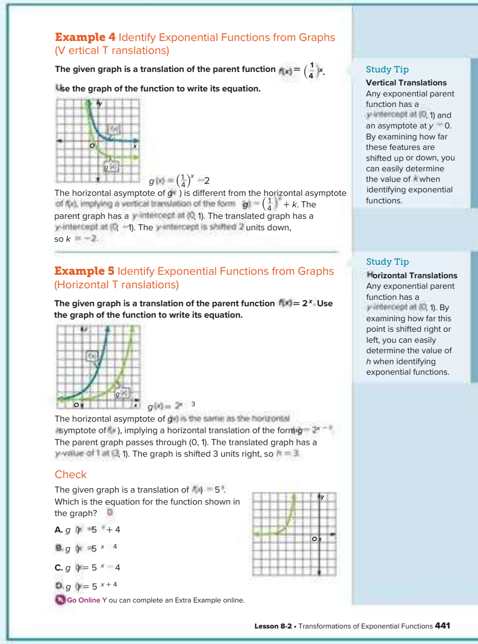
1 Explain Correspondences Encourage students to explain the relationships between the parent function and the translated function used in this example.

Questions for Mathematical Discourse

AL What is the base in the function $g(x)$? 2

OL How can you use a corresponding point on each graph to determine the horizontal translation? Sample answer: Identify points with the same y-coordinate on each function. The difference in the x-coordinates indicates the horizontal shift.

BL Do either $f(x)$ or $g(x)$ ever have an output value of 0? Explain. No; sample answer: Though both $f(x)$ and $g(x)$ get infinitely close to a value of 0 as x decreases, they will never reach that value. This is because the negative values of x in the exponent will make the output value of the exponential function closer to zero, but can never make it equal to zero.


Example 4 Identify Exponential Functions from Graphs (Vertical Translations)

The given graph is a translation of the parent function $f(x) = \left(\frac{1}{4}\right)^x$. Use the graph of the function to write its equation.

$g(x) = \left(\frac{1}{4}\right)^x - 2$

The horizontal asymptote of $g(x)$ is different from the horizontal asymptote of $f(x)$, implying a vertical translation of the form $g(x) = \left(\frac{1}{4}\right)^x + k$. The parent graph has a y-intercept at (0, 1). The translated graph has a y-intercept at (0, −1). The y-intercept is shifted 2 units down, so $k = -2$.

Study Tip
Vertical Translations Any exponential parent function has a y-intercept at (0, 1) and an asymptote at $y = 0$. By examining how far these features are shifted up or down, you can easily determine the value of k when identifying exponential functions.

Example 5 Identify Exponential Functions from Graphs (Horizontal Translations)

The given graph is a translation of the parent function $f(x) = 2^x$. Use the graph of the function to write its equation.

$g(x) = 2^{x-3}$

The horizontal asymptote of $g(x)$ is the same as the horizontal asymptote of $f(x)$, implying a horizontal translation of the form $g(x) = 2^{x-h}$. The parent graph passes through (0, 1). The translated graph has a y-value of 1 at (3, 1). The graph is shifted 3 units right, so $h = 3$.

Study Tip
Horizontal Translations Any exponential parent function has a y-intercept at (0, 1). By examining how far this point is shifted right or left, you can easily determine the value of h when identifying exponential functions.

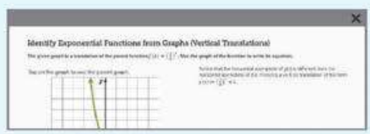
Check
The given graph is a translation of $f(x) = 5^x$. Which is the equation for the function shown in the graph? D

A. $g(x) = 5^x + 4$
B. $g(x) = 5^{x-4}$
C. $g(x) = 5^x - 4$
D. $g(x) = 5^{x+4}$

Go Online You can complete an Extra Example online.

Lesson 8-2 • Transformations of Exponential Functions 441

Interactive Presentation



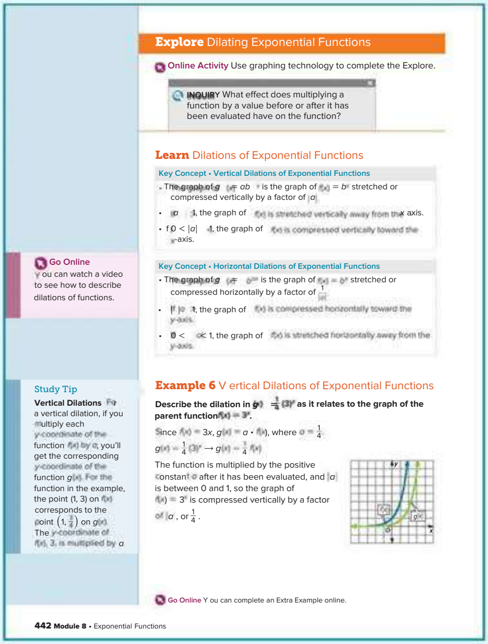
Example 4

TYPE

Students complete the statements to identify the translation in the function.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Explore Dilating Exponential Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY What effect does multiplying a function by a value before or after it has been evaluated have on the function?

Learn Dilations of Exponential Functions

Key Concept • Vertical Dilations of Exponential Functions

- The graph of $g(x) = ab^x$ is the graph of $f(x) = b^x$ stretched or compressed vertically by a factor of $|a|$.
- If $|a| > 1$, the graph of $f(x)$ is stretched vertically away from the x-axis.
- If $0 < |a| < 1$, the graph of $f(x)$ is compressed vertically toward the x-axis.

Key Concept • Horizontal Dilations of Exponential Functions

- The graph of $g(x) = b^{ax}$ is the graph of $f(x) = b^x$ stretched or compressed horizontally by a factor of $\frac{1}{|a|}$.
- If $|a| > 1$, the graph of $f(x)$ is compressed horizontally toward the y-axis.
- If $0 < |a| < 1$, the graph of $f(x)$ is stretched horizontally away from the y-axis.

Go Online You can watch a video to see how to describe dilations of functions.

Study Tip

Vertical Dilations For a vertical dilation, if you multiply each y-coordinate of the function $f(x)$ by a, you'll get the corresponding y-coordinate of the function $g(x)$. For the function in the example, the point (1, 3) on $f(x)$ corresponds to the point $\left(1, \frac{3}{4}\right)$ on $g(x)$. The y-coordinate of $f(x)$, 3, is multiplied by a.

Example 6 Vertical Dilations of Exponential Functions

Describe the dilation in $g(x) = \frac{1}{4}(3)^x$ as it relates to the graph of the parent function $f(x) = 3^x$.

Since $f(x) = 3x$, $g(x) = a \cdot f(x)$, where $a = \frac{1}{4}$.

$g(x) = \frac{1}{4}(3)^x \rightarrow g(x) = \frac{1}{4}f(x)$

The function is multiplied by the positive constant a after it has been evaluated, and $|a|$ is between 0 and 1, so the graph of $f(x) = 3^x$ is compressed vertically by a factor of $|a|$, or $\frac{1}{4}$.

Go Online You can complete an Extra Example online.

442 Module 8 • Exponential Functions

Learn Dilations of Exponential Functions

Objective

Students identify the effects on the graphs of exponential functions by replacing $af(x)$ with $f(ax)$.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of dilations of exponential functions in this Learn.

What Students Are Learning

A function $g(x)$ is a vertical dilation of $f(x)$ when it can be mapped with the relationship $g(x) = af(x)$. This means that each y-coordinate of the function $f(x)$ is multiplied by a to get the corresponding y-coordinate of the function $g(x)$. A horizontal dilation $g(x)$ of $f(x)$ can be mapped with the relationship $g(x) = f(ax)$.

Example 6 Vertical Dilations of Exponential Functions

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. In this example, students should use clear mathematical language to describe the dilation.

Questions for Mathematical Discourse

AL What is the asymptote of $g(x)$? of $f(x)$? $y = 0$; $y = 0$

OL What is the y-intercept of $f(x)$? of $g(x)$? 1; $\frac{1}{4}$

BL Is there any input value that will result in $f(x) = g(x)$? Explain. No; sample answer: Each value of $g(x)$ will be one-fourth the value of $f(x)$. Because there is no output of 0 for this function, there is no value where one-fourth of the output will be equal to the original output.

Common Error

The graph of $f(x)$ and $g(x)$ in Example 6 are dilations that both have the asymptote of $y = 0$. In the graph, it looks like these two functions touch each other on the left side of the graph, but it is important for students to realize that this is an illusion based on the scale of the graph. Test values or experiment with zooming in on graphing calculators to convince students that at any interval in which you look at the values, $f(x)$ and $g(x)$ will be different by a factor of a.

Interactive Presentation

Example 6

SWIPE

Students move through slides to see how to graph a dilation of an exponential function.

Example 7 Horizontal Dilations of Exponential Functions

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the equations and graphs of the functions in this example.

Questions for Mathematical Discourse

AL What is the base of $g(x)$? The exponent? $\frac{5}{3}$; $2x$

OL What is the y-intercept of $f(x)$? Of $g(x)$? 1; 1

BL How can you tell from the base of $f(x)$ and $g(x)$ that the functions will be increasing? The base is $\frac{5}{3}$, which is greater than 1, so the function will increase.

Example 8 Describe Dilations of Exponential Functions

MP Teaching the Mathematical Practices

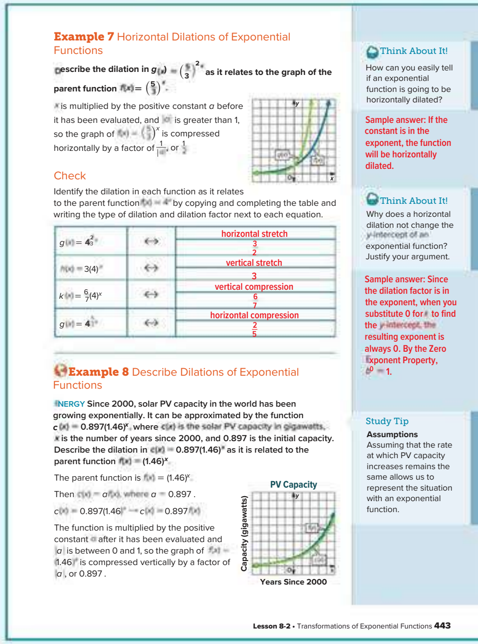
4 Make Assumptions In the Study Tip, have students point out where an assumption or approximation was made in the solution.

Questions for Mathematical Discourse

AL What does $x = 0$ represent in this situation? the year 2000

OL Do you expect the graph of $c(x)$ to grow more or less rapidly than $f(x)$? Explain. Less rapidly; sample answer: Because each output is being multiplied by a value less than one, each y-value of $c(x)$ will be less than the y-value of $f(x)$.

BL What will be the solar PV capacity in the year 2030 to the nearest hundredth, according to the function $c(x)$? 76,450.11



Example 7 Horizontal Dilations of Exponential Functions

Describe the dilation in $g(x) = \left(\frac{5}{3}\right)^{2x}$ as it relates to the graph of the parent function $f(x) = \left(\frac{5}{3}\right)^x$.

x is multiplied by the positive constant a before it has been evaluated, and $|a|$ is greater than 1, so the graph of $f(x) = \left(\frac{5}{3}\right)^x$ is compressed horizontally by a factor of $\frac{1}{|a|}$, or $\frac{1}{2}$.

Check

Identify the dilation in each function as it relates to the parent function $f(x) = 4^x$ by copying and completing the table and writing the type of dilation and dilation factor next to each equation.

$g(x) = 4^{\frac{2}{3}x}$	⟷	horizontal stretch; $\frac{3}{2}$
$h(x) = 3(4)^x$	⟷	vertical stretch; 3
$k(x) = \frac{6}{7}(4)^x$	⟷	vertical compression; $\frac{6}{7}$
$g(x) = 4^{\frac{5}{2}x}$	⟷	horizontal compression; $\frac{2}{5}$

Example 8 Describe Dilations of Exponential Functions

ENERGY **Since 2000, solar PV capacity in the world has been growing exponentially. It can be approximated by the function $c(x) = 0.897(1.46)^x$, where $c(x)$ is the solar PV capacity in gigawatts, x is the number of years since 2000, and 0.897 is the initial capacity. Describe the dilation in $c(x) = 0.897(1.46)^x$ as it is related to the parent function $f(x) = (1.46)^x$.**

The parent function is $f(x) = (1.46)^x$.

Then $c(x) = af(x)$, where $a = 0.897$.

$c(x) = 0.897(1.46)^x \rightarrow c(x) = 0.897f(x)$

The function is multiplied by the positive constant a after it has been evaluated and $|a|$ is between 0 and 1, so the graph of $f(x) = (1.46)^x$ is compressed vertically by a factor of $|a|$, or 0.897.

Think About It! How can you easily tell if an exponential function is going to be horizontally dilated?

Sample answer: If the constant is in the exponent, the function will be horizontally dilated.

Think About It! Why does a horizontal dilation not change the y-intercept of an exponential function? Justify your argument.

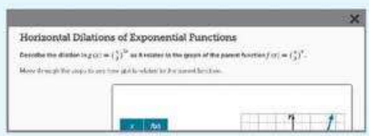
Sample answer: Since the dilation factor is in the exponent, when you substitute 0 for x to find the y-intercept, the resulting exponent is always 0. By the Zero Exponent Property, $b^0 = 1$.

Study Tip

Assumptions Assuming that the rate at which PV capacity increases remains the same allows us to represent the situation with an exponential function.

Lesson 8-2 • Transformations of Exponential Functions 443

Interactive Presentation



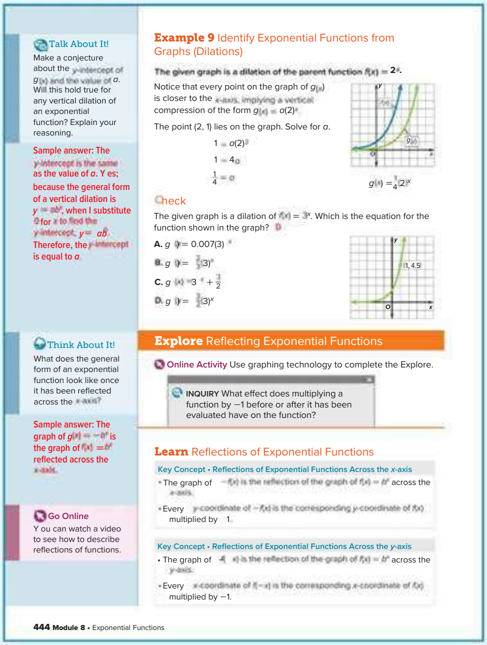
Example 7

SWIPE

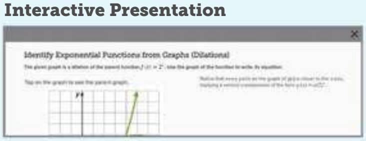
Students move through slides to see how to graph a dilation of an exponential function.

TYPE

Students answer questions about horizontal dilations of exponential functions.



Interactive Presentation



Example 9

TYPE

Students make a conjecture about the relationship between the y-intercept and the value of a.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 9 Identify Exponential Functions from Graphs (Dilations)

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL What is the base in function $f(x)$? 2

OL What is the y-intercept of $g(x)$? $\frac{1}{4}$

BL Why does it make sense that $a < 1$? Sample answer: The transformed function is below the parent function, so you can tell that it is a vertical compression. It makes sense that the values would be a fraction of the original.

Learn Reflections of Exponential Functions

Objective

Students identify the effects on the graphs of exponential functions by replacing $-af(x)$ with $f(-ax)$.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the parent function and the reflected function used in this Learn.

Example 10 Vertical Reflections of Exponential Functions

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of $g(x)$ to identify the transformations in the function.

Questions for Mathematical Discourse

AL What number is multiplied by $f(x)$ to obtain $g(x)$? -3

OL What is the range of $f(x)$? Of $g(x)$? $f(x) > 0$; $g(x) < 0$

BL What about the function $g(x)$ would need to be different for it to be a vertical reflection without any vertical stretching? Instead of multiplying by -3, it would need to be multiplied by -1.

Example 11 Horizontal Reflections of Exponential Functions

Teaching the Mathematical Practices

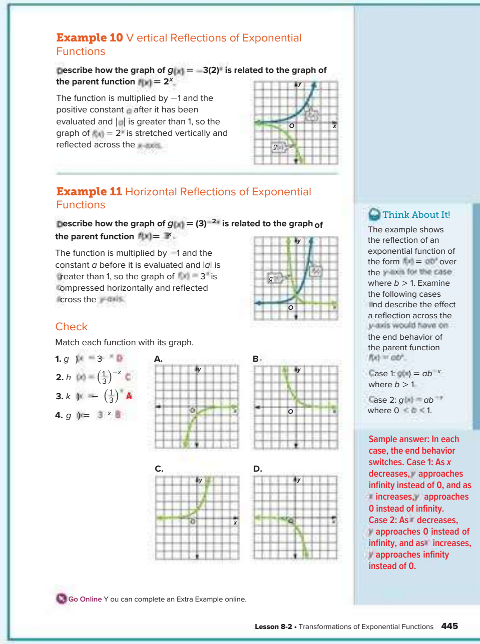
3 Analyze Cases The Think About It! feature guides students to examine the cases of reflections of exponential functions. Encourage students to familiarize themselves with each case.

Questions for Mathematical Discourse

AL What is different about the two functions? Sample answer: The exponent of $f(x)$ is just x, but the exponent of $g(x)$ is $-2x$.

OL Does $g(x)$ represent exponential growth or decay? decay

BL How can you rewrite $g(x)$ as a function with an exponent of x? Show your work. $g(x) = 3^{-2x} = (3^{2})^{-x} = \left(\frac{1}{9}\right)^x$



Example 10 Vertical Reflections of Exponential Functions

Describe how the graph of $g(x) = -3(2)^x$ is related to the graph of the parent function $f(x) = 2^x$.

The function is multiplied by -1 and the positive constant a after it has been evaluated and $|a|$ is greater than 1, so the graph of $f(x) = 2^x$ is stretched vertically and reflected across the x-axis.

Example 11 Horizontal Reflections of Exponential Functions

Describe how the graph of $g(x) = (3)^{-2x}$ is related to the graph of the parent function $f(x) = 3^x$.

The function is multiplied by -1 and the constant a before it is evaluated and $|a|$ is greater than 1, so the graph of $f(x) = 3^x$ is compressed horizontally and reflected across the y-axis.

Check

Match each function with its graph.

1. $g(x) = 3^{-x}$ D
2. $h(x) = \left(\frac{1}{3}\right)^{-x}$ C
3. $k(x) = -\left(\frac{1}{3}\right)^x$ A
4. $g(x) = 3^{-x}$ B

A. B. C. D.

Go Online You can complete an Extra Example online.

Think About It!

The example shows the reflection of an exponential function of the form $f(x) = ab^x$ over the y-axis for the case where $b > 1$. Examine the following cases and describe the effect a reflection across the y-axis would have on the end behavior of the parent function $f(x) = ab^x$.

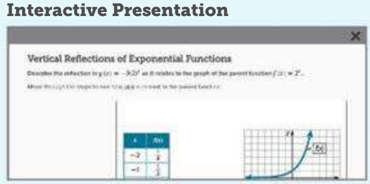
Case 1: $g(x) = ab^{-x}$ where $b > 1$

Case 2: $g(x) = ab^{-x}$ where $0 < b < 1$.

Sample answer: In each case, the end behavior switches. Case 1: As x decreases, y approaches infinity instead of 0, and as x increases, y approaches 0 instead of infinity. Case 2: As x decreases, y approaches 0 instead of infinity, and as x increases, y approaches infinity instead of 0.

Lesson 8-2 • Transformations of Exponential Functions 445

Interactive Presentation



Example 10

SWIPE

Students move through slides to see how to graph a reflection of an exponential function.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Go Online You can watch a video to see how to graph transformations of exponential functions using a graphing calculator.

Think About It!

Write an exponential function that is the parent function $f(x) = 2^x$ stretched vertically and translated 3 units left and 6 units up.

Sample answer: $g(x) = 4(2)^{x+3} + 6$

Learn Transformations of Exponential Functions

The general form of an exponential function is $f(x) = ab^{x-h} + k$, where a, h, and k are parameters that dilate, reflect, or translate a parent function with base b.

- The value of $|a|$ stretches or compresses (dilates) the parent graph.
- When a is negative, the graph is reflected across the x-axis.
- The value of h shifts (translates) the parent graph right or left.
- The value of k shifts (translates) the parent graph up or down.

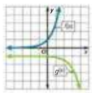
Example 12 Multiple Transformations of Exponential Functions

Describe how the graph of $g(x) = -\frac{1}{2}(3)^{x-2} - 1$ is related to the graph of the parent function $f(x) = 3^x$.

$a < 0$ and $0 < |a| < 1$, so the graph of $f(x) = 3^x$ is reflected across the x-axis and compressed vertically by a factor of $|a|$, or $\frac{1}{2}$.

$h > 0$, so the graph is then translated h units right, or 2 units right.

$k < 0$, so the graph is then translated $|k|$ units down, or 1 unit down.



$g(x) = -\frac{1}{2}(3)^{x-2} - 1$ is the graph of the parent function compressed vertically, reflected across the x-axis, and translated 2 units right and 1 unit down.

Check

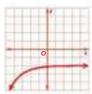
Part A

Describe how the graph of $g(x) = -4\left(\frac{1}{2}\right)^{x+5} - 2$ is related to the graph of the parent function $f(x) = \left(\frac{1}{2}\right)^x$.

The graph of $f(x) = \left(\frac{1}{2}\right)^x$ is reflected across the ___?___ (x-axis) and ___?___ (stretched) vertically. The graph is translated 5 units ___?___ (left) and 2 units ___?___ (down).

Part B

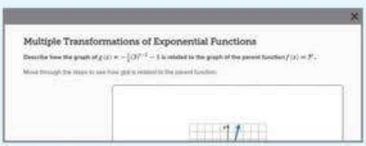
Sketch the graph of $g(x) = -4\left(\frac{1}{2}\right)^{x+5} - 2$.



Go Online You can complete an Extra Example online.

446 **Module 8** • Exponential Functions

Interactive Presentation



Example 12

TYPE

Students write an exponential function of a transformed function.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Learn Transformations of Exponential Functions

Objective

Students use transformations to identify exponential functions from graphs and write equations of exponential functions.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of transformations of exponential functions in this Learn.

Example 12 Multiple Transformations of Exponential Functions

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the parent function and the transformed function used in this example.

Questions for Mathematical Discourse

AL What is the exponent in $g(x)$? $x - 2$ What is the value of h? 2

OL How many different transformations are performed on $g(x)$? Explain. 4; Sample answer; There are values for a, h and k in the transformed equation. Also, $a < 0$, which means there is a reflection and a dilation.

BL Which part of $g(x)$ represents a reflection? Explain. Sample answer: The value of $a = -\frac{1}{2}$ represents a vertical reflection and a compression of the graph by one-half.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–20
2	exercises that use a variety of skills from this lesson	21–43
2	exercises that extend concepts learned in this lesson to new contexts	44, 45
3	exercises that emphasize higher-order and critical-thinking skills	46–50

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–45 odd, 46–50
- Extension: The Natural Base, *e*
- ALEKS Exponential Functions

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–49 odd
- Remediation, Review Resources: Transformations of Linear Functions
- Personal Tutors
- Extra Examples 1–12
- ALEKS Equations of Lines

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–19 odd
- Remediation, Review Resources: Transformations of Linear Functions
- ALEKS Equations of Lines
- Arrive**MATH** Take Another Look

Practice

Go Online Y ou can complete your homework online.

Examples 1–3, 6–7, 10–12

Describe the transformation of $g(x)$ as it relates to the parent function $f(x)$.

1. $f(x) = 6^x$; $g(x) = 6^x + 8$
translated up 8 units

2. $f(x) = 5^x$; $g(x) = -5^x$
reflected across the *x*-axis

3. $f(x) = 3^x + 1$; $g(x) = 3^{2x} + 1$
compressed horizontally

4. $f(x) = 4^x - 3$; $g(x) = 4^{0.5x} - 3$
stretched horizontally

5. $f(x) = 2.3^x$; $g(x) = -2.3^{x-1}$
reflected across the *x*-axis; translated 1 unit right

6. $f(x) = 2^x$; $g(x) = 2^{-x} + 1$
reflected across the *y*-axis; translated 1 unit up

7. $f(x) = 5^x + 2$; $g(x) = 5^{-x} + 6$
reflected across the *y*-axis; translated 4 units up

8. $f(x) = 1.4^x - 1$; $g(x) = -1.4^x + 6$
reflected across the *x*-axis; translated 7 units up

9. $f(x) = 3^x + 1$; $g(x) = 2(3^x + 1)$
stretched vertically; shifted 2 units up

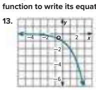
10. $f(x) = -4^x$; $g(x) = \frac{1}{3}(-4^x)$
compressed vertically

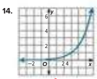
11. $f(x) = 4^x$; $g(x) = 4^{x-3}$
translated right 3 units

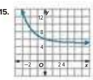
12. $f(x) = \left(\frac{1}{2}\right)^x + 5$; $g(x) = \left(\frac{1}{2}\right)^x$
translated down 5 units

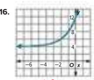
Examples 4–5, 9

Each graph is a transformation of the parent function $y = 2^x$. Use the graph of the function to write its equation.

13.

$y = -2^x$

14.

$y = 2^{x-3}$

15.

$y = 2^{-x} + 5$

16.

$y = 2^{x+3} + 4$

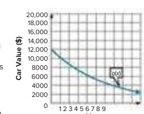
Example 8

17. SAVING Celia invests $2000 in a savings account that earns 1.25% interest per year compounded annually. The amount of money in her bank account after x years can be modeled by $g(x) = 2000(1.0125)^x$. Describe the dilation in $g(x)$ as it relates to the parent function $f(x) = 1.0125^x$.
stretched vertically by a factor of 2000

18. CAFFEINE Suppose an 8-ounce cup of coffee contains 100 milligrams of caffeine. The rate at which caffeine is eliminated from an adult's body is 11% per hour. The function $f(x) = 100(0.89)^x$ can be used to model the amount of caffeine left in a person's bloodstream after x hours of consuming the cup of coffee. Suppose the function $g(x) = 25(0.89)^x$ represents the amount of caffeine left in a person's bloodstream after x hours of consuming an 8-ounce cup of green tea. Describe $g(x)$ as a transformation of $f(x)$. compressed vertically by a factor of $\frac{1}{4}$

19. VISITORS The number of visitors to a new skateboarding park can be modeled by the exponential function $g(x) = 20(2^x)$, where x represents the number of months since the park's grand opening. Explain how the number of visitors during that first month is a dilation of the parent function $f(x) = 2^x$. stretched vertically by a factor of 20

20. DEPRECIATION Depreciation is the decrease in the value of an item resulting from its age or wear. When an item loses about the same percent of its value each year, an exponential function can be used to model its decreasing value over time. The function $g(x) = 12{,}000(0.85)^x$ can be used to model the value of a $12,000 car as it depreciates at an annual rate of 15% over x years. $g(x)$ is a dilation of the parent function $f(x) = 0.85^x$. The graph shows the function $g(x)$.



a. Write the equation of a function $h(x)$ that represents the depreciation of a $20,000 car depreciating at the same rate over x years. $h(x) = 20{,}000(0.85)^x$

b. Describe $h(x)$ as it relates to the parent function. stretched vertically by a factor of 20,000

c. What is the difference between the values of the $12,000 car and the $20,000 car after 5 years? They differ by approximately $3500.

Mixed Exercises

Describe the transformation of $g(x)$ as it relates to the parent function $f(x) = 2^x$.

21. $g(x) = 2^x + 6$
translated up 6 units

22. $g(x) = 3(2)^x$
stretched vertically

23. $g(x) = -\frac{1}{4}(2)^x$
reflected across the *x*-axis; compressed vertically

24. $g(x) = -3 + 2^x$
translated down 3 units

25. $g(x) = 2^{-x}$
reflected across the *y*-axis

26. $g(x) = -5(2)^x$
reflected across the *x*-axis; stretched vertically

Write a function $g(x)$ to represent the transformation of the parent function $f(x)$.

27. $f(x) = 2^x$ translated 3 units up
$g(x) = 2^x + 3$

28. $f(x) = 8^x$ translated 1 unit down
$g(x) = 8^x - 1$

29. $f(x) = 5^x$ translated 2 units right
$g(x) = 5^{x-2}$

30. $f(x) = 3^x$ translated 4 units left
$g(x) = 3^{x+4}$

31. $f(x) = 6^x + 7$ translated 2 units down
$g(x) = 6^x + 5$

32. $f(x) = -2^x + 3$ translated 5 units right
$g(x) = -2^{x-5} + 3$

33. $f(x) = 4^x$ is compressed vertically by a factor of $\frac{1}{2}$
$g(x) = \frac{1}{2}(4^x)$

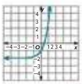
34. $f(x) = 3^x$ is stretched vertically by a factor of 5
$g(x) = 5(3^x)$

35. $f(x) = 2^x$ is compressed horizontally by a factor of $\frac{1}{3}$
$g(x) = 2^{3x}$

36. $f(x) = 5^x$ is stretched horizontally by a factor of 4
$g(x) = 5^{0.25x}$

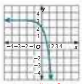
Each graph is a transformation of the parent function $f(x) = 5^x$. Use the graph of the function to write its equation.

37.



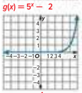
$g(x) = 5^x - 2$

38.



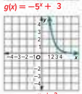
$g(x) = -5^x + 3$

39.



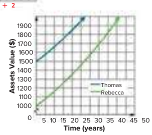
$g(x) = 5^{x-4}$

40.



$g(x) = 5^{-x+2}$

41. ASSETS Thomas and Rebecca each put $1000 into a bank account that earns 1.5% interest per year compounded annually. Thomas also has an antique toy automobile. The graph shows the amount of their assets over time.



a. Describe the graph of Thomas' assets as a transformation of Rebecca's assets. translated up 500 units

b. Use the graph to extrapolate the value of Thomas' antique toy automobile. $500

The graph of $g(x)$ is a transformation of the parent function $f(x)$. Graph $g(x)$ and describe the transformation in each function as it relates to the parent function.

42. $f(x) = 3^x$
$g(x) = 5 \cdot 3^{x+2} - 4$
See margin.

43. $f(x) = 2^x$
$g(x) = -3 \cdot 2^{-x} + 1$
See margin.

44. CONSTRUCT ARGUMENTS Name the coordinates of the point at which the graphs of $g(x) = 2^x + 3$ and $h(x) = 5^x + 3$ intersect. Explain your reasoning. The parent function $f(x) = b^x$ goes through the point (0, 1). Both of these functions have a vertical translation up 3. This means the graphs will intersect at (0, 4).

45. STRUCTURE Describe the similarities between the graph of $f(x) = 4^{x+2}$ and the graph of $g(x) = 16 \cdot 4^x$. Use the properties of exponents to justify your answer. The graphs of these two exponential functions are the same. $f(x) = 4^{x+2} = 4^x \cdot 4^2 = 16 \cdot 4^x = g(x)$

Higher-Order Thinking Skills

46. ANALYZE What would happen to the shape of the graph of an exponential function if the function is multiplied by a number between 0 and −1? What would happen to its shape if the exponent is multiplied by a number between 0 and −1? Justify your argument. See margin.

47. FIND THE ERROR Jennifer claims that the graph of $g(x) = 2(2^x)$ is a graph that rises more rapidly than its parent function $f(x) = 2^x$. James claims that it is actually the parent graph shifted to the left 2 units. Who is correct? Explain your reasoning. See margin.

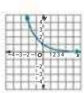
48. WRITE A deficit is a negative amount of some quantity, such as money. A deficit that is growing exponentially can be modeled by $y = ab^{c(x-h)} + k$. Describe the constraints on a, b, and c. A deficit that is exponentially growing is modeled where $a < 0$ and either $b > 1$ and $c < 0$ or $0 < b < 1$ and $c < 0$.

49. WHICH ONE DOESN'T BELONG? Consider each pair of transformations of the function $f(x)$ to $g(x)$. Which one does not belong? Justify your conclusion.

$f(x) = 3^{x+2}$ $g(x) = 3^{x-1} + 2$	$f(x) = 2^x$ $g(x) = 2^{x+3} + 2$	$f(x) = 4^{x+1}$ $g(x) = 4^{x+4} + 2$

The first pair; $g(x)$ is shifted right 3 units instead of left 3 units.

50. CREATE The graph shows a parent function $f(x)$.



a. Write a function to represent the parent function $f(x)$. $f(x) = \left(\frac{1}{2}\right)^x$

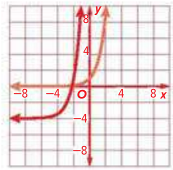
b. Write a function to represent a transformation of the parent function $g(x)$. Sample answer: $g(x) = 4\left(\frac{1}{2}\right)^x - 1$

c. Describe the transformation. Sample answer: $g(x)$ is stretched vertically by a factor of 4 and shifted down 1 unit.

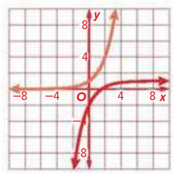
d. Graph the transformed function $g(x)$. See margin.

Answers

42. The graph is shifted left 2 units, stretched vertically by a factor of 5, and then shifted down 4 units.



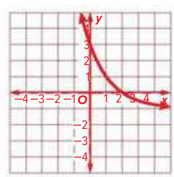
43. The graph has been reflected over the x-axis and reflected over the y-axis. It has been stretched vertically by a factor of 3 and shifted up 1 unit.



46. Sample answer: In each case, transformations. When multiplying the function by a number between 0 and negative 1, reflect the graph over the x-axis and flatten its shape. When multiplying the exponent by a number between 0 and negative one, reflect over the y-axis and stretch its shape.

47. Jennifer is correct. Sample answer: As it is written, the function is multiplied by 2, which causes the graph to rise more rapidly than the parent graph, so Jennifer is correct. However, $g(x) = 2(2^x)$ is equivalent to $g(x) = 2^{x+1}$. This graph is the parent graph of $f(x) = 2^x$ shifted to the left one unit, but it still rises at the same rate.

50d. Sample answer:



Writing Exponential Functions

F.LE.2, F.LE.5

LESSON GOAL

Students create exponential functions and solve problems involving exponential growth and decay.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Writing an Exponential Function to Model Population Growth

Develop:

Constructing Exponential Functions
- Write an Exponential Function Given Two Points
- Write an Exponential Function Given a Graph
- Write an Exponential Function Given a Description

Solving Problems Involving Exponential Growth
- Exponential Growth
- Compound Interest

Solving Problems Involving Exponential Decay
- Exponential Decay

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Construct Linear Functions	●	●		●
Extension: Continuously Compounding Interest		●	●	●

Language Development Handbook

Assign page 45 of the *Language Development Handbook* to help your students build mathematical language related to exponential growth and decay.

ELL You can use the tips and suggestions on page T45 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (including reading these from a table).

F.LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

Standards for Mathematical Practice:

6 Attend to precision.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students identified the effects of transformations on the graphs of exponential functions.
F.IF.7e, F.BF.3

Now
Students create exponential functions and solve problems involving exponential growth and decay.
F.LE.2, F.LE.5

Next
Students will use the properties of exponents to transform expressions for exponential functions.
A.SSE.3c, F.IF.8b

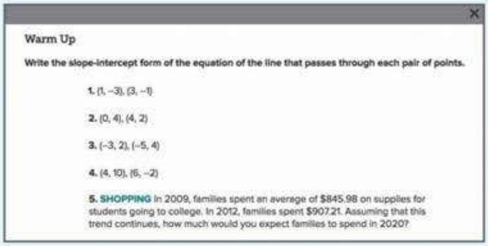
Rigor

The Three Pillars of Rigor

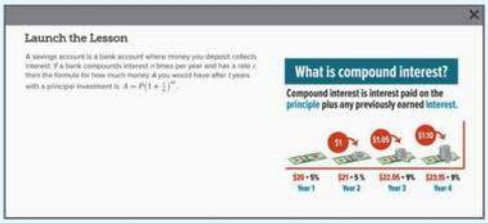
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge Working through the Explore and Learn activities can help students build a bridge to conceptual understanding. When students understand how to create exponential functions and solve problems involving exponential growth and decay, they can move to procedural fluency and apply the math to problems in everyday life.

Interactive Presentation



Warm Up



Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- writing linear functions

Answers:

1. $y = x - 4$
2. $y = -\frac{1}{2}x + 4$
3. $y = -x - 1$
4. $y = -6x + 34$
5. $1070.49

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of exponential functions.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then discuss the questions below with the class.

Mathematical Background

This lesson introduces methods for writing exponential functions to fit a variety of exponential situations. Exponential behavior is defined by the rate of change. The equation for exponential growth represents the growth rate, r, in a function of the form $y = a(1 + r)^t$, where t represents time from the beginning of the growth. The equation for exponential decay represents the decay rate, r, in a function of the form $y = a(1 - r)^t$, where t represents time from the beginning of the decay.

Explore Writing an Exponential Function to Model Population Growth

Objective

Students explore writing exponential equations to model real-world situations.

Teaching the Mathematical Practices

4 Interpret Mathematical Results In this Explore activity, point out that to solve the problem, students should interpret their mathematical results in the context of the problem.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

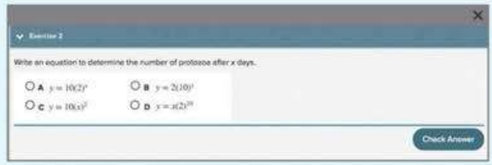
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. The students are presented with a situation in which a population of protozoa double every day. As students answer questions related to the growth of this population, they will make connections to their understanding of exponential functions. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore



Explore

MULTIPLE CHOICE

Students select an equation to determine the number of protozoa after *x* days.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Writing an Exponential Function to Model Population Growth (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How could you determine the number of days if you counted 50 protozoa? Sample answer: Because there are more than 40 protozoa, but less than 80, it must be sometime during day 3.
- What situation could be described by $y = 2(10)^x$? Sample answer: Two bacteria whose population increase by 10 times each day.

Inquiry

How can you find an equation that models the population growth of a colony of organisms that grows exponentially? Sample answer: Find the initial size of the colony of organisms to determine the y-intercept. The original size of the population at time $x = 0$ is a in the equation $y = ab^x$. Find the size of the population y at another time x, and use those values of x and y to determine the value for b.

Go Online to find additional teaching notes and answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Constructing Exponential Functions

Objective

Students construct exponential functions by using a graph, a description, or two points.

Teaching the Mathematical Practices

1 Special Cases Work with students to examine the three methods for writing exponential functions. Encourage students to familiarize themselves with each method, and to know the best time to use each one.

Example 1 Write an Exponential Function Given Two Points

Teaching the Mathematical Practices

4 Make Assumptions In the Study Tip, have students point out where an assumption was made in the solution.

Questions for Mathematical Discourse

AL Is the exponential function increasing or decreasing? Explain. Increasing; sample answer: From the two points, you can tell that the higher input value results in a higher output value, so it is an increasing exponential function.

OL What about the equations $6 = ab^1$ and $24 = ab^3$ suggests that the first equation should be solved for the variable a first? Sample answer: The exponent in the first equation is 1, while the exponent in the second equation is 3.

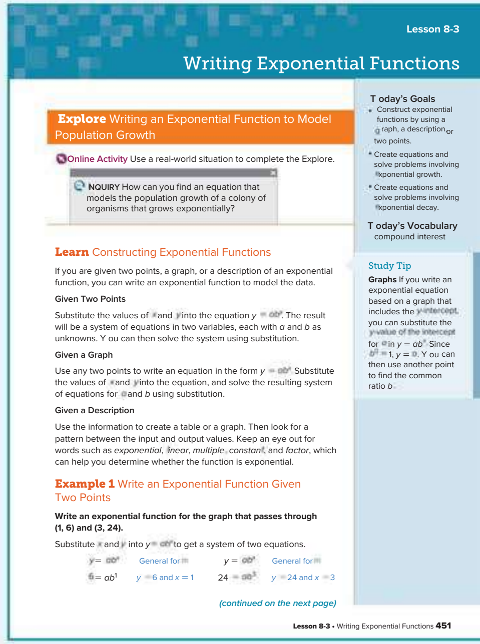
BL What is the value of this equation when $x = 4$? 48

Common Error

Students may realize that the equation $4 = b^2$ could result in a solution of $b = -2$. If this comes up, encourage students to consider this case, which would result in $a = -3$ and an equation of $y = (-3)(-2)^x$. Plot points in this equation to see if it is an exponential function, and view the results using graphing software or a graphing calculator. Students should be able to identify that the negative base is the reason why this function is not an exponential function. At that point, go back to the definition of an exponential function and remind students that there is a limitation of $b > 0$ in that definition.

Go Online

- F ind additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.


Lesson 8-3

Writing Exponential Functions

Explore Writing an Exponential Function to Model Population Growth

Online Activity Use a real-world situation to complete the Explore.

NQUIRY How can you find an equation that models the population growth of a colony of organisms that grows exponentially?

Learn Constructing Exponential Functions

If you are given two points, a graph, or a description of an exponential function, you can write an exponential function to model the data.

Given Two Points

Substitute the values of x and y into the equation $y = ab^x$. The result will be a system of equations in two variables, each with a and b as unknowns. Y ou can then solve the system using substitution.

Given a Graph

Use any two points to write an equation in the form $y = ab^x$. Substitute the values of x and y into the equation, and solve the resulting system of equations for a and b using substitution.

Given a Description

Use the information to create a table or a graph. Then look for a pattern between the input and output values. Keep an eye out for words such as *exponential*, *linear*, *multiple*, *constant*, and *factor*, which can help you determine whether the function is exponential.

Example 1 Write an Exponential Function Given Two Points

Write an exponential function for the graph that passes through (1, 6) and (3, 24).

Substitute x and y into $y = ab^x$ to get a system of two equations.

$y = ab^x$	General form	$y = ab^x$	General form
$6 = ab^1$	$y = 6$ and $x = 1$	$24 = ab^3$	$y = 24$ and $x = 3$

(continued on the next page)

T oday's Goals
- Construct exponential functions by using a graph, a description, or two points.
- Create equations and solve problems involving exponential growth.
- Create equations and solve problems involving exponential decay.

T oday's Vocabulary
compound interest

Study Tip
Graphs If you write an exponential equation based on a graph that includes the y-intercept, you can substitute the y-value of the intercept for a in $y = ab^x$. Since $b^0 = 1$, $y = a$. Y ou can then use another point to find the common ratio b.

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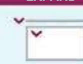
Interactive Presentation


Learn

EXPAND


Students tap to see how you can construct an exponential function given two points, a graph, or a description.

Solve the system of equations using substitution.

Solve the first equation for a.

$6 = ab^1$ — First equation

$\frac{6}{b} = a$ — Division Property of Equality

Substitute $\frac{6}{b}$ for a in the second equation to find b.

$24 = ab^3$ — Second equation

$24 = \frac{6}{b}b^3$ — Substitute $\frac{6}{b}$ for a

$24 = \frac{6b^3}{b}$ — Multiply.

$24 = 6b^2$ — Quotient of Powers

$4 = b^2$ — Division Property of Equality; then simplify

$2 = b$ — Definition of exponent

Substitute 2 for b in either equation to find a.

$6 = ab^1$ — Second equation

$6 = a(2)^1$ — Substitute 2 for b

$3 = a$ — Simplify.

Write the equation.

$y = 3 \cdot 2^x$

Study Tip

Assumptions The graphs of multiple functions pass through the points (1, 6) and (3, 24). For example, the linear equation $y = 9x - 3$ also passes through these points. However, because you are told to find the function that passes through those points, you can assume that the exponential relationship is the correct one.

Check

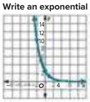
Write an exponential function that passes through (−1, 20) and (1, 5).

$y = \underline{\ ?\ } \cdot \underline{\ ?\ }^x$

10   0.5

Example 2 Write an Exponential Function Given a Graph

Write an exponential function for the graph.



x	y
−2	10
−1	5
0	2.5
1	1.25

You can use any two points to write a system of two equations using $y = ab^x$.

$2.5 = ab^0$

$1.25 = ab^1$

Since there is a zero exponent in the first equation, solve it for a.

$a = 2.5$

Then substitute this value into the second equation to find b.

$b = 0.5$

The graph can be modeled by the function $y = 2.5(0.5)^x$.

Think About It!

Use the graph to estimate the value of y when $x = 2$. Then use the function to find y when $x = 2$.

Sample answer: From the graph it appears that y is about 0.5 when $x = 2$. Using the function, when $x = 2$, $y = 2.5(0.5)^2$ or 0.625.

Go Online You can complete an Extra Example online.

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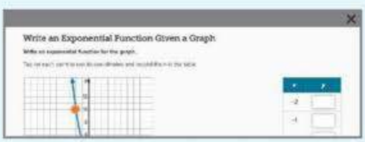
Interactive Presentation



Example 2

TAP

Students tap to reveal the coordinates of points on the graph.

TYPE

Students answer questions about the graph of the exponential function.

DIFFERENTIATE

Language Development Activity ELL

Beginning/Intermediate Have students work in small groups. This strategy allows every student to have an opportunity to speak several times. Ask a question or give a prompt about writing exponential functions, such as "Name one way that writing an exponential function is similar to, or different from, writing a linear function." Then pass a stick or other object to the student. The student speaks, everyone listens, and then the student passes the object to the next person. The next student speaks, everyone listens, and then the student passes the object on. Repeat until everyone has had one or two turns.

Example 2 Write an Exponential Function Given a Graph

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the graph, table, and function used in this example.

Questions for Mathematical Discourse

AL What is the asymptote of the graphed function? $y = 0$

OL Why is it helpful to use an equation where $x = 0$ when writing the system of equations? Sample answer: The exponent of zero results in $a(b^0) = a(1) = a$, which makes it easy to solve for the unknown value of a.

BL Describe the behavior of the y-values as the x-values increase by 1. Sample answer: The y-values decrease by half.

Example 3 Write an Exponential Function Given a Description

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about exponential functions to solving a real-world problem.

Questions for Mathematical Discourse

AL What does the y-intercept represent in this situation? Sample answer: The value of the prize at the start of the contest.

OL How can you check that the situation represents exponential growth?
Sample answer: Check the ratio of consecutive terms. The ratio is $b = 1.1$, so it is an example of exponential growth because $b > 0$.

BL Write an equation that could be used to find when the prize equals \$2000. $2000 = 1000(1.1)^x$

Learn Solving Problems Involving Exponential Growth

Objective

Students create equations and solve problems involving exponential growth by using the exponential growth formula.

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In this lesson, guide students to see what information they can gather about the equations just from looking at them.

Essential Question Follow-Up

Students have begun constructing exponential functions to describe situations.

Ask:

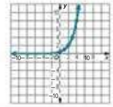
Why does an exponential growth function model some situations?
Sample answer: Situations in which a quantity increases by a regular percentage or proportion represent exponential growth. Examples of exponential growth in the real world include monetary situations and populations that increase due to a consistent birth rate.

Check

Part A Use the graph to estimate the value of y when $x = 4$. $y = 8$

Part B Write an exponential function that models the graph. $y = 0.5(2)^x$

Part C Use the function in Part B to find y when $x = 4$. $y = 8$



Example 3 Write an Exponential Function Given a Description

CONTEST **A radio station is giving away \$1000 to the first listener who answers a question correctly. If the question goes unanswered for one hour, the prize increases by 10% until it is answered correctly. Write a function to describe this situation.**

Hour (h)	Prize Total (P)
0	1000
1	1100
2	1210
3	1331

This is an example of exponential growth, so $b > 1$. Divide each value of P by the preceding term to find a common ratio of 1.1.

The value of a can be found by identifying the y-intercept.

$a = 1000$

Write the function that best models the situation.

$P = 1000(1.1)^h$

Talk About It!
Why is the common ratio 1.1 and not 0.1? Explain.

Sample answer: The prize is increasing by a rate of 10%, which signifies a 110% increase.

Learn Solving Problems Involving Exponential Growth

Key Concept • Equation for Exponential Growth

$y = a(1 + r)^t$

y is the final amount.
t is time.
a is the initial amount.
r is the rate of growth expressed as a decimal, $r > 0$.

Key Concept • Equation for Compound Interest

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

A is the current amount.
r is the annual interest rate expressed as a decimal, $r > 0$.
n is the number of times the interest is compounded each year.
t is time in years.
P is the principal, or the initial amount.

Think About It!
Why is the constant 1 in the exponential growth formula? What does it represent?

Sample answer: The constant 1 is added to a rate between 0 and 1 so that the multiplier is greater than 1. The 1 represents the initial amount that is carried forward.

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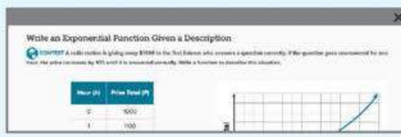
Interactive Presentation



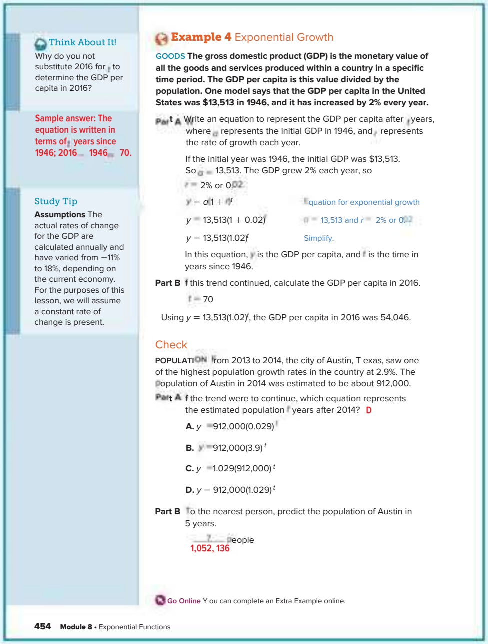
Example 3

SELECT

Students select the value of a and the function that best models the situation.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Think About It!
Why do you not substitute 2016 for t to determine the GDP per capita in 2016?

Sample answer: The equation is written in terms of t years since 1946; 2016 − 1946 = 70.

Study Tip
Assumptions The actual rates of change for the GDP are calculated annually and have varied from −11% to 18%, depending on the current economy. For the purposes of this lesson, we will assume a constant rate of change is present.

Example 4 Exponential Growth

GOODS **The gross domestic product (GDP) is the monetary value of all the goods and services produced within a country in a specific time period. The GDP per capita is this value divided by the population. One model says that the GDP per capita in the United States was $13,513 in 1946, and it has increased by 2% every year.**

Part A Write an equation to represent the GDP per capita after t years, where a represents the initial GDP in 1946, and r represents the rate of growth each year.

If the initial year was 1946, the initial GDP was $13,513. So $a = 13{,}513$. The GDP grew 2% each year, so $r = 2\%$ or 0.02.

$y = a(1 + r)^t$ Equation for exponential growth

$y = 13{,}513(1 + 0.02)^t$ $a = 13{,}513$ and $r = 2\%$ or 0.02

$y = 13{,}513(1.02)^t$ Simplify.

In this equation, y is the GDP per capita, and t is the time in years since 1946.

Part B If this trend continued, calculate the GDP per capita in 2016.

$t = 70$

Using $y = 13{,}513(1.02)^t$, the GDP per capita in 2016 was 54,046.

Check

POPULATION From 2013 to 2014, the city of Austin, Texas, saw one of the highest population growth rates in the country at 2.9%. The population of Austin in 2014 was estimated to be about 912,000.

Part A If the trend were to continue, which equation represents the estimated population t years after 2014? D

A. $y = 912{,}000(0.029)^t$

B. $y = 912{,}000(3.9)^t$

C. $y = 1.029(912{,}000)^t$

D. $y = 912{,}000(1.029)^t$

Part B To the nearest person, predict the population of Austin in 5 years.

___?___ people
1,052, 136

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 4

TYPE

Students complete the calculations to evaluate an exponential growth function.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Example 4 Exponential Growth

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to write the equation and solve the problem in this example, students will need to analyze the mathematical relationships in the problem to draw a conclusion.

Questions for Mathematical Discourse

AL What is represented by $t = 0$ in this situation? the starting year of 1946

OL Why is the expression $(1 + r)$ used in the equation? Sample answer: The model says the amount increases by 2% each year, so you know that this is an exponential growth function. If the 1 is not included, the values will decrease instead of increase.

BL Predict the GDP per capita in 2046 using this equation, rounded to the nearest dollar. $97,897

Common Error

Models of situations that move through time are usually based on a start time from which the model is extrapolated. This means the input value of $t = 0$ usually has to be translated into another time measurement that the students will be familiar with, such as a year, day, or hour of the day. Students will need to translate the time given in the description into an elapsed time in the proper units to use the model to predict the value.

Apply Example 5 Compound Interest

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- What basic exponential equation can you use to solve this problem?
- How can you determine a reasonable estimate for the amount in Maria's account after 5 years?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

DIFFERENTIATE

Enrichment Activity BL

Have students flip 50 pennies and count the number of heads. Then have students remove those pennies that landed on heads and repeat the activity. Students should record their results and make a plot of the trial number versus the number of heads counted in that trial. Have students graph their data and then explain why, in theory, their data should be modeled by the equation $y = \left(\frac{1}{2}\right)^x$.

Apply Example 5 Compound Interest

COLLEGE PLANNING **Maria invests $5500 into a college savings account that pays 3.25% compounded quarterly. How much money will there be in the account after 5 years?**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to determine how much money will be in Maria's account after 5 years. How can I write an equation to represent this situation? I can apply what I have learned about different types of functions

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will substitute the information I know into the compound interest equation and simplify. Then I will find the amount of money in Maria's account after 5 years. I will use the properties of exponents to simplify my equation.

3 What is your solution?

Use your strategy to solve the problem.

Write an equation to represent the amount of money in Maria's account after t years.

$A = 5500(1.008125)^{4t}$

How much money will be in Maria's account after 5 years?

$6466.22

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: My solution is reasonable because if I find the amount of money in the account after each quarter, I get the same answer as I do when I use the compound interest equation.

Check

BANKING Twin brothers Amare and Jermaine each received $1000 for graduation. Amare invests his money in an account that pays 2.25% compounded daily. Jermaine invests his money in an account that pays 2.25% compounded annually.

Part A Which brother will have more money at the end of 10 years? A

A. Amare
B. Jermaine
C. The accounts will be equal.

Part B To the nearest cent, how much more money? $? 3.11

Go Online You can complete an Extra Example online.

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Interactive Presentation

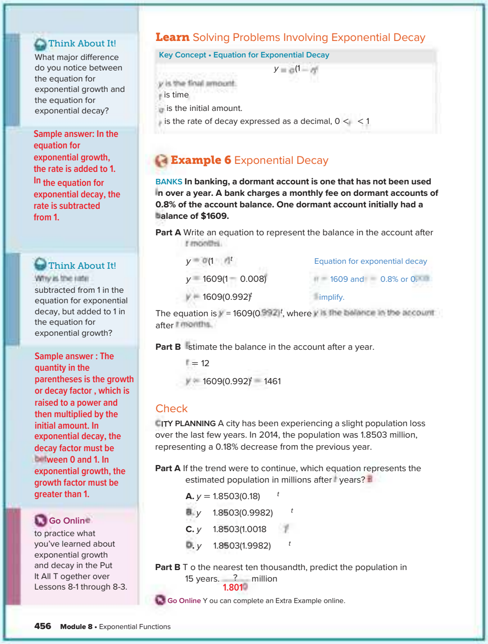
Check

TYPE

Students answer a question about compound interest and checking their solution.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Think About It!
What major difference do you notice between the equation for exponential growth and the equation for exponential decay?

Sample answer: In the equation for exponential growth, the rate is added to 1. In the equation for exponential decay, the rate is subtracted from 1.

Think About It!
Why is the rate subtracted from 1 in the equation for exponential decay, but added to 1 in the equation for exponential growth?

Sample answer : The quantity in the parentheses is the growth or decay factor , which is raised to a power and then multiplied by the initial amount. In exponential decay, the decay factor must be between 0 and 1. In exponential growth, the growth factor must be greater than 1.

Go Online to practice what you've learned about exponential growth and decay in the Put It All T ogether over Lessons 8-1 through 8-3.

Learn Solving Problems Involving Exponential Decay

Key Concept • Equation for Exponential Decay

$y = a(1 - r)^t$

y is the final amount.
t is time
a is the initial amount.
r is the rate of decay expressed as a decimal, $0 < r < 1$

Example 6 Exponential Decay

BANKS **In banking, a dormant account is one that has not been used in over a year. A bank charges a monthly fee on dormant accounts of 0.8% of the account balance. One dormant account initially had a balance of $1609.**

Part A Write an equation to represent the balance in the account after t months.

$y = a(1 - r)^t$ Equation for exponential decay
$y = 1609(1 - 0.008)^t$ $a = 1609$ and $r = 0.8\%$ or 0.008
$y = 1609(0.992)^t$ Simplify.

The equation is $y = 1609(0.992)^t$, where y is the balance in the account after t months.

Part B Estimate the balance in the account after a year.

$t = 12$
$y = 1609(0.992)^t = 1461$

Check

CITY PLANNING A city has been experiencing a slight population loss over the last few years. In 2014, the population was 1.8503 million, representing a 0.18% decrease from the previous year.

Part A If the trend were to continue, which equation represents the estimated population in millions after t years? B

A. $y = 1.8503(0.18)^t$
B. $y = 1.8503(0.9982)^t$
C. $y = 1.8503(1.0018)^t$
D. $y = 1.8503(1.9982)^t$

Part B T o the nearest ten thousandth, predict the population in 15 years. __?__ million 1.8010

Go Online Y ou can complete an Extra Example online.

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Interactive Presentation

Example 6

TYPE

Students complete the calculation to estimate the balance of the account.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Learn Solving Problems Involving Exponential Decay

Objective

Students create equations and solve problems involving exponential decay by using the exponential decay formula.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Example 6 Exponential Decay

MP Teaching the Mathematical Practices

4 Apply Mathematics In this example, students will apply what they have learned about exponential decay to solving a real-world problem.

Questions for Mathematical Discourse

AL What is represented by $t = 0$ in this situation? Sample answer: The time when the account has been inactive for less than one year.

OL What value of t represents 1 year? $t = 12$

BL What domain is appropriate to the function in this situation? Explain. The set of whole numbers; sample answer: The variable t represents the number of months with the fee charged every month, so they will be non-negative integer values only.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–21
2	exercises that use a variety of skills from this lesson	22–28
2	exercise that extends concepts learned in this lesson to new contexts	29
3	exercises that emphasize higher-order and critical-thinking skills	30–35

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–29 odd, 30–35
- Extension: Continuously Compounding Interest
- ALEKS® Exponential Functions

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–35 odd
- Remediation, Review Resources: Construct Linear Functions
- Personal Tutors
- Extra Examples 1–6
- ALEKS® Tables and Graphs of Lines

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–21 odd
- Remediation, Review Resources: Construct Linear Functions
- ALEKS® Tables and Graphs of Lines
- Arrive**MATH** Take Another Look

Practice

Go Online Y ou can complete your homework online.

Example 1

Write an exponential function for a graph that passes through the points.

1. (2, 16) and (3, 32) $y = 4 \cdot 2^x$

2. (1, 1) and (3, 0.25) $y = 2 \cdot 0.5^x$

3. (2, 90) and (4, 810) $y = 10 \cdot 3^x$

4. (−2, 4) and (1, 0.5) $y = 0.5^x$

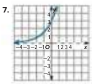
5. (1, 12) and (3, 192) $y = 3 \cdot 4^x$

6. (1, 18) and (3, 72) $y = 9 \cdot 2^x$

Example 2

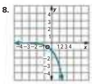
Write an exponential function for the graph.

7.



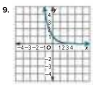
$y = 3 \cdot 2^x$

8.



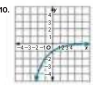
$y = -3^x$

9.



$y = \left(\frac{1}{4}\right)^x$

10.



$y = -\left(\frac{1}{2}\right)^x$

Example 3

11. BIOLOGY A certain species of bacteria in a laboratory culture begins with 50 cells and doubles in number every 30 minutes. Write a function to model the situation.
$f(x) = 50 \cdot 2^x$, where x is the number of 30-minute time periods

12. DEPRECIATION Amrita bought a new delivery van for $32,500. The value of this van depreciates at a rate of 12% each year. Write a function to model the value of the van after x years of ownership.
$f(x) = 32{,}500 \cdot (0.88)^x$

13. COMMUNICATION Cell phone usage grew about 23% each year from 2010 to 2016. If cell phone usage in 2010 was 43 million, write a function to model U.S. cell phone usage over that time period.
$f(x) = 43 \cdot (1.23)^x$, where x is the number of years since 2010

Examples 4–6

14. INVESTING Robyn invests $1500 at 4.85% compounded quarterly. Write an equation to represent the amount of money she will have in t years. $A = 1500\left(1 + \frac{0.0485}{4}\right)^{4t}$

15. POPULATION The population of New York City increased from 8,192,426 in 2010 to 8,550,405 in 2015. The annual rate of population increase for the period was about 0.9%.

a. Write an equation for the population, P, t years after 2010. $P = 8{,}192{,}426(1.009)^t$

b. Use the equation to predict the population of New York City in 2025. about 9,370,872

16. SAVINGS A company has a bonus incentive for its employees. The company pays employees an initial signing bonus of $1000 and invests that amount for the employees. Suppose the investment earns 8% interest compounded quarterly.

a. If an employee receiving this incentive withdraws the balance of the account after 5 years, how much will be in the account? about $1485.95

b. If an employee receiving this incentive withdraws the balance of the account after 35 years, how much will be in the account? about $15,996.47

17. MANUFACTURING A textile company bought a piece of weaving equipment for $60,000. It is expected to depreciate at an average rate of 10% per year.

a. Write an equation for the value of the piece of equipment z after t years. $Z = 60{,}000(0.90)^t$

b. Find the value of the piece of equipment after 6 years. about $31,886

18. HIGHER EDUCATION The table lists the average annual costs of attending a four-year college in the United States during a recent year.

College Sector	Tuition and Fees	Room and Board
Four-year Public	$9,410	$10,138
Four-year Private	$32,410	$11,516

Source: College Board

Rayelle's parents plan to invest $15,000 in a mutual fund earning an average of 4.5 percent interest, compounded monthly. After 15 years, for how many years will this investment be able to cover the tuition, fees, room, and board for Rayelle at a public college if costs stay the same? Round your answer to the nearest month.
1 year and 6 months

19. DEPRECIATION The value of a home theater system depreciates by about 7% each year. Aeryn purchases a home theater system for $3000. What is its value 4 years after purchase? Round your answer to the nearest hundred. **$2200**

20. MONEY Hans opens a savings account by depositing $1200. The account earns 0.2 percent interest compounded weekly. How much will be in the account in 10 years if he makes no more deposits? Assume that there are exactly 52 weeks in a year, and round your answer to the nearest cent. **$1224.24**

21. POPULATION In 2016 the U.S. Census Bureau estimated the population of the United States at 322 million. If the annual rate of growth was about 0.81%, find the expected population at the time of the 2030 census. Round your answer to the nearest ten million. **360 million**

Mixed Exercises

Write an exponential function for a graph that passes through the points.

22. (2, 1.4) and (4, 5.6) $y = 0.35 \cdot 2^x$

23. (1, 10.4) and (4, 665.6) $y = 2.6 \cdot 4^x$

24. (1, 42) and (3, 2688) $y = 5.25 \cdot 8^x$

25. POPULATION The population of Camden, New Jersey, has been decreasing by 0.12% a year on average. If this trend continues, and the population was 79,318 in 2006, estimate Camden's population in 2025. **about 77,529**

26. MEDICINE When doctors prescribe medication, they have to consider the rate at which the body filters a drug from the bloodstream. Suppose it takes the human body 6 days to filter out half of a certain vaccine. The amount of the vaccine remaining in the bloodstream x days after an injection is given by the equation $y = y_0(0.5)^{\frac{x}{6}}$, where y_0 is the initial amount. Suppose a doctor injects a patient with 20 μg (micrograms) of the vaccine.
   a. How much of the vaccine will remain after 1 day? Round your answer to the nearest tenth, if necessary. **17.8 μg**
   b. How much of the vaccine will remain after 12 days? Round your answer to the nearest tenth, if necessary. **5 μg**
   c. After how many days will the amount of vaccine be less than 1 μg? **after 26 days**

27. USE TOOLS Graham invested money to save for a car. After x years, the value of Graham's investment can be modeled by the equation $y = 2400(0.95)^x$. How much did Graham originally invest? Is the value of his investment increasing or decreasing? Explain your reasoning. Use technology to find when the investment will be worth half of its starting value. **See margin.**

Lesson 8-3 • Writing Exponential Functions 459

28. USE A MODEL There is a leak in a container that holds a certain nontoxic gas. Each hour, it loses 10% of its volume.
   a. Write an equation that models the amount of gas left in the container after x hours, assuming there were 300 cubic centimeters in the container before the leak. Then use your equation to determine the amount of gas left in the container after 11 hours. Round your answer to the nearest tenth. $h(x) = 300(0.9)^x$; **approximately 94.1 cm³**
   b. Dewanda believes a graph of this function should be a scatter plot instead of a continuous curve. Do you agree? Explain how this relates to the domain of the function. **See margin.**

29. STRUCTURE A wildlife researcher is studying the population of deer in a forest.

Years of Study	0	1	2	3
Population	128	160	200	250

   a. The table shows the estimated number of deer in the forest over a 3-year period. Write an exponential function that fits this data and can be used to predict the deer population in future years. $P(t) = 128(1.25)^t$
   b. The average rate of change is the change in the value of the dependent variable divided by the change in the value of the independent variable. What was the average rate of change in population during those three years? **an increase of approximately 41 deer per year**
   c. If the population growth follows the model from **part a**, do you expect the deer population to continue to increase by the value you came up with in **part b**? Explain. **No; the amount of increase is exponential, not linear.**
   d. Use the values in the table to show how you know the function is exponential, not linear. **See margin.**

Higher-Order Thinking Skills

30. ANALYZE Determine the growth rate (as a percent) of a population that quadruples every year. Justify your argument. **300%; Solving $y = (1 + r)^t$ for $y = 4$, $a = 1$, and $t = 1$ gives $r = 3$ or 300%.**

31. PERSEVERE Santos invested $1200 into an account with an interest rate of 8% compounded monthly. Use a calculator to approximate how long it will take for Santos's investment to reach $2500. **about 9.2 years**

32. ANALYZE The amount of water in a container doubles every minute. After 8 minutes, the container is full. After how many minutes was the container half-full? Justify your argument. **See margin.**

33. WRITE What should you consider when using exponential models to make decisions? **See margin.**

34. WRITE Compare and contrast the exponential growth formula and the exponential decay formula. **See margin.**

35. CREATE Honovi purchased a new car for $25,000 and has $5000 left to invest.
   a. Choose an interest rate between 4% and 7% for Honovi's investment, and find the length of time it would take for the investment to double. **Sample answer: 5%; about 14.2 years**
   b. Choose an annual depreciation rate from 8% to 10% for the new car that Honovi purchased, and find the length of time it would take for the car's value to be equal to one-half of the purchase price. **Sample answer: 10%; about 6.6 years**
   c. Using the rates from part **a** and part **b**, find the length of time it would take for the investment to be equal to the value of the car. What is the value at that time? **Sample answer: about 10.4 years; about $8320**

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Answers

27. Sample answer: The equation can be rewritten in the form $y = a(1 + r)^x$ to find the amount of original investment, a, and the rate of increase or decrease. Because $a = 2400$, he invested $2400. Because $1 + r = 0.95$ and is less than 1, his investment is decreasing in value. We can use a graphing calculator to find that the investment will be worth $1200 in about 13.5 years.

28b. No; sample answer: The gas leaks continuously. The domain of the function is restricted to nonnegative real numbers.

29d. Sample answer: There is no common difference over equal intervals (differences are 32, 40 and 50). There is a common factor (factor is 1.25 in each case.)

32. 7; Sample answer: Because the amount of water doubles every minute, the container would be half full a minute before it was full.

33. Sample answer: Exponential models can grow without bound, which is usually not the case for the situation that is being modeled. For instance, a population cannot grow without bound due to space and food constraints. Therefore, the situation that is being modeled should be carefully considered when used to make decisions.

34. Sample answer: The exponential growth formula is $y = a(1 + r)^t$, where a is the initial amount, t is time, y is the final amount, and r is the rate of change expressed as a decimal. The exponential decay formula is basically the same except the rate is subtracted from 1 and r represents the rate of decay.

Lesson 8-4

Transforming Exponential Expressions

A.SSE.3c, F.IF.8b

LESSON GOAL

Students use the properties of exponents to transform expressions for exponential functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Transforming Expressions

• Write Equivalent Exponential Expressions

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Powers of Monomials	●	●		●
Extension: Present Value and Future Value		●	●	●

Language Development Handbook

Assign page 46 of the *Language Development Handbook* to help your students build mathematical language related to transforming expressions for exponential functions.

ELL You can use the tips and suggestions on page T46 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.SSE.3c Use the properties of exponents to transform expressions for exponential functions.

F.IF.8b Use the properties of exponents to interpret expressions for exponential functions.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

4 Model with mathematics.

6 Attend to precision.

Coherence

Vertical Alignment

Previous

Students created exponential functions and solved problems involving exponential growth and decay.

F.LE.2, F.LE.5

Now

Students use the properties of exponents to transform expressions for exponential functions.

A.SSE.3c, F.IF.8b

Next

Students will relate exponential functions to geometric sequences.

F.BF.2, F.LE.2

Rigor

The Three Pillars of Rigor

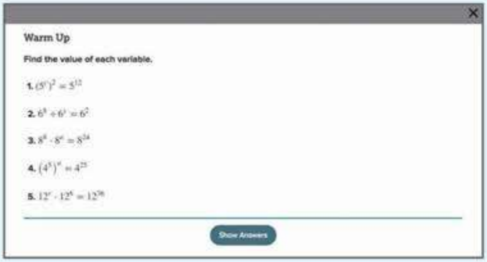
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of exponential functions and apply that understanding to solving problems related to exponential growth and decay. They build fluency by using points, graphs, and situations to construct exponential functions.

Mathematical Background

Exponential functions represent rates of increase or decrease in physical situations. A rate can be evident from an exponential function written in a certain form, but it is possible to translate between rates by converting the exponential expression into another form to highlight the amount over a different time frame.

Interactive Presentation



Warm Up

Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- evaluating expressions with exponents

Answers:

1. 6
2. 6
3. 16
4. 5
5. 30

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can learn how writing an equivalent exponential expression can be used to determine the best interest rate.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Example 1 Write Equivalent Exponential Expressions

Teaching the Mathematical Practices

4 Interpret Mathematical Results In this example, point out that to solve the problem, students should interpret their mathematical results in the context of the problem.

Questions for Mathematical Discourse

AL Which interest rate is compounded monthly? Savewell Bank's interest of 0.15%

OL How does the compounding vary between banks? Sample answer: Savewell's interest is compounded monthly while Second Local's is compounded annually. So, Savewell's interest will compound 12 times for every 1 time of Second Local's.

BL Would the effective monthly or annual interest rates be the same if a different value of a was used? Explain. Y es; sample answer: Because the interest rate affects the base of the exponential expression, multiplying by a different constant a does not affect the interest rate.

Go Online

- F ind additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 8-4

Transforming Exponential Expressions

Example 1 Write Equivalent Exponential Expressions

BANKING **Savewell Bank offers a savings account with 0.15% interest compounded monthly, and Second Local Bank offers a savings account with 2% interest compounded annually.**

T o compare the two accounts, we need to compare rates with the same compounding frequency. One way to do this is to compare the approximate monthly interest rates offered by each bank, which is also called the *effective* monthly interest rate.

Part A Compare monthly rates.

Write a function to represent the amount *A* that would be earned after *t* years with Second Local Bank. Then write an equivalent function that represents monthly compounding.

For convenience, let the initial amount of the investment be $1.

$y = a(1 + r)^t$ Equation for exponential growth

$A(t) = 1(1 + 0.02)^t$ $y = A(t)$, $a = 1$, $r = 2\%$ or 0.02

$A(t) = 1.02^t$ Simplify.

Then write a function that represents 12 compoundings per year, a power of $12t$, instead of 1 compounding per year, a power of $1t$.

$A(t) = 1.02^t$ Original function

$A(t) = 1.02^{\left(\frac{1}{12} \cdot 12\right)t}$ 1 year $= \frac{1 \text{ year}}{12 \text{ months}} \cdot 12$ months

$A(t) = \left(1.02^{\frac{1}{12}}\right)^{12t}$ Power of a Power

$A(t) \approx 1.00165^{12t}$ $(1.02)^{\frac{1}{12}} = \sqrt[12]{1.02}$ or about 1.00165

The effective monthly interest rate offered by Second Local Bank is about 1.00165 or about 0. 165% per month. It is slightly more than the 0. 15% offered by Savewell Bank. So, Second Local Bank is a better choice.

Part B Compare annual rates.

Write a function to represent the amount *A* earned after *t* months by Savewell Bank. Then write an equivalent function that represents annual compounding.

$y = a(1 + r)^t$ Equation for exponential growth

$A(t) = 1(1 + 0.0015)^t$ $y = A(t)$, $a = 1$, $r = 0.15\%$ or 0.0015

$A(t) = 1.0015^t$ Simplify.

(continued on the next page)

T oday's Goal
- Use the properties of exponents to transform expressions for exponential functions.

Go Online Y ou can watch a video to see how to compare savings accounts.

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Interactive Presentation

Example 1

WATCH

Students watch a video that introduces a problem about writing equivalent exponential expressions.

$A(t) = 1.0015^t$ represents the amount earned with a savings account at Savewell Bank after t months.

Write an equivalent function that represents 1 compounding per year. Since there are 12 months in a year, the exponent should be $\frac{1}{12}t$.

$A(t) = 1.0015^{1t}$	Original function
$A(t) = 1.0015^{(12 \cdot \frac{1}{12})t}$	1 year = 12 months $\cdot \frac{1\text{ year}}{12\text{ months}}$
$A(t) = (1.0015^{12})^{\frac{1}{12}t}$	Power of a Power
$A(t) = (1.0181)^{\frac{1}{12}t}$	$1.0015^{12} \approx 1.0181$

From this expression, we can determine that the effective annual interest rate of Savewell Bank is about 0.0181, or about 1.81%, which is less than the 2% interest rate offered by Second Local Bank.

Talk About It

Does the result in **Part B** make sense compared to the result of **Part A**? Explain.

Y es; sample answer: Like Part A, this confirms that Second Local Bank is a better choice. Whether the banks are compared using their monthly or annual percentage rates, Second Local Bank's rate is greater .

Check

SAVINGS T areq is planning to invest money into a savings account. Oak Hills Financial offers 3.1% interest compounded annually. First City Bank has savings accounts with a quarterly compounded interest rate of 0.7%.

Part A Write the expression $A(t)$ to represent the amount that T areq earns after t quarters through Oak Hills Financial.

$A(t) \approx$ ___?___ 1.0077^{4t}

What is the effective quarterly interest rate of Oak Hills Financial, rounded to the nearest hundredth? 0.77%

Part B Write the expression $A(t)$ to represent the amount that T areq earns after t years through First City Bank.

$A(t) \approx$ ___?___ $1.028^{\frac{1}{4}t}$

What is the effective annual rate of First City Bank, rounded to the nearest hundredth? 2.83%

___?___ Oak Hills Financial is the better bank for T areq's savings account.

Go Online Y ou can complete an Extra Example online.

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Interactive Presentation

Check

CHECK

Students complete the Check online to determine whether they are ready to move on.

Common Error

While converting expressions, it is worth revisiting the conversion of percentage rates to decimals. Remind students that the rates in an exponential function are written as decimals. Discuss other times when converting between different equivalent representations has been used in mathematics, such as converting units or money or balancing equations to solve them.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–5
2	exercises that use a variety of skills from this lesson	6–12
3	exercises that emphasize higher-order and critical-thinking skills	13–16

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–11 odd, 13–16
- Extension: Present Value and Future Value
- ALEKS Exponential Functions

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–15 odd
- Remediation, Review Resources: Powers of Monomials
- Personal Tutors
- Extra Example 1
- ALEKS Product, Power, and Quotient Rules

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–5 odd
- Remediation, Review Resources: Powers of Monomials
- *Quick Review Math Handbook*: Transforming Exponential Expressions
- Arrive**MATH** Take Another Look
- ALEKS Product, Power, and Quotient Rules

Answers

1b. Bank B has the better plan because the effective quarterly interest rate is 0.8%, which is greater than the quarterly interest rate of about 0.52% for Bank A.

1c. About 3.2%; sample answer: This confirms the result of part **b** because 3.2% is greater than the annual interest rate at Bank A, so Bank B has the better plan.

2b. Sample answer: The ring is increasing in value at a faster rate because the growth rate is 0.33% per month, which is greater than the growth rate of about 0.26% per month for the necklace.

2c. About 4.0%; sample answer: This confirms the result of part **b** because 4.0% is greater than the annual rate of increase of the necklace, so the ring is increasing in value at a faster rate.

3. Bank A; Bank A has a quarterly interest rate of 0.95%. Bank B has a quarterly interest rate of about 0.92%. Bank A's quarterly interest rate is higher.

Practice

Go Online You can complete your homework online.

Example 1

1. INVESTING Kimiyo is planning to invest money in a savings account. She is comparing the interest rates of savings accounts at two banks. Bank A offers a savings account with 2.1% interest compounded annually. Bank B offers a savings account with a quarterly compounded interest rate of 0.8%.
   a. Write a function to represent the amount A that Kimiyo would earn after t years through Bank A, assuming an initial investment of $1. Then write an equivalent function that represents quarterly compounding. $A(t) = (1.021)^t$; $A(t) = (1.0052)^{4t}$
   b. Which is the better plan? Explain. See margin.
   c. What is the approximate effective annual interest rate at Bank B? How does your result relate to your answer to part **b**? See margin.
2. COLLECTIONS Keandra is comparing the growth rates in the value of two items in a collection. The value of a necklace increases by 3.2% per year. The value of a ring increases by 0.33% per month.
   a. Write a function to represent the value A of the necklace after t years, assuming an initial value of $1. Then write an equivalent function that represents monthly compounding. $A(t) = (1.032)^t$; $A(t) = (1.0026)^{12t}$
   b. Which item is increasing in value at a faster rate? Explain. See margin.
   c. What is the approximate annual rate of growth of the ring? How does your result relate to your answer to part **b**? See margin.
3. SAVINGS Amir is trying to decide between two savings account plans at two different banks. He finds that Bank A offers a quarterly compounded interest rate of 0.95%, while Bank B offers 3.75% interest compounded annually. Which is the better plan? Explain. See margin.
4. BACTERIA The scientist found that Bacteria A has a growth rate of 0.99% per minute, while Bacteria B has a growth rate of 0.018% per second. Determine which bacterium has a faster growth rate. Explain. See margin.
5. POPULATION The population of Species A is decreasing at a rate of about 0.25% per quarter. The population of Species B is decreasing at a rate of about 1.34% per year. Determine which species has a population that is decreasing at a faster rate. Explain See margin.

Mixed Exercises

6. POPULATION The table shows the population of two small towns that experience increases in population.
   a. Write a function that can be used to estimate the population $P(t)$ of Town A t years after 2012. $P(t) = 8{,}000(1.06)^t$
   b. Write a function that can be used to estimate the population $P(t)$ of Town B t years after 2012. $P(t) = 9{,}500(1.05)^t$
   c. Use your equations and properties of exponents to find the approximate effective monthly increase in the populations of Town A and Town B. Town A: about 0.49%; Town B: about 0.41%

Year	Population Town A	Population Town B
2012	8,000	9,500
2013	8,480	9,975
2014	8,989	10,474
2015	9,528	10,997
2016	10,100	11,547

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7. ACCOUNTS Dominic is trying to decide between two checking account plans. Plan A offers a monthly compounded interest rate of 0.05%, while Plan B offers 0.5% interest compounded annually. Which is the better plan? Plan A

8. CAR DEPRECIATION Juana is deciding between two cars to purchase. Car A depreciates annually at a rate of 3.5%, while Car B depreciates monthly at a rate of 0.32%. Which car has a better effective rate of depreciation? See margin.

9. INVESTMENT As a wedding gift, Dotty and Brad received $10,000 cash from Dotty's grandparents. The couple is trying to decide where to invest the money. Account A offers 2.3% interest compounded semi-annually. Account B offers 4.2% interest compounded annually. Which account has the better rate? Explain. See margin.

10. SAVINGS Hernando is deciding between two certificate of deposit accounts. Account Y offers 4.5% interest compounded annually. Account Z offers 1.13% interest compounded quarterly. Which is the better deal? Explain. See margin.

11. FINANCE Gita is deciding between two retirement accounts. Account A offers 0.5% interest compounded monthly. Account B offers 2.5% interest compounded annually. Which is the better deal? Explain. See margin.

12. WILDLIFE The table shows that the population of hawks in two different nature preserves has been decreasing.

Year	Hawk Population (Nature Preserve A)	Hawk Population (Nature Preserve B)
2013	114	120
2014	111	115
2015	108	110
2016	105	106

a. Write a function that can be used to estimate the population $P(t)$ of the hawks in Nature Preserve A t years after 2013. $P(t) = 114(0.975)^t$

b. Write a function that can be used to estimate the population $P(t)$ of the hawks in Nature Preserve B t years after 2013. $P(t) = 120(0.96)^t$

c. Use your equations and properties of exponents to find the approximate effective quarterly decrease in population of hawks in Nature Preserve A and Nature Preserve B. Nature Preserve A: about 0.76%; Nature Preserve B: about 1.02%

Higher-Order Thinking Skills

13. PERSEVERE The rate at which an object cools is related to the temperature of the surrounding environment. At the time of an experiment, Mrs. Haubner's lab temperature was 72°F . The approximate temperature of the water at time t in minutes in Mrs. Haubner's lab is predicted by the function $T(t) = 72 + (212 - 72)2.72^{-0.4t}$, where −0.4° per minute is defined as the rate of cooling. Rewrite this function so that the coefficient of t in the exponent is 1. $T(t) = 72 + 140(0.67)^t$

14. WRITE Explain why it is important for a consumer to compare rates in the same unit before making a purchase. See margin.

15. CREATE Write a scenario that compares two accounts with interest rates compounded at different rate units. Then determine which account has the better rate. See margin.

16. FIND THE ERROR Marsha is opening a savings account. Eagle Savings Bank is offering her an account with a 0.13% monthly interest rate, while Admiral Savings Bank is offering an account with a 1% annual interest rate. Marsha believes the account at Admiral Savings bank is better because 1% is a greater interest rate than 0.13%. Why is Marsha incorrect? Explain your reasoning. See margin.

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Answers

4. Bacteria B; Bacteria A grows at a rate of 0.99% per minute. Bacteria B grows at a rate of about 1.09% per minute. Bacteria B has a faster growth rate.

5. Species B; the population of Species A is decreasing at a rate of about 0.25% per quarter. The population of Species B is decreasing at a rate of about 0.33% per quarter. The population of Species B is decreasing at a faster rate.

8. Car A; Car A has a monthly depreciation rate of about 0.29%. Car B has a monthly depreciation rate of 0.32%. Car A's monthly depreciation rate is lower.

9. Account A; Account A has a semi-annual interest rate of 2.3%. Account B has a semi-annual interest rate of about 2.1%. Account A's semi-annual interest rate is greater.

10. Account Z; Account Y has a quarterly interest rate of about 1.11%. Account Z has a quarterly interest rate of 1.13%. Account Z's quarterly interest rate is greater.

11. Account B; Account A has a monthly interest rate of 0.5%. Account B has a monthly interest rate of about 0.21%. Account B's monthly interest rate is greater.

14. Sample answer: To determine the better rate, compare rates with the same compounding frequency. Looking at only rates can be misleading if the rates have different compounding frequencies.

15. Sample answer: Bank A offers a savings account with a 0.6% interest rate compounded quarterly. Bank B offers a savings account with a 2% interest rate compounded annually. Bank A offers the better interest rate because it has a higher effective annual interest rate of about 2.4%.

16. Sample answer: The two interest rates are not being compounded at the same frequency. The 1% annual interest rate actually comes out to a 0.08% monthly interest rate, so Eagle Savings Bank is the better choice.

Lesson 8-5

Geometric Sequences

F.BF.2, F.LE.2

LESSON GOAL

Students write and graph equations of geometric sequences.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Modeling Geometric Sequences

Develop:

Geometric Sequences
- Geometric Sequences
- Identify Geometric Sequences
- Find Terms of Geometric Sequences

Geometric Sequences as Exponential Functions
- Find the *n*th Term of a Geometric Sequence
- Use a Geometric Sequence

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Arithmetic Sequences	●	●		●
Extension: Pay It Forward		●	●	●

Language Development Handbook

Assign page 47 of the *Language Development Handbook* to help your students build mathematical language related to writing and graphing equations of geometric sequences.

ELL You can use the tips and suggestions on page T47 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Functions

Standards for Mathematical Content:

F.BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (including reading these from a table).

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

3 Construct viable arguments and critique the reasoning of others.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous
Students related linear functions to arithmetic sequences.
F.LE.1a, F.LE.3

Now
Students write and graph equations of geometric sequences.
F.BF.2, F.LE.2

Next
Students will write arithmetic and geometric sequences recursively.
F.IF.3, F.BF.2

Rigor

The Three Pillars of Rigor

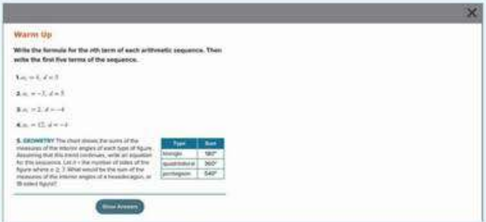
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge Working through the Explore and Learn activities can help students build a bridge to conceptual understanding. When students understand how to write and graph equations of geometric sequences, they can move to procedural fluency and apply the math to problems in everyday life.

Mathematical Background

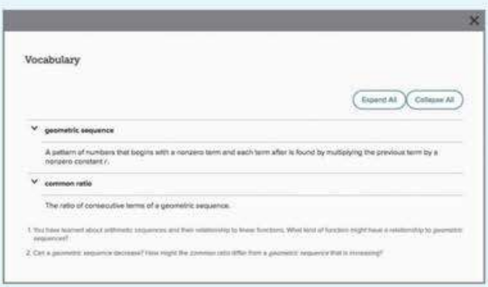
A geometric sequence is a pattern of numbers that begins with a nonzero term *a* and is continued by multiplying each term by a nonzero constant, *r*. The *n*th term of a geometric sequence is represented by the equation $a_n = a_1 r^{n-1}$.

Interactive Presentation



Warm Up

Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- writing explicit formulas to represent arithmetic sequences

Answers:

1. $a_n = 3n + 1$; 4, 7, 10, 13, 16
2. $a_n = 5n - 8$; −3, 2, 7, 12, 17
3. $a_n = -4n + 6$; 2, −2, −6, −10, −14
4. $a_n = -4n + 16$; 12, 8, 4, 0, −4
5. $a_n = 180(n - 2)$; 2520°

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students learn how the frequencies associated with the keys of the piano can be represented by a geometric sequence.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Modeling Geometric Sequences

Objective

Students use data to explore modeling real-world situations with geometric sequences.

MP Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in the height of the ball after each bounce in this Explore activity.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

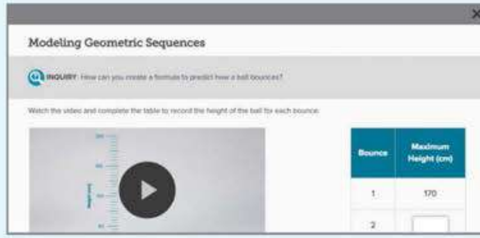
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

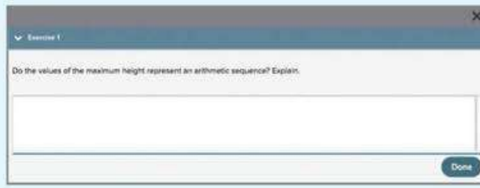
Students will complete guiding exercises throughout the Explore activity. Students watch a video and record data about the height of a ball bouncing. They will explore the relationship between the heights of the ball bounces, discovering that the relationship is a geometric sequence. Then students will answer the Inquiry question.

(continued on the next page)

Interactive Presentation



Explore



Explore

TYPE

Students complete a table to record the height of a bouncing ball and answer questions about the ratios.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Modeling Geometric Sequences (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Can a bouncing ball reach the same maximum height on every bounce? Explain. No; sample answer: The ball is losing some energy with each bounce, so the maximum height of the ball will decrease.
- Why do you think an average ratio was used for the formula? Sample answer: Because this is experimental data, the ratios are not the same between each pair of values. An average is a good way to use the information.

Inquiry

How can you create a formula to predict how a ball bounces? Sample answer: Write an explicit equation for the geometric sequence of the tennis ball's height by substituting values into the formula $a_n = a_1 r^{n-1}$.

Go Online to find additional teaching notes and answers for the guiding exercises.

Learn Geometric Sequences

Objective

Students identify and generate geometric sequences by using the common ratio.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Example 1 Geometric Sequences

MP Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in the common ratio in this example.

Questions for Mathematical Discourse

- **AL** How do you determine each term in a geometric sequence? Multiply the previous term by the common ratio.
- **OL** What operation can you perform with the first two terms to identify the common ratio? Divide the second term by the first term.
- **BL** How do you know the common ratio will be a negative number? Sample answer: The sequence is negative, positive, negative, positive, and so on, and the only way to achieve that is multiplying by a negative number.

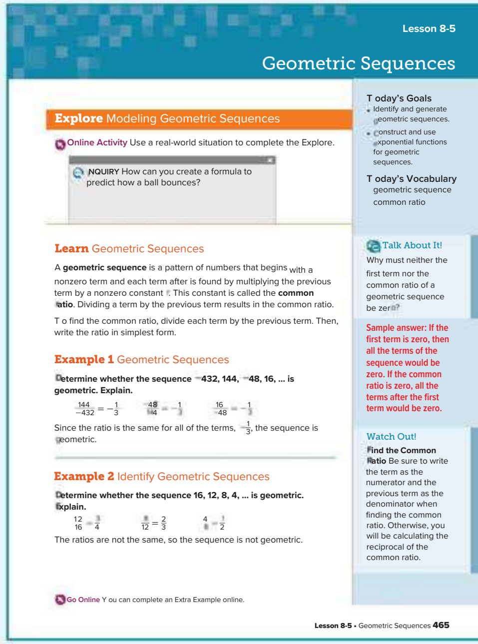
Example 2 Identify Geometric Sequences

MP Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to construct the argument that this sequence is not a geometric sequence.

Questions for Mathematical Discourse

- **AL** What is the ratio of the first two terms? $\frac{3}{4}$
- **OL** Will this sequence ever reach 0? Explain. Y es; sample answer: Each term decreases by 4, so the fifth term in the sequence will be 0.
- **BL** What type of sequence is shown? Explain. Arithmetic; sample answer: There is a common difference of −4 rather than a common ratio.


Lesson 8-5

Geometric Sequences

Explore Modeling Geometric Sequences

Online Activity Use a real-world situation to complete the Explore.

NQUIRY How can you create a formula to predict how a ball bounces?

Learn Geometric Sequences

A **geometric sequence** is a pattern of numbers that begins with a nonzero term and each term after is found by multiplying the previous term by a nonzero constant . This constant is called the **common atio**. Dividing a term by the previous term results in the common ratio.

T o find the common ratio, divide each term by the previous term. Then, write the ratio in simplest form.

Example 1 Geometric Sequences

Determine whether the sequence −432, 144, −48, 16, ... is geometric. Explain.

$\frac{144}{-432} = -\frac{1}{3}$ $\frac{-48}{144} = -\frac{1}{3}$ $\frac{16}{-48} = -\frac{1}{3}$

Since the ratio is the same for all of the terms, $-\frac{1}{3}$, the sequence is geometric.

Example 2 Identify Geometric Sequences

Determine whether the sequence 16, 12, 8, 4, ... is geometric. Explain.

$\frac{12}{16} = \frac{3}{4}$ $\frac{8}{12} = \frac{2}{3}$ $\frac{4}{8} = \frac{1}{2}$

The ratios are not the same, so the sequence is not geometric.

Go Online Y ou can complete an Extra Example online.

T oday's Goals
- Identify and generate geometric sequences.
- Construct and use exponential functions for geometric sequences.

T oday's Vocabulary
geometric sequence
common ratio

Talk About It!
Why must neither the first term nor the common ratio of a geometric sequence be zero?

Sample answer: If the first term is zero, then all the terms of the sequence would be zero. If the common ratio is zero, all the terms after the first term would be zero.

Watch Out!
Find the Common Ratio Be sure to write the term as the numerator and the previous term as the denominator when finding the common ratio. Otherwise, you will be calculating the reciprocal of the common ratio.

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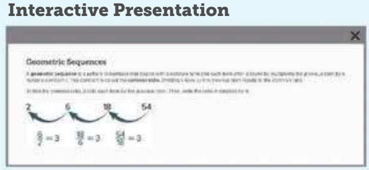
Interactive Presentation



Learn

TYPE

Students answer a question to determine why the first term and common ratio of a geometric sequence cannot be zero.

Check

Determine whether each sequence is geometric. If so, determine its common ratio

a.

n	1	2	3	4	…
a_n	8	20	50	125	…

This sequence __?__ a geometric sequence. It has a common ratio of __?__. **is**, **2.5**

b. −0.7 , 0.07 , −0.007 , 0.0007 , …

This sequence __?__ a geometric sequence. It has a common ratio of __?__. **is**, **−0.1**

Example 3 Find Terms of Geometric Sequences

Find the next three terms in each geometric sequence.

a. 64, 16, 4, 1, …

Step 1 Find the common ratio.

$\frac{16}{64} = \frac{1}{4}$ $\quad \frac{4}{16} = \frac{1}{4}$ $\quad \frac{1}{4} = \frac{1}{4}$

The common ratio is $\frac{1}{4}$.

Step 2 Multiply by the common ratio.

$1 \times \frac{1}{4} = \frac{1}{4}$ $\quad \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$ $\quad \frac{1}{16} \times \frac{1}{4} = \frac{1}{64}$

The next three terms are $\frac{1}{4}$, $\frac{1}{16}$, and $\frac{1}{64}$.

b.

n	1	2	3	4	…
a_n	8	12	18	27	…

Step 1 Find the common ratio.

$\frac{12}{8} = 1.5$ $\quad \frac{18}{12} = 1.5$ $\quad \frac{27}{18} = 1.5$

The common ratio is 1.5.

Step 2 Multiply by the common ratio.

$27 \times 1.5 = 40.5$ $\quad 40.5 \times 1.5 = 60.75$ $\quad 60.75 \times 1.5 = 91.125$

The next three terms are 40.5, 60.75, and 91.125.

Check

Find the next three terms in each geometric sequence.

a. 729, 243, 81, __?__, __?__, __?__, … **27, 9, 3**

b. −4, −44, −484, __?__, __?__, __?__, … **−5,324, −58,564, −644,204**

Go Online You can complete an Extra Example online.

Think About It! How could you determine a term prior to any given term of a geometric sequence?

Sample answer: Divide the term by the common ratio.

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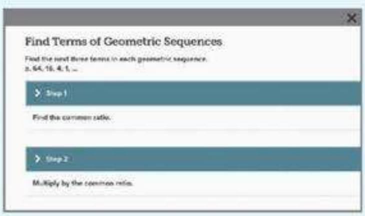
Interactive Presentation



Example 3

TYPE

Students calculate the common ratio of a geometric sequence.

CHECK

Students complete the Check online to determine whether they are ready to move on.

DIFFERENTIATE

Language Development Activity ELL

Put students in groups of mixed language and math abilities. Have groups discuss the differences between arithmetic and geometric sequences. Suggest that they help each other organize clear, concise, and accurate notes about these and other concepts taught in this lesson.

Example 3 Find Terms of Geometric Sequences

MP Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in this example.

Questions for Mathematical Discourse

AL In part **a,** what appears to be happening to each term? Sample answer: Each term is divided by 4.

OL In part **a,** will the term 0 ever appear? Explain. No; sample answer: Because each term is found by dividing the previous term by 4 $\left(\text{or multiplying by } \frac{1}{4}\right)$, the values will continue to get smaller but will never reach 0.

BL For both parts, how do you know the common ratio will be a positive number? Sample answer: All numbers in the sequences are positive, so the common ratio has to be a positive number.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Geometric Sequences as Exponential Functions

Objective

Students construct and use exponential functions for geometric sequences by computing common ratios and applying the explicit formula for the *n*th term.

MP Teaching the Mathematical Practices

3 Reason Inductively In this Learn, students will use inductive reasoning to make plausible arguments.

Example 4 Find the *n*th T erm of a Geometric Sequence

MP Teaching the Mathematical Practices

7 Look for a Pattern Help students to see the pattern in the sequence in this example.

Questions for Mathematical Discourse

AL What value do you substitute for a_1 in the equation? 512

OL What value do you substitute for r? Explain. $\frac{1}{2}$; Sample answer: We divided the terms to find the common ratio of $\frac{1}{2}$.

BL Why is it important to enclose the $\frac{1}{2}$ in parentheses? Sample answer: The variable r is raised to a power. When r is a fraction, like in this example, parentheses are used to make sure that both the numerator and denominator are raised to a power.

Learn Geometric Sequences as Exponential Functions

Key Concept • *n*th Term of a Geometric Sequence

The *n*th term a_n of a geometric sequence with first term a_1 and common ratio r is given by the following formula, where n is any positive integer and $a_1, r \neq 0$.

$$a_n = a_1 r^{n-1}$$

Example 4 Find the *n*th T erm of a Geometric Sequence

Use an explicit formula to find the 11th term of each geometric sequence.

512, 256, 128, 64, ...

The first term of the sequence is 512. So, $a_1 = 512$.

Find the common ratio.

$\frac{256}{512} = \frac{1}{2}$ $\frac{128}{256} = \frac{1}{2}$ $\frac{64}{128} = \frac{1}{2}$

The common ratio is $\frac{1}{2}$.

Use the common ratio to find the 11th term of the sequence.

$a_n = a_1 r^{n-1}$	Formula for the *n*th term
$a_n = 512\left(\frac{1}{2}\right)^{n-1}$	$a_1 = 512$ and $r = \frac{1}{2}$
$a_{11} = 512\left(\frac{1}{2}\right)^{11-1}$	T o find the eleventh term, $n = 11$.
$= 512\left(\frac{1}{2}\right)^{10}$	Simplify.
$= 512\left(\frac{1}{1024}\right)$	$\left(\frac{1}{2}\right)^{10} = \left(\frac{1}{1024}\right)$
$= \frac{1}{2}$	Simplify.

Check

Write the equation for the *n*th term of the geometric sequence.

n	1	2	3	4	...
a_n	729	243	81	27	...

$a_n =$ __?__ $729\left(\frac{1}{3}\right)^{n-1}$

Find the 8th term of the sequence. $\frac{1}{3}$

Go Online You can complete an Extra Example online.

Think About It! When finding the *n*th term of a geometric sequence, why is r raised to the $n-1$ power instead of to the *n*th power?

Sample answer: The second term a_2 is multiplied by r. The third term a_3 is multiplied by $r \cdot r$, or r^2. For each term, r is raised to one less than the value of n.

Watch Out! **Exponents** Remember that the base, which is the common ratio, is raised to $n-1$ instead of n.

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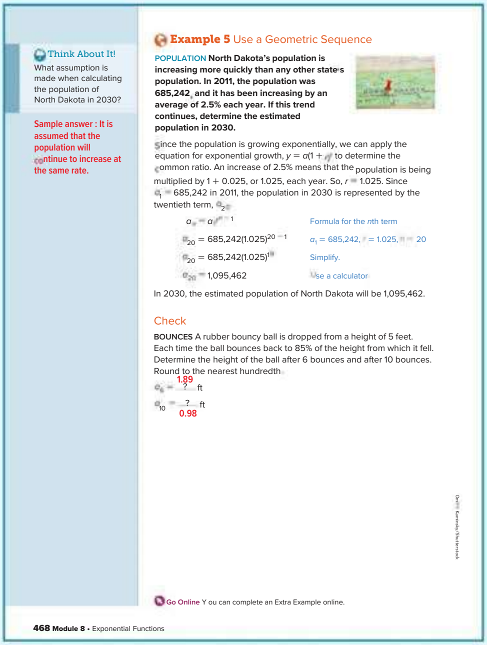
Interactive Presentation

Learn

TYPE

Students answer a question todetermine whether they understand the formula for finding the *n*th term of a geometric sequence.



Think About It!

What assumption is made when calculating the population of North Dakota in 2030?

Sample answer : It is assumed that the population will continue to increase at the same rate.

Example 5 Use a Geometric Sequence

POPULATION North Dakota's population is increasing more quickly than any other state's population. In 2011, the population was 685,242, and it has been increasing by an average of 2.5% each year. If this trend continues, determine the estimated population in 2030.

Since the population is growing exponentially, we can apply the equation for exponential growth, $y = a(1 + r)^t$ to determine the common ratio. An increase of 2.5% means that the population is being multiplied by 1 + 0.025, or 1.025, each year. So, $r = 1.025$. Since $a_1 = 685{,}242$ in 2011, the population in 2030 is represented by the twentieth term, a_{20}.

$a_n = a_1r^{n-1}$	Formula for the *n*th term
$a_{20} = 685{,}242(1.025)^{20-1}$	$a_1 = 685{,}242$, $r = 1.025$, $n = 20$
$a_{20} = 685{,}242(1.025)^{19}$	Simplify.
$a_{20} = 1{,}095{,}462$	Use a calculator

In 2030, the estimated population of North Dakota will be 1,095,462.

Check

BOUNCES A rubber bouncy ball is dropped from a height of 5 feet. Each time the ball bounces back to 85% of the height from which it fell. Determine the height of the ball after 6 bounces and after 10 bounces. Round to the nearest hundredth.

a_6 = ___?___ ft 1.89

a_{10} = ___?___ ft 0.98

Dmitry Kaminsky/Shutterstock

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 5

TYPE

Students answer a question about the assumptions made during the calculations.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 5 Use a Geometric Sequence

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about geometric sequences to solving a real-world problem.

Questions for Mathematical Discourse

AL What does a_1 represent in this geometric sequence? the initial population of 685,242 in 2011

OL How can you check your solution? Sample answer: Multiply 685,242 by 1.025. Then multiply the product by 1.025 and continue this process until I reach the 20th term of the sequence.

BL Why is the year 2030 represented by the 20th term when it is only 19 years after 2011? Sample answer: The first term is for the year 2011. This means we can think of the "0th" term as 2010, making it 20 years between 2010 and 2030.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–31
2	exercises that use a variety of skills from this lesson	32–41
2	exercises that extend concepts learned in this lesson to new contexts	42–45
3	exercises that emphasize higher-order and critical-thinking skills	46–52

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–45 odd, 46–52
- Extension: Pay It Forward
- ALEKS Geometric Sequences

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–51 odd
- Remediation, Review Resources: Arithmetic Sequences
- Personal Tutors
- Extra Examples 1–5
- ALEKS Arithmetic Sequences

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–31
- Remediation, Review Resources: Arithmetic Sequences
- *Quick Review Math Handbook*: Geometric Sequences as Exponential Functions
- ArriveMATH Take Another Look
- ALEKS Arithmetic Sequences

Answers

1. The ratios are not the same, so the sequence is not geometric.
2. The ratios are not the same, so the sequence is not geometric.
3. Because the ratio is the same for all of the terms, 5, the sequence is geometric.
4. Because the ratio is the same for all of the terms, $\frac{1}{2}$, the sequence is geometric.
5. The ratios are not the same, so the sequence is not geometric.
6. The ratios are not the same, so the sequence is not geometric.

Practice

Go Online You can complete your homework online.

Examples 1 and 2

Determine whether each sequence is geometric. Explain. See margin.

1. 4, 1, 2, ...
2. 10, 20, 30, 40, ...
3. 4, 20, 100, ...
4. 212, 106, 53, ...
5. −10, −8, −6, −4, ...
6. 5, −10, 20, 40, ...
7. −96, −48, −24, −12, ...
8. 7, 13, 19, 25, ...
9. 3, 9, 81, 6561, ...
10. 108, 66, 141, 99, ...
11. $\frac{3}{8}, -\frac{1}{8}, -\frac{5}{8}, -\frac{9}{8}, \ldots$
12. $\frac{7}{3}$, 14, 84, 504, ...

Example 3

Find the next three terms in each geometric sequence.

13. 2, −10, 50, ... −250, 1250, −6250
14. 36, 12, 4, ... $\frac{4}{3}, \frac{4}{9}, \frac{4}{27}$
15. 4, 12, 36, ... 108, 324, 972
16. 400, 100, 25, ... $\frac{25}{4}, \frac{25}{16}, \frac{5}{64}$
17. −6, −42, −294, ... −2058; −14,406; −100,842
18. 1024, −128, 16, ... $-2, \frac{1}{4}, -\frac{1}{32}$
19. 2, 6, 18, ... 54, 162, 486
20. 2500, 500, 100, ... $20, 4, \frac{4}{5}$
21. $\frac{4}{5}, \frac{2}{5}, \frac{1}{5}, \ldots$ $\frac{1}{10}, \frac{1}{20}, \frac{1}{40}$
22. −4, 24, −144, ... 864; −5184; 31, 104
23. 72, 12, 2, ... $\frac{1}{3}, \frac{1}{18}, \frac{1}{108}$
24. −3, −12, −48, ... −192, −768, −3072

Example 4

Use an explicit formula to find the 10th term of each geometric sequence.

25. 1, 9, 81, 729, ... 387,420,489
26. 2, 8, 32, 128, ... 524,288
27. −9, 27, −81, 243, ... 177, 147
28. 6, −24, 96, −384, ... −1,572,864

Example 5

29. MUSEUMS The table shows the annual visitors to a museum in millions. Write an equation for the projected number of visitors after *n* years. $a_n = 4 \cdot \left(\frac{3}{2}\right)^{n-1}$

Year	Visitors (millions)
1	4
2	6
3	9
4	$13\frac{1}{2}$
n	?

30. WORLD POPULATION The CIA estimates that the world population is growing at a rate of 1.167% each year. The world population in 2015 was about 7.3 billion.
   a. Write an equation for the world population after *n* years. $a_n = 7{,}300{,}000{,}000 \cdot 1.01167^{n-1}$
   b. Find the estimated world population in 2025. ≈ 8.1 billion

31. DEPRECIATION Te'Andra has a computer system that she bought for $5000. Each year, the computer system loses one-fifth of its then-current value. How much money will the computer system be worth after 6 years? $1310.72

Mixed Exercises

32. POPULATION The table shows the projected population of the United States through 2060. Does this table show an *arithmetic sequence*, a *geometric sequence*, or neither? Explain. Neither, there is no common ratio or difference.

Year	Population
2020	334,503,000
2030	359,402,000
2040	380,219,000
2050	398,328,000
2060	416,795,000

Source: U.S. Census Bureau

33. SAVINGS ACCOUNTS A bank offers a savings account that earns 0.5% interest each month.
   a. Write an equation for the balance of the savings account after *n* months. $a_n = P \cdot 1.005^n$
   b. Given an initial deposit of $500, what will the account balance be after 15 months? $538.84

34. Write an equation for the *n*th term of the geometric sequence 3, −24, 192, ... Then find the 9th term of this sequence. $a_n = 3(-8)^{n-1}$; 50,331,648

35. Write an equation for the *n*th term of the geometric sequence $\frac{9}{16}, \frac{3}{8}, \frac{1}{4}, \ldots$. Then find the 7th term of this sequence. $a_n = \frac{9}{16}\left(\frac{2}{3}\right)^{n-1}; \frac{4}{81}$

36. Write an equation for the *n*th term of the geometric sequence 1000, 200, 40, ...Then find the 5th term of this sequence $a_n = 1000\left(\frac{1}{5}\right)^{n-1}; \frac{8}{5}$

37. Write an equation for the *n*th term of the geometric sequence $-8, -2, -\frac{1}{2}, \ldots$.

Find the 8th term of this sequence. $a_n = -8\left(\frac{1}{4}\right)^{n-1}$; $-\frac{1}{2048}$

38. Write an equation for the *n*th term of the geometric sequence 32, 48, 72, … .

Find the 6th term of this sequence. $a_n = 32\left(\frac{3}{2}\right)^{n-1}$; 243

39. USE A SOURCE Research the average annual salary for a 25-year-old and the average rate of increase in salary per year. Then write an equation for the *n*th year of employment. Find the 20th term of this sequence, and explain what it means. See margin.

40. STRUCTURE For each of the geometric sequences below, fill in the missing terms, write the corresponding exponential equation, and use the exponential equation to determine the 10th term in the sequence.

a. 0.5, 6, 72, 864, 10,368; $f(x) =$ $0.5(12)^{x-1}$; 10th term: 2,579,890, 176

b. 5, 10, 20, 40, 80; $g(x) =$ $5(2)^{x-1}$; 10th term: 2560

41. REASONING Find the previous three terms of the geometric sequence, −192, − 768, −3072, … . −3, −12, −48

42. STATE YOUR ASSUMPTION Consider two different geometric sequences. Each starts with the same constant. The common ratio producing subsequent terms in the first is positive and is the reciprocal of the common ratio producing subsequent terms in the second. How would the graphs of the two sequences compare? Think about intercepts, asymptotes, and end behavior. Then graph an example of the situation. See margin.

43. REASONING You have just been offered a part-time job. The employer offers two different methods of payment. They are shown in the table.

Month	Method 1 Payment	Method 2 Payment
1	$100.00	$0.01
2	$108.00	$0.02
3	$116.00	$0.04
4	$124.00	$0.08

a. Describe the two different methods of payment being offered. See margin.

b. What kind of mathematical equations can you use to model each situation? How do you know? Write each equation. See margin.

c. You are planning to work at this job for two years. Your manager promises to raise your salary the way it is described in the table, as long as you meet the minimum performance rating each month. Which payment plan would you choose? Explain your reasoning. See margin.

44. CONSTRUCT ARGUMENTS The terms of a geometric sequence are defined by the equation $a_n = 512(0.5)^x$. A second sequence contains the terms $b_3 = 7168$ and $b_7 = 28$.

a. Determine which sequence has the greater common ratio. See margin.

b. What is the initial term of each sequence? Explain your reasoning. See margin.

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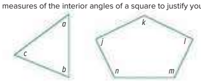
45. REGULARITY The sum of the interior angles of a triangle is 180°. The interior angles of a pentagon add to 540°. Is the relationship between the number of sides in a polygon and the sum of interior angles a geometric sequence? Use the sum of the measures of the interior angles of a square to justify your answer. See margin.



$a + b + c = 180°$ $j + k + l + m + n = 540°$

Higher-Order Thinking Skills

46. PERSEVERE Write a sequence that is both geometric and arithmetic. Explain your answer. See margin.

47. FIND THE ERROR Haro and Matthew are finding the ninth term of the geometric sequence –5, 10, –20, … . Is either of them correct? Explain your reasoning. See margin.

Haro

$r = \frac{10}{-5}$ or -2

$a_9 = -5(-2)^{9-1}$

$= -5(512)$

$= -2560$

Matthew

$r = \frac{10}{-5}$ or -2

$a_9 = -5 \cdot (-2)^{9-1}$

$= -5 \cdot -256$

$= 1280$

48. ANALYZE Write a sequence of numbers that form a pattern but are neither arithmetic nor geometric. Justify your argument. Sample answer: 1, 4, 9, 16, 25, 36, . . .; This is the sequence of squares of counting numbers.

49. WRITE How are graphs of geometric sequences and exponential functions similar? How are they different? See Mod. 8. Answer Appendix.

50. WRITE Summarize how to find a specific term of a geometric sequence. See Mod. 8. Answer Appendix.

51. CREATE Give a counterexample for the following statement: As *n* increases in a geometric sequence, the value of a_n will move farther away from zero. Sample answer: In the geometric sequence 6, 3, 1.5, . . .; the value of *r* is 0.5 and the absolute value of a_{n+1} will be closer to zero than the value of a_n.

52. CREATE Write a geometric sequence. Then explain why your sequence is geometric. Sample answer: 32, 16, 8, 4, . . .; Since the ratio is the same for all of the terms, $\frac{1}{2}$, the sequence is geometric.

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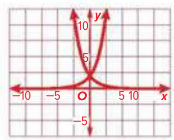
Answers

7. Because the ratio is the same for all of the terms, $\frac{1}{2}$, the sequence is geometric.
8. The ratios are not the same, so the sequence is not geometric.
9. The ratios are not the same, so the sequence is not geometric.
10. The ratios are not the same, so the sequence is not geometric.
11. The ratios are not the same, so the sequence is not geometric.
12. Because the ratio is the same for all of the terms, 6, the sequence is geometric.

39. Sample answer: The average annual salary is about $39,416, and the average annual rate of increase is about 3%. $a_n = 39{,}416(1.03)^{n-1}$; $\approx 69{,}116.19$; This means that after 20 years of employment the average annual salary will be about $69,116.19.

42. Sample answer: The two graphs should have the same *y*-intercept because their first term is the same. One graph would be the reflection of the other across the *y*-axis, so the graphs would have the same horizontal asymptote (assuming an infinite domain), although one would approach the asymptote as it grew in a positive direction and the other would approach the asymptote as it grew in a negative direction.



43a. The first method provides a starting salary of $100 and an $8 per month raise. The second method provides a starting salary of $0.01 and doubles it each month.

43b. The first situation is linear because there is a common difference of $8. The equation is $y = 8x + 92$. The second situation is exponential because it is a geometric sequence with a common ratio of 2. The equation is $y = 0.01(2)^{x-1}$.

43c. Sample answer: As long as I do not need money immediately, I would use the second method. In the last month, I would make $y = 0.01(2)^{23} =$ $83,886.08 due to the fact that the payment is growing exponentially. In the last month in the first method, I would make $y = 8(24) + 92 = \$284$.

44a. The common ratio of the 2nd sequence is $\left(\frac{28}{7168}\right)^{\frac{1}{4}} = \frac{1}{4}$ The first sequence has the greater common ratio of 0.5.

44b. The initial term of the first sequence is $a_0 = 512$. In the second sequence, we know that $b_n = b_0(0.25)^n$. Using b_3 and solving for b_0, we find $b_0 = 458{,}752$.

45. Sample answer: If the values fit a geometric sequence, then $r = \sqrt{\frac{540}{180}} = \sqrt{3}$. This would mean that the interior angles of a square would have a sum of $180\sqrt{3} \approx 312°$. Because the sum of the angles in a square is 360°, this is not a geometric sequence.

46. Sample answer: 1, 1, 1, 1, ...; The common ratio is 1 making it a geometric sequence, but the common difference is 0 making it an arithmetic sequence as well.

47. Neither; Haro calculated the exponent incorrectly. Matthew did not calculate $(-2)^8$ correctly.

Lesson 8-6

Recursive Formulas

F.IF.3, F.BF.2

LESSON GOAL

Students write arithmetic and geometric sequences recursively.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Using Recursive Formulas
- Recursive Formula for an Arithmetic Sequence
- Recursive Formula for a Geometric Sequence

Explore: Writing Recursive Formulas from Sequences

Develop:

Writing Recursive Formulas
- Write a Recursive Formula Using a List
- Write a Recursive Formula Using a Graph
- Write Recursive and Explicit Formulas
- Translate Between Recursive and Explicit Formulas

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Geometric Sequences	●	●		●
Extension: The Fibonacci Sequence		●	●	●

Language Development Handbook

Assign page 48 of the *Language Development Handbook* to help your students build mathematical language related to writing arithmetic and geometric sequences recursively.

ELL You can use the tips and suggestions on page T48 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is the subset of integers.

F.BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Standards for Mathematical Practice:

4 Model with mathematics.

7 Look for and make use of structure.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students wrote and graphed equations of geometric sequences.
F.BF.2, F.LE.2

Now
Students write arithmetic and geometric sequences recursively.
F.IF.3, F.BF.2

Next
Students will relate exponential functions to logarithmic functions.
F.LE.2 (Course 3)

Rigor

The Three Pillars of Rigor

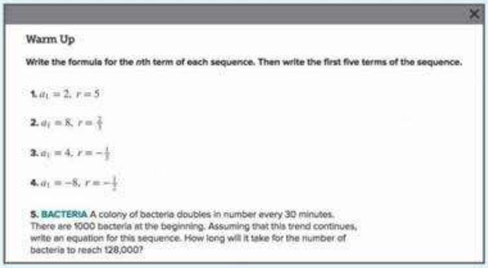
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students continue to expand their understanding of sequences as functions, and they build fluency by writing recursive formulas for arithmetic and geometric sequences.

Mathematical Background

Arithmetic sequences involve a series of numbers with a common difference between consecutive terms, while geometric sequences involve a series of numbers with a common ratio between consecutive terms. Explicit formulas for a sequence are written in reference to the initial value, but sequences can also be defined by recursive formulas that increment the sequence based on the previous value.

Interactive Presentation



Warm Up

Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- writing explicit formulas to represent geometric sequences

Answers:

1. $a_n = 2(5)^{n-1}$; 2, 10, 50, 250, 1250
2. $a_n = 8\left(\frac{2}{3}\right)^{n-1}$; 8, $\frac{16}{3}$, $\frac{32}{9}$, $\frac{64}{27}$, $\frac{128}{81}$
3. $a_n = 4\left(-\frac{1}{3}\right)^{n-1}$; 4, $-\frac{4}{3}$, $\frac{4}{9}$, $-\frac{4}{27}$, $\frac{4}{81}$
4. $a_n = -8\left(-\frac{1}{2}\right)^{n-1}$; −8, 4, −2, 1, $-\frac{1}{2}$
5. $a_n = 1000(2)^n$; $3\frac{1}{2}$ hours

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world example of recursive formulas.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Writing Recursive Formulas from Sequences

Objective

Students use a sketch to explore writing recursive formulas for geometric sequences.

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this Explore, students will need to use a sketch. Work with students to explore and deepen their understanding of formulas for geometric sequences.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

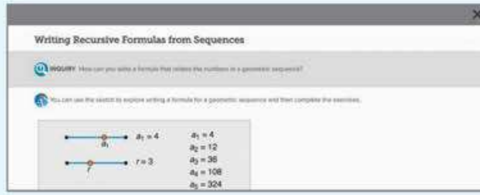
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students use a sketch to analyze various geometric sequences and identify relationships based on the first term and common ratio. Student use their results to determine how to write recursive formulas. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

Explore

WEB SKETCHPAD

Students use a sketch to explore geometric sequences and recursive formulas.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Writing Recursive Formulas from Sequences (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why is it necessary to know *a* when using a recursive formula? Sample answer: The formula will tell you what to multiply each term by to get the next term, but you need to know where to start.
- How is a geometric sequence different from an exponential function, in terms of *r*? Sample answer: *r* can be negative in a geometric sequence, but the rate in an exponential function cannot.

Inquiry

How can you write a formula that relates the numbers in a geometric sequence? Sample answer: Find the relationship between the terms of the sequence, or the common ratio. Then substitute the common ratio into the formula $a_n = r \cdot a_{n-1}$

Go Online to find additional teaching notes and answers for the guiding exercises.

Learn Using Recursive Formulas

Objective

Students calculate terms in sequences by using recursive formulas.

MP Teaching the Mathematical Practices

2 Different Properties Help students to see the difference between explicit and recursive formulas, and to know the best time to use which one.

Essential Question Follow-Up

Students have studied explicit and recursive formulas for arithmetic and geometric sequences.

Ask:

Why is it useful to know both recursive and explicit formulas for sequences? Sample answer: When a pattern can be represented by a sequence, sometimes it is useful to predict it based on the total number of terms, and sometimes it is useful to iterate the steps from the previously-known term.

Example 1 Recursive Formula for an Arithmetic Sequence

Questions for Mathematical Discourse

AL What is the common difference? −9

OL Why are we given a_1? Sample answer: We have to have a number to start from when generating the sequence.

BL Can you find the twentieth term using the recursive formula without finding terms six through nineteen? Explain. No; sample answer: This kind of formula requires that you know the previous term. To find the twentieth term, you would have to find all terms before it.

Example 2 Recursive Formula for a Geometric Sequence

MP Teaching the Mathematical Practices

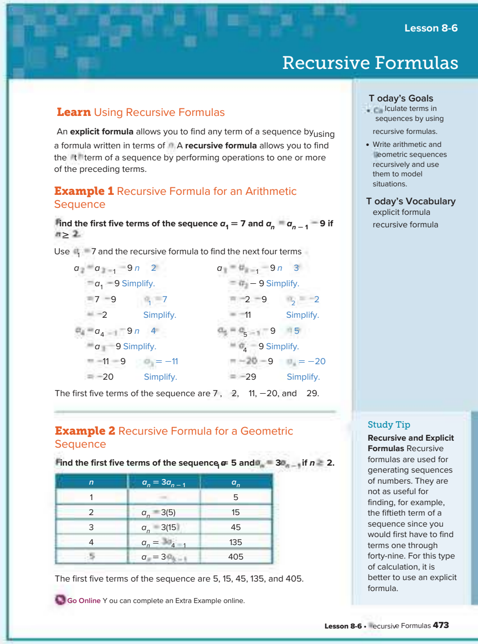
7 Look for a Pattern Help students to see the pattern in this example.

Questions for Mathematical Discourse

AL What is the common ratio in this geometric sequence? 3

OL How can you easily find the value of a_6? Sample answer: Because we have calculated the value of the 5th term, we can multiply by 3 to get the 6th term.

BL Is a_2 defined for this recursive formula? Explain. No; sample answer: The formula is defined for a_1 and then for a_n when $n \geq 2$, so $n = -2$ is not within the defined domain of the geometric sequence.



Lesson 8-6

Recursive Formulas

Learn Using Recursive Formulas

An **explicit formula** allows you to find any term of a sequence by using a formula written in terms of n. A **recursive formula** allows you to find the nth term of a sequence by performing operations to one or more of the preceding terms.

Example 1 Recursive Formula for an Arithmetic Sequence

Find the first five terms of the sequence $a_1 = 7$ and $a_n = a_{n-1} - 9$ if $n \geq 2$.

Use $a_1 = 7$ and the recursive formula to find the next four terms.

$a_2 = a_{2-1} - 9$ $n = 2$
$= a_1 - 9$ Simplify.
$= 7 - 9$ $a_1 = 7$
$= -2$ Simplify.

$a_3 = a_{3-1} - 9$ $n = 3$
$= a_2 - 9$ Simplify.
$= -2 - 9$ $a_2 = -2$
$= -11$ Simplify.

$a_4 = a_{4-1} - 9$ $n = 4$
$= a_3 - 9$ Simplify.
$= -11 - 9$ $a_3 = -11$
$= -20$ Simplify.

$a_5 = a_{5-1} - 9$ $n = 5$
$= a_4 - 9$ Simplify.
$= -20 - 9$ $a_4 = -20$
$= -29$ Simplify.

The first five terms of the sequence are 7, −2, −11, −20, and −29.

Example 2 Recursive Formula for a Geometric Sequence

Find the first five terms of the sequence $a_1 = 5$ and $a_n = 3a_{n-1}$ if $n \geq 2$.

n	$a_n = 3a_{n-1}$	a_n
1	—	5
2	$a_n = 3(5)$	15
3	$a_n = 3(15)$	45
4	$a_n = 3a_{4-1}$	135
5	$a_n = 3a_{5-1}$	405

The first five terms of the sequence are 5, 15, 45, 135, and 405.

Go Online You can complete an Extra Example online.

Today's Goals
- Calculate terms in sequences by using recursive formulas.
- Write arithmetic and geometric sequences recursively and use them to model situations.

Today's Vocabulary
explicit formula
recursive formula

Study Tip
Recursive and Explicit Formulas Recursive formulas are used for generating sequences of numbers. They are not as useful for finding, for example, the fiftieth term of a sequence since you would first have to find terms one through forty-nine. For this type of calculation, it is better to use an explicit formula.

Lesson 8-6 • Recursive Formulas 473

Interactive Presentation

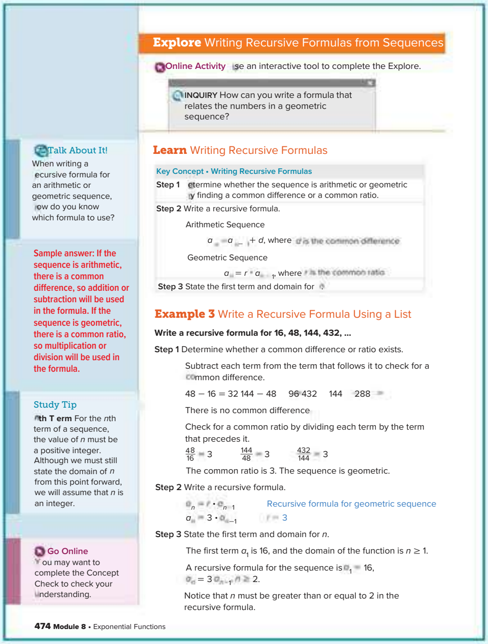
Learn

DRAG & DROP

Students complete a geometric sequence by dragging the correct terms.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Explore Writing Recursive Formulas from Sequences

Online Activity Use an interactive tool to complete the Explore.

INQUIRY How can you write a formula that relates the numbers in a geometric sequence?

Learn Writing Recursive Formulas

Key Concept • Writing Recursive Formulas

Step 1 Determine whether the sequence is arithmetic or geometric by finding a common difference or a common ratio.

Step 2 Write a recursive formula.

Arithmetic Sequence

$a_n = a_{n-1} + d$, where d is the common difference

Geometric Sequence

$a_n = r \cdot a_{n-1}$, where r is the common ratio

Step 3 State the first term and domain for n.

Example 3 Write a Recursive Formula Using a List

Write a recursive formula for 16, 48, 144, 432, ...

Step 1 Determine whether a common difference or ratio exists.

Subtract each term from the term that follows it to check for a common difference.

48 − 16 = 32 144 − 48 = 96 432 − 144 = 288

There is no common difference.

Check for a common ratio by dividing each term by the term that precedes it.

$\frac{48}{16} = 3$ $\frac{144}{48} = 3$ $\frac{432}{144} = 3$

The common ratio is 3. The sequence is geometric.

Step 2 Write a recursive formula.

$a_n = r \cdot a_{n-1}$ Recursive formula for geometric sequence

$a_n = 3 \cdot a_{n-1}$ $r = 3$

Step 3 State the first term and domain for n.

The first term a_1 is 16, and the domain of the function is $n \geq 1$.

A recursive formula for the sequence is $a_1 = 16$, $a_n = 3a_{n-1}$, $n \geq 2$.

Notice that n must be greater than or equal to 2 in the recursive formula.

Talk About It! When writing a recursive formula for an arithmetic or geometric sequence, how do you know which formula to use?

Sample answer: If the sequence is arithmetic, there is a common difference, so addition or subtraction will be used in the formula. If the sequence is geometric, there is a common ratio, so multiplication or division will be used in the formula.

Study Tip

nth Term For the nth term of a sequence, the value of n must be a positive integer. Although we must still state the domain of n from this point forward, we will assume that n is an integer.

Go Online You may want to complete the Concept Check to check your understanding.

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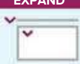
Interactive Presentation

Example 3

EXPAND



Students tap to see the steps of writing a recursive formula.

Learn Writing Recursive Formulas

Objective

Students write arithmetic or geometric sequences recursively and use them to model situations.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

DIFFERENTIATE

Reteaching Activity AL ELL

Divide the class into groups of two or three students. Have each student write a sequence on one note card and the recursive formula for the sequence on another note card. Check that students have written the recursive formulas correctly. Repeat the process for 10 sequences. Then, have the students lay the cards face down. Each student should take turns flipping over two cards, attempting to find a match between a sequence and its recursive formula.

Example 3 Write a Recursive Formula Using a List

Questions for Mathematical Discourse

AL Which type of sequence has a common ratio? geometric

OL What is the common ratio of this geometric sequence? 3

BL How would the sequence be different if the common ratio was −3? Sample answer: The terms in the sequence would alternate between positive and negative.

Common Error

In Step 1, point out to the students that the common difference and common ratio must be checked between all of the consecutive terms provided to be sure the sequence applies to the entire sequence.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Example 4 Write a Recursive Formula Using a Graph

Teaching the Mathematical Practices

1 Check Answers Mathematically proficient students continually ask themselves, "Does this make sense?" Point out that in the Think About It! feature, students need to determine how they can check their answer. Point out that they should ask themselves whether their answer makes sense and whether they have answered the problem question.

Questions for Mathematical Discourse

AL Do the graphed points form a straight line or a curve? straight line

OL Which type of sequence is represented by a linear equation? arithmetic

BL Will this sequence have negative values? Explain. Yes; sample answer: The sequence has a negative common difference, representing it as a decreasing function on the graph. This line will pass over the x-axis into negative values.

Example 5 Write Recursive and Explicit Formulas

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about recursive and explicit formulas to solving a real-world problem.

Questions for Mathematical Discourse

AL Does the number of infected persons each day illustrate a common difference or common ratio? common ratio

OL What type of sequence does the situation represent? geometric

BL Describe how the recursive formula would change if the number of infected persons had a common difference of 4 instead of a common ratio of 4. Sample answer: I would add 4 to a_{n-1} to find the next term instead of multiplying by 4. The first term and domain would remain the same; $a_1 = 3$, $a_n = a_{n-1} + 4$, and $n \geq 2$.

Check

SOCIAL MEDIA The table shows the total number of views at the end of each day for a video.

Day	Views
1	100
2	9000
3	17,900
4	26,800

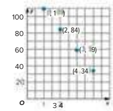
Write a recursive formula for the sequence.

$a_1 =$ ___?___ 100

$a_n = a_{n-1}$ ___?___ + 8900

Example 4 Write a Recursive Formula Using a Graph

Write a recursive formula for the graph.



Step 1 Find a common difference or common ratio, or determine that neither exists.

$84 - 109 = -25$

$59 - 84 = -25$

$34 - 59 = -25$

The common difference is -25. The sequence is arithmetic.

Step 2 Write a recursive formula.

$a_n = a_{n-1} + d$ Recursive formula for arithmetic sequence

$a_n = a_{n-1} + (-25)$ $d = -25$

Step 3 State the first term and domain for n.

The first term a_1 is 109, and $n \geq 2$.

A recursive formula for the sequence is $a_1 = 109$, $a_n = a_{n-1} - 25$, $n \geq 2$.

Think About It!

How can you make sure that your recursive formula is correct?

Sample answer: Use the formula to determine terms two, three, and four and compare them to the given terms.

Math History Minute

Hungarian mathematician **Rózsa Péter** (1905–1977) was the first Hungarian female mathematician to become an Academic Doctor of Mathematics. She helped to establish the modern field of recursive function theory, and she was the author of *Playing with Infinity: Mathematical Explorations and Excursions*.

Example 5 Write Recursive and Explicit Formulas

MOVIES **The premise of a movie is that a new virus is spreading, turning infected persons into zombie-like creatures. The table outlines the total number of infected persons at the end of each day.**

Day	Infected Persons
1	3
2	12
3	48
4	192
5	768

a. Write a recursive formula for the sequence.

Step 1 Find a common difference or common ratio.

$12 - 3 = 9$ $48 - 12 = 36$ $192 - 48 = 144$

There is no common difference. Check for a common ratio by dividing each term by the term that precedes it.

$\frac{12}{3} = 4$ $\frac{48}{12} = 4$ $\frac{192}{48} = 4$ $\frac{768}{192} = 4$

There is a common ratio of 4. The sequence is geometric.

(continued on the next page)

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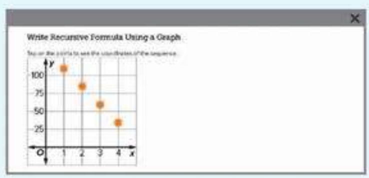
Interactive Presentation



Example 4

TAP

Students tap on each point to see the coordinates of the sequence.

Step 2 Write a recursive formula.

$a_n = r a_{n-1}$ Recursive formula for geometric sequence

$a_n = 4a_{n-1}$ $r = 4$

Step 3 State the first term and domain for n.

The first term a_1 is 3, and $n \geq 1$. A recursive formula for the sequence is $a_1 = 3$, $a_n = 4a_{n-1}$, $n \geq 2$.

b. Write an explicit formula for the sequence.

Steps 1 and 2 The common ratio is 4.

Step 3 Use the formula for the nth term of a geometric sequence.

$a_n = a_1 r^{n-1}$ Formula for the nth term

$= 3(4)^{n-1}$ $a_1 = 3$ and $r = 4$

An explicit formula for the sequence is $a_n = 3(4)^{n-1}$.

Think About It!
For the sequence in part b, find a_2.

1263.75

Example 6 Translate Between Recursive and Explicit Formulas

A recursive formula is useful when finding a number of successive terms in a sequence. An explicit formula is useful when finding the nth term of a sequence. Therefore, it may be necessary to translate between the two forms.

a. Write a recursive formula for $a_n = 0.5n + 2$.

$a_n = 0.5n + 2$ is an explicit formula for an arithmetic sequence with $d = 0.5$ and $a_1 = 0.5(1) + 2$ or 2.5.

Therefore, a recursive formula for a is $a_1 = 2.5$, $a_n = a_{n-1} + 0.5$, $n \geq 2$.

b. Write an explicit formula for $a_1 = 1011$, $a_n = 1.25a_{n-1}$, $n \geq 2$.

$a_n = 1.25a_{n-1}$ is a recursive formula for a geometric sequence with $a_1 = 1011$ and $r = 1.25$.

Therefore, an explicit formula for a_n is $a_n = 1011 \cdot (1.25)^{n-1}$.

Study Tip
Geometric Sequences Recall that the formula for the nth term of a geometric sequence is $a_n = a_1 r^{n-1}$.

Check

Part A Write a recursive formula for $a_n = \frac{1}{2} + (n - 1)10$.

$a_1 = \underline{?}$ 0.5

$a_n = 1.25a_{n-1} \underline{?}$ +10

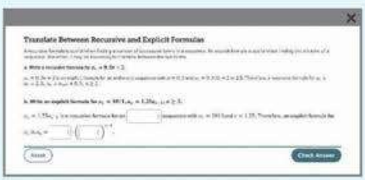
Part B Write an explicit formula for $a_1 = -60$, $a_n = 1.5a_{n-1}$, $n \geq 2$.

$a_n = \underline{?}(\underline{?})^{n-1}$
−60 1.5

Go Online You can complete an Extra Example online.

476 **Module 8** • Exponential Functions

Interactive Presentation



Example 6

SELECT

Students select phrases and values to translate between recursive and explicit formulas.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 6 Translate Between Recursive and Explicit Formulas

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the sequences and formulas in this example to translate between recursive and explicit formulas.

Questions for Mathematical Discourse

AL How can you tell if the given formula is recursive or explicit? Sample answer: If the formula uses a_{n-1} and lists a value for a_1, then it is recursive. If only n is used, the formula is explicit.

OL In part **a**, what is the common difference? 0.5

BL When converting from an explicit formula, why do you have to find the value of a_1? Sample answer: The explicit formula has been simplified from $a_n = (n - 1)d + a_1$, so the constant no longer equals the first term.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–29
2	exercises that use a variety of skills from this lesson	30–33
2	exercises that extend concepts learned in this lesson to new contexts	34–36
3	exercises that emphasize higher-order and critical-thinking skills	37–42

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–35 odd, 37–42
- Extension: The Fibonacci Sequence
- ALEKS Geometric Sequences

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–41 odd
- Remediation, Review Resources: Geometric Sequences
- Personal Tutors
- Extra Examples 1–6
- ALEKS Geometric Sequences

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–29 odd
- Remediation, Review Resources: Geometric Sequences
- *Quick Review Math Handbook*: Recursive Formulas
- Arrive**MATH** Take Another Look
- ALEKS Geometric Sequences

Practice

Go Online Y ou can complete your homework online.

Examples 1 and 2

Find the first five terms of each sequence.

1. $a_1 = 23, a_n = a_{n-1} + 7, n \geq 2$
23, 30, 37, 44, 51

2. $a_1 = 48, a_n = -0.5a_{n-1} + 8, n \geq 2$
48, −16, 16, 0, 8

3. $a_1 = 8, a_n = 2.5a_{n-1}, n \geq 2$
8, 20, 50, 125, 312.5

4. $a_1 = 12, a_n = 3a_{n-1} - 21, n \geq 2$
12, 15, 24, 51, 132

5. $a_1 = 13, a_n = -2a_{n-1} - 3, n \geq 2$
13, −29, 55, −113, 223

6. $a_1 = \frac{1}{2}, a_n = a_{n-1} + \frac{3}{2}, n \geq 2$
$\frac{1}{2}, 2, \frac{7}{2}, 5, \frac{13}{2}$

Example 3

Write a recursive formula for each sequence.

7. 12, −1, −14, −27, ...
$a_1 = 12, a_n = a_{n-1} - 13, n \geq 2$

8. 27, 41, 55, 69, ...
$a_1 = 27, a_n = a_{n-1} + 14, n \geq 2$

9. 2, 11, 20, 29, ...
$a_1 = 2, a_n = a_{n-1} + 9, n \geq 2$

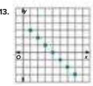
10. 100, 80, 64, 51.2, ...
$a_1 = 100, a_n = 0.8a_{n-1}, n \geq 2$

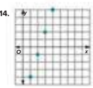
11. 40, −60, 90, −135, ...
$a_1 = 40, a_n = -1.5a_{n-1}, n \geq 2$

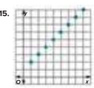
12. 81, 27, 9, 3, ...
$a_1 = 81, a_n = \frac{1}{3}a_{n-1}, n \geq 2$

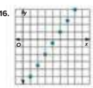
Example 4

Write a recursive formula for each graph.

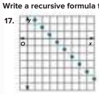
13.

$a_1 = 3, a_n = a_{n-1} - 1, n \geq 2$

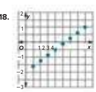
14.

$a_1 = -4, a_n = a_{n-1} + 3, n \geq 2$

15.

$a_1 = 2, a_n = a_{n-1} + 1, n \geq 2$

16.

$a_1 = -5, a_n = a_{n-1} + 1.5, n \geq 2$

Write a recursive formula for each graph.

17.

$a_1 = \frac{5}{2}, a_n = a_{n-1} - 1, n \geq 2$

18.


$a_1 = -3.25, a_n = a_{n-1} + 0.75, n \geq 2$

Example 5

19. VIRAL VIDEOS A viral video got 175 views in one hour, 350 views in two hours, 525 views in three hours, 700 views in four hours, and so on.

a. Find the next 5 terms in the sequence. 875, 1050, 1225, 1400, 1575

b. Write a recursive formula for the sequence. $a_1 = 175, a_n = a_{n-1} + 175, n \geq 2$

c. Write an explicit formula for the sequence. $a_n = 175n$

20. PAPER A piece of paper is folded several times. The number of sections into which the piece of paper is divided after each fold is shown.

Number of Folds	Sections
1	2
2	4
3	8
4	16
5	32

a. Write a recursive formula for the sequence. $a_1 = 2, a_n = 2a_{n-1}, n \geq 2$

b. Write an explicit formula for the sequence. $a_n = 2^n$

21. SNOW A snowman begins to melt as the temperature rises. The height of the snowman in feet after each hour is shown.

Hour	Height (ft)
1	6.0
2	5.4
3	4.86
4	4.374

a. Write a recursive formula for the sequence. $a_1 = 6, a_n = 0.9a_{n-1}, n \geq 2$

b. Write an explicit formula for the sequence. $a_n = 6(0.9)^{n-1}$

Example 6

For each recursive formula, write an explicit formula. For each explicit formula, write a recursive formula.

22. $a_n = 3(4)^{n-1}$
$a_1 = 3, a_n = 4a_{n-1}, n \geq 2$

23. $a_1 = -2, a_n = a_{n-1} - 12, n \geq 2$
$a_n = -12n + 10$

24. $a_1 = 38, a_n = \frac{1}{2}a_{n-1}, n \geq 2$
$a_n = 38\left(\frac{1}{2}\right)^{n-1}$

25. $a_n = -7n + 52$
$a_1 = 45, a_n = a_{n-1} - 7, n \geq 2$

26. $a_1 = 38, a_n = a_{n-1} - 17, n \geq 2$
$a_n = -17n + 55$

27. $a_n = 5n - 16$
$a_1 = -11, a_n = a_{n-1} + 5, n \geq 2$

28. $a_n = 50(0.75)^{n-1}$
$a_1 = 50, a_n = 0.75a_{n-1}, n \geq 2$

29. $a_1 = 16, a_n = 4a_{n-1}, n \geq 2$
$a_n = 16(4)^{n-1}$

Mixed Exercises

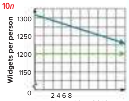
30. CLEANING An equation for the cost a_n in dollars that a carpet cleaning company charges for cleaning n rooms is $a_n = 50 + 25(n - 1)$. Write a recursive formula to represent the cost a_n. $a_1 = 50$, $a_n = a_{n-1} + 25$, $n \geq 2$

31. SAVINGS A recursive formula for the balance of a savings account a_n in dollars at the beginning of year n is $a_1 = 500$, $a_n = 1.05a_{n-1}$, $n \geq 2$. Write an explicit formula to represent the balance of the savings account a_n. $a_n = 500(1.05)^{n-1}$

32. USE TOOLS In 2010, County A had a population of 1.3 million people. The largest factory in the area produced 1700 million widgets per year. The population of County A is projected to grow at 1.2% per year, and the number of widgets produced is expected to grow by 10 million per year.

a. Develop explicit formulas for the population and annual widget production, in millions, as functions of the number of years n after 2010.
population: $p_n = 1.3(1.012)^n$; widget production: $g_n = 1700 + 10n$

b. The graph of $y = 1700 + \frac{10x}{1.3}(1.012)^x$ represents the annual widget production per person for County A from 2010 to 2020, where x is the number of years after 2010. The next-highest widget-producing county produces widgets at a constant rate of 1200 widgets per person. Use a graphing calculator to extend the graph and find the year when County A will no longer be the leader in widget production. Explain your results. See margin.



33. USE A MODEL Ramon has been tracing his family tree with his parents. He claims that he has over 250 great- great- great- great- great- great-grandparents. Is this possible? Write both an explicit and recursive formula for this situation. See margin.

34. REASONING Carl Friedrich Gauss, a German mathematician of the 1700s, was asked as a young boy for the sum of the integers from 1 to 100, and he unhesitatingly replied with the correct answer.

a. Identify the type of the sequence 1, 2, 3, ... 100, and explore a way to find its sum based on grouping pairs of numbers from each end of the sequence. Then explain how Guass was able to find the sum so quickly. See margin.

b. Find an explicit formula for the sum S of n terms of an arithmetic sequence whose first term is a_1 and whose nth, or last, term is a_n. See margin.

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35. REGULARITY The first ten numbers in the Fibonacci sequence can be defined by $a_{n+1} = a_n + a_{n-1}$, and each ratio $\frac{a_n}{a_{n-1}}$ can be computed using a spreadsheet (see column C).

Fibonacci sequence

	A	B	C
1	1	1	
2	2	1	1
3	3	2	2
4	4	3	1.5
5	5	5	1.666667
6	6	8	1.6
7	7	13	1.625
8	8	21	1.615385
9	9	34	1.619048
10	10	55	1.617647

Sheet 1 Sheet 2 Sheet 3

a. Which spreadsheet formulas could have been used to calculate the entries in cells B3 and C2?
Sample answer: B3 = B2 + B1 and C2 = B2 ÷ B1

b. Compute the ratio $\frac{a_n}{a_{n-1}}$ up to $n = 50$. What do you observe? See margin.

36. STRUCTURE There is a famous puzzle called the "Tower of Hanoi." There are three pegs, and a certain number of disks of varying sizes can be set on each peg. The puzzle starts with the disks in a stack on the left-most peg, with the largest disk on the bottom and the disks getting smaller as they are stacked. The goal is to move the disks from the left-most peg to the right-most peg while obeying three rules. First, only one disk can be moved at a time. Second, only the top disk on any peg can be moved. Third, at no time can a larger disk be placed on a smaller disk.

a. If a_n is the number of moves it takes to solve a puzzle consisting of n disks, discuss why the recursive formula $a_n = a_{n-1} + 1 + a_{n-1}$ makes sense. See margin.

b. Simplify the recursive formula. What is a_1? Why? See margin.

Higher-Order Thinking Skills

37. FIND THE ERROR Pati and Linda are working on a math problem that involves the sequence 2, −2, 2, −2, 2, ... Pati thinks that the sequence can be written as a recursive formula. Linda believes that the sequence can be written as an explicit formula. Is either of them correct? Explain your reasoning. See margin.

38. PERSEVERE Find a_1 for the sequence in which $a_4 = 1104$ and $a_n = 4a_{n-1} + 16$. 12

39. ANAL YZE Determine whether the following statement is *true* or *false*. Justify your argument. *There is only one recursive formula for every sequence.* See margin.

40. PERSEVERE Find a recursive formula for 4, 9, 19, 39, 79, ... $a_1 = 4$, $a_n = 2a_{n-1} + 1$, $n \geq 2$

41. WRITE Explain the difference between an explicit formula and a recursive formula. See margin.

42. CREATE Give a counterexample for the following statement:
In a recursive sequence, if $a_1 = a_2$, then $a_2 = a_3$, and so on. See margin.

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Answers

32b. 2024; Sample answer: Graph $Y1 = \left(\frac{1700 + 10x}{1.3(1.012)^x}\right)$ and $Y2 = 1200$, and observe where the graphs intersect, (13.7, 1200).

33. Ramon has 2 parents, 4 grandparents, 8 great-grandparents, and so on. We can write a geometric sequence to count the number of ancestors in a given generation. The recursive formula is $a_1 = 2$, $a_n = 2a_{n-1}$, $n \geq 2$. The explicit formula is $a_n = 2^n$. Ramon's claim is about the 8th generation back: $a_8 = 2^8 = 256$. Ramon is correct.

34a. Arithmetic; Group $1 + 2 + 3 + \ldots + 98 + 99 + 100$ as $(1 + 100) + (2 + 99) + (3 + 98) + \ldots + (50 + 51)$. There are 50 sums, each equal to 101, so the whole sum must be $50(101) = 5050$.

34b. Sample answer: The sum is equal to $(a_1 + a_n) + (a_1 + 1 + a_n - 1) + \ldots$, and there are $\frac{n}{2}$ of these pairs. So the sum is $S = \frac{n}{2}(a_1 + a_n)$.

35b. The ratio approaches a constant value of 1.618034.... For larger values of n, the Fibonacci numbers behave like a geometric sequence with a common ratio of 1.618034....

36a. It takes a_{n-1} moves to move the top $n - 1$ disks from the left-most peg to the middle peg. It then takes 1 move to move the largest disk from the left-most peg to the right-most peg. Finally, it takes a_{n-1} moves to move the $n - 1$ disks from the middle peg to the right-most peg.

36b. $a_n = 2a_{n-1} + 1$; $a_1 = 1$; It takes 1 move to move a single disk from the left-most peg to the right-most peg.

37. Both; sample answer: The sequence can be written as the recursive formula $a_1 = 2$, $a_n = (-1)a_{n-1}$, $n \geq 2$. The sequence can also be written as the explicit formula $a_n = 2(-1)^{n-1}$.

39. False; sample answer: A recursive formula for the sequence 1, 2, 3, ... can be written as $a_1 = 1$, $a_n = a_{n-1} + 1$, $n \geq 2$ or as $a_1 = 1$, $a_2 = 2$, $a_n = a_{n-2} + 2$, $n \geq 3$.

41. Sample answer: In an explicit formula, the nth term a_n is given as a function of n. In a recursive formula, the nth term a_n is found by performing operations to one or more of the terms that precede it.

42. Sample answer: In the recursive sequence, $a_1 = 3$, $a_2 = 3$, $a_{n+2} = a_n + a_{n+1}$, $n \geq 1$, the values of a_1, a_2, and a_3 are 3, 3, and 6, respectively. $a_1 = a_2$ but $a_2 \neq a_3$.

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why is it important to identify whether a relationship is represented by a straight line or a curve?
- Why does an exponential growth function model some situations?
- Why is it useful to know both recursive and explicit formulas for sequences?

Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to exponential functions.

LS **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Linear and Exponential Relationships** and **Quadratic Functions and Modeling**.

- Build Linear and Exponential Functions Models
- Interpret Expressions for Functions
- Construct and Compare Linear, Quadratic, and Exponential Models and Solve Problems

Module 8 • Exponential Functions

Review

Essential Question

When and how can exponential functions represent real-world situations?

Exponential functions can be used in real life to represent situations that grow or decay. One example is representing compound interest.

Module Summary

Lesson 8-1

Exponential Functions

- Functions of the form $y = ab^x$, where $a \neq 0$ and $b > 1$, are exponential growth functions.
- Functions of the form $y = ab^x$, where $a \neq 0$ and $0 < b < 1$, are exponential decay functions.
- The graphs of exponential functions have an asymptote.

Lessons 8-2 through 8-4

Transforming and Writing Exponential Functions

- The graph $f(x) = b^x$ is a parent graph of an exponential function.
- The graph of $g(x) = b^x + k$ is the graph of $f(x) = b^x$ translated vertically.
- The graph of $g(x) = b^{x-h}$ is the graph of $f(x) = b^x$ translated horizontally.
- The graph $g(x) = ab^x$ is the graph of $f(x) = b^x$ stretched or compressed vertically by a factor of $|a|$.
- The graph $g(x) = b^{ax}$ is the graph of $f(x) = b^x$ stretched or compressed horizontally by a factor of $\frac{1}{|a|}$.
- When an exponential function $f(x)$ is multiplied by -1, the result is a reflection across the x- or y-axis.
- In the equation $y = a(1 + r)^t$, y is the final amount, a is the initial amount, r is the rate of change expressed as a decimal, and t is time.

Lesson 8-5

Geometric Sequences

- A geometric sequence is a pattern of numbers that begins with a nonzero term and each term after is found by multiplying the previous term by a nonzero constant r.
- The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by the formula $a_n = a_1 r^{n-1}$, where n is any positive integer, $a_1 \neq 0$, and $r \neq 0$.

Lesson 8-6

Recursive Functions

- An explicit formula allows you to find any term a_n of a sequence by using a formula written in terms of n.
- To write a recursive formula for an arithmetic or geometric sequence, determine whether the sequence is arithmetic or geometric by finding a common difference or a common ratio.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

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Test Practice

1. **GRAPH** The table shows the function $y = 2^x - 1$. (Lesson 8-1)

x	y
0	0
1	1
2	3
3	7

Graph the function.

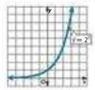
2. **MUL TIPLE CHOICE** The table shows the number of text messages Ernesto sent each month. (Lesson 8-1)

Month	Text Messages
April	2
May	6
June	18
July	54

What type of behavior is shown in the table?

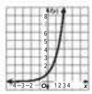
A. linea
B. piece-wise
C. exponential
D. none of the abov

3. **OPEN RESPONSE** Describe the end behavior of the graph of the exponential function shown on the graph. (Lesson 8-1)



As x increases, y increases; and, as x decreases, y approaches 0.

4. **MUL TIPLE CHOICE** Consider the graph. Which function represents the reflection of the parent function $f(x) = 3^x$ across the y-axis? (Lesson 8-2)



A. $f(x) = -3^x$
B. $f(x) = 3^{-x}$
C. $f(x) = 3^{-x}$
D. $f(x) = -2(3)^x$

Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Put It All Together: Lessons 8-1 through 8-3
Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

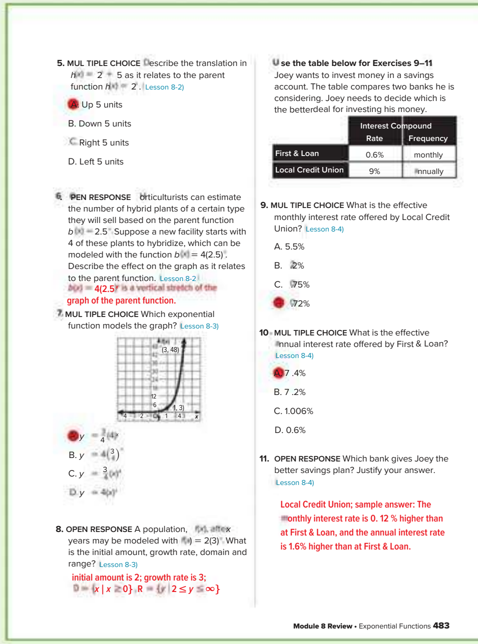
Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–20 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Graph	Students create a graph on an online coordinate plane.	1
Multiple Choice	Students select one correct answer.	2, 4, 5, 7, 9, 10, 12, 14
Table Item	Students complete a table by entering the correct values.	17
Open Response	Students construct their own response.	3, 6, 8, 11, 13, 15, 16

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
A.SSE.3c	8-4	8–11
F.IF.3	8-6	16, 17
F.IF.4	8-1	3
F.IF.7e	8-1	1
F.LE.1c	8-1	2
F.LE.2	8-3, 8-5	7, 12, 13
F.LE.5	8-3	8
F.BF.2	8-5, 8-6	14, 15
F.BF.3	8-2	4–6



5. MULTIPLE CHOICE Describe the translation in $h(x) = 2^x + 5$ as it relates to the parent function $h(x) = 2^x$. (Lesson 8-2)

A. Up 5 units

B. Down 5 units

C. Right 5 units

D. Left 5 units

6. OPEN RESPONSE Horticulturists can estimate the number of hybrid plants of a certain type they will sell based on the parent function $b(x) = 2.5^x$. Suppose a new facility starts with 4 of these plants to hybridize, which can be modeled with the function $b(x) = 4(2.5)^x$. Describe the effect on the graph as it relates to the parent function. (Lesson 8-2)

$b(x) = 4(2.5)^x$ is a vertical stretch of the graph of the parent function.

7. MULTIPLE CHOICE Which exponential function models the graph? (Lesson 8-3)

A. $y = \frac{3}{4}(4)^x$

B. $y = 4\left(\frac{3}{4}\right)^x$

C. $y = \frac{3}{4}(x)^4$

D. $y = 4(x)^{[illegible]}$

8. OPEN RESPONSE A population, $f(x)$, after x years may be modeled with $f(x) = 2(3)^x$. What is the initial amount, growth rate, domain and range? (Lesson 8-3)

initial amount is 2; growth rate is 3; $D = \{x \mid x \geq 0\}$, $R = \{y \mid 2 \leq y \leq \infty\}$

Use the table below for Exercises 9–11

Joey wants to invest money in a savings account. The table compares two banks he is considering. Joey needs to decide which is the betterdeal for investing his money.

	Interest Rate	Compound Frequency
First & Loan	0.6%	monthly
Local Credit Union	9%	annually

9. MULTIPLE CHOICE What is the effective monthly interest rate offered by Local Credit Union? (Lesson 8-4)

A. 5.5%

B. [illegible].2%

C. [illegible].75%

D. [illegible].72%

10. MULTIPLE CHOICE What is the effective annual interest rate offered by First & Loan? (Lesson 8-4)

A. 7.4%

B. 7.2%

C. 1.006%

D. 0.6%

11. OPEN RESPONSE Which bank gives Joey the better savings plan? Justify your answer. (Lesson 8-4)

Local Credit Union; sample answer: The monthly interest rate is 0.12% higher than at First & Loan, and the annual interest rate is 1.6% higher than at First & Loan.

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12. MULTIPLE CHOICE Whitney invests \$3000 in an account earning 4.5% interest that is compounded annually. How much money will be in Whitney's account after 10 years? (Lesson 8-3)

A. \$1893.02

B. \$4658.91

C. \$4700.98

D. \$123,254.07

13. OPEN RESPONSE Attendance for local baseball games has been increasing by an average of 10% per year for the last few years. In 2018, the average attendance was 100 people.

Predict the average number of people attending local baseball games in 2022 if this trend continues. Round to the nearest whole number. (Lesson 8-5)

146 people

14. MULTIPLE CHOICE What equation can be written for the nth term of this geometric sequence? (Lesson 8-5)

n	1	2	3	4
a_n	100	−50	25	−12.5

A. $a_n = 100(2)^{n-1}$

B. $a_n = 100(-2)^{n-1}$

C. $a_n = 100\left(-\frac{1}{2}\right)^{n-1}$

D. $a_n = 100\left(\frac{1}{2}\right)^{n-1}$

15. OPEN RESPONSE The table shows the number of pages Aaron read in his book each day. Write a recursive formula for the sequence. (Lesson 8-6)

Day	1	2	3	4
Pages Read	20	35	50	65

$a_1 = 20, a_n = a_{n-1} + 15, n \geq 2$

16. OPEN RESPONSE What are the first five terms of the sequence for $a_1 = -2$ and $a_n = 2a_{n-1} + 5$ if $n \geq 2$. (Lesson 8-6)

−2, 1, 7 , 19, 43

17. OPEN RESPONSE Copy and complete the table for the geometric sequence (Lesson 8-6)

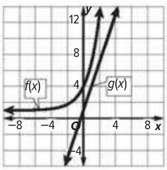
$a_1 = 3$ and $a_n = 4a_{n-1}$ if $n \geq 2$

n	formula	a_n
1	—	3
2	$a_n = 4(3)$	? 12
3	$a_n = 4($?$)$ 12	? 48
4	$a_n = 4($?$)$ 48	? 192

Lesson 8-1

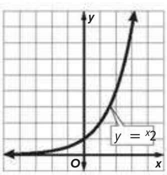
16. Sample answer: Let $a = 3$, $b = 2$, and $c = 1$.

x	$f(x) = 3 \times 2^x + 1$	$g(x) = 3x + 1$
−5	1.09375	−14
−4	1.1875	−11
−3	1.375	−8
−2	1.75	−5
−1	2.5	−2
0	4	1
1	7	4
2	13	7
3	25	10
4	49	13
5	97	16



The y-intercept of $f(x)$ is 4 and the y-intercept of $g(x)$ is 1. $f(x)$ does not have an x-intercept. The x-intercept of $g(x)$ is $-\frac{1}{3}$. As x increases, both $f(x)$ and $g(x)$ increase. As x decreases, $f(x)$ gets closer to 1 and $g(x)$ decreases. All function values for $f(x)$ are positive, while $g(x)$ has positive values for $x > \frac{1}{3}$ and negative values for $x < -\frac{1}{3}$. Neither $f(x)$ nor $g(x)$ has maximum or minimum points, and neither has symmetry.

19. Sample answer: The number of teams competing in a basketball tournament can be represented by $y = 2^x$, where the number of teams competing is y and the number of rounds is x. The y-intercept of the graph is 1. The graph increases rapidly for $x > 0$. With an exponential model, each team that joins the tournament will play all of the other teams. If the scenario were modeled with a linear function, each team that joined would play a fixed number of teams.



Lesson 8-5

49. Sample answer: When graphed, the terms of a geometric sequence lie on a curve that can be represented by an exponential function. They are different in that the domain of a geometric sequence is the set of natural numbers, while the domain of an exponential function is all real numbers. Thus, geometric sequences are discrete, while exponential functions are continuous.

50. Sample answer: First find the common ratio. Then use the formula $a_n = a_1 \cdot r^{n-1}$. Substitute the first term for a_1 and the common ratio for r. Let n represent the numbered term in the sequence. Then solve the equation.

Module 9

Statistics

Module Goals

- Students represent data using numerical statistics and graphical methods.
- Students analyze the shapes of distributions.
- Students summarize and interpret categorical data using frequency tables.

Focus

Domain: Numbers and Quantity, Statistics and Probability

Standards for Mathematical Content:

S.ID.1 Represent data with plots on the real number line (dot plots, histograms, and box plots).

S.ID.2 Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.

S.ID.3 Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

Also addresses N.Q.1, S.ID.5

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Coherence

Vertical Alignment

Previous

Students used statistics to describe and draw inferences about one or two populations of data.

6.SP, 7.SP

Now

Students use appropriate statistics to represent, compare, and analyze data.

S.ID.2, S.ID.3

Next

Students will approximate data by using a normal distribution.

S.ID.4 (Course 3)

Rigor

The Three Pillars of Rigor

Students will use the three pillars of rigor to help them meet standards. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Independent Practice.

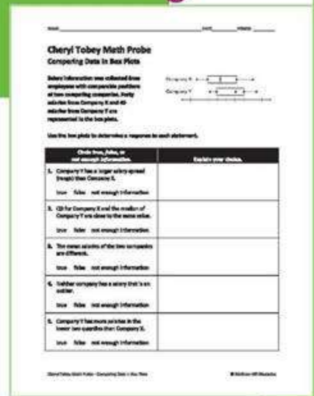
1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

EXPLORE → LEARN → EXAMPLE & PRACTICE

Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
9-1 Measures of Center	Prep for S.ID.2	1	0.5
9-2 Representing Data	N.Q.1, S.ID.1	1	0.5
9-3 Using Data	Prep for S.IC.1, Prep for S.IC.6	1	0.5
9-4 Measures of Spread	N.Q.1, S.ID.1	1	0.5
9-5 Distributions of Data	S.ID.3	1	0.5
9-6 Comparing Sets of Data	S.ID.2, S.ID.3	2	1
9-7 Summarizing Categorical Data	S.ID.5	1	0.5
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	**11**	**5.5**

Module Resource


Cheryl Tobey Math Probe
Comparing Data in Box Plots

Formative Assessment Math Probe
Comparing Data in Box Plots

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will determine which statement correctly represents the situation described and explain their choices.

Targeted Concepts Data distributions displayed in box plots can be analyzed and compared even without a specific scale.

Targeted Misconceptions

- Students may incorrectly interpret spread (range) as the maximum value.
- Students may incorrectly see quartiles as showing actual data values instead of spread.
- Students may interpret the middle line in a box plot as representing the mean instead of the median.
- Students may not know what an outlier is and/or how to find one using scale increments given on a box plot.

Use the Probe after Lesson 9-4.

Answers: 1. false 2. true 3. not enough information 4. true 5. false

Collect and Assess Student Answers

If the student selects these responses...	**Then** the student likely...
1. true	misinterprets Company Y's higher maximum with a larger spread.
2. false	does not know what Q3 and/or the median are and/or how to find them on a box plot.
3. true or false	does not know which measure of center is given in a box plot or does not understand how to find the median and/or mean.
4. false	does not know what an outlier is or how to determine if there are outliers in a box plot without scale values. Some students who answer *true* for this are still unsure and answer *true* because it "looks" like there are no outliers.
5. true	associates the larger lower box in Company Y (Q1 – Median) as having more salaries than the corresponding Company X's smaller lower box instead of associating the size of the boxes with spread.
1, 2, 4, and 5: not enough information	does not understand how to find information to compare two data sets represented in box plots and/or is confused by not having specific scale values.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS Graphical Displays
- Lesson 9-3, Learn, Examples 2 and 3

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

IGNITE!

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How do you summarize and interpret data? Sample answer: By using statistics, you can analyze data to find meaningful results. Calculating measures of center and spread and making a dot plot, bar graph, or histogram can be used to interpret the data.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

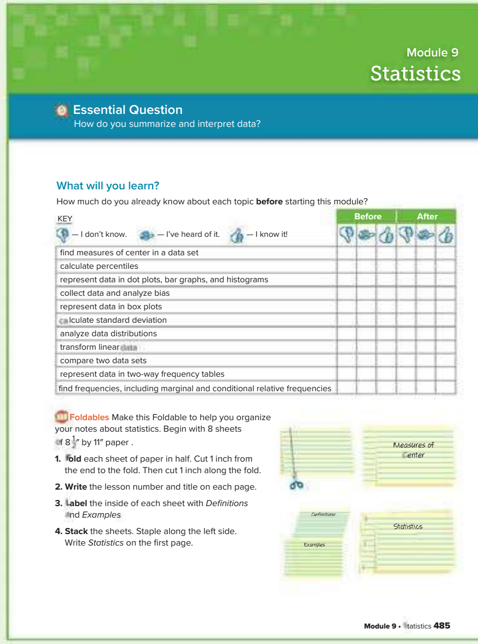
Focus Students write notes about statistics for each lesson in this module.

Teach Have students make and label their Foldables as illustrated. For each lesson, have students record definitions and examples on the appropriate sheets.

When to Use It Encourage students to add to their Foldables as they work through the module, and to use them to review for the Module Assessment.

Launch the Module

For this module, the Launch the Module video uses social media to provide a context in which to discuss the topics covered in this chapter: measures of center, histograms, measures of spread, standard deviation, and two-way frequency tables. Students learn how these statistics would be helpful in a variety of contexts, and are told that they will learn how to use and interpret statistics in real-world situations.



Module 9
Statistics

Essential Question
How do you summarize and interpret data?

What will you learn?
How much do you already know about each topic **before** starting this module?

KEY
— I don't know. — I've heard of it. — I know it!

	Before			After		
	I don't know	I've heard of it	I know it	I don't know	I've heard of it	I know it
find measures of center in a data set						
calculate percentiles						
represent data in dot plots, bar graphs, and histograms						
collect data and analyze bias						
represent data in box plots						
calculate standard deviation						
analyze data distributions						
transform linear data						
compare two data sets						
represent data in two-way frequency tables						
find frequencies, including marginal and conditional relative frequencies						

Foldables Make this Foldable to help you organize your notes about statistics. Begin with 8 sheets of $8\frac{1}{2}$″ by 11″ paper.

1. **Fold** each sheet of paper in half. Cut 1 inch from the end to the fold. Then cut 1 inch along the fold.
2. **Write** the lesson number and title on each page.
3. **Label** the inside of each sheet with *Definitions* and *Examples*
4. **Stack** the sheets. Staple along the left side. Write *Statistics* on the first page.

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Interactive Presentation

What Vocabulary Will Y ou Learn?

- bar graph
- bias
- box plot
- categorical data
- conditional relative frequency
- distribution
- dot plot
- extreme values
- five-number summary
- histogram
- interquartile range
- joint frequencies
- linear transformation
- lower quartile
- measurement data
- measures of center
- measures of spread
- median
- negatively skewed distribution
- outlier
- percentile
- population
- positively skewed distribution
- quartile
- range
- relative frequency
- sample
- standard deviation
- statistic
- symmetric distribution
- two-way frequency table
- two-way relative frequency table
- univariate data
- upper quartile
- variable
- variance

Are Y ou Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review

Example 1

Add the set of values.

12.5, 3.4, 1.75, 9

12.5
3.4
1.75 Align the numbers at the decimal.
+ 9

26.65

The sum is 26.65.

Example 2

Write the fraction $\frac{33}{80}$ as a percent. Round to the nearest tenth.

$\frac{33}{80} \approx 0.413$ Simplify and round.

$0.413 \cdot 100 = 41.3$ Multiply the decimal by 100.

$\frac{33}{80} \approx 41.3\%$ Write as a percent.

Quick Check

Add each set of values.

1. 13.2, 15, 17 .68 45.88
2. 4.5, 1.95, 2.36, 8.1 16.91
3. $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{9}{10}$ $3\frac{3}{20}$
4. −8, −4, 1, 5 −6

Write each fraction as a percent. Round to the nearest tenth.

5. $\frac{14}{17}$ 82.4%
6. $\frac{7}{8}$ 87 .5%
7. $\frac{107}{125}$ 85.6%
8. $\frac{625}{1024}$ 61.0%

How did you do?

Which exercises did you answer correctly in the Quick Check?

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define A sample is a subset of a population.

Example There are 120 students in the 9th grade. A sample of 8 students is randomly selected to be interviewed for a TV show.

Ask What is the population? the entire class of 120 students What is the sample? the 8 students selected to be interviewed

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- finding mean, median, and mode
- making inferences about populations
- finding measures of spread
- collecting data
- completing frequency tables

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the **Data Analysis and Probability** section to ensure student success in this module.

Mindset Matters

Attitude Ownership

Part of developing a growth mindset involves acknowledging progress in growth thinking and sharing it with others. It's important for students to own their mindset, be proud of their growth, and see themselves as someone who has a growth mentality—not just in math, but with learning in general.

How Can I Apply It?

Have students complete a math mindset project to share how they have grown throughout the year. They might choose the delivery method, such as a **poster, blog post, video, or podcast.** Students can share their mindset journey with their classmates as part of a class discussion, or they might post their projects for others to see.

Lesson 9-1

Measures of Center

LESSON GOAL

Students represent sets of data using measures of center and percentiles.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Mean, Median, and Mode

• Measures of Center

Explore: Finding Percentiles

Develop:

Percentiles

• Find Percentiles

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Measures of Center	●	●		●
Extension: Choosing the Best Measure of Center		●	●	●

Language Development Handbook

Assign page 49 of the *Language Development Handbook* to help your students build mathematical language related to representing sets of data using measures of center and percentiles.

ELL You can use the tips and suggestions on page T49 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Statistics and Probability

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

4 Model with mathematics.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students used measures of center for numerical data to draw inferences about a population.

7.SP.4

Now

Students find measures of center and percentiles.

Next

Students will represent data using dot plots, histograms, and box plots.

S.ID.1

Rigor

The Three Pillars of Rigor

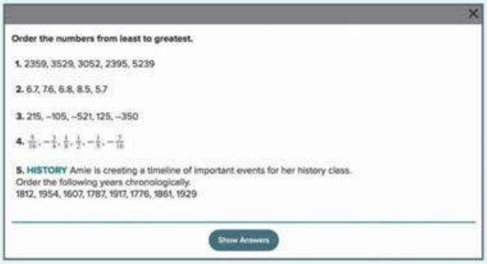
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand their understanding of and fluency with mean, median, and mode (first studied in Grade 6) to prepare for comparing measures of center and spread in data distributions. They apply their understanding of measures of center by solving real-world problems.

Mathematical Background

The *mean* of a set of data is the average of the data values. To find the mean, add the numbers and divide the sum by the number of addends. The *mode* is the number that occurs most often in a data set. Some data sets have no mode; others may have one, or more than one, mode. The *median* is the middle number in a data set that has been ordered from least to greatest. When there are two middle numbers, the median is the mean of these two numbers. An *outlier* is a number that is distant from most of the other data. A *percentile* indicates what percent of the total number of data values fall below a particular number.

Interactive Presentation



Warm Up

Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- ordering real numbers from least to greatest

Answers:

1. 2359, 2395, 3052, 3529, 5239
2. 5.7, 6.7, 6.8, 7.6, 8.5
3. −521, −350, −105, 125, 215
4. $-\frac{3}{4}, -\frac{7}{16}, -\frac{1}{5}, \frac{1}{8}, \frac{5}{16}, \frac{1}{2}$
5. 1607, 1776, 1787, 1812, 1861, 1917, 1929, 1954

Launch the Lesson

Teaching the Mathematical Practices

3 Construct Arguments Encourage students to consider the different methods the manager might use and construct an argument for using one method over the others.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use this practice?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Finding Percentiles

Objective

Students explore how to describe data with percentiles.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

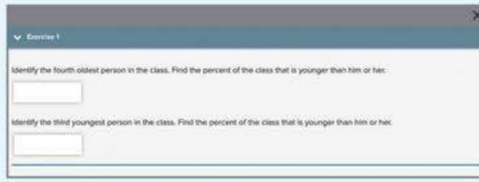
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will complete an activity in which they order the students in their class according to age and then identify the percentages of students that are younger than students in particular positions in the lineup. Students learn that these percents are called *percentiles*. Then they answer a series of questions about the data and about various percentiles in relation to their data. Then, students will answer the Inquiry Question.

(*continued on the next page*)

Interactive Presentation

Explore



Explore

TYPE

Students complete the calculations to find percentiles.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry question and can view a sample answer.

Explore Finding Percentiles (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What is the difference between a percentile and a percent? Sample answer: A percent is a ratio that compares a number to 100. A percentile is a statistic that tells you the percent of the data that falls below a certain data value.
- What would be the purpose of arranging the data vertically from greatest to least? Sample answer: It would make it easier to see the number of data values that fall below a particular value.

Inquiry

How can you describe a data value based on its position in the data set? Sample answer: Use a percentile, which indicates the percent of the data that lies below that value.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Mean, Median, and Mode

Objective

Students represent sets of data by using measures of center.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

What Students Are Learning

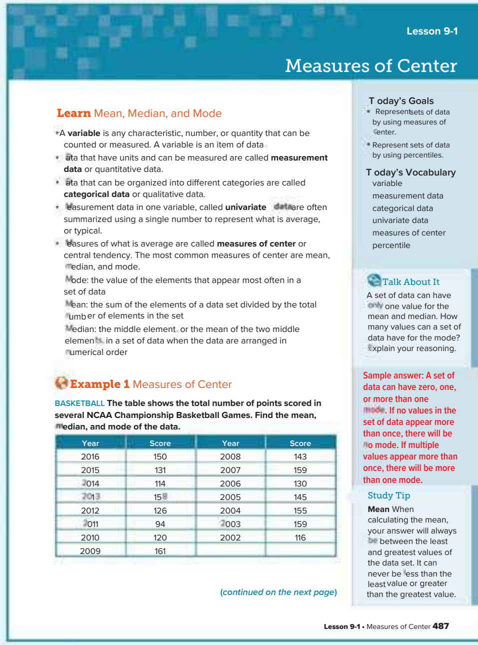
Students are learning important definitions pertaining to data and measures of central tendency. They will be using these concepts throughout the module.

Common Misconception

Some students may believe that the mode is not a measure of central tendency, citing that the mode can be, for example, the least value in a data set. Explain that the mode is considered to be a measure of central tendency. It represents what can be described as a typical measurement in the data set.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 9-1

Measures of Center

Learn Mean, Median, and Mode

- A **variable** is any characteristic, number, or quantity that can be counted or measured. A variable is an item of data.
- Data that have units and can be measured are called **measurement data** or quantitative data.
- Data that can be organized into different categories are called **categorical data** or qualitative data.
- Measurement data in one variable, called **univariate data** are often summarized using a single number to represent what is average, or typical.
- Measures of what is average are called **measures of center** or central tendency. The most common measures of center are mean, median, and mode.

Mode: the value of the elements that appear most often in a set of data

Mean: the sum of the elements of a data set divided by the total number of elements in the set

Median: the middle element or the mean of the two middle elements in a set of data when the data are arranged in numerical order

Example 1 Measures of Center

BASKETBALL **The table shows the total number of points scored in several NCAA Championship Basketball Games. Find the mean, median, and mode of the data.**

Year	Score	Year	Score
2016	150	2008	143
2015	131	2007	159
2014	114	2006	130
2013	158	2005	145
2012	126	2004	155
2011	94	2003	159
2010	120	2002	116
2009	161		

(continued on the next page)

Today's Goals
- Represent sets of data by using measures of center.
- Represent sets of data by using percentiles.

Today's Vocabulary
variable
measurement data
categorical data
univariate data
measures of center
percentile

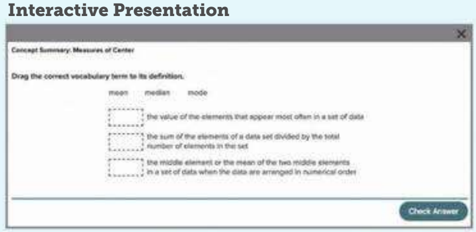
Talk About It
A set of data can have only one value for the mean and median. How many values can a set of data have for the mode? Explain your reasoning.

Sample answer: A set of data can have zero, one, or more than one mode. If no values in the set of data appear more than once, there will be no mode. If multiple values appear more than once, there will be more than one mode.

Study Tip
Mean When calculating the mean, your answer will always be between the least and greatest values of the data set. It can never be less than the least value or greater than the greatest value.

Lesson 9-1 • Measures of Center 487

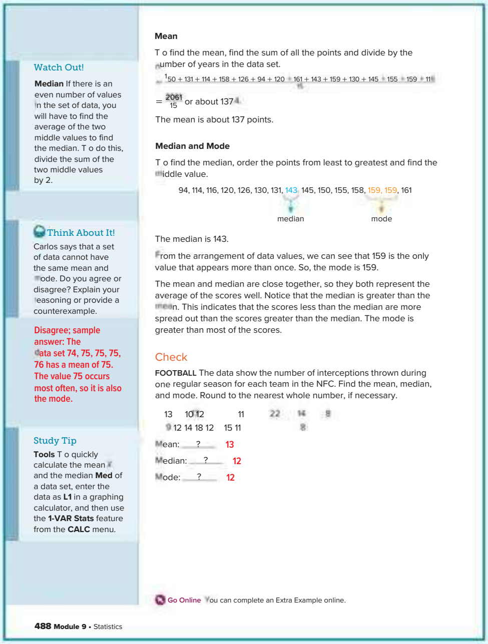
Interactive Presentation



Learn

DRAG & DROP

Students drag vocabulary terms to match them with their definitions.



Watch Out!

Median If there is an even number of values in the set of data, you will have to find the average of the two middle values to find the median. To do this, divide the sum of the two middle values by 2.

Think About It!

Carlos says that a set of data cannot have the same mean and mode. Do you agree or disagree? Explain your reasoning or provide a counterexample.

Disagree; sample answer: The data set 74, 75, 75, 75, 76 has a mean of 75. The value 75 occurs most often, so it is also the mode.

Study Tip

Tools To quickly calculate the mean and the median **Med** of a data set, enter the data as **L1** in a graphing calculator, and then use the **1-VAR Stats** feature from the **CALC** menu.

Mean

To find the mean, find the sum of all the points and divide by the number of years in the data set.

$\frac{150 + 131 + 114 + 158 + 126 + 94 + 120 + 161 + 143 + 159 + 130 + 145 + 155 + 159 + 116}{15}$

$= \frac{2061}{15}$ or about 137.4

The mean is about 137 points.

Median and Mode

To find the median, order the points from least to greatest and find the middle value.

94, 114, 116, 120, 126, 130, 131, 143, 145, 150, 155, 158, 159, 159, 161

median: 143; mode: 159, 159

The median is 143.

From the arrangement of data values, we can see that 159 is the only value that appears more than once. So, the mode is 159.

The mean and median are close together, so they both represent the average of the scores well. Notice that the median is greater than the mean. This indicates that the scores less than the median are more spread out than the scores greater than the median. The mode is greater than most of the scores.

Check

FOOTBALL The data show the number of interceptions thrown during one regular season for each team in the NFC. Find the mean, median, and mode. Round to the nearest whole number, if necessary.

13 10 12 11 22 14 8
9 12 14 18 12 15 11 8

Mean: ? 13

Median: ? 12

Mode: ? 12

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 1

TYPE

Students answer a question to show they understand measures of center.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 1 Measures of Center

Teaching the Mathematical Practices

3 Analyze Cases Work with students to look at the Think About It! feature. Ask students to determine whether Carlos's reasoning is correct or incorrect. If Carlos's reasoning is incorrect, have students identify a counterexample that disproves his claim.

Questions for Mathematical Discourse

AL How do you find the mean? Add the scores and divide by the number of scores.

OL What does the mean indicate about the data in the context of the situation? Sample answer: It indicates that the average number of points scored in the games is about 137 points.

BL Create a set of data that has the same mean, median, and mode. Sample answer: {10, 12, 15, 15, 15, 18, 20}

Common Error

Some students may confuse the mean with the median. Tell them they can distinguish the two by noticing that the word *median* contains a "d", as does the word *middle*. So the median is the middle number.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Percentiles

Objective

Students represent sets of data by using percentiles.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Important to Know

A data value that represents the 50th percentile is not necessarily equal to the mean or the median. Students should not make this assumption.

Common Misconception

You may want to discuss the differences between *percent* and *percentile*. For example, a test score at the 65th percentile means that 65% of the scores are either the same as the score at the 65th percentile or less than the score at that rank. It does not mean a test score of 65%.

Explore Finding Percentiles

Online Activity Use a real-world situation to complete the Explore.

INQUIRY How can you describe a data value based on its position in the data set?

Go Online You can watch a video to see how to find percentile rank.

Learn Percentiles

A **percentile** is a measure that is often used to report test data, such as standardized test scores. It tells us what percent of the total scores were below a given score.

- Percentiles measure rank from the bottom.
- There is no 0 percentile rank. The lowest score is at the 1st percentile.
- There is no 100th percentile rank. The highest score is at the 99th percentile.

Key Concept • Finding Percentiles

To find the percentile rank of an element of a data set, use these steps.

Step 1 Order the data values from greatest to least.

Step 2 Find the number of data values less than the chosen element. Divide that number by the total number of values in the data set.

Step 3 Multiply the value from Step 2 by 100.

Study Tip

Percent vs Percentile *Percent* and *percentile* mean two different things. For example, a score at the 40th percentile means that 40% of the scores are either the same as the score at the 40th percentile or less than the score at that rank. It does not mean that the person scored 40% of the possible points.

Example 2 Find Percentiles

FIGURE SKATING The table shows the total points scored by each country in the team figure skating event in the 2014 Olympic Winter Games. Find the United States' percentile rank.

Country	Score
Canada	65
China	20
France	22
Germany	17
Great Britain	8
Italy	52
Japan	51
Russia	75
Ukraine	10
United States	60

(continued on the next page)

Go Online You can complete an Extra Example online.

Interactive Presentation

Learn

TYPE

Students answer a question to show they understand percentiles.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Step 1 Order the data.

Order the data values from greatest to least.

Country	Score
Russia	75
Canada	65
United States	60
Italy	52
Japan	51
France	22
China	20
Germany	17
Ukraine	10
Great Britain	8

Step 2 Divide.

Divide the number of teams with scores lower than the United States by the total number of teams.

$$\frac{\text{number of teams below the United States}}{\text{Total number of teams}} = \frac{7}{10}$$

Step 3 Multiply by 100.

$\frac{7}{10} \cdot 100$ or 70

The United States figure skating team scored at the 70th percentile in the 2014 Olympics.

Study Tip

Percentiles The team with the highest score is at the 99th percentile rank, and the team with the lowest score is at the 1st percentile rank.

Think About It!

Which team scored at the 40th percentile?

France

Check

DRUM CORPS The table shows the scores of the corps that competed in the Drum Corps International World Championship World Class Finals in 2015

Corps	Score
Bluecoats	96.925
Blue Devils	97 .650
Blue Knights	91.850
Blue Stars	85.150
Boston Crusaders	86.800
Carolina Crown	97 .075
Crossmen	85.025
Madison Scouts	88.750
Phantom Regiment	90.325
Santa Clara Vanguard	93.850
The Cadets	95.900
The Cavaliers	88.325

The Cavaliers scored at the __?__ 25th percentile. The __?__ Bluecoats scored at the 75th percentile.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 2

EXPAND

Students can tap to see the steps for finding percentiles.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Find Percentiles

Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the quantities used in this problem.

Questions for Mathematical Discourse

AL What is the purpose of arranging the scores vertically from greatest to least? to see the number of scores at or below a particular score

OL What is the least score? 8 At what percentile rank is the least score? 1st percentile What is the score at the 99th percentile? 75

BL Suppose the best possible score is 100 points. What percent of the total number of points did Canada receive? 65% What percentile rank is Canada? 80th percentile

Common Error

Students may state that Russia's score represents the 90th percentile. Tell students that although 90% of the data falls below Russia's score, convention is to rank the greatest data value as the 99th percentile.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are having trouble calculating percentiles,
THEN partner them with a student who has a better understanding of how to calculate percentiles, and have them work through several problems together.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–12
2	exercises that use a variety of skills from this lesson	13–23
2	exercises that extend concepts learned in this lesson to new contexts	24–31
3	exercises that emphasize higher-order and critical-thinking skills	32–39

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–31 odd, 32–39
- Extension: Choosing the Best Measure of Center
- ALEKS Data Analysis

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–31
- Remediation, Review Resources: Measures of Center
- Personal Tutors
- Extra Examples 1–2
- ALEKS Ordering Numbers from Least to Greatest

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–11 odd
- Remediation, Review Resources: Measures of Center
- Arrive**MATH** Take Another Look
- ALEKS Ordering Numbers from Least to Greatest

Answers

13. Sample answer: The mean could be slightly higher because on a few Saturday nights throughout the year, there were a very large number of people at the movies, which caused the mean to increase but did not affect the median.

14. Sample answer: The mode time it takes to fly from New York City to Chicago is the most frequent, but there could have been a few flights that were much longer due to delays, which affects the mean but does not affect the mode.

Practice

Go Online Y ou can complete your homework online.

Example 1

Find the mean, median, and mode for each data set.

1. {17, 11, 8, 15, 28, 20, 10, 16}
mean: 15.625, median: 15.5, mode: none

2. {2.5, 6.4, 7.0, 5.3, 1.1, 6.4, 3.5, 6.2, 3.9, 4.0}
mean: 4.63, median: 4.65, mode: 6.4

3.

2 1 1 5 7				
3 2 4 6 2				

mean: 3.3, median: 2.5, mode: 2

4.

50 30 40 10			
20 80 60 90			
10 30 110 70			

mean: 50, median: 45, mode: 10, 30

5. number of students helping at a booth each hour: 3, 5, 8, 1, 4, 11, 3
mean: 5 students; median: 4 students; mode: 3 students

6. weight in pounds of boxes loaded onto a semi-truck: 201, 201, 200, 199, 199
mean: 200 lbs.; median: 200 lbs.; mode: 201 lbs. and 199 lbs.

7. car speeds in miles per hour observed by a highway patrol officer: 60, 53, 53, 52, 53, 55, 55, 57 mean: 54.75 mph; median: 54 mph; mode: 53 mph

8. number of songs downloaded by students last week in Ms. Turner's class: 3, 7, 21, 23, 63, 27, 29, 95, 23
mean: about 32 songs; median: 23 songs; mode: 23 songs

9. ratings of an online video: 2, 5, 3.5, 4, 4.5, 1, 1, 4, 2, 1.5, 2.5, 2, 3, 3.5
mean: about 2.8; median: 2.75; mode: 2

Example 2

MARCHING BAND **A competition was recently held for 12 high school marching bands. Each band received a score from 0 through 100, with 100 being the highest.**

Band	Score Band	Score
Freeport	78 Madison	69
Ross	85 Monmouth 67	
Hamilton	88 Carlisle	65
Groveport 94 Dupont		48
Lakehurst 56 Cave City 90		
Benton	77 Monroe	80

10. Find Hamilton High School's percentile rank.
75th percentile

11. Find Monmouth High School's percentile rank.
25th percentile

12. Find Freeport High School's percentile rank.
50th percentile

Mixed Exercises

13. REASONING The mean number of people at the movies on Saturday nights throughout the year is 425, and the median is 412. Explain why the mean could be slightly higher. See margin.

14. REASONING The mode length of time it takes to fly from New York City to Chicago is 2 hours 35 minutes, and the mean is 3 hours 15 minutes. Explain why the mode could be slightly lower. See margin.

Lesson 9-1 • Measures of Center **491**

15. FOOTBALL Find the mean, median, and mode for the data set. The weights in pounds of 5 offensive linemen of a football team: 217, 212, 285, 245, 301.
mean: 252; median: 245; mode: none

16. WEB SITES The ratings for a new recipe Web site varied from very low, 1 point, to very high, 10 points, with half of the scores receiving a rating of 7. If a new rating of 7 were added to the data set, how would the mode be affected? Explain.
The mode would not be affected because half of the data values are 7, so the mode is already 7. Adding a data value of 7 would mean that mode is still 7.

17. Find a mean of {16, 19, 22, 27, 33, 19, 25}. 23

18. SINGING In a singing competition that involved 50 contestants, Reina's score ranked higher than 40 of the contestants. In what percentile did Reina score?
80th percentile

19. SPORTS The table shows the number of points scored by a basketball team during their first several games. Find the mean, median, and mode of the number of points scored. mean: 51.5, median: 51, mode: none

Game	1 2 3 4 5 6 7 8							
Points	43 50 52 47 55 61 48 56							

20. PERFORMANCE At a bodybuilding competition, Shawnte earned a score of 42 points. There were 19 competitors who received a lower score than Shawnte and 5 competitors who earned a higher score. What was Shawnte's percentile rank in the bodybuilding competition? 76th percentile

21. GRADES On her first four quizzes, Rachael has earned scores of 21, 24, 23, and 17 points. What score must Rachael earn on her fifth and final quiz so that both the mean and median of her quiz scores is 21? 20 points

22. QUIZZES Sequon scored 95, 86, 81, 83, and 95 on his math quizzes this quarter. Find the mode of his quiz scores. 95

23. BAND Out of the 30 bands at the competition, Coastal High School's band scored higher than 27 others. Find the percentile rank for Coastal High School's band. 90th percentile

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24. SHOPPING The table shows the prices of comparable laptop computers at different retailers.

a. Find the mean, median, and mode of the prices.
mean: 392, median: 384, mode: 425

b. Why are the mean and median much lower than the mode?
See margin.

c. After deliberation, Nikki is interested in buying a laptop from either retailer C, F, or J. What are the percentile ranks for the laptop at each of these retailers?
C: 20^{th} percentile; F: 30^{th} percentile; J: 40^{th} percentile

Retailer	Price ($)
A	389
B	425
C	350
D	499
E	475
F	360
G	319
H	425
I	299
J	379

25. NOVELS The table shows the lengths (in words) of the seven novels on the required reading list of Miguel's language arts class.

a. Find the mean, median, and mode for the data set.
mean = 109,633; median = 66,556, no mode

b. Predict which novels will be lower than the 50^{th} percentile in length. Verify your prediction.
See margin.

c. If *Lord of the Flies* comes with an additional 10,020-word online reading assignment, then how will the median length be affected? How will the mean be affected? See margin.

Novel	Number of Words
Old Y eller	35,968
Lord of the Flies	59,900
Moby Dick	206,052
Jane Eyre	183,858
Great Expectations	183,349
Call of the Wild	31,750
The Color Purple	66,556

26. VOLUNTEERING The table shows the number of hours different students spent volunteering as part of a community outreach program. Find the mean, median, and mode of the data set. mean: 40, median: 35, mode: 25

Volunteer Hours				
25	30	35	40	35
25	50	45	25	90

27. USE A SOURCE Research the total medal counts for Canada, France, Japan, Russia, Brazil, and Great Britain at the 2016 Rio de Janeiro Olympics. Make a table of the data you collect. Then find the percentile rank of each country.
See margin.

28. CONSTRUCT ARGUMENTS The table shows the number of pet adoptions each week for a shelter over a two-month period. If there are 65 pets adopted next week, how are the mean, median, and mode affected? See margin.

Pet Adoptions			
7	19	26	20
23	21	24	20

29. STATE YOUR ASSUMPTION A data set has a mean of 37, a median of 36.5, and a mode of 37. What assumption(s) can you make about the dataset? See margin.

30. BOWLING The table shows Lucinda's score for each of her last ten bowling games.

a. Find the mean, median, and mode of the scores. Round to the nearest whole number. mean: 229, median: 223, mode: 220

b. Why is the mean slightly higher than the median? See margin.

Game	Score
1	220
2	235
3	255
4	210
5	240
6	220
7	225
8	220
9	250
10	210

31. DANCE COMPETITION At a dance competition, Pascal earned a score of 73 points. There were 12 competitors who received a lower score than Pascal and 3 competitors who earned a higher score. What was Pascal's percentile rank in the dance competition? 75^{th} percentile

Higher-Order Thinking Skills

32. CREATE Create a data set that has a mean of 11, a median of 10, and a mode of 8.
Sample answer: 8, 8, 9, 10, 11, 15, 16

33. WRITE Describe how an outlier value that is greater than the numbers in the data set affects each measure of center. See margin.

34. ANALYZE Determine whether the statement is *true* or *false*. If it is false, explain how to make the statement true. See margin.

To find percentile rank, divide the selected value by the total of all the values.

35. PERSEVERE Describe the effect on the mean, median, and mode of a set when all the items in the set are multiplied by the same number.
The mean, median, and mode will all be multiplied by the number.

36. WHICH ONE DOESN'T BELONG? Analyze each situation. Which situation is NOT best described by the median of the data? Explain. See margin.

An art gallery has many items for sale that are reasonably priced, but it also carries luxury priced paintings.	Most of the students volunteered 2 hours each week, but James volunteered 8 hours per week.	The amusement park had about the same number of attendees each day. On the annual bring-a-friend-for-free day, the number of attendees tripled.

37. FIND THE ERROR Julio is studying botany and has been tracking the growth of 10 tomato plants each week. The first week, the plants measured the following growth: 1 in., 1.5 in., 2.2 in., 0.5 in., 1 in., 1.25 in., 1.4 in., 2 in., 2.1 in., 1.9 in. In his research paper, Julio includes the median growth value for the week. Has Julio chosen the best measure of center to describe the plant growth? Explain.
Julio should have chosen the mean because all the growth values are close together.

38. STRUCTURE Explain how you determine that a data set is best described by the mean. See margin.

39. WRITE Explain in your own words the process for finding a percentile rank. See margin.

Answers

24b. Most retailers sell laptop computers for $425, but the majority of prices are much lower. This means the mode is high, but the mean and median are much lower.

25b. Sample answer: The novels lower than the 50^{th} percentile would be those consisting of words in the thirty-thousands and in the upper fifty-thousands. My prediction is correct because those three books are in the 47^{th} percentile, which is just under the 50^{th} percentile.

25c. The median will change from 66,556 to 69,920, a difference of 3,364 words. The mean will change from 109,633 to 111,065, a difference of 1,432 words.

27. Canada: 20^{th} percentile; France: 50^{th} percentile; Japan: 40^{th} percentile; Russia: 60^{th} percentile; Brazil: 10^{th} percentile; Great Britain: 70^{th} percentile

Olympic Medal Counts	
Country	**Total Medals**
Australia	29
Brazil	19
Canada	22
China	70
France	42
Great Britain	67
Japan	41
New Zealand	18
Russia	56
United States	121

28. The mean is 20, the median is 20.5, and the mode is 20. The new mean is 25, so the mean increases. The new median is 21, so the median increases, but not by a lot. The new mode is 20, so the mode was not affected.

29. Sample answer: The data is tightly clustered around 37 because all three measures of center are close.

30b. Half the bowling scores are above 223 and half are below 223, but the scores below 223 are close to 223, whereas the scores above 223 are not as close to 223.

33. Because the mean is an average of all the numbers in the data set, it is most affected by outliers. An outlier on the high end will cause the mean to increase. The median is the middle value in the data set, so adding one high number should not affect the median much unless the data set has values that are widely spread. The mode is the most frequent number, so the outlier will have no effect on the mode unless the outlier is the same as the mode.

34. False; list the numbers from greatest to least, then divide the number of values below the selected value by the total number of values.

36. The second choice, the hours students volunteered, would not be described by the median. Because most students volunteer 2 hours, the mode is the best representation.

38. When looking at the data set, if there are no outliers and the numbers are relatively close together, then the mean is the best descriptor.

39. Sample answer: To find a percentile rank, order the data set in decreasing order. Count the number of items below the item you are ranking, and divide that by the total number of items. Multiply this answer by 100 to arrive at the percentile rank.

Lesson 9-2

Representing Data

N.Q.1, S.ID.1

LESSON GOAL

Students represent data using dot plots, histograms, and bar graphs.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Dot Plots
- Make a Dot Plot
- Make a Dot Plot by Using a Scaled Number Line

Bar Graphs and Histograms
- Determine an Appropriate Graph for Discrete Data
- Determine an Appropriate Graph for Continuous Data

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Find the Mode	●	●		●
Extension: Segmented Bar Charts		●	●	●

Language Development Handbook

Assign page 50 of the *Language Development Handbook* to help your students build mathematical language related to representing data using dot plots, histograms, and bar graphs.

ELL You can use the tips and suggestions on page T50 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min: 0.5 day

45 min: 1 day

Focus

Domains: Numbers and Quantity, Statistics and Probability

Standards for Mathematical Content:

N.Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

S.ID.1 Represent data with plots on the real number line (dot plots, histograms, and box plots).

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

4 Model with mathematics.

Coherence

Vertical Alignment

Previous
Students analyzed dot plots, histograms, and box plots.
6.SP.4

Now
Students represent data using dot plots, histograms, and bar graphs.
N.Q.1, S.ID.1

Next
Students will use statistics appropriate to the shape of the data distribution to compare centers and spread of two or more data sets.
S.ID.2, S.ID.3

Rigor

The Three Pillars of Rigor

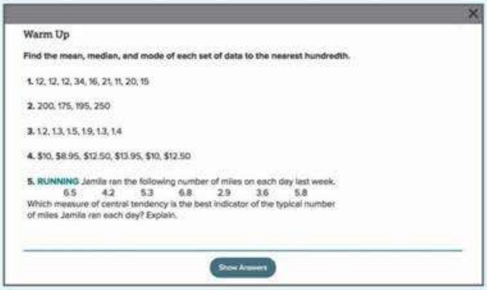
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students apply their understanding of data distributions by solving real-world problems. They build fluency by making dot plots, bar graphs, and histograms.

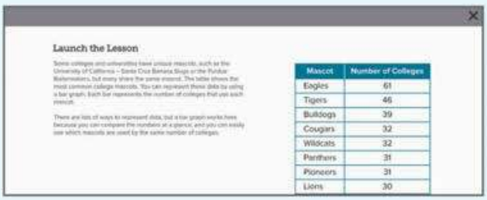
Mathematical Background

There are many ways to represent data graphically. The type of data, and the purpose of the display, typically dictate which type of display would be most appropriate. A *dot plot* is used for small sets of data that fall into discrete categories. A *bar graph* is useful for comparing data. A *histogram* is similar to a bar graph, but each bar represents a range of data values.

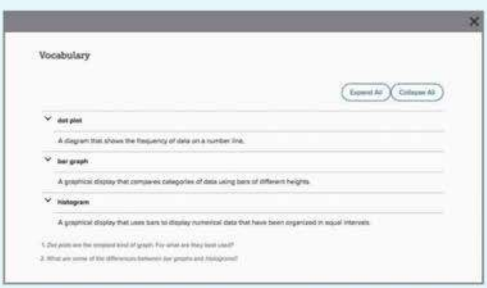
Interactive Presentation



Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skills for this lesson:

- finding mean, median, and mode
- determining which measure of central tendency is the best indicator to use in a given situation

Answers:

1. 17, 15, 12
2. 205, 197.5, no mode
3. 1.43, 1.35, 1.3
4. $11.32, $11.25, $10 and $12.50
5. Sample answer: Median; because it is in the middle.

Launch the Lesson

Teaching the Mathematical Practices

4 Use Tools Encourage students to consider the advantages of having a visual display of the data.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Dot Plots

Objective

Students represent sets of data by using dot plots.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Things to Remember

The scale for a dot plot must be chosen so that every value in the data set is represented on the number line.

Common Misconception

A common misconception some students may have is that the values on the number line must be the values in the data set. Correct this misconception by demonstrating that the number line must contain equal intervals, including numbers for which there are no data values.

Example 1 Make a Dot Plot

Teaching the Mathematical Practices

1 Explain Correspondences Guide students as they use the information in this example to plot data to represent the situation.

Questions for Mathematical Discourse

- **AL** What are the minimum and maximum values needed for the number line? 9 and 15
- **OL** Based on the dot plot, what is the mode of the data? 15
- **BL** What is the median of the data? 13

Common Error

Some students may accidentally omit a data point. Encourage them to count the number of dots they have plotted and make sure it is equal to the number of data values in the set.

Go Online

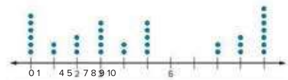
- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 9-2

Representing Data

Learn Dot Plots

One way to represent data is by using a **dot plot**, which is a diagram that shows the frequency of data on a number line.



Key Concept • Making Dot Plots

Step 1 Write the data points in order from least to greatest.

Step 2 Make a number line that starts at the least data point and ends at the greatest data point. Choose an appropriate scale.

Step 3 Plot the dots on the number line. Stack the points when there is more than one data point with the same number.

Step 4 If appropriate, include a label for the number line and title for the dot plot.

Example 1 Make a Dot Plot

Represent the data as a dot plot.

11, 12, 14, 15, 12, 13, 15, 13, 9, 15, 12, 13, 15, 15, 11

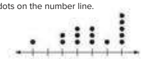
Step 1 Write the data points in order from least to greatest.

9, 11, 11, 12, 12, 12, 13, 13, 13, 14, 15, 15, 15, 15, 15

Step 2 Make a number line.

The data are whole numbers ranging from 9 to 15. So, make a number line starting at 9 with intervals of 1.

Step 3 Plot the dots on the number line.



Step 4 If appropriate, include a label for the number line and title for the dot plot.

Because no information is given regarding what these data represent, no title is needed for this dot plot.

Go Online You can complete an Extra Example online.

Today's Goals

- Represent sets of data by using dot plots.
- Determine whether discrete or continuous graphical representations are appropriate, and represent sets of data by using bar graphs or histograms.

Today's Vocabulary

dot plot
bar graph
histogram

Think About It!

What would be the benefit of representing data in a dot plot?

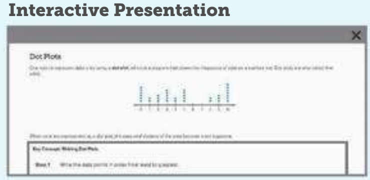
Sample answer: A dot plot would make certain statistics more obvious. The range, mode, and possibly the median, could be easily determined from the graph. A dot plot also makes clusters and gaps in the data more obvious.

Study Tip

Accuracy Count the listed data and check that the number of data points matches the sum of the frequency table. Missing just one or two pieces of data can change the values of statistics.

Lesson 9-2 • Representing Data 495

Interactive Presentation

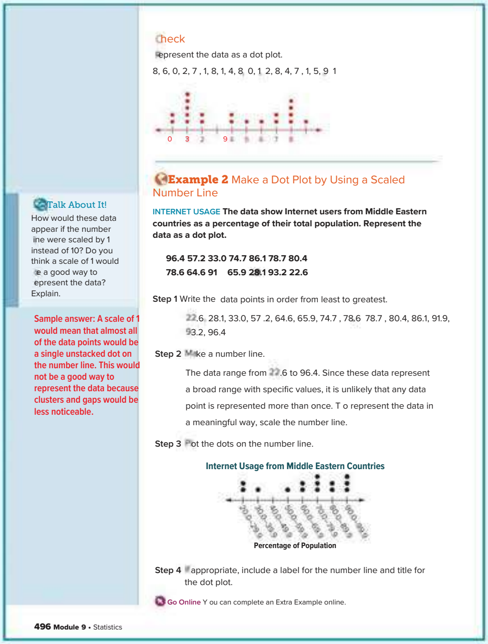


Learn

TYPE

Students answer a question to show they understand how to create a dot plot.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**



Check

Represent the data as a dot plot.

8, 6, 0, 2, 7 , 1, 8, 1, 4, 8 0, 1 2, 8, 4, 7 , 1, 5, 9 1

Example 2 Make a Dot Plot by Using a Scaled Number Line

INTERNET USAGE **The data show Internet users from Middle Eastern countries as a percentage of their total population. Represent the data as a dot plot.**

96.4 57.2 33.0 74.7 86.1 78.7 80.4
78.6 64.6 91 65.9 28.1 93.2 22.6

Step 1 Write the data points in order from least to greatest.

22.6, 28.1, 33.0, 57 .2, 64.6, 65.9, 74.7 , 78.6 78.7 , 80.4, 86.1, 91.9, 93.2, 96.4

Step 2 Make a number line.

The data range from 22.6 to 96.4. Since these data represent a broad range with specific values, it is unlikely that any data point is represented more than once. T o represent the data in a meaningful way, scale the number line.

Step 3 Plot the dots on the number line.

Step 4 If appropriate, include a label for the number line and title for the dot plot.

Go Online Y ou can complete an Extra Example online.

Talk About It!
How would these data appear if the number line were scaled by 1 instead of 10? Do you think a scale of 1 would be a good way to represent the data? Explain.

Sample answer: A scale of 1 would mean that almost all of the data points would be a single unstacked dot on the number line. This would not be a good way to represent the data because clusters and gaps would be less noticeable.

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Interactive Presentation

Example 2

TAP

Students move through the steps to make a dot plot.

TYPE

Students answer a question to show they understand how to make a dot plot by using a scaled number line.

Example 2 Make a Dot Plot by Using a Scaled Number Line

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about dot plots to solve a real-world problem.

Questions for Mathematical Discourse

AL Why should you use intervals to construct this dot plot? Sample answer: Because no number is repeated more than once, intervals will allow for stacking of the data. Also, the data is very widely spread, so using intervals will make the dot plot more compact.

OL Can the least and greatest data values be determined from the dot plot? Explain. No; sample answer: When intervals are used for the number line, you do not know the exact values of the data points.

BL How do you know if you have chosen a good scale for a dot plot? Sample answer: There are stacked dots, and the dot plot is fairly compact without too many large gaps.

Common Error

Some students may try to begin the number line scale at the least data value. Help students to see that the first interval can start at any number, as long as the least data value falls in that first interval.

Learn Bar Graphs and Histograms

Objective

Students determine whether discrete or continuous graphical representations are appropriate, and then represent sets of data by using bar graphs or histograms.

Teaching the Mathematical Practices

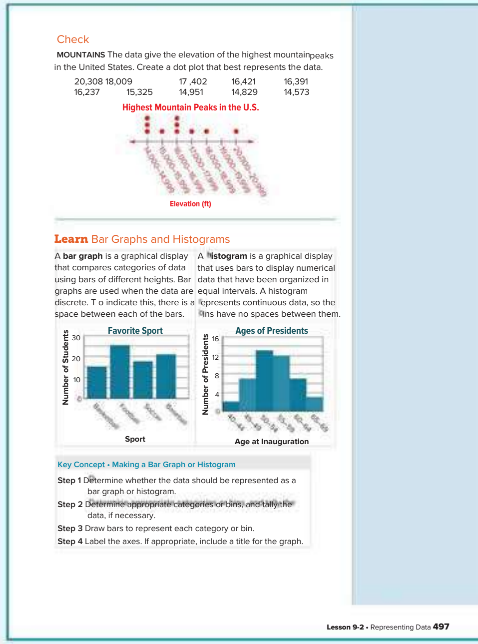
7 Use Structure Help students to explore the structure of bar graphs and histograms in this Learn.

What Students Are Learning

Students are learning about bar graphs and their characteristics. They will use what they learn to compare bar graphs to histograms and decide which type of display is appropriate for a given set of data. They also will learn how to construct each type of display.

Common Misconception

A common misconception that students may have is that a histogram is another name for a bar graph. Explain that this is not the case and that the differences between the two types of displays will be explained in this lesson.



Check

MOUNTAINS The data give the elevation of the highest mountain peaks in the United States. Create a dot plot that best represents the data.

20,308	18,009	17,402	16,421	16,391
16,237	15,325	14,951	14,829	14,573

Highest Mountain Peaks in the U.S.

Elevation (ft)

Learn Bar Graphs and Histograms

A **bar graph** is a graphical display that compares categories of data using bars of different heights. Bar graphs are used when the data are discrete. To indicate this, there is a space between each of the bars.

A **histogram** is a graphical display that uses bars to display numerical data that have been organized in equal intervals. A histogram represents continuous data, so the bins have no spaces between them.

Favorite Sport (Number of Students; Sport)

Ages of Presidents (Number of Presidents; Age at Inauguration)

Key Concept • Making a Bar Graph or Histogram

Step 1 Determine whether the data should be represented as a bar graph or histogram.

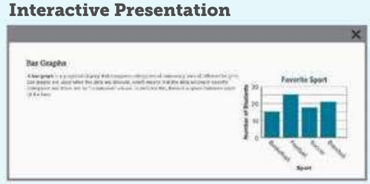
Step 2 Determine appropriate categories or bins, and tally the data, if necessary.

Step 3 Draw bars to represent each category or bin.

Step 4 Label the axes. If appropriate, include a title for the graph.

Lesson 9-2 • Representing Data 497

Interactive Presentation



Learn

SELECT

Students categorize statements as a description of a bar graph or histogram.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Apply Example 3 Determine an Appropriate Graph for Discrete Data

OLYMPICS **The table shows the total number of Olympic medals won by U.S. athletes competing in selected events from the first Summer Olympics in 1896 through 2012. Make a graph of the data to show the total medals won for each sport.**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Determine whether the data should be represented as a bar graph or histogram. These data represent discrete, categorical data, so use a bar graph.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

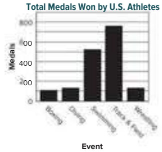
I'll tally the number of medals won for each sport and then I'll create the bar graph. Each event will be represented by one bar. I have learned how to make a bar graph.

3 What is your solution?

Use your strategy to solve the problem.

Event	Gold Silver Bronze Total			
Boxing	49 2	3	39	111
Diving	48 41		43	132
Swimming	230 164		126	520
Track & Field	319 247		193	759
Wrestling	52 43		34	129

Total Medals Won by U.S. Athletes



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Apply Example 3 Determine an Appropriate Graph for Discrete Data

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

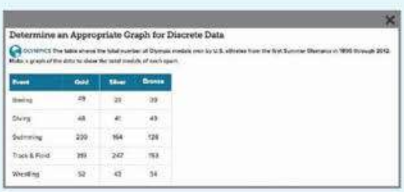
If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- Why is a bar graph a good choice to represent this data?
- What is a disadvantage of using the graph in this example?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Interactive Presentation



Apply Example 3

TYPE

Students will complete the table to tally the total number of medals won in each event.

Example 4 Determine an Appropriate Graph for Continuous Data

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about graphing continuous data to solve a real-world problem.

Questions for Mathematical Discourse

AL What is the best interval to use for the histogram? one minute

OL Why is a histogram the best choice for this data? Sample answer: The data is continuous.

BL How would the graph change if the interval used was 2 minutes? Sample answer: The data would stack even higher, and there would only be one gap at 1:32:00-1:33:59.

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: Because the graph is supposed to represent the total number of medals in each event, it makes sense for the data to be organizedin categories. Categorical data should be represented by a bar graph.

Check

VIDEO GAMES The table shows the number of active video game players in each country. Make a graph that best displays the data.

Country	Australia	Brazil	France	Germany	Italy	Poland	Spain	Turkey	UK	US
Players (millions)	9.5	40.2	25.3	38.5	18.6	11.8	21.8	33.6	157	

Example 4 Determine an Appropriate Graph for Continuous Data

MARATHON The results of the top finishers of the 2015 New York City Marathon, wheelchair division, are given below. Determine whether the data are *discrete* or *continuous*. Then make a graph.

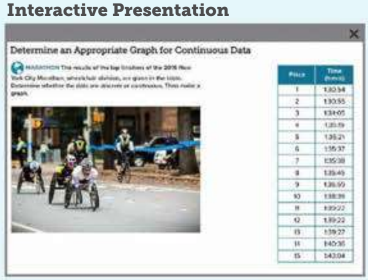
1:30:54 1:30:55 1:34:05 1:35:19 1:35:21 1:35:37 1:35:38 1:36:45
1:36:59 1:38:39 1:39:22 1:39:22 1:39:27 1:39:27 1:40:36 1:43:04

Step 1 Because racers can finish with any time, the data are continuous and you can use a histogram.

(continued on the next page)

Lesson 9-2 • Representing Data 499

Interactive Presentation



Example 4

TAP

Students move through the steps to determine an appropriate graph for continuous data.

TYPE

Students complete the calculations to determine an appropriate graph.

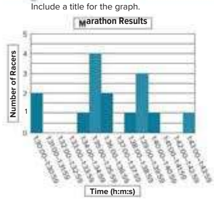
Think About It!
Describe the histogram. What does it show you about the racers' times? What do the gaps in the graph represent?

Sample answer: The histogram has two data points in the first bin, one data point in the last bin, and two clusters in the middle. It shows that after the first two winners, the racers came in groups. The gaps between the clusters represent the times at which no racers finished.

Step 2 Because the data are spread over several minutes, group the data by the minute. Then, tally each interval.

Time (h:m:s)	Frequency	Time (h:m:s)	Frequency
1:30:00–1:30:59	2	1:37:00–1:37:59	0
1:31:00–1:31:59	0	1:38:00–1:38:59	1
1:32:00–1:32:59	0	1:39:00–1:39:59	3
1:33:00–1:33:59	0	1:40:00–1:40:59	1
1:34:00–1:34:59	1	1:41:00–1:41:59	0
1:35:00–1:35:59	4	1:42:00–1:42:59	0
1:36:00–1:36:59	2	1:43:00–1:43:59	1

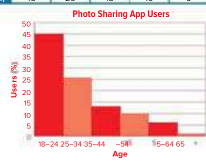
Steps 3 and 4 Draw a bar to represent each bin. Label the axes. Include a title for the graph.



Check

PHOTO SHARING The table shows the users of a photo sharing app by age group. Make a graph that best displays the data.

Age	18–24	–34	–44	45–54	55–64	65+
Users (%)	45	26	13	10	6	1



Go Online You can complete an Extra Example online.

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Interactive Presentation

Check

CHECK

Students complete the Check online to determine whether they are ready to move on.

Essential Question Follow-Up

Students have been creating dot plots, bar graphs, and histograms for real-world data sets.

Ask:

Why is it useful to know how to create and interpret different types of data displays? Sample answer: Not all data can be displayed on the same type of graph. Because the type of display chosen is dependent on the type of data, it is important to know about the different types of data displays.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–5
2	exercises that use a variety of skills from this lesson	6–10
2	exercises that extend concepts learned in this lesson to new contexts	11–12
3	exercises that emphasize higher-order and critical-thinking skills	13–17

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–11 odd, 13–17
- Extension: Segmented Bar Charts
- ALEKS Graphical Displays

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–11 odd
- Remediation, Review Resources: Find the Mode
- Personal Tutors
- Extra Examples 1–4
- ALEKS Finding Mean, Median, and Mode

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–5 odd
- Remediation, Review Resources: Find the Mode
- *Quick Review Math Handbook*: Representing Data
- Arrive**MATH** Take Another Look
- ALEKS Finding Mean, Median, and Mode

Answers

1. **Summer Reading Program**

Number of Books Read

Practice

Go Online Y ou can complete your homework online.

Examples 1 and 2

1. READING The table shows the number of books read by students in a summer reading program. Make a dot plot of the data. **See margin.**

Number of Books Read				
3	8	5	6	5
4	5	5	4	5
5	6	8	8	4

2. QUIZ SCORES Represent the quiz scores as a dot plot. Scale the number line as needed.

   50, 45, 24, 28, 27 , 38, 21, 22, 23, 42, 41, 35, 37 , 25, 43 **See margin.**

Examples 3 and 4

3. SURVEY A survey was conducted among students in Mr. Dalton's science class to determine a field trip destination. The results are shown in the table at the right. Make a graph to display the data. **See margin.**

Destination	Number of Votes
zoo	6
museum	4
observatory	11
state park	7

4. MOVIES In a survey, students were asked to name their favorite type of movie. Of those surveyed, 8 chose action movies, 6 chose comedies, 5 chose horror movies, 3 chose dramas, and 7 chose science fiction movies (sci-fi). Determine whether the data are discrete or continuous. Then make a graph. **The data are discrete and categorical. See margin for graph.**

5. CONCERT The table shows the number of attendees by age at a concert. Determine whether the data should be shown in a *bar graph* or *histogram*. Then make an appropriate graph for the data. **See margin.**

Concert Attendees	
0–10	400
10–20	1440
20–30	2400
30–40	2000
40–50	960
50–60	560
60–70	240

Mixed Exercises

6. PRIZES The table shows the number of prizes won by customers at a carnival game each of the past several days. Determine whether the data are discrete or continuous. Then make an appropriate graph for the data. **See margin.**

Prizes Won				
37	29	53	32	42
21	41	45	17	27
44	34	24	34	31
19	51	48	35	54
46	38	39	49	25

7. JOGGING The number of miles Lisa jogged each of the last 10 days are 3, 4, 6, 2, 5, 8, 7 , 6, 4, and 5.
   a. Choose the most appropriate type of data display and graph the data. **See margin.**
   b. How many days did Lisa jog at least 4 miles? **8**
   c. What was the greatest number of miles she jogged in a day? **8**

8. MOVIES The number of movies that are released theatrically each year are shown in the table.
   a. Select an appropriate display for the data. Explain your reasoning. **Bar graph; the data are discrete.**

Year	2010	2011	2012	2013	2014	2015	2016
Number of Movies Released	563	609	678	661	709	708	718

Source: comScore, MPAA

   b. Make a graph of the data. **See Mod. 9 Answer Appendix.**

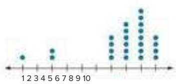
9. RUNNING The ages of the participants in a 10K race at Masonville are 65, 47 , 23, 70, 41, 55, 32, 29, 56, 39, 12, 57 , 25, 33, 15, 18, 35, 22, 63, 49, 23, 30, 37 , 40, and 50.
   a. Construct an appropriate data display for the data. See Mod. 9 Answer Appendix.
   b. How many participants are less than 30 years old? 8
   c. In what interval is the most frequent age? 30–39

10. ORCHESTRA The ages of the members of an orchestra are 39, 43, 31, 53, 41, 25, 35, 46, 27, 34, 37, 26, 51, 29, 36, 40, 33, 28, 48, 26, 42, and 38 years. Make a graph of the data. See Mod. 9 Answer Appendix.

11. PRECISION A scientific research study tracks the growth of an insect in millimeters. The growth data for each insect in the study during week 1 are 1.1, 1.25, 1.3, 1.67, 1.9, 2.35, 2.1, 2.3, 1.5, 1.7, 2.25, 2.1, 2.45, 1.37, 1.83. The scientist is preparing a histogram to show the distribution of growth across the population. How should the scientist break down his data into categories? Sample answer: The scientist should break down the data into increments of two-tenths starting at 1 and going through 2.6.

12. PETS The pets owned by Liza's classmates are rabbit: 2, dog: 6, cat: 3, horse: 2, bird: 5, mouse: 1, fish: 3, and other: 1.
   a. Make a dot plot of the data. See Mod. 9 Answer Appendix.
   b. How many types of pets are represented by the dot plot? 8
   c. Which pet is the most popular? Dog

13. ANALYZE Make two conclusions about a product that received the ratings shown in the dot plot. Justify your conclusions. See Mod. 9 Answer Appendix.

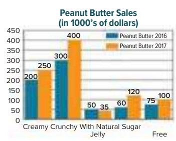


14. REGULARITY Explain when a histogram is the best model for data, and describe the process of creating a histogram. See Mod. 9 Answer Appendix.

Higher-Order Thinking Skills

15. WRITE Explain why it may be necessary to scale the number line of a dot plot. See Mod. 9 Answer Appendix.

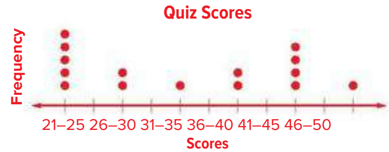
16. PERSEVERE Using the data provided in the double bar graph about peanut butter, what are two conclusions the grocery store could infer? See Mod. 9 Answer Appendix.



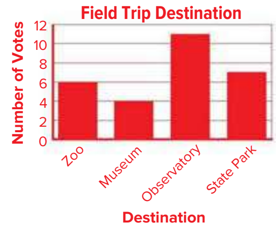
17. STRUCTURE How is a bar graph similar to a histogram? How is it different? See Mod. 9 Answer Appendix.

Answers

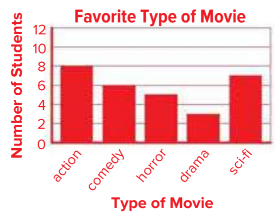
2.



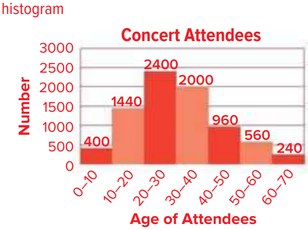
3.



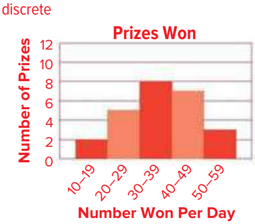
4.



5. histogram



6. discrete



7a.

Lesson 9-3
Using Data

LESSON GOAL

Students analyze data collection and representation methods to determine bias or identify misleading information.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Phrasing Questions

Develop:

Collecting Data
- Sample Bias
- Question Bias

Using Statistics and Representations
- Data Summaries
- Data Representation

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	BL	OL	ELL
Remediation: Statistical Questions	●	●		●
Extension: Sampling Methods		●	●	●

Language Development Handbook

Assign page 51 of the *Language Development Handbook* to help your students build mathematical language related to analyzing data collection and representation methods to determine bias or misleading information.

ELL You can use the tips and suggestions on page T51 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Statistics and Probability

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students analyzed and represented data using dot plots, histograms, and box plots.
6.SP.4, S.ID.1

Now
Students analyze data collection and representation methods to determine bias or identify misleading information.

Next
Students will use statistics appropriate to the shape of the data distribution to compare centers and spread of two or more data sets.
S.ID.2, S.ID.3

Rigor

The Three Pillars of Rigor

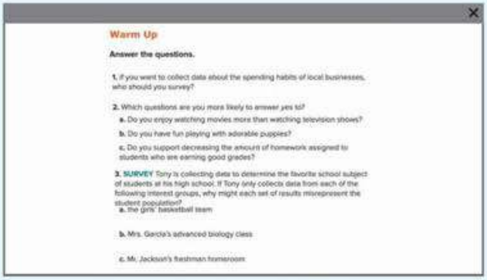
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students begin to develop an understanding of collecting data to be used in data distributions. They apply their understanding to solving problems involving sampling and bias.

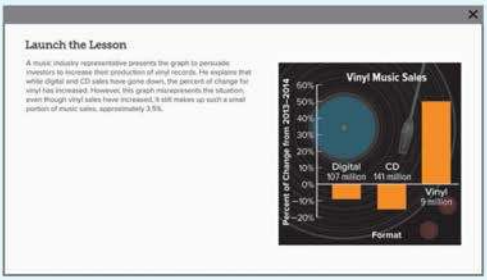
Mathematical Background

A *sample* is a portion of a group, and the population is the group from which the sample is taken. After selecting a sample, you can conduct a survey, an observational study, or an experiment to estimate the characteristics of the population and make predictions. It is important to analyze collection and representation methods to check for bias or misleading information.

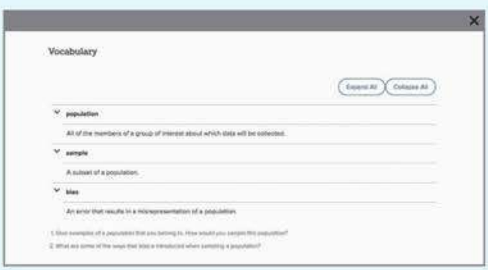
Interactive Presentation



Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- surveying

Answers:

1. entrepreneurs who own businesses in your community
2. b and c

3a. The data were collected from only females.

3b. The data were collected from students who excel in science.

3c. The data were collected from only freshman students.

Launch the Lesson

Teaching the Mathematical Practices

6 Use Quantities Encourage students to think about the quantities indicated by the graph and what information the graph does and does not provide about music sales. Have them discuss how the graph may be misleading.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Phrasing Questions

Objective

Students explore how the phrasing of questions can lead to bias.

MP Teaching the Mathematical Practices

4 Make Assumptions Have students explain an assumption or approximation that was made to solve the problem.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

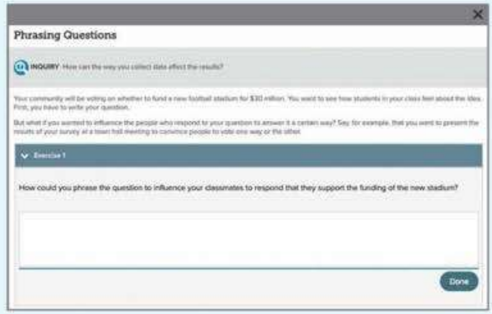
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will explore the different ways that the wording of a survey question may influence responses. They will use different wording to create their own survey questions, use their questions to collect data, and then answer a series of questions about their observations. Then, students will answer the Inquiry question.

(*continued on the next page*)

Interactive Presentation



Explore

TAP

Students move through the exercises to explore how phrasing questions can influence a response.

TYPE

Students answer questions to show they understand how to properly phrase questions.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry question and can view a sample answer.

Explore Phrasing Questions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How did you write your first question to garner support for the new stadium? Sample answer: I used words that would make people think that if they voted "yes," it would be beneficial for them.
- Did the people you surveyed respond the way you anticipated they would? See students' responses.

Inquiry

How can the way you collect data affect the results?
Sample answer: Using positive or negative language in a survey question can influence the way people respond.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Collecting Data

Objective
Students identify potential bias in sampling methods and questions.

Teaching the Mathematical Practices
1 Seek Information Help students to see how to collect data without bias in this Learn.

Important to Know
The importance of avoiding bias is key to obtaining meaningful data that can be used to accurately estimate a characteristic of a population. It is important to avoid bias in the way the sample is selected as well as in the way survey questions are worded.

Example 1 Sample Bias

Teaching the Mathematical Practices
4 Apply Mathematics In this example, students apply what they have learned about sample bias to solving a real-world problem.

Questions for Mathematical Discourse
AL Which population is included in the sample? American households with a landline

OL Why is the potential bias that is identified a concern? Sample answer: The data collected is incomplete and is not a good representation of all voters.

BL What question did the pollsters' data actually answer? Sample answer: How do people with landline phones plan to vote?

Common Error
Students are often confused when it comes to determining whether or not a study is biased. Encourage them to consider such factors as how the survey participants were chosen, how the questions were asked, and how many participants were included in the study.

Go Online
- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 9-3

Using Data

Explore Phrasing Questions

Online Activity Use a real-world situation to complete the Explore.

INQUIRY How can the way you collect data affect the results?

Learn Collecting Data

A **population** consists of all the members of a group of interest about which data will be collected. Since it may be impractical to examine every member of a population, a subset of the group, called a **sample**, is sometimes selected to represent the population. The sample can then be analyzed to draw conclusions about the entire population.

Sample data are often used to estimate a characteristic of a population. Therefore, a sample should be selected so that it closely represents the entire population. Also, the larger the sample size, or the more samples taken, the better it represents the population.

A **bias** is an error that results in a misrepresentation of a population. If a sample favors one conclusion over another, the sample is biased and the data are invalid.

Example 1 Sample Bias

POLLS **Before the 2010 elections for members of the U.S. House of Representatives, pollsters called American households on their landline phones to see how they planned to vote. What kind of sample bias might have affected the poll?**

Step 1 Identify the intended population.

The population is all likely voters.

Step 2 Identify the sample method.

The data for this poll were collected over landline phones, so the sample consists of likely voters who have a landline.

Step 3 Determine potential bias.

Because not all likely voters have landline phones, the results could be skewed because not all likely voters are available for this sample.

Go Online You can complete an Extra Example online.

Today's Goals
- Identify potential bias in sampling methods and questions.
- Identify potential bias in statistics and representations of data.

Today's Vocabulary
population
sample
bias
statistic

Talk About It!
Some polls use both landlines and cell phones. How might this alleviate the issue of bias with the landline-only sampling method?

Sample answer: A larger percentage of the population has a landline or a cell phone than has just a landline, so more voters can be reached. However, not everyone has one of those modes of communication, so it does not eliminate the issue of bias.

Lesson 9-3 • Using Data **503**

Interactive Presentation

Learn

SWIPE

Students drag the slider to see a sample of a population.

Check

SOCIAL MEDIA Shia wants to determine the age of the average internet user. He posts a poll to his friends on a social media site, asking their age. Is this a good sample? If not, what kind of sample bias might have affected the poll? D

A. This is a good sample.

B. This is not a good sample. He asked only people on the Internet.

C. This is not a good sample. He asked only about users' ages.

D. This is not a good sample. He asked only his friends on a specific Web site.

Think About It!
Rewrite the question to try to remove the bias.

Sample answer: "Do you support a ban on soft drinks?"

Example 2 Question Bias

SOFT DRINKS **A survey organization wants to see what percent of New York City citizens support a ban on soft drinks. The question posed is, "Do you support a ban on soft drinks, which contribute to heart disease and tooth decay?"**

Part A Identify bias.

Step 1 Identify the purpose of the question: To find the percent of New York citizens who support a ban on soft drinks.

Step 2 Identify potential bias in the question: The question lists some of the health risks of soft drinks. This might make respondents more likely to respond that they do support a ban.

Part B Identify interests.

The bias in the question might make respondents more likely to support a ban. This bias could serve the interests of health groups who want to ban soft drinks or companies who sell competing drinks, like juices.

Study Tip
Assumptions When you identify and try to remove bias, you will make assumptions about what information is important for the question and what might create undue bias. For example, you need to consider whether details of the plan are relevant and whether the question includes persuasive language.

Check

FILM One of your friends wants to determine whether people in your class prefer to watch movies or television. She asks, "Do you prefer to watch movies or television?"

Part A Does this question potentially bias the results? A

A. No; the question is as neutral as possible.

B. Yes; your friend asked only people in your class.

C. Yes; your friend provided only two options.

D. Yes; the framing of the question influences the respondent to choose *television*.

Part B Whose interests might be served by asking the question in this way? No one; the question is as neutral as possible.

Go Online You can complete an Extra Example online.

504 Module 9 • Statistics

Interactive Presentation

Example 2

TAP

Students move through the steps to see an example of question bias.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Question Bias

Teaching the Mathematical Practices

4 Make Assumptions In the Study Tip, have students point out where an assumption or approximation was made in the solution.

Questions for Mathematical Discourse

AL What is the survey question trying to determine? whether people in New York City support a ban on soft drinks

OL What key words in the question might encourage a respondent to answer one way over another? *heart disease* and *tooth decay*

BL Give an example of another biased survey question about the soda ban. Sample answer: Do you support a ban on soda because soda is a primary cause of childhood obesity?

DIFFERENTIATE

Enrichment Activity BL

Have students write two questions that they will use to survey their fellow students about the same issue. Have them write one question that they feel is biased and one that they feel is unbiased. Have students choose two different samples from the same population and administer their surveys. Then have them compare the results of their surveys with their expected results and share their observations with the class.

Essential Question Follow-Up

Students have been exploring bias in sampling and questioning techniques.

Ask:

How are statistics used in the real world to sway opinions? Sample answer: Studies may use biased samples or biased questions to make the resulting data appear more or less favorable.

Learn Using Statistics and Representations

Objective

Students identify potential bias in statistics and representations of data.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of statistics of a data set in this Learn.

What Students Are Learning

Students are learning that statistics and representations may be used to misrepresent data, and that this is often intentional.

Common Misconception

The information in this part of the lesson may come as a surprise to students whose conception of mathematics and statistics is that "the numbers don't lie." This quote is used often enough to make students feel that statistics are factual. Use the examples in this part of the lesson to help students see that statistics can be, and often are, manipulated to misrepresent what is true.

Example 3 Data Summaries

Questions for Mathematical Discourse

AL Which measure of center makes it appear that the students did better? mode Worse? mean

OL What is affecting the mean and keeping it from being an accurate measure of the data? the two 0 scores

BL How does removing the two 0 scores affect the mean? Sample answer: The mean will be closer to the median and will be a better indicator of the performance of the students who took the exam.

Common Error

If students recalculate the mean of the data to not include the outliers (in order to make a comparison), make sure that they remember to reduce the number of data values by which they divide.

Learn Using Statistics and Representations

A **statistic** is a measure that describes a characteristic of a sample. Like data, statistics and representations of data are nonneutral. When the average of a set of data is discussed, it uses a measure of center: mean, median, or mode. However, depending on the data set and what information is being conveyed, one measure of center might not give the whole picture of the data. Even if the data is being discussed in whole, it can also be misrepresented. For example, a person might manipulate the scales of the axes of a graph or how the data are represented graphically to misrepresent the data.

Example 3 Data Summaries

TEACHING **A teacher wants to tell his students how the average student did on an exam, so he looks at the scores in his gradebook. Two students scored a 0 because they stopped showing up for class in the last month and did not take the exam. He uses the mean, 71, as the measure of center. Does the mean accurately represent these data?**

0, 0, 82, 83, 85, 87 , 88, 88, 91, 91, 91

Step 1 Identify the other measures of center. Round your answer to the nearest unit.

median = 87

mode = 91

Step 2 Analyze the measures of center and how they align with the information the teacher wants to convey.

Mean: The mean, 71, is affected by the two 0 scores. However, no one who showed up for the exam scored below an 82, so the mean does not do a good job of indicating the performance of the students who took the exam.

Median: The median, 87 , is not affected by the extreme values. It provides a more accurate average for how students performed on the test because it includes the scores of the two students who did not take the exam at all.

Mode: The mode, 91, is both the score most students received and the highest score received on the exam, but it does not accurately portray how students performed on average.

Because the teacher wants to discuss the performance of students who took the exam, the mean is not the best measure of center. It indicates that all students who took the exam performed worse than their actual scores.

Go Online Y ou can complete an Extra Example online.

Math History Minute

With M. A. Girschick, David Blackwell (1919–2010) authored the classic book *Theory of Games and Statistical Decisions*. In 1965, he became the first African American president of the American Statistical Society.

Lesson 9-3 • Using Data 505

Interactive Presentation

Example 3

TYPE

Students complete the calculations for the median and mode of the data.

TAP

Students tap to see a data summary.

Check

READING Karen writes down the number of books she has read for each of her classes so far: 5, 6, 4, 5, 5, 6, 5, 4, 5. Using the mean, 5, she says that she reads around 5 books on average for an English course. Does the mean accurately represent the data for the situation? Explain. D

A. No; the mean is overly influenced by the low numbers.

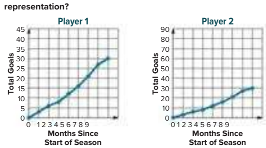
B. No; the mean is overly influenced by the high numbers.

C. No; the mean doesn't tell how many pages are in the average book.

D. Y es; the mean accurately represents the data for the situation.

Example 4 Data Representation

SOCCER **A group compares two soccer players who play the same position for different teams. They make a graph of the number of goals scored throughout the season for each player. Do the graphs misrepresent the data? Whose interests might be served by the representation?**



Part A Identify misleading representations.

Step 1 Identify the purpose of the graphs. The purpose of the graphs is to compare the number of goals each soccer player scored.

Step 2 Identify differences in the graphs. The graphs appear to be the same in terms of the data being represented, but the *y*-axis for the second player goes up to 90 in increments of 10, whereas the first goes up to 45 in increments of 5.

Step 3 Identify how this affects the representation of the data. Although the numbers are the same, the scale for Player 1 makes it look like he is scoring more goals than Player 2.

Part B Identify interests.
The misleading representation of the data makes it appear that Player 1 scores more goals than Player 2. This bias could serve the interests of the team, sponsors of Player 1's team, or Player 1's agent.

Go Online Y ou can complete an Extra Example online.

Think About It!
If the two graphs had the same scales but were comparing a goalkeeper and a forward, how might the data be misleading when comparing the skill of the two players?

Sample answer: Because goalkeepers are not primarily goal scorers, it would be an unfair basis for comparison.

Use a Source
Find a graph or set of statistics online. Ask yourself, are the data accurately represented? Whose interests are served by the graph or statistics?

Answers will vary.

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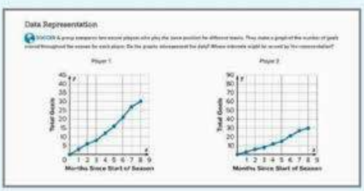
Interactive Presentation



Example 4

TAP

Students tap to identify misleading representations.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 4 Data Representation

Teaching the Mathematical Practices

5 Use a Source Guide students to find external information to answer the questions posed in the Use a Source feature.

Questions for Mathematical Discourse

AL What appears to be the maximum of each graph? about 30

OL How are the graphs different? Sample answer: The scales on the vertical axes are different.

BL Which scale is more appropriate for the data? Explain your reasoning. Sample answer: The scale used for Player 1 is more appropriate because neither player appears to have scored more than 30 goals during the season. There is no reason to have the scale go up to 90 goals.

Common Error

Students often read a graph without looking at the scales on the axes. Remind them that the scales make a difference in how the graph looks.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–8
2	exercises that use a variety of skills from this lesson	9–14
3	exercises that emphasize higher-order and critical-thinking skills	15–22

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–13 odd, 15–22
- Extension: Sampling Methods

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–13 odd
- Remediation, Review Resources: Statistical Questions
- Personal Tutors
- Extra Examples 1–4
- ALEKS Analyzing Survey Questions

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–7 odd
- Remediation, Review Resources: Statistical Questions
- Arrive**MATH** Take Another Look
- ALEKS Analyzing Survey Questions

Answers

1. Sample answer: The intended population is all students. By asking only students leaving basketball practice, Awan is not getting a representative example of the entire student body.

3. Sample answer: The first sentence states a positive outcome of music education, which may bias the respondent toward support. This bias may serve people trying to keep music education in schools.

6. Sample answer: If there is an outlier, the median is the better measure to use because the mean is affected by the outlier and pulled in its direction. Therefore, the median is closer to the true center.

7. Sample answer: The scale for Vendor 1 starts at 40, and because of the size of the bars, it looks like their sales doubled in one year, when they increased about 50%. Vendor 2 had the same sales figures as Vendor 1 but it appears that they had more sales than Vendor 1.

Practice

Example 1

1. **SPORTS** Awan wants to know what the favorite sport is among students. To find out, he asks everyone he sees leaving school after basketball practice. Identify the intended population and determine the potential sample bias. See margin.

2. **STORES** Raya wants to conduct a survey at a nearby mall to determine which are the mall's most popular stores. How could she choose a sample that is unbiased? Sample answer: Raya could survey people from various locations in the mall.

Example 2

3. **MUSIC** Shea is shopping online, and a survey question pops up that says, "Music education enriches student learning. Do you support music education in schools?" See margin.
   a. Identify potential bias in the question.
   b. Identify whose interests may be served by the question.

4. **CANDIDATES** There are three candidates for mayor. To investigate how the townspeople feel about the candidates, a newspaper posts a poll that lists the three candidates and asks which candidate people support. The poll appears on the same page as an opinion piece in support of one of the candidates.
   a. Identify potential bias in the question.
   b. Identify whose interests may be served by the question. Sample answer: The opinion piece creates bias because of its location on the same page as the poll. This could serve the interests of the candidate in the opinion piece.

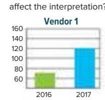
Example 3

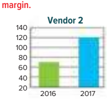
5. **BUTTERFLIES** Tania recorded the number of butterflies she saw on her daily runs each day for a week. The numbers are: 1, 8, 2, 2, 5, 6, and 4. Find the mean, median, and mode of the data. Which measure(s) are appropriate to accurately summarize the data? Mean: 4, median: 4, mode: 2; The mean and median are appropriate measures to use to accurately summarize the data.

6. **OUTLIERS** In a data set with an outlier, which measure of center, mean or median, is the better measure to use to describe the center of the data? Explain your reasoning. See margin.

Example 4

7. **SALES** The graphs show the number of T-shirts sold at a baseball tournament for two years by two different vendors. The tournament director wants to compare the vendors. Do the graphs misrepresent the data? How does that difference affect the interpretation? See margin.





8. **SCALE** If the same set of data is graphed with a scale of 0 to 10 on the y-axis and then with a scale of 0–100 on the y-axis, what effect does that have on the representation of the data? See margin.

Lesson 9-3 • Using Data 507

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

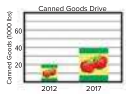
Mixed Exercises

9. SCIENCE A school wants to know which area of science; physics, biology, or chemistry, is most interesting to its students. Would it be better to survey students in a class that is an elective or required to get a sample with the least bias? Explain your reasoning. **See margin.**

10. TAX Before surveying people about whether they favor or oppose a proposed tax, the surveyors want to present information about the tax. Suppose the surveyors give facts about the tax without giving opinions. How could the facts given by the surveyors introduce bias? **Sample answer: Surveyors could pick which facts to share, which could introduce bias.**

11. CONSTRUCT ARGUMENTS The weights, in pounds, of several dolphins at a sea animal care facility are 185, 222, 755, 801, 835, 990, and 1104. Which measure of center best represents the data? Justify your conclusion. **See margin.**

12. FOOD DRIVE The chart shows the number of canned goods collected by Valley High School in 2012 and 2017. Is the graph misleading? Explain. **See margin.**



13. REASONING The number of participants at reading club for six weeks are 11, 12, 10, 13, 10, and 10. Without calculating the measures of center, how would adding an outlier of 24 participants affect which measure of center most appropriately represents the data? **See margin.**

14. PRECISION A community garden has 8 tomato plants with heights ranging from 0.4 to 0.9 meters. Regina found the median to be 0.7 meters, which she rounded and reported as 1 meter. Is Regina's report of the median accurate? Explain your reasoning. **No; 1 m is greater than the maximum height.**

Higher-Order Thinking Skills

15. REGULARITY Describe a general method for assessing a sample for bias. **See margin.**

16. STRUCTURE How are the median and mean scores affected if all data values in a set are increased by a specific value, such as 10? **See margin.**

17. CREATE Create two sets of data and display them in a graph or chart that shows bias toward one of the sets of data. **See students' work.**

18. WRITE Write two scenarios that have different examples of sample bias. Have a classmate rewrite your statements without bias. **See students' work.**

19. CREATE Think of a topic about which you can survey the teachers at your school. Conduct the survey. Explain whether your survey question(s) introduce bias. **See students' work.**

20. ANALYZE Is a biased sample *sometimes*, *always*, or *never* valid? Justify your argument. **See margin.**

21. PERSEVERE If the mean, median, and mode of a data set are equal, the data set is symmetric. If a data set has a mean that is less than its median, what does that tell you about the data set? **Sample answer: There are more extreme values in the lower end, which causes the mean to be lower than the median.**

22. FIND THE ERROR Two students collected data on the sizes of box turtle shells. Olivia measured 8 turtles from a pond near her school. Caleb measured 2 turtles from each of 4 ponds around town. Which is more likely to be free of sample bias? Explain your reasoning. **See margin.**

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Answers

8. Sample answer: Graphing the same data using different scales changes the appearance of the data. Using a greater scale, 0-100, makes the data look flatter, indicating a weaker relationship; using a smaller scale, 0-10, makes the data look steeper, indicating a stronger relationship.
9. Sample answer: The required class would be better because it is more likely to contain a representative sample of students. The elective class might not be representative of the whole student body because these courses are chosen for reasons such as personal preference or future career aspirations.
11. Median; sample answer: The two lowest weights are much lower than the others, so the mean will be affected by those outliers.
12. Yes; sample answer: The can for 2017 is 2 times as wide and 2 times as high as 2012, which implies that the school raised 4 times more canned goods, when they raised double the canned goods.
13. Sample answer: The original data are very close together, so it is likely that the measures of center will all be the same or very close. Adding an outlier of 24 to the data set will cause the mean to go up, but the median and mode would likely stay unchanged or very close to the original number. So, in this case the median or mode would best represent the center of data.
15. Sample answer: To assess a sample for bias, identify the intended population and sample method; then, based on this information, assess whether there is potential sample bias.
16. Sample answer: The mean and median are affected the same as the data values, so if data are increased by 10, the measures increase by 10.
20. Sample answer: The biased sample can sometimes be true because there exists a small probability that the selected sample from which the results are obtained represent the characteristics of the group.
22. Sample answer: Caleb's data is more likely to be free of bias because his sample is drawn from multiple sites, so the turtles should be more representative of the population.

Measures of Spread

N.Q.1, S.ID.1

LESSON GOAL

Students represent sets of data using measures of spread.

1 LAUNCH

Launch the lesson with a Warm Up and an introduction.

2 EXPLORE AND DEVELOP

Explore: Using Measures of Spread to Describe Data

Develop:

Range and Interquartile Range
- Range
- Make a Box Plot
- Interquartile Range

Standard Deviation
- Calculate Standard Deviation

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	BL		ELL
Remediation: Compare Populations	●	●		●
Extension: Chebyschev's Theorem		●	●	●

Language Development Handbook

Assign page 52 of the *Language Development Handbook* to help your students build mathematical language related to representing sets of data using measures of spread.

ELL You can use the tips and suggestions on page T52 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Statistics and Probability

Standards for Mathematical Content:

N.Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

S.ID.1 Represent data with plots on the real number line (dot plots, histograms, and box plots).

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

4 Model with mathematics.

Coherence

Vertical Alignment

Previous
Students analyzed and represented data using dot plots, histograms, and box plots.
7.SP.1, S.ID.1

Now
Students represent sets of data using measures of spread.
N.Q.1, S.ID.1

Next
Students will analyze the shapes of distributions to determine appropriate statistics and identify extreme data points. **S.ID.3**

Rigor

The Three Pillars of Rigor

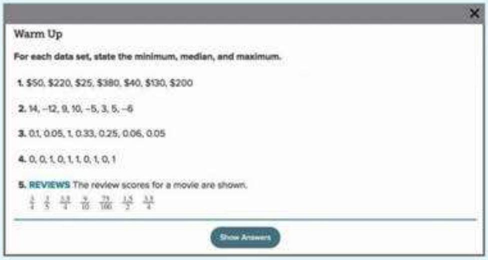
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students apply their understanding of data distributions by solving real-world problems. They build fluency by making box plots and finding variance and standard deviation.

Mathematical Background

Measures of variation describe the spread of the data in a data set. The range describes the overall spread and is the difference between the greatest and least data values. Quartiles and interquartile range provide information about how the data is distributed. The variance and standard deviation describe the spread around the mean. Two data sets can have the same range and mean, but the spread around the mean can be quite different.

Interactive Presentation



Warm Up

Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding the minimum, median, and maximum data values in a set of data

Answers:

1. \$25, \$130, \$380
2. −12, 4, 14
3. 0.05, 0.1, 1
4. 0, 0.5, 1
5. $\frac{3}{5}, \frac{3}{4}, \frac{75}{100}$, or $\frac{1.5}{2}, \frac{9}{10}$

Launch the Lesson

Teaching the Mathematical Practices

2 Attend to Quantities Encourage students to consider why the statistic Mr. Frond announced misled the student and why she says that she would have preferred knowing the mean.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Using Measures of Spread to Describe Data

Objective

Students use a sketch to explore how standard deviation can be used to describe data sets.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

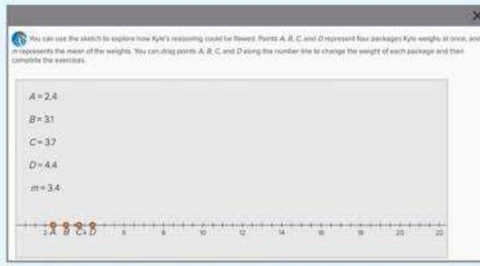
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use a sketch to enable them to create two different sets of real-world data that have the same mean. They will then answer questions about the data and follow a series of steps to calculate the standard deviation for each set of data. Finally, students will analyze and compare the standard deviations. Then, students will answer the Inquiry question.

(continued on the next page)

Interactive Presentation

Explore



Explore

WEB SKETCHPAD

Students use a sketch to explore measures of spread.

TYPE

Students complete the calculations to determine measures of spread.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry question and can view a sample answer.

Explore Using Measures of Spread to Describe Data (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- In the context of this situation, which is more useful, the mean or the standard deviation? Explain. Sample answer: The standard deviation is more useful because the weights all need to be very close to the mean, not just produce the mean when calculated.
- In what type of situation is knowing the mean sufficient? Sample answer: a situation in which all the data are very close to the mean

Inquiry

Why might you describe a data set with more than the mean?
Sample answer: Data sets may have the same mean but be very different from each other. Other statistics can provide more information about the spread of the data.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Range and Interquartile Range

Objective

Students determine measures of spread, including the range and interquartile range, of a set of data.

MP Teaching the Mathematical Practices

2 Create Representations Students learn how a box plot can be used to represent the five-number summary of a data set.

Important to Know

Two data sets may have the same mean, but the spread of the data in one of the data sets may be very different from that of the other data set. A box plot is a very useful tool for analyzing spread, and double box plots are often used to compare the spreads of two related data sets.

Common Misconception

A common misconception that some students may have is that Q_2 is the mean of the data. Correct this thinking and help students understand that because the median divides the data into two equal-sized groups, each containing 50% of the data, it represents Q_2.

Example 1 Range

Questions for Mathematical Discourse

AL What is the highest score? 99 the lowest score? 62

OL What does the range represent? Sample answer: The difference between the greatest data value and the least data value. It represents the overall spread of the data.

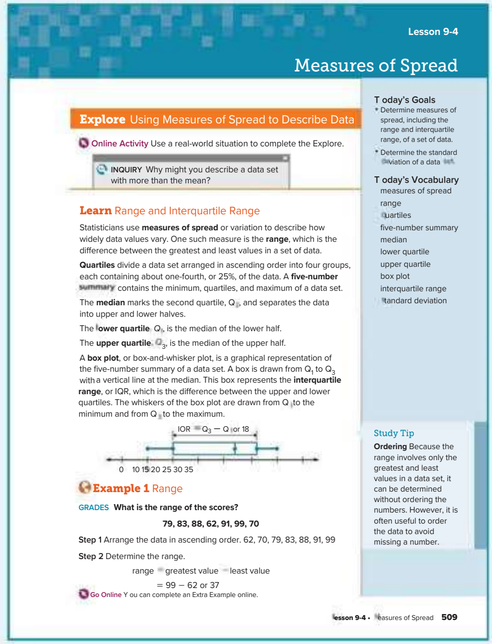
BL Create a set of data with 9 values, a mode of 6, and a range of 5. Sample answer: {4, 3, 6, 5, 4, 6, 6, 7, 8}

Common Error

If students get the wrong answer, it may be because they incorrectly identify the greatest data value and/or the least data value. Encourage students to either arrange the data in order so that they don't miss a number or to circle the numbers in the data list that they plan to use for their calculation and then double-check that those numbers truly are the maximum and minimum values.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.


Lesson 9-4

Measures of Spread

Explore Using Measures of Spread to Describe Data

Online Activity Use a real-world situation to complete the Explore.

INQUIRY Why might you describe a data set with more than the mean?

Learn Range and Interquartile Range

Statisticians use **measures of spread** or variation to describe how widely data values vary. One such measure is the **range**, which is the difference between the greatest and least values in a set of data.

Quartiles divide a data set arranged in ascending order into four groups, each containing about one-fourth, or 25%, of the data. A **five-number summary** contains the minimum, quartiles, and maximum of a data set.

The **median** marks the second quartile, Q_2, and separates the data into upper and lower halves.

The **lower quartile**, Q_1, is the median of the lower half.

The **upper quartile**, Q_3, is the median of the upper half.

A **box plot**, or box-and-whisker plot, is a graphical representation of the five-number summary of a data set. A box is drawn from Q_1 to Q_3 with a vertical line at the median. This box represents the **interquartile range**, or IQR, which is the difference between the upper and lower quartiles. The whiskers of the box plot are drawn from Q_1 to the minimum and from Q_3 to the maximum.

Example 1 Range

GRADES **What is the range of the scores?**

79, 83, 88, 62, 91, 99, 70

Step 1 Arrange the data in ascending order. 62, 70, 79, 83, 88, 91, 99

Step 2 Determine the range.

range = greatest value − least value

= 99 − 62 or 37

Go Online You can complete an Extra Example online.

Today's Goals
- Determine measures of spread, including the range and interquartile range, of a set of data.
- Determine the standard deviation of a data set.

Today's Vocabulary
measures of spread
range
quartiles
five-number summary
median
lower quartile
upper quartile
box plot
interquartile range
standard deviation

Study Tip

Ordering Because the range involves only the greatest and least values in a data set, it can be determined without ordering the numbers. However, it is often useful to order the data to avoid missing a number.

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Interactive Presentation


Example 1

TAP

Students tap to explore range and interquartile range.

N.Q.1, S.ID.1

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 2 Make a Box Plot

Think About It!
If Q_1 were located between two numbers, how would you determine its value?

Sample answer: To determine Q_1, find the mean of the two numbers on either side.

BOX OFFICE **A financial analyst for a movie studio wants to determine how much most of the top-earning movies have grossed to compare his studio's recent grosses. The worldwide grosses, in millions of dollars, for the top 10 highest-grossing films of all time are given. Determine the five-number summary and draw a box plot of the data to see the spread of the data.**

2788	2187	2060	1670	1520
1516	1405	1342	1277	1215

Part A Determine the five-number summary.

Step 1 Arrange the data in ascending order.

1215, 1277 , 1342, 1405, 1516, 1520, 1670, 2060, 2187 , 2788

Step 2 Determine the five-number summary of the data.

1215, 1277, 1342, 1405, 1516, 1520, 1670, 2060, 2187, 2788

min $Q_1 = 1342$ $Q_2 = \frac{1516 + 1520}{2}$ or 1518 $Q_2 = 2060$ max

Part B Construct a box plot.

Step 1 Construct a number line.

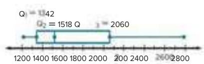
Because the minimum is 1215 and the maximum is 2788, your number line must include those values.

Step 2 Draw the box.

Draw and label a box from Q_1 to Q_3, with a vertical line at the median.

Step 3 Draw the whiskers.

Draw a line from the minimum to Q_1. Draw a line from Q_3 to the maximum.



Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 2

EXPAND

Students tap to see how to make a box plot.

TYPE

Students answer a question to show they understand how to construct a box plot.

Example 2 Make a Box Plot

Teaching the Mathematical Practices

1 Explain Correspondences Guide students as they use the information in this example to plot data to represent the situation.

Questions for Mathematical Discourse

AL What statistics make up the five-number summary? minimum, maximum, median, lower quartile, upper quartile

OL When drawing a box plot, how do you know where the box starts and ends? Sample answer: Calculate the first quartile and the third quartile. Then draw the box so that it starts at the first quartile and ends at the third quartile.

BL Why does it not make sense for the number line to start at 0? Sample answer: If the number line started at 0, and included the maximum value, the graph would end up in a very small portion and would not be very clear or helpful.

Common Error

Some students may forget that the vertical line inside the box represents the median of the data and may, in error, draw the line in the middle of the box. Point out this common error so that students will avoid making it.

Example 3 Interquartile Range

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about interquartile range to solving a real-world problem.

Questions for Mathematical Discourse

AL What is a quartile? one of four equal groups into which data can be divided

OL What is the median? −15 the lower quartile? −18 the upper quartile? −13

BL What does the interquartile range represent? Sample answer: The difference between the upper quartile and the lower quartile, or the middle 50% of the data.

Common Error

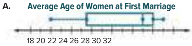
Some students may count the median as a data value in the lower half, and then again in the upper half of the data when calculating the first and third quartiles. Correct this error, and tell students to use the median as a dividing point between the two halves but not as a data point in either half.

Check

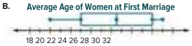
MARRIAGE The average age at which women first get married differs by country. The ages for eight countries are shown. Select the box plot for the data. B

25, 26, 21, 27 , 31, 31, 29, 20

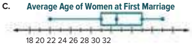
A. Average Age of Women at First Marriage



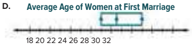
B. Average Age of Women at First Marriage



C. Average Age of Women at First Marriage



D. Average Age of Women at First Marriage



Example 3 Interquartile Range

AUDIO **Sarah wants to upload a song that she recorded and share it with her friends. She wants to know whether her song, which is currently −12 decibels on average, is the right volume compared to other songs online so listeners do not have to adjust their volume. She writes down the average volume of seven songs. Find the interquartile range of these average volumes.**

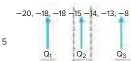
−18, −20, −8, −13, −14, −15, −18

Step 1 Order the data: −20, −18, −18, −15, −14, −13, −8

Step 2 Determine Q_1 and Q_3.

Step 3 Determine the *IQR*.

$IQR = Q_3 - Q_1 = -13 - (-18)$ or 5



Check

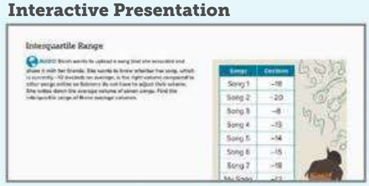
WRITING Cora is writing a novel and tracks the number of pages she writes each day for a week. The number of pages she wrote each day for a week is shown. Find the interquartile range of the data set. 5

6, 5, 0, 3, 8, 1, 4

Go Online Y ou can complete an Extra Example online.

Lesson 9-4 • Measures of Spread 511

Interactive Presentation



Example 3

CHECK

Students complete the Check online to determine whether they are ready to move on.

Think About It!
If all the data in a set were the same value, then what would the standard deviation be?

0

Learn Standard Deviation

In a data set, the **standard deviation** shows how the data deviate from the mean. A majority of the data in a set, approximately two-thirds, is contained within 1 standard deviation below and above the mean. So, if two data sets have the same mean, but one has a greater standard deviation, then the data in that set is more spread out from the mean.

Key Concept • Standard Deviation

Step 1 Find the mean $\bar{x}$.

Step 2 Find the square of the difference between each data value x_n and the mean, $(\bar{x} - x_n)^2$.

Step 3 Find the sum of all the values in Step 2.

Step 4 Divide the sum by the number of values in the set of data n. This value is the variance.

Step 5 Take the square root of the variance.

Formula $\sigma = \sqrt{\frac{(\bar{x} - x_1)^2 + (\bar{x} - x_2)^2 + \cdots + (\bar{x} - x_n)^2}{n}}$

Example 4 Calculate Standard Deviation

UNIVERSITY **The number of students accepted at the main campus of each university in the Big Ten East Division are shown. Find and interpret the standard deviation of the data set.**

27,300	**12,333**	**15,570**	**21,610**
17,413	**25,772**	**18,230**	

Step 1 Find the mean $\bar{x}$.

$$\bar{x} = \frac{27{,}300 + 12{,}333 + 15{,}570 + 21{,}610 + 17{,}413 + 25{,}772 + 18{,}230}{7} \text{ or } 19{,}747$$

Step 2 Find the square of the differences, $(\bar{x} - x_n)^2$.

$$(19{,}747 - 27{,}300)^2 = 57{,}047{,}809$$
$$(19{,}747 - 12{,}333)^2 = 54{,}967{,}396$$
$$(19{,}747 - 15{,}570)^2 = 17{,}447{,}329$$
$$(19{,}747 - 21{,}610)^2 = 3{,}470{,}769$$
$$(19{,}747 - 17{,}413)^2 = 5{,}447{,}556$$
$$(19{,}747 - 25{,}772)^2 = 36{,}300{,}625$$
$$(19{,}747 - 18{,}230)^2 = 2{,}301{,}289$$

Think About It!
If another data point less than 14,719 or greater than 24,775 is added to the data set, how would that change the standard deviation?

The standard deviation would increase.

Step 3 Find the sum.

$$57{,}047{,}809 + 54{,}967{,}396 + \ldots + 2{,}301{,}289 = 176{,}982{,}773$$

Step 4 Divide by the number of values. This value is the variance.

$$\frac{176{,}982{,}773}{7} \approx 25{,}283{,}253$$

Step 5 Find the standard deviation.

$$\sqrt{25{,}283{,}253} \approx 5028$$

Go Online
You can complete an Extra Example online.

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Interactive Presentation

Learn

TYPE

Students answer a question to show they understand how to find standard deviation.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Learn Standard Deviation

Objective

Students determine the standard deviation of a data set.

Teaching the Mathematical Practices

4 Use Tools Students follow a set of steps to learn how to use the formula for finding the standard deviation of a set of data.

About the Key Concept

Calculating the standard deviation involves finding the differences between each data value and the mean, finding the average of the squares of those differences, and taking the square root of the result. The resulting value is the standard deviation, which is a measure of how much the data deviate from the mean.

Example 4 Calculate Standard Deviation

Questions for Mathematical Discourse

AL Why do you first need to find the mean? Sample answer: Y ou need the mean to find the differences in the next step.

OL Explain how to find the sum of the squares of the differences. Subtract each data value from the mean, square each difference, and then add the results.

BL What is the interval that contains values that lie within one standard deviation of the mean in this example? Explain. 14,719 to 24,775; You find the values that are 5028 more and 5028 less than the mean of 19,747.

Common Error

Some students may think that they have made an error if the calculation of the standard deviation results in a number that is not close to the mean. Remind students that the standard deviation is not a measure of center but is instead a measure of spread, and that the value indicates how much the data deviate from the mean.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–20
2	exercises that use a variety of skills from this lesson	21–26
3	exercises that emphasize higher-order and critical-thinking skills	27–30

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–25 odd, 27–30
- Extension: Chebyschev's Theorem
- ALEKS Data Analysis

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–29 odd
- Remediation, Review Resources: Compare Populations
- Personal Tutors
- Extra Examples 1–4
- ALEKS Making Inferences About Population

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–19 odd
- Remediation, Review Resources: Compare Populations
- *Quick Review Math Handbook:* Statistics and Parameters
- Arrive**MATH** Take Another Look
- ALEKS Making Inferences About Population

Practice

Go Online Y ou can complete your homework online.

Example 1

Find the range of each data set.

1. 12, 27 , 43, 52, 43, 18, 45, 53, 26 41

2. 132, 127 , 129, 130, 141, 125, 138, 129 16

3. 56, 101, 78, 49, 55, 108, 111, 64 62

4. 5.9, 6.2, 3.9, 3.7 , 8.5, 6.2, 9.0, 8.7 , 4.5, 9.3 5.6

5. EXERCISE Kent tracked his daily number of minutes of exercise. Find the range of the data set. 20

Number of Minutes of Exercise					
30	35	25	28	40	38
36	29	34	45	42	39

Example 2

Determine the five-number summary and draw a box plot of the data. 6–10. See margin.

6. prices in dollars of smartphones: 311, 309, 312, 314, 399, 312

7. attendance at an event for the last nine years: 68, 99, 73, 65, 67 , 62, 80, 81, 83

8. books a student checks out of the library: 17 , 9, 10, 17 , 18, 5, 2

9. ounces of soda dispensed into 36-ounce cups: 36.1, 35.8, 35.2, 36.5, 36.0, 36.2, 35.7 , 35.8, 35.9, 36.4, 35.6

10. ages of riders on a roller coaster: 45, 17 , 16, 22, 25, 19, 20, 21, 32, 37 , 19, 21, 24, 20, 18, 22, 23, 19

Example 3

Find the interquartile range of each data set.

11. 43, 36, 51, 68, 50, 27 , 38, 81, 33 25

12. 201, 225, 217 , 240, 232, 252, 228, 231 15

13. 94, 87 , 105, 99, 118, 97 , 102, 85 13

14. 8.4, 7 .1, 6.3, 6.8, 9.2, 7 .3, 8.8, 7 .9, 5.3, 8.2 1.6

15. HEART RATE A nurse tracked the heart rates of several patients. Find the interquartile range (IQR) of the data set: 108, 88, 119, 75, 96, 88, 100, 99, 125, 81. 20

Example 4

Find the standard deviation.

16. {10, 9, 11, 6, 9} 1.67

17. {6, 8, 2, 3, 2, 9} 2.83

18. {23, 18, 28, 36, 15} 7.46

19. {44, 35, 40, 37 , 43, 38, 40} 2.97

20. PARKING A city councilor wants to know how much revenue the city would earn by installing parking meters on Main Street. He counts the number of cars parked on Main Street each weekday: {64, 79, 81, 53, 63}. Find the standard deviation. 10.55

Mixed Exercises

21. REASONING A hockey team keeps track of how many goals it scores each game: {2, 4, 0, 3, 7 , 2}. Find and interpret the standard deviation of the data. See margin.

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22. FOOTBALL The table shows information about the number of carries a running back had over a number of years. Find and interpret the standard deviation of the number of carries. See margin.

Year	Number of Carries
2006	31
2007	90
2008	105
2009	115
2010	162

23. MOVIES The manager at a movie theater kept track of the age of each person in a matinee movie: 67 , 62, 65, 38, 69, 67 , 59, 41, 43, 36, 45, 22, 69, 68, 18, 15, 9, 60, 64.

a. Determine the five-number summary for the data set. 9, 36, 59, 67, 69

b. Draw a box plot of the data. See margin.

24. GAS PRICES Renee is planning a road trip to her aunt's house. T o estimate how much the trip will cost, she goes online and finds the price of a gallon of gasoline for 5 randomly selected gas stations along the route: $2.09, $2. 19, $3.99, $2.39, $2.29.

a. Determine the five-number summary for the data set. 2.09, 2. 14, 2.29, 3. 19, 3.99

b. Draw a box plot of the data. See margin.

Find the range, five-number summary, interquartile range, and standard deviation for each data set. Then draw a box plot of the data.

25. SEASHELLS Jorja collected the following number of seashells for the last nine trips to the beach: 5, 11, 7 , 12, 13, 17 , 3, 15, 18 See margin.

26. SHOE SIZE The following shoe sizes of students at a high school were randomly recorded for one hour: 6, 8, 8.5, 10, 12, 6.5, 7 , 8, 8.5, 7 .5, 9, 11.5, 10, 13, 5.5, 6.5, 5, 9.5. See margin.

Higher-Order Thinking Skills

27. FIND THE ERROR Jennifer and Megan are determining one way to decrease the size of the standard deviation of a set of data. Is either correct? Explain your reasoning. See margin.

Jennifer	Megan
Remove the outliers from the data set.	Add data values to the data set that are equal to the mean.

28. ANAL YZE Determine whether the statement *Two random samples taken from the same population will have the same mean and standard deviation* is *sometimes, always,* or *never* true. Justify your argument. See margin.

29. CREATE Write your own survey question and collect data about your question from 8 classmates. Use that data to find the range, five-number summary, interquartile range, and standard deviation for the data set. Then draw a box plot of the data. See students' work.

30. WRITE What does the interquartile range tell you about how data clusters around the median of the data? See margin.

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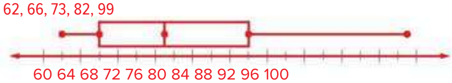
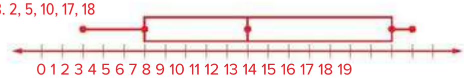
Answers

6. 309, 311, 312, 314, 399

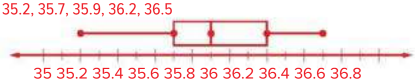
7. 62, 66, 73, 82, 99



8. 2, 5, 10, 17, 18



9. 35.2, 35.7, 35.9, 36.2, 36.5

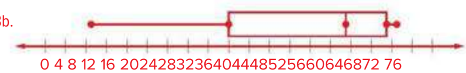


10. 16, 19, 21, 24, 45

21. 2.16; Because the standard deviation is large compared to the mean of 3, the number of goals scored each game is not relatively close to the mean.

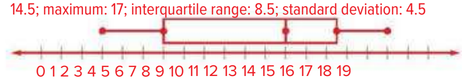
22. The standard deviation is approximately 42.3 carries, which is large compared to the mean of 100.6. This suggests that the number of carries per season is not relatively close to the mean of 100.6 carries.

23b.



24b.

25. range: 14; minimum: 3; lower quartile: 6; median: 12; upper quartile: 14.5; maximum: 17; interquartile range: 8.5; standard deviation: 4.5



26. range: 8; minimum: 5; lower quartile: 6.5; median: 8.25; upper quartile: 10; maximum: 13; interquartile range: 3.5; standard deviation: 2.19

27. Both; sample answer: When an outlier is removed from a set of data, the spread and standard deviation of the data will decrease. When more values that are equal to the mean of a data set are added to the data set, the mean will be stronger and outliers will have less influence.

28. Sometimes; sample answer: If the samples are truly random, they would rarely contain identical elements, and the mean and standard deviation would differ. If the sample produces identical elements, the mean and standard deviation would be the same.

30. Sample answer: The interquartile range represents the middle 50% of data values. Because this data is not affected by outliers, it accurately shows whether the data is closely centered around the median or spread out.

Distributions of Data

S.ID.3

LESSON GOAL

Students analyze the shapes of distributions to determine appropriate statistics and identify extreme data points.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Shapes of Distributions
- Analyze Distribution by Using Technology
- Choose Appropriate Statistics by Using a Histogram
- Choose Appropriate Statistics by Using a Box Plot

Extreme Data Points
- Choose Appropriate Statistics with Extreme Data Points

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example

Resources	AL	OL	BL	ELL
Remediation: Measures of Variation	●	●		●
Extension: Happy Birthday		●	●	●

Language Development Handbook

Assign page 53 of the *Language Development Handbook* to help your students build mathematical language related to analyzing the shapes of distributions to determine appropriate statistics and identifying extreme data points.

ELL You can use the tips and suggestions on page T53 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Statistics and Probability

Standards for Mathematical Content:

S.ID.3 Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

Standards for Mathematical Practice:

4 Model with mathematics.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous
Students represented sets of data using measures of spread.
6.SP.4, 6.SP.5c, N.Q.1, S.ID.1

Now
Students analyze the shapes of distributions to determine appropriate statistics and identify extreme data points.
S.ID.3

Next
Students will use statistics appropriate to the shapes of the distributions to compare the measures of center and spread of two data sets.
S.ID.2, S.ID.3

Rigor

The Three Pillars of Rigor

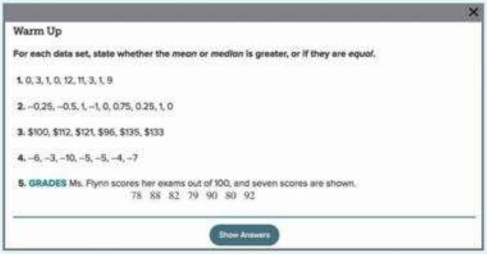
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand their understanding of and fluency with measures of center and spread to explore shapes of data distributions. They apply their understanding of data distributions by solving real-world problems.

Mathematical Background

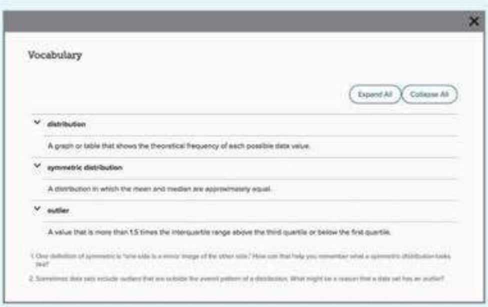
A distribution of data shows the frequency of each possible data value. The shape of a distribution can be determined by looking at its histogram or box-and-whisker plot. When describing a distribution, use the mean and standard deviation if the graph is symmetric and the five-number summary if the distribution is skewed.

Interactive Presentation



Warm Up

Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- comparing the mean and median of a set of data

Answers:

1. mean
2. equal
3. median
4. median
5. mean

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world situation in which it would be helpful to analyze its data distribution.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Learn Shapes of Distributions

Objective

Students interpret differences in the shape of distributions by examining histograms and box plots.

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the graphs used in this Learn.

Important to Know

When data are symmetric, the mean and the median are located near the center of the data and are close in value. When data are skewed, the median will be closer to the side of the data that contains more data values, and the mean will be pulled away from the median, toward the other direction.

Common Misconception

A common misconception some students may have is that the term *negatively skewed* indicates a distribution in which the mean and the median lie closer to the left side of the graph, and the term *positively skewed* indicates a distribution in which the mean and the median lie closer to the right side of the graph. This misconception is actually the opposite of the true distributions for each case. Use the visuals in this lesson to correct this thinking.

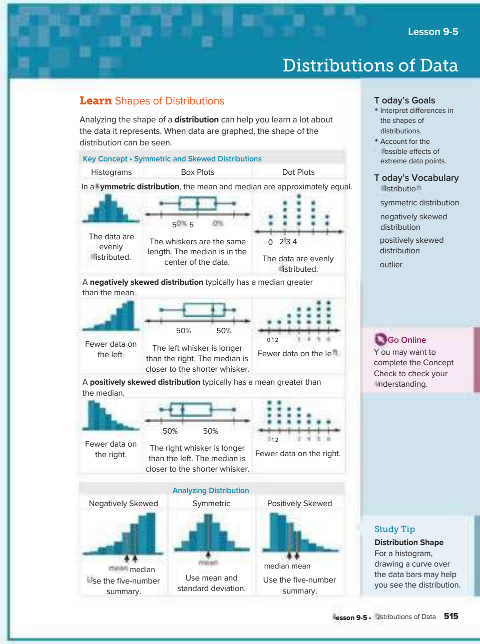
Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

DIFFERENTIATE

Language Development Activity ELL

Intermediate Instruct a small group of students to write a paragraph describing what is happening in each figure illustrating the types of distributions. Students' paragraphs should describe each part of the diagrams in their own words. Ask for volunteers to read their paragraphs. Have students ask for clarification as needed.


Lesson 9-5

Distributions of Data

Learn Shapes of Distributions

Analyzing the shape of a **distribution** can help you learn a lot about the data it represents. When data are graphed, the shape of the distribution can be seen.

Key Concept • Symmetric and Skewed Distributions

Histograms	Box Plots	Dot Plots
In a **symmetric distribution**, the mean and median are approximately equal.		
The data are evenly distributed.	The whiskers are the same length. The median is in the center of the data.	The data are evenly distributed.
A **negatively skewed distribution** typically has a median greater than the mean		
Fewer data on the left	The left whisker is longer than the right. The median is closer to the shorter whisker.	Fewer data on the left.
A **positively skewed distribution** typically has a mean greater than the median.		
Fewer data on the right.	The right whisker is longer than the left. The median is closer to the shorter whisker.	Fewer data on the right.

Analyzing Distribution

Negatively Skewed	Symmetric	Positively Skewed
mean median	mean	median mean
Use the five-number summary.	Use mean and standard deviation.	Use the five-number summary.

Today's Goals
- Interpret differences in the shapes of distributions.
- Account for the possible effects of extreme data points.

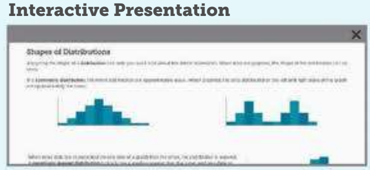
Today's Vocabulary
distribution
symmetric distribution
negatively skewed distribution
positively skewed distribution
outlier

Go Online
You may want to complete the Concept Check to check your understanding.

Study Tip
Distribution Shape For a histogram, drawing a curve over the data bars may help you see the distribution.

Lesson 9-5 • Distributions of Data 515

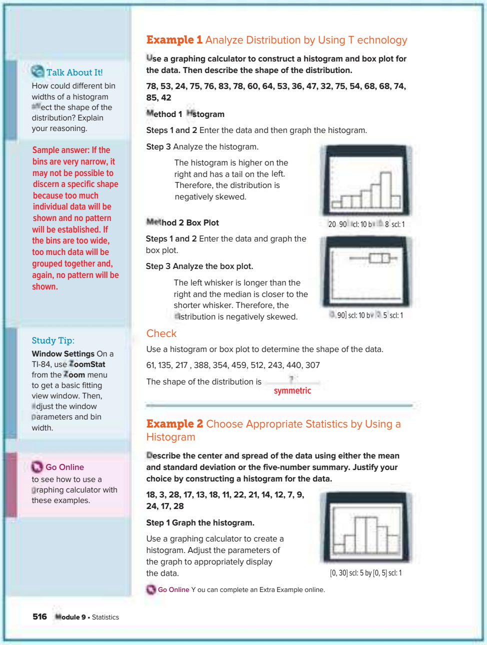
Interactive Presentation



Learn

TAP

Students tap to compare symmetric and skewed data.



Talk About It!

How could different bin widths of a histogram affect the shape of the distribution? Explain your reasoning.

Sample answer: If the bins are very narrow, it may not be possible to discern a specific shape because too much individual data will be shown and no pattern will be established. If the bins are too wide, too much data will be grouped together and, again, no pattern will be shown.

Study Tip:

Window Settings On a TI-84, use **ZoomStat** from the **Zoom** menu to get a basic fitting view window. Then, adjust the window parameters and bin width.

Go Online to see how to use a graphing calculator with these examples.

Example 1 Analyze Distribution by Using T echnology

Use a graphing calculator to construct a histogram and box plot for the data. Then describe the shape of the distribution.

78, 53, 24, 75, 76, 83, 78, 60, 64, 53, 36, 47, 32, 75, 54, 68, 68, 74, 85, 42

Method 1 Histogram

Steps 1 and 2 Enter the data and then graph the histogram.

Step 3 Analyze the histogram.

The histogram is higher on the right and has a tail on the left. Therefore, the distribution is negatively skewed.

20 90 cl: 10 b 8 scl: 1

Method 2 Box Plot

Steps 1 and 2 Enter the data and graph the box plot.

Step 3 Analyze the box plot.

The left whisker is longer than the right and the median is closer to the shorter whisker. Therefore, the distribution is negatively skewed.

90] scl: 10 b 5 scl: 1

Check

Use a histogram or box plot to determine the shape of the data.

61, 135, 217 , 388, 354, 459, 512, 243, 440, 307

The shape of the distribution is ? symmetric

Example 2 Choose Appropriate Statistics by Using a Histogram

Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a histogram for the data.

18, 3, 28, 17, 13, 18, 11, 22, 21, 14, 12, 7, 9, 24, 17, 28

Step 1 Graph the histogram.

Use a graphing calculator to create a histogram. Adjust the parameters of the graph to appropriately display the data.

[0, 30] scl: 5 by [0, 5] scl: 1

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 1

TAP

Students tap to select a calculator to help analyze a distribution.

TYPE

Students answer a question to show they understand analyzing a distribution using technology.

Example 1 Analyze Distribution by Using Technology

Teaching the Mathematical Practices

5 Analyze Graphs Help students analyze the graph they have generated using graphing calculators. Point out that to see the entire graph, students may need to adjust the viewing window.

Questions for Mathematical Discourse

AL Describe the shape of the graph. Sample answer: The graph is high on the right and goes lower and lower as it moves to the left.

OL What type of distribution is this? negatively skewed

BL Can you tell from the box plot that the data is negatively skewed as well? Explain. Yes; sample answer: There is a longer whisker to the left, showing that most of the data is on the right side of the graph.

Example 2 Choose Appropriate Statistics by Using a Histogram

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL Describe the shape of the histogram. symmetric

OL Which statistics best represent the data? mean and standard deviation

BL Why would the mean and standard deviation not be the best statistics to use for skewed data? When the data is skewed, there are values far from the mean, making it less representative of the data than the median.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Example 3 Choose Appropriate Statistics by Using a Box Plot

MP Teaching the Mathematical Practices

5 Analyze Graphs In this example, students will analyze a box plot that they generate using a graphing calculator.

Questions for Mathematical Discourse

AL What does the line inside the box plot represent? the median of the data

OL Why does this box plot indicate that you should use a five-number summary to describe the data? Sample answer: The whiskers are very different lengths, showing that most of the data have lower values, but there are high values that will raise the mean.

BL What would a histogram for this data look like? Sample answer: The histogram would also be positively skewed. So the bars on the left would be taller than those on the right.

This graph shows that the frequency of the data in the middle is high while frequency of data to the left and right are low. Therefore, the distribution is symmetric.

Step 2 Calculate statistics.

The distribution is symmetric, so the mean and standard deviation are good statistics to represent the data.

T o display the statistics, press **STAT**, access the **CALC** menu, select **1-VAR Stats**, and Press **ENTER**

The mean $\bar{x}$ is about 16.4 with a standard deviation of about 7 0

Think About It!

Why is mean an appropriate statistic to represent the center for these data?

Sample answer: Since the shape of the distribution of these data is symmetric, there is approximately the same amount of data greater than and less than the mean.

Example 3 Choose Appropriate Statistics by Using a Box Plot

Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a box plot for the data.

202, 148, 21, 60, 74, 140, 462, 157, 225, 23, 88, 241, 59, 139, 351

Step 1 Graph the box plot.

Use a graphing calculator to create a box plot. Adjust the parameters of the graph to appropriately display the data.

[0, 500] scl: 100 by [0, 5] scl: 1

The right whisker is longer than the left and the median is slightly closer to the right whisker. So, this distribution is positively skewed.

Step 2 Calculate statistics.

The distribution is positively skewed, so use the five-number summary.

T o display the statistics, press stat, access the **CALC** menu, select **1-VAR Stats** and press enter. Use the down arrow key to display more statistics.

Maximum: 462

Minimum: 21

Median: 140

Lower Quartile: 60

Upper Quartile: 225

Go Online Y ou can complete an Extra Example online.

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Interactive Presentation

Example 3

TAP

Students tap to select a calculator to help create a box plot for the data.

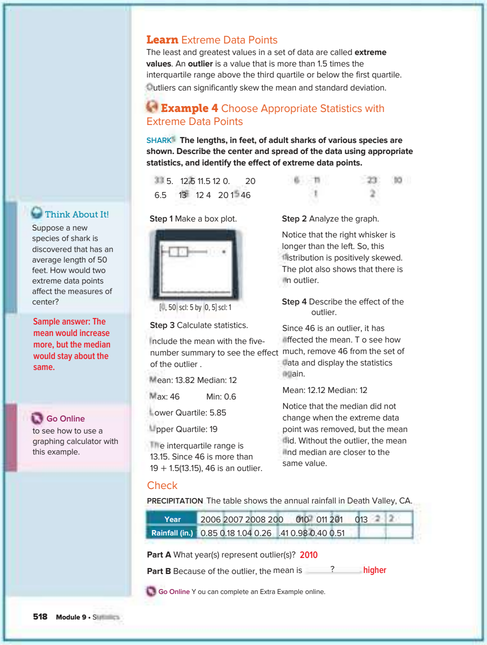
CHECK

Students complete the Check online to determine whether they are ready to move on.

DIFFERENTIATE

Enrichment Activity BL ELL

IF students are having difficulty understanding how extreme data points affect the statistical measures associated with the data, **THEN** pair them with students that have a better grasp of the concept, and have them go back and review the material on this slide together. Have the students discuss the results obtained when using the sketch, focusing on how and why each measure is or is not affected by different outliers.



Learn Extreme Data Points

The least and greatest values in a set of data are called **extreme values**. An **outlier** is a value that is more than 1.5 times the interquartile range above the third quartile or below the first quartile. Outliers can significantly skew the mean and standard deviation.

Example 4 Choose Appropriate Statistics with Extreme Data Points

SHARKS **The lengths, in feet, of adult sharks of various species are shown. Describe the center and spread of the data using appropriate statistics, and identify the effect of extreme data points.**

33 5. 12.5 11.5 12 0. 20 6 11 23 10
6.5 13 12 4 20 15 46 1 2

Step 1 Make a box plot.

[0, 50] scl: 5 by [0, 5] scl: 1

Step 2 Analyze the graph.

Notice that the right whisker is longer than the left. So, this distribution is positively skewed. The plot also shows that there is an outlier.

Step 3 Calculate statistics.

Include the mean with the five-number summary to see the effect of the outlier .

Mean: 13.82 Median: 12

Max: 46 Min: 0.6

Lower Quartile: 5.85

Upper Quartile: 19

The interquartile range is 13.15. Since 46 is more than 19 + 1.5(13.15), 46 is an outlier.

Step 4 Describe the effect of the outlier.

Since 46 is an outlier, it has affected the mean. T o see how much, remove 46 from the set of data and display the statistics again.

Mean: 12.12 Median: 12

Notice that the median did not change when the extreme data point was removed, but the mean did. Without the outlier, the mean and median are closer to the same value.

Think About It! Suppose a new species of shark is discovered that has an average length of 50 feet. How would two extreme data points affect the measures of center?

Sample answer: The mean would increase more, but the median would stay about the same.

Go Online to see how to use a graphing calculator with this example.

Check

PRECIPITATION The table shows the annual rainfall in Death Valley, CA.

Year	2006	2007	2008	200	010	011	201	013	2	2
Rainfall (in.)	0.85	0.18	1.04	0.26	.41	0.98	0.40	0.51		

Part A What year(s) represent outlier(s)? 2010

Part B Because of the outlier, the mean is ___?___ higher

Go Online Y ou can complete an Extra Example online.

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Learn Extreme Data Points

Objective

Students account for the possible effects of extreme data points.

Teaching the Mathematical Practices

4 Use Tools Students will use an interactive sketch to investigate how the mean, median, and standard deviation of a data set are affected by an extreme data point.

What Students Are Learning

Students will explore how extreme data points (outliers) affect the mean, standard deviation, and median of a set of data. They will use a sketch to change the value of the outlier and observe the effect on the related statistics. They will then record their observations in a table.

Example 4 Choose Appropriate Statistics with Extreme Data Points

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this example, students will need to use a graphing calculator. Work with students to explore and deepen their understanding of extreme data points.

Questions for Mathematical Discourse

AL What does the point that is separated from the whiskers represent? Sample answer: There is a value that is much higher than the rest, or the outlier of the data.

OL Which statistics best represent the data? the five-number summary

BL Could there be an outlier for this data that lies below the minimum value? Explain. No; sample answer: The minimum value is close to 0, and negative values do not make sense in this context.

Exit Ticket

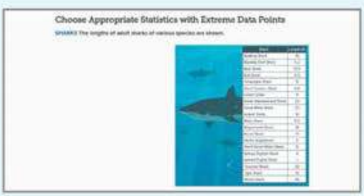
Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Interactive Presentation



Example 4

TAP

Students tap to select a calculator to help create a box plot for the data.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–8
2	exercises that use a variety of skills from this lesson	9–18
2	exercises that extend concepts learned in this lesson to new contexts	19–23
3	exercises that emphasize higher-order and critical-thinking skills	24–27

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–23 odd, 24–27
- Extension: Happy Birthday

IF students score 66%–89% on the Checks, **THEN** assign: OL

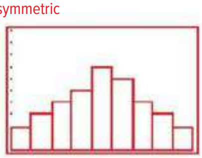
- Practice, Exercises 1–27 odd
- Remediation, Review Resources: Measures of Variation
- Personal Tutors
- Extra Examples 1–4
- ALEKS Finding Measure of Spread

IF students score 65% or less on the Checks, **THEN** assign: AL

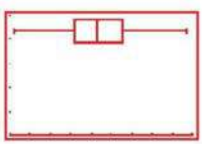
- Practice, Exercises 1–7 odd
- Remediation, Review Resources: Measures of Variation
- *Quick Review Math Handbook:* Distributions of Data
- Arrive**MATH** Take Another Look
- ALEKS Finding Measures of Spread

Answers

1. symmetric



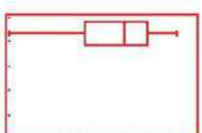
[24, 78] scl: 6 by [0, 10] scl: 1



[24, 78] scl: 6 by [0, 5] scl: 1

2. negatively skewed

[33, 57] scl: 3 by [0, 10] scl: 1



[33, 57] scl: 3 by [0, 5] scl: 1

Practice

Go Online Y ou can complete your homework online.

Example 1

Use a graphing calculator to construct a histogram and a box plot for the data. Then describe the shape of the distribution. See margin.

1. 55, 65, 70, 73, 25, 36, 33, 47, 52, 54, 55, 60, 45, 39, 48, 55, 46, 38
50, 54, 63, 31, 49, 54, 68, 35, 27, 45, 53, 62, 47, 41, 50, 76, 67, 49

2. 42, 48, 51, 39, 47, 50, 48, 51, 54, 46, 49, 36, 50, 55, 51, 43, 46, 37
50, 52, 43, 40, 33, 51, 45, 53, 44, 40, 52, 54, 48, 51, 47, 43, 50, 46

Example 2

Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a histogram for the data. See margin.

3. 32, 44, 50, 49, 21, 12, 27, 41, 48, 30, 50, 23, 37, 16, 49, 53, 33, 25
35, 40, 48, 39, 50, 24, 15, 29, 37, 50, 36, 43, 49, 44, 46, 27, 42, 47

4. 82, 86, 74, 90, 70, 81, 89, 88, 75, 72, 69, 91, 96, 82, 80, 78, 74, 94
85, 77, 80, 67, 76, 84, 80, 83, 88, 92, 87, 79, 84, 96, 85, 73, 82, 83

Example 3

Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a box plot for the data. See margin.

5. 47, 16, 70, 80, 28, 33, 91, 55, 60, 45, 86, 54, 30, 98, 34, 87, 44, 35
64, 58, 27, 67, 72, 68, 31, 95, 37, 41, 97, 56, 49, 71, 84, 66, 45, 93

6. 64, 36, 32, 65, 41, 38, 50, 44, 39, 34, 47, 35, 46, 36, 53, 35, 68, 40
36, 62, 34, 38, 59, 46, 63, 38, 67, 39, 59, 43, 39, 36, 60, 47, 52, 45

Example 4

7. FL YING The various prices of a flight from Los Angeles to New Y ork are shown. \$182, \$234, \$264, \$271, \$277, \$314, \$317, \$455 See margin.
   a. Make a box plot of the data.
   b. Calculate the statistics that best represent the data.
   c. Describe the effect of the outlier.

8. EXERCISE Y oshiko tracked her minutes of exercise each day for 10 days as shown. 57, 60, 53, 59, 57, 61, 61, 54, 62, 10 See margin.
   a. Make a box plot of the data.
   b. Calculate the statistics that best represent the data.
   c. Describe the effect of the outlier.

Mixed Exercises

USE TOOLS **Use a graphing calculator to construct a histogram and a box plot for the data. Then describe the shape of the distribution.** 9–11. See Mod. 9 Answer Appendix.

9. 14, 71, 63, 42, 24, 76, 34, 77, 37, 69, 54, 64, 47, 74, 59, 43, 76, 56
78, 52, 18, 54, 39, 28, 56, 74, 68, 36, 20, 49, 67, 47, 69, 68, 72, 69

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10. 53, 34, 36, 38, 43, 49, 52, 36, 39, 37, 58, 45, 37, 38, 46, 52, 45, 39
55, 39, 40, 55, 38, 40, 42, 38, 45, 36, 46, 39, 35, 41, 49, 43, 52, 34

11. 51, 19, 46, 64, 29, 51, 58, 30, 55, 31, 34, 31, 50, 37, 40, 39, 40, 41
42, 32, 24, 48, 43, 45, 38, 43, 58, 47, 34, 36, 50, 54, 46, 28, 60, 22

12. TRACK Daryn recorded the number of laps he walked around the track each week. Use a graphing calculator to construct a histogram for the data, and describe the shape of the distribution.

    17, 21, 23, 26, 27, 28, 28, 27, 33, 34, 33, 27, 29, 22, 19, 28, 35 See Mod. 9 Answer Appendix.

13. GOLF Mr. Swatsky's geometry class's miniature golf scores are shown below. Use a graphing calculator to construct a box plot for the data, and describe the shape of the distribution. See Mod. 9 Answer Appendix.

Scores
36, 38, 38, 39, 40, 42, 44, 46, 46, 47, 48, 48, 50,
52, 52, 53, 54, 55, 56, 56, 56, 60, 57, 58, 63

14. HAIR LENGTH Ruth recorded the lengths, in centimeters, of hair of students in her school. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by creating a box plot for the data. See Mod. 9 Answer Appendix.

    40, 39, 37, 26, 25, 40, 35, 34, 26, 39, 42, 33, 26, 25, 34, 38, 41, 34
    37, 39, 32, 30, 22, 38, 36, 28, 27, 39, 34, 26, 36, 38, 25, 39, 23, 8

15. PRESIDENTS The ages of the presidents of the United States at the time of their inaugurations are shown. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by creating a box plot for the data. See Mod. 9 Answer Appendix.

Ages of Presidents
57, 61, 57, 57, 58, 57, 61, 54, 68, 51, 49, 64, 50, 48, 65,
52, 56, 46, 54, 49, 51, 47, 55, 55, 54, 42, 51, 56, 55, 51,
54, 51, 60, 62, 43, 55, 56, 61, 52, 69, 64, 46, 54, 47

16. AUTOMOTIVE A service station tracks the number of cars they service per day.

Cars Serviced
40, 47, 37, 42, 46, 31, 50, 41, 17, 43, 36, 45, 21, 43, 45, 23, 49, 50,
48, 26, 42, 46, 35, 52, 27, 51, 31, 44, 35, 27, 46, 39, 33, 50, 45, 50

   a. Use a graphing calculator to construct a histogram for the data, and describe the shape of the distribution. See Mod. 9 Answer Appendix.
   b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice. Sample answer: The distribution is skewed, so use the five-number summary. min: 17, max: 52, med: 42.5, Q1: 34, Q3: 46.5

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17. COMMUTE The number of miles that Armando drove each week during a 15-week period is shown.

Distance (miles)
62, 110, 92, 430, 73, 84, 525, 123, 86, 290, 114, 98, 103, 312, 71

a. Use a graphing calculator to construct a box plot. Describe the center and spread of the data. Sample answer: The distribution is skewed, so use the five-number summary. min: 62, max: 525, med: 103, Q1: 84, Q3: 290

b. Armando visited four colleges during this period, and these visits account for the four highest weekly totals. Remove these four values from the data set. Use a graphing calculator to construct a box plot that reflects this change. Then describe the center and spread of the new data set. See Mod. 9 Answer Appendix.

c. Calculate and compare the mean and median for the original data set to the mean and median for the data set from **part b**. Original: mean 171.5, median 103; altered: mean about 92.4, median 92. The means differ by about 79.1, while the medians differ by 11.

18. ELEVATION The table contains data about 10 elevations in the United States. See Mod. 9 Answer Appendix.

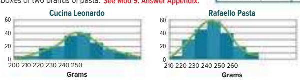
a. Use a graphing calculator to construct a box plot for the data, and describe the shape of the distribution.

b. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

c. If there is an outlier, describe its effect on the statistics.

Elevations in the US	
Mt McKinley, AK	20,237
Mt Whitney, CA	14,494
Mt Elbert, CO	14,433
Mt Rainier, WA	14,410
Gannett Peak, WY	13,804
Mauna Kea, HI	13,796
Kings Peak, UT	13,528
Wheeler Peak NM	13,161
Boundary Peak, NV 13,140	
Granite Peak, MT	12,799

19. USE A MODEL The histograms show the weight of sample boxes of two brands of pasta. See Mod 9. Answer Appendix.



a. Do the two packages of pasta likely have the same advertised weight? Which manufacturer's quantity control appears better? Explain your answers based on the distributions.

b. Infer the two population distribution shapes by analyzing the smooth curves across the tops of the histograms. Describe the shapes you observed.

20. STRUCTURE The United States has been sending astronauts up in the Space Shuttle since 1981. The table provides data regarding the duration of Space Shuttle flights from 1981 to 1985, and then from 2005 to 2011.

Length of Flights from 1981–1985 (days)	Length of Flights from 2005–2011 (days)
Days: 2, 2, 8, 7, 5, 5, 6, 6, 10, 8, 7, 6, 8, 8, 3, 7, 7, 7, 8, 7, 4, 7, 7	Days: 14, 13, 12, 13, 14, 13, 15, 13, 16, 14, 15, 13, 13, 16, 14, 11, 14, 15, 12, 13, 16, 13

Choose and calculate the statistics appropriate for the distribution of the data sets. Use the statistics to compare the two sets. See Mod. 9 Answer Appendix.

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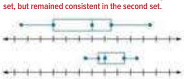
21. REASONING Gerardo live streams with 15 of his friends. Most of his streams have lasted 10–15 days so far, however he has two streams that have lasted 93 days. Describe what Gerardo's data distribution would look like currently and how it would be affected if he lost his longest streams. Currently, Gerardo's distribution would be positively skewed. If he lost his longest streams, the data would represent a symmetric distribution.

22. CONSTRUCT ARGUMENTS Examine the two box plots shown. Without knowing the data points but assuming the same scale, what conclusion can be made? Justify your argument. The interquartile range tells you that the data varied in the first set, but remained consistent in the second set.



23. SUPREME COURT The table gives the ages of the Supreme Court Justices in 2017. See Mod. 9 Answer Appendix.

a. Use a graphing calculator to construct a histogram for the data, and describe the shape of the distribution.

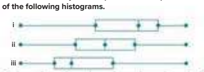
b. Describe the center and spread of the data using appropriate statistics. Justify your choice.

c. If there is an outlier, describe its effect on the statistics.

Supreme Court Justices	
Neil Gorsuch	49
Elena Kagan	57
Sonia Sotomayor	62
Samuel Anthony Alito	67
Stephen G. Breyer	78
Ruth Bader Ginsburg	84
Clarence Thomas	68
Anthony M Kennedy	80
John G. Roberts Jr.	62

Higher-Order Thinking Skills

24. PERSEVERE **Identify the box plot that corresponds to each of the following histograms.**



A. iii

B. i

C. ii

25. ANAL YZE Research and write a definition for a *bimodal distribution*. How can the measures of center and spread of a bimodal distribution be described? See Mod. 9 Answer Appendix.

26. CREATE Give an example of a set of real-world data with a distribution that is symmetric and one with a distribution that is not symmetric. See Mod. 9 Answer Appendix.

27. WRITE Explain why the mean and standard deviation are used to describe the center and spread of a symmetrical distribution and the five-number summary is used to describe the center and spread of a skewed distribution. See Mod. 9 Answer Appendix.

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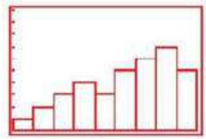
Answers

3. Sample answer: The distribution is skewed, so use the five-number summary. The range is 53 – 12, or 41. The median is 39.5, and half of the data are between 28 and 48.



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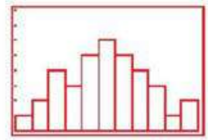
4. Sample answer: The distribution is symmetric, so use the mean and standard deviation. The mean is 82 with a standard deviation of about 7.4



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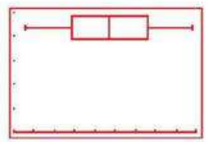
5. Sample answer: The distribution is symmetric, so use the mean and standard deviation. The mean is about 58.7 with standard deviation of about 22.8.



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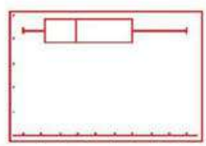
6. Sample answer: The distribution is skewed, so use the five-number summary. The minimum is 32, the maximum is 68, the median is 43.5, and half of the data are between 37 and 56.



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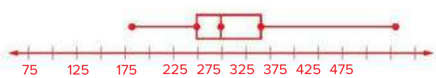
7. a.



b. min: 182, Q1: 249, median: 274, Q3: 315.5, max: 455

c. The outlier mainly affects the mean. When the outlier is removed, the median decreases, but only $3 to $271. However, the mean changes from $289 to $266, which is more representative of the data as a whole.

8. a.

b. min: 10, Q1: 54, median: 58, Q3: 61, max: 62

c. The outlier affects the mean. When the outlier is removed, the median increases by 1; however, the mean increases from 53.4 to 58.2, which is more representative of the data as a whole.

Lesson 9-6

Comparing Sets of Data

S.ID.2, S.ID.3

LESSON GOAL

Students use statistics appropriate to the shapes of the distributions to compare the measures of center and spread of two data sets.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore:

- Transforming Sets of Data by Using Addition
- Transforming Sets of Data by Using Multiplication

Develop:

Linear Transformations of Data

- Transformations Using Addition
- Transformations Using Multiplication
- Compare Symmetric Distributions of Data
- Compare Skewed Distributions of Data

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Statistical Questions	●	●		●
Extension: Mean Absolute Deviation		●	●	●

Language Development Handbook

Assign page 54 of the *Language Development Handbook* to help your students build mathematical language related to measuring the center and spread of two data sets.

ELL You can use the tips and suggestions on page T54 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Statistics and Probability

Standards for Mathematical Content:

S.ID.2 Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.

S.ID.3 Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

Standards for Mathematical Practice:

4 Model with mathematics.

5 Use appropriate tools strategically.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students analyzed the shapes of distributions to determine appropriate statistics and identify extreme data points.
S.ID.3

Now
Students use statistics appropriate to the shapes of the distributions to compare the measures of center and spread of two data sets.
S.ID.2, S.ID.3

Next
Students will summarize and interpret categorical data using frequency tables.
S.ID.5

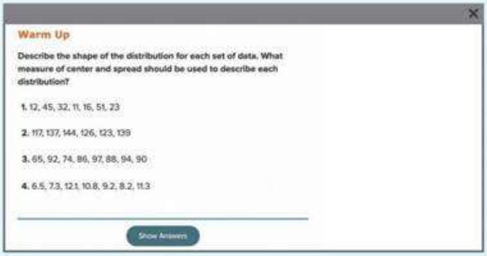
Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand their understanding of and fluency with the shapes of data distributions to compare measures of center and spread in two or more data distributions. They apply their understanding of comparing data distributions by solving real-world problems.

Interactive Presentation



Warm Up

Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skills for this lesson:

- analyzing the distribution of a set of data
- determining appropriate measures of center and spread

Answers:

1. positively skewed; median; five-number summary
2. symmetric; mean; standard deviation
3. negatively skewed; median; five-number summary
4. symmetric; mean; standard deviation

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics Encourage students to consider how they can apply what they have learned about statistical measures and representations to the situation described in the video. Have them discuss how the manager might use these concepts to make good business decisions.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then, discuss the question below with the class.

Mathematical Background

If a real number k is added to every value in a set of data, the mean, median, and mode of the data set can be found by adding k to the mean, median, and mode of the original data set. The range and standard deviation will be the same. If every value in a set of data is multiplied by a constant k, $k > 0$, then the mean, median, mode, range, and standard deviation of the new data set can be found by multiplying each original statistic by k.

Explore Transforming Sets of Data by Using Addition

Objective

Students use a calculator to explore how using addition to transform a set of data affects the measures of center and spread.

Teaching the Mathematical Practices

8 Look for a Pattern Help students see the pattern in this Explore activity.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use a graphing calculator to find the mean, median, mode, range, and standard deviation of a given set of data. Then they will add 3 to each data value and recalculate the statistics. They will compare their results. Then, students will answer the Inquiry question.

(*continued on the next page*)

Interactive Presentation

Explore

Explore

TYPE

Students complete the calculations to find statistics on a transformed set of data.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry question and can view a sample answer.

Explore Transforming Sets of Data by Using Addition (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why does the median increase by 3? Sample answer: Each data value increases by 3. So the middle number increases by 3.
- Suppose the range of a data set is 6.6. If 0.4 is added to each data value, what would be the range of the new data set? 6.6

Inquiry

How can you find the measures of center and spread of a set of data that has been transformed using addition? Sample answer: Add the number that has been added to the data values to the measures of center. The measures of spread will be the same as those for the original data set.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Explore Transforming Sets of Data by Using Multiplication

Objective

Students use a calculator to explore how using multiplication to transform a set of data affects the measures of center and spread.

Teaching the Mathematical Practices

5 Compare Predictions with Data Point out that in this Explore activity, students should use a graphing calculator to compare their predictions with the data.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore activity is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use a graphing calculator to find the mean, median, mode, range, and standard deviation of a given set of data. Then they will multiply each data value by 2 and recalculate the statistics. They will compare their results. Then, students will answer the Inquiry question.

(continued on the next page)

Interactive Presentation

Explore

Explore

TYPE

Students complete the calculations to find the statistics on a transformed set of data.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry question and can view a sample answer.

Explore Transforming Sets of Data by Using Multiplication (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why does the mode double? Sample answer: The number that was the mode is now twice the value that it was before.
- Suppose the range of a data set is 2.5. If each data value is multiplied by 4, what would be the range of the new data set? 10

Inquiry

How can you find the measures of center and spread of a set of data that has been transformed using multiplication? Sample answer: Multiply the measures of center and spread by the number by which the data values have been multiplied.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Linear T ransformations of Data

Objective

Students describe the effects that linear transformations have on measures of center and spread.

Teaching the Mathematical Practices

7 Use Structure Help students explore the structure of linear transformations of data in this Learn.

About the Key Concept

In the Key Concept, students will learn how transforming a set of data affects the measures of center and spread of the data. Students consider transformations by addition and transformations by multiplication.

Common Misconception

A common misconception some students may have is that transforming a data set by addition will affect not only the measures of center, but also the range and the standard deviation. Remind students that the range and the standard deviation are measures of spread, and help them to see that the spread of the data in the new data set will be exactly the same as the spread of the original set of data.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 9-6

Comparing Sets of Data

Today's Goal
- Describe the effects that linear transformations have on measures of center and spread.

Today's Vocabulary
linear transformation

Explore Transforming Sets of Data by Using Addition

Online Activity Use graphing technology to complete the Explore.

INQUIRY How can you find the measures of center and spread of a set of data that has been transformed using addition?

Explore Transforming Sets of Data by Using Multiplication

Online Activity Use graphing technology to complete the Explore.

INQUIRY How can you find the measures of center and spread of a set of data that has been transformed using multiplication?

Learn Linear T ransformations of Data

A **linear transformation** is one or more operations performed on a set of data that can be written as a linear function. Common linear transformations are adding a constant to or multiplying a constant by every value in the set of data.

Key Concept • Linear Transformations of Data

Transformations Using Addition	Transformations Using Multiplication
A real number is added to every value in a set of data, $\neq 0$.	Every value in a set of data is multiplied by a constant k, $k > 0$.
Measures of Center	
The mean, median, and mode of the new set of data can be found by adding to the mean, median, and mode of the original set of data.	The mean, median, and mode of the new set of data can be found by multiplying each original statistic by k.
Measures of Spread	
The range and standard deviation of the new set of data will be unchanged.	The range and standard deviation of the new set of data can be found by multiplying each original statistic by k.

Lesson 9-6 • Comparing Sets of Data 523

Interactive Presentation

Learn

TAP

Students tap to compare transformations using addition and multiplication.

Example 1 Transformations Using Addition

Find the mean, median, mode, range, and standard deviation of the data set obtained after adding 6 to each value.

8, 11, 3, 6, 15, 3, 5, 7, 14, 3, 5, 4

Method 1 Add 6 to the measures of center and spread of the original set of data.

Find the mean, median, mode, range, and standard deviation of the original data set.

Mean 7 | Mode 3 | Standard Deviation 4
Median 5.5 | Range 12

Add 6 to the mean, median, and mode. The range and standard deviation are unchanged.

Mean 13 | Mode 9 | Standard Deviation 4
Median 11.5 | Range 12

Method 2 Add 6 to each data value of the original set of data.
Add 6 to each data value.

14, 17, 9, 12, 21, 9, 11, 13, 20, 9, 11, 10

Mean 13 | Mode 9 | Standard Deviation 4
Median 11.5 | Range 12

Study Tip
1-Var Stats To quickly calculate the mean $\bar{x}$, median **Med**, standard deviation σ, and range of a data set, enter the data as **L1** in a graphing calculator, and then use the **1-VAR Stats** feature from the **CALC** menu. Subtract **minX** from **maxX** to find the range.

Think About It!
Compare and contrast the two methods.

Sample answer: In Method 1, add after finding the statistics of the data set, and in Method 2 add before. Method 1 requires adding to only three values — the mean, median, and mode — while Method 2 requires adding to every value in the original set.

Example 2 Transformations Using Multiplication

Find the mean, median, mode, range, and standard deviation of the data set obtained after multiplying each value by 4.

12, 18, 20, 12, 14, 18, 11, 21, 13, 18, 11, 24

Find the measures of center and spread for the original data set.

Mean	Median	Mode	Range	Standard Deviation
16	16	18	13	4.2

Multiply the measures of center and spread by 4.

Mean	Median	Mode	Range	Standard Deviation
64	64	72	52	16.8

Think About It!
Use a calculator and the data set to examine the effect of multiplying by a constant k, $k<0$. How can the mean, median, mode, range, and standard deviation of the new data set be found?

Sample answer: The mean, median, and mode of the new set of data can be found by multiplying each original statistic by k. The range and the standard deviation can be found by multiplying each original statistic by $|k|$.

Check
Find the mean, median, mode, range, and standard deviation of the data set obtained after multiplying each value by 0.6. Round to the nearest tenth, if necessary. 24.6, 24.6, 23.4, 12.6, 3.7

45, 33, 43, 51, 39, 48, 34, 39, 30, 39, 47, 44

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 1

TAP
Students tap to choose a method for transforming data using addition.

TYPE
Students answer a question to show they understand transformations using addition.

CHECK
Students complete the Check online to determine whether they are ready to move on.

Example 1 Transformations Using Addition

MP Teaching the Mathematical Practices

1 Special Cases Work with students to evaluate the two methods shown. Encourage students to familiarize themselves with both methods, and to know the best time to use each one.

Questions for Mathematical Discourse

AL What happens to the sum of the data values if each value is increased by 6? The sum increases by 72 because each of the 12 values has been increased by 6.

OL Why is the range of a set of data unaffected when the data are all increased by the same value? Sample answer: The least value and greatest value are increased by the same amount, so the difference between the two remains the same.

BL Each value in a data set with n values and a mean of x is increased by y. What is the mean of the new data set? $x + y$

Common Error

Students may add the constant to the range and to the standard deviation. Check that students understand *why* this would be incorrect.

Example 2 Transformations Using Multiplication

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL How do you determine the mean of this data? Add all the data values and divide by 12.

OL Is it necessary to multiply every data value by 4 to solve the problem? Explain. No; sample answer: The mean, median, mode, range, and standard deviation of the original data set can be multiplied by 4 instead.

BL Why is the range of a set of data affected when the data are all multiplied by the same value? Explain verbally and algebraically. Sample answer: The least value and greatest value are both multiplied by the same amount, so the difference between the two will also be multiplied by the same amount. $4x - 4y = 4(x - y)$

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Example 3 Compare Symmetric Distributions of Data

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to solve the problem in this example, students will need to analyze the mathematical relationships in the problem to draw a conclusion.

Questions for Mathematical Discourse

AL How do the three histograms compare? Sample answer: They all are relatively symmetric, although the distributions have different shapes and lie in different areas of the window.

OL Which statistics best represent the data? mean and standard deviation

BL How can you use the histograms to support the fact that the standard deviations are so close in value but the means are so different? Sample answer: The histograms show that the centers of the data are different, with breakfast the lowest and lunch the highest, but they also show that for each of the three times of day, the spread of the data is about the same.

Common Error

Some students may describe the data for breakfast and lunch as being asymmetric because the bars are not clustered in the middle of the graph, as they are in the dinner graph. Help students to recognize that, while the location of the bars on the horizontal axis is determined by the values of the data, the shape is determined by the relative heights of the bars.

Example 3 Compare Symmetric Distributions of Data

RESTAURANTS **The numbers of customers eating at a restaurant during breakfast, lunch, and dinner each day are shown below.**

Breakfast: 71, 58, 65, 48, 44, 56, 68, 64, 51, 67 , 74, 62, 59, 53, 62, 73, 54, 49, 63, 55

Lunch: 115, 105, 87 , 108, 117 , 110, 92, 101, 114, 91, 109, 96, 100, 98, 103, 111, 95, 94, 102, 106

Dinner 76, 62, 91, 76, 79, 68, 65, 89, 81, 76, 90, 82, 79, 74, 71, 73, 84, 87 , 81, 64

Part A Construct a histogram or box plot for each set of data. Then describe the shape of each distribution.

Method 1 Histogram

Enter the data in L1, L2, and L3. From the **STAT PLOT** menu, enter L1 as the **Xlist** for Plot 1, **L2** for Plot 2, and **L3** for Plot 3. Select [histogram] as the plot type for each Plot. View each histogram by turning on Plot 1, Plot 2, and then Plot 3. Use the same window dimensions and bin width for each graph.

Breakfast | Lunch | Dinner

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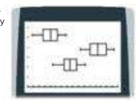
For each time of day, the distribution is high in the middle and low on the left and right. Therefore, all of the distributions are symmetric.

Method 2 Box Plot

Enter the data using the same process. Select [box plot] as the plot type for each set of data. T o view all of the box plots at once, turn on Plot 1, Plot 2, and Plot 3 and graph.

For each time of day, the lengths of the whiskers are approximately equal, and the median is in the middle of the data. The left and right sides are approximately mirror images of one another. Therefore, all of the distributions are symmetric.



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(continued on the next page)

Go Online Y ou can complete an Extra Example online.

Study Tip

Window and Bin Settings When setting the window dimensions for multiple sets of data, try setting the minimum and maximum as the least and greatest values of all the sets. When selecting a bin width, consider the context of the situation. For example, if the data does not include fractional numbers, as would be the case with number of people, use a whole number as the bin width.

Lesson 9-6 • Comparing Sets of Data **525**

Interactive Presentation

Example 3

TAP

Students tap to select a calculator to compare symmetric distributions of data.

Talk About It!
Describe how this data could be used to make decisions about the restaurant.

Sample answer: During the least busy time, breakfast, fewer employees could be scheduled to work since there are the fewest number of customers at this time. More employees could be scheduled for lunch and dinner shifts.

Go Online to see how to use a graphing calculator with this example.

Part B **Compare the data sets using the means and standard deviation.**

All of the distributions are symmetric, so use the means and standard deviation to describe the centers and spreads.

Breakfast | Lunch | Dinner

The means vary, with breakfast having the lowest average number of customers and lunch having the highest average number of customers. However, the standard deviations are approximately equal. This means that, while the average number of customers for each time of day is very different, the number of customers for each time of day generally varies by the same amount from day to day.

Check

DOGS The weights, in pounds, for a sample of the three most popular breeds of dogs are shown below.

Labrador Retriever: 75, 59, 63, 68, 67 , 59, 69, 63, 60, 76, 70, 74, 67 , 68, 71, 65, 62, 74, 66, 78

German Shepherd: 53, 61, 58, 74, 85, 80, 72, 57 , 64, 69, 81, 75, 73, 64, 76, 68, 66, 51, 67 , 73

Golden Retriever: 2, 59, 67 , 72, 64, 67 , 69, 76, 63, 64, 73, 69, 71, 75, 59, 64, 69, 59, 74, 68

Part A Use a graphing calculator to construct a histogram or box plot for each set of data. Then complete the statement about the shape of each distribution.

All of the distributions are ___?___ symmetric

Part B Compare the data sets using the means and standard deviations. What conclusion(s) can you make about the sets of data? Select all that apply. A, C, E, F

A. The average weight of each breed is about the same.
B. The weights of all three breeds are very close to their means.
C. The weights of the German shepherds vary more than the other breeds.
D. On average, the golden retrievers weigh much more than the other breeds
E. The means of the weights differ by less than 1.5 pounds.
F. The weights of the Labrador retrievers and golden retrievers are generally closer to their means than the German shepherds' weights are to their mean.

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Interactive Presentation

Check

CHECK

Students complete the Check online to determine whether they are ready to move on.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are having difficulty using their graphing calculators to construct the graphs and obtain the related statistics,
THEN pair the students who are struggling with students who are meeting with success, and have them work through several examples together. Encourage students who were struggling to create a list of helpful hints that they can use when they are working on their own.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Example 4 Compare Skewed Distributions of Data

Teaching the Mathematical Practices

5 Analyze Graphs Help students analyze the graph they have generated using graphing calculators. Point out that to see the entire graph, students may need to adjust the viewing window.

Questions for Mathematical Discourse

AL What do the histograms indicate about the similarities and the differences between the two sets of data? Sample answer: They are both negatively skewed, but the data for the girls is much greater than the data for the boys.

OL Why are the mean and standard deviation not appropriate measures for describing the data? Sample answer: The mean and standard deviation are good descriptions of the data only when the data are symmetric. These data are skewed, so it is better to use the five-number summary.

BL Compare the medians of the two data sets, and tell what they indicate about the data in the context of the situation. Sample answer: The median for the girls is 2 million greater than the median for the boys. This means that the median number of girls participating in tennis during that time period was 2 million more than the median number of boys.

Common Error

Some students may use the wrong statistics to describe the data, forgetting to use the shape of the distribution to decide between using the mean and standard deviation, and the five-number summary. Remind students about the importance of graphing the data so that they can make the correct determination of which statistics to use.

Example 4 Compare Skewed Distributions of Data

SPORTS **The numbers of high school boys and girls, in hundred thousands, participating in tennis from 2001–2015 are shown below.**

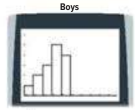
Boys (hundred thousands)	Girls (hundred thousands)
144, 139, 145, 153, 149, 153, 157 , 156, 157 , 163, 161, 160, 157 , 161, 157	164, 160, 163, 168, 169, 174, 177 , 172, 178, 182, 182, 181, 181, 184, 183

Part A Construct a histogram or box plot for each set of data. Then describe the shape of each distribution.

Method 1 Histogram

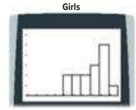
Enter the data in L1 and **L2**. From the **STATPLOT** menu, enter L1 as the **Xlist** for Plot 1 and L2 for Plot 2. Select [histogram icon] as the plot type for each Plot. View each histogram by turning on Plot 1, and then Plot 2. Use the same window dimensions and bin width for each graph.

Boys



[139, 189] scl: 5 by [0, 6] scl: 1

Girls



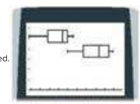
[139, 189] scl: 5 by [0, 6] scl: 1

Both distributions are high on the right and have tails on the left. Therefore, both distributions are negatively skewed.

Method 2 Box Plot

Enter the data using the same process. Select Δ as the plot type for each set of data. T o view both box plots at once, turn on Plot 1 and Plot 2 and graph.

For each distribution, the left whisker is longer than the right, and the median is closer to the right whisker. Therefore, both distributions are negatively skewed.



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Part B Compare the data sets using the five-number summaries.

Both distributions are skewed, so use the five-number summary to compare the data.

(continued on the next page)

Go Online Y ou can complete an Extra Example online.

Lesson 9-6 • Comparing Sets of Data **527**

Interactive Presentation

Example 4

TAP

Students select a calculator to create a histogram of the data.

Go Online to see how to use a graphing calculator with this example.

Think About It!
Compare two other statistics from the five-number summaries. What does this tell you about the number of girls and boys that participated in tennis?

Sample answer: The maximum for the number of boys that participated is 163, while the minimum for the number of girls that participated is 160. This means that the greatest number of boys that ever participated was only 3000 more than the fewest number of girls that ever participated.

The upper quartile for the number of boys that participated in tennis is 160, while the minimum number of girls that participated is 160. This means there were only 160,000 or more boys participating in tennis for 25% of the years, while at least 160,000 girls participated every year.

We can conclude that many more girls participated in tennis from 2001 to 2015 than boys.

Boys

Girls

Check

FUNDRAISING The number of raffle tickets sold by Darius and Makya each day are shown below.

Darius: 5, 1, 15, 4, 10, 23, 9, 3, 17 , 2, 6, 21, 5, 13, 28, 10, 14, 7 , 5, 19, 9, 22, 10, 8, 15, 9, 13, 19, 22, 30

Makya: 18, 1, 17 , 10, 19, 3, 7 , 20, 9, 22, 12, 13, 16, 18, 16, 5, 17 , 15, 6, 11, 18, 14, 16, 18, 1, 16, 18, 23, 15, 10

Part A Use a graphing calculator to construct a histogram or box plot for each set of data. Then complete the statement about the shape of each distribution.

The distribution of Darius' raffle ticket sales is __positively skewed__ ?.

The distribution of Makya's raffle ticket sales is __negatively skewed__ ?.

Part B Compare the data sets using the five-number summaries. What conclusion(s) can you make about the sets of data? Select all that apply. B, C, E, F

A. The median number of tickets Darius sold is much higher than the median number of tickets Makya sold.

B. The median number of tickets Darius sold is the same as the lower quartile of Makya's sales.

C. The data from Darius' sales is spread over a wider range than the data from Makya's sales.

D. The median number of tickets each student sold was the same.

E. The fewest number of tickets each student sold in a day was 1.

F. The upper 50% of Darius' data spans from 10 to 30, while the upper 75% of Makya's data spans from 10 to 23.

Go Online You can complete an Extra Example online.

528 Module 9 • Statistics

Interactive Presentation

Check

CHECK

Students complete the Check online to determine whether they are ready to move on.

Essential Question Follow-Up

Students are using technology to create data displays that they then use to compare data sets.

Ask:

How are histograms and box plots useful for comparing real-world data? Sample answer: They provide a picture of each data set, which makes it easy to compare the shapes of the distributions and to identify and compare important statistics about the data sets.

DIFFERENTIATE

Enrichment Activity BL

Have students use the Internet to find data about two cities in the United States that they can use for a comparison-of-data display. This data could be population data, median household incomes, weather data, or something similar. Ask students to make box plots for each data set and compare them. Their analyses should include a comparison using either the means and standard deviations or the five-number summaries. Have students summarize their observations in the context of the data situation and share their work with the class.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–10
2	exercises that use a variety of skills from this lesson and extend concepts learned in this lesson to new contexts	11–28
3	exercises that emphasize higher-order and critical-thinking skills	29–33

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–27 odd, 29–33
- Extension: Mean Absolute Deviation

IF students score 66%–89% on the Checks, **THEN** assign: OL

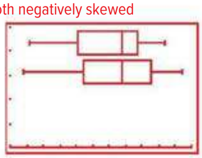
- Practice, Exercises 1–33 odd
- Remediation, Review Resources: Statistical Questions
- Personal Tutors
- Extra Examples 1–4
- ALEKS

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–9 odd
- Remediation, Review Resources: Statistical Questions
- *Quick Review Math Handbook:* Comparing Sets of Data
- Arrive**MATH** Take Another Look
- ALEKS

Answers

9a. both negatively skewed



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Practice

Go Online Y ou can complete your homework online.

Example 1

Find the mean, median, mode, range, and standard deviation of each data set that is obtained after adding the given constant to each value.

1. 52, 53, 49, 61, 57, 52, 48, 60, 50, 47; +8
60.9; 60; 60; 14; 4.7

2. 101, 99, 97, 88, 92, 100, 97, 89, 94, 90; +(−13)
81.7; 82.5; 84; 13; 4.5

3. 27, 21, 34, 42, 20, 19, 18, 26, 25, 33; +(−4)
22.5; 21.5; no mode; 24; 7.4

4. 72, 56, 71, 63, 68, 59, 77, 74, 76, 66; +16
84.2; 88.5; no mode; 21; 6.8

Example 2

Find the mean, median, mode, range, and standard deviation of each data set that is obtained after multiplying each value by the given constant.

5. 11, 7, 3, 13, 16, 8, 3, 11, 17, 3; ×4
36.8; 38; 12; 56; 20.0

6. 64, 42, 58, 40, 61, 67, 58, 52, 51, 49; ×0.2
10.8; 11; 11.6; 5.4; 1.7

7. 33, 37, 38, 29, 35, 37, 27, 40, 28, 31; ×0.8
26.8; 27.2; 29.6; 10.4; 3.5

8. 1, 5, 4, 2, 1, 3, 6, 2, 5, 1; ×6.5
19.5; 16.3; 6.5; 32.5; 11.6

Examples 3 and 4

9. BASEBALL The total wins per season for the first 17 seasons of the Marlins are shown. The total wins over the same time period for the Cubs are also shown.

Marlins
64, 51, 67, 80, 92, 54, 64, 79, 76, 79, 91,
83, 83, 78, 71, 84, 87

Cubs
84, 49, 73, 76, 68, 90, 67, 65, 88, 67, 88,
89, 79, 66, 85, 97, 83

a. Use a graphing calculator to construct a box plot for each set of data. Then describe the shape of each distribution. See margin.

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice. See margin.

10. HEALTH CLUBS To plan their future equipment purchases, the Northville Health Club randomly chooses 8 patrons and tracks how many minutes they spend on the treadmill.

Minutes on Treadmill Last Week	Minutes on Treadmill This Week
30	20
30	30
45	45
20	45
60	30
30	60
30	50
45	45

a. Use a graphing calculator to construct a histogram for each set of data. Then describe the shape of each distribution. See margin.

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice. See margin.

Lesson 9-6 • Comparing Sets of Data 529

Mixed Exercises

Find the mean, median, mode, range, and standard deviation of each data set that is obtained after adding or multiplying each value by the given constant(s).

11. 98, 95, 97, 89, 88, 95, 90, 81, 87, 95; +2
93.5; 94.5; 97; 17; 5.1

12. 32, 30, 27, 29, 25, 33, 38, 26, 23, 31; ×1.6
47; 47.2; no mode; 24; 6.7

13. 14, 17, 13, 9, 15, 7, 12, 16, 8, 9; ×5
60; 62.5; 45; 50; 16.9

14. 5, 12, 7, 3, 8, 5, 7, 1, 4, 7, 3, 9; +22
27.9; 28; 29; 11; 2.9

15. 12, 15, 16, 12, 12, 15, 17, 19, 22, 27, 42, 42; +5
25.9; 21.5; 17; 30; 10.3

16. 49, 43, 26, 39, 40, 30, 33, 64, 26, 45, 23, 26; ×3, +(−8)
103, 100, 70, 123, 34.7

17. 71, 72, 68, 70, 72, 67, 68, 72, 65, 70; ×0.2
13.9; 14; 14.4; 1.4; 0.5

18. 112, 91, 108, 129, 80, 99, 78, 80; +(−15)
82.1; 80; 65; 51; 17.1

19. 57, 38, 42, 51, 39, 44, 33, 55; +(−7), ×2
75.8, 72, no mode, 48, 16.1

20. 55, 50, 58, 52, 56, 57, 50, 55, 50; ×2, +5
112.3; 115; 105; 16; 6.0

21. BOWLING The scores of 15 bowlers are shown in the table.

Score
211, 123, 183, 176, 224, 115, 109, 136, 152, 177, 127, 196, 143, 166, 170

a. Find the mean, median, mode, range, and standard deviation of the scores.
160.5; 166; no mode; 115; 33.9

b. The handicap of the bowling team will add 56 points to each score. Find the statistics of the scores while including the handicap. 216.5; 222; no mode; 115; 33.9

22. COMPETITION The distances that 18 participants threw a football are shown in the table.

Distance (feet)
96, 94, 114, 85, 96, 109, 90, 109, 67, 82, 98, 79, 69, 70, 106, 96, 112, 84

a. Find the mean, median, mode, range, and standard deviation of the participants' distances. 92; 95; 96; 47; 14.5

b. Find the statistics of the participants' distances in yards. 30.7; 31.7; 32; 15.7; 4.8

23. TEMPERATURE The monthly average high temperatures for Lexington, Kentucky, are shown in the table.

Temperature (°F)
40, 45, 55, 65, 74, 82, 86, 85, 78, 67, 55, 44

a. Find the mean, median, mode, range, and standard deviation of the temperatures. 64.7, 66, 55, 46, 15.9

b. Find the statistics of the temperatures in degrees Celsius. Recall that $C = \frac{5}{9}(F - 32)$. 18. 1, 18.9, 12.8, 25.6, 8.9

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24. FANTASY SPORTS The weekly total points of Scott's and Azumi's fantasy baseball teams are shown in the tables.

Scott's Team
109, 99, 121, 137, 131, 141, 77,
83, 139, 92, 42, 133, 98, 153, 124,
102, 113, 117, 112, 128, 107, 147

Azumi's Team
113, 121, 98, 104, 106, 123, 175,
141, 109, 129, 49, 110, 112, 144,
106, 119, 127, 88, 132, 93, 137, 123

a. Use a graphing calculator to construct a box plot for each set of data. Then describe the shape of each distribution. See margin.

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice. See margin.

c. How does eliminating the outliers of each data set affect the statistics and comparison from **part b**? See margin.

25. BUSINESS Saeed owns an electronics store. He is revising his pricing for phone accessories. His current prices for an assortment of accessories are listed at the right. He has also determined that the mean price for the same assortment of accessories at a rival store is $10.99.

Saeed's Price Data ($)
14.99 4.49 9.99
18.49 12.99 6.99
8.49 21.99 13.49
13.99 9.99 10.99
12.49 4.49 12.99

a. Saeed wants to match his rival's prices. Make a table to list the new prices. Explain. See margin.

b. Compare the mean and standard deviation of the current prices to the new prices. See margin.

26. REASONING Two different samples on the shell diameter of a species of snail are shown.

Sample A (mm)
45 35 37
40 42 40
28 38 31

Sample B (mm)
26 44 40
27 35 28
26 39 31

a. Use the median and interquartile range to compare the samples. See Mod. 9 Answer Appendix.

b. Based on your findings and on the data points in each sample, which sample appears to be more representative? Explain your reasoning. See Mod. 9 Answer Appendix.

27. STRUCTURE Height data samples of 17-year-old male and female students are shown. Use the mean and standard deviation to compare the samples. See Mod. 9 Answer Appendix.

Heights of Male Students (inches)	Heights of Female Students (inches)
71 69 67 67 62 69	
68 69 70 65 71 66	
72 74 68 63 65 68	
71 69 72 66 63 70	

28. CONSTRUCT ARGUMENTS Francisca is planning a two-week vacation to one of two cities and wants to base her decision on the weather history for the same dates as her vacation. She has collected the number of days that it has rained during this two-week period for each city over the past 10 years. The results are shown.

City A
5 0
7 6
5 6
6 6
3 2

City B
4 4
6 5
3 7
4 3
5 7

a. Determine the shape of each distribution, and use the appropriate statistics to find the center and spread for each set of data. See Mod. 9 Answer Appendix.

b. Which city do you think Francisca should visit on her vacation? Justify your argument. See Mod. 9 Answer Appendix.

Higher-Order Thinking Skills

29. WRITE Compare and contrast the benefits of displaying data using histograms and box plots. See Mod. 9 Answer Appendix.

30. ANALYZE If every value in a set of data is multiplied by a constant k, $k < 0$, then how can the mean, median, mode, range, and standard deviation of the new data set be found? See Mod. 9 Answer Appendix.

31. PERSEVERE A salesperson has 15 SUVs priced between $33,000 and $37,000 and 5 luxury cars priced between $44,000 and $48,000. The average price for all of the vehicles is $39,250. The salesperson decides to reduce the prices of the SUVs by $2000 per vehicle. What is the new average price for all of the vehicles? $37,750

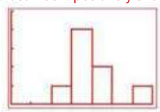
32. ANALYZE If k is added to every value in a set of data, and then each resulting value is multiplied by a constant m, $m > 0$, how can the mean, median, mode, range, and standard deviation of the new data set be found? Justify your argument. See Mod. 9 Answer Appendix.

33. WRITE Explain why the mean and standard deviation are used to compare the center and spread of two symmetrical distributions, and the five-number summary is used to compare the center and spread of two skewed distributions or a symmetric distribution and a skewed distribution. See Mod. 9 Answer Appendix.

Answers

9b. Sample answer: The distributions are skewed, so use the five-number summaries. The medians for both teams are 79. The upper quartile and maximum for the Marlins are 83.5 and 92. The upper quartile and maximum for the Cubs are 88 and 97. This means that the upper 50% of data for the Cubs is slightly higher than the upper 50% of data for the Marlins. Overall, we can conclude that the Cubs were slightly more successful than the Marlins during this time period.

10a. last week: positively skewed; this week: negatively skewed



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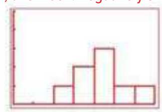

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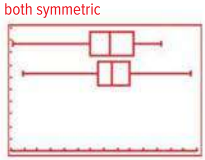
10b. Sample answer: The distributions are skewed, so use the five-number summaries. The median for last week is 30, and for this week is 45. The lower quartile and minimum for both weeks are 30 and 20. The maximum for both weeks is 60. The upper quartile for this week is 47, and for last week it is 45. This means that the middle 50% of data for this week is higher than the middle 50% of data for last week. Overall, we can conclude that the median time spent on the treadmill was higher this week than last week.

24a. both symmetric



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24b. Sample answer: The distributions are symmetric, so use the means and standard deviations. Scott's mean: about 113.9 with standard deviation of about 25.4, Azumi's mean: about 116.3 with standard deviation of about 23.9. Azumi's totals are slightly higher and more consistent than Scott's.

24c. Sample answer: Scott's new mean: about 117.3 and standard deviation 20.5, Azumi's new mean: about 116.8 with standard deviation 15.2. Scott's totals are slightly higher than Azumi's, but Azumi's are slightly more consistent than Scott's.

25a. Sample answer: The mean of Saeed's prices is $11.79, which is $0.80 more than his rival's mean price. The new prices come from subtracting $0.80 from each price, which will reduce the mean price to be the same as his rival's.

New Prices				
14.19	3.69	9.19	17.69	12.19
6.19	7.69	21.19	12.69	13.19
9.19	10.19	11.69	3.69	12.19

25b. Current prices: $\mu = 11.79$, $\sigma = 4.60$
New prices: $\mu = 10.99$, $\sigma = 4.60$
The mean has dropped by 0.8, but the standard deviation has remained constant.

Lesson 9-7

S.ID.5

Summarizing Categorical Data

LESSON GOAL

Students summarize and interpret categorical data using frequency tables.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Categorical Data

Develop:

Two-Way Frequency Tables
- Use a Two-Way Frequency Table

Two-Way Relative Frequency Tables
- Use a Two-Way Relative Frequency Table
- Use a Two-Way Conditional Relative Frequency Table

You may want your students to complete the Checks online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL BL		ELL
Remediation: Two-Way Tables	● ●			●
Extension: Conditional Probability		● ●		●

Language Development Handbook

Assign page 55 of the *Language Development Handbook* to help your students build mathematical language related to summarizing and interpreting categorical data using frequency tables.

ELL You can use the tips and suggestions on page T55 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Statistics and Probability

Standards for Mathematical Content:

S.ID.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

4 Model with mathematics.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students used appropriate measures of center and spread based on the shape of the distribution.
7.SP.4, S.ID.2, S.ID.3

Now
Students will approximate data by using a normal distribution.
S.ID.4 (Course 3)

Rigor

The Three Pillars of Rigor

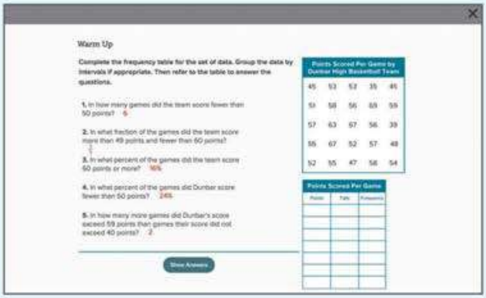
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop understanding of two-way frequency tables and build fluency by making frequency tables and interpreting frequencies. They apply their understanding of two-way frequency tables by solving real-world problems.

Mathematical Background

A two-way frequency table is used to show the frequencies of data from a survey or experiment classified according to two categories, with the rows indicating one category and the columns indicating the other. To create a two-way relative frequency table, divide each of the values by the total number of observations and replace them with their corresponding decimals or percents.

Interactive Presentation



Warm Up

Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- completing frequency tables

Answers:

1. 6
2. $\frac{3}{5}$
3. 16%
4. 24%
5. 2

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of two-way frequency tables.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Categorical Data

Objective

Students explore using a two-way table to organize data.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Ideas for Use

Recommended Use Present the Inquiry question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore activity is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

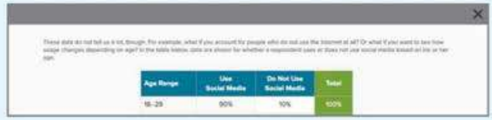
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will explore how a two-way frequency table can provide detailed information about the results of a survey. They will answer a series of questions related to a two-way frequency table displaying data on social media use. Then, students will answer the Inquiry question.

(continued on the next page)

Interactive Presentation

Explore



Explore

Interactive Presentation

Explore

TYPE

Students answer questions about data presented in a two-way table.

TYPE

Students respond to the Inquiry question and can view a sample answer.

Explore Categorical Data (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What information does the second table show that the first does not? social media usage by age
- What would be another type of two-way frequency table that could be created for social media usage? Sample answer: social media usage by gender

Inquiry

What is the advantage of organizing data in a two-way table?
Sample answer: The table displays how people's responses are or are not related to other characteristics of the people responding to the question.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Two-Way Frequency Tables

Objective

Students organize and determine categorical data in a two-way frequency table.

Teaching the Mathematical Practices

7 Use Structure Students will use the structure of a two-way frequency table to explore how it represents data.

What Students Are Learning

Students are learning how to read a two-way frequency table. They learn how to identify the subcategories, joint frequencies, and marginal frequencies and what each of these represents in the given context.

Common Misconception

Some students may have a misconception about what a two-way frequency table indicates about a set of data. If this is students' first experience working with such tables, it may be helpful to spend some time discussing what each value in the table indicates about the data (in context).

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 9-7

Summarizing Categorical Data

Today's Goals
- Organize categorical data in a two-way frequency table.
- Determine and interpret the values in a two-way relative frequency table.

Today's Vocabulary
- two-way frequency table
- joint frequencies
- marginal frequencies
- relative frequency
- two-way relative frequency table
- conditional relative frequency

Explore Categorical Data

Online Activity Use a real-world situation to complete the Explore.

INQUIRY What is the advantage of organizing data in a two-way table?

Learn Two-Way Frequency Tables

A **two-way frequency table** or *contingency table* is used to show the frequencies of data from a survey or experiment classified according to two categories, with the rows indicating one category and the columns indicating the other.

Suppose you are constructing a two-way frequency table based on two categories, grade level and employment. The table is constructed below for sample values.

Grade	Employed	Unemployed	Totals
Junior	8	12	20
Senior	15	10	25
Totals	23	22	45

Subcategories: The subcategories are the column and row headers that represent the two different types of categories. In this case, Employed, Unemployed, Junior, and Senior are the subcategories.

Joint frequencies: **Joint frequencies** are the values for every combination of subcategories. So, 8 is a joint frequency that represents the number of students who are employed and juniors.

Marginal frequencies: **Marginal frequencies** are the totals of each subcategory. So, 20 is a marginal frequency that represents the total number of juniors.

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Interactive Presentation

Learn

TAP

Students tap buttons to see the breakdown of data in a two-way frequency table.

Study Tip

Check For each cell, you can see if your calculations are correct by calculating the value for that cell using different data from the table. For example, you could calculate the total number of female participants in the study either by adding the number of women named Casey or Riley, or by subtracting the number of men from the total number of people named Casey or Riley. Either way, you should get the same number.

Use a Source

Create your own two-way frequency table. Find data online that divides a group of subjects into two categories, with each subject fitting into one subcategory of each. For example, in the data shown the categories are whether each person is male or female and whether each person's name is Casey or Riley. Determine the subcategories, enter the given data, and fill in any cells for which values are not provided.

Example 1 Use a Two-Way Frequency Table

NAMES Unisex names are names often used for both males and females. Until 2013, the top two unisex names in the U.S. were Casey and Riley, with 176,544 Caseys and 154,861 Rileys. There are 104,161 males with the name Casey and 75,882 females with the name Riley. Organize the data in a two-way frequency table.

Steps 1 and 2 Enter the given data in a table. Then use the information given to fill in the rest of the cells.

Top Unisex Names in the U.S.			
	Casey	Riley	Totals
Male	104,161	78,979	183,140
Female	72,383	75,882	148,265
Totals	176,544	154,861	331,405

Male Rileys: 154,861 − 75,882 = 78,979

Total Males: 104,161 + 78,979 = 183,140

Female Caseys: 176,544 − 104,161 = 72,383

Total Females: 72,383 + 75,882 = 148,265

Totals: 176,544 + 154,861 = 331,405

Check

TECHNOLOGY Pew Research Center released a survey that asked whether participants thought technological advancements in the future will make people's lives better or worse. Of the people interviewed, 423 earned less than $50,000 per year and 328 earned $50,000 or more. Of those earning less than $50,000 per year, 262 thought that people's lives would get better, and 240 of those who earned $50,000 or more thought the same. Copy and complete the two-way frequency table.

Will technological advancements in the future make people's lives better or worse?			
	Better	Worse	Totals
< $50,000	262	161	423
≥ $50,000	240	88	328
Totals	502	249	751

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 1

SWIPE

Students move through the slides to complete a two-way frequency table.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Use a Two-Way Frequency Table

Teaching the Mathematical Practices

5 Use a Source Guide students to find external information to answer the questions posed in the Use a Source feature.

Questions for Mathematical Discourse

AL How can you determine the number of males named Riley? Sample answer: You can subtract the number of females named Riley from the total number of Rileys. 154,861 − 75,882 = 78,979

OL What are two ways to find the number that goes in the last row of the last column? Sample answer: Add the numbers in the two cells above it, or add the numbers in the two cells to the left of it.

BL What does the first number in the last column represent? the total number of males included in the sample

Common Error

Students may place one of the given data values in the wrong cell, which then affects the calculations of values for other cells. Prompt students to reread the problem after they have placed the given data into the table, and check that their entries are in the correct cells.

DIFFERENTIATE

Enrichment Activity AL BL

Have students work in pairs to create a survey that can be used to gather data that can be represented in a two-way relative frequency table. Have them survey their classmates, gather the data, construct the table, and summarize their findings. Then have students share their results with the class.

Learn Two-Way Relative Frequency Tables

Objective

Students determine and interpret the values in a two-way relative frequency table.

MP Teaching the Mathematical Practices

7 Use Structure Students will focus on the structure of two-way relative frequency tables and two-way conditional relative frequency tables to understand how they can be constructed from the data in a two-way frequency table.

Important to Know

Because the relative frequencies are often numbers that have been rounded to the nearest percent, the totals in each row and column, as calculated by division, may not be equal to the sum of the percents in the related row or column.

Common Misconception

Some students may misconceive the meaning of the percents in a relative or conditional relative frequency table. It may be helpful to spend some time having students write statements that summarize what each percent in the given table represents.

Example 2 Use a Two-Way Relative Frequency Table

Questions for Mathematical Discourse

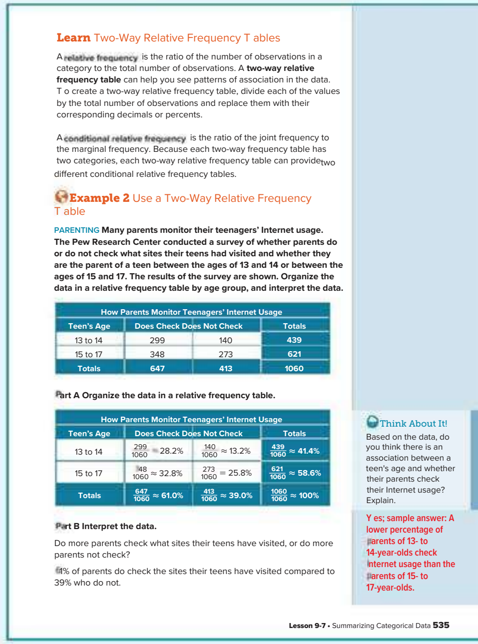
AL What is the total number of parents that were surveyed? 1060

OL What must the sum of the joint frequencies always be? about 100% Why? Sample answer: The sum represents the entire sample.

BL What does the first joint relative frequency (28.2%) represent? the percent of parents surveyed that check the Internet usage of their 13-to-14-year-old children

Common Error

Some students may have difficulty using the table to complete statements like the one in **Part B**. Help students to see how they can use the row and column headings in the table to help guide them to the cells whose entries will enable them to complete the statement correctly.



Learn Two-Way Relative Frequency T ables

A **relative frequency** is the ratio of the number of observations in a category to the total number of observations. A **two-way relative frequency table** can help you see patterns of association in the data. T o create a two-way relative frequency table, divide each of the values by the total number of observations and replace them with their corresponding decimals or percents.

A **conditional relative frequency** is the ratio of the joint frequency to the marginal frequency. Because each two-way frequency table has two categories, each two-way relative frequency table can provide two different conditional relative frequency tables.

Example 2 Use a Two-Way Relative Frequency T able

PARENTING **Many parents monitor their teenagers' Internet usage. The Pew Research Center conducted a survey of whether parents do or do not check what sites their teens had visited and whether they are the parent of a teen between the ages of 13 and 14 or between the ages of 15 and 17. The results of the survey are shown. Organize the data in a relative frequency table by age group, and interpret the data.**

How Parents Monitor Teenagers' Internet Usage			
Teen's Age	Does Check	Does Not Check	Totals
13 to 14	299	140	439
15 to 17	348	273	621
Totals	647	413	1060

Part A Organize the data in a relative frequency table.

How Parents Monitor Teenagers' Internet Usage			
Teen's Age	Does Check	Does Not Check	Totals
13 to 14	$\frac{299}{1060} \approx 28.2\%$	$\frac{140}{1060} \approx 13.2\%$	$\frac{439}{1060} \approx 41.4\%$
15 to 17	$\frac{348}{1060} \approx 32.8\%$	$\frac{273}{1060} = 25.8\%$	$\frac{621}{1060} \approx 58.6\%$
Totals	$\frac{647}{1060} \approx 61.0\%$	$\frac{413}{1060} \approx 39.0\%$	$\frac{1060}{1060} \approx 100\%$

Part B Interpret the data.

Do more parents check what sites their teens have visited, or do more parents not check?

61% of parents do check the sites their teens have visited compared to 39% who do not.

Think About It! Based on the data, do you think there is an association between a teen's age and whether their parents check their Internet usage? Explain.

Y es; sample answer: A lower percentage of parents of 13- to 14-year-olds check Internet usage than the parents of 15- to 17-year-olds.

Lesson 9-7 • Summarizing Categorical Data **535**

Interactive Presentation

Learn

TAP

Students tap buttons to see the breakdown of data in a two-way relative frequency table.

Think About It!

If the condition for the relative frequency table were whether a person had voted or not rather than age group, the joint frequency for people between the ages of 18 and 24 who voted would be 52.5%. If someone claims that this indicates 52.5% of all people who voted in the 2012 election were between the ages of 18 and 24, would they be correct? Justify your argument.

No; sample answer: 52.5% represents the number of voters who were between the ages of 18 and 24 among voting aged 18 to 24 and voters aged 75 and over, not the entire voting population.

Example 3 Use a Two-Way Conditional Relative Frequency Table

VOTING **According to the U.S. Census Bureau, voter turnout describes how many eligible voters show up to vote in an election. The table shows the number of eligible voters who did and did not vote in 2012 for the oldest and youngest eligible age groups. Organize the data in a conditional relative frequency table by age group, and interpret the data.**

Voter Turnout			
Age Group	Voted	Did Not Vote	Totals
18 to 24	12,515	13,275	25,790
75 and over	11, 344	5380	6,724
Totals	23,859	18,655	42,514

Part A Organize the data in a conditional relative frequency table by age group.

Step 1 Determine which marginal frequencies to use.

The conditional relative frequency relates the number of voters or nonvoters to the age group, so the relevant marginal frequencies are the total numbers of voters for each age group.

Step 2 Determine the ratios of the joint frequencies to the marginal frequencies.

Voter Turnout			
Age Group	Voted	Did Not Vote	Totals
18 to 24	$\frac{12,515}{25,790} \approx 48.5\%$	$\frac{13,275}{25,790} \approx 51.5\%$	$\frac{25,790}{25,790} = 100\%$
75 and over	$\frac{11,344}{16,724} \approx 67.8\%$	$\frac{5380}{16,724} \approx 32.2\%$	$\frac{16,724}{16,724} = 100\%$

Part B Interpret the data.

Which age group has the higher voter turnout?

The percent of eligible voters aged 18 to 24 that voted is 48.5%, and the percent for those aged 75 and over is 67.8%. Based on the data, there is an association between age and whether a person voted. People aged 18 to 24 were more likely to not have voted than people aged 75 and over.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 3

EXPAND

Students tap to show how to organize data in a two-way conditional relative frequency table.

TYPE

Students answer a question to show they understand conditional relative frequency tables.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Use a Two-Way Conditional Relative Frequency Table

Teaching the Mathematical Practices

7 Use Structure Help students use the structure of conditional frequency tables in this example to organize and interpret the data.

Questions for Mathematical Discourse

AL What does the entry 13,275 represent? the number of 18- to 24-year olds that did not vote

OL Why do you eliminate the bottom row of the table in **Part A**? Sample answer: Eliminate the bottom row of the table because the problem asks for a conditional relative frequency table by age group, and the age groups are represented by the rows. The totals in the bottom row are for the columns and are not relevant for the conditional relative frequency table.

BL In Step 2, how do you decide what number to divide each entry by? Explain. Sample answer: To organize the data by age group, divide each entry by the total number of people in that age group.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–10
2	exercises that use a variety of skills from this lesson	11–32
2	exercises that extend concepts learned in this lesson to new contexts	33, 34
3	exercises that emphasize higher-order and critical-thinking skills	35–38

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–31 odd, 35–38
- Extension: Conditional Probability
- ALEKS Data Analysis

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–37 odd
- Remediation, Review Resources: Two-Way Tables
- Personal Tutors
- Extra Examples 1–3
- ALEKS Completing Frequency Tables

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–9 odd
- Remediation, Review Resources: Two-Way Tables
- *Quick Review Math Handbook*: Two-Way Frequency Tables
- Arrive**MATH** Take Another Look
- ALEKS Completing Frequency Tables

Answers

1.

	Small	Large	Total
Cherry	35	20	55
Grape	25	15	40
Watermelon	15	15	30
Total	75	50	125

Practice

Go Online Y ou can complete your homework online.

Example 1

TREATS **The owner of a snow cone stand keeps track of the sizes and flavors sold one afternoon. He sold 125 snow cones in all. Of these, 40% were large snow cones, 32% were grape, and 12% were small watermelon snow cones. The stand sold 15 more cherry snow cones than grape. The most popular snow cone of the day was small cherry, with a total of 35 sales.**

1. Construct a two-way frequency table to organize the data. See margin.

2. How many large grape snow cones were sold? 15

3. How many watermelon snow cones were sold in all? 30

4. How many more small snow cones were sold than large snow cones? 25

Example 2

FOREIGN LANGUAGE **Christy surveyed several students at her school and asked each person what foreign language he or she is studying. The results are shown in the table.**

	Male Female Total		
Spanish	18	20	38
French	16	12	28
German	6	8	14
T otal	40	40	80

5. Construct a relative frequency table by converting the data in the table to percentages. Round to the nearest tenth, if necessary. See margin.

6. Find the joint relative frequency of a female student who is studying French. 15%

7. Interpret the data. Sample answer: Most of the students are studying Spanish.

Example 3

CLASS PRESIDENT **In a poll for senior class president, 68 of the 145 male students said they planned to vote for Santiago. Out of 139 female students, 89 planned to vote for his opponent, Measha.**

8. Construct a conditional relative frequency table based on voter preference. Show your calculations. See margin.

9. What does each conditional relative frequency represent? Sample answer: Each conditional relative frequency represents the proportion of each candidate's support from each gender.

10. What is the probability that a vote for Measha will come from a female student? How is this different from the probability that a female student intends to vote for Measha? $\frac{89}{166} \approx 54\%$; Sample answer: This probability represents the proportion of Measha's support that is female rather than the proportion of female students voting for Measha, which is 64%.

Mixed Exercises

VETERINARIAN **The two-way frequency table shows the number of dogs and cats that were seen at a veterinarian's office and the primary purpose of their visit.**

	Dog	Cat	Total
Exam	12	5	17
Shots	6	3	9
Grooming 7		2	9
T otal	25	10	35

11. How many dogs were seen for an exam today? 12

12. How many more dogs than cats were seen at the veterinarian's office? 15

BIRD WATCHING **A group of bird-watchers has been tracking the number of tree swallows, cardinals, and goldfinches in a region. Over the weekend, a total of 40 birds were observed. Of those, 45% were male, 37.5% were cardinals, and 12.5% were male tree swallows. Twice as many female cardinals were observed as male cardinals. There were 5 female goldfinches spotted.**

13. Construct a two-way frequency table to organize the data. See margin.

14. How many more female tree swallows were seen than male cardinals? 2

15. How many male goldfinches and female cardinals were seen? 18

16. How many more female birds were seen than male birds? 4

SCHOOL ACTIVITIES **The two-way frequency table shows the number of students who participate in school sports or clubs at Monroe High School.**

	Sports or No Sports Clubs or Clubs		Total
Freshmen	48	60	108
Sophomores 60		72	132
Juniors	51	69	120
Seniors	57	63	120
T otal	216	264	480

17. Construct a relative frequency table by converting the data in the table to percentages. Round to the nearest tenth, if necessary. See margin.

18. Find the joint relative frequency of a sophomore who participates in school sports or clubs. 12.5%

19. What percentage of freshmen do not participate in school sports or clubs? Round to the nearest tenth percent, if necessary. 55.6%

20. What percentage of seniors participate in school sports or clubs? Round to the nearest tenth percent, if necessary. 47.5%

SCHOOL MASCOT **The freshmen and sophomores at Lakeview High School are tasked with adopting a new school mascot next school year. The district asked a representative group of students to vote for one of the three mascot finalists and to indicate to which grade they belong. The results are shown in the table.**

School Mascot Vote Results			
	Freshmen	Sophomores	Total
Panthers	30	36	66
Hornets	17	33	50
Lions	28	31	59
T otal	75	100	175

21. How many students voted for Panthers? 66

22. How many students voted for Lions? 59

23. How many sophomores were in the representative group? 100

24. Of the students who voted for Hornets, how many of them are freshmen? 17

25. Of the students who voted for Lions, how many of them are sophomores? 31

26. T o the nearest whole, what percent of all the students voted for Lions? 34%

27. T o the nearest whole, what percent of all the students voted for Panthers? 38%

28. T o the nearest whole, what percent of all the students who voted were freshmen? 43%

THANKSGIVING PIE **An online poll collected a sample of Thanksgiving pie preferences for different U.S. regions.**

Region	Apple	Sweet Potato	Pumpkin	Totals
West	77	4	13	
Midwest	32		54	
South		63	24	
Northeast	92	2		
T otal	213	75	117	

29. PRECISION Copy and complete the table. Then find each relative frequency to the nearest tenth of a percent. See margin.

30. USE A MODEL Assuming the poll is representative of the whole population, what is a reasonable estimate of the probability that a family will be from the northeast and will be eating pumpkin pie on Thanksgiving? A reasonable estimate is the corresponding relative frequency, 6.4%.

31. STRUCTURE Construct a table of conditional relative frequencies based on pie preference. Round each percent to the nearest tenth. Interpret the meaning of the probabilities in the context of the problem. See margin.

32. REGULARITY If we had found the conditional relative frequencies by dividing by the total replies from each region, what would be the meaning of the probability in each cell? Sample answer: The percent in each cell would represent the probability of a person from that region preferring that type of pie. For example, the entry in the row for West and the column for Apple would be 77/ 94 ≈ 81.9% and that means that there is an 81.9% probability that a person from the West prefers apple pie.

VEHICLES **The table shows the relative frequencies of drive systems for different vehicle types in a school parking lot. There are 215 vehicles in the lot.**

Vehicle Type	2WD	AWD	Totals
Hatchbacks	42%	4%	
Sedans	28%	6%	
SUVs	1%	19%	
T otal			215

33. USE TOOLS Construct a table to show the joint and marginal frequencies. See Mod. 9 Answer Appendix.

34. REASONING Without calculating individual frequencies, how many times greater will the conditional relative frequencies based on drive systems for AWD be than the relative frequencies for AWD, and why? See Mod. 9 Answer Appendix.

Higher-Order Thinking Skills

35. PERSEVERE Len conducted a survey among a random group of 1000 families in his home state of California. He wanted to determine whether there is an association between gasoline prices and distances traveled on family vacations. He collected the following information. According to Len's two-way frequency table, does there appear to be an association between gasoline prices and vacation distances traveled? Explain. See Mod. 9 Answer Appendix.

	\$1.75–\$3.24 per gallon	\$3.25-\$4.74 per gallon	Total
Less than 250 miles	109	255	364
More than 250 miles	329	86	415
No vacation travel	34	187	221
Total	472	528	1000

36. CREATE Select your own data for a two-way frequency table, write a question related to the data in the table, and provide the solution. See Mod. 9 Answer Appendix.

37. WRITE Compare two-way relative frequency tables and two-way conditional relative frequency tables. See Mod. 9 Answer Appendix.

38. FIND THE ERROR Magdalena took a survey of students in her school to find out what snack was most popular.

Favorite Snack Vote Results			
	Freshmen	Sophomores	Total
Fruit Snack	65	61	126
Granola	27	21	48
Y ogurt	21	18	39
T otal	113	100	213

a. Interpret the data based on the conditional relative frequency related to age groups. See Mod. 9 Answer Appendix.

b. Magdalena claims that fruit snack is the most popular snack for freshmen and sophomores, and Ben claims that a higher percentage of sophomores prefer fruit snack than do freshmen. Is either correct? Explain your reasoningSee Mod. 9 Answer Appendix.

Answers

5.

	Male	Female	Total
Spanish	22.5%	25%	47.5%
French	20%	15%	35%
German	7.5%	10%	17.5%
Total	50%	50%	100%

8.

	Santiago	Measha
Female	$\frac{50}{118} \approx 42\%$	$\frac{89}{166} \approx 54\%$
Male	$\frac{68}{118} \approx 58\%$	$\frac{77}{166} \approx 46\%$
Total	100%	100%

13.

	Male	Female	Total
Tree Swallow	5	7	12
Cardinal	5	10	15
Goldfinch	8	5	13
Total	18	22	40

17.

	Sports or Clubs	No Sports or Clubs	Total
Freshmen	10%	12.5%	22.5%
Sophomores	12.5%	15%	27.5%
Juniors	10.6%	14.4%	25%
Seniors	11.9%	13.1%	25%
Total	45%	55%	100%

29.

	Apple	Sweet Potato	Pumpkin	Totals
West	77 ≈ **19.0%**	4 ≈ **1.0%**	13 ≈ **3.2%**	94 ≈ 23.2%
Midwest	32 ≈ **7.9%**	6 ≈ **1.5%**	54 ≈ **13.3%**	92 ≈ 22.7%
South	12 ≈ **3.0%**	63 ≈ **15.6%**	24 ≈ **5.9%**	99 ≈ 24.4%
Northeast	2 ≈ **22.7%**	2 ≈ **0.5%**	26 ≈ **6.4%**	120 ≈ 29.6%
Total	213 ≈ 52.6%	75 ≈ 18.5%	117 ≈ 28.9%	405 = 100%

31. Sample answer: The conditional relative frequencies based on pie preference give the probability of a person preferring a particular pie choice being from one of the U.S. regions. For example, there is an 84% probability that a person who prefers sweet potato pie is from the south.

	Apple	Sweet Potato	Pumpkin
West	36.2%	5.3%	11.1%
Midwest	15.0%	8.0%	46.2%
South	5.6%	84%	20.5%
Northeast	43.2%	2.7%	22.2%
Total	100%	100%	100%

Lesson 9-8
Normal Distributions

MAFS.912.S-ID.1.4

LESSON GOAL

Construct probability distributions and use normal distributions to analyze data.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Probability Distributions
- Analyze a Probability Distribution

The Normal Distribution
- Approximate Data by Using a Normal Distribution
- Use the Empirical Rule to Analyze Data

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Measures of Spread	●	●		●
Extension: Sample Deviation of Sample Data		●	●	●

Language Development Handbook

A variety of resources are available to support students as they build mathematical language and understanding of key math concepts, including:

- Scaffolds and supports in the *Language Development Handbook*
- Activities designed to build mathematical discourse

ELL You can use these resources as well as point-of-use ELL tips and strategies to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Statistics

Standards for Mathematical Content:

MAFS.912.S-ID.1.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

Standards for Mathematical Practice:

5 Use appropriate tools strategically.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students summarized and interpreted categorical data using frequency tables. **S.ID.5, MAFS.912.S-ID.2.5**

Now
Students construct probability and normal distributions.
MAFS.912.S-ID.1.4

Next
Students will graph and analyze rational functions.
MAFS.912.F-IF.3.7d

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of distributions by extending to normal distributions and build fluency by estimating population percentages. They apply their understanding of normal distributions by solving real-world problems.

Mathematical Background

The graphs of all normally distributed variables have essentially the same shape. With appropriate labeling of the mean and the points that are one standard deviation from the mean, the same normal curve can represent any normal distribution.

Interactive Presentation

Warm Up

Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skills for this lesson:

- finding the variance
- finding the standard deviation

Answers:

1. 17.5
2. 19
3. 20.5
4. 22
5. 23.5
6. 25
7. 26.5

Launch the Lesson

Teaching the Mathematical Practices

8 Look for and express repeated reasoning Encourage students to look for the repeated reasoning with the normal distribution and large data sets, such as heights, weights, and test scores.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Probability Distributions

Objective

Students construct probability distributions.

Teaching the Mathematical Practices

5 Use appropriate tools strategically Students will use technology to construct a probability distribution.

About the Key Concept

A random variable is a variable with possible values that are the outcomes of a random event. A probability distribution is a mapping of those outcomes to their probabilities of occurrence. It is usually shown as a histogram or bar graph. The probability distribution of a random variable *X* must satisfy these conditions: the probability of each value of the random variable *X* must be between 0 and 1, and the sum of the probabilities of all of the values of *X* must equal 1. A discrete random variable is finite and can be counted, and are represented by a bar graph. A continuous random variable can take on any value, and are represented by a histogram.

Go Online

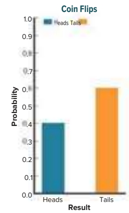
- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 9-8

Normal Distributions

Learn Probability Distributions

A **random variable** is a variable with possible values that are the outcomes of a random event. A **probability distribution** is a mapping of those outcomes to their probabilities of occurrence. It is usually shown as a histogram or bar graph. For example, if the random variable X represents the outcomes of 20 coin flips, a bar graph representing the results 8 heads and 12 tails might have bars showing $\frac{8}{20}$, or 0.4 for heads and $\frac{12}{20}$, or 0.6 for tails.



Key Concept • Conditions for Probability Distributions

The probability distribution of a random variable X must satisfy these conditions.

- The probability of each value of the random variable X must be between 0 and 1.
- The sum of the probabilities of all of the values of X must equal 1.

There are two types of random variables and distributions. A **discrete random variable** is finite and can be counted. The outcomes for flipping a coin are discrete because there are only two—heads or tails. A discrete distribution can be represented by a bar graph.

A **continuous random variable** can take on any value. The outcomes for teens' ages are continuous. A continuous distribution can be represented by a histogram.

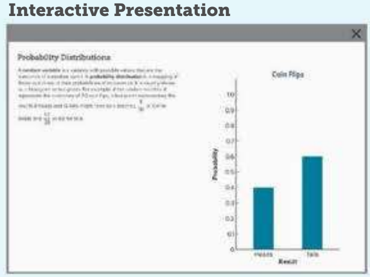
Today's Vocabulary
random variable
probability distribution
discrete random variable
continuous random variable
normal distribution

Think About It!
Why must the probability of each value of a random variable be between 0 and 1?

Sample answer: A probability of 0 means that there is 0% chance that an event will occur. A probability cannot be negative because it cannot get any less certain than 0%. A probability of 1 means that it is certain, or there is a 100% chance, that the event will happen. A probability greater than 1 implies that it is more certain than 100%, which is impossible.

Lesson 9-8 • Normal Distributions 541

Interactive Presentation



Learn

TYPE

Students explain why the probability of each value must be between 0 and 1.

Study Tip

Discrete vs. Continuous Variables representing height, weight, and capacity will always be continuous variables because they can take on any positive value.

Example 1 Analyze a Probability Distribution

GROCERIES **A grocery store chain measures the lengths of time the employees take to scan and bag customers' items. The frequency of each time interval is given. Construct a probability distribution to represent the data.**

Time (X)	Frequency
0–0:59	12
1–1:59	27
2–2:59	38
3–3:59	25
4–4:59	9

Step 1 Construct a relative frequency table.

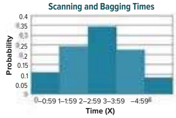
The relative frequency table converts the frequencies to probabilities. Divide each frequency by the total number of measurements, 111. Round your answers to the nearest thousandth.

Time (X)	Frequency	Relative Frequency
0–0:59	12	0.108
1–1:59	27	0.243
2–2:59	38	0.342
3–3:59	25	0.225
4–4:59	9	0.081

Step 2 Graph the probability distribution.

The bars are not separated on the graph because the distribution is continuous.

Scanning and Bagging Times



Talk About It!

If a histogram that measures frequency is symmetric about the mean, will its probability distribution necessarily be symmetric? Explain your reasoning.

Yes; sample answer: If the mean occurs the most frequently, and instances of the variable occur less frequently as you get further from the mean, then the relative frequency will still be the highest bar, with bars getting shorter as you get farther from the mean.

Learn The Normal Distribution

The **normal distribution** is the most common continuous probability distribution. It is bell-shaped and symmetric about the mean.

Go Online You can complete an Extra Example online.

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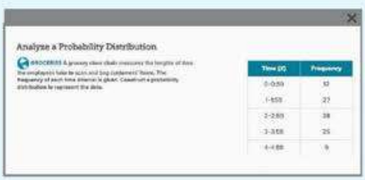
Interactive Presentation



Example 1

TYPE

Students enter the relative frequencies to complete the probability distribution table.

TYPE

Students determine if the probability distribution for a histogram that is symmetric about the mean will also be symmetric.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Analyze a Probability Distribution

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to construct the probability distribution and its graph, students will need to use pencil and paper, a calculator, or a statistical package.

Leveled Discussion Questions

AL What does a relative frequency represent? Sample answer: The number of data values in an interval out of the total number of data values, which gives the probability of data being within each interval.

OL Why can the endpoint of one interval not be the same as the beginning point of the next interval? Sample answer: Because if a data value was the value of the endpoint, it could be placed in both intervals and be counted twice.

BL In this situation, why do you think the probability distribution is symmetric about the mean? Sample answer: Some customers will have only a few items and other customers may have a large amount, but most customers will probably buy similar amounts of items.

Common Error

Encourage students to round correctly when constructing probability distributions. Since the probability of each interval should sum to 1, rounding errors may affect this rule of a probability distribution.

Learn The Normal Distribution

Objective

Students determine if data sets are normally distributed and apply the Empirical Rule.

Teaching the Mathematical Practices

5 Use appropriate tools strategically Students will use technology to analyze the area under a normal distribution for a given interval.

8 Look for and express repeated reasoning Students will understand how the mean, standard deviation, and a normal distribution relate in order to solve a problem.

About the Key Concept

The normal distribution is the most common continuous probability distribution. The graph of a normal distribution is continuous, bell-shaped, and symmetric with respect to the mean. The mean, median, and mode are equal and located at the center. The curve approaches, but never touches, the *x*-axis. The total area under the curve is equal to 1 or 100%. The area under the curve between two values for *X* represents the probability that a data point will fall in that interval.

Common Misconception

A common misconception some students may have is that any symmetric distribution is normally distributed. Symmetric does not imply normal as data sets with two peaks can be symmetric without be normally distributed. Reinforce that normal distributions are bell-shaped and symmetric about the mean.

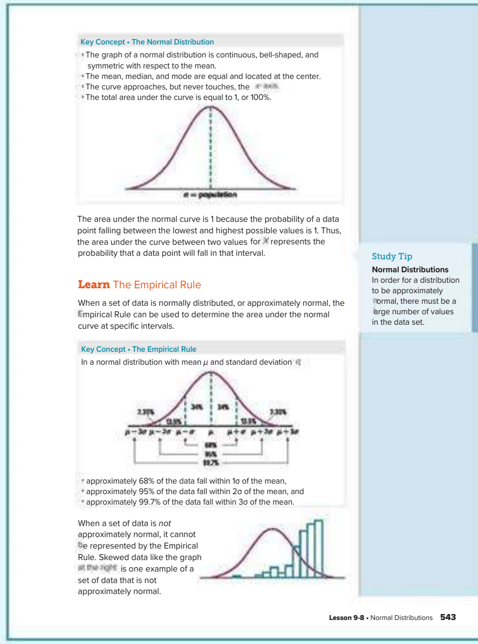
Learn The Empirical Rule

Objective

Students determine if data sets are normally distributed and apply the Empirical Rule.

About the Key Concept

When a set of data is normally distributed, or approximately normal, the Empirical Rule can be used to determine the area under the normal curve at specific intervals. Approximately 68% of the data fall within 1σ of the mean, approximately 95% of the data fall within 2σ of the mean, and approximately 99.7% of the data fall within 3σ of the mean.



Key Concept • The Normal Distribution

- The graph of a normal distribution is continuous, bell-shaped, and symmetric with respect to the mean.
- The mean, median, and mode are equal and located at the center.
- The curve approaches, but never touches, the x-axis.
- The total area under the curve is equal to 1, or 100%.

The area under the normal curve is 1 because the probability of a data point falling between the lowest and highest possible values is 1. Thus, the area under the curve between two values for X represents the probability that a data point will fall in that interval.

Study Tip

Normal Distributions In order for a distribution to be approximately normal, there must be a large number of values in the data set.

Learn The Empirical Rule

When a set of data is normally distributed, or approximately normal, the Empirical Rule can be used to determine the area under the normal curve at specific intervals.

Key Concept • The Empirical Rule

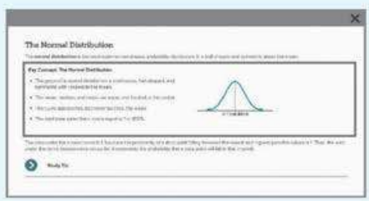
In a normal distribution with mean μ and standard deviation σ,

- approximately 68% of the data fall within 1σ of the mean,
- approximately 95% of the data fall within 2σ of the mean, and
- approximately 99.7% of the data fall within 3σ of the mean.

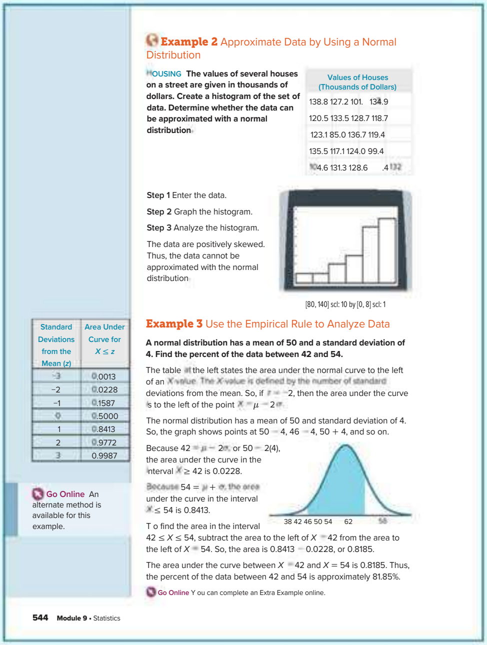
When a set of data is *not* approximately normal, it cannot be represented by the Empirical Rule. Skewed data like the graph at the right is one example of a set of data that is not approximately normal.

Lesson 9-8 • Normal Distributions 543

Interactive Presentation



Learn



Example 2 Approximate Data by Using a Normal Distribution

HOUSING **The values of several houses on a street are given in thousands of dollars. Create a histogram of the set of data. Determine whether the data can be approximated with a normal distribution**

Values of Houses (Thousands of Dollars)			
138.8	127.2	101.	134.9
120.5	133.5	128.7	118.7
123.1	85.0	136.7	119.4
135.5	117.1	124.0	99.4
104.6	131.3	128.6	.4 132

Step 1 Enter the data.

Step 2 Graph the histogram.

Step 3 Analyze the histogram.

The data are positively skewed. Thus, the data cannot be approximated with the normal distribution

[80, 140] scl: 10 by [0, 8] scl: 1

Example 3 Use the Empirical Rule to Analyze Data

A normal distribution has a mean of 50 and a standard deviation of 4. Find the percent of the data between 42 and 54.

Standard Deviations from the Mean (z)	Area Under Curve for $X \le z$
−3	0.0013
−2	0.0228
−1	0.1587
0	0.5000
1	0.8413
2	0.9772
3	0.9987

The table at the left states the area under the normal curve to the left of an X-value. The X-value is defined by the number of standard deviations from the mean. So, if $z = -2$, then the area under the curve is to the left of the point $X = \mu - 2\sigma$.

The normal distribution has a mean of 50 and standard deviation of 4. So, the graph shows points at 50 − 4, 46 − 4, 50 + 4, and so on.

Because $42 = \mu - 2\sigma$, or $50 - 2(4)$, the area under the curve in the interval $X \ge 42$ is 0.0228.

Because $54 = \mu + \sigma$, the area under the curve in the interval $X \le 54$ is 0.8413.

38 42 46 50 54 62 58

To find the area in the interval $42 \le X \le 54$, subtract the area to the left of $X = 42$ from the area to the left of $X = 54$. So, the area is 0.8413 − 0.0228, or 0.8185.

The area under the curve between $X = 42$ and $X = 54$ is 0.8185. Thus, the percent of the data between 42 and 54 is approximately 81.85%.

Go Online An alternate method is available for this example.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 2

TAP

Students move through the steps to determine if the data are approximately normally distributed.

Example 2 Approximate Data by Using a Normal Distribution

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to construct the probability distribution and its graph, students will need to use pencil and paper and a graphing calculator.

Leveled Discussion Questions

AL What does a normal curve look like? Sample answer: bell shaped and symmetric about the mean

OL Why are positively skewed data not considered normally distributed? Sample answer: Because a normal curve is bell shaped and symmetric about the mean. If a distribution is skewed, there will not be an even number of data points to either side of the mean.

BL Suppose a histogram reveals a distribution seems to be symmetric about the mean. How can we further verify the distribution is approximately normal? Sample answer: Find the percent of data within one standard deviation, two standard deviations, and three standard deviations. Those percentages should be roughly 68%, 95% and 99.7%.

Example 3 Use the Empirical Rule to Analyze Data

Leveled Discussion Questions

AL How many standard deviations away from the mean is 42? 2 How many standard deviations away from the mean is 54? 1

OL To find the percent of data less than 42, why do we use −2 on the table? Sample answer: −2 represents the number of standard deviations 42 is below the mean.

BL Between what other two values will approximately 81.85% of the data be? 46 and 58

Exit Ticket

Recommended Use

At the end of class, have students respond to the Exit Ticket prompt using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, have students respond to the Exit Ticket prompt verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–12
2	exercises that use a variety of skills from this lesson	13–14
3	exercises that emphasize higher-order and critical-thinking skills	15–18

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–13 odd, 15–18
- Extension: Sample Deviation of Sample Data
- ALEKS

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Measures of Spread
- Personal Tutors
- Extra Examples 1–3
- ALEKS Population Standard Deviation

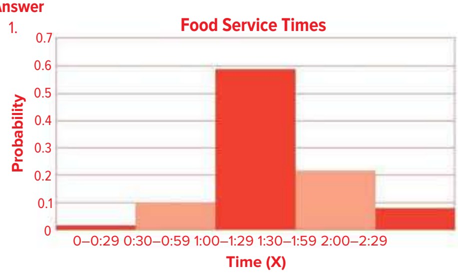
AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–11 odd
- Remediation, Review Resources: Measures of Spread
- Arrive**MATH** Take Another Look
- ALEKS Population Standard Deviation

Answer

1.



Practice

Go Online Y ou can complete your homework online.

Example 1 1–4. See margin.

Construct a probability distribution to represent each set of data.

1. FOOD SERVICE The table shows the length of time each customer spent in the drive-thru line one day at a fast-food restaurant.

Time (X)	Frequency
0–0:29	4
0:30–0:59	22
1:00–1:29	131
1:30–1:59	49
2:00–2:29	18

2. FOOD The table shows the numbers of packages of the five most popular flavors of bagels sold in the U.S. in a recent year.

Flavor (X)	Packages (millions)
plain	136
cinnamon raisin	56
everything	40
blueberry	38
100% whole wheat	21

3. SOCIAL MEDIA The table shows the responses teens had to the question "How many new friends have you met online?"

Time (X)	Frequency
0	456
1	22
2–5	233
6 or more	307

4. GOVERNMENT The table shows the ages of the U.S. presidents at their inauguration.

Age (X)	Presidents
41–45	2
46–50	8
51–55	16
56–60	9
61–65	7
66–70	3

Example 2

Determine whether each set of data can be approximated with a normal distribution. Explain your reasoning.

5.

Values of Used Cars (thousands of dollars)
4.8 7.2 10.1 4.9 13.9
12.7 13.1 4.2 11.8 12.6
13.1 5.6 11.7 13.4 11.3
9.5 7.7 12.1 9.4 5.6

No; the data are not approximately symmetric.

6.

Speeds of Cars on I–71 (mph)
65 66 61 69 68
68 71 62 66 65
67 60 72 67 65
68 62 66 67 68
60 66 69 71 66

Yes; the data are approximately symmetric.

7.

Men's Shot Put Distances (m)
21.30 19.49 18.58 20.08 19.70
18.91 18.21 18.97 19.26 18.49
18.31 18.73 19.53 18.81 19.63
18.94 17.57 17.09 20.38 18.89
18.60 17.19 18.63 18.52 18.67

Yes; the data are approximately symmetric.

8.

Women's 400 m Relay Times (s)
42.91 44.41 44.58 43.34 43.45
44.73 43.08 45.09 44.71 44.63
44.44 44.27 43.85 43.76 44.65
44.85 43.73 45.12 44.47 44.92
44.54 44.51 44.68 44.61 44.71

No; the data are positively skewed.

Example 3

A normal distribution has a mean of 455 and a standard deviation of 24.

9. Find the percent of the data between 407 and 455. 47.72%
10. What percent of the data are greater than 479? 15.87%
11. Find the percent of the data that are less than 407. 2.28%
12. What percent of the data are between 431 and 503? 81.85%

Mixed Exercises

13. BIRTHS The table shows the numbers of births in the United States in a recent year.
    a. Complete the relative frequency column.
    b. Construct a probability distribution to represent the data. See margin.

Births	States	Relative Frequency
0–19,999	12	0.24
20,000–39,999	9	0.18
40,000–59,999	6	0.12
60,000–79,999	7	0.14
80,000–99,999	4	0.08
100,000–119,999	3	0.06
120,000–139,999	3	0.06
140,000–159,999	2	0.04
160,000 or more	4	0.08

Source: Annie E. Casey Foundation

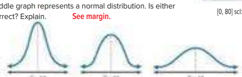
14. REACTION TIME In a test of 1200 teenagers, the reaction times to a visual cue were normally distributed with a mean of 0.25 second and a standard deviation of 0.05 second.
    a. About how many teenagers had reaction times between 0.15 and 0.35 second? 1140
    b. What is the probability that a teenager selected at random had a reaction time greater than 0.3 second? about 16%

Higher-Order Thinking Skills

15. ANALYZE The graphing calculator screen shows the graph of a normal distribution for a large set of data that has a mean of 50 and a standard deviation of 10. If every data point in the set is increased by 5 points, describe how the mean, standard deviation, and graph of the data changes. See margin.

[0, 80] scl: 10 by [0, 0.05] scl: 0.01

16. FIND THE ERROR Courtney says that the graphs all represent normal distributions with the same mean but different standard deviations. Michael says that only the middle graph represents a normal distribution. Is either correct? Explain. See margin.



17. CREATE Create a probability distribution in which one possible value of the random variable is twice as likely to occur as one other possible value of the random variable.
    Sample answer: Rolling a 1 or a 2 is twice as likely as rolling a 3 on a die.
18. PERSEVERE The boxes of cereal in a shipment are normally distributed with a mean weight of 17.1 ounces and a standard deviation of 0.2 ounce. Nine of the boxes weigh more than 17.5 ounces. How many boxes are in the shipment? 395

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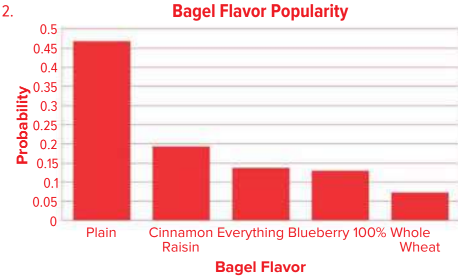
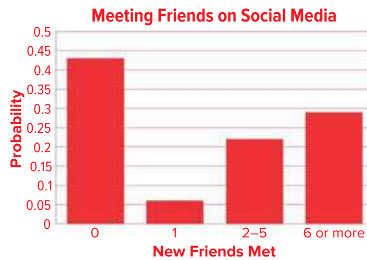
Answers

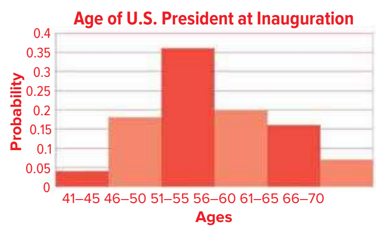
2.



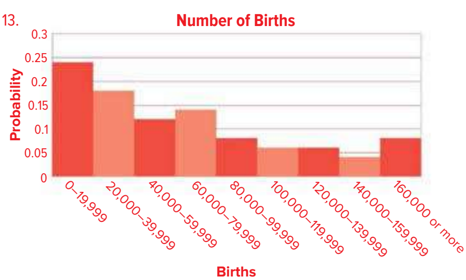
3.



4.



13.



15. The mean would increase by 25; the standard deviation would not change; and the graph would be translated 25 units to the right.

16. Courtney; all three graphs are normal distributions with the same mean. The first graph has the least standard deviation, the standard deviation of the middle graph is slightly greater, and the standard deviation of the last graph is greatest.

Module 9 • Statistics

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why is it useful to know how to create and interpret different types of data displays?
- How are statistics used in the real world to sway opinions?
- How are histograms and box plots useful for comparing real-world data?

Then have students write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to statistics.

LS **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Descriptive Statistics**.

- Interpreting Categorical and Quantitative Data
- Summarize, Represent, and Interpret Data on Two Categorical and Quantitative Variables

Module 9 • Statistics

Review

Essential Question

How do you summarize and interpret data?

By using statistics, you can analyze data to find meaningful results. Calculating measures of center and spread and making a dot plot, bar graph, or histogram can help you interpret the data.

Module Summary

Lessons 9-1 and 9-4

Measures of Center and Spread

- The mean of a data set is the sum of the elements of the data set divided by the total number of elements in the set.
- The median of a data set is the middle element or the mean of the two middle elements in the set of data when the data are arranged in numerical order.
- The mode of a data set is the value of the elements that appear most often in the set of data.
- The formula for standard deviation, with mean $\bar{x}$ and n terms is $\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}}$

Lessons 9-2 and 9-3

Representing and Using Data

- Dot plots, bar graphs, and histograms are commonly used to represent data.
- Bar graphs are used with discrete data and histograms are used with continuous data.
- A population is all members of a group of interest about which data will be collected. A sample is a subset of the population.
- A bias is an error that results in a misrepresentation of a population.

Lesson 9-5

Distributions of Data

- In a symmetric distribution, the mean and median are approximately equal.
- A negatively skewed distribution typically has a median greater than the mean. A positively skewed distribution typically has a mean greater than the median.
- An outlier is a value that is more than 1.5 times the interquartile range above the third quartile or below the first quartile.

Lesson 9-6

Comparing Sets of Data

- A linear transformation is one or more operations performed on a set of data that can be written as a linear function.
- Common linear transformations are adding a constant to or multiplying a constant by every value in the set of data.

Lesson 9-7

Two-Way Frequency Tables

- A two-way frequency table shows the frequencies of data classified according to two categories.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

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Test Practice

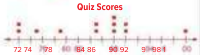
1. GRAPH Make a dot plot of the quiz scores of Ms. Perez's third period class.

Quiz Scores			
85	88	75	100
90	90	88	72
72	79	88	85

(Lesson 9-2)

Sample answer:

Quiz Scores



2. OPEN RESPONSE When is it a good idea to scale the number line when making a dot plot?

(Lesson 9-2)

Sample answer: Scaling the number line of a dot plot provides a meaningful way to represent data with a broad range of specific values. It is a much better representation of the data than a number line that consists of single, unstacked dots.

3. MULTI-SELECT Which of the statements are true regarding dot plots, bar graphs, and histograms? Select all that apply.

Lesson 9-2

A. Dot plots use a number line and dots to represent very large amounts of data.

B. Bar graphs are used to represent data that is continuous.

C. Histograms are used to represent data that is continuous.

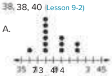
D. Bar graphs are used to represent data that is discrete.

E. Histograms are used to represent data that is discrete.

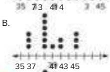
4. MULTIPLE CHOICE Which dot plot correctly models these data values?

36, 38, 42, 36, 36, 40, 42, 38, 38, 39, 40, 38, 38, 38, 40 (Lesson 9-2)

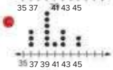
A.



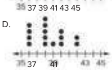
B.



C.



D.



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Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

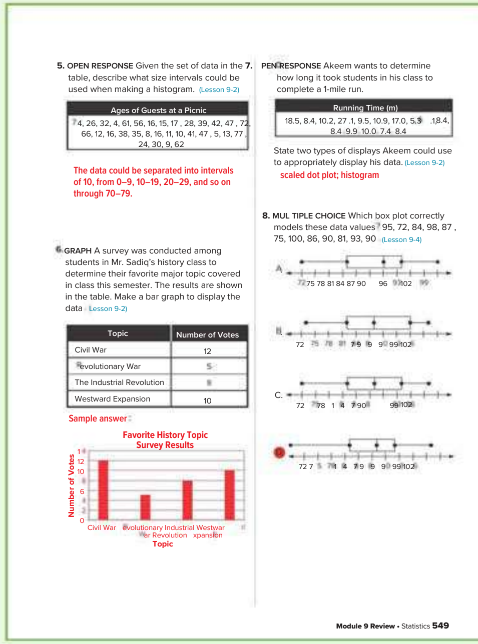
Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1-14 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Graph	Students create a graph online.	1, 6, 9
Open Response	Students construct their own response.	2, 5, 7, 10, 12, 14
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	3
Multiple Choice	Students select one correct answer.	4, 8, 11, 13

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
S.ID.1	9-2, 9-4	1, 3, 4, 6–9
S.ID.2	9-6	12
S.ID.3	9-5	11
S.ID.5	9-7	13, 14
N.Q.1	9-2, 9-4	2, 5, 10



5. **OPEN RESPONSE** Given the set of data in the table, describe what size intervals could be used when making a histogram. (Lesson 9-2)

Ages of Guests at a Picnic
74, 26, 32, 4, 61, 56, 16, 15, 17, 28, 39, 42, 47, 72, 66, 12, 16, 38, 35, 8, 16, 11, 10, 41, 47, 5, 13, 77, 24, 30, 9, 62

The data could be separated into intervals of 10, from 0–9, 10–19, 20–29, and so on through 70–79.

6. **GRAPH** A survey was conducted among students in Mr. Sadiq's history class to determine their favorite major topic covered in class this semester. The results are shown in the table. Make a bar graph to display the data. (Lesson 9-2)

Topic	Number of Votes
Civil War	12
Revolutionary War	5
The Industrial Revolution	8
Westward Expansion	10

Sample answer:

7. **OPEN RESPONSE** Akeem wants to determine how long it took students in his class to complete a 1-mile run.

Running Time (m)
18.5, 8.4, 10.2, 27.1, 9.5, 10.9, 17.0, 5.3, 8.4, 8.4, 9.9, 10.0, 7.4, 8.4

State two types of displays Akeem could use to appropriately display his data. (Lesson 9-2)

scaled dot plot; histogram

8. **MULTIPLE CHOICE** Which box plot correctly models these data values 95, 72, 84, 98, 87, 75, 100, 86, 90, 81, 93, 90 (Lesson 9-4)

A.

B.

C.

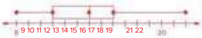
D.

Module 9 Review • Statistics 549

9. GRAPH The table shows the number of pages each student read in one night.

Pages Read
13, 15, 8, 22, 11, 17 , 15, 9, 14, 16, 13

Create a box plot to represent the set of data. (Lesson 9-4)



10. OPEN RESPONSE Fifteen people in their fifties were surveyed about the number of apps they have on their cell phone. (There was an assumption that all 15 of them owned a cell phone). The results are listed, below.

6, 0, 11, 8, 9, 6, 7 , 3, 1, 2, 10, 7 , 22, 5, 13
(Lesson 9-4)

A box plot to represent this data would have to begin at ? 0 because that is the minimum value, and would have to extend to ? 22 because that is the maximum value.

The most appropriate scale to display the data in the box plot should be ? 1 or 2.

11. MULTIPLE CHOICE The table shows the annual snowfall amounts for several towns.

Town	Snowfall (in.)
Westfield	246
Brattleboro	73
Cambridge	54
Danville	73
Shelburne	86
Lowell	67

Which measure(s) of center and measure(s) of spread best describe the set of data? (Lesson 9-5)

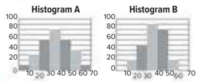
A. mean

B. median

C. standard deviation

D. five-number summary

12. OPEN RESPONSE *True* or *false*: Histogram B has more variability than Histogram A. (Lesson 9-6) False



13. MULTIPLE CHOICE The junior varsity dance team is selecting the color of their new uniforms. The team consists of 28 freshmen and sophomores. Of the 16 freshmen, 7 want red uniforms and 9 want black uniforms. Only 4 of the sophomores want black uniforms. How many total team members want red uniforms? (Lesson 9-7)

A. 7

B. 13

C. 15

D. 21

14. OPEN RESPONSE The table shows the frequencies of positions for different offensive players on a school football team. There are 38 offensive players on the team.

Position	Senior	Junior	Sophomore
Quarterback	1	1	0
Running Back	2	1	1
Receiver	3	2	2
Lineman	13	6	6

Suppose a junior player was picked at random. What is the probability that player is a receiver? (Lesson 9-7) $\frac{1}{5}$

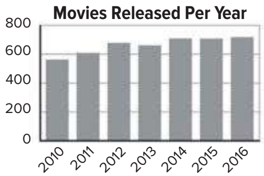
15. OPEN RESPONSE A normal distribution has a mean of 347 .2 and a standard deviation of 13.9. (Lesson 9-8)

The data that is less than 319.4 represents 2.5 ? % of the data.

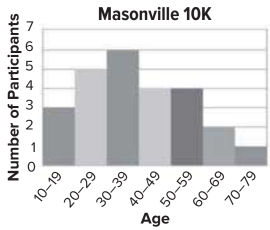
The data that is greater than 361.1 represents 16 ? % of the data.

Lesson 9-2

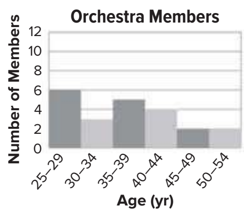
8b.



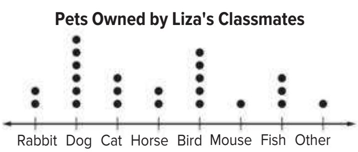
9a.



10.



12a.



13. Sample answer: 1) Because the data is clustered around ratings 7–10, it can be concluded that the product is well-liked by most customers and may have minor inconsistencies that certain people did not like. 2) Because there are only two low ratings, it can be concluded that dissatisfaction with the product could be a result of personal preference or manufacturer defect in a specific item.

14. A histogram is the best model for a data set when there is continuous data. To create a histogram, first determine an appropriate scale for the data set, draw bars for each scale, label the axes, and include a title, if appropriate.

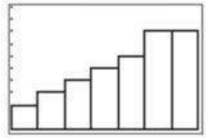
15. Sample answer: If the range of the data is broad with specific, unrepeating values, then it makes the dot plot more meaningful if the range is divided up into equal intervals.

16. Sample answer: 1) The grocery store could infer that consumers are gaining interest in more natural products because the natural peanut butter had the largest sales growth. 2) They could also infer that the convenience of having the jelly already in the jar with the peanut butter is not a significant priority for consumers because the sales have decreased significantly.

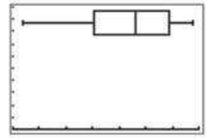
17. Sample answer: Bar graphs and histograms are similar because each displays data with bars. They are different because a bar graph is best used with data that are discrete and a histogram represents data that are continuous. For this reason, the bars in a bar graph do not touch and represent single values while the bars in a histogram touch and represent a range of values.

Lesson 9-5

9. negatively skewed

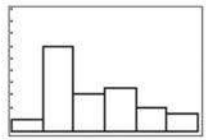


[10, 80] scl: 10 by [0, 10] scl: 1

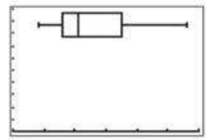


[10, 80] scl: 10 by [0, 10] scl: 1

10. positively skewed

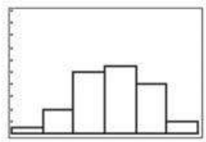


[30, 60] scl: 5 by [0, 20] scl: 2

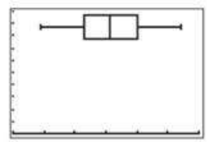


[30, 60] scl: 5 by [0, 20] scl: 2

11. symmetric

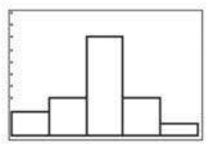


[10, 70] scl: 10 by [0, 20] scl: 2



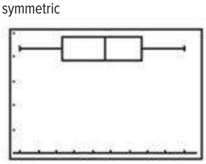
[10, 70] scl: 10 by [0, 20] scl: 2

12. symmetric



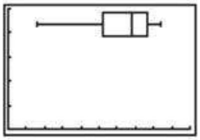
[15, 40] scl: 5 by [0, 10] scl: 1

13. symmetric



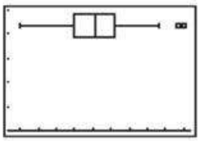
[35, 65] scl: 3 by [0, 5] scl: 1

14. Sample answer: The distribution is skewed, so use the five-number summary. The range is 42 − 8, or 34. The median is 34, and half of the data are between 26 and 38.5.



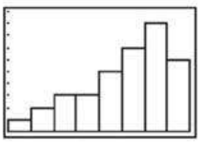
[0, 50] scl: 5 by [0, 5] scl: 1

15. Sample answer: The distribution is approximately symmetric, so use the mean and standard deviation. The mean is about 54.7 years with standard deviation of about 6.2 years.



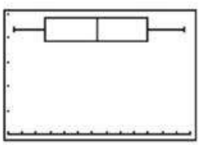
[40, 70] scl: 3 by [0, 5] scl: 1

16a. negatively skewed



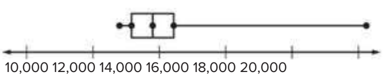
[15, 55] scl: 5 by [0, 10] scl: 1

17b. Sample answer: The distribution is symmetric, so use the mean and standard deviation. The mean is about 92.4 with standard deviation of about 18.4.



[60, 125] scl: 5 by [0, 5] scl: 1

18a.



The data is positively skewed.

b. The data is skewed so use the five-number summary; min: 12,799, Q_1: 13,161, median: 13,800, Q_3: 14,433, max: 20,237.

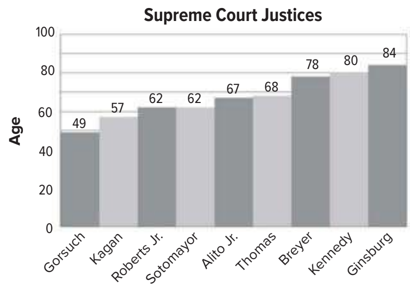
c. Mt. McKinley is an outlier. When the outlier is removed, the median decreases slightly from 13,800 to 13,796, however, the mean decreases from 14,380 to 13,729, which is more representative of the data as a whole.

19a. Sample answer: 225–230 g would be a reasonable advertised weight for either brand, so it is quite likely that they have the same advertised weight. Rafaello appears to have better control over the exact quantity in each package because its distribution is grouped more closely about the mean.

19b. Sample answer: Both distributions have an inverted, symmetric U-shape with "tails" on either side. Leonardo's distribution is lower and wider.

20. Sample answer: Because the distributions are skewed, compare using five-number summary. 1981–1985: min = 2, Q_1 = 5, median = 7, Q_3 = 8, IQR = 3, max = 10; 2005–2011: min = 11, Q_1 = 13, median = 13.5, Q_3 = 15, IQR = 2, max = 16. From 1981–1985, all the flights were shorter. From 2005–2011, all flights were longer, and the durations were more closely and evenly grouped around the median.

23a.



negatively skewed

23b. The data is skewed, so use the five-number summary; min: 49, Q_1: 59.5, median: 67, Q_3: 79, max: 84.

23c. There are no outliers in the data.

25. Sample answer: A bimodal distribution is a distribution of data that is characterized by having data divided into two clusters, thus producing two modes and having two peaks. The distribution can be described by summarizing the center and spread of each cluster of data.

26. Sample answer: The average high temperature over the course of a year for a city may have a symmetrical distribution. The attendance at a baseball stadium over the course of a season may be skewed.

27. Sample answer: In a symmetrical distribution, the majority of the data are located near the center of the distribution. The mean of the distribution is also located near the center of the distribution. Therefore, the mean and standard deviation should be used to describe the data. In a skewed distribution, the majority of the data lie either on the right or left side of the distribution. Because the distribution has a tail or may have outliers, the mean is pulled away from the majority of the data. The median is less affected. Therefore, the five-number summary should be used to describe the data.

Lesson 9-6

26a. Sample answer: For sample A, m = 38 mm, IQR = $Q_3 - Q_1$ = 41 – 33 = 8 mm. For sample B, m = 31 mm, IQR = $Q_3 - Q_1$ = 39.5 − 26.5 = 13 mm. Sample A has a higher median but a lower IQR. Therefore, sample A tends to be less spread out than B, but also tends to be larger than sample B.

26b. Sample answer: Sample A is more closely and more evenly grouped around its median. Sample B is skewed toward smaller diameters. So sample A is more representative.

27. Sample answer: Male students, $\bar{x}$ = 70.0 in., σ = 2.0 in. Female students, $\bar{x}$ = 66.3 in., σ = 2.7 in. Sample answer: For male students, the mean is 70.0 in., and the standard deviation is 2.0 in. For female students, the mean is 66.3 in., and the standard deviation is 2.7 in. On average, males are taller. However, because the standard deviation of males is smaller than that of females, the heights of females are more spread out.

28a. The distribution for each set of data is skewed. For City A, the median is 5.5 and IQR = 6 − 3 = 3. For City B, the median is 4.5 and IQR = 6 − 4 = 2.

28b. Sample answer: I would advise Francisca to visit City B because there is less risk with the number of rainy days. With the spread of City A being as large as it is, there is potential for the number of rainy days to be far more than desired for a vacation.

29. Sample answer: Histograms show the frequency of values occurring within set intervals. This makes the shape of the distribution easy to recognize. However, no specific values of the data set can be identified from looking at the histogram, and the overall spread of the data can be difficult to determine. The box plot shows the data divided into four sections. This aids when comparing the spread of one set of data to another. However, the box plots are limited because they cannot display the data any more specifically than showing it divided into four sections.

30. Sample answer: The mean, median, and mode of the new data set can be found by multiplying each original statistic by k. The range and the standard deviation can be found by multiplying each original statistic by $|k|$.

32. Sample answer: The mean, median, and mode of the new data set can be found by adding k to each original statistic and then multiplying each resulting value by m. Because the range and the standard deviation are not affected when a constant is added to a set of data, they can be found by multiplying each original value by the constant m.

33. Sample answer: When two distributions are symmetric, determine how close the averages are and how spread out each set of data is. The mean and standard deviation are the best values to use for this comparison. When distributions are skewed, determine which direction the data is skewed and the degree to which the data is skewed. The mean and standard deviation cannot provide information in this regard, but get this information by comparing the range, quartiles, and medians found in the five-number summaries. So if one or both sets of data are skewed, it is best to compare their five-number summaries.

Lesson 9-7

33.

Vehicle Type	2WD	AWD	Total
Hatchbacks	90	9	99
Sedans	60	13	73
SUVs	2	41	43
Total	152	63	215

34. $\frac{215}{63} \approx 3.41$; Sample answer: The conditional relative frequencies use the same numerators as the relative frequencies but have denominators of 63 instead of 215. So, the percents will be greater by a factor of $\frac{215}{63}$ or about 3.41.

35. Sample answer: Yes, there does appear to be an association. When the gasoline prices are higher, the distances traveled appear to be lower; when the gasoline prices are lower, the distances traveled appear to be higher.

36. Sample answer:

	Male	Female	Total
Purchased Class Ring	100	125	225
Did NOT Purchase Class Ring	150	35	185
Total	250	160	410

Find the joint relative frequency of a male who did not purchase a class ring. 36.6%

37. Sample answer: A relative frequency is the ratio of the number in a category to the overall total of both categories. A conditional relative frequency is the ratio of the joint frequency to the marginal frequency. Therefore, it is important to understand what relationship is being analyzed because each two-way relative frequency table can provide two different conditional relative frequency tables.

38a. Sample answer: Both freshmen and sophomores like fruit snacks the most and yogurt the least, and, by percentage, their preferences are almost equal.

38b. They are both correct. Fruit snack has a higher relative frequency for both grades than the other snacks. Also, 61% of sophomores and 57.5% of freshmen prefer fruit snacks.

Module 10

Tools of Geometry

Module Goals

- Students understand the basic elements of geometry, including points, lines, segments, planes, and angles.
- Students measure distances and compute midpoints on number lines and the coordinate plane.

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Also addresses G.CO.12 and G.MG.1.

Standards for Mathematical Practice:

All standards for Mathematical Practice will be addressed in this module.

Be Sure to Cover

To completely cover G.CO.12, go online to assign the following constructions.

- Copy a Line Segment (Lesson 10-3)
- Bisect a Segment (Lesson 10-7)

Coherence

Vertical Alignment

Previous

Students graphed points on a number line and graphed points and lines on a coordinate plane.

6.NS.6c, 8.EE.5, 8.EE.8a

Now

Students derive and use the distance, slope, and midpoint formulas to verify geometric relationships, and students construct segments and lines using a variety of tools.

G.CO.12, G.GPE.6

Next

Students will represent transformations in the plane and make formal geometric constructions using a variety of tools and methods.

G.CO.9, G.CO.12

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. After they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Practice.

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION
EXPLORE	LEARN	EXAMPLE & PRACTICE

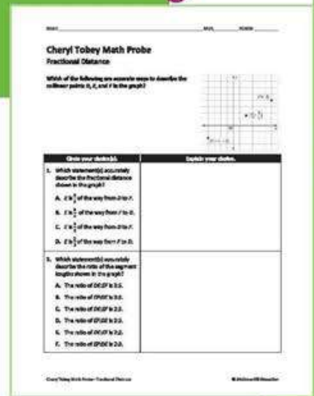
Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
10-1 The Geometric System		1	0.5
10-2 Points, Lines, and Planes	G.CO.1, G.MG.1	1	0.5
10-3 Line Segments	G.CO.1, G.CO.12	1	0.5
10-4 Distance	G.CO.1	1	0.5
10-5 Locating Points on a Number Line	G.GPE.6	1	0.5
10-6 Locating Points on a Coordinate Plane	G.GPE.6	1	0.5
10-7 Midpoints and Bisectors	G.GPE.6, G.CO.12	2	1
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	11	5.5

CHERYL TOBEY MATH PROBES

Formative Assessment Math Probe
Fractional Distance

Module Resource


Cheryl Tobey Math Probe
Fractional Distance

Answers: 1. A and D; 2. F

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will select statements that accurately describe fractional distances and explain their choices.

Targeted Concepts The position of a point between two other points can be used to analyze and compare segment lengths using ratios.

Targeted Misconceptions

- Students may incorrectly use the directed line, often going in alphabetical order or the order in which the letters first appear on the line without regard to the fractional distance described.
- Students may confuse the fractional distance with the ratio comparing the two smaller segment lengths. Often students see ratios as fractions and only consider part to whole relationships.

Use the Probe after Lesson 10-5.

Collect and Assess Student Answers

If the student selects these responses...	**Then** the student likely...
1. B and/or C	used the wrong fractional distance with the line direction. **Example:** Three fifths is the correct fractional distance from *D* to *F*, not from *F* to *D*.
2. A, B, C, D	used a fractional distance for the ratio (part to whole) instead of a part to part ratio to describe the relationship between segments *DE* and *EF*. **Example:** For Item 2A, the student uses the fractional distance $\frac{3}{5}$ to describe the relationship between *DE* and *EF* instead of 3:2 (3 partitions to 2 partitions).
3. E	is confusing the direction of the line and comparing using the ratio 2:3.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS® Distances and Midpoints on a Number Line, Distances and Midpoints in the Coordinate Plane
- Lesson 10-5, all Learns, all Examples

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

IGNITE!

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How are points, lines, and segments used to model the real world? Sample answer: Points, lines, and segments allow something that is abstract to be seen as a drawing. It in turn allows for certain calculations to solve for missing measures.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

Focus Students read about the basic elements of geometry and compute distances and midpoints on number lines and coordinate planes.

Teach Throughout the module, have students take notes under the tabs of their Foldables while working through each lesson. They should include definitions of terms and key concepts. Encourage students to record examples of each type of basic element of geometry. Also encourage them to record formulas for distance and midpoint.

When to Use It Use appropriate tabs as students cover each lesson in this module. Students should add to the vocabulary tab during each lesson.

Launch the Module

For this module, the Launch the Module video uses playing a game of chess to model the basic rules of geometry, such as finding the distance between two points. Students learn about using rules of geometry in astronomy and traveling.

Module 10

Tools of Geometry

Essential Question

How are points, lines, and segments used to model the real world?

What Will Y ou Learn?

How much do you already know about each topic **before** starting this module?

KEY
— I don't know. — I've heard of it — I know it!

	Before			After		
	I don't know	I've heard of it	I know it	I don't know	I've heard of it	I know it
analyze axiomatic systems and identify types of geometry						
analyze figures to identify points, lines, planes, and intersections of lines and planes						
find measures of line segments						
apply the Distance Formula to find lengths of line segments						
find points that partition directed line segments on number lines						
find points that partition directed line segments on the coordinate plane						
find midpoints and bisect line segments						

Foldables Make this Foldable to help you organize your notes about geometric concepts. Begin with four sheets of 11 × 17 paper.

1. **Fold** the four sheets of paper in half.
2. **Cut** along the top fold of the papers. Staple along the side to form a book.
3. **Cut** the right sides of each paper to create a tab for each lesson
4. **Label** each tab with a lesson number

Module 10 • Tools of Geometry 551

Interactive Presentation

What Vocabulary Will Y ou Learn?

- analytic geometry
- axiom
- axiomatic system
- betweenness of points
- bisect
- collinear
- congruent
- congruent segments
- coplanar
- defined term
- definition
- directed line segment
- distance
- equidistant
- fractional distance
- intersection
- line
- line segment
- midpoint
- plane
- point
- postulate
- segment bisector
- space
- synthetic geometry
- theorem
- undefined terms

Are Y ou Ready?

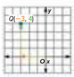
Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review

Example 1

Graph and label the point Q(−3, 4) in the coordinate plane.

Start at the origin. Because the *x*-coordinate is negative, move 3 units to the left. Then move 4 units up because the *y*-coordinate is positive. Draw a dot and label it *Q*.



Example 2

Evaluate the expression $[-2-(-7)]^2+(1-8)^2$.

Follow the order of operations.

$[-2-(-7)]^2+(1-8)^2$

$= 5^2+(-7)^2$ Subtract in parentheses.

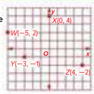
$= 25+49$ Evaluate exponents.

$= 74$ Add.

Quick Check

Graph and label each point on the coordinate plane.

1. $W(-5, 2)$
2. $X(0, 4)$
3. $Y(-3, -1)$
4. $Z(4, -2)$



Evaluate each expression.

5. $(4-2)^2+(7-3)^2$ 20
6. $(-5-3)^2+(3-4)^2$ 65
7. $[-1-(-9)]^2+(5-3)^2$ 68
8. $[-3-(-4)]^2+[-1-(-6)]^2$ 26

How did you do?

Which exercises did you answer correctly in the Quick Check?

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define Betweenness of points refers to the relationship between points on a line. Point *C* is between *A* and *B* if and only if *A*, *B*, and *C* are collinear and $AC + CB = AB$.

Example Point *F* is between points *D* and *E*. $DF = 3$ centimeters and $FE = 5$ centimeters.

Ask How are *DF* and *FE* related to *DE*? What is the measure of $\overline{DE}$? The sum of *DF* and *FE* should equal *DE*. $DE = 8$ cm

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- adding integers
- subtracting integers
- solving one-step equations
- solving multi-step equations
- measuring line segments on a coordinate plane
- converting fractions and decimals
- adding rational numbers

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the **Segments and Angles** section to ensure student success in this module.

Mindset Matters

Promote Growth Over Speed

Learning requires time and effort—time to think, reason, make mistakes, and learn from your mistakes and the mistakes of others. Ultimately, it's about the deep connections students make in their thinking and reasoning that matter more than the speed at which a problem is solved.

How Can I Apply It?

Have students complete the **Rate Yourself** chart before starting the module, discuss their mistakes and progress as you work through each lesson, and then reflect on their growth at the end of the module.

Lesson 10-1
The Geometric System

LESSON GOAL

Students analyze axiomatic systems and identify types of geometry.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Using a Game to Explore Axiomatic Systems

Develop:

The Axiomatic System of Geometry
- Apply an Axiomatic System

Types of Geometry
- Identify Types of Geometry

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Add Integers	●	●		●
Extension: Writing Good Definitions		●	●	●

Language Development Handbook

Assign page 56 of the *Language Development Handbook* to help your students build mathematical language related to axiomatic systems and types of geometry.

ELL You can use the tips and suggestions on page T56 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students graphed points on a number line and graphed points and lines on a coordinate plane.
6.NS.6c, 8.EE.5, 8.EE.8a

Now
Students learn about axiomatic systems and apply axioms to draw correct conclusions and identify examples of synthetic and analytic geometry.

Next
Students will identify points, lines, and planes and intersections of lines and planes.
G.CO.1

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students begin to develop an understanding of the different geometrical systems, and they are introduced to some of the terms they will use throughout the course.

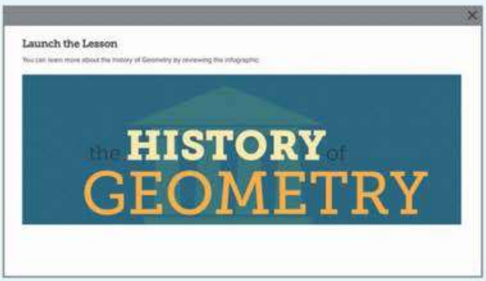
Mathematical Background

Axiomatic systems start with a set of *undefined terms* that are never formally explained by means of more basic concepts. In geometry, these undefined terms are points, lines, and planes. These undefined terms are then used to write *definitions*, which assign properties to other mathematical objects, like line segments and angles. *Axioms* or *postulates*, statements that are accepted as true without proof, are assumed to be true in the axiomatic system. *Theorems* are statements that can then be proved using the undefined terms, defined terms, axioms, and other theorems.

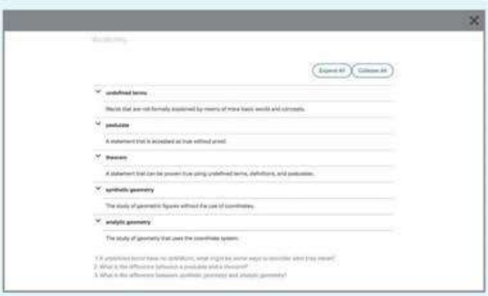
While there are many different types of geometry, this lesson introduces the two used most in this course: synthetic geometry and analytic geometry.

Interactive Presentation

Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- adding integers

Answers:

1. −7
2. −5
3. 5
4. 3

Launch the Lesson

Teaching the Mathematical Practices

3 Construct Arguments In this Launch the Lesson, students can use the infographic to learn about how an axiomatic system is used to construct arguments.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards? and How can I use these practices?,* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Using a Game to Explore Axiomatic Systems

Objective

Students analyze and apply the properties of an axiomatic system in a game.

Teaching the Mathematical Practices

4 Apply Mathematics In this Explore, students can see a real-world situation represented by an axiomatic system.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Alternate Use

Read aloud the Inquiry Question to set up the Explore activity, or have a student read it aloud. After having students complete the activity, lead the class in discussion to complete the exercises and answer the Inquiry Question.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students are given the definition of an axiomatic system. Then they read the set of rules for a game. Next, students complete the guiding exercises. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore

TYPE

Students respond to guiding exercises and explain why they agree or disagree.

Interactive Presentation

Explore

TYPE

Students can respond to the Inquiry Question and can view a sample answer.

Explore Using a Game to Explore Axiomatic Systems (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why is it hard to play a game when the rules are incomplete? Sample answer: If you don't have all of the information, you could miss something important or play incorrectly.
- Why would it be important for a game to have rules that don’t contradict each other? Contradictory rules would cause confusion.

Inquiry

What characteristics must an axiomatic system have? Sample answer: The axioms within a system must be complete. They should not be contradictory or repetitive. Axioms need these characteristics to prove other facts.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn The Axiomatic System of Geometry

Objective

Students learn about axiomatic systems and apply axioms to draw correct conclusions.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of the axiomatic system of geometry in this Learn.

Common Misconception

Students sometimes struggle with understanding that postulates cannot be proven and must be taken as true. Share some postulates with students and point out why it seems they can be assumed to be true.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

DIFFERENTIATE

Language Development Activity ELL

Beginning Before students read the lesson, use pattern blocks to illustrate how to categorize objects. Use color and shape to differentiate the blocks.

Intermediate Have partners make and use flashcards to check each other's pronunciation and understanding of vocabulary.

Advanced Have students scan the lesson for content vocabulary words in context. Help them pronounce the vocabulary words correctly. Discuss vocabulary meanings with them.

Advanced High After reading each example of the lesson, use an Interactive Question-Response to discuss it. Have students record the main idea and details of the paragraphs in their notes.

Lesson 10-1

The Geometric System

Explore Using a Game to Explore Axiomatic Systems

Online Activity Use a real-world situation to complete the Explore.

INQUIRY What are the characteristics of a good set of rules?

Learn The Axiomatic System of Geometry

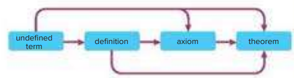
Geometry is an axiomatic system based on logical reasoning and axioms.

The Axiomatic System of Geometry

An **axiomatic system** has a set of axioms from which theorems can be derived.

undefined terms	words, usually readily understood, that are not formally explained by means of more basic words and concepts
definition	assigns properties to a mathematical object
defined term	a term that has a definition and can be explained using undefined terms and/or defined terms
axiom or **postulate**	statement that is accepted as true without proof
theorem	statement or conjecture that can be proven true using undefined terms, definitions, and axioms

Undefined terms are used to write definitions. Undefined terms and definitions are used to create axioms. Undefined terms, definitions, and axioms are used to prove theorems.



One real-world axiomatic system that is probably familiar to you is the set of rules to a game. The rules are the axioms, and they are used to evaluate the legality of each play.

Today's Goals
- Apply axioms to draw conclusions.
- Identify examples of synthetic and analytic geometry.

Today's Vocabulary
axiomatic system
undefined terms
definition
defined term
axiom
postulate
theorem
synthetic geometry
analytic geometry

Math History Minute

Thales (c. 624 546 B.C was a Greek mathematician, philosopher , and astronomer , and is the first known individual attributed with a mathematical discovery. He inspired Euclid, Plato, and Aristotle, who considered him to be the first philosopher in the Greek tradition

Lesson 10-1 • The Geometric System 553

Interactive Presentation

Learn

TAP

Students tap on each button to see a definition.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | **3 APPLICATION**

Example 1 Apply an Axiomatic System

ANIMALS **In the fictional country of Rythoth, blue animals are from the mountains, and red animals are from the valleys. These animals are categorized into three distinct classes: mammals, birds, and reptiles. Mammals are covered by hair or fur , birds are covered by feathers, and reptiles are covered by scales.**

Part A Categorize the animals.

Write the name of each animal in the corresponding categories in the table.

Birthplace	Mammal	Bird	Reptile
Mountains	Rorx	Pax	Awub
Valleys	Klub	Prit	Zog

Part B Use axioms.

Use the previously given axioms and the table you filled in to draw three conclusions about the species of animals shown.

- The Rorx is a mammal from the mountains of Rythoth.
- The Zog is a reptile from the valleys of Rythoth.
- The Prit is a bird from the valleys of Rythoth.

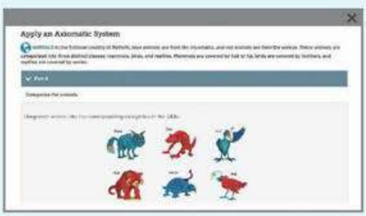
Talk About It!
What conclusion cannot be made from the provided axioms?

Sample answer: The axioms cannot be used to conclude why animals from the mountains are blue and animals from the valleys are red.

Study Tip
Theorems Theorems, or conclusions, made from a set of axioms must be true in every situation. It takes only one example that contradicts the conjecture to show that a theorem or conclusion is not true.

554 **Module 10** • Tools of Geometry

Interactive Presentation



Example 1

DRAG & DROP

Students drag objects to classify them according to definitions and axioms.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Apply an Axiomatic System

Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated axioms to draw conclusions.

Questions for Mathematical Discourse

AL How would you label the Klub according to its color, class, and body covering? red; mammal; fur

OL In **Part A**, can any of these animals be classified into multiple cells of the grid? Explain. No; each animal only fits into one class, has one body covering and is one color.

BL What additional characteristic can be used to label each animal? number of legs

Common Error

Students may confuse the different types of properties of the animals. Help them keep straight the types of animal and the colors, which are linked to the regions where the animals live by the axioms.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Types of Geometry

Objective

Students identify examples of synthetic and analytic geometry.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between synthetic and analytic geometry in this Learn.

Important to Know

There are actually many axiomatic systems of geometry, not just the ones mentioned here. For example, spherical geometry is a synthetic geometry that is useful for modeling air travel and mapping continents in the real world. Polar coordinates are included in analytic geometry and sometimes taught in trigonometry.

Check

PLANETS The fictional galaxy of Yogul contains at least 20 planets including Mothera, Sothera, and Kothera. An animal can live on any planet in the Yogul galaxy that contains its biome. Lizards and scorpions live in the desert. Frogs and monkeys live in tropical forests. Bears and foxes can be found in the tundra. The biomes of each planet are permanent and will not change over time.

Color	Biome
	desert
	tropical forest
	tundra

Use the axioms given to determine what conclusions can be made about the planets of Yogul. Select all that apply. **A, F**

A. Bears and foxes can live on Sothera.

B. Lizards and scorpions can only live on Mothera.

C. Only frogs and monkeys can survive on Kothera.

D. Bears and foxes can survive on Sothera at temperatures as low as −20°F.

E. All animals can live on Kothera.

F. Scorpions and lizards can live on Mothera.

Learn Types of Geometry

There are several types of geometry that are built upon different sets of postulates including synthetic geometry and analytic geometry.

Synthetic geometry is the study of geometric figures without the use of coordinates. Synthetic geometry is sometimes called *pure geometry* or *Euclidean geometry*.

Analytic geometry is the study of geometry using a coordinate system. Analytic geometry is sometimes called *coordinate geometry* or *Cartesian geometry*.

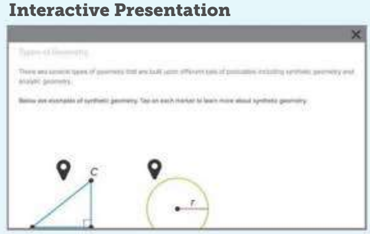
Think About It!
What is an advantage of using analytic geometry instead of synthetic geometry?

Sample answer: When you are using analytic geometry, you can determine lengths by using the coordinate grid.

Go Online You can complete an Extra Example online.

Lesson 10-1 • The Geometric System 555

Interactive Presentation



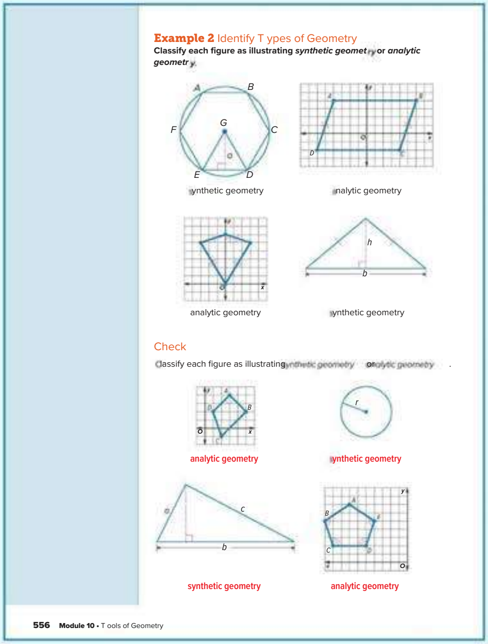
Learn

TAP

Students tap to reveal information about types of geometry.

TYPE

Students answer a question about the different types of geoemetry.


Example 2 Identify T ypes of Geometry

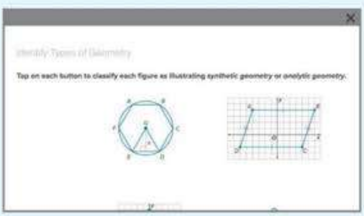
Classify each figure as illustrating *synthetic geometry* or *analytic geometry*.

Check

Classify each figure as illustrating synthetic geometry or analytic geometry.

556 Module 10 • Tools of Geometry

Interactive Presentation



Example 2

TAP

Students tap on each button to reveal solutions.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Example 2 Identify T ypes of Geometry

MP Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use definitions to classify figures as illustrating synthetic or analytic geometry.

Questions for Mathematical Discourse

AL What is a coordinate system? Every point is numerically specified on the plane.

OL How can you determine which type of geometric figure is studied in analytic geometry? Sample answer: All geometric figures in analytic geometry are on coordinate systems.

BL How can you create two of your own geometric figures, one for each type of geometry? Explain. Sample answer: When sketching a figure using synthetic geometry, identify congruent sides, and label the bases and the height. When sketching a figure using analytic geometry, plot the endpoints on a coordinate plane and label congruent sides and angles.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–10
2	exercises that use a variety of skills from this lesson	13–14
2	exercises that extend concepts learned in this lesson to new contexts	11–12, 15–18
3	exercises that emphasize higher-order and critical-thinking skills	19–23

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–17 odd, 19–23
- Extension: Writing Good Definitions

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–23 odd
- Remediation, Review Resources: Add Integers
- Personal Tutors
- Extra Examples 1–3
- ALEKS Addition and Subtraction with Integers

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–15 odd
- Remediation, Review Resources: Add Integers
- ALEKS Addition and Subtraction with Integers

Answers

1. Sample answer: Kelsey's jersey number is greater than 5 and less than 11. Marie and Kelsey are on the same team. Kylie's team scored 26 points.
2. Sample answer: Mercedes bought 5 short sleeve T-shirts and 5 long sleeve T-shirts. Quinn paid $80.00. Rachel bought 5 long sleeve T-shirts. Hector bought 5 black sweatshirts and 5 yellow short sleeve T-shirts.
3. Sample answer: Tom mulched the yard of all three clients this week. Ms. Martinez paid Tom $115 this week. Mr. Hansen paid Tom to mow his lawn and mulch his yard. Mrs. Johnson used all of Tom's services this week.
4. Sample answer: Mateo bought a small vanilla cupcake and a small cupcake with vanilla icing. Bethany's cupcake had strawberry icing.

Practice

Go Online Y ou can complete your homework online.

Example 1

1. BASKETBALL The Badgers' basketball team has 10 players. During practice, half of the players wear red jerseys numbered 1–5, and the other half wear yellow jerseys numbered 6–10. The yellow team wins the practice game 32-26.
   - Kylie wears number 5 and scores 9 points.
   - Kelsey's team wins the game.
   - Marie and Kylie are on opposing teams.

   Use the axioms to make three conclusions about the game played. See margin.

2. PRINTING Rico's T-shirt Company sells customized short sleeve T-shirts, long sleeve T-shirts, and sweatshirts. Each type of shirt sells in multiples of 5. It costs $25.00 for 5 short sleeve T-shirts, $30.00 for 5 long sleeve T-shirts, and $40.00 for 5 sweatshirts. Short sleeve and long sleeve T-shirts can be made in any color except navy or black. Sweatshirts are only made in navy and black.
   - Mercedes bought green shirts for $55.00.
   - Quinn bought 10 navy sweatshirts.
   - Rachel paid $30.00 for several red shirts.
   - Hector bought black and yellow shirts for $65.00.

   Use the axioms to make four conclusions about the shirts sold. See margin.

3. LANDSCAPING Tom owns a landscaping business. He charges $40 for a yard cleanup, $50 to mow a lawn, and $75 to mulch a yard. On average, it takes Tom 25 minutes for a yard cleanup, 40 minutes to mow a lawn, and 2 hours to mulch a yard. Tom's clients are Mr. Hansen, Ms. Martinez, and Mrs. Johnson.
   - Mr. Hansen paid $125 for lawn services this week.
   - Tom spent more than an hour at Ms. Martinez' house this week.
   - Mrs. Johnson wrote Tom a check for $165 for the week.
   - Tom made $405 from his three clients this week.

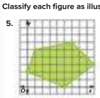
   Use the axioms to make four conclusions about the landscaping that Tom did. See margin.

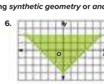
4. CUPCAKES Olivia's Cupcake Shoppe sells small and large cupcakes in three flavors.
   - Niamh paid $3 for a cupcake with buttercream icing.
   - Bethany bought a small vanilla cupcake.
   - Mateo paid $3.50 for a cupcake with strawberry icing and a chocolate cupcake.

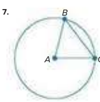
   Use the axioms to make two conclusions about the cupcakes that were purchased. See margin.

Example 2

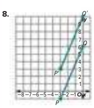
Classify each figure as illustrating *synthetic geometry* or *analytic geometry*.

5. 
analytic geometry

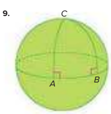
6. 
analytic geometry

7. 

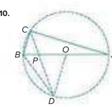
synthetic geometry

8. 

analytic geometry

9. 

synthetic geometry

10. 

synthetic geometry

Mixed Exercises

11. RESTAURANT Damon sells three types of salads at his restaurant: cobb, wedge, and spinach. Each salad is served with 2 dinner rolls. The price of the cobb salad is $7.99, the price of the wedge salad is $8.99, and the price of the spinach salad is $5.99. Grilled chicken can be added to any salad for an additional $2.00.
    - Malik spent $7.99 on a salad.
    - Pedro and Deandra each spent $8.99 on their salads.
    - Rafael ate a wedge salad.
    - Drake did not add chicken to his salad.

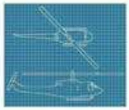
    Use the axioms to make a conclusion about the salads that are eaten. Sample answer: Pedro and Rafael ate the same type of salad.
12. CLASSROOM Mrs. Fields teaches high school geometry. Her classroom tools include a compass, straightedge, pencil, and protractor. Does Mrs. Fields likely teach *analytic geometry* or *synthetic geometry*? Explain your reasoning. Sample answer: The tools Mrs. Fields uses in her classroom are better suited for synthetic geometry, which is not done in the coordinate plane.
13. REASONING Theo is stuck on a problem on a test. The problem is asking him to use a given formula to find the distance between two points on a graph. Is Theo using analytic *geometry* or *synthetic geometry*? Explain your reasoning. Sample answer: Theo is likely doing analytic geometry, because he is using a graph with points.
14. USE A SOURCE Survey a group of students in your classroom about favorite colors. Write three axioms about the data you collected. Then use your axioms to write a conclusion. Explain your reasoning. See margin.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

15. STATE YOUR ASSUMPTION Sydney is an engineer. She is using a blueprint for a project that is drawn on a grid, as shown. Is Sydney likely using *analytic geometry* or *synthetic geometry*? Explain any assumptions that you make.

Sample answer: Because Sydney's plan is on a grid, she is likely using analytic geometry; that is, assuming the grid is used as a coordinate system.



16. Mr. Sail assigns a project where students identify shapes that represent real-world objects. Is this an example of *analytic geometry* or *synthetic geometry*? Explain your reasoning. **Sample answer: This is an example of synthetic geometry because a coordinate plane is not used.**

17. CONSTRUCT ARGUMENTS Consider the following axiomatic system for bus routes. **See margin.**
- Each bus route lists the stops in the order at which they are visited by the bus.
- Each route visits at least four distinct stops.
- No route visits the same stop twice, except for the first stop, which is always the same as the last stop.
- There is a stop called Downtown, which is visited by each route.
- Every stop other than Downtown is visited by at most two routes.

The city has stops at Downtown, King St, Maxwell Ave, Stadium District, State St, Grace Blvd, and Charlotte Ave. Are the following three routes a model for the axiomatic system? Justify your argument.
ROUTE 1: Downtown, King St, Stadium District, State St, Downtown
ROUTE 2: Stadium District, State St, Grace Blvd, Maxwell Ave, Downtown, Stadium District
ROUTE 3: King St, Stadium District, Downtown, Maxwell Ave, Stadium District, King St

18. SHOPPING The Clothing Shop is having a sale. All clothes are 20% off, and all accessories are 30% off.
- Jaisa bought two necklaces.
- Sheree bought a shirt and a purse.

Use the axioms to make one conclusion about Jaisa or Sheree's purchases.
Sample answer: Jaisa saved 30% on her purchase.

Higher-Order Thinking Skills

19. WRITE Write a comparison of the rules and plays of a game and the elements of an axiomatic system. Then choose a game or sport for which you know the rules. Explain a rule from the game or sport and a play from the game. Does the play violate or fall within the rule? Explain. **See margin.**

20. CREATE Given the following list of axioms, draw a model to properly represent the information. **See margin.**
- There exist five points.
- Each line contains only these five points.
- There exist two lines.
- Each line contains at least two points.

21. WHICH ONE DOESN'T BELONG? Three-point geometry is a finite subset of geometry with the following four axioms:
- There exists exactly three distinct points.
- Each pair of distinct points are on exactly one line.
- Not all the points are on the same line.
- Each pair of distinct lines intersect in at least one point.

Which of the following does not satisfy all the axioms of three-point geometry? Justify your conclusion.

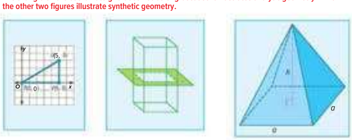
Sample answer: The second figure does not satisfy all the axioms. The axioms do not specify that the line segments connecting the points need to be straight, so the first and third figures would work.

22. FIND THE ERROR Grant read the following axioms for a video game he is playing.
- There are four keys hidden on each level.
- Each level ends when the player collects the third key.
- The game has 10 levels.

From these axioms, Grant concluded:
- to complete the game, he will need to find 30 keys.
- there are 40 keys in the game.
- he can collect all 40 keys in the game.

Are Grant's conclusions correct? Explain your reasoning. **Sample answer: Grant's third conclusion is incorrect because the second axiom says that each level will end when the third key is collected; therefore, Grant couldn't collect more than 30 keys.**

23. WHICH ONE DOESN'T BELONG? Using your understanding of analytic and synthetic geometry, which of the following figures does not belong? Justify your conclusion. **Sample answer: The triangle on the coordinate grid does not belong because it illustrates analytic geometry while the other two figures illustrate synthetic geometry.**



Answers

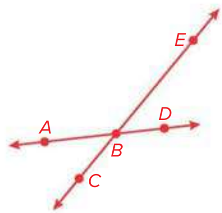
14. Sample answer: Suppose we have the following axioms:
- There are exactly five colors chosen: red, orange, yellow, green and blue.
- Given any two colors, there is exactly one child who likes these two colors.
- Every student likes exactly two different colors among the five.

Conclusion: There were 10 students surveyed. The following are the color combinations chosen: red-orange, red-yellow, red-green, red-blue, orange-yellow, orange-green, orange-blue, yellow-green, yellow-blue, green blue.

17. The three routes are not a model for the axiomatic system. Axioms 1, 2, and 4 are satisfied. Axiom 3 is not satisfied because Route 3 visits Stadium District twice and it is not the first/last stop. Axiom 5 is not satisfied because all three routes visit Stadium District.

19. Sample answer: The rules of a game are like the axioms of an axiomatic system. They establish what can happen within the game. Plays are like theorems. They are tested against the rules or axioms to see whether they are legal in the game. In basketball, it is a rule that during playing time 5 players from each team shall be on the playing court. A play in which 6 players are on the court is a violation because the rules allow exactly 5 players.

20. Sample answer:



Lesson 10-2

Points, Lines, and Planes

G.MG.1, G.CO.1

LESSON GOAL

Students analyze figures to identify points, lines, planes, and intersections of lines and planes.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Points, Lines, and Planes
- Name Lines and Planes
- Model Points, Lines, and Planes

Explore: Intersections of Three Planes

Intersections of Lines and Planes
- Draw Geometric Figures
- Interpret Drawings
- Model Intersections

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Subtract Integers	●	●		●
Extension: Fano Plane		●	●	●

Language Development Handbook

Assign page 57 of the *Language Development Handbook* to help your students build mathematical language related to points, lines, planes, and the intersections of lines and planes.

ELL You can use the tips and suggestions on page T57 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G.MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

3 Construct viable arguments and critique the reasoning of others.

6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students drew points, lines, line segments, and rays, and identified these in two-dimensional figures.

Now
Students analyze figures to identify points, lines, and planes and identify intersections of lines and planes.
G.CO.1, G.MG.1

Next
Students will calculate measures of line segments and apply the definition of congruent line segments to find missing values.
G.CO.1, G.CO.12

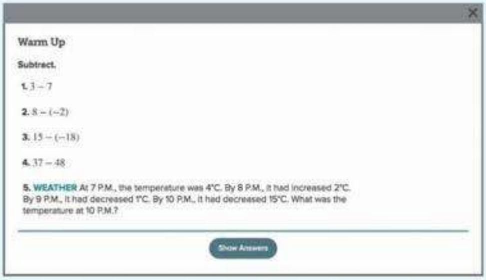
Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students begin to develop an understanding of points and lines, and they apply their understanding by using these shapes and their measures to model real-world objects.

Interactive Presentation



Warm Up

Warm Up

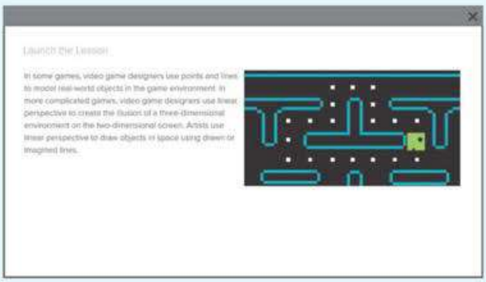
Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- subtracting real numbers

Answers:

1. −4
2. 10
3. 33
4. −11
5. −10°C


Launch the Lesson

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of points, lines, and planes.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Mathematical Background

In geometry, a *point* is a location without shape or size. A *line* contains points and has no thickness or width. Points on the same line are *collinear*, and there is exactly one line through any two points. The intersection of two lines is a point.

A *plane* is a flat surface made of points. A plane has no depth and extends infinitely in all directions. Points on the same plane are *coplanar*, and the intersection of two planes is a line.


Today's Vocabulary

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Intersections of Three Planes

Objective

Students construct the intersection of three planes and identify the appropriate geometric definition that describes the intersection.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to understand the content in the Explore activity, students will need to make and use concrete models. Work with students to explore and deepen their understanding of points, lines, and planes.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students construct a paper model of three planes that intersect at a point using heavy paper, scissors, and tape. Then students complete guiding exercises on what they learn from this model. Next students construct a paper model of three planes that intersect in a line. Students then complete guiding exercises on the second model. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore

Explore

TAP

Students tap to watch a video demonstrating how to build a model.

TYPE

Students type answers to guiding exercises

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Intersections of Three Planes (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What are some examples of real-world objects that could model planes? Sample answers: tables, walls, desk
- What are the limitations of these objects as the model? Sample answer: The walls don't continue in all directions infinitely, so I have to imagine them continuing.

Inquiry

What figures can be formed by the intersection of three planes? Sample answer: Three planes can intersect in a point or a line.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Points, Lines, and Planes

Objective

Students identify points, lines, and planes.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Common Misconception

Students may want to name a line using all of the labeled points. Remind them that only two letters are needed to name a line. This means there are often many possible correct names for a single line. **Ask:** How many different names can be written for a line with four labeled points? 12

Essential Question Follow-Up

Students learn about the undefined terms *point*, *line*, and *plane*.

Ask:

Why are the terms *point*, *line*, and *plane* undefined? Sample answer: because they are the most basic building blocks of geometry, they cannot be explained using simpler terms.

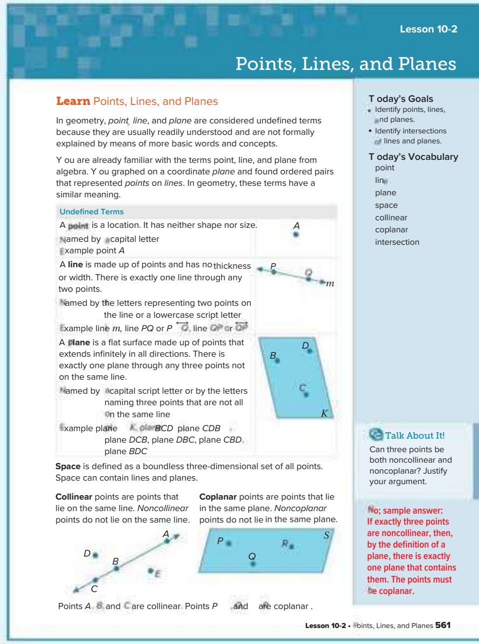
Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

DIFFERENTIATE

Reteaching Activity AL

Explain how points, lines, and planes exist in nature. For example, planes can model leaves, lily pads, and the surface of a pond; lines can model spider webs, sunbeams, tree trunks, the edge of a riverbed, and the veins of a leaf.



Lesson 10-2

Points, Lines, and Planes

Learn Points, Lines, and Planes

In geometry, *point*, *line*, and *plane* are considered undefined terms because they are usually readily understood and are not formally explained by means of more basic words and concepts.

You are already familiar with the terms point, line, and plane from algebra. You graphed on a coordinate *plane* and found ordered pairs that represented *points* on *lines*. In geometry, these terms have a similar meaning.

Undefined Terms

A **point** is a location. It has neither shape nor size.
Named by a capital letter
Example point *A*

A **line** is made up of points and has no thickness or width. There is exactly one line through any two points.
Named by the letters representing two points on the line or a lowercase script letter
Example line *m*, line *PQ* or $\overleftrightarrow{PQ}$, line *QP* or $\overleftrightarrow{QP}$

A **plane** is a flat surface made up of points that extends infinitely in all directions. There is exactly one plane through any three points not on the same line.
Named by a capital script letter or by the letters naming three points that are not all on the same line
Example plane *K*, plane *BCD*, plane *CDB*, plane *DCB*, plane *DBC*, plane *CBD*, plane *BDC*

Space is defined as a boundless three-dimensional set of all points. Space can contain lines and planes.

Collinear points are points that lie on the same line. *Noncollinear* points do not lie on the same line.

Coplanar points are points that lie in the same plane. *Noncoplanar* points do not lie in the same plane.

Points *A*, *B*, and *C* are collinear. Points *P*, *Q*, and *R* are coplanar.

Today's Goals
- Identify points, lines, and planes.
- Identify intersections of lines and planes.

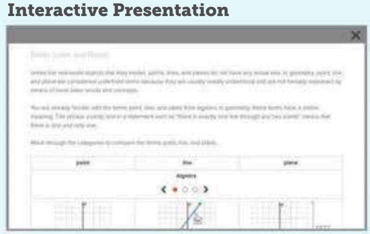
Today's Vocabulary
point
line
plane
space
collinear
coplanar
intersection

Talk About It!
Can three points be both noncollinear and noncoplanar? Justify your argument.

No; sample answer: If exactly three points are noncollinear, then, by the definition of a plane, there is exactly one plane that contains them. The points must be coplanar.

Lesson 10-2 • Points, Lines, and Planes 561

Interactive Presentation



Learn

TAP

Students tap to reveal information on points, lines, and planes.

TYPE

Students answer a question to show they understand points, lines, and planes.

G.MG.1, G.CO.1

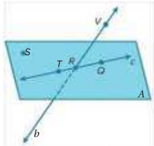
1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Example 1 Name Lines and Planes

Use the figure to name each of the following.

a. a line containing point Q

The line can be named as line c or any two of the three points on the line can be used to name the line.



Using the choices provided, write the additional names for line c below.

$\overleftrightarrow{TR}$ $\overleftrightarrow{RV}$ $\overleftrightarrow{VT}$ $\overleftrightarrow{RQ}$ $\overleftrightarrow{QR}$ $\overleftrightarrow{RT}$ $\overleftrightarrow{TS}$ $\overleftrightarrow{VR}$ $\overleftrightarrow{QT}$

$\overleftrightarrow{TR}$ $\overleftrightarrow{RT}$ $\overleftrightarrow{TQ}$ $\overleftrightarrow{RQ}$ $\overleftrightarrow{QR}$

Study Tip

Additional Planes Although not drawn, there is another plane that contains point S and point T. Because points S, T, and V are noncollinear, points S and T are in plane STV.

b. a plane containing point S and point T

One plane that can be named is plane A. You can also use the letters of any three noncollinear points to name this plane. Plane *TRS* and plane *TQS* can be used to name this plane.

Circle another correct name for plane A.

(plane *QST*) plane *STV* plane *QVS* plane *VST*

Example 2 Model Points, Lines, and Planes

STUDENT DESK **Name the geometric terms modeled by the objects in the picture.**

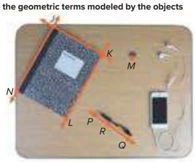
The notebook models plane *JKL*.

The edges of the notebook model line *JK* and line *KL*.

The quarter models point *M* in space.

Points *N*, *L*, and *K* are coplanar.

Points *P*, *Q*, and *R* are collinear.



McGraw-Hill Education

Go Online You can complete an Extra Example online.

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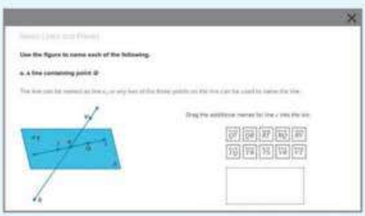
Interactive Presentation



Example 1

DRAG & DROP

Students drag possible line names to the solution area.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Name Lines and Planes

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of points, lines, and planes in this example to name lines and planes.

Questions for Mathematical Discourse

AL In part **a**, what other points are located on the line containing point *Q*? *T* and *R*

OL How do you name a line? Either use two letters representing points on the line or use the lowercase script letter that identifies the line.

BL In part **a**, is $\overleftrightarrow{TR}$ the same line as $\overleftrightarrow{RT}$? Explain. Yes; the line is named by two points on the line, *T* and *R*.

Common Error

Students may think that different names refer to different lines.

Example 2 Model Points, Lines, and Planes

Teaching the Mathematical Practices

5 Use a Source Guide students to find external information to answer the questions posed in the Use a Source feature.

Questions for Mathematical Discourse

AL What geometric figure is named by a single letter? a point

OL Why are two letters used to name the line that models the top edge of the notebook? Sample answer: Because the top corners of the notebook are modeled by two points and there exists exactly one line through any two points, the line that models the top edge of the notebook is named using two letters.

BL Does the pen lie on plane *JKL*? Explain. Yes; sample answer: Because a plane extends infinitely in all directions, the pen lies on the plane.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Intersections of Lines and Planes

Objective

Students identify intersections of lines and planes.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Example 3 Draw Geometric Figures

Teaching the Mathematical Practices

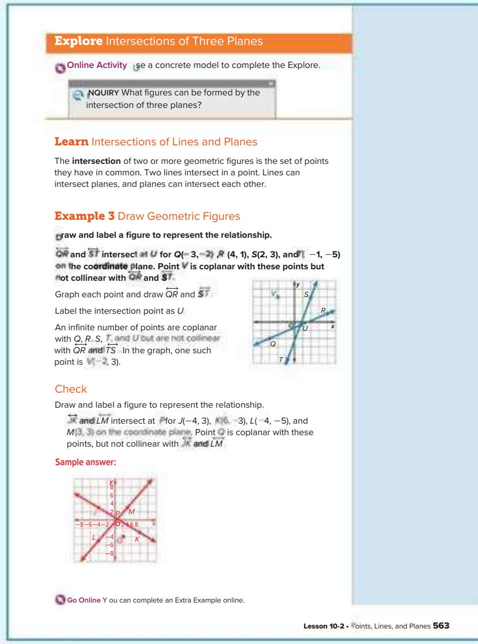
3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. In Example 3, ask students to justify their conclusions.

Questions for Mathematical Discourse

AL What is the meaning of *coplanar* and *collinear*? Coplanar means on the same plane; *collinear* means on the same line.

OL How do you know that point *V* is coplanar with the other points? Because it is graphed on the same coordinate plane, it is coplanar with the other points.

BL Is it possible for two points to be collinear but not coplanar? No; if two points fall on the same line, then they are on the same plane.



Explore Intersections of Three Planes

Online Activity Use a concrete model to complete the Explore.

INQUIRY What figures can be formed by the intersection of three planes?

Learn Intersections of Lines and Planes

The **intersection** of two or more geometric figures is the set of points they have in common. Two lines intersect in a point. Lines can intersect planes, and planes can intersect each other.

Example 3 Draw Geometric Figures

Draw and label a figure to represent the relationship.

$\overleftrightarrow{QR}$ and $\overleftrightarrow{ST}$ intersect at U for $Q(-3, -2)$, $R(4, 1)$, $S(2, 3)$, and $T(-1, -5)$ on the coordinate plane. Point V is coplanar with these points but not collinear with $\overleftrightarrow{QR}$ and $\overleftrightarrow{ST}$.

Graph each point and draw $\overleftrightarrow{QR}$ and $\overleftrightarrow{ST}$.

Label the intersection point as U.

An infinite number of points are coplanar with Q, R, S, T, and U but are not collinear with $\overleftrightarrow{QR}$ and $\overleftrightarrow{TS}$. In the graph, one such point is $V(-2, 3)$.

Check

Draw and label a figure to represent the relationship.

$\overleftrightarrow{JK}$ and $\overleftrightarrow{LM}$ intersect at P for $J(-4, 3)$, $K(6, -3)$, $L(-4, -5)$, and $M(3, 3)$ on the coordinate plane. Point Q is coplanar with these points, but not collinear with $\overleftrightarrow{JK}$ and $\overleftrightarrow{LM}$.

Sample answer:

Go Online You can complete an Extra Example online.

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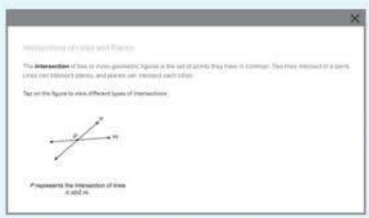
Interactive Presentation



Learn

TAP

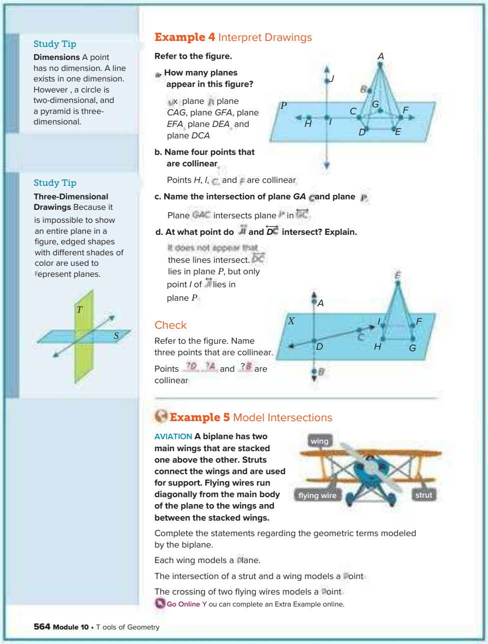
Students tap to view different types of intersections.

TYPE

Students answer a question to show they understand intersections of lines and planes.

G.MG.1, G.CO.1

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**



Study Tip

Dimensions A point has no dimension. A line exists in one dimension. However , a circle is two-dimensional, and a pyramid is three-dimensional.

Study Tip

Three-Dimensional Drawings Because it is impossible to show an entire plane in a figure, edged shapes with different shades of color are used to represent planes.

Example 4 Interpret Drawings

Refer to the figure.

a. How many planes appear in this figure?

six: plane P, plane CAG, plane GFA, plane EFA, plane DEA, and plane DCA

b. Name four points that are collinear.

Points H, I, C, and F are collinear.

c. Name the intersection of plane GAC and plane P.

Plane GAC intersects plane P in $\overleftrightarrow{GC}$.

d. At what point do $\overleftrightarrow{IJ}$ and $\overleftrightarrow{DC}$ intersect? Explain.

It does not appear that these lines intersect. $\overleftrightarrow{DC}$ lies in plane P, but only point I of $\overleftrightarrow{IJ}$ lies in plane P.

Check

Refer to the figure. Name three points that are collinear.

Points ?D, ?A, and ?B are collinear.

Example 5 Model Intersections

AVIATION **A biplane has two main wings that are stacked one above the other. Struts connect the wings and are used for support. Flying wires run diagonally from the main body of the plane to the wings and between the stacked wings.**

Complete the statements regarding the geometric terms modeled by the biplane.

Each wing models a plane.

The intersection of a strut and a wing models a point.

The crossing of two flying wires models a point.

Go Online You can complete an Extra Example online.

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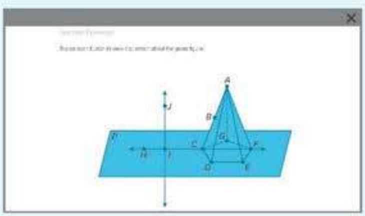
Interactive Presentation



Example 4

TAP

Students tap to reveal information about a diagram.

DRAG & DROP

Students drag terms to match the objects they represent.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 4 Interpret Drawings

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of points, lines, and planes in this example to identify collinear points and name planes.

Questions for Mathematical Discourse

AL When looking for planes, which figure would you look at and why? The pyramid; sample answer: Because the pyramid is three-dimensional, the faces of the pyramid are planes.

OL What three points are considered to be coplanar? Sample answer: Points E, F, and A are coplanar because they lie on the same plane.

BL Does $\overleftrightarrow{IJ}$ intersect plane CDE? If so, name the intersection. yes; point I

Example 5 Model Intersections

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to identify the geometric terms modeled in this example, students will need to analyze the parts of a biplane and determine how they are related mathematically.

Questions for Mathematical Discourse

AL What geometric objects are modeled by the wires? lines

OL What other part of the biplane can represent a line? a strut

BL Is a strut a part of the top wing? Use geometric terms to explain your reasoning. No; sample answer: The strut is a line that intersects the plane of the top wing.

Common Misconception

Students may assume that a line must be drawn between two or more points for the points to be collinear. **Ask:** Draw a point A on your paper and place your pencil above it. If the tip of your pencil is point B, are points A and B collinear? yes Any two points are said to be collinear because it is *possible* to draw a line through them.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–31
2	exercises that use a variety of skills from this lesson	32–44
3	exercises that emphasize higher-order and critical-thinking skills	45–50

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–43 odd, 45–50
- Extension: Fano Plane
- ALEKS Points, Lines, and Planes

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–49 odd
- Remediation, Review Resources: Subtract Integers
- Personal Tutors
- Extra Examples 1–5
- ALEKS Addition and Subtraction with Integers

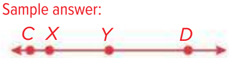
AL

IF students score 65% or less on the Checks,
THEN assign:

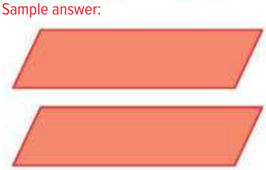
- Practice, Exercises 1–31 odd
- Remediation, Review Resources: Subtract Integers
- *Quick Review Math Handbook*: Points, Lines, and Planes
- ALEKS Addition and Subtraction with Integers

Answers

20. Sample answer:



21. Sample answer:

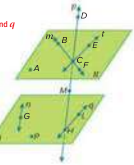


Practice

Go Online You can complete your homework online.

Example 1

Refer to the figure for Exercises 1–7.



1. Name the lines that are only in plane Q. Sample answer: n and q
2. How many planes are labeled in the figure? 2
3. Name the plane containing the lines m and t. plane R
4. Name the intersection of lines m and t. point C
5. Name a point that is not coplanar with points A, B, and C. Sample answer: point P
6. Are points F, M, G, and P coplanar? Explain. No; sample answer: Point F lies in plane R, points G and P lie in plane Q, and point M lies between planes R and Q.
7. Does line n intersect line q? Explain. Yes; sample answer: Line n intersects line q when the lines are extended.

Example 2

Name the geometric terms modeled by each object or phrase.

8. roof of a house

plane

9. a tabletop

plane

10. bridge support beam

line

11. a chessboard

plane

12.

two planes intersecting in a line

13.

point on a line

14. a wall and the floor
two planes intersecting in a line

15. the edge of a table line

16. two connected walls
two planes intersecting in a line

17. a blanket
plane

18. a telephone pole
line

19. a tablet computer
plane

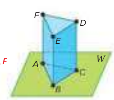
Example 3

USE TOOLS **Draw and label a figure for each relationship.**

20. Points X and Y lie on $\overleftrightarrow{CD}$. See margin.
21. Two planes do not intersect. See margin.
22. Line m intersects plane R at a single point. See margin.
23. Three lines intersect at point J but do not all lie in the same plane. See margin.
24. Points A(2, 3), B(2, −3), C, and D are collinear, but A, B, C, D, and F are not. See margin.

Example 4

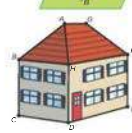
Refer to the figure for Exercises 25–28.



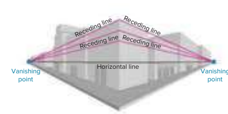
25. How many planes are shown in the figure? 5
26. How many of the planes contain points F and E? 2
27. Name four points that are coplanar. A, B, E, F or B, C, D, E or A, C, D, F
28. Are points A, B, and C coplanar? Explain. Yes; sample answer: Points A, B, and C lie in plane W.

Example 5

29. BUILDING The roof and exterior walls of a house represent intersecting planes. Using the image, name all the lines that are formed by the intersecting planes. AB, AG, AH, BC, BH, CD, DE, DH, EF, FG, FH



30. If the surface of a lake represents a plane, what geometric term is represented by the intersection of a fishing line and the lake's surface? point
31. ART Perspective drawing is a method that artists use to create paintings and drawings of three-dimensional objects. The artist first draws the horizon line and two vanishing points along the horizon. Buildings or other objects are created by drawing receding lines and vertical lines.
 a. Where do the receding lines and horizon lines intersect? The lines intersect at the vanishing points.
 b. Identify examples of planes within this picture. Sample answer: The walls of the building and the ground form planes.



Mixed Exercises

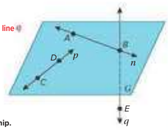
USE TOOLS **Draw and label a figure for each relationship.**

32. $\overleftrightarrow{LM}$ and $\overleftrightarrow{NP}$ are coplanar but do not intersect. See margin.

33. $\overleftrightarrow{FG}$ and $\overleftrightarrow{JK}$ intersect at $P(4, 3)$, where point F is at $(-2, 5)$ and point J is at $(7, 9)$. See margin.

34. Lines s and t intersect, and line v does not intersect either one. See margin.

Refer to the figure for Exercises 35–38.



35. Name a line that contains point E. Sample answer: line q

36. Name a point contained in line n. A or B

37. What is another name for line p? $\overleftrightarrow{CD}$ or $\overleftrightarrow{DC}$

38. Name the plane containing lines n and p. Sample answer: plane G

USE TOOLS **Draw and label a figure for each relationship.**

39. Point K lies on $\overleftrightarrow{RT}$. See margin.

40. Plane J contains line s. See Mod. 10 Answer Appendix.

41. $\overleftrightarrow{YP}$ lies in plane B and contains point C, but does not contain point H. See Mod. 10 Answer Appendix.

42. Lines q and f intersect at point Z in plane U. See Mod. 10 Answer Appendix.

43. Name the geometric term modeled by the object. lines perpendicular to a plane

44. Name the geometric term modeled by a partially-opened folder. planes intersecting in a line

Higher-Order Thinking Skills

45. CREATE Sketch three planes that intersect in a point. See Mod. 10 Answer Appendix.

46. ANALYZE Is it possible for two points on the surface of a prism to be neither collinear nor coplanar? Justify your argument. No; sample answer: There is exactly one line through any two points and exactly one plane through any three points that are not on the same line. Therefore, any two points on the prism must be collinear and coplanar.

47. FIND THE ERROR Camille and Hiroshi are trying to determine the greatest number of lines that can be drawn using any two of four random points. Is either correct? Explain your reasoning. Sample answer: Hiroshi is correct. After you draw the line from the first point to the other three, one of the lines from the second point is already drawn.

Camille	Hiroshi
Because there are four points, 4 • 3 or 12 lines can be drawn between the points.	You can draw 3 • 2 • 1 or 6 lines between the points.

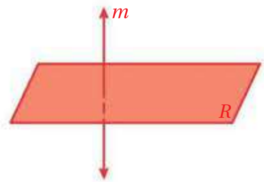
48. PERSEVERE What is the greatest number of planes determined using any three of the points A, B, C, and D if no three points are collinear? 4

49. WRITE A *finite plane* is a plane that has boundaries or does not extend indefinitely. The sides of a cereal box shown are finite planes. Give a real-life example of a finite plane. Is it possible to have a real-life object that is an infinite plane? Explain your reasoning. Sample answer: A table is a finite plane. It is not possible to have a real-life object that is an infinite plane because all real-life objects have boundaries.

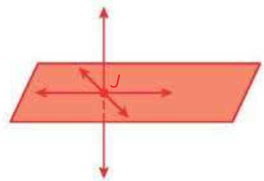
50. CREATE Sketch three planes that intersect in a line. See Mod. 10 Answer Appendix.

Answers

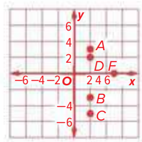
22. Sample answer:



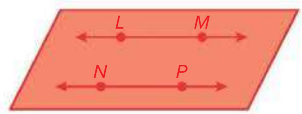
23. Sample answer:



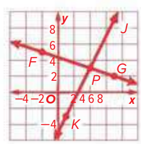
24. Sample answer:



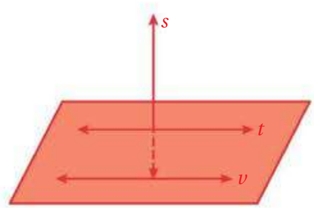
32. Sample answer:



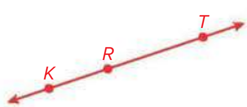
33. Sample answer:



34. Sample answer:



39. Sample answer:



Lesson 10-3

Line Segments

G.CO.1, G.CO.12

LESSON GOAL

Students find measures of line segments.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Using Tools to Determine Betweenness of Points

Develop:

Betweenness of Points
- Find Measurements by Adding
- Find Measurements by Subtracting
- Write and Solve Equations to Find Measurements
- Use Betweenness of Points

Line Segment Congruence
- Write and Solve Equations by Using Congruence

You may want your students to complete the Checks online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Solving One-Step Equations	●	●		●
Extension: Around the World		●	●	●

Language Development Handbook

Assign page 58 of the Language Development Handbook to help your students build mathematical language related to finding the measures of line segments.

ELL You can use the tips and suggestions on page T58 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

Standards for Mathematical Practice:

5 Use appropriate tools strategically.

6 Attend to precision.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students solved one-step equations to find missing values.
6.EE.7, A.REI.3

Now
Students apply betweenness of points to calculate measures of line segments and apply the definition of congruent line segments to find missing values.
G.CO.1, G.CO.12

Next
Students will apply the Distance Formula to find the length of line segments.
G.CO.1

Rigor

The Three Pillars of Rigor

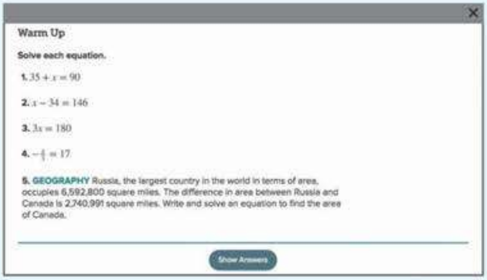
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of line segments, and they build fluency by making constructions related to segments. They apply their understanding by solving real-world problems related to line segments.

Mathematical Background

A line cannot be measured because it extends infinitely in each direction. A line segment, however, has two endpoints and can be measured. Two segments with the same measure are said to be congruent. The symbol for congruence is $\cong$. An equal number of tick marks also indicates that segments are congruent.

Interactive Presentation



Warm Up

Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

• solving one-step equations

Answers:

1. 55
2. 180
3. 60
4. −51
5. $c + 2{,}740{,}991 = 6{,}592{,}800$; 3,851,809 mi^2

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see real-world objects that can be modeled by congruent line segments.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?,* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Using Tools to Determine Betweenness of Points

Objective

Students derive the definition of betweenness of points using graphing and measuring tools.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to complete the activity in the Explore, students will need to use pencil, paper, and a ruler. Work with students to explore and deepen their understanding of dividing segments.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

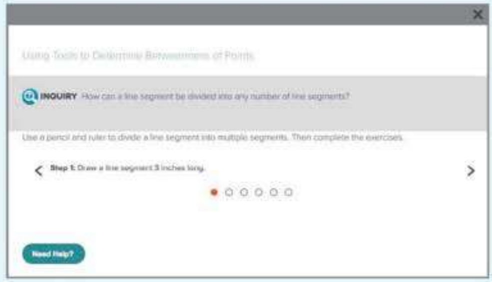
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students tap buttons to see the instructions for the activity. They use pencil, paper, and a ruler to draw line segments and divide them into multiple parts. Students then complete the guiding exercises by measuring the lengths of each part of the line and finding the sum of the measures. Then students write equations that can be used to find the lengths of the entire segments. Next students compare the results of their equations to the actual lengths of the segments and state conclusions based on their comparison. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

TAP

Students tap to explore betweenness of points.

Interactive Presentation

Explore

TYPE

a|

Students will respond to the Inquiry Question and can view a sample answer.

Explore Using Tools to Determine Betweenness of Points (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What would happen if you had drawn the point at a different location on the line segment? Sample answer: The lengths of the line segments created by the new point would be different, but the sum of the lengths would still equal the total length of the original line segment.
- Compare the lengths with a partner. How is his/her example different than yours? How is it similar? Sample answer: The other person's segments have different measurements. The sum of the lengths of their line segments equals the sum of the lengths of my line segments.

Inquiry

How can a line segment be divided into any number of line segments? Sample answer: There are an infinite number of points between the endpoints of a line segment. New line segments can be created by connecting any of the points on the original line segment.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Betweenness of Points

Objective

Students apply betweenness of points to calculate measures of line segments.

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the drawing and the notation of a line or line segment.

Common Misconception

Review how to use a ruler. For metric rulers, explain how centimeters and millimeters are marked. For standard rulers, some students may need to be shown how an inch ruler is divided using marks for $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$. These fractions often need to be added and reduced to get a measurement in inches.

Example 1 Find Measurements by Adding

MP Teaching the Mathematical Practices

2 Attend to Precision Point out that it is important to note the meaning of the quantities used in this problem.

Questions for Mathematical Discourse

AL How can you name all three line segments in the diagram? Sample answer: $\overline{XY}$, $\overline{YZ}$, and $\overline{XZ}$

OL What is XZ in meters? 0.151 m

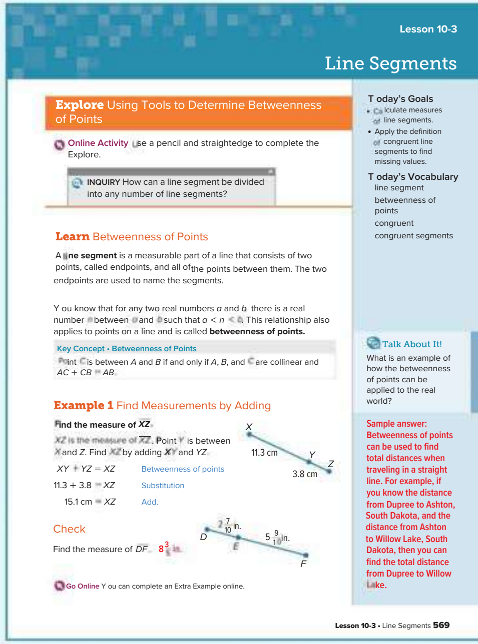
BL Suppose point W is between Y and Z such that $YW = 1.2x$ and $WZ = 3x - 0.4$. If $YZ = 4.6$, what is YW? $YW = 1.2$

DIFFERENTIATE

Reteaching Activity AL ELL
Have students create a game that requires measurement to determine a winner. Many competitive games and sports use measurement to compare athletes and determine the winner. Examples include bocce ball and discus throw.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 10-3

Line Segments

Explore Using Tools to Determine Betweenness of Points

Online Activity Use a pencil and straightedge to complete the Explore.

INQUIRY How can a line segment be divided into any number of line segments?

Learn Betweenness of Points

A **line segment** is a measurable part of a line that consists of two points, called endpoints, and all of the points between them. The two endpoints are used to name the segments.

You know that for any two real numbers a and b there is a real number n between a and b such that $a < n < b$. This relationship also applies to points on a line and is called **betweenness of points.**

Key Concept • Betweenness of Points
Point C is between A and B if and only if A, B, and C are collinear and $AC + CB = AB$.

Example 1 Find Measurements by Adding

Find the measure of $\overline{XZ}$.

XZ is the measure of $\overline{XZ}$. Point Y is between X and Z. Find XZ by adding XY and YZ.

$XY + YZ = XZ$ Betweenness of points
$11.3 + 3.8 = XZ$ Substitution
$15.1 \text{ cm} = XZ$ Add.

Check

Find the measure of $\overline{DF}$. $8\frac{3}{5}$ in.

Go Online You can complete an Extra Example online.

Today's Goals
- Calculate measures of line segments.
- Apply the definition of congruent line segments to find missing values.

Today's Vocabulary
line segment
betweenness of points
congruent
congruent segments

Talk About It!
What is an example of how the betweenness of points can be applied to the real world?

Sample answer: Betweenness of points can be used to find total distances when traveling in a straight line. For example, if you know the distance from Dupree to Ashton, South Dakota, and the distance from Ashton to Willow Lake, South Dakota, then you can find the total distance from Dupree to Willow Lake.

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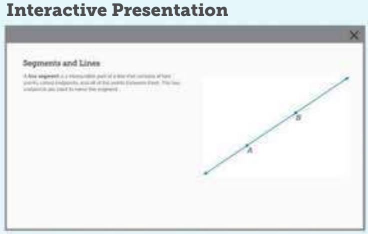
Interactive Presentation



Learn

TAP

Students tap to reveal information on betweenness of points.

Problem-Solving Tip

Draw a Diagram Draw a diagram to help you see and correctly interpret a situation that has been described in words.

Example 2 Find Measurements by Subtracting

Find the measure of $\overline{QR}$.

Point Q is between points P and R.

$PQ + QR = PR$ Betweenness of Points

$6\frac{5}{8} + QR = 13\frac{3}{4}$ Substitution

$QR = 7\frac{1}{8}$ ft Subtract $6\frac{5}{8}$ from each side and simplify.

Check

Find the measure of $\overline{PQ}$. Round your answer to the nearest tenth, if necessary.

6.5 ? cm

Think About It!

How can you check your solution for x?

Sample answer: You can substitute the value of x into the original equation to check your solution. If the two sides of the equation are not equal, then you have made an error.

$4x - 12 = x + 2x + 3$

$4(15) - 12 \stackrel{?}{=} 15 + 2(15) + 3$

$60 - 12 \stackrel{?}{=} 15 + 30 + 3$

$48 = 48$

Example 3 Write and Solve Equations to Find Measurements

Find the value of x and BC if B is between A and C, $AC = 4x - 12$, $AB = x$, and $BC = 2x + 3$.

Step 1 Plot two points and label them A and C. Connect the points.

Step 2 Plot point B between points A and C.

Step 3 Label segments AB, BC, and AC with their given measures.

Step 4 Use betweenness of points to write an equation and solve for x.

$AC = AB + BC$ Betweenness of points

$4x - 12 = x + 2x + 3$ Substitution

$4x - 12 = 3x + 3$ Combine like terms.

$x - 12 = 3$ Subtract $3x$ from each side. Simplify.

$x = 15$ Add 12 to each side. Simplify.

Now find BC.

$BC = 2x + 3$ Given

$= 2(15) + 3$ $x = 15$

$= 33$

Think About It!

Once you find BC, how could you find AC without evaluating $AC = 4x - 12$?

Sample answer: You could add the value of BC to the value of AB, which is the same as the value of x. $33 + 15 = 48$.

Go Online You can complete an Extra Example online.

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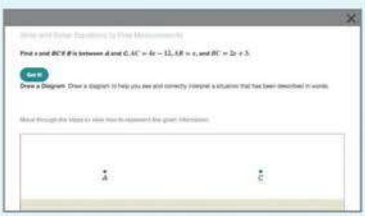
Interactive Presentation



Example 3

TAP

Students tap to reveal steps in the problem.

Example 2 Find Measurements by Subtracting

Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the quantities used in this problem.

Questions for Mathematical Discourse

AL What inequality statement compares $\overline{PR}$ and $\overline{QR}$? Sample answer: $\overline{PR} > \overline{QR}$

OL Suppose $PQ = 4\frac{1}{3}$ ft. What is QR? $9\frac{5}{12}$ ft

BL If point D is between points P and Q and $DQ = 6$ ft, how far from point P would point D have to be? $\frac{5}{8}$ ft

Common Error

Students may add instead of subtracting or subtract fractional quantities incorrectly. Guide them to set up and compute their answers correctly.

Example 3 Write and Solve Equations to Find Measurements

Teaching the Mathematical Practices

8 Attend to Details Mathematically proficient students continually ask themselves, “Does this make sense?” Point out that in this example, students will evaluate the reasonableness of their answer.

Questions for Mathematical Discourse

AL Of the three line segments, which is the longest? AC

OL Which two segment lengths should be added together to equal the third? AB and BC should be added together to equal AC.

BL If AB is ten more than twice the length of BC, and AC is two less than four times the length of BC, how long is BC? 12 units

Common Error

Students may incorrectly set up the equation. Help them model their equation based on a correctly drawn figure.

Apply Example 4 Use Betweenness of Points

Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the quantities used in this problem.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- H ow can you write phrases such as *10 feet more than six times the distance* as expressions?
- W hat do you notice about the height of the Space Needle?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Apply Example 4 Use Betweenness of Points

SPACE NEEDLE Darrell is visiting the Space Needle in Seattle, Washington. He knows that the total height of the Space Needle is 605 feet. The distance from the ground to the observation deck is 10 feet more than six times the distance from the observation deck to the top of the Space Needle. Help Darrell find the distance from the ground to the observation deck.

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to find the distance from the ground to the observation deck. How does the distance from the ground to the observation deck compare to the total height of the Space Needle? I can express the information that I am given in the exercise as an equation, solve for any missing information, and then use that information to find the answer.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will express the information that I am given into an equation that represents the total height of the Space Needle. I have learned how to convert written information into expressions, and I have learned how to solve equations.

3 What is your solution?

Use your strategy to solve the problem.

What equation represents the distance from the ground to the top of the Space Needle?

$x + 6x + 10 = 605$

What is the distance from the ground to the observation deck?

520 ft

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: 520 feet seems reasonable for the distance from the ground to the observation deck. The distance from the observation deck to the top of the Space Needle is 85 feet. The combined heights are realistic compared to the total height.

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Interactive Presentation

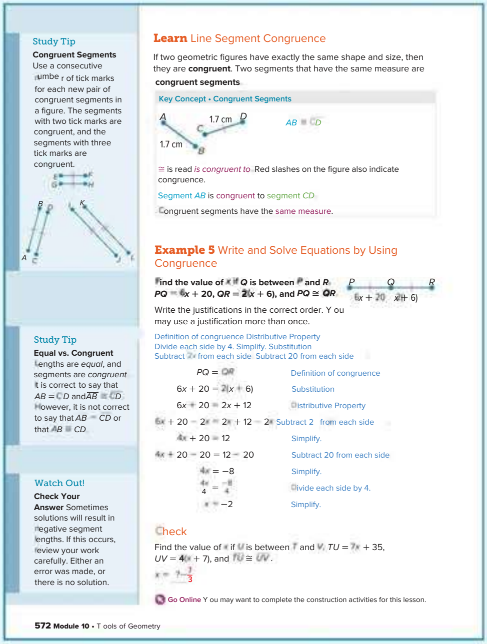
Apply Example 4

DRAG & DROP

Students drag justifications to complete the solution.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Study Tip

Congruent Segments Use a consecutive number of tick marks for each new pair of congruent segments in a figure. The segments with two tick marks are congruent, and the segments with three tick marks are congruent.

Learn Line Segment Congruence

If two geometric figures have exactly the same shape and size, then they are **congruent**. Two segments that have the same measure are **congruent segments**.

Key Concept • Congruent Segments

A, 1.7 cm, C, D, 1.7 cm, B — $\overline{AB} \cong \overline{CD}$

≅ is read *is congruent to*. Red slashes on the figure also indicate congruence.

Segment *AB* is congruent to segment *CD*.

Congruent segments have the same measure.

Example 5 Write and Solve Equations by Using Congruence

Find the value of *x* if *Q* is between *P* and *R*, *PQ* = 6*x* + 20, *QR* = 2(*x* + 6), and $\overline{PQ} \cong \overline{QR}$.

Write the justifications in the correct order. You may use a justification more than once.

Definition of congruence, Distributive Property, Divide each side by 4, Simplify, Substitution, Subtract 2*x* from each side, Subtract 20 from each side

$PQ = QR$	Definition of congruence
$6x + 20 = 2(x + 6)$	Substitution
$6x + 20 = 2x + 12$	Distributive Property
$6x + 20 - 2x = 2x + 12 - 2x$	Subtract 2x from each side
$4x + 20 = 12$	Simplify.
$4x + 20 - 20 = 12 - 20$	Subtract 20 from each side
$4x = -8$	Simplify.
$\frac{4x}{4} = \frac{-8}{4}$	Divide each side by 4.
$x = -2$	Simplify.

Study Tip

Equal vs. Congruent Lengths are *equal*, and segments are *congruent*. It is correct to say that $AB = CD$ and $\overline{AB} \cong \overline{CD}$. However, it is not correct to say that $\overline{AB} = \overline{CD}$ or that $AB \cong CD$.

Watch Out!

Check Your Answer Sometimes solutions will result in negative segment lengths. If this occurs, review your work carefully. Either an error was made, or there is no solution.

Check

Find the value of *x* if *U* is between *T* and *V*, *TU* = 7*x* + 35, *UV* = 4(*x* + 7), and $\overline{TU} \cong \overline{UV}$.

$x = ?-\frac{7}{3}$

Go Online You may want to complete the construction activities for this lesson.

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Learn Line Segment Congruence

Objective

Students apply the definition of congruent line segments to find missing values.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Example 5 Write and Solve Equations by Using Congruence

Teaching the Mathematical Practices

1 Check Answers Mathematically proficient students continually ask themselves, “Does this make sense?” Point out that in this example, students need to check their answer. Point out that they should ask themselves whether their solution makes sense and whether they have answered the problem question.

Questions for Mathematical Discourse

AL How do you determine whether a quotient is negative or positive when dividing with negative numbers? The quotient of a positive dividend and a positive divisor is positive; the quotient of a negative dividend and a negative divisor is positive; the quotient of a negative dividend and a positive divisor is negative; and the quotient of a positive dividend and a negative divisor is negative.

OL What is *PQ*? 68 units

BL If your result was a negative segment length, what should you do? Explain. Negative lengths are not possible. Distance is always positive. You should look for a mistake in your solution.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Interactive Presentation

Example 5

DRAG & DROP

Students drag justifications to complete the equation.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–33
2	exercises that use a variety of skills from this lesson	34–38
2	exercises that extend concepts learned in this lesson to new contexts	39–46
3	exercises that emphasize higher-order and critical-thinking skills	47–51

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–33 odd, 47–51
- Extension: Solving One-Step Equations
- ALEKS Points, Lines, and Planes

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–51, odd
- Remediation, Review Resources: Solving One-Step Equations
- Personal Tutors
- Extra Examples 1–5
- ALEKS One-Step Linear Equations

IF students score 65% or less on the Checks, **THEN** assign: AL

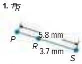
- Practice, Exercises 1–33 odd
- Remediation, Review Resources: Solving One-Step Equations
- *Quick Review Math Handbook*: Linear Measure
- ALEKS One-Step Linear Equations

Practice

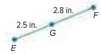
Go Online Y ou can complete your homework online.

Examples 1 and 2

Find the measure of each segment.

1. $\overline{PR}$


2. 1 mm

2. $\overline{EF}$


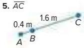
5.3 in.

3. $\overline{JL}$

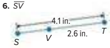

1. 1 cm

4. $\overline{HJ}$

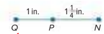

9. 1 ft

5. $\overline{AC}$


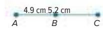
2.0 m

6. $\overline{SV}$


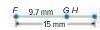
1.5 in.

7. $\overline{NQ}$


$2\frac{1}{4}$ in.

8. $\overline{AC}$


10. 1 cm

9. $\overline{GH}$


5.3 mm

Example 3

Find the value of the variable and *YZ* if *Y* is between *X* and *Z*.

10. $XY = 11$, $YZ = 4c$, $XZ = 83$
$c = 18$; $YZ = 72$

11. $XY = 6b$, $YZ = 8b$, $XZ = 175$
$b = 12.5$; $YZ = 100$

12. $XY = 7a$, $YZ = 5a$, $XZ = 6a + 24$
$a = 4$; $YZ = 20$

13. $XY = 5.5$, $YZ = 2c$, $XZ = 8.9$
$c = 1.7$; $YZ = 3.4$

14. $XY = 5n$, $YZ = 2n$, $XZ = 91$
$n = 13$; $YZ = 26$

15. $XY = 4w$, $YZ = 6w$, $XZ = 12w - 8$
$w = 4$; $YZ = 24$

16. $XY = 11d$, $YZ = 9d - 2$, $XZ = 5d + 28$
$d = 2$; $YZ = 16$

17. $XY = 4n + 3$, $YZ = 2n - 7$, $XZ = 20$
$n = 4$; $YZ = 1$

18. $XY = 3a - 4$, $YZ = 6a + 2$, $XZ = 5a + 22$
$a = 6$; $YZ = 38$

19. $XY = 3k - 2$, $YZ = 7k + 4$, $XZ = 4k + 38$
$k = 6$; $YZ = 46$

20. $XY = 4x$, $YZ = x$, and $XZ = 25$
$x = 5$; $YZ = 5$

21. $XY = 4x$, $YZ = 3x$, and $XZ = 42$
$x = 6$; $YZ = 18$

22. $XY = 12$, $YZ = 2x$, and $XZ = 28$
$x = 8$; $YZ = 16$

23. $XY = 2x + 1$, $YZ = 6x$, and $XZ = 81$
$x = 10$; $YZ = 60$

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Example 4

24. RAILROADS A straight railroad track is being built to connect two cities. The measured distance of the track between the two cities is 160.5 miles. A mail stop is 28.5 miles from the first city. How far is the mail stop from the second city? 132 mi

25. CARPENTRY A carpenter has a piece of wood that is 78 inches long. He wants to cut it so that one piece is five times as long as the other piece. What are the lengths of the two pieces? 13 in. and 65 in.

26. WALKING Marshall lives 2300 yards from school and 1500 yards from the pharmacy. The school, pharmacy, and his home are all collinear, as shown in the figure.

What is the distance from the pharmacy to the school? 800 yd

27. COFFEE SHOP Chenoa wants to stop for coffee on her way to school. The distance from Chenoa's house to the coffee shop is 3 miles more than twice the distance from the coffee shop to Chenoa's school. The total distance from Chenoa's house to her school is 5 times the distance from the coffee shop to her school.

a. What is the distance from Chenoa's house to the coffee shop? Write your answer as a decimal, if necessary. 6 mi

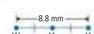
b. What assumptions did you make when solving this problem?
Sample answer: I assumed the three locations were in a straight line.

Example 5

Find the measure of each segment.

28. $\overline{MO}$


9.2 cm

29. $\overline{WY}$


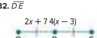
4.4 mm

30. $\overline{FG}$


4.2 cm

31. $\overline{QT}$


10.8 in.

32. $\overline{DE}$


26 units

33. $\overline{UX}$


66 units

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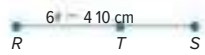
Mixed Exercises

34. Find the length of $\overline{UW}$ if W is between U and V, $UV = 16.8$ centimeters, and $VW = 7.9$ centimeters. 8.9 cm

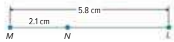
35. Find the value of x if $RS = 24$ centimeters. 3



36. Find the length of $\overline{LO}$ if M is between L and O, $LM = 7x - 9$, $MO = 14$ inches, and $LO = 10x - 7$. 33 in.

37. Find the value of x if $\overline{PQ} \cong \overline{RS}$, $PQ = 9x - 7$, and $RS = 29$. 4

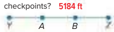
38. Find the measure of $\overline{NL}$. 3.7 cm



39. PRECISION If point P is between A and M, write a true statement.
Sample answer: $AP + PM = AM$

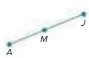
40. HIKING A hiking trail is 20 kilometers long. Park organizers want to build 5 rest stops for hikers with one on each end of the trail and the other 3 spaced evenly between. How much distance will separate successive rest stops? 5 km

41. RACE The map shows the route of a race. You are at Y, 6000 feet from the first checkpoint A. The second checkpoint B is located at the midpoint between A and the end of the race Z. The total race is 3.1 miles. How far apart are the two checkpoints? 5184 ft



42. FIELD TRIP The marching band at Jefferson High School is taking a field trip from Lansing, Michigan, to Detroit, Michigan. The bus driver was told to stop 53 miles into the trip. If the rest of the trip is 41 miles and the entire journey can be represented by the expression $3x + 16$, find the value of x. 26

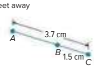
43. DISTANCE Madison lives between Anoa and Jamie as depicted on the line segment. The distance between Anoa's house and Madison's house is represented by $3x + 2$ miles, the distance between Madison's house and Jamie's house is represented by $3x + 4$ miles, and the distance between Anoa's house and Jamie's house is represented by $9x - 3$ miles. Find the value of x. Then find the distance between Madison's house and Jamie's house. $x = 3$; 13 mi



44. FIREFIGHTING A firefighter training course is taking place in a high-rise building. The high-rise building where they practice is 48 stories high. If the emergency happens on the top floor and the firefighters have already gone 29 stories, how many stories do they still need to go? 19 stories

45. CAFE You are waiting at the end of a long straight line at Coffee Express. Your friend Denzel is $r + 12$ feet in front of you. Denzel is $2r + 4$ feet away from the front of the line. If Denzel is in the exact middle of the line, how many feet away are you from the front of the line? 40 ft

46. REASONING For $\overline{AC}$, write and solve an equation to find AB. $1.5 + AB = 3.7$; $AB = 2.2$ cm



Higher-Order Thinking Skills

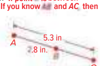
47. PERSEVERE Point K is between points J and L. If $JK = x^2 - 4x$, $KL = 3x - 2$, and $JL = 28$, find JK and KL. $JK = 12$, $KL = 16$

48. ANALYZE Determine whether the statement *If point M is between points C and D, then CD is greater than either CM and MD* is *sometimes, always,* or *never* true. Justify your argument. Always; sample answer: If point M is between points C and D, then $CM + MD = CD$. Because measures cannot be negative, CD, which represents the whole, must always be greater than either of the measures of its parts, CM or MD.

49. PERSEVERE Point C is located between points B and D. Also, $BC = x + 7$, $CD = 3y + 4$, $BD = 38$, and $BD = 2x + 8y$. Find the values of x and y. $x = 3$; $y = 4$

50. WRITE If point B is between points A and C, explain how you can find AC if you know AB and BC. Explain how you can find BC if you know AB and AC. If point B is between points A and C and you know AB and BC, then add AB and BC to find AC. If you know AB and AC, then subtract AB from AC to find BC.

51. CREATE Sketch line segment AC. Plot point B between A and C. Use a ruler to find AC and AB. Then write and solve an equation to find BC. Sample answer: $2.8 + BC = 5.3$; $BC = 2.5$ in



Lesson 10-4

Distance

G.CO.1

LESSON GOAL

Students apply the Distance Formula to find lengths of line segments.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Distance on a Number Line
- Find Distance on a Number Line
- Determine Segment Congruence

Explore: Using the Pythagorean Theorem to Find Distances

Distance on the Coordinate Plane
- Find Distance on the Coordinate Plane
- Calculate Distance in the Real World

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	BL	ELL	
Remediation: Solving Multi-Step Equations	●	●		●
Extension: Taxicab Geometry		●	●	●

Language Development Handbook

Assign page 59 of the *Language Development Handbook* to help your students build mathematical language related to applying the Distance Formula to find the lengths of line segments.

ELL You can use the tips and suggestions on page T59 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
2 Reason abstractly and quantitatively.
6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students used the Pythagorean Theorem to find the distance between two points on the coordinate plane.
8.G.8

Now
Students apply the Distance Formula to find the length of a line segment.
G.CO.1

Next
Students will determine the coordinates of a point that is a given fractional distance less than one from one end of a line segment to the other.
G.GPE.6

Rigor

The Three Pillars of Rigor

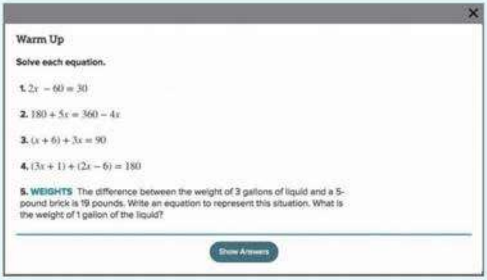
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of distance along a line. They apply their understanding by solving real-world problems related to linear distance.

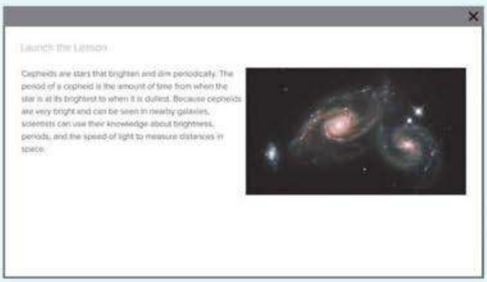
Mathematical Background

The coordinates of the endpoints of a segment can be used to find the length of the segment. On a number line, the distance between the endpoints is the absolute value of their difference. On a coordinate plane, you can use the Distance Formula or the Pythagorean Theorem to calculate the distance between two points.

Interactive Presentation



Warm Up



Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- solving multi-step equations

Answers:

1. 45
2. 20
3. 21
4. 37
5. $3g - 5 = 19$; 8 lb

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of measures of line segments.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then, discuss the question below with the class.

Explore Using the Pythagorean Theorem to Find Distances

Objective

Students use dynamic geometry software to calculate the distance between two points on the coordinate plane using the Pythagorean Theorem.

MP Teaching the Mathematical Practices

5 Decide When to Use Tools Mathematically proficient students can make sound decisions about when to use mathematical tools such as dynamic geometry software. Help them see why using these tools will help to solve problems and what the limitations are of using the tools.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

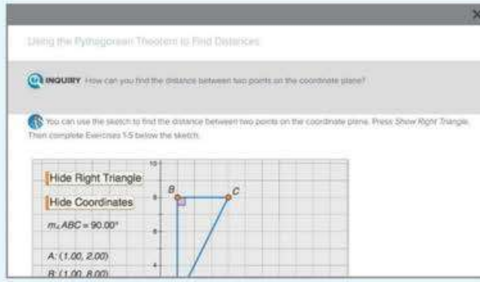
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students tap to reveal the parts of the diagram that is used to compute the distance between two points on the coordinate plane. Then students complete the guiding exercises. As they do this, students use the Pythagorean Theorem to compute the distance between the two given points. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

WEB SKETCHPAD

Students use a sketch to explore using the Pythagorean Theorem to find distances on the coordinate plane.

TAP

Students tap to reveal parts of the diagram.

Interactive Presentation

Explore

TAP

Students tap to select the correct answer to an exercise.

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Using the Pythagorean Theorem to Find Distances (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Which side of a triangle is the hypotenuse? The side opposite the right angle is the hypotenuse.
- Why is distance found using the Pythagorean Theorem? Can it be found just by counting spaces of the coordinate plane? The line is diagonal, so you cannot just count squares. You need to use the Pythagorean Theorem instead.

Inquiry

How can you find the distance between two points on the coordinate plane? Sample answer: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Distance on a Number Line

Objective

Students apply the Distance Formula to find the length of a line segment on a number line.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Essential Question Follow-Up

Students learn how to compute distances on a number line.

Ask:

Why is it important to know how to compute distances on a number line? Sample answer: You can compute distances along a straight line in the real world.

Example 1 Find Distance on a Number Line

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL What is absolute value? Absolute value is the distance between two numbers.

OL When using the distance formula to find the distance between points F and C, does it matter which endpoint is used for x_1 and which endpoint is used for x_2? Explain. No; distance is the absolute value of the difference between two points.

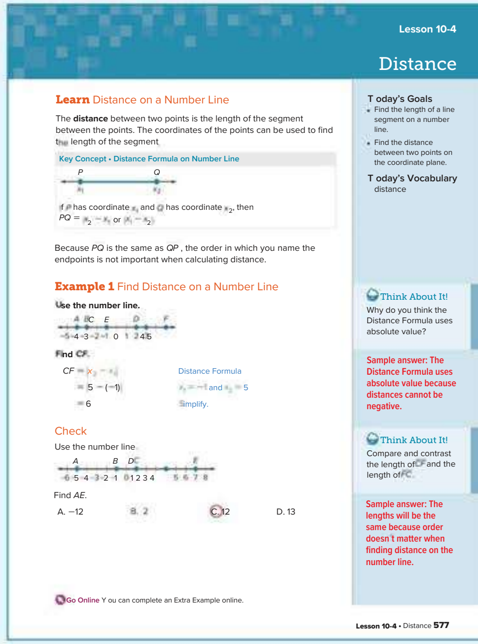
BL What line segments are congruent to $\overline{CF}$? $\overline{AD}$ and $\overline{BE}$

Common Error

Students may incorrectly subtract negative numbers. Remind them that subtracting a negative number is the same as adding the absolute value of that number.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.


Lesson 10-4

Distance

Learn Distance on a Number Line

The **distance** between two points is the length of the segment between the points. The coordinates of the points can be used to find the length of the segment

Key Concept • Distance Formula on Number Line

If P has coordinate x_1 and Q has coordinate x_2, then $PQ = |x_2 - x_1|$ or $|x_1 - x_2|$

Because PQ is the same as QP, the order in which you name the endpoints is not important when calculating distance.

Example 1 Find Distance on a Number Line

Use the number line.

Find CF.

$CF = |x_2 - x_1|$ Distance Formula

$= |5 - (-1)|$ $x_1 = -1$ and $x_2 = 5$

$= 6$ Simplify.

Check

Use the number line.

Find AE.

A. −12 B. 2 C. 12 D. 13

Go Online You can complete an Extra Example online.

Today's Goals
- Find the length of a line segment on a number line.
- Find the distance between two points on the coordinate plane.

Today's Vocabulary
distance

Think About It!
Why do you think the Distance Formula uses absolute value?

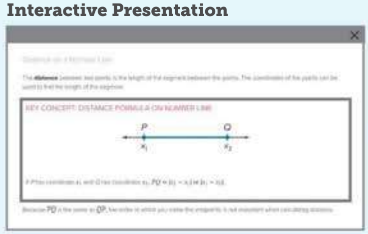
Sample answer: The Distance Formula uses absolute value because distances cannot be negative.

Think About It!
Compare and contrast the length of $\overline{CF}$ and the length of $\overline{FC}$.

Sample answer: The lengths will be the same because order doesn't matter when finding distance on the number line.

Lesson 10-4 • Distance 577

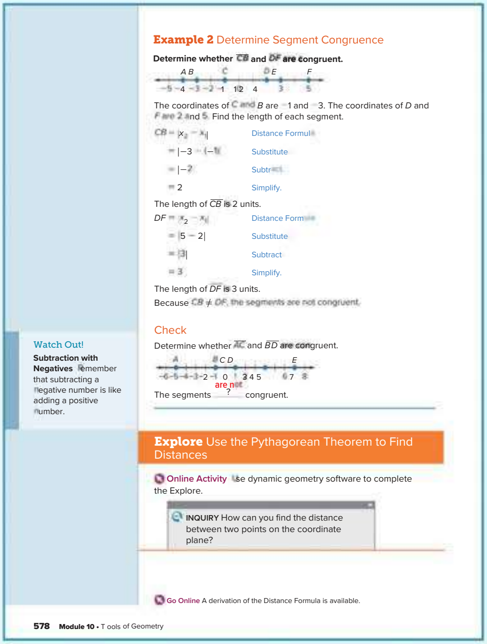
Interactive Presentation



Learn

TYPE

Students respond to the Think About It! question.



Example 2 Determine Segment Congruence

Determine whether $\overline{CB}$ and $\overline{DF}$ are congruent.

The coordinates of C and B are −1 and −3. The coordinates of D and F are 2 and 5. Find the length of each segment.

$CB = |x_2 - x_1|$ Distance Formula
$= |-3 - (-1)|$ Substitute
$= |-2|$ Subtract.
$= 2$ Simplify.

The length of $\overline{CB}$ is 2 units.

$DF = |x_2 - x_1|$ Distance Formula
$= |5 - 2|$ Substitute
$= |3|$ Subtract
$= 3$ Simplify.

The length of $\overline{DF}$ is 3 units.

Because $CB \neq DF$, the segments are not congruent.

Watch Out! **Subtraction with Negatives** Remember that subtracting a negative number is like adding a positive number.

Check

Determine whether $\overline{AC}$ and $\overline{BD}$ are congruent.

The segments are not congruent.

Explore Use the Pythagorean Theorem to Find Distances

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How can you find the distance between two points on the coordinate plane?

Go Online A derivation of the Distance Formula is available.

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Example 2 Determine Segment Congruence

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

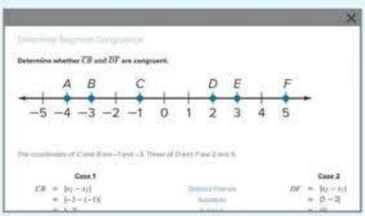
Questions for Mathematical Discourse

AL How do you subtract negative numbers? Add the opposite of the 2nd number to the 1st number.

OL Is $|2 - (-6)| = |-6 - 2|$? Explain. Yes; both equal 8.

BL If each of the four coordinates was its opposite, would the two segments be congruent? No; the distances would still equal 2 and 3.

Interactive Presentation



Example 2

SELECT

Students select the solution from a list of choices.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Learn Distance on the Coordinate Plane

Objective

Students apply the Distance Formula to find the distance between two points on the coordinate plane.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Common Misconception

After subtracting in the Distance Formula, students will need to square a negative number. Remind them that the square of a negative number is a positive number.

DIFFERENTIATE

Enrichment Activity BL

Have each student pair or group of three create a scenario that uses adding and subtracting negative numbers in real-life situations. Scenarios may include buying something at a store, borrowing money, paying debts, or depositing money. Ask pairs/groups to share their scenarios with the class.

Example 3 Find Distance on the Coordinate Plane

Teaching the Mathematical Practices

1 Understand the Approaches of Others Work with students to look at the Alternate Method. Ask students to compare and contrast the original method and the alternate method.

Questions for Mathematical Discourse

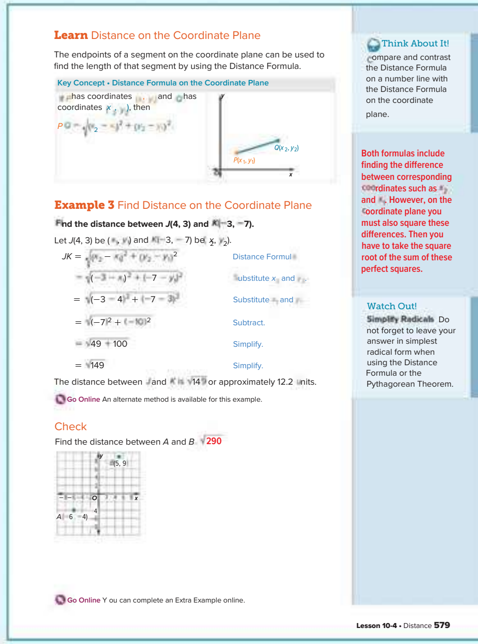
AL Explain why the distance is not $\sqrt{-149}$. When you square a negative value, the result is a positive value, so $\sqrt{(-7)^2 + (-10)^2} = \sqrt{49 + 100} = \sqrt{149}$.

OL Does it matter if you use (4, 3) or (−3, −7) for (x_1, y_1)? Explain. No; sample answer: The distance between the points will be the same no matter which point you use.

BL You can also use a right triangle and the Pythagorean Theorem to find the distance. If $J(4, 3)$ and $K(-3, -6)$ are two vertices of the right triangle, name two other points that could be used to form a right triangle. (−3, 3) or (4, −7)

Common Error

Students may take the square roots of the addends in the sum before adding, when they should add first and then take the square root. Make sure that students understand that they cannot separate a square root into two square roots at a plus sign.



Learn Distance on the Coordinate Plane

The endpoints of a segment on the coordinate plane can be used to find the length of that segment by using the Distance Formula.

Key Concept • Distance Formula on the Coordinate Plane

If P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2), then

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 3 Find Distance on the Coordinate Plane

Find the distance between $J(4, 3)$ and $K(-3, -7)$.

Let $J(4, 3)$ be (x_1, y_1) and $K(-3, -7)$ be (x_2, y_2).

$JK = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{(-3 - x_1)^2 + (-7 - y_1)^2}$	Substitute x_2 and y_2.
$= \sqrt{(-3 - 4)^2 + (-7 - 3)^2}$	Substitute x_1 and y_1.
$= \sqrt{(-7)^2 + (-10)^2}$	Subtract.
$= \sqrt{49 + 100}$	Simplify.
$= \sqrt{149}$	Simplify.

The distance between J and K is $\sqrt{149}$ or approximately 12.2 units.

Go Online An alternate method is available for this example.

Check

Find the distance between A and B. $\sqrt{290}$

Go Online You can complete an Extra Example online.

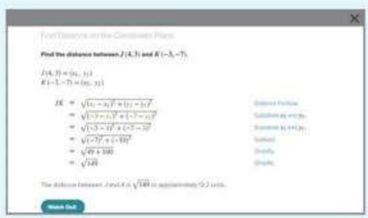
Think About It!
Compare and contrast the Distance Formula on a number line with the Distance Formula on the coordinate plane.

Both formulas include finding the difference between corresponding coo rdinates such as x_2 and x_1. However, on the coordinate plane you must also square these differences. Then you have to take the square root of the sum of these perfect squares.

Watch Out!
Simplify Radicals Do not forget to leave your answer in simplest radical form when using the Distance Formula or the Pythagorean Theorem.

Lesson 10-4 • Distance 579

Interactive Presentation



Example 3

TAP

Students tap to reveal an Alternative Method.

TYPE

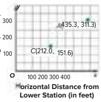
Students respond to the Think About It! question.

Example 4 Calculate Distance

INCLINE Chelsea and Amie are sitting in separate cars on the Monongahela Incline. Chelsea is traveling up Mount Washington and Amie is traveling down. When the two girls notice each other, Chelsea has a horizontal distance of 212.0 feet from the lower station and is at a height of 151.6 feet. Amie has a horizontal distance of 435.3 feet from the lower station and is at a height of 311.3 feet. What is the distance between the two girls?

Step 1 Draw a diagram.

Draw a diagram to represent the situation. Label the x-axis as the "Horizontal Distance from Lower Station (in feet)." Label the y-axis as the "Height (in feet)." Use a scale of 50 on the x-axis and the y-axis.



Step 2 Use the Distance Formula.

$(x_1, y_1) = (212.0, 151.6)$ and $(x_2, y_2) = (435.3, 311.3)$

$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{(435.3 - 212.0)^2 + (311.3 - 151.6)^2}$	Substitute.
$= \sqrt{223.3^2 + 159.7^2}$	Subtract
$= \sqrt{49{,}862.89 + 25{,}504.09}$	Square each term.
$= \sqrt{75{,}366.98}$	Add.
≈ 274.5	Take the positive square root.

Chelsea and Amie are approximately 274.5 feet apart.

Think About It!

Does your answer seem reasonable? Why or why not?

Yes; sample answer: Compared to the height and horizontal distances between Chelsea and Amie, my answer of 274.5 feet is realistic.

Check

SNOWBOARDING Manuel wants to go snowboarding with his friend. The closest ski and snowboard resort is approximately 20 miles west and 50 miles north of his house. Manuel picks up his friend who lives 15 miles south and 10 miles east of Manuel's house. How far away are the two boys from the resort?

71.6 ? mi

Go Online You can complete an Extra Example online.

580 Module 10 • Tools of Geometry

Interactive Presentation

Example 4

DRAG & DROP

Students drag points to draw a diagram to represent real-world situations.

CHECK

Students complete the Check online to determine whether they are ready to move on.

DIFFERENTIATE

Reteaching Activity AL ELL

IF Students are struggling to remember the Distance Formula and/or its steps

THEN use a graphic organizer to help students organize and label their steps.

Example 4 Calculate Distance

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This example asks students to justify their conclusions.

Questions for Mathematical Discourse

AL What ordered pair represents Chelsea's position when the girls notice each other? (212.0, 151.6)

OL What ordered pair represents a distance of 100 feet from point *C*? Sample answer: (312, 151.6)

BL If you made a right triangle to find the distance, what is the length of each leg of the triangle? 223.3 ft and 159.7 ft

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–30
2	exercises that use a variety of skills from this lesson	31–46
2	exercises that extend concepts learned in this lesson to new contexts	47–48
3	exercises that emphasize higher-order and critical-thinking skills	49–54

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–47 odd, 49–54
- Extension: Taxicab Geometry
- ALEKS Distances and Midpoints on a Number Line, Distances and Midpoints in the Coordinate Plane

IF students score 66%–89% or more on the Checks, **THEN** assign: OL

- Practice, Exercises 1–53 odd
- Remediation, Review Resources: Solving Multi-Step Equations
- Personal Tutors
- Extra Examples 1–4
- ALEKS Multi-Step Linear Equations

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–29 odd
- Remediation, Review Resources: Solving Multi-Step Equations
- *Quick Review Math Handbook*: Distance and Midpoints
- ALEKS Multi-Step Linear Equations

Practice

Go Online Y ou can complete your homework online.

Example 1

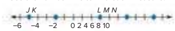
Use the number line to find each measure.

1. JL 5
2. JK 3
3. KP 9
4. NP 2
5. JP 12
6. LN 5

Use the number line to find each measure.

7. JK 3
8. LK 2
9. FG 3
10. JG 6
11. EH 9
12. LF 14

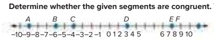
Use the number line to find each measure.



13. LN 6
14. JL 8

Example 2

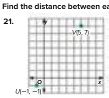
Determine whether the given segments are congruent. Write *yes* or *no*.



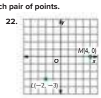
15. $\overline{AB}$ and $\overline{EF}$ yes
16. $\overline{BD}$ and $\overline{DF}$ yes
17. $\overline{AC}$ and $\overline{CD}$ no
18. $\overline{AC}$ and $\overline{DE}$ yes
19. $\overline{BE}$ and $\overline{CF}$ no
20. $\overline{CD}$ and $\overline{DF}$ no

Example 3

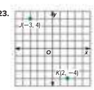
Find the distance between each pair of points.

21. 

10 units

22. 

$\sqrt{45}$ or about 6.7 units

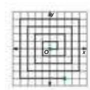
23. 

$\sqrt{89}$ or about 9.4 units

24. $A(2, 6)$, $N(5, 10)$ 5 units
25. $R(3, 4)$, $T(7, 2)$ $\sqrt{20}$ or about 4.5 units
26. $X(-3, 8)$, $Z(-5, 1)$ $\sqrt{53}$ or about 7.3 units

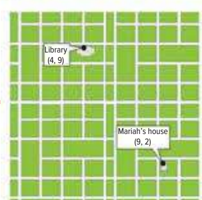
Example 4

27. SPIRALS Denise traces the spiral shown in the figure. The spiral begins at the origin. What is the shortest distance between Denise's starting point and her ending point? $\sqrt{20}$ or approximately 4.5 units

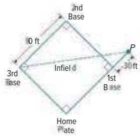


28. ZOOLOGY A tiny songbird called the blackpoll warbler migrates each fall from North America. A tracking study showed one bird flew from Vermont at map coordinates (63, 45) to Venezuela at map coordinates (67, 10) in three days. If each map coordinate represents 75 kilometers, how far did the bird travel? 2642 km

29. CONSTRUCT ARGUMENTS Mariah is training for a sprint-distance triathlon. She plans on cycling from her house to the library, shown on the grid with a scale in miles. If the cycling portion of the triathlon is 12 miles, will Mariah have cycled at least $\frac{2}{3}$ of that distance during her bike ride? Justify your argument. Y es; sample answer: The distance between Mariah's house and the library is $\sqrt{74}$ or about 8.6 miles. Because $\frac{2}{3}$ of 12 miles is 8 miles, Mariah's bike ride is more than $\frac{2}{3}$ of the cycling portion of the triathlon.



30. SPORTS The distance between each base on a baseball infield is 90 feet. The third baseman throws a ball from third base to point P. To the nearest foot, how far did the player throw the ball? **150 ft**



Mixed Exercises

Find the distance between each pair of points. Round to the nearest tenth, if necessary.

31. $M(-4, 9)$, $N(-5, 3)$ **$\sqrt{37}$ or about 6.1 units**

32. $C(2, 4)$, $D(5, 7)$ **$\sqrt{18}$ or about 4.2 units**

33. $A(5, 1)$, $B(3, 6)$ **$\sqrt{29}$ or about 5.4 units**

34. $V(4, 4)$, $X(5, 8)$ **$\sqrt{17}$ or about 4.1 units**

35. $S(6, 4)$, $T(3, 2)$ **$\sqrt{13}$ or about 3.6 units**

36. $M(1, 8)$, $N(3, 3)$ **$\sqrt{29}$ or about 5.4 units**

37. $W(-8, 1)$, $Y(0, 6)$ **$\sqrt{89}$ or about 9.4 units**

38. $B(3, 4)$, $C(5, -5)$ **$\sqrt{5}$ or about 2.2 units**

39. $R(6, 11)$, $T(3, -7)$ **$3\sqrt{37}$ or about 18.2 units**

40. $A(-3, 8)$ and $B(-1, 4)$ **$\sqrt{20}$ or about 4.5 units**

41. $M(4, -3)$ and $N(-2, 1)$ **$\sqrt{52}$ or about 7.2 units**

42. $X(-3, 5)$ and $Y(4, 2)$ **$\sqrt{58}$ or about 7.6 units**

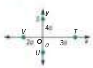
43. Use the number line to determine whether $\overline{SV}$ and $\overline{UX}$ are congruent. Write *yes* or *no*. **no**

Name the point(s) that satisfy the given condition.

44. two points on the x-axis that are 10 units from (1, 8) **(−5, 0), (7, 0)**

45. two points on the y-axis that are 25 units from (−24, 3) **(0, −4), (0, 10)**

46. Refer to the figure. Are $\overline{VT}$ and $\overline{SU}$ congruent? **$\overline{VT}$ and $\overline{SU}$ are congruent.**



47. KNITTING Mei is knitting a scarf with diagonal stripes. Before she began, she laid out the pattern on a coordinate grid where each unit represented 2 inches. On the grid, the first stripe began at (2, 0) and ended at (5, 4). All the stripes are the same length. How many inches long is each stripe on the scarf? **10 in.**

48. ART A terracotta bowl artifact has a triangular pattern around the top, as shown. All the triangles are about the same size and can be represented on a coordinate plane with vertices at points (0, 6.8), (4.5, 6.8), and (2.25, 0). If each unit represents 1 centimeter, what is the approximate perimeter of each triangle, to the nearest tenth of a centimeter? **18.8 cm**

Higher-Order Thinking Skills

49. ANALYZE Consider rectangle $QRST$ with $QR = ST = 4$ centimeters and $RS = QT = 2$ centimeters. If point U is on $\overline{QR}$ such that $QU = UR$ and point V is on $\overline{RS}$ such that $RV = VS$, then is $\overline{QU}$ congruent to $\overline{RV}$? Justify your argument.
No; sample answer: We know that $QU + UR = QR = 4$ and $QU = UR$, so $QU = 2$. Further, we know that $RV + VS = RS = 2$, and $RV = VS$, so $RV = 1$. Because QU is not equal to RV, we know that $\overline{QU}$ is not congruent to $\overline{RV}$.

50. WRITE Explain how the Pythagorean Theorem and the Distance Formula are related. **See margin.**

51. PERSEVERE Point P is located on the segment between point $A(1, 4)$ and point $D(7, 13)$. The distance from A to P is twice the distance from P to D. What are the coordinates of point P? **(5, 10)**

52. CREATE Plot points Y and Z on a coordinate plane. Then use the Distance Formula to find YZ. **See margin.**

53. PERSEVERE Suppose point A is located at (1, 3) on a coordinate plane. If AB is 10 and the x-coordinate of point B is 9, explain how to use the Distance Formula to find the y-coordinate of point B. **See margin.**

54. WRITE Explain how to use the Distance Formula to find the distance between points (a, b) and (c, d).
Sample answer: Substitute (a, b) for (x_1, y_1) and (c, d) for (x_2, y_2): $D = \sqrt{(c-a)^2 + (d-b)^2}$.

Answers

50. Sample answer: The Pythagorean Theorem relates the lengths of the legs of a right triangle to the length of the hypotenuse using the formula $c^2 = a^2 + b^2$. If you take the square root of the formula, you get $c = \sqrt{a^2 + b^2}$. Think of the hypotenuse of the triangle as the distance between the two points, the a value as the horizontal distance $x_2 - x_1$, and the b value as the vertical distance $y_2 - y_1$. If you substitute, the Pythagorean Theorem becomes the Distance Formula, $c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

52. Sample answer: Plot point Y at (2, 6) and Z at (−2, 8). Substitute (2, 6) for (x_1, y_1) and (−2, 8) for (x_2, y_2) in the Distance Formula: $D = \sqrt{(-2 - 2)^2 + (8 - 6)^2}$. Solve for D: $D = \sqrt{(2)^2 + (-4)^2} = \sqrt{4 + 16} = \sqrt{20}$ or about 4.5

53. Sample answer: Substitute 10 for d, (1, 3) for (x_1, y_1), and (9, y) for (x_2, y_2) in the Distance Formula: $10 = \sqrt{(y - 3)^2 + (9 - 1)^2}$. Solve for y:
$100 = (y - 3)^2 + (9 - 1)^2$
$= (y - 3)^2 + 8^2$
$= (y - 3)^2 + 64$
$36 = (y - 3)^2$
$6 = y - 3$ or $-6 = y - 3$
$9 = y$ or $-3 = y$. So, the y-coordinate of point B is 9 or −3.

Lesson 10-5

Locating Points on a Number Line

G.GPE.6

LESSON GOAL

Students find points that partition directed line segments on number lines.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Locating Points on a Number Line with Fractional Distance

Develop:

Locating Points on a Number Line with Fractional Distance
- Locate a Point at a Fractional Distance
- Locate a Point at a Fractional Distance in the Real World

Locating Points on a Number Line with a Given Ratio
- Locate a Point on a Number Line Given a Ratio
- Partition a Directed Line Segment

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL	
Remediation: Find Distance on the Coordinate Plane	•	•			•
Extension: Relationships Among Lines			•	•	•

Language Development Handbook

Assign page 60 of the *Language Development Handbook* to help your students build mathematical language related to finding points that partition directed line segments on number lines.

ELL You can use the tips and suggestions on page T60 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
5 Use appropriate tools strategically.
7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students found the distance between two points on a coordinate plane by applying the Distance Formula.
G.CO.1

Now
Students find points that partition directed line segments on number lines.
G.GPE.6

Next
Students will find the midpoints of line segments.
G.GPE.6

Rigor

The Three Pillars of Rigor

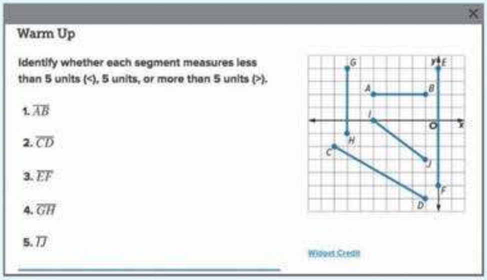
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of how a point on a directed line segment can partition the segment in a given ratio. They build fluency by locating points on the coordinate plane when given a ratio or fractional distance, and they apply their understanding by solving real-world problems.

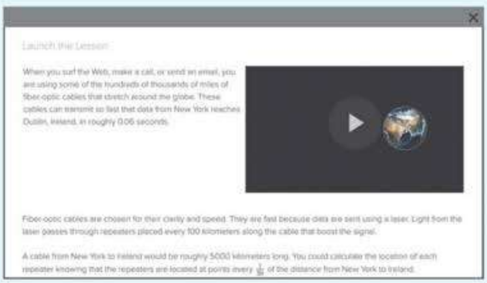
Mathematical Background

To find the coordinate of a point that divides a directed line segment into a ratio of $a:b$, first add a and b to find the total number of partitions on the directed line segment. Then make sure that there are a partitions to the left of the point and b partitions to the right of the point. Later, you can use this mathematical reasoning to develop the Midpoint Formula.

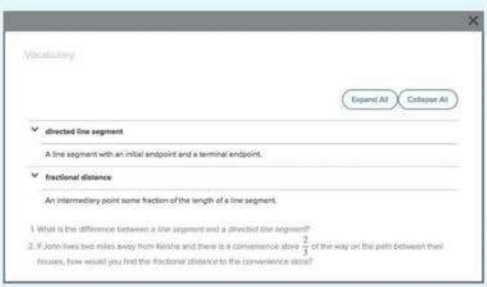
Interactive Presentation



Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- measuring line segments on the coordinate plane

Answers:

1. <
2. >
3. >
4. 5
5. 5

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of proportional reasoning.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Locating Points on a Number Line with Fractional Distance

Objective

Students use dynamic geometry software to find the point on a directed line segment on a number line that is a given fractional distance from the initial point.

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In this Explore, guide students to see what information they can gather about the expression just from looking at it.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

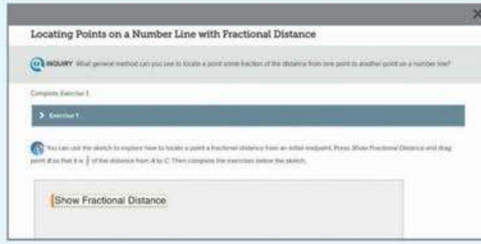
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students answer the guiding exercises and tap to reveal steps in the solution. Next students complete the guiding exercises to display their understanding of finding the coordinate using fractional distance. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

WEB SKETCHPAD

Students use a sketch to explore points on a number line.

TYPE

Students type answers to the guiding exercises.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Locating Points on a Number Line with Fractional Distance (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What does the word *fractional* mean? Sample answer: in pieces
- How do you think you would find *fractional* distance? Sample answer: Divide a distance into parts and then count however many are needed.
- Given point *A* at 2 and point *C* at 12, how would you find the coordinate of point *B* such that *B* is $\frac{2}{5}$ of the distance from *A* to *C*? I would divide $\overline{AC}$ into five equal parts. Then I would place point B two parts from point A.

Inquiry

What general method can you use to locate a point some fraction of the distance from one point to another point on a number line? Sample answer: Multiply the difference between the two coordinates by the given fraction. If you are locating the point a fractional distance to the right of one endpoint, then add the product to that endpoint. If you are locating the point a fractional distance to the left of one endpoint, then subtract the product from that endpoint.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Locating Points on a Number Line with Fractional Distance

Objective

Students find a point on a directed line segment on a number line that is a given fractional distance from the initial point.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

About the Key Concept

Notice how the process of locating a point at a fractional distance on a number line is related to finding the length of a line segment.

Common Misconception

Students frequently switch the initial endpoint and the terminal endpoint in the equation that they use to calculate the location of the point in the directed line segment. Remind students to be sure to differentiate between the initial endpoint (x_1) and terminal endpoint (x_2).

Essential Question Follow-Up

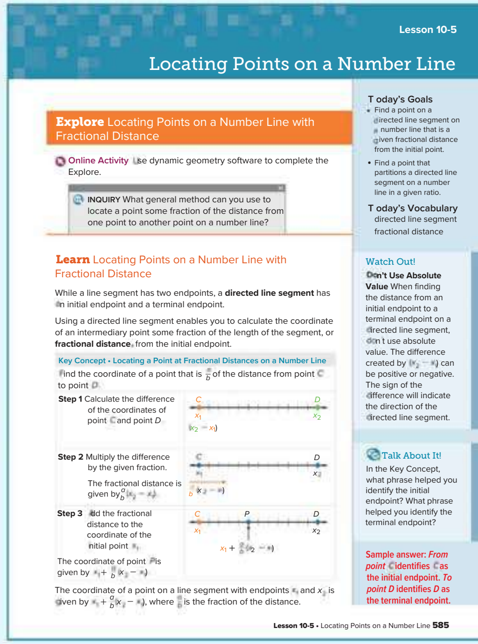
Students learn about fractional distance along directed line segments.

Ask:

Why might locating a fractional distance along a line segment be useful in applying points, lines, and planes in the real world? Sample answer: You might need to know where to locate pit stops or water stations along a race course.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 10-5

Locating Points on a Number Line

Explore Locating Points on a Number Line with Fractional Distance

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY What general method can you use to locate a point some fraction of the distance from one point to another point on a number line?

Learn Locating Points on a Number Line with Fractional Distance

While a line segment has two endpoints, a **directed line segment** has an initial endpoint and a terminal endpoint.

Using a directed line segment enables you to calculate the coordinate of an intermediary point some fraction of the length of the segment, or **fractional distance**, from the initial endpoint.

Key Concept • Locating a Point at Fractional Distances on a Number Line

Find the coordinate of a point that is $\frac{a}{b}$ of the distance from point C to point D.

Step 1 Calculate the difference of the coordinates of point C and point D. $(x_2 - x_1)$

Step 2 Multiply the difference by the given fraction. The fractional distance is given by $\frac{a}{b}(x_2 - x_1)$.

Step 3 Add the fractional distance to the coordinate of the initial point x_1. The coordinate of point P is given by $x_1 + \frac{a}{b}(x_2 - x_1)$

The coordinate of a point on a line segment with endpoints x_1 and x_2 is given by $x_1 + \frac{a}{b}(x_2 - x_1)$, where $\frac{a}{b}$ is the fraction of the distance.

Today's Goals
- Find a point on a directed line segment on a number line that is a given fractional distance from the initial point.
- Find a point that partitions a directed line segment on a number line in a given ratio.

Today's Vocabulary
directed line segment
fractional distance

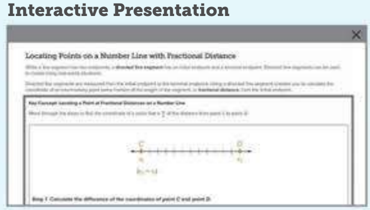
Watch Out!
Don't Use Absolute Value When finding the distance from an initial endpoint to a terminal endpoint on a directed line segment, don't use absolute value. The difference created by $(x_2 - x_1)$ can be positive or negative. The sign of the difference will indicate the direction of the directed line segment.

Talk About It!
In the Key Concept, what phrase helped you identify the initial endpoint? What phrase helped you identify the terminal endpoint?

Sample answer: From point C identifies C as the initial endpoint. To point D identifies D as the terminal endpoint.

Lesson 10-5 • Locating Points on a Number Line 585

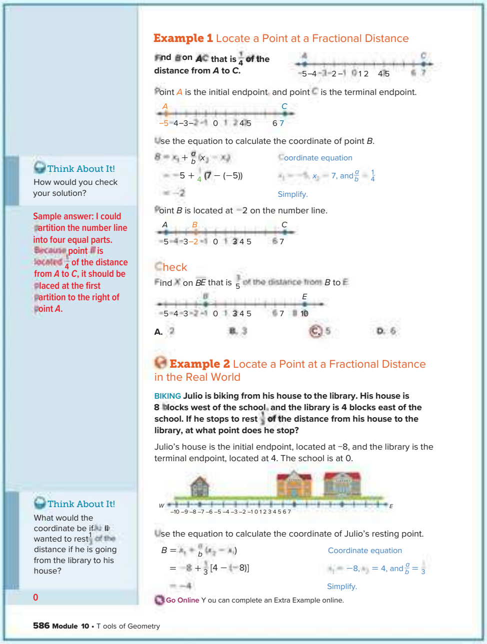
Interactive Presentation



Learn

TAP

Students tap to reveal steps in a solution process.


Example 1 Locate a Point at a Fractional Distance

Find B on $\overline{AC}$ that is $\frac{1}{4}$ of the distance from A to C.

Point A is the initial endpoint, and point C is the terminal endpoint.

Use the equation to calculate the coordinate of point B.

$B = x_1 + \frac{a}{b}(x_2 - x_1)$ Coordinate equation

$= -5 + \frac{1}{4}(7 - (-5))$ $x_1 = -5$, $x_2 = 7$, and $\frac{a}{b} = \frac{1}{4}$

$= -2$ Simplify.

Point B is located at −2 on the number line.

Check

Find X on $\overline{BE}$ that is $\frac{3}{5}$ of the distance from B to E.

A. 2 B. 3 C. 5 D. 6

Think About It! How would you check your solution?

Sample answer: I could partition the number line into four equal parts. Because point B is located $\frac{1}{4}$ of the distance from A to C, it should be placed at the first partition to the right of point A.

Example 2 Locate a Point at a Fractional Distance in the Real World

BIKING Julio is biking from his house to the library. His house is 8 blocks west of the school, and the library is 4 blocks east of the school. If he stops to rest $\frac{1}{3}$ of the distance from his house to the library, at what point does he stop?

Julio's house is the initial endpoint, located at −8, and the library is the terminal endpoint, located at 4. The school is at 0.

Use the equation to calculate the coordinate of Julio's resting point.

$B = x_1 + \frac{a}{b}(x_2 - x_1)$ Coordinate equation

$= -8 + \frac{1}{3}[4 - (-8)]$ $x_1 = -8$, $x_2 = 4$, and $\frac{a}{b} = \frac{1}{3}$

$= -4$ Simplify.

Think About It! What would the coordinate be if Julio wanted to rest $\frac{1}{3}$ of the distance if he is going from the library to his house?

0

Go Online You can complete an Extra Example online.

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Example 1 Locate a Point at a Fractional Distance

Teaching the Mathematical Practices

8 Attend to Details Mathematically proficient students continually ask themselves, "Does this make sense?" Point out that in this example, students will evaluate the reasonableness of their answer.

Questions for Mathematical Discourse

AL What is the midpoint of $\overline{AC}$? 1

OL What point is $\frac{1}{4}$ of the distance from A to C? 3

BL If Y lies $\frac{1}{6}$ the distance from A to C, where does it fall on the number line? −3

Example 2 Locate a Point at a Fractional Distance in the Real World

Teaching the Mathematical Practices

4 Apply Mathematics Students apply what they have learned about locating points at a fractional distance to solve a real-world problem.

Questions for Mathematical Discourse

AL Which building is located in the opposite position of Julio's resting point? The library is located at positive four on the number line which is opposite of −4.

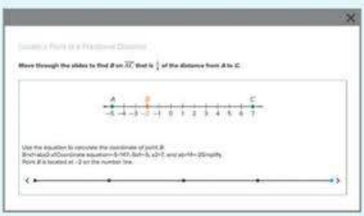
OL What point is $\frac{1}{6}$ of the distance from Julio's house to the library? −6

BL What points are $\frac{1}{6}$ of the distance from the midpoint of the segment from Julio's house to the library? −4 and 0

Common Error

Students might mix up the endpoints of the directed line segment. Remind students that in this problem, the order of the endpoints matters and that they should use them correctly.

Interactive Presentation



Example 1

TAP

Students tap to reveal steps in a solution process.

TYPE

Students answer a question to show that they understand how to locate a point at a fractional distance.

CHECK

Students complete the Check online to determine whether they are ready to move on.

DIFFERENTIATE

Enrichment Activity BL

Have students plan a road trip with four to five cities. Ask students to research the distances between the cities and to calculate the total distance from their starting city to their final destination. Ask students to describe the location of each city using fractional distance.

Learn Locating Points on a Number Line with a Given Ratio

Objective

Students find a point that partitions a directed line segment on a number line in a given ratio.

MP Teaching the Mathematical Practices

3 Analyze Cases The Concept Check guides students to examine the cases of the ratio used in the Section Formula. Encourage students to familiarize themselves with all of the cases.

Example 3 Locate a Point on a Number Line When Given a Ratio

MP Teaching the Mathematical Practices

1 Check Answers Mathematically proficient students continually ask themselves, “Does this make sense?” Point out that in this example, students need to check their answer. Point out that they should ask themselves whether their answer makes sense and whether they have answered the problem question.

Questions for Mathematical Discourse

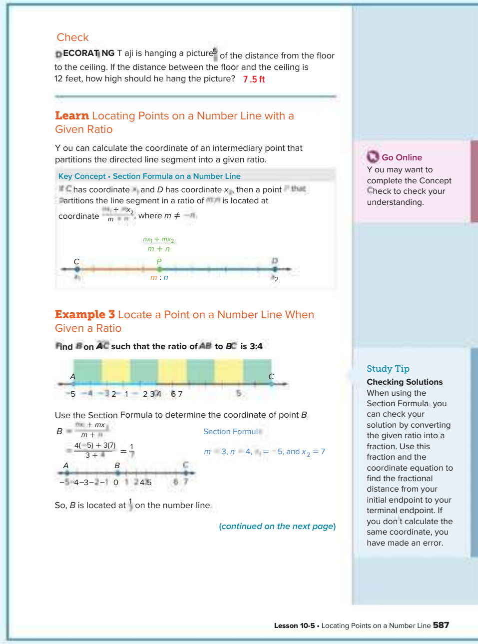
AL What formula would you used to determine the coordinate of point *B*? Section Formula: $B = \frac{nx_1 + mx_2}{m + n}$

OL What would be the coordinate of point *D* such that the ratio of *AD* to *DC* is 2:3? $-\frac{1}{5}$

BL What would be the coordinate of point *E* such that the ratio of *BE* to *EC* is 3:1? Use a method other than the one described in this lesson and explain your method. Find the midpoint *M* of *AB* and then find the midpoint of *MB*.

Common Error

Students may substitute incorrectly into the formula for the location of a point on a number line given a ratio. Make sure they understand how the quantities relate in the formula.



Check

DECORATING Taji is hanging a picture $\frac{5}{8}$ of the distance from the floor to the ceiling. If the distance between the floor and the ceiling is 12 feet, how high should he hang the picture? 7.5 ft

Learn Locating Points on a Number Line with a Given Ratio

You can calculate the coordinate of an intermediary point that partitions the directed line segment into a given ratio.

Key Concept • Section Formula on a Number Line

If *C* has coordinate x_1 and *D* has coordinate x_2, then a point *P* that partitions the line segment in a ratio of *m*:*n* is located at coordinate $\frac{nx_1 + mx_2}{m + n}$, where $m \neq -n$.

Example 3 Locate a Point on a Number Line When Given a Ratio

Find *B* on $\overline{AC}$ such that the ratio of *AB* to *BC* is 3:4

Use the Section Formula to determine the coordinate of point *B*

$B = \frac{nx_1 + mx_2}{m + n}$ Section Formula

$= \frac{4(-5) + 3(7)}{3 + 4} = \frac{1}{7}$ $m = 3, n = 4, x_1 = -5$, and $x_2 = 7$

So, *B* is located at $\frac{1}{7}$ on the number line.

(continued on the next page)

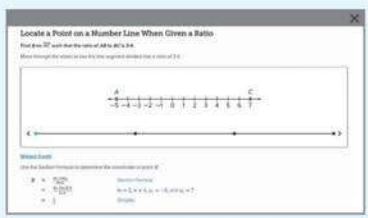
Go Online You may want to complete the Concept Check to check your understanding.

Study Tip

Checking Solutions When using the Section Formula, you can check your solution by converting the given ratio into a fraction. Use this fraction and the coordinate equation to find the fractional distance from your initial endpoint to your terminal endpoint. If you don't calculate the same coordinate, you have made an error.

Lesson 10-5 • Locating Points on a Number Line 587

Interactive Presentation



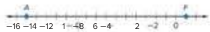
Example 3

TAP Students move through the slides to see the line segment divided.

CHECK Students complete the Check online to determine whether they are ready to move on.

Check

Find P on $\overline{AF}$ such that the ratio of AP to PF is 1:3.



P is located at -11 on the number line.

Example 4 Partition a Directed Line Segment

ROAD TRIP Jorge is traveling 2563 miles from New York City to San Francisco by car. He plans on stopping for gas when the ratio of the distance he has already traveled to the distance he still has to travel is 2:5. How far has Jorge traveled when he stops for gas?

Use the Section Formula to determine how far Jorge will travel before he stops for gas.

$P = \frac{nx_1 + mx_2}{m + n}$ Section Formula

$= \frac{5(0) + 2(2563)}{2 + 5} = 732.3$ $m = 2, n = 5, x_1 = 0$, and $x_2 = 2563$

When Jorge has traveled 732.3 miles from New Y ork City, the ratio of the distance he has traveled to the distance that he still has to travel is 2:5.

Think About It!
How can you use estimation to check your answer?

Sample answer: When the ratio of the distance Jorge has traveled to the distance he has left to travel is 2:5, Jorge has traveled $\frac{2}{7}$ of the distance. Use 2800 miles for the total distance because it is divisible by 7 . Jorge will stop for gas when he has traveled $\frac{2}{7}$ of the 2800 miles, or about 800 miles. Compared to the estimate, the solution is reasonable.

Check

ERRANDS Eduardo travels 30 miles from his house to the bike shop. When Eduardo goes to the bike shop, he always stops at a local pizza place that is along the way. The ratio of the distance Eduardo travels from his house to the pizza place to the distance he travels from the pizza place to the bike shop is 2:3.

How far is the pizza place from Eduardo's house?

12 ? mi

Go Online Y ou can complete an Extra Example online.

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Example 4 Partition a Directed Line Segment

MP Teaching the Mathematical Practices

5 Use Estimation Point out that in this example, students need to include an estimate and check against the estimate at the end.

Questions for Mathematical Discourse

AL Let A be New York, B be San Francisco, and C be where Jorge stops for gas. The ratio of AC to CB is 2:5. Into how many equal sections can you divide $\overline{AB}$ to find point C? 7

OL Where does Jorge stop for gas? Use the graph to estimate the answer. 700 mi

BL Find the distance from New York where Jorge stops again for gas if the ratio of the distance traveled to the distance left to go is 3:1. 1922.3 mi

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Interactive Presentation

Example 4

TAP

Students tap to estimate a given distance.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–25
2	exercises that use a variety of skills from this lesson	26–28
3	exercises that emphasize higher-order and critical-thinking skills	29–32

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–25 odd, 29–32
- Extension: Relationships Among Lines
- ALEKS Distances and Midpoints on a Number Line, Distances and Midpoints in the Coordinate Plane

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–31 odd
- Remediation, Review Resources: Distance on the Coordinate Plane
- Personal Tutors
- Extra Examples 1–4
- ALEKS Applying the Pythagorean Theorem

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–25 odd
- Remediation, Review Resources: Distance on the Coordinate Plane
- ALEKS Applying the Pythagorean Theorem

Important to Know

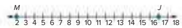
Digital Exercise Alert Exercise 29 requires a construction. Students will need to complete the construction by using a compass and straightedge.

Practice

Go Online You can complete your homework online.

Examples 1 and 3

Refer to the number line.

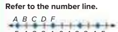


1. Find the coordinate of point B that is $\frac{1}{4}$ of the distance from M to J. 6
2. Find the coordinate of point C that is $\frac{7}{8}$ of the distance from M to J. 16
3. Find the coordinate of point D that is $\frac{7}{16}$ of the distance from M to J. 9
4. Find the coordinate of point X such that the ratio of MX to XJ is 3:1. 14
5. Find the coordinate of point X such that the ratio of MX to XJ is 2:3. 8.4
6. Find the coordinate of point X such that the ratio of MX to XJ is 1:1. 10

Refer to the number line.

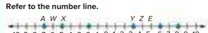
7. Find the coordinate of point G that is $\frac{2}{3}$ of the distance from B to D. −1
8. Find the coordinate of point H that is $\frac{1}{5}$ of the distance from C to F. −2.2
9. Find the coordinate of point J that is $\frac{1}{6}$ of the distance from A to E. −5.5
10. Find the coordinate of point K that is $\frac{4}{5}$ of the distance from A to F. 2.6
11. Find the coordinate of point X such that the ratio of AX to XF is 1:3. −4
12. Find the coordinate of point X such that the ratio of BX to XF is 3:2. 1
13. Find the coordinate of point X such that the ratio of CX to XE is 1:1. −1
14. Find the coordinate of point X such that the ratio of FX to XD is 5:3. 2.5

Refer to the number line.



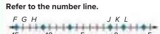
15. Find the coordinate of point X on $\overline{AF}$ that is $\frac{1}{3}$ of the distance from A to F. −2
16. Find the coordinate of point Y on $\overline{AC}$ that is $\frac{1}{4}$ of the distance from A to C. −4

Refer to the number line.



17. Which point on $\overline{AE}$ is $\frac{2}{3}$ of the distance from A to E? Y
18. Point X is what fractional distance from E to A? $\frac{2}{3}$
19. Find the coordinate of point M on $\overline{AE}$ that is $\frac{1}{5}$ of the distance from A to E. −4

Refer to the number line.



20. The ratio of FX to XK is 1:1. Which point is located at X? H
21. Find the coordinate of Q on $\overline{FL}$ such that the ratio of FQ to QL is 12:7. −3

Examples 2 and 4

22. TRAVEL Caroline is taking a road trip on I-70 in Kansas. She stops for gas at mile marker 36. Her destination is at mile marker 353 in T opeka, but she decides to stop at an attraction $\frac{3}{4}$ of the way after stopping for gas. At about which mile marker did Caroline stop to visit the attraction? 274

23. HIKING A hiking trail is 24 miles from start to finish. There are two rest areas located along the trail.

a. The first rest area is located such that the ratio of the distance from the start of the trail to the rest area and the distance from the rest area to the end of the trail is 2:9. T o the nearest hundredth of a mile, how far is the first rest area from the starting point of the trail? **4.36 mi**

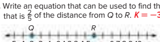
b. Kadisha claims that the distance she has walked and that the distance she has left to walk has a ratio of 5:7. How many miles has Kadisha walked? **10 mi**

24. Melany wants to hang a canvas, which is 8 feet wide, on his wall. Where on the canvas should Melany mark the location of the hangers if the canvas requires a hanger every $\frac{1}{5}$ of its length, excluding the edges? Justify your answer.
Sample answer: The canvas requires hangers every $\frac{1}{5}$ of its length or every 1.6 feet, excluding the endpoints. So the canvas needs hangers at 1.6 feet, 3.2 feet, 4.8 feet, and 6.4 feet from the edge.

25. MIGRATION Many American White Pelicans migrate each year, with hundreds of them stopping to rest in various locations along the way. The ratio of the distance some flocks travel from their summer home to one stopover to the distance from the stopover to the winter home is 3:4. If the total distance that the pelicans migrate is 1680 miles, how long is the distance from the summer home to the stopover? **720 mi**

Mixed Exercises

26. Write an equation that can be used to find the coordinate of point K that is $\frac{2}{5}$ of the distance from Q to R. **$K = -3 + \frac{2}{5}[4 - (-3)]$**



27. SOCIAL MEDIA Tito is posting a photo and needs to resize it to fit. The photo's width should fill $\frac{4}{5}$ of the width of the page. On Tito's screen, the total width of the page is 3 inches. How wide should the photo be? **$2\frac{2}{5}$ in.**

28. NEONATAL At birth, the ratio of a baby's head length to the length of the rest of its body is 1:3. If a baby's total body length is 22 inches, how long is the baby's head? **$5\frac{1}{2}$ in.**

Higher-Order Thinking Skills

29. CREATE Draw a segment and label it $\overline{AB}$. Using only a compass and a straightedge, construct a segment $\overline{CD}$ such that $CD = 5\frac{1}{4}AB$. Explain and then justify your construction. **Sample answer: Draw $\overline{AB}$. Next, draw a construction line and place point C on it. From C, strike 6 arcs in succession of length AB. On the sixth segment of length AB, perform a segment bisector two times to create a $\frac{1}{4}AB$ length. Label the endpoint D.**

30. WRITE Naoki wants to center a canvas, which is 8 feet wide, on his bedroom wall, which is 17 feet wide. Where on the wall should Naoki mark the location of the nails, if the canvas requires nails every $\frac{1}{5}$ of its length, excluding the edges? Explain your solution process. **Sample answer: I found the midpoint of the wall and the midpoint of the canvas using the midpoint formula. $M_w = \frac{17+0}{2} = 8.5$, and $M_C = \frac{8+0}{2} = 4$. Because the midpoint of the canvas aligns with the midpoint of the wall, I know that one edge of the canvas will be at least $8.5 - 4 = 4.5$ feet from the corner of the wall, and the other canvas edge will be at $8.5 + 4 =$ 12.5 feet from the corner of the wall. The canvas requires nails every $\frac{1}{5}$ of its length or ever 1.6 feet, excluding the endpoints. So, the canvas needs a nail 6. 1 ft, 7.7 ft, 9.3 ft, and 10.9 ft from the corner of the wall.**

31. ANALYZE Determine whether the following statement is *sometimes, always,* or *never* true. Justify your argument.

If $\overline{XY}$ is on a number line and point W is $\frac{2}{5}$ of the distance from X to Y, then the coordinate of point W is greater than the coordinate of point X.
Sometimes; sample answer: If the coordinate of X is 0 and the coordinate of Y is negative, then the coordinate of W will be negative and less than the coordinate of X. If the coordinate of X is positive and the coordinate of Y is greater than the coordinate of X, then the coordinate of W will be greater than the coordinate of X.

32. PERSEVERE On a number line, point A is at 5, and point B is at −10. Point C is on $\overline{AB}$ such that the ratio of AC to CB is 1:3. Find D on $\overline{BC}$ that is $\frac{3}{8}$ of the distance from B to C. **$-\frac{185}{32}$ or about −5.78**

Lesson 10-6

Locating Points on a Coordinate Plane

G.GPE.6

LESSON GOAL

Students find points that partition directed line segments on the coordinate plane.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Applying Fractional Distance

Develop:

Locating Points on the Coordinate Plane with Fractional Distance
- Fractional Distances on the Coordinate Plane

Locating Points on the Coordinate Plane with a Given Ratio
- Locate a Point on the Coordinate Plane When Given a Ratio
- Partition a Directed Line Segment on the Coordinate Plane

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Rational Numbers	●	●		●
Extension: Fractional Distances		●	●	●

Language Development Handbook

Assign page 61 of the *Language Development Handbook* to help your students build mathematical language related to finding points that partition directed line segments on the coordinate plane.

ELL You can use the tips and suggestions on page T61 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
4 Model with mathematics.
5 Use appropriate tools strategically.
6 Attend to precision.
7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students used the Distance Formula to find the distance between two points on the coordinate plane.
G.CO.1

Now
Students determine the coordinates of a point on a directed line segment that partitions the segment in a given ratio on the coordinate plane.
G.GPE.6

Next
Students will find midpoints and bisect line segments.
G.GPE.6

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

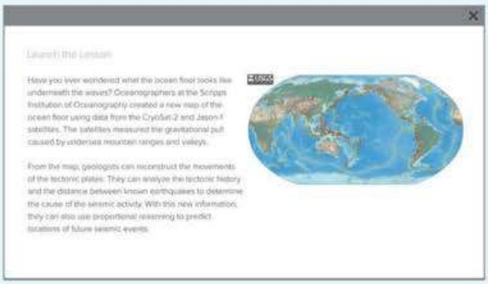
Conceptual Bridge In this lesson, students expand on their understanding of how a point on a directed line segment can partition the segment in a given ratio. They build fluency by locating points on the coordinate plane when given a ratio or fractional distance, and they apply their understanding by solving real-world problems.

Mathematical Background

To find the coordinate of a point that divides a directed line segment into a ratio of $a:b$, first add a and b to find the total number of partitions on the directed line segment. Then make sure that there are a partitions to the left of the point and b partitions to the right of the point in both the horizontal and vertical directions. Later, you can use this mathematical reasoning to develop the Midpoint Formula.

Interactive Presentation

Warm Up



Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- converting fractions and decimals

Answers:

1. $-8\frac{3}{4}$
2. 3.4
3. −3.5
4. $\frac{3}{250}$
5. 0.85

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of proportional reasoning.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Explore Applying Fractional Distance

Objective

Students locate points that partition a directed line segment on a coordinate plane a given fractional distance from an initial point.

Teaching the Mathematical Practices

4 Make Assumptions Have students explain an assumption or approximation that was made to solve the problem.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

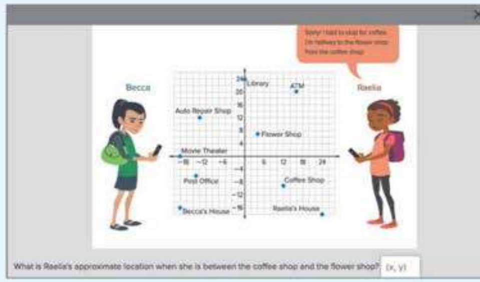
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students watch a video about two people who are texting each other with their locations as they are on their way to meet. Students then use the information given in the texts to compute the coordinates of the locations for the guiding exercises. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore



Explore

WATCH

Students watch a video to complete the problem.

TYPE

Students complete the guiding exercises to solve the problem.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Explore Applying Fractional Distance (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Which fractional distance was the easiest to approximate? Why? Sample answer: It's easiest to approximate half the distance because it's easier to tell where the middle appears to be.
- Why does it matter where the starting point is for a fractional distance other than $\frac{1}{2}$? Sample answer: One-half is just the middle, so it doesn't matter which point you start from. But if you are going some other fractional distance, like one-fourth, you will be closer to one point or another. That means you need to know where you started from.

Inquiry

How do we use fractional distances in the real world? Sample answer: We use fractional distances to describe distance traveled and to estimate arrival times. We can also use fractional distances to hang art and arrange furniture.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Locating Points on the Coordinate Plane with Fractional Distance

Objective

Students find a point on a directed line segment on the coordinate plane that is a given fractional distance from the initial point.

Teaching the Mathematical Practices

6 Use Definitions In this Learn, students will use definitions to examine claims.

About the Key Concept

Notice how the formulas for the *x*- and *y*-coordinates are related to the formula for locating a point at a fractional distance on a number line.

Example 1 Fractional Distances on the Coordinate Plane

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. Guide students to see what information they can gather about the expression just from looking at it.

Questions for Mathematical Discourse

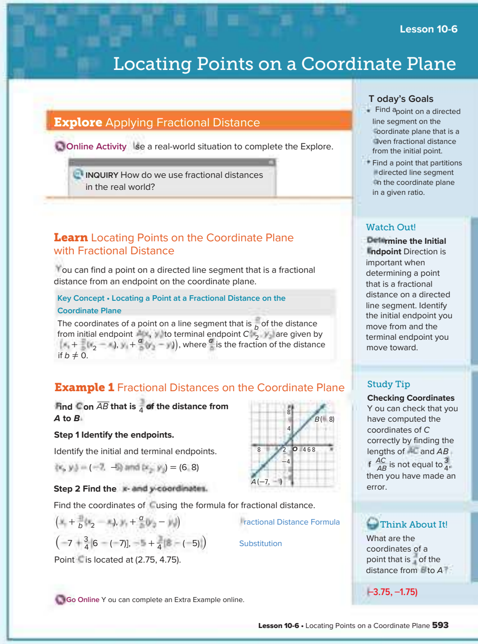
AL How can you locate a point at a fractional distance on the coordinate plane? The coordinates of a point on a line segment that is $\frac{a}{b}$ of the distance from initial endpoint $A(x_1, y_1)$ to terminal endpoint $C(x_2, y_2)$ are given by $\left(x_1 + \frac{a}{b}(x_2 - x_1), y_1 + \frac{a}{b}(y_2 - y_1)\right)$, where $\frac{a}{b}$ is the fraction of the distance if $b \neq 0$.

OL How do you know which values to use when calculating fractional distance on the coordinate plane? The values for x_1 and y_1 are obtained from the initial end point, x_2 and y_2 are obtained from the terminal end point.

BL What is an easy mistake that people could make when calculating fractional distance on the coordinate plane? A common mistake may be that people don't always use each *x* and *y* in the correct order.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.


Lesson 10-6

Locating Points on a Coordinate Plane

Explore Applying Fractional Distance

Online Activity Use a real-world situation to complete the Explore.

INQUIRY How do we use fractional distances in the real world?

Learn Locating Points on the Coordinate Plane with Fractional Distance

You can find a point on a directed line segment that is a fractional distance from an endpoint on the coordinate plane.

Key Concept • Locating a Point at a Fractional Distance on the Coordinate Plane

The coordinates of a point on a line segment that is $\frac{a}{b}$ of the distance from initial endpoint $A(x_1, y_1)$ to terminal endpoint $C(x_2, y_2)$ are given by $\left(x_1 + \frac{a}{b}(x_2 - x_1), y_1 + \frac{a}{b}(y_2 - y_1)\right)$, where $\frac{a}{b}$ is the fraction of the distance if $b \neq 0$.

Example 1 Fractional Distances on the Coordinate Plane

Find *C* on $\overline{AB}$ that is $\frac{3}{4}$ of the distance from *A* to *B*.

Step 1 Identify the endpoints.

Identify the initial and terminal endpoints.

$(x_1, y_1) = (-7, -5)$ and $(x_2, y_2) = (6, 8)$

Step 2 Find the *x*- and *y*-coordinates.

Find the coordinates of *C* using the formula for fractional distance.

$\left(x_1 + \frac{a}{b}(x_2 - x_1), y_1 + \frac{a}{b}(y_2 - y_1)\right)$ Fractional Distance Formula

$\left(-7 + \frac{3}{4}[6 - (-7)], -5 + \frac{3}{4}[8 - (-5)]\right)$ Substitution

Point *C* is located at (2.75, 4.75).

Go Online You can complete an Extra Example online.

Today's Goals
- Find a point on a directed line segment on the coordinate plane that is a given fractional distance from the initial point.
- Find a point that partitions a directed line segment on the coordinate plane in a given ratio.

Watch Out!
Determine the Initial Endpoint Direction is important when determining a point that is a fractional distance on a directed line segment. Identify the initial endpoint you move from and the terminal endpoint you move toward.

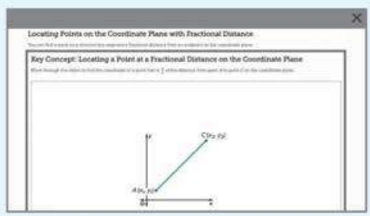
Study Tip
Checking Coordinates You can check that you have computed the coordinates of *C* correctly by finding the lengths of $\overline{AC}$ and *AB*. If $\frac{AC}{AB}$ is not equal to $\frac{3}{4}$, then you have made an error.

Think About It!
What are the coordinates of a point that is $\frac{3}{4}$ of the distance from *B* to *A*?

(−3.75, −1.75)

Lesson 10-6 • Locating Points on a Coordinate Plane 593

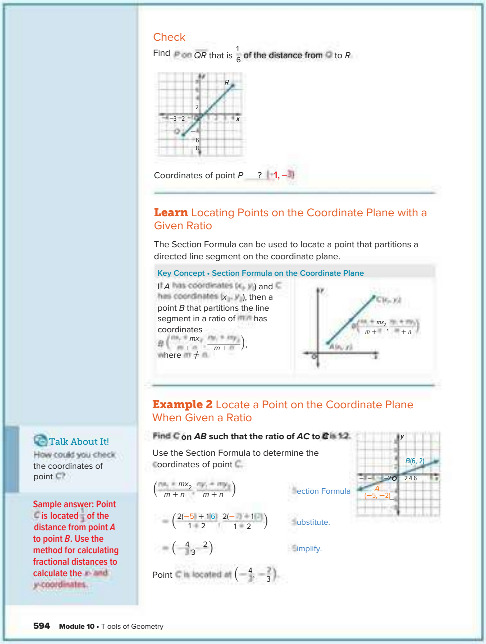
Interactive Presentation



Learn

TAP Students move through the steps to locate a point at a fractional distance.

CHECK Students complete the Check online to determine whether they are ready to move on.



Check

Find P on $\overline{QR}$ that is $\frac{1}{6}$ of the distance from Q to R.

Coordinates of point P ___?___ $(-1, -3)$

Learn Locating Points on the Coordinate Plane with a Given Ratio

The Section Formula can be used to locate a point that partitions a directed line segment on the coordinate plane.

Key Concept • Section Formula on the Coordinate Plane

If A has coordinates (x_1, y_1) and C has coordinates (x_2, y_2), then a point B that partitions the line segment in a ratio of $m:n$ has coordinates $B\left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$, where $m \neq n$.

Example 2 Locate a Point on the Coordinate Plane When Given a Ratio

Find C on $\overline{AB}$ such that the ratio of AC to CB is 1:2.

Use the Section Formula to determine the coordinates of point C.

$\left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$ Section Formula

$= \left(\frac{2(-5) + 1(6)}{1+2}, \frac{2(-2) + 1(2)}{1+2}\right)$ Substitute.

$= \left(-\frac{4}{3}, -\frac{2}{3}\right)$ Simplify.

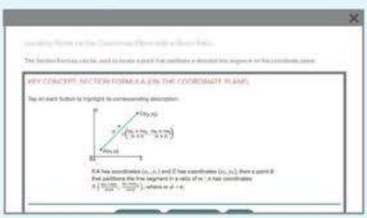
Point C is located at $\left(-\frac{4}{3}, -\frac{2}{3}\right)$.

Talk About It! How could you check the coordinates of point C?

Sample answer: Point C is located $\frac{1}{3}$ of the distance from point A to point B. Use the method for calculating fractional distances to calculate the x- and y-coordinates.

594 Module 10 • Tools of Geometry

Interactive Presentation



Learn

TAP

Students tap to reveal information about locating points on the coordinate plane.

Common Misconception

Students may forget that the order of endpoints is important in locating points that are a fractional distance along a directed line segment. Thus they frequently will switch the initial endpoint and the terminal endpoint in the equation that they use to calculate the location of the point in the directed line segment. Remind students to be sure to differentiate between the initial endpoint (x_1) and the terminal endpoint (x_2).

Essential Question Follow-Up

Students learn how to locate points on the coordinate plane at fractional distances along a directed line segment.

Ask:

Why might it be important to locate points at fractional distances on the coordinate plane? Sample answer: For distances in the real world, it might be useful to find a fractional distance between locations for various stops along a route.

Learn Locating Points on the Coordinate Plane with a Given Ratio

Objective

Students find a point that partitions a directed line segment on the coordinate plane in a given ratio.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of the Section Formula in this Learn.

Example 2 Locate a Point on the Coordinate Plane When Given a Ratio

Teaching the Mathematical Practices

4 Interpret Mathematical Results In this example, point out that to solve the problem, students should interpret their mathematical results in the context of the problem.

Questions for Mathematical Discourse

AL What is the Section Formula? $C = \left(\frac{mx_1 + mx_2}{m+n}, + \frac{my_1 + my_2}{m+n}\right)$

OL How do you know which values to use in the Section Formula? The values for x_1 and y_1 are obtained from the initial endpoints, and x_2 and y_2 are obtained from the final endpoints.

BL How can you avoid making a common mistake when substituting into the Section Formula? Be sure not to switch the order of the x and y coordinates.

DIFFERENTIATE

Reteaching Activity AL ELL
Have students create a Venn Diagram to compare the Fractional Distance Formula to the Section Formula. Ask students to share what they wrote down with the class.

Example 3 Partition a Directed Line Segment on the Coordinate Plane

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about partitioning directed line segments to solving a real-world problem.

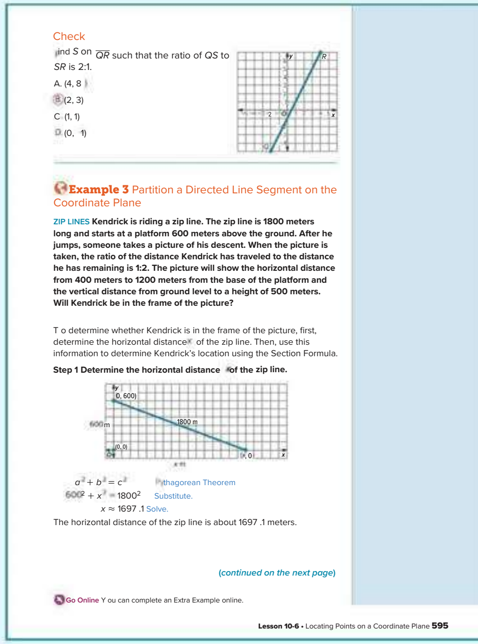
Questions for Mathematical Discourse

AL How do you know that the Pythagorean Theorem can be used to find the horizontal distance of the zip-line? Sample answer: Two of the three sides in a right triangle are provided in the diagram.

OL How high is Kendrick when the photo is taken? Use the graph to estimate the distance. 400 m

BL If a second photo is taken when the ratio of Kendrick's distance traveled to distance remaining is 7:10, how far has Kendrick traveled horizontally? Estimate the distance. 600 m

(continued on the next page)


Check

Find S on $\overline{QR}$ such that the ratio of QS to SR is 2:1.

A. (4, 8)
B. (2, 3)
C. (1, 1)
D. (0, 1)

Example 3 Partition a Directed Line Segment on the Coordinate Plane

ZIP LINES Kendrick is riding a zip line. The zip line is 1800 meters long and starts at a platform 600 meters above the ground. After he jumps, someone takes a picture of his descent. When the picture is taken, the ratio of the distance Kendrick has traveled to the distance he has remaining is 1:2. The picture will show the horizontal distance from 400 meters to 1200 meters from the base of the platform and the vertical distance from ground level to a height of 500 meters. Will Kendrick be in the frame of the picture?

T o determine whether Kendrick is in the frame of the picture, first, determine the horizontal distance x of the zip line. Then, use this information to determine Kendrick's location using the Section Formula.

Step 1 Determine the horizontal distance x of the zip line.

$a^2 + b^2 = c^2$ Pythagorean Theorem

$600^2 + x^2 = 1800^2$ Substitute.

$x \approx 1697.1$ Solve.

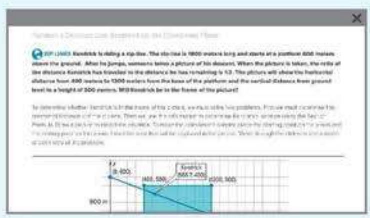
The horizontal distance of the zip line is about 1697 .1 meters.

(continued on the next page)

Go Online Y ou can complete an Extra Example online.

Lesson 10-6 • Locating Points on a Coordinate Plane 595

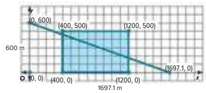
Interactive Presentation


Example 3

TAP

Students move through the slides to solve the problem.

Step 2 Model the area captured by the photograph.



Step 3 Determine Kendrick's location on the zip line.

Use the Section Formula to calculate Kendrick's coordinates.

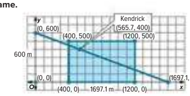
$\left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$ Section Formula

$= \left(\frac{2(0) + 1(1697.1)}{1+2}, \frac{2(600) + 1(0)}{1+2}\right)$ Substitute.

$= (565.7, 400)$ Simplify.

Kendrick is at (565.7 , 400) when the picture is taken.

Step 4 Graph Kendrick's location to determine whether he is in the frame.



Y es. Kendrick is in the frame when the picture is taken.

Check

TRAVEL Andre is traveling from Jeffersonville to Springfield. He plans to stop for a break when the distance he has traveled and the distance he has left to travel have a ratio of 3:7 . Where should Andre stop for his break?

A. (13, 12.5) B. (22, 12.5) C. (−3, 6.5) D. (−12, 6.5)

Go Online Y ou can complete an Extra Example online.

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Common Error

Students may multiply the coordinates of the starting point of the line segment by the corresponding part of the ratio, rather than the part of the ratio that corresponds to the endpoint. Make sure that they notice that the coordinates should be multiplied by the opposite part of the ratio, not the corresponding part of the ratio.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–15
2	exercises that use a variety of skills from this lesson	16–20
3	exercises that emphasize higher-order and critical-thinking skills	21–25

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–15 odd, 21–25
- Extension: Fractional Distances
- ALEKS Distances and Midpoints on a Number Line, Distances and Midpoints in the Coordinate Plane

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–25 odd
- Remediation, Review Resources: Rational Numbers
- Personal Tutors
- Extra Examples 1–3
- ALEKS Converting Fractions to Decimals

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–15 odd
- Remediation, Review Resources: Rational Numbers
- ALEKS Converting Fractions to Decimals

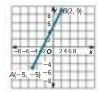
Practice

Go Online Y ou can complete your homework online.

Example 1

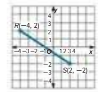
Find the coordinates of point *X* on the coordinate plane for each situation.

1. Point X on $\overline{AB}$ is $\frac{1}{5}$ of the distance from A to B.



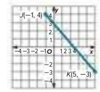
(−3.6, −2.2)

2. Point X on $\overline{RS}$ is $\frac{1}{6}$ of the distance from R to S.



$\left(-3, 1\frac{1}{3}\right)$

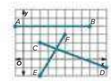
3. Point X on $\overline{JK}$ is $\frac{1}{3}$ of the distance from J to K.



$\left(1, 1\frac{2}{3}\right)$

Example 2

Refer to the coordinate grid.



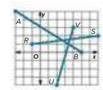
4. Find point X on $\overline{AB}$ such that the ratio of AX to XB is 1:3. $\left(\frac{5}{4}, 4\right)$

5. Find point Y on $\overline{CD}$ such that the ratio of DY to YC is 2:1. $\left(\frac{14}{3}, 1\right)$

6. Find point Z on $\overline{EF}$ such that the ratio of EZ to ZF is 2:3. $\left(\frac{16}{5}, 0\right)$

Examples 1 and 2

Refer to the coordinate grid.



7 . Find point C on $\overline{AB}$ that is $\frac{1}{5}$ of the distance from A to B. $\left(-\frac{7}{5}, 4\right)$

8. Find point Q on $\overline{RS}$ that is $\frac{5}{8}$ of the distance from R to S. $\left(4, \frac{13}{8}\right)$

9. Find point W on $\overline{UV}$ that is $\frac{1}{7}$ of the distance from U to V. $\left(\frac{16}{7}, -3\right)$

10. Find point D on $\overline{AB}$ that is $\frac{3}{4}$ of the distance from A to B. $\left(3, \frac{5}{4}\right)$

11. Find point Z on $\overline{RS}$ such that the ratio of RZ to ZS is 1:3. $\left(1, \frac{5}{4}\right)$

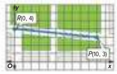
12. Find point G on $\overline{AB}$ such that the ratio of AG to GB is 3:2. $\left(\frac{9}{5}, 2\right)$

13. Find point E on $\overline{UV}$ such that the ratio of UE to EV is 3:4. $\left(\frac{20}{7}, -1\right)$

Lesson 10-6 • Locating Points on a Coordinate Plane **597**

Example 3

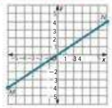
14. MAPS Leila is walking from the park at point P to a restaurant at point *R*. She wants to stop for a break when the distance she has traveled and the distance she has left to travel has a ratio of 3:5. At which point should Leila stop for her break? **(6.25, 3.375)**



15. CITY PLANNING The United States Capitol is located at (2, 4) on a coordinate grid. The White House is located at (−10, 16) on the same coordinate grid. Find two points on the straight line between the United States Capitol and the White House such that the ratio is 1:3. **(−7, 11) and (−1, 1)**

Mixed Exercises

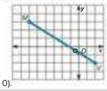
Refer to the coordinate grid.



16. Find X on $\overline{MN}$ that is $\frac{3}{4}$ of the distance from M to N. **(3, 2)**

17. Find Y on $\overline{MN}$ such that the ratio of MY to YN is 1:3. **(−3, −2)**

Point *D* is located on $\overline{MV}$. The coordinates of D are $\left(0, -\frac{3}{4}\right)$.



18. What ratio relates *MD* to *DV*? **3:1**

19. What fraction of the distance from M to V is *MD*? $\frac{3}{4}$

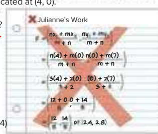
20. What ratio relates *DV* to *MD*? **1:3**

Higher-Order Thinking Skills

21. FIND THE ERROR Point W is located at (0, 7), and point *X* is located at (4, 0). Julianne wants to find point P on $\overline{WX}$ such that WP to PX is 2:3.

a. What error did Julianne make when solving this problem? **Julianne substituted the wrong values for (x_1, y_1) and (x_2, y_2).**

b. What are the correct coordinates of point P? **(1.6, 4.2)**



22. ANALYZE Is the point one-third of the distance from (x_1, y_1) to (x_2, y_2) *sometimes*, *always*, or *never* the point $\left(\frac{x_1 + x_2}{3}, \frac{y_1 + y_2}{3}\right)$? Justify your argument.
sometimes; only when the segment lies on the x- or y-axis

23. WRITE Point *P* is located on the segment between point *A*(1, 4) and point *D*(7, 13). The distance from *A* to *P* is twice the distance from P to *D*. Explain how to find the fractional distance that P is from *A* to D. What are the coordinates of point *P*? **Sample answer: Because the distance from *A* to P is twice the distance from P to *D*, the distance from *A* to P could be 2 and the distance from P to D could be 1. Therefore, the fractional distance that P is from *A* to D is $\frac{2}{2+1}$ or $\frac{2}{3}$. The coordinates of point P are (5, 10).**

24. PERSEVERE Point C(6, 9) is located on the segment between point *A*(4, 8) and point B. Point C is $\frac{1}{4}$ of the distance from A to *B*. What are the coordinates of point B? **(12, 12)**

25. CREATE Draw a line on a coordinate plane. Label two points on the line F and *G*. Locate a third point on the line between points F and G and label this point H. The point H on $\overline{FG}$ is what fractional distance from F to G? **See margin.**

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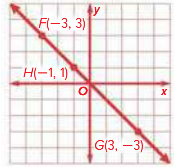
Answers

25. Sample answer:



H is $\frac{1}{3}$ of the distance from *F* to *G*.

Lesson 10-7

Midpoints and Bisectors

G.GPE.6, G.CO.12

LESSON GOAL

Students find midpoints and bisect line segments.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Midpoints

Develop:

Midpoints on a Number Line
- Find the Midpoint on a Number Line
- Midpoints in the Real World

Midpoints on the Coordinate Plane
- Find the Midpoint on the Coordinate Plane
- Find Missing Coordinates

Bisectors
- Find Missing Measures
- Find the Total Length

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Add Rational Numbers	●	●		●
Extension: Archimedes' Law of the Lever		●	●	●

Language Development Handbook

Assign page 62 of the *Language Development Handbook* to help your students build mathematical language related to midpoints and bisecting line segments.

ELL You can use the tips and suggestions on page T62 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Geometry

Standards for Mathematical Content:

G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

G.CO.12 Make formal geometric constructions with a variety of tools and methods. (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.)

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students partitioned segments in a given ratio on the coordinate plane.
G.GPE.6

Now
Students find midpoints and bisect line segments.
G.GPE.6, G.CO.12

Next
Students will prove theorems about lines and angles, and use theorems about lines and angles to solve problems.
G.CO.1, G.CO.12

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of fractional distances to midpoints and segment bisectors. They build fluency by finding midpoints, and they apply their understanding by solving real-world problems related to midpoints.

Mathematical Background

The midpoint of a segment is the point halfway between its endpoints. The midpoint divides a segment in a ratio of 1 : 1. The midpoint of a segment with endpoints a and b on a number line is the sum of a and b divided by 2. The Midpoint Formula is used to find the midpoint of a segment on the coordinate plane.

Interactive Presentation

Warm Up

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- adding rational numbers

Answers:

1. 0.9
2. -37.43
3. $-\frac{1}{6}$
4. $-\frac{3}{8}$
5. $1614.12


Launch the Lesson

Launch the Lesson

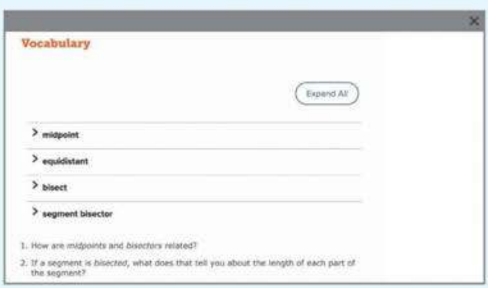
Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of locating the midpoint of a line segment.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align to the standards covered in this lesson.



Today's Vocabulary

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Midpoints

Objective

Students use paper folding to find the midpoint of a number line.

MP Teaching the Mathematical Practices

3 Construct Arguments In this Explore, students will use stated assumptions, definitions, and previously established results to draw conclusions.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students begin the Explore by watching a video. This video leads them through a paper-folding activity where they find the midpoint of a segment on the paper. Then students complete the guiding exercises leading them to discover the Midpoint Formula. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore

WATCH Students watch a video to learn about using paper folding to find the midpoint of a segment.

TYPE Students type to complete the guiding exercises.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Midpoints (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How is a number line a helpful tool? Sample answer: Number lines provide a visual representation.
- What are some skills that you need to have to be able to find midpoints? Sample answer: Y ou need to be able to add positive and negative numbers.

Inquiry

What general formula can you use to find the midpoint of a line segment? Sample answer: If the line segment has endpoints x_1 and x_2, then you can find the midpoint using the formula $M = \frac{x_1 + x_2}{2}$.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Midpoints on a Number Line

Objective

Students find the coordinate of a midpoint on a number line by using the Midpoint Formula.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the Midpoint Formula used in this example and the formula for locating a point on a number line given a fractional distance or ratio.

Important to Know

To find the Midpoint Formula, you can use the Fractional Distance Formula with a fractional distance of $\frac{1}{2}$. The fractional distance is $\frac{1}{2}$ because the midpoint is exactly halfway between the endpoints.

Example 1 Find the Midpoint on a Number Line

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This example asks students to justify their conclusions.

Questions for Mathematical Discourse

AL How do you determine which points represent x_1 and x_2? Sample answer: Choose the location of both endpoints on a segment.

OL What is the relationship between the distance and the midpoint? Sample answer: The midpoint is found by dividing the distance into two equal parts and identifying the point in the middle.

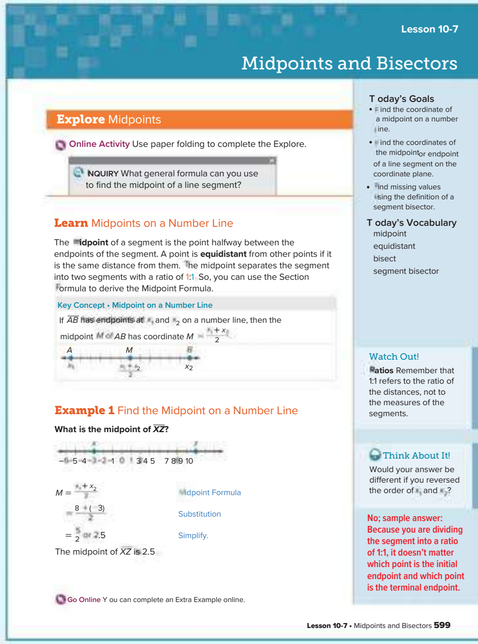
BL How is finding the midpoint of a segment like finding the mean between two numbers? Sample answer: You add two numbers and divide by 2 to find both. The midpoint is the mean of the two endpoints.

Common Error

A common mistake is that students subtract the coordinates in the Midpoint Formula because subtraction is used in the distance and the slope formulas. Remind students that the midpoint is the mean of each coordinate and that to find the mean or the average, the sum is divided by the number of terms.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.


Lesson 10-7

Midpoints and Bisectors

Explore Midpoints

Online Activity Use paper folding to complete the Explore.

INQUIRY What general formula can you use to find the midpoint of a line segment?

Learn Midpoints on a Number Line

The **midpoint** of a segment is the point halfway between the endpoints of the segment. A point is **equidistant** from other points if it is the same distance from them. The midpoint separates the segment into two segments with a ratio of 1:1. So, you can use the Section Formula to derive the Midpoint Formula.

Key Concept • Midpoint on a Number Line

If $\overline{AB}$ has endpoints at x_1 and x_2 on a number line, then the midpoint M of $\overline{AB}$ has coordinate $M = \frac{x_1 + x_2}{2}$.

Example 1 Find the Midpoint on a Number Line

What is the midpoint of $\overline{XZ}$?

$M = \frac{x_1 + x_2}{2}$ Midpoint Formula

$= \frac{8 + (-3)}{2}$ Substitution

$= \frac{5}{2}$ or 2.5 Simplify.

The midpoint of $\overline{XZ}$ is 2.5.

Go Online You can complete an Extra Example online.

Today's Goals
- Find the coordinate of a midpoint on a number line.
- Find the coordinates of the midpoint or endpoint of a line segment on the coordinate plane.
- Find missing values using the definition of a segment bisector.

Today's Vocabulary
midpoint
equidistant
bisect
segment bisector

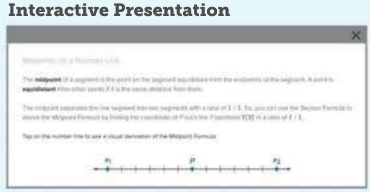
Watch Out!
Ratios Remember that 1:1 refers to the ratio of the distances, not to the measures of the segments.

Think About It!
Would your answer be different if you reversed the order of x_1 and x_2?

No; sample answer: Because you are dividing the segment into a ratio of 1:1, it doesn't matter which point is the initial endpoint and which point is the terminal endpoint.

Lesson 10-7 • Midpoints and Bisectors 599

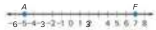
Interactive Presentation



Learn

TAP Students tap to reveal a Study Tip.

DRAG & DROP Students drag a midpoint to its correct location on a number line.



Check

What is the midpoint of $\overline{AF}$?

1

Example 2 Midpoints in the Real World

SIGNS Aponi works at a vintage clothing store. She wants to hang a new sign so it is centered above the dressing-room doors. Given that the dressing-room doors have the same width, find the point along the wall that Aponi should hang the new sign.

$M = \frac{x_1 + x_2}{2}$ Midpoint Formula

$= \frac{7.5 + (13.5)}{2}$ Substitution

$= \frac{21}{2}$ or 10.5 Simplify.

Aponi should hang the sign 10.5 feet from the left side of the wall.

Think About It!

How else could Aponi have located the midpoint?

Sample answer: Aponi could have found the distance between the sides of the dressing room doors and then found half of that distance. Next, she would add that quotient to the left side of the dressing room doors or subtract the quotient from the right side of the dressing room doors to find the midpoint.

Check

DISTANCE Jorge travels from his school on 38^{th} Street to the library on 62^{nd} Street. He stops halfway there to take a break. Where does Jorge stop to rest?

Jorge stops at ? 50th street

Go Online You can complete an Extra Example online.

600 Module 10 • Tools of Geometry

Interactive Presentation

Example 2

TAP

Students tap to reveal steps in a solution.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Midpoints in the Real World

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about midpoints to solve a real-world problem.

Questions for Mathematical Discourse

AL Is there enough information to determine the midpoint of the back wall? Explain. No; there is unmeasured space to the right of the dressing room.

OL How wide is each dressing room door? 3 ft

BL If the wall space to the right of the dressing room is $\frac{1}{5}$ the width of the space to the left of the dressing room, how many feet mark the midpoint of the wall? 7.5 ft

DIFFERENTIATE

Enrichment Activity BL

Have students share with a partner the different methods they used to find the midpoint, or center. After a couple of minutes, bring the class together to share their ideas. Write on the board so students can see how many students used similar methods and different methods that other students used.

Learn Midpoints on the Coordinate Plane

Objective

Students find the coordinates of the midpoint or endpoint of a line segment on the coordinate plane by using the Midpoint Formula.

MP Teaching the Mathematical Practices

8 Notice Regularity In this lesson, help students see the regularity in the way that midpoint coordinates are computed on number lines and coordinate planes.

Example 3 Find the Midpoint on the Coordinate Plane

MP Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to construct an argument.

Questions for Mathematical Discourse

AL What is *AM* and *MB*? Both lengths are congruent and equal $\sqrt{26}$.

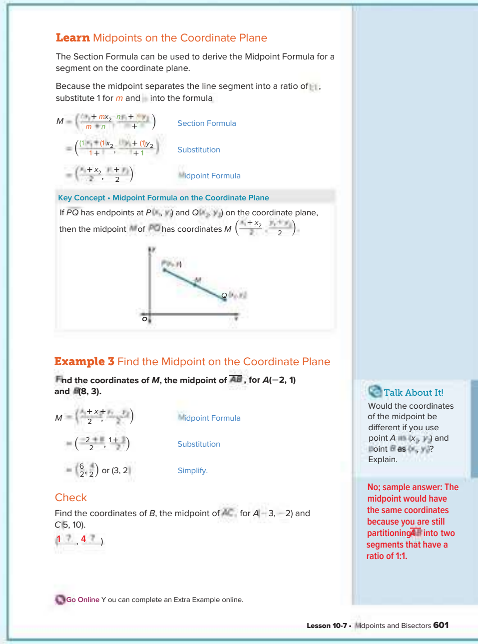
OL What are two additional ordered pairs, points *X* and *Y*, for which *M* is also a midpoint? Sample answer: $X(-2, -1)$ and $Y(8, 5)$

BL How do you compare the two formulas for the midpoint of a line segment? Sample answer: Both involve adding the two endpoints and dividing by 2. Because one is on the coordinate plane, there are two coordinates for the endpoint.

DIFFERENTIATE

Enrichment Activity BL

Have students sketch three different segments that each have (0,0) as the midpoint. Write the coordinates of the endpoints of each segment. What do you notice about the coordinates? Sample answer: In each pair, the *x*-coordinates and the *y*-coordinates are opposites.



Learn Midpoints on the Coordinate Plane

The Section Formula can be used to derive the Midpoint Formula for a segment on the coordinate plane.

Because the midpoint separates the line segment into a ratio of 1:1, substitute 1 for *m* and *n* into the formula.

$M = \left(\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}\right)$ Section Formula

$= \left(\frac{(1)x_1 + (1)x_2}{1 + 1}, \frac{(1)y_1 + (1)y_2}{1 + 1}\right)$ Substitution

$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ Midpoint Formula

Key Concept • Midpoint Formula on the Coordinate Plane

If $\overline{PQ}$ has endpoints at $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the coordinate plane, then the midpoint *M* of $\overline{PQ}$ has coordinates $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example 3 Find the Midpoint on the Coordinate Plane

Find the coordinates of *M*, the midpoint of $\overline{AB}$, for $A(-2, 1)$ and $B(8, 3)$.

$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ Midpoint Formula

$= \left(\frac{-2 + 8}{2}, \frac{1 + 3}{2}\right)$ Substitution

$= \left(\frac{6}{2}, \frac{4}{2}\right)$ or (3, 2) Simplify.

Check

Find the coordinates of *B*, the midpoint of $\overline{AC}$, for $A(-3, -2)$ and $C(5, 10)$.

(1 ?, 4 ?)

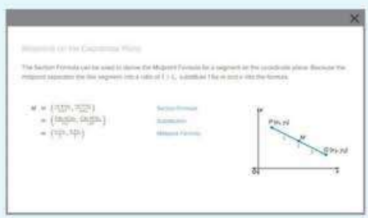
Talk About It! Would the coordinates of the midpoint be different if you use point *A* as (x_2, y_2) and point *B* as (x_1, y_1)? Explain.

No; sample answer: The midpoint would have the same coordinates because you are still partitioning $\overline{AB}$ into two segments that have a ratio of 1:1.

Go Online You can complete an Extra Example online.

Lesson 10-7 • Midpoints and Bisectors 601

Interactive Presentation



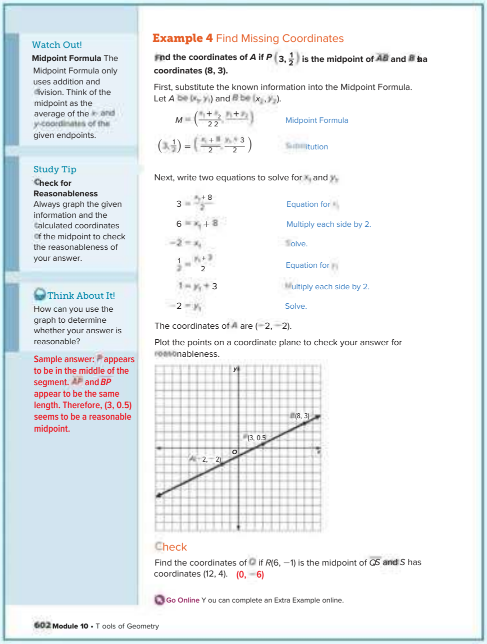
Learn

TAP Students tap to see steps in the derivation of the Midpoint Formula and select a solution to a problem.

TYPE Students type to answer questions.

G.GPE.6, G.CO.12

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION



Watch Out!

Midpoint Formula The Midpoint Formula only uses addition and division. Think of the midpoint as the average of the x- and y-coordinates of the given endpoints.

Study Tip

Check for Reasonableness Always graph the given information and the calculated coordinates of the midpoint to check the reasonableness of your answer.

Think About It!

How can you use the graph to determine whether your answer is reasonable?

Sample answer: P appears to be in the middle of the segment. $\overline{AP}$ and $\overline{BP}$ appear to be the same length. Therefore, (3, 0.5) seems to be a reasonable midpoint.

Example 4 Find Missing Coordinates

Find the coordinates of A if $P\left(3, \frac{1}{2}\right)$ is the midpoint of $\overline{AB}$ and B has coordinates (8, 3).

First, substitute the known information into the Midpoint Formula. Let A be (x_1, y_1) and B be (x_2, y_2).

$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ Midpoint Formula

$\left(3, \frac{1}{2}\right) = \left(\frac{x_1 + 8}{2}, \frac{y_1 + 3}{2}\right)$ Substitution

Next, write two equations to solve for x_1 and y_1.

$3 = \frac{x_1 + 8}{2}$ Equation for x_1

$6 = x_1 + 8$ Multiply each side by 2.

$-2 = x_1$ Solve.

$\frac{1}{2} = \frac{y_1 + 3}{2}$ Equation for y_1

$1 = y_1 + 3$ Multiply each side by 2.

$-2 = y_1$ Solve.

The coordinates of A are (−2, −2).

Plot the points on a coordinate plane to check your answer for reasonableness.

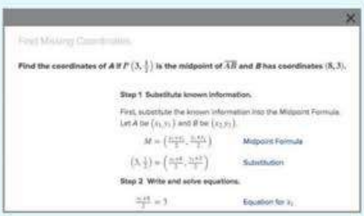
Check

Find the coordinates of Q if R(6, −1) is the midpoint of $\overline{QS}$ and S has coordinates (12, 4). (0, −6)

Go Online You can complete an Extra Example online.

602 Module 10 • Tools of Geometry

Interactive Presentation



Example 4

TAP

Students tap to reveal steps in the solution.

WEB SKETCHPAD

Students use a sketch to model the problem.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 4 Find Missing Coordinates

Teaching the Mathematical Practices

8 Attend to Details Mathematically proficient students continually ask themselves, “Does this make sense?” Point out that in this example, students will evaluate the reasonableness of their answer.

Questions for Mathematical Discourse

AL What formula will you use to solve this problem? $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

OL Does P have to fall on $\overline{AB}$? Explain. Yes; by definition, the midpoint is part of the line segment.

BL Suppose you found the coordinates of A to be (−4, 7). How would the check tell you the answer is incorrect? Sample answer: Point P would not be on the line segment $\overline{AB}$, so the answer would be incorrect.

Common Error

Students may try to find the midpoint of $\overline{BP}$ rather than find the coordinates of A. Make sure that students understand how they should use the formula based on what is given in the problem.

DIFFERENTIATE

Language Development Activity AL ELL

Mark the back of a meterstick with one endpoint and the midpoint of a segment. Hold the meterstick up for students so they can see your marks but not the centimeter marks. Ask a volunteer to mark on the back of the stick about where they visualize the other endpoint of the segment. Have a second volunteer verify the first student’s mark or add another mark. Place a pen upright on the endpoint so it shows exactly where the endpoint of the segment is and compare to the students’ marks.

Learn Bisectors

Objective

Students apply the definition of a segment bisector to find missing values.

Teaching the Mathematical Practices

3 Analyze Cases This Learn guides students to examine cases of types of segment bisectors. Encourage students to familiarize themselves with all of the cases.

Example 5 Find Missing Measures

Teaching the Mathematical Practices

2 Create Representations Guide students to write an equation that models the situation in this example. Then use the equation to solve the problem.

Questions for Mathematical Discourse

AL What happens to a line segment that is bisected? It is cut into two shorter line segments that have equal length.

OL How do you determine the length of RQ? Explain. Because $RT = 11$, and $RT = TQ$, $11 + 11 = 22$. Therefore, $RQ = 22$.

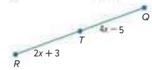
BL Why did we write $2x + 3 = 4x - 5$? Sample answer: The line segment was bisected into two shorter line segments of equal length.

Learn Bisectors

Because the midpoint separates the segment into two congruent segments, we can say that the midpoint **bisects** the segment. Any segment, line, plane, or point that bisects a segment is called a **segment bisector**.

Example 5 Find Missing Measures

Find the measure of $\overline{RT}$ if T is the midpoint of $\overline{RQ}$.



Because T is the midpoint, $RT = TQ$. Use this equation to solve for x.

$RT = TQ$	Definition of midpoint
$2x + 3 = 4x - 5$	Substitution
$3 = 2x - 5$	Subtract $2x$ from each side.
$8 = 2x$	Add 5 to each side.
$4 = x$	Divide each side by 2.

Substitute 4 for x in the equation for RT.

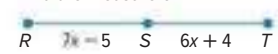
$RT = 2x + 3$	Equation for RT
$= 2(4) + 3$	Substitution
$= 11$	Simplify.

Think About It!
Is there a way to find the length of $\overline{TQ}$ without calculating when you know the length of $\overline{RT}$? Why or why not?

Yes; sample answer: Because T is the midpoint of $\overline{RQ}$, the lengths of $\overline{RT}$ and $\overline{TQ}$ are the same. Thus, TQ is equal to RT.

Check

Find the measure of $\overline{RS}$ if S is the midpoint of $\overline{RT}$.



A. 56

B. 58

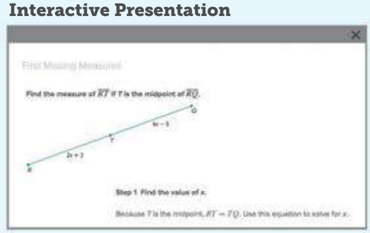
C. 112

D. 116

Go Online You can complete an Extra Example online.

Lesson 10-7 • Midpoints and Bisectors 603

Interactive Presentation



Example 5

TAP

Students tap to reveal steps in the solution and enter solutions.

Example 6 Find the Total Length

Find the measure of $\overline{AC}$ if B is the midpoint of $\overline{AC}$.

Because B is the midpoint, $AB = BC$. Use this equation to solve for x.

$AB = BC$	Definition of midpoint
$5x - 3 = 2x + 9$	Substitution
$3x - 3 = 9$	Subtract $2x$ from each side
$3x = 12$	Add 3 to each side.
$x = 4$	Divide each side by 3.

The length of $\overline{AC}$ is equal to the sum of $\overline{AB}$ and $\overline{BC}$. So, to find the length of $\overline{AC}$, **substitute 4** for x in the expression $5x - 3 + 2x + 9$.

$AC = 5x - 3 + 2x + 9$	Length of $\overline{AC}$
$= 5(4) - 3 + 2(4) + 9$	$x = 4$
$= 20 - 3 + 8 + 9$	Multiply.
$= 34$	Simplify.

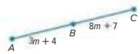
The measure of $\overline{AC}$ is 34.

Think About It!
What concept are we using when we say that $AC = AB + BC$?

betweenness of points

Check

Find the measure of $\overline{AC}$ if B is the midpoint of $\overline{AC}$. Round your answer to the nearest tenth, if necessary.



4.4

Pause and Reflect

Did you struggle with anything in this lesson? If so, how did you deal with it?

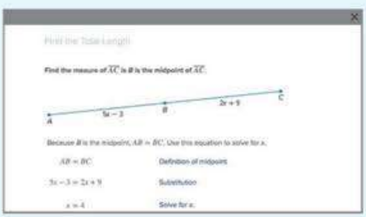
See students' observations.

Go Online
You may want to complete the construction activities for this lesson.

Go Online You can complete an Extra Example online.

604 Module 10 • Tools of Geometry

Interactive Presentation



Example 6

TYPE

Students type to enter the solution and answer a question.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 6 Find the Total Length

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL What do you know about AB and BC? Sample answer: Because B is the midpoint of $\overline{AC}$, then $AB = BC$.

OL What is AB? 17

BL Suppose $AB = -9y - 2$ and $BC = 14 - 5y$. When you solve the equation, $y = -4$. Is this possible? Explain. Yes; sample answer: Although y is a negative number, when you substitute it into the expressions, the length is a positive number.

Common Error

Students may think they are finished with the problem when they find the solution to the equation. Remind them to check the problem statement to make sure they have solved the problem.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–48
2	exercises that use a variety of skills from this lesson	49–52
2	exercises that extend concepts learned in this lesson to new contexts	53–58
3	exercises that emphasize higher-order and critical-thinking skills	59–61

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–47 odd, 59–61
- Extension: Archimedes' Law of the Lever
- ALEKS Distances and Midpoints on a Number Line, Distances and Midpoints in the Coordinate Plane

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–61 odd
- Remediation, Review Resources: Add Rational Numbers
- Personal Tutors
- Extra Examples 1–6
- ALEKS Addition and Subtraction with Fractions; Addition and Subtraction

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–47 odd
- Remediation, Review Resources: Add Rational Numbers
- ALEKS Addition and Subtraction with Fractions; Addition and Subtraction

Practice

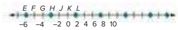
Go Online You can complete your homework online.

Example 1

Use the number line to find the coordinate of the midpoint of each segment.

1. $\overline{KM}$ −2
2. $\overline{JP}$ −1
3. $\overline{LN}$ 0.5
4. $\overline{MP}$ 2.5
5. $\overline{LP}$ 1.5
6. $\overline{JN}$ −2

Use the number line to find the coordinate of the midpoint of each segment.



7. $\overline{FK}$ 3
8. $\overline{HK}$ 6
9. $\overline{EF}$ −4.5
10. $\overline{FG}$ −1.5
11. $\overline{JL}$ 8.5
12. $\overline{EL}$ 2.5

USE TOOLS **Use the number line to find the coordinate of the midpoint of each segment.**

13. $\overline{DE}$ 9
14. $\overline{BC}$ 1
15. $\overline{BD}$ 3
16. $\overline{AD}$ $1\frac{1}{2}$

Example 2

17. HOME IMPROVEMENT Callie wants to build a fence halfway between her house and her neighbor's house. How far away from Callie's house should the fence be built? 9 yd

18. DINING Calvino's home is located at the midpoint between Fast Pizza and Pizza Now. Fast Pizza is a quarter mile away from Calvino's home. How far away is Pizza Now from Calvino's home? How far apart are the two pizzerias? Pizza Now is a quarter mile from Calvino's home; the two pizzerias are a half mile apart.

Lesson 10-7 • Midpoints and Bisectors 605

Example 3

Find the coordinates of the midpoint of a segment with the given endpoints.

19. (5, 11), (3, 1) (4, 6)
20. (7, −5), (3, 3) (5, −1)
21. (−8, −11), (2, 5) (−3, −3)
22. (7, 0), (2, 4) (4.5, 2)
23. (−5, 1), (2, 6) (−1.5, 3.5)
24. (−4, −7), (12, −6) (4, −6.5)
25. (2, 8), (8, 0) (5, 4)
26. (9, −3), (5, 1) (7, −1)
27. (22, 4), (15, 7) (18.5, 5.5)
28. (12, 2), (7, 9) (9.5, 5.5)
29. (−15, 4), (2, −10) (−6.5, −3)
30. (−2, 5), (3, −17) (0.5, −6)
31. (2.4, 14), (6, 6.8) (4.2, 10.4)
32. (−11.2, −3.4), (−5.6, −7.8) (−8.4, −5.6)

Example 4

Find the coordinates of the missing endpoint if B is the midpoint of $\overline{AC}$.

33. $C(-5, 4)$, $B(-2, 5)$ $A(1, 6)$
34. $A(1, 7)$, $B(-3, 1)$ $C(-7, -5)$
35. $A(-4, 2)$, $B(6, -1)$ $C(16, -4)$
36. $C(-6, -2)$, $B(-3, -5)$ $A(0, -8)$
37. $A(4, -0.25)$, $B(-4, 6.5)$ $C(-12, 13.25)$
38. $C\left(\frac{5}{3}, -6\right)$, $B\left(\frac{8}{3}, 4\right)$ $A\left(\frac{11}{3}, 14\right)$

Examples 5 and 6

Suppose M is the midpoint of $\overline{FG}$. Find each missing measure.

39. $FM = 5y + 13$, $MG = 5 - 3y$, $FG = ?$ 16
40. $FM = 3x - 4$, $MG = 5x - 26$, $FG = ?$ 58
41. $FM = 8a + 1$, $FG = 42$, $a = ?$ 2.5
42. $MG = 7x - 15$, $FG = 33$, $x = ?$ 4.5
43. $FM = 3n + 1$, $MG = 6 - 2n$, $FG = ?$ 8
44. $FM = 12x - 4$, $MG = 5x + 10$, $FG = ?$ 40
45. $FM = 2k - 5$, $FG = 18$, $k = ?$ 7
46. $FG = 14a + 1$, $FM = 14.5$, $a = ?$ 2
47. $MG = 13x + 1$, $FG = 15$, $x = ?$ 0.5
48. $FG = 11x - 15.6$, $MG = 10.9$, $x = ?$ 3.4

Mixed Exercises

Find the coordinates of the missing endpoint if P is the midpoint of $\overline{NQ}$.

49. $N(2, 0)$, $P(5, 2)$ $Q(8, 4)$
50. $N(5, 4)$, $P(6, 3)$ $Q(7, 2)$
51. $Q(3, 9)$, $P(-1, 5)$ $N(-5, 1)$

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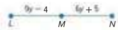
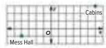
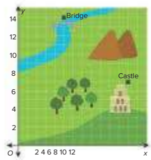
52. Find the value of y if M is the midpoint of $\overline{LN}$. 3



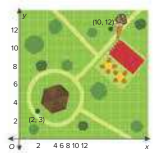
53. CAMPING Troop 175 is designing a new campground by first mapping everything on a coordinate grid. They found locations for the mess hall and their cabins. They want the bathrooms to be halfway between these two places. What are the coordinates of the location of the bathrooms? (1, 1)



54. GAME DESIGN A computer software designer is creating a new video game. The designer wants to create a secret passage that is halfway between the castle and the bridge. Where should the secret passage be located? (8.5, 10.5)

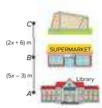


55. SCAVENGER HUNT Pablo is going to ask Bianca to prom by sending her on a scavenger hunt. At the end of the scavenger hunt, Pablo will be standing halfway between the gazebo and the ice cream shop in town. Where should Pablo stand? (6, 7.5)

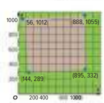


56. WALKING Javier walks from his home at point K to the Internet café at point O. If the school at point W is exactly halfway between Javier's house and the Internet café, how far does Javier walk? 256 m

57. SCHOOL LIFE Bryan is at the library doing a research paper. He leaves the library at point A and walks to the soccer field for a game at point C. The supermarket at point B is exactly halfway between the library and the soccer field. After Bryan's first soccer game, he walks to the supermarket to buy a snack, and then he walks back to the soccer field for his second game. Not including the time spent at the soccer game, how far does Bryan walk? 48 m



58. REASONING A drone flying over a field of corn identifies a dry area. The coordinates of the vertices of the area are shown. To what coordinates should the portable irrigation system be sent to water the dry area? Explain your reasoning. Sample answer: (516, 672); This is the midpoint of the segment with endpoints at (144, 289) and (888, 1055). This is the approximate center of the dry area, so the irrigation system should be placed here.



Higher-Order Thinking Skills

59. PERSEVERE Describe a method of finding the midpoint of a segment that has one endpoint at (0, 0). Derive the midpoint formula, give an example using your method, and explain why your method works.
Sample answer: The midpoint of a segment is the average of the coordinates of the endpoints. Divide each coordinate of the endpoint that is not located at the origin by 2. For example, if the segment has coordinates (0, 0) and (−10, 6), then the midpoint is located at $\left(-\frac{10}{2}, \frac{6}{2}\right)$ or (−5, 3). Using the Midpoint Formula, if the endpoints of the segment are (0 ,0) and (a, b), then the midpoint is $\left(\frac{a+0}{2}, \frac{b+0}{2}\right)$ or $\left(\frac{a}{2}, \frac{b}{2}\right)$.

60. WRITE Explain how the Midpoint Formula is a special case of the Section Formula.
The Midpoint Formula is a special case of the Section Formula where the segments into which the larger segment is divided are in a 1:1 ratio.

61. CREATE Construct $\overline{AC}$ given $\overline{AB}$ if B is the midpoint of $\overline{AC}$.
Sample answer:

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why are the terms *point*, *line*, and *plane* undefined?
- Why is it important to know how to compute distances on a number line?
- Why might locating a fractional distance along a line segment be useful in applying points, lines, and planes in the real world?
- Why might it be important to locate points at fractional distances on the coordinate plane?

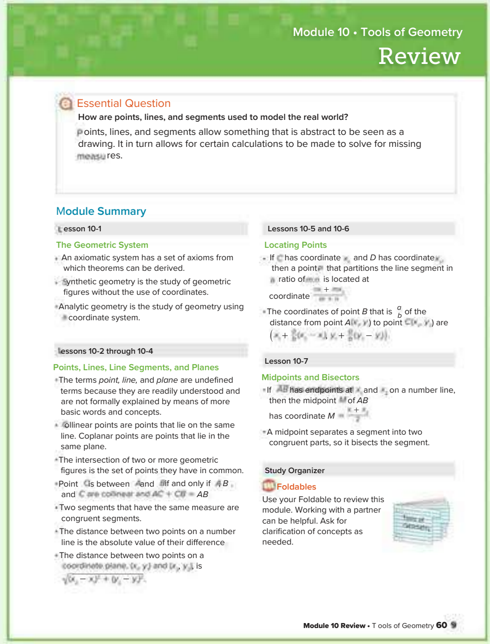
Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to points, lines, and planes, and distances and midpoints.

LearnSmart Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Congruence, Proof, and Constructions.**

- Make Geometric Constructions



Module 10 • Tools of Geometry

Review

Essential Question

How are points, lines, and segments used to model the real world?

Points, lines, and segments allow something that is abstract to be seen as a drawing. It in turn allows for certain calculations to be made to solve for missing measures.

Module Summary

Lesson 10-1

The Geometric System

- An axiomatic system has a set of axioms from which theorems can be derived.
- Synthetic geometry is the study of geometric figures without the use of coordinates.
- Analytic geometry is the study of geometry using a coordinate system.

Lessons 10-2 through 10-4

Points, Lines, Line Segments, and Planes

- The terms *point, line,* and *plane* are undefined terms because they are readily understood and are not formally explained by means of more basic words and concepts.
- Collinear points are points that lie on the same line. Coplanar points are points that lie in the same plane.
- The intersection of two or more geometric figures is the set of points they have in common.
- Point C is between A and B if and only if A, B, and C are collinear and $AC + CB = AB$.
- Two segments that have the same measure are congruent segments.
- The distance between two points on a number line is the absolute value of their difference.
- The distance between two points on a coordinate plane, (x_1, y_1) and (x_2, y_2), is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Lessons 10-5 and 10-6

Locating Points

- If C has coordinate x_1 and D has coordinate x_2, then a point P that partitions the line segment in a ratio of $m:n$ is located at coordinate $\frac{nx_1 + mx_2}{m + n}$.
- The coordinates of point B that is $\frac{a}{b}$ of the distance from point $A(x_1, y_1)$ to point $C(x_2, y_2)$ are $\left(x_1 + \frac{a}{b}(x_2 - x_1), y_1 + \frac{a}{b}(y_2 - y_1)\right)$.

Lesson 10-7

Midpoints and Bisectors

- If $\overline{AB}$ has endpoints at x_1 and x_2 on a number line, then the midpoint M of AB has coordinate $M = \frac{x_1 + x_2}{2}$.
- A midpoint separates a segment into two congruent parts, so it bisects the segment.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

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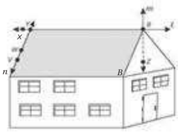
Test Practice

1. MUL TI-SELECT Select all real-world objects that model a line. (Lesson 10-2)
   A. electric tablet
   B. pool stick
   C. scoop of ice cream
   D. light pole
   E. emoji

2. MUL TI-SELECT Use the figure to name all planes containing point W. (Lesson 10-2)

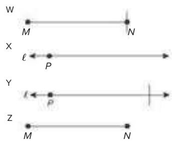


   A. plane *VWY*
   B. plane *VWX*
   C. plane *RYV*
   D. plane *VWZ*
   E. plane *RYX*

3. OPEN RESPONSE What geometric figures do the pages of the book represent? (Lesson 10-2)

intersecting planes; line m

4. ULTIPLE CHOICE Which sequence dentifies the correct order for completing the construction to copy a line segment sing a compass and straightedge? (Lesson 10-3)



   A. X Y , Z, W
   B. W, Z, X, Y
   C. W, Y , X, Z
   D. Z, X, W, Y

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Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

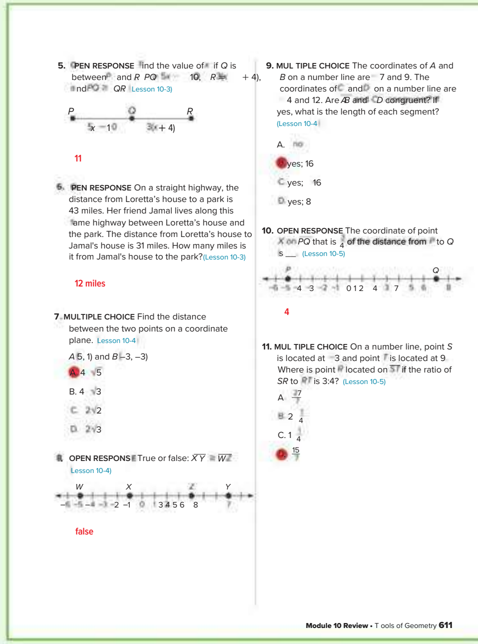
Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–16 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	4, 7, 9, 11–12, 15–17
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	2
Table Item	Students complete a table by entering the correct values.	1
Open Response	Students construct their own response.	3, 5–6, 8, 10, 13–14

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
G.CO.1, G.MG.1	10-2	1–3
G.CO.1, G.CO.12	10-3	4–6
G.CO.1	10-4	7–9
G.GPE.6	10-5	10, 11
G.GPE.6	10-6	12, 13
G.GPE.6, G.CO.12	10-7	14–17


5. OPEN RESPONSE Find the value of x if Q is between P and R, PQ = 5x − 10, RQ = 3(x + 4), and PQ ≅ QR. (Lesson 10-3)

P Q R
5x − 10   3(x + 4)

11

6. OPEN RESPONSE On a straight highway, the distance from Loretta's house to a park is 43 miles. Her friend Jamal lives along this same highway between Loretta's house and the park. The distance from Loretta's house to Jamal's house is 31 miles. How many miles is it from Jamal's house to the park? (Lesson 10-3)

12 miles

7. MULTIPLE CHOICE Find the distance between the two points on a coordinate plane. (Lesson 10-4)

A(5, 1) and B(−3, −3)

A. 4√5
B. 4√3
C. 2√2
D. 2√3

8. OPEN RESPONSE True or false: XY ≅ WZ (Lesson 10-4)

W X Z Y

false

9. MULTIPLE CHOICE The coordinates of A and B on a number line are −7 and 9. The coordinates of C and D on a number line are −4 and 12. Are AB and CD congruent? If yes, what is the length of each segment? (Lesson 10-4)

A. no
B. yes; 16
C. yes; 16
D. yes; 8

10. OPEN RESPONSE The coordinate of point X on PQ that is 1/4 of the distance from P to Q is ___. (Lesson 10-5)

4

11. MULTIPLE CHOICE On a number line, point S is located at −3 and point T is located at 9. Where is point R located on ST if the ratio of SR to RT is 3:4? (Lesson 10-5)

A. 27/7
B. 2 1/4
C. 1 1/4
D. 15/7

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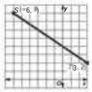
12. **MULTIPLE CHOICE** Find point R on $\overline{ST}$ such that the ratio of SR to RT is 1:2. (Lesson 10-6)



A. $R(-5, 6)$

B. $R(-3, 6)$

C. $R(1.5, 5)$

D. $R(0, 4)$

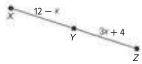
13. **OPEN RESPONSE** Alonso plans to go to the animal shelter to adopt a dog and then take the dog to Precious Pup Grooming Services. The shelter is located at $(-1, 9)$ on the coordinate plane, while Precious Pup Grooming Services is located at $(11, 0)$ on the coordinate plane. Find the location of Alonso's home if it is $\frac{1}{3}$ of the distance from the shelter to Precious Pup Grooming Services. (Lesson 10-6)

(3, 6)

14. **OPEN RESPONSE** Find the coordinates of A if $M(6, -1)$ is the midpoint of $\overline{AB}$ and B has the coordinates $(8, -7)$. (Lesson 10-7)

(4, 5)

15. **MULTIPLE CHOICE** Find the measure of $\overline{YZ}$ if Y is the midpoint of $\overline{XZ}$. (Lesson 10-7)



A. 2

B. 10

C. 16

D. 20

16. **MULTIPLE CHOICE** Find the y-coordinate of the point M, the midpoint of $\overline{AB}$, for $A(-3, 3)$ and $B(5, 7)$. (Lesson 10-7)

A. −1

B. 1

C. 2

D. 5

17. **MULTIPLE CHOICE** Points A and B are plotted on a number line. What is the location of M, the midpoint of $\overline{AB}$, for A at -9 and B at 28? (Lesson 10-7)

A. M is located at 18.5 on the number line.

B. M is located at 14 on the number line.

C. M is located at 9.5 on the number line.

D. M is located at $\frac{10}{3}$ on the number line.

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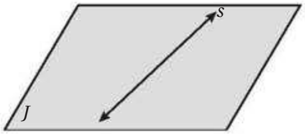
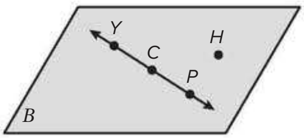
Lesson 10-2

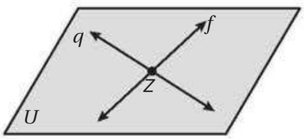
40. Sample answer:



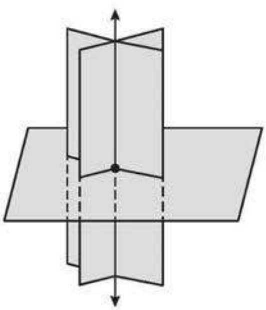
41. Sample answer:



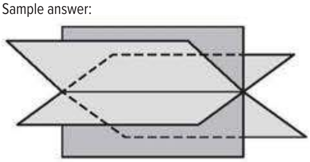
42. Sample answer:



45. Sample answer:



50. Sample answer:



MODULE 10 ANSWER APPENDIX

Module 11

Angles and Geometric Figures

Module Goals

- Students find measures of angles.
- Students find measures of two- and three-dimensional figures.
- Students use precision and accuracy when reporting measurements.

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G.MG.1 Use geometric shapes, their measures, and their properties to describe objects.

Also addresses G.CO.2, G.CO.12, G.GPE.7, and G.GMD.3.

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this Module.

Be Sure to Cover

To completely cover G.CO.12, go online to assign the following activities:

- Bisect an Angle (Construction, Lesson 11-1)
- Copy an Angle (Construction, Lesson 11-1)
- Construct a Perpendicular Bisector of a Segment (Construction, Lesson 11-2)
- Construct a Perpendicular Line Through a Point on the Line (Construction, Lesson 11-2)
- Construct a Perpendicular Line Through a Point Not on the Line (Construction, Lesson 11-2)
- Representing Transformations (Tracing Activity, Lesson 11-4)

Coherence

Vertical Alignment

Previous

Students studied angles and two- and three-dimensional figures in Grades 7-8.

6.G, 7.G, 8.G

Now

Students represent transformations in the plane and make formal geometric constructions using a variety of tools and methods.

G.CO.2, G.CO.12

Next

Students will prove theorems about lines and angles.

G.CO.9

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Practice.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

EXPLORE → LEARN → EXAMPLE & PRACTICE

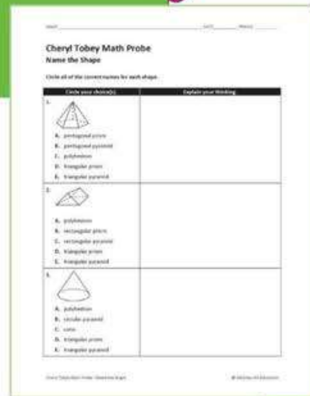
Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
11-1 Angles and Congruence	G.CO.1, G.CO.12	2	1
11-2 Angle Relationships	G.CO.1, G.CO.12	2	1
11-3 Two-Dimensional Figures	G.GPE.7, G.MG.1	1	0.5
11-4 Transformations in the Plane	G.CO. 2	3	1.5
11-5 Three-Dimensional Figures	G.MG.1, G.GMD.3	1	0.5
11-6 Two-Dimensional Representations of Three-Dimensional Figures	G.MG.1	1	0.5
11-7 Precision and Accuracy	N.Q.3	2	1
11-8 Representing Measurements	N.Q.3	1	0.5
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	16	8

CHERYL TOBEY MATH PROBES

Formative Assessment Math Probe
Name the Shape

Module Resource



Answers: 1. A and E
2. A and B
3. E

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will determine the correct names for three-dimensional shapes and explain their thinking.

Targeted Concepts Understand that polyhedra are solid (three-dimensional) figures with polygonal faces, and recognize the difference between prisms and pyramids.

Targeted Misconceptions

- Students may see all solid figures as polyhedra, including ones with nonpolygonal faces.
- Students may incorrectly interchange the labels *pyramid* and *prism*, especially when a triangular prism is not “sitting” on its base.
- Students may name solid figures by a shape other than the base.

Use the Probe after Lesson 11-5.

Collect and Assess Student Answers

If the student selects these responses...	**Then** the student likely...
3. A, D	does not recognize the term polyhedron, does not understand that polyhedra have polygonal faces, and/or does not recognize that a pyramid is a polyhedron.
1. B, D **2.** C, E **3.** B	has confused a pyramid with a prism and/or vise a versa. This often happens with a triangular prism when the solid is “sitting” on one of its rectangular sides instead of a base.
1. B, C **2.** D, E **3.** B, C	uses a side other than the base to identify a name for the figure.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS Solids and Cross Sections
- Lesson 11-5, Learn, Examples 1–2

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

![IGNITE!]

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How are angles and two-dimensional figures used to model the real world? Sample answer: Architects use two-dimensional figures to design structures that use the space effectively. Angles are used in aviation, architecture, design, and are found in nature.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

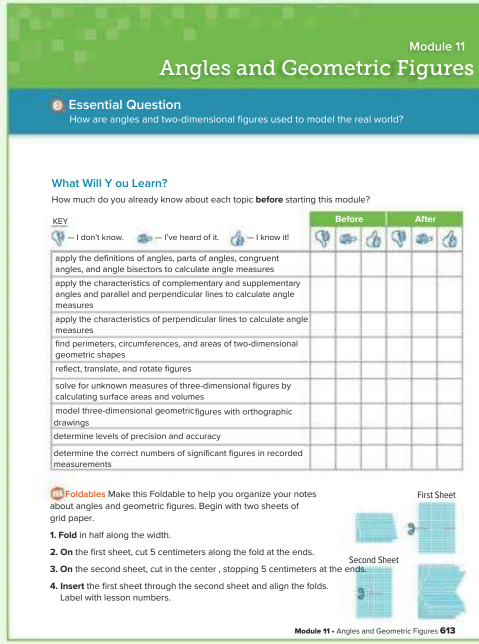
Focus Students read about angles, two- and three-dimensional objects, accuracy, and significant figures.

Teach Throughout the module, have students take notes under the tabs of their Foldables while working through each lesson. They should include definitions of terms and key concepts. Encourage students to record examples from each lesson in their Foldable.

When to Use It Use the appropriate tabs as students cover each lesson in this module.

Launch the Module

For this module, the Launch the Module video uses real-world things to model angles, two-dimensional objects, and three-dimensional objects. Students learn about using two- and three-dimensional objects in architecture and art.



Module 11

Angles and Geometric Figures

Essential Question

How are angles and two-dimensional figures used to model the real world?

What Will Y ou Learn?

How much do you already know about each topic **before** starting this module?

KEY

— I don't know. — I've heard of it. — I know it!

	Before			After		
	I don't know	I've heard of it	I know it!	I don't know	I've heard of it	I know it!
apply the definitions of angles, parts of angles, congruent angles, and angle bisectors to calculate angle measures						
apply the characteristics of complementary and supplementary angles and parallel and perpendicular lines to calculate angle measures						
apply the characteristics of perpendicular lines to calculate angle measures						
find perimeters, circumferences, and areas of two-dimensional geometric shapes						
reflect, translate, and rotate figures						
solve for unknown measures of three-dimensional figures by calculating surface areas and volumes						
model three-dimensional geometric figures with orthographic drawings						
determine levels of precision and accuracy						
determine the correct numbers of significant figures in recorded measurements						

Foldables Make this Foldable to help you organize your notes about angles and geometric figures. Begin with two sheets of grid paper.

1. **Fold** in half along the width.
2. **On** the first sheet, cut 5 centimeters along the fold at the ends.
3. **On** the second sheet, cut in the center , stopping 5 centimeters at the ends.
4. **Insert** the first sheet through the second sheet and align the folds. Label with lesson numbers.

First Sheet

Second Sheet

Module 11 • Angles and Geometric Figures **613**

Interactive Presentation

What Vocabulary Will Y ou Learn?

- accuracy
- adjacent angles
- angle
- angle bisector
- angle of rotation
- approximate error
- area
- base of a pyramid or cone
- bases of a prism or cylinder
- center of rotation
- circumference
- complementary angles
- component form
- concave
- cone
- congruent angles
- convex
- cylinder
- edge of a polyhedron
- equiangular polygon
- equilateral polygon
- exterior
- face of a polyhedron
- geometric model
- image
- interior
- line of reflection
- linear pair
- net
- opposite rays
- orthographic drawing
- perimeter
- perpendicular
- Platonic solid
- polygon
- Polyhedron
- precision
- Preimage
- prism
- pyramid
- ray
- reflection
- regular polygon
- regular polygon
- regular polyhedron
- rigid motion
- rotation
- sides
- significant figures
- sphere
- straight angle
- supplementary angles
- surface area
- transformation
- translation
- translation vector
- vertex
- vertex of a polyhedron
- vertical angles
- volume

Are Y ou Ready?

Complete the Quick Review to see if you are ready to start this module.
Then complete the Quick Check.

Quick Review

Example 1	Example 2
Solve $5x + 2 = 90$.	**Evaluate $2(3)(4) + 2(3)(5) + 2(4)(5)$.**
$5x + 2 = 90$ Original equation.	$2(3)(4) + 2(3)(5) + 2(4)(5)$ Original expression
$5x = 88$ Subtract 2 from each side.	$= 24 + 30 + 40$ Multiply.
$x = 17.6$ Divide each side by 5.	$= 94$ Add.

Quick Check

Solve each equation.	Evaluate each expression.
1. $3x - 9 = 180$ 63	**5.** $6(15)(22)$ 1980
2. $2x + 10x - 9 = 90$ 8.25	**6.** $0.5(8)(9)$ 36
3. $15x + 42 = 12x + 51$ 3	**7.** $2(6)(7) + 2(6)(10) + 2(7)(10)$ 344
4. $9x + 1 = 17x - 31$ 4	**8.** $0.5(5)(12) + 0.5(5)(12) + 5(14) + 12(14) + 13(14)$ 480

How Did Y ou Do?
Which exercises did you answer correctly in the Quick Check?

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define Complementary angles are two angles with measures that have a sum of 90°.

Example $m\angle ABC = 48°$ and $m\angle CBD = 42°$

Ask Do the measures of the two angles add up to 90°? Y es; $48° + 42° = 90°$

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- subtracting rational numbers
- classifying angles
- using angle pairs
- finding perimeter and area
- identifying three-dimensional figures
- evaluating expressions with absolute value
- converting measurements

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You can use the ALEKS pie report to see which students know the topics in the **Angles, Introduction to Perimeter and Area,** and **Solids and Cross Sections** modules—who is ready to learn these topics and who isn't quite ready to learn them yet—and then adjust your instruction as appropriate.

Mindset Matters

Model Constructive Feedback

For students to grow, they need to receive timely, constructive feedback that references a specific skill or area. You can also model what appropriate feedback looks and sounds like so that students can collaborate and give one another constructive feedback in a way that is positive and helpful.

How Can I Apply It?

Use the **Questions for Mathematical Discourse** in the Teacher Edition to ask students questions and to share feedback on their thinking. This is a great opportunity to model feedback for the class so that students can give one another feedback during collaborative activities.

Lesson 11-1

Angles and Congruence

G.CO.1, G.CO.12

LESSON GOAL

Students identify and use different kinds of angles.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Angles Formed by Intersecting Lines

Develop:

Angles
- Identify Angles

Congruent Angles
- Congruent Angles and Angle Bisectors

Special Angle Pairs
- Vertical Angles and Angle Pairs

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Subtract Rational Numbers	●	●		●
Extension: Using a Compass		●	●	●

Language Development Handbook

Assign page 63 of the *Language Development Handbook* to help your students build mathematical language related to angles.

ELL You can use the tips and suggestions on page T63 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students analyzed angles formed by two parallel lines cut by a transversal.
8.G.5

Now
Students identify and use different kinds of angles.
G.CO.1, G.CO.12

Next
Students will find measures of angles using complementary and supplementary angles.
G.CO.1, G.CO.12

Rigor

The Three Pillars of Rigor

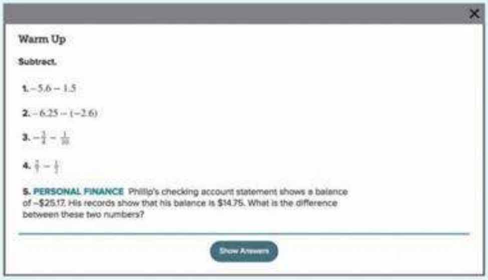
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop a precise understanding of angles, and they build fluency by making constructions related to angles. They apply their understanding by solving real-world problems about pairs of angles.

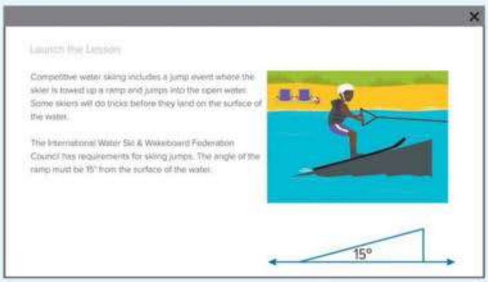
Mathematical Background

This lesson introduces the definition of an angle and special types of angle pairs. Adjacent angles are two angles that lie in the same plane, have a common vertex and a common side, but have no common interior points. Vertical angles are two non-adjacent angles formed by two intersecting lines. All vertical angles are congruent. A linear pair is a pair of adjacent angles whose noncommon sides are opposite rays.

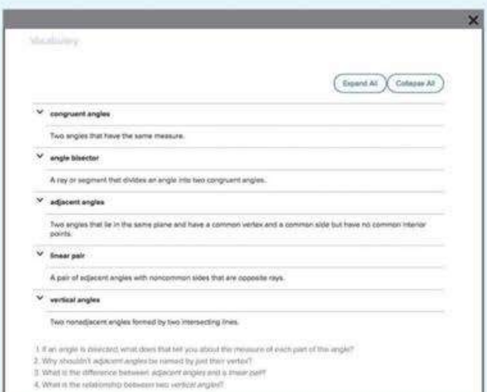
Interactive Presentation



Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- subtracting rational numbers

Answers:

1. −7.1
2. −3.65
3. $-\frac{17}{20}$
4. $-\frac{3}{14}$
5. \$39.92

Launch the Lesson

Teaching the Mathematical Practices

4 Model with Mathematics In this Launch the Lesson, students can see a real-world application of angles.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Angles Formed by Intersecting Lines

Objective

Students use dynamic geometry software to discover the angle relationships created by intersecting lines.

MP Teaching the Mathematical Practices

4 Model with Mathematics Throughout the Explore, encourage students to identify relationships between angles formed by intersecting lines.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

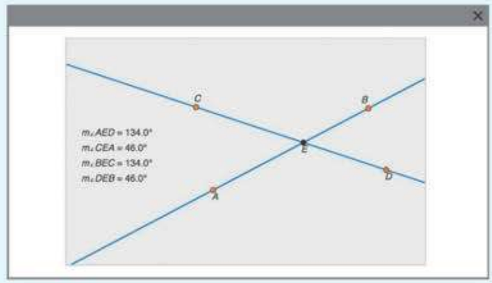
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use a sketch to investigate angle relationships when angles are formed by intersecting lines. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore



Explore

WEB SKETCHPAD

Students use a sketch to complete an activity in which they explore angle relationships.

TYPE

Students answer questions about the angle relationships present.

Interactive Presentation

Explore

TYPE

a|

Students respond to the Inquiry Question and can view a sample answer.

Explore Angles Formed by Intersecting Lines (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What relationships help us determine the measurements of the set of angles created by two intersecting lines? Sample answer: angle-based relationships help us determine the measurements of the set of angles.
- What are the angle sets created by intersecting lines called? Sample answer: adjacent angles, linear pair, vertical angles

Inquiry

What angle relationships are formed by two intersecting lines? Sample answer: The sum of two adjacent angle measures is 180°. Two angles across from one another are congruent.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Angles

Objective

Students apply the definitions of angles and parts of angles to analyze figures.

Teaching the Mathematical Practices

5 Decide When to Use Tools Mathematically proficient students can make sound decisions about when to use mathematical tools such as a protractor. Help them see why using these tools will help to solve problems and what the limitations are of using the tools.

Common Misconception

Students may think that a straight angle is a straight line with a measure of 0°. Have students draw a straight line containing three named points. Have them use a protractor to measure the straight angle having one of the points as its vertex. Students should notice that the angle measures 180°.

Essential Question Follow-Up

Students have begun identifying angles.

Ask:

Why are angles important in the real world? Sample answer: In architecture, angles are important to make buildings structurally sound. In art, angles can change the viewing angle to change the perspective.

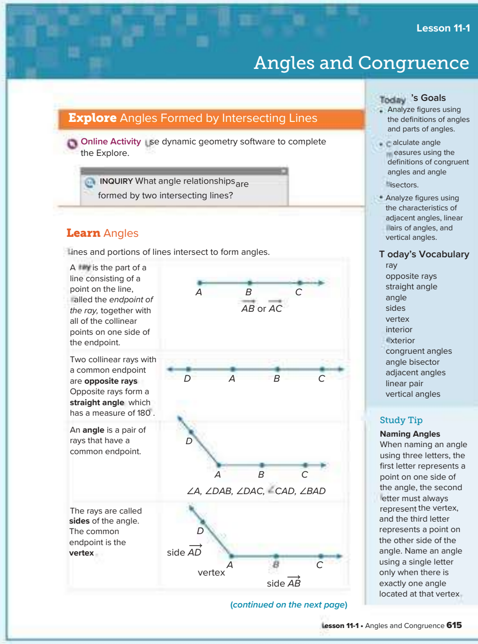
Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are having a hard time identifying or naming angles, **THEN** draw an angle with the vertex labeled *B* and two points, one on each ray, labeled *A* and *C*. Point out that the angle consists of two rays that meet at the vertex. Remind students that when naming angles, the vertex is always the middle letter, a point on one ray is the first letter, and a point on the other ray is the last letter. The example angle would be named $\angle ABC$.



Lesson 11-1

Angles and Congruence

Explore Angles Formed by Intersecting Lines

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY What angle relationships are formed by two intersecting lines?

Learn Angles

Lines and portions of lines intersect to form angles.

A ray is the part of a line consisting of a point on the line, called the *endpoint of the ray*, together with all of the collinear points on one side of the endpoint.

$\overrightarrow{AB}$ or $\overrightarrow{AC}$

Two collinear rays with a common endpoint are **opposite rays**. Opposite rays form a **straight angle**, which has a measure of 180°.

An **angle** is a pair of rays that have a common endpoint.

$\angle A$, $\angle DAB$, $\angle DAC$, $\angle CAD$, $\angle BAD$

The rays are called **sides** of the angle. The common endpoint is the **vertex**.

Today's Goals
- Analyze figures using the definitions of angles and parts of angles.
- Calculate angle measures using the definitions of congruent angles and angle bisectors.
- Analyze figures using the characteristics of adjacent angles, linear pairs of angles, and vertical angles.

Today's Vocabulary
ray
opposite rays
straight angle
angle
sides
vertex
interior
exterior
congruent angles
angle bisector
adjacent angles
linear pair
vertical angles

Study Tip

Naming Angles When naming an angle using three letters, the first letter represents a point on one side of the angle, the second letter must always represent the vertex, and the third letter represents a point on the other side of the angle. Name an angle using a single letter only when there is exactly one angle located at that vertex.

(continued on the next page)

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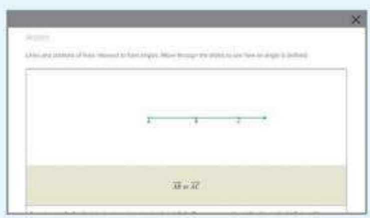
Interactive Presentation



Learn

TAP Students tap through the slides to see how an angle is defined.

TAP Students tap through the slides to see how an angle divides a plane into three distinct parts.

WATCH Students watch a video on how to use a protractor to measure and draw angles.

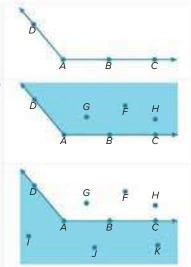
Go Online You can watch a video to see how to use a protractor to measure and draw angles.

An angle divides a plane into three distinct parts.

Points *D*, *A*, *B*, and *C* lie on the angle.

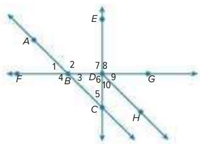
Points *G*, *F*, and *H* lie in the **interior** of the angle.

Points *I*, *J*, and *K* lie in the **exterior** of the angle.



Example 1 Identify Angles

Use the figure to identify the angles or parts of angles that satisfy each given condition.



a. Name all the angles that have *D* as a vertex.
$\angle EDF$, $\angle EDG$, $\angle FDC$, $\angle GDC$, $\angle GDH$, $\angle CDH$, $\angle 6$, $\angle 7$, $\angle 8$, $\angle 9$, $\angle 10$

b. Name the sides of $\angle 2$. $\overrightarrow{BA}$ and $\overrightarrow{BG}$

c. Name a point in the interior of $\angle FDE$. *A*

d. Name a point or points in the exterior of $\angle FDE$. *C*, *H*, and *G*

Go Online You can complete an Extra Example online.

Think About It!
Can a point be in the interior of one angle and in the exterior of another angle? If so, give an example.

Yes; sample answer: *E* is in the interior of $\angle ABG$ and is in the exterior of $\angle FDH$.

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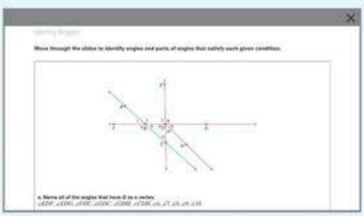
Interactive Presentation



Example 1

TAP

Students move through the slides to identify angles and parts of angles.

TYPE

Students determine whether a point can be in the interior of one angle and also be in the exterior of another angle.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Identify Angles

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL What is the significance of the vertex of an angle? The vertex is a common endpoint of the two sides of the angle.

OL What is the vertex of $\angle 9$? *D*

BL What is another name for $\angle 3$? Sample answers: $\angle GBC$, $\angle CBG$, $\angle DBC$, or $\angle CBD$

Common Error

Often students name angles incorrectly because they do not place the vertex as the center letter. Remind students that when naming an angle with three letters, the letters should follow the shape of the angle.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Congruent Angles

Objective

Students apply the definitions of congruent angles and angle bisectors to calculate angle measures.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of congruent angles in this Learn.

What Students Are Learning

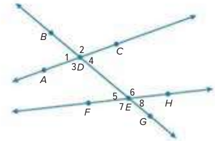
Congruent angles have the same measure. An angle bisector is a ray or segment that divides an angle into two congruent angles.

Common Misconception

Students may assume that the measure of an angle depends on the lengths of the line segments shown on the sides of the angle. Remind students that the sides of an angle are rays that extend infinitely, and thus, the sides have no length. The points on the rays are useful only in naming the angle, not for determining the measures of the angle.

Check

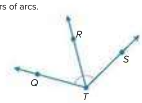
Use the figure to identify the angles or parts of angles that satisfy the given condition. Which angle has sides $\overrightarrow{DB}$ and $\overrightarrow{DC}$? Select all that apply.



A. ∠2
B. ∠3
C. ∠ADB
D. ∠BDC
E. ∠CDB
F. ∠EDC

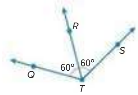
Learn Congruent Angles

The measure of an angle is the measure in degrees of the space between the sides of an angle. Angles that have the same measure are **congruent angles**. Congruent angles are indicated on the figure by matching numbers of arcs.



$\angle QTS \cong \angle STR$

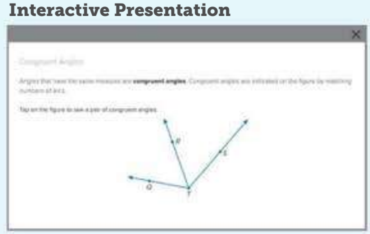
A ray or segment that divides an angle into two congruent parts is an **angle bisector**. In the figure, $\overrightarrow{TR}$ bisects $\angle QTS$.



$m\angle QTR \cong m\angle STR$

Lesson 11-1 • Angles and Congruence 617

Interactive Presentation



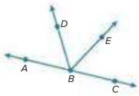
Learn

TAP

Students tap the given figure to see congruent angles.

Example 2 Congruent Angles and Angle Bisectors

In the figure, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays and $\overrightarrow{BD}$ bisects $\angle ABE$. If $m\angle ABD = (4x + 14)°$ and $m\angle DBE = (8x - 32)°$, find $m\angle DBE$.



We can solve for this in two steps. First, solve for x. Then find $m\angle DBE$.

Step 1: Because $\overrightarrow{BD}$ bisects $\angle ABE$, $\angle ABD \cong \angle DBE$. By the definition of congruence, these angles have the same measure.

$m\angle ABD = m\angle DBE$	Definition of congruent angles
$4x + 14 = 8x - 32$	Substitution
$14 = 4x - 32$	Subtract $4x$ from each side.
$46 = 4x$	Add 32 to each side.
$11.5 = x$	Divide each side by 4.

Step 2: Because we are asked to find $m\angle DBE$, we substitute 11.5 for x in the expression.

$m\angle DBE = 8x - 32$	Given
$= 8(11.5) - 32$	Substitute.
$= 92 - 32$	Multiply.
$= 60$	Subtract.
$m\angle DBE = 60°$	

Talk About It!

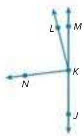
Suppose $\overrightarrow{BE}$ is an angle bisector of $\angle DBC$. What is $m\angle EBC$? Explain your solution process.

Sample answer: If $\overrightarrow{BE}$ is an angle bisector of $\angle DBC$, $m\angle DBE = m\angle EBC$. By substitution, $m\angle EBC = 60°$.

Check

In the figure, $\overrightarrow{KJ}$ and $\overrightarrow{KM}$ are opposite rays, and $\overrightarrow{KN}$ bisects $\angle JKL$. If $m\angle JKN = (8x - 13)°$ and $m\angle NKL = (6x + 11)°$, find $m\angle JKN$.

$m\angle JKN =$? ° 83



Go Online You can complete an Extra Example online.

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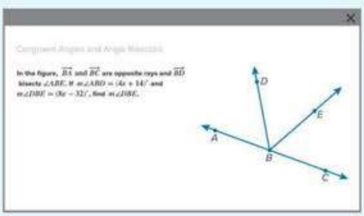
Interactive Presentation



Example 2

TAP

Students move through the steps to solve for the missing value.

TYPE

Students discuss their solution process when given a different piece of information.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Example 2 Congruent Angles and Angle Bisectors

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL Why isn't $\angle B$ an appropriate way to name any of the angles in this diagram? Because B is the vertex of multiple angles, it doesn't specify just one angle.

OL Why can the equation $4x + 14 = 8x - 32$ be used to solve for x? Sample answer: Because $\overrightarrow{BD}$ is the angle bisector of $\angle ABE$, we know the two angles are congruent. So, their measures must be equal.

BL What is the measure of $\angle ABE$? Show all work. 120°; $m\angle ABE = 2(60) = 120°$

Common Error

Students may confuse the kind of figure that can be bisected. Remind students that a line of reflection must exist. When the figure is folded along this line, each point on one side maps to a corresponding point on the other side of the line. A ray cannot be bisected.

Learn Special Angle Pairs

Objective

Students apply the characteristics of adjacent angles, linear pairs of angles, and vertical angles to analyze figures.

MP Teaching the Mathematical Practices

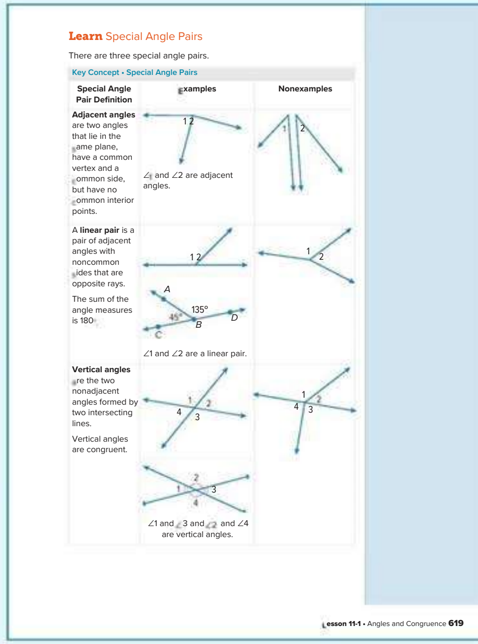
3 Analyze Cases This Learn guides students to examine cases of special angle pairs. Encourage students to familiarize themselves with all of the cases.

Important to Know

Adjacent angles are two angles that lie in the same plane with a common vertex and common side. A linear pair is a special type of adjacent angles with noncommon sides that are opposite rays. Two nonadjacent angles formed by intersecting lines are called vertical angles, and these angles are congruent.

Common Misconception

Students may assume that all adjacent angles are also linear pairs. Have students draw two adjacent angles and a linear pair. Then compare and contrast the two different types of angle pairs.



Learn Special Angle Pairs

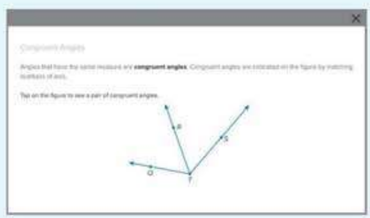
There are three special angle pairs.

Key Concept • Special Angle Pairs

Special Angle Pair Definition	Examples	Nonexamples
Adjacent angles are two angles that lie in the same plane, have a common vertex and a common side, but have no common interior points.	∠1 and ∠2 are adjacent angles.	
A **linear pair** is a pair of adjacent angles with noncommon sides that are opposite rays. The sum of the angle measures is 180	∠1 and ∠2 are a linear pair.	
Vertical angles are the two nonadjacent angles formed by two intersecting lines. Vertical angles are congruent.	∠1 and ∠3 and ∠2 and ∠4 are vertical angles.	

Lesson 11-1 • Angles and Congruence 619

Interactive Presentation



Learn

TAP

Students move through the definitions to see examples and non-examples of special angle pairs.

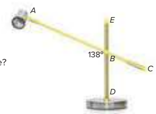
DIFFERENTIATE

Reteaching Activity AL ELL

IF students have difficulty using the special angle relationships,
THEN have students draw two intersecting lines. Instruct them to write one or two sentence describing each relationship and to provide an example. For extra practice, have students analyze the figures found throughout the lesson and determine which angle relationships are or are not present in them.

Example 3 Vertical Angles and Angle Pairs

HOME DECOR The office lamp is made using two intersecting metal bars.



a. How many pairs of adjacent angles do you see in the figure? List two pairs.

4; Sample answer: ∠*DBA* and ∠*ABE*, ∠*ABE* and ∠*EBC*

b. Identify two pairs of vertical angles in the figure.

∠*DBA* and ∠*EBC*, ∠*ABE* and ∠*CBD*

c. How many linear pairs do you see in the figure? List each pair.

4; ∠*DBA* and ∠*ABE*, ∠*ABE* and ∠*EBC*, ∠*EBC* and ∠*CBD*, ∠*CBD* and ∠*DBA*

d. Find *m*∠*EBC*.

Because ∠*ABD* and ∠*EBC* are formed by intersecting line segments, they are vertical angles. Because vertical angles are congruent, *m*∠*EBC* is the same as *m*∠*ABD*, 138°.

e. Find *m*∠*ABE*.

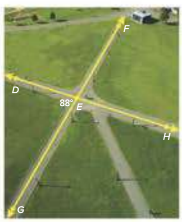
Because ∠*ABE* and ∠*ABD* form a linear pair, their measures add to 180°. Thus, $m\angle ABE = 180 - m\angle ABD = 180 - 138 = 42°$.

Think About It!
Can vertical angles also be adjacent angles? Explain.

No; sample answer: Adjacent angles have a common side. Vertical angles have a common vertex but not a common side.

Check

PARK A city planner is designing a park. He wants to place two pathways that intersect near the center of the park. If $m\angle GED = 88°$, identify the true statement(s).



firina/123RF, Glowimages/Getty Images

(A.) $m\angle DEF = 92°$

B. $m\angle DEG = 92°$

(C.) $m\angle FEH = 88°$

D. $m\angle DEH = 92°$

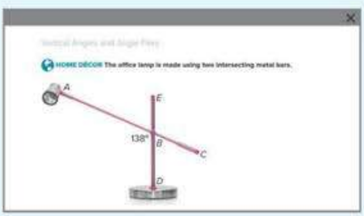
E. $m\angle GEH = 88°$

Go Online You may want to complete the construction activities for this lesson.

Go Online You can complete an Extra Example online.

620 **Module 11** • Angles and Geometric Figures

Interactive Presentation



Example 3

TAP

Students move through the slides and complete the exercises.

TYPE

Students explain whether vertical angles can also be adjacent angles.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Vertical Angles and Angle Pairs

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about special angle pairs to solving a real-world problem.

Questions for Mathematical Discourse

AL Why are angles *ABE* and *CBD* not considered adjacent? Although they share the same vertex, they do not share the same side.

OL Can $m\angle EBC = 42°$? Explain. No; because angles *ABD* and *EBC* are vertical, they have the same angle measure.

BL Can the two angles of a linear pair be congruent? Explain. Yes; sample answer: If both angles are 90°, then they will be congruent angles.

Common Error

Students may not correctly identify pairs of vertical angles or not realize that vertical angles are congruent. Remind students that vertical angles share a vertex but are nonadjacent.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–17
2	exercises that use a variety of skills from this lesson	18–48
3	exercises that emphasize higher-order and critical-thinking skills	49–50

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–47 odd, 49–50
- Extension: Using a Compass
- ALEKS Angles

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–47 odd
- Remediation, Review Resources: Subtract Rational Numbers
- Personal Tutors
- Extra Examples 1–3
- ALEKS Addition and Subtraction with Fractions; Decimals: Addition and Subtraction

AL

IF students score 65% or less on the Checks,
THEN assign:

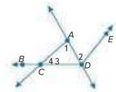
- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Subtract Rational Numbers
- *Quick Review Math Handbook*: Angle Measure
- ALEKS Addition and Subtraction with Fractions; Decimals: Addition and Subtraction

Practice

Go Online Y ou can complete your homework online.

Example 1

Use the figure to identify angles and parts of angles that satisfy each given condition.



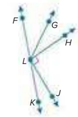
1. Name the vertex of ∠1. *A*

2. Name the sides of ∠4. $\overrightarrow{CA}$, $\overrightarrow{CD}$

3. What is another name for ∠3? ∠*ADC*, ∠*CDA*

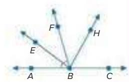
4. What is another name for ∠*CAD*? ∠1, ∠*DAC*

Example 2



5. In the figure, $\overrightarrow{LF}$ and $\overrightarrow{LK}$ are opposite rays. $\overrightarrow{LG}$ bisects ∠*FLH*. If $m\angle FLG = 14x + 5$ and $m\angle HLG = 17x - 1$, find $m\angle FLH$. 66°

In the figure, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays. $\overrightarrow{BH}$ bisects ∠*EBC* and $\overrightarrow{BE}$ bisects ∠*ABF*.



6. If $m\angle ABE = 2n + 7$ and $m\angle EBF = 4n - 13$, find $m\angle ABE$. 27°

7. If $m\angle EBH = 6x + 12$ and $m\angle HBC = 8x - 10$, find $m\angle EBH$. 78°

8. If $m\angle ABF = 7b - 24$ and $m\angle ABE = 2b$, find $m\angle EBF$. 16°

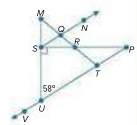
9. If $m\angle EBC = 31a - 2$ and $m\angle EBH = 4a + 45$, find $m\angle HBC$. 61°

10. If $m\angle ABF = 8w - 6$ and $m\angle ABE = 2(w + 11)$, find $m\angle EBF$. 47°

11. If $m\angle EBC = 3r + 10$ and $m\angle ABE = 2r - 20$, find $m\angle EBF$. 56°

Example 3

Refer to the figure.



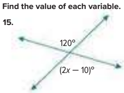
12. Name two adjacent angles. Sample answer: ∠*MQN* and ∠*NQR*

13. Name two vertical angles. Sample answer: ∠*SRQ* and ∠*TRP*

14. Find $m\angle SUV$. 122°

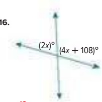
Find the value of each variable.

15.



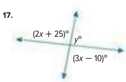
65

16.



12

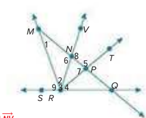
17.



$x = 35$, $y = 85$

Mixed Exercises

Refer to the figure to name the vertex of each angle.



18. ∠1 *M*

19. ∠2 *R*

20. ∠4 *Q*

21. ∠7 *P*

Use the figure to name the sides of each angle.

22. ∠*QPT* $\overrightarrow{PQ}$ and $\overrightarrow{PT}$

23. ∠*MNV* $\overrightarrow{NM}$ and $\overrightarrow{NV}$

24. ∠6 $\overrightarrow{NM}$ and $\overrightarrow{NR}$

25. ∠3 $\overrightarrow{RP}$ and $\overrightarrow{RQ}$ or $\overrightarrow{RT}$ and $\overrightarrow{RQ}$

Use the figure to write another name for each angle.

26. ∠9 ∠*MRS*, ∠*SRM*

27. ∠*QPT* ∠*TPQ*

28. ∠*MQS*
∠4, ∠*SQM*, ∠*MQR*, ∠*RQM*, ∠*NQS*, ∠*SQN*,
∠*NQR*, ∠*RQN*, ∠*PQR*, ∠*RQP* , ∠*PQS*, ∠*SQP*

29. ∠5 ∠*TPN*, ∠*NPT*, ∠*TPM*, ∠*MPT*

Use the figure above to name each angle, point, or pair of angles.

30. a point in the interior of ∠*VRQ*
P, *T*

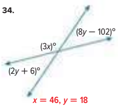
31. a point in the exterior of ∠*MRT*
S, *Q*

32. a pair of angles that share exactly one point
Sample answer: ∠6, ∠8

33. a pair of angles that share more than one point
Sample answer: ∠*MPR*, ∠*PRQ*

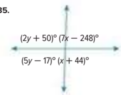
Find the value of each variable.

34.



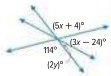
$x = 46, y = 18$

35.



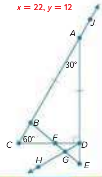
$x = 48, y = 21$

36.



$x = 22, y = 12$

Name an angle or angle pair that satisfies each condition.



37. two adjacent angles **Sample answer: ∠HGE, ∠DGE**

38. two vertical angles **Sample answer: ∠BFC, ∠DFE**

39. a linear pair that has vertex *F* **Sample answer: ∠BFC, ∠BFD**

Use the picture at the right.

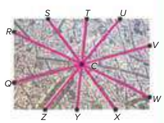
40. Name four rays. **Sample answer: $\overrightarrow{ST}$, $\overrightarrow{TU}$, $\overrightarrow{TV}$, $\overrightarrow{TW}$**

41. Name three angles. **Sample answer: ∠STV, ∠VTW, ∠UTW**

TRAFFIC **In the traffic circle around the Arc de Triomphe in Paris, France, there are eight lanes of traffic. Tell whether each angle pair satisfies the given condition.**

42. vertical angles

- **a.** ∠ZCY and ∠TCU **yes**   **b.** ∠XCW and ∠SCT **no**
- **c.** ∠QCR and ∠WCV **yes**   **d.** ∠TCU and ∠UCT **no**



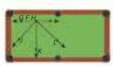
43. linear pair

- **a.** ∠RCU and ∠WCU **yes**   **b.** ∠QCR and ∠SCR **no**
- **c.** ∠VCX and ∠WCY **no**   **d.** ∠ZCR and ∠UCW **no**

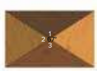
44. adjacent angles

- **a.** ∠WCU and ∠RCU **yes**   **b.** ∠QCS and ∠SCR **no**
- **c.** ∠VCW and ∠QCR **no**   **d.** ∠VCX and ∠VCU **yes**

45. POOL Felipe uses a computer program to model the paths of pool balls. ∠GFH is a straight angle that represents the rail of the pool table. If $\overrightarrow{FK}$ bisects ∠JFL, and m∠JFL = 90°, what is m∠LFK? **45°**



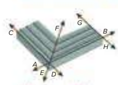
46. WOODWORKING Oliver makes rectangular blocks like the one shown and then glues them together to make a plaque. Find m∠1, m∠2, and m∠3, so he can cut the pieces of the plaque. **113°; 67°; 113°**



47. WOODWORKING Naomi cuts two pieces of baseboard molding to meet in a corner at a 90° angle.

a. T o what degree should she set her table saw for the cut? **45°**

b. Which ray represents the angle bisector of the molding angle? **$\overrightarrow{EF}$**



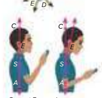
48. TEXTING Moving your head forward to look at a screen can stress your spine. Experts recommend aligning your ears with your shoulders and arms. **They form a linear pair.**

a. In the forward posture, what is the relationship between ∠CSE and ∠ESA?

b. In a correct posture, what is the relationship between $\overrightarrow{SE}$ and $\overrightarrow{SA}$? **They are opposite rays.**

c. If you are standing so that m∠CSE = 26°, what is m∠ESA? **154°**

d. Standing so that m∠CSE ≥ 15° puts more than 27 pounds of pressure on your spine. If there is 34 pounds of pressure on your spine, what inequality describes m∠ESA? **m∠ESA ≤ 165°**

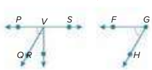


Correct Forward
posture posture

Higher-Order Thinking Skills

49. PERSEVERE $\overrightarrow{MP}$ bisects ∠LMN, $\overrightarrow{MQ}$ bisects ∠LMP, and $\overrightarrow{MR}$ bisects ∠QMP. If m∠RMP = 21, find m∠LMN. Explain your reasoning. **168°; sample answer: If m∠RMP = 21° and $\overrightarrow{MR}$ bisects ∠QMP, then m∠QMP = 2(21) or 42°. If m∠QMP = 42° and $\overrightarrow{MQ}$ bisects ∠LMP, then m∠LMP = 2(42) or 84°. If m∠LMP = 84° and $\overrightarrow{MP}$ bisects ∠LMN, then m∠LMN = 2(84) = 168°.**

50. ANAL YZE Maria constructed a copy of ∠PVQ and labeled it ∠FGH.



a. Are ∠FGH and ∠QVS a linear pair? Explain. **No; sample answer: Because even though they are supplementary, they are not adjacent angles.**

b. Maria must also copy ∠QVS. Sal says she can create a copy of ∠QVS if she extends $\overrightarrow{GH}$ past G. Mona says Maria can create a copy of ∠QVS by extending $\overrightarrow{GF}$ past G. Who is correct? Justify your argument.

Both Sal and Mona are correct. Extending either line will create a linear pair. Because ∠FGH is in both linear pairs and the sum of the angles in both pairs must be 180°, the other angle in each pair will have the same measure and be congruent.

Lesson 11-2
Angle Relationships

G.CO.1, G.CO.12

LESSON GOAL

Students find measures of angles using complementary and supplementary angles and identify what can and cannot be assumed about angles in a diagram.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Complementary and Supplementary Angles

Develop:

Complementary and Supplementary Angles
- Complementary and Supplementary Angles

Perpendicularity
- Perpendicular Lines

Explore: Interpreting Diagrams

Develop:

Interpreting Diagrams
- Interpreting Diagrams

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Vertical and Adjacent Angles	●	●		●
Extension: Runway Angles		●	●	●

Language Development Handbook

Assign page 64 of the *Language Development Handbook* to help your students build mathematical language related to angle relationships.

ELL You can use the tips and suggestions on page T64 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students wrote simple equations to find missing angle measures formed by complementary or supplementary angles. **7.G.5**

Now
Students find measures of angles using complementary and supplementary angles.
G.CO.1, G.CO.12

Next
Students will find measures of two-dimensional objects.
G.GPE.7, G.MG.1

Rigor

The Three Pillars of Rigor

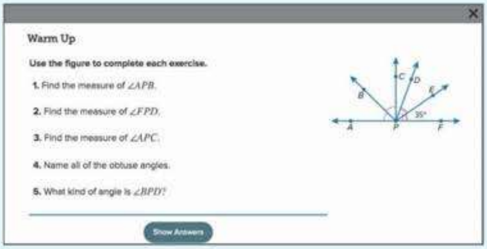
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop a precise understanding of angle relationships, and they build fluency by making constructions related to angles. They apply their understanding by solving real-world problems about pairs of angles.

Mathematical Background

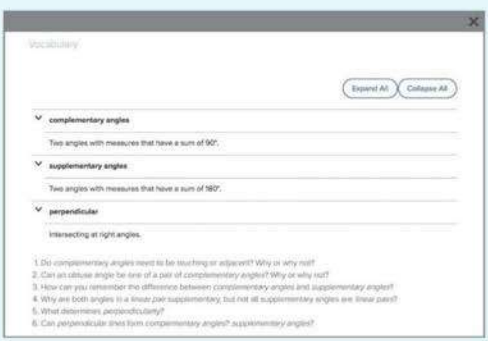
Dynamic geometry software is used to help students explore angle relationships, such as complementary and supplementary angles, by manipulating associated points and then examining the relationships.

Interactive Presentation



Warm Up

Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- classifying angles

Answers:

1. 45°
2. 70°
3. 90°
4. ∠*APE*, ∠*APD*, ∠*FPB*, and ∠*BPE*
5. acute

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of complementary angles.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?,* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Complementary and Supplementary Angles

Objective

Students use dynamic geometry software to explore the relationships between complementary and supplementary angles.

MP Teaching the Mathematical Practices

3 Make Conjectures In this Explore, students will make conjectures and then build a logical progression of statements to validate the conjectures. Once students have made their conjectures, guide the students to validate them.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

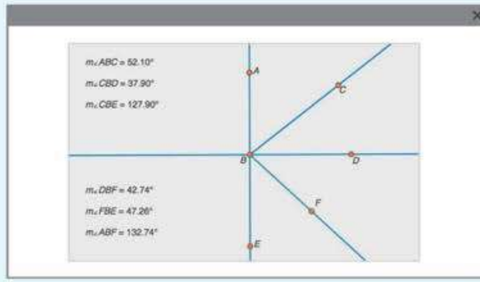
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use a sketch to explore complementary and supplementary angles. They will answer questions leading them to the formal definitions of each angle pair. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore



Explore

WEB SKETCHPAD

Students use a sketch to complete an activity in which they explore complementary and supplementary angles.

TYPE

Students answer questions about the angle relationships.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Complementary and Supplementary Angles (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How are complementary angles and right angles similar, and how are they different? Sample answer: The sum of the measures of two complementary angles equals 90°. A right angle also measures 90°. Only one right angle is needed to form a 90° angle, but two complementary angles are needed to form a 90° angle.
- How are supplementary angles and straight angles similar, and how are they different? Sample answer: The sum of the measures of two supplementary angles equals 180°. A straight angle also measures 180°. Only one straight angle is needed to form a 180° angle, but two supplementary angles are needed to from a 180° angle.

Inquiry

How do complementary angles compare to supplementary angles? Sample answer: The sum of the measures of two complementary angles is 90°. The sum of the measures of two supplementary angles is 180°, twice the sum of two complementary angles.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Explore Interpreting Diagrams

Objective

Students use dynamic geometry software to discover what can and cannot be assumed about angles in a diagram.

MP Teaching the Mathematical Practices

3 Reason Inductively In this Explore, students will use inductive reasoning to make plausible arguments.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

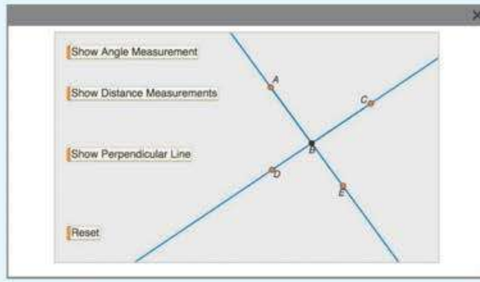
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use a sketch to explore how to interpret a diagram. They will answer questions about angle measurements formed during the exploration. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore



Explore

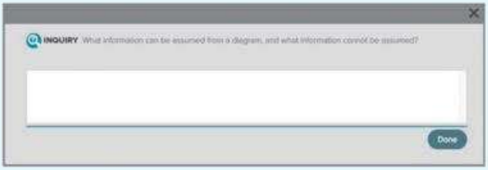
WEB SKETCHPAD

Students use a sketch to interpret a diagram.

TYPE

Students answer questions about angle measures and assumptions.

Interactive Presentation



Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Interpreting Diagrams (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What is the measure of a right angle? 90°
- Why should you not make assumptions about the information presented in diagrams? Sample answer: If you assume the measure of a segment or angle or the relationships between segment and angle pairs based on how they appear in a diagram, you may be assuming measures or relationships that are not true.

Inquiry

What information can be assumed from a diagram, and what information cannot be assumed? Sample answer: Y ou can assume figures are coplanar if they appear to be so. You can assume that points are collinear. You can assume that points or lines of intersection are represented, and you can assume the relationship among angle pairs. You cannot assume that two line segments or angles are congruent just because they appear to have the same measure. You cannot assume that an angle is a right angle unless it is marked with a right angle symbol.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Complementary and Supplementary Angles

Objective

Students apply the characteristics of complementary and supplementary angles to calculate angle measures.

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to solve the problem in this Learn, students will need to analyze the mathematical relationships in the problem to draw a conclusion.

What Students are Learning

Complementary angles are two angles whose measures have a sum of 90°. Supplementary angles are two angles whose measures have a sum of to 180°. Angles do not have to be adjacent to be complementary or supplementary angles. All linear pairs are supplementary angles.

Common Misconception

Students often believe only adjacent angles can be complementary or supplementary. Point students back to the example showing nonadjacent complementary and supplementary angles.

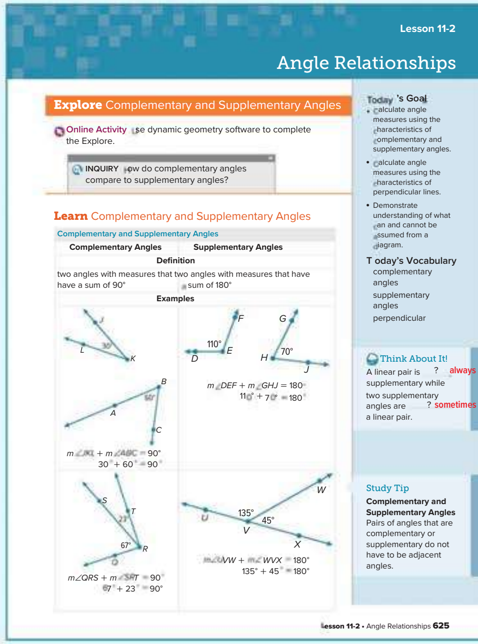
DIFFERENTIATE

Enrichment Activity BL

Can vertical angles ever be complementary or supplementary? Explain. Yes; sample answer: Vertical angles are complementary when each angle measures 45°. They are supplementary when each angle measures 90°.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 11-2

Angle Relationships

Explore Complementary and Supplementary Angles

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How do complementary angles compare to supplementary angles?

Learn Complementary and Supplementary Angles

Complementary and Supplementary Angles

Complementary Angles	Supplementary Angles
Definition	
two angles with measures that have a sum of 90°	two angles with measures that have a sum of 180°
Examples	
$m\angle JKL + m\angle ABC = 90°$ $30° + 60° = 90°$	$m\angle DEF + m\angle GHJ = 180°$ $110° + 70° = 180°$
$m\angle QRS + m\angle SRT = 90°$ $67° + 23° = 90°$	$m\angle UVW + m\angle WVX = 180°$ $135° + 45° = 180°$

Today's Goal
- Calculate angle measures using the characteristics of complementary and supplementary angles.
- Calculate angle measures using the characteristics of perpendicular lines.
- Demonstrate understanding of what can and cannot be assumed from a diagram.

Today's Vocabulary
complementary angles
supplementary angles
perpendicular

Think About It!
A linear pair is ___?___ always supplementary while two supplementary angles are ___?___ sometimes a linear pair.

Study Tip
Complementary and Supplementary Angles Pairs of angles that are complementary or supplementary do not have to be adjacent angles.

Lesson 11-2 • Angle Relationships 625

Interactive Presentation

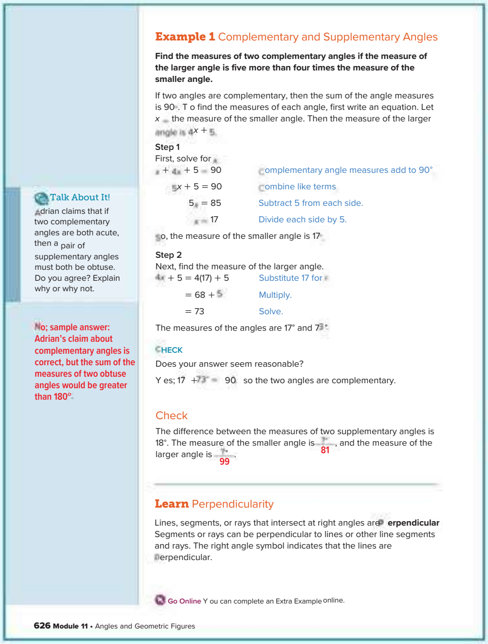
Learn

TAP

Students tap to compare the relationships between complementary and supplementary angles.

SELECT

Students select terms to complete a statement about supplementary angles.



Example 1 Complementary and Supplementary Angles

Find the measures of two complementary angles if the measure of the larger angle is five more than four times the measure of the smaller angle.

If two angles are complementary, then the sum of the angle measures is 90°. To find the measures of each angle, first write an equation. Let x = the measure of the smaller angle. Then the measure of the larger angle is $4x + 5$.

Step 1

First, solve for x.

$x + 4x + 5 = 90$ Complementary angle measures add to 90°

$5x + 5 = 90$ Combine like terms

$5x = 85$ Subtract 5 from each side.

$x = 17$ Divide each side by 5.

So, the measure of the smaller angle is 17°.

Step 2

Next, find the measure of the larger angle.

$4x + 5 = 4(17) + 5$ Substitute 17 for x.

$= 68 + 5$ Multiply.

$= 73$ Solve.

The measures of the angles are 17° and 73°.

CHECK

Does your answer seem reasonable?

Yes; $17 + 73 = 90$ so the two angles are complementary.

Talk About It!

Adrian claims that if two complementary angles are both acute, then a pair of supplementary angles must both be obtuse. Do you agree? Explain why or why not.

No; sample answer: Adrian's claim about complementary angles is correct, but the sum of the measures of two obtuse angles would be greater than 180°.

Check

The difference between the measures of two supplementary angles is 18°. The measure of the smaller angle is ___?° 81, and the measure of the larger angle is ___?° 99.

Learn Perpendicularity

Lines, segments, or rays that intersect at right angles are **perpendicular**. Segments or rays can be perpendicular to lines or other line segments and rays. The right angle symbol indicates that the lines are perpendicular.

Go Online You can complete an Extra Example online.

626 Module 11 • Angles and Geometric Figures

Interactive Presentation

Example 1

TAP

Students move through the steps to find the measures of two complementary angles.

TYPE

Students type to answer a question about supplementary and complementary angles.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Complementary and Supplementary Angles

Teaching the Mathematical Practices

8 Attend to Details Mathematically proficient students continually ask themselves, "Does this make sense?" Point out that in this example, students will evaluate the reasonableness of their answer.

Questions for Mathematical Discourse

AL Can complementary angles also be adjacent? Explain. Yes; if two angles share a common side and a common vertex, and have a sum of 90°, then they are adjacent and complementary.

OL What expression represents the sum of the two angles? $x + 4x + 5$

BL Suppose the angles had instead been supplementary angles. What is the value of x? Explain. $x = 35$; $x + 4x + 5 = 180$, so $5x = 175$, which means $x = 35$.

Common Error

Students often mix up whether the sum is 90° or 180° when using the definition of complementary and supplementary angles. A quick way to remember is: 'c' comes before 's' in the alphabet, as 90 comes before 180 on the number line.

Learn Perpendicularity

Objective

Students apply the characteristics of perpendicular lines to calculate angle measures.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of perpendicular lines in this Learn.

Key Concept

Lines, segments, or rays that intersect at 90° angles are perpendicular. The right angle symbol indicates that lines are perpendicular.

Example 2 Perpendicular Lines

Teaching the Mathematical Practices

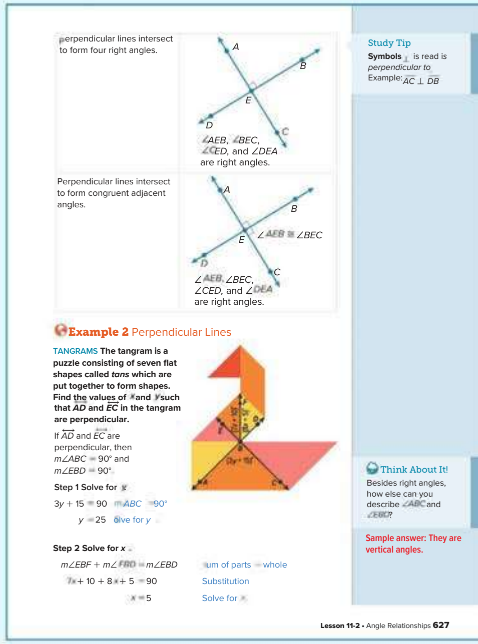
6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL What is true of the angles formed by perpendicular lines? All four angles are congruent; they each measure 90°.

OL What is the relationship between ∠*EBF* and ∠*FBD*? They are complementary angles; the sum of these two angles is 90°.

BL Besides right angles and vertical angles, how else can you describe ∠*ABC* and ∠*EBD*? They are supplementary angles.



erpendicular lines intersect to form four right angles.

∠AEB, ∠BEC, ∠CED, and ∠DEA are right angles.

Perpendicular lines intersect to form congruent adjacent angles.

∠AEB ≅ ∠BEC

∠AEB, ∠BEC, ∠CED, and ∠DEA are right angles.

Study Tip

Symbols ⊥ is read is *perpendicular to*

Example: $\overline{AC} \perp \overline{DB}$

Example 2 Perpendicular Lines

TANGRAMS **The tangram is a puzzle consisting of seven flat shapes called *tans* which are put together to form shapes. Find the values of x and y such that $\overleftrightarrow{AD}$ and $\overleftrightarrow{EC}$ in the tangram are perpendicular.**

If $\overleftrightarrow{AD}$ and $\overleftrightarrow{EC}$ are perpendicular, then $m\angle ABC = 90°$ and $m\angle EBD = 90°$.

Step 1 Solve for y.

$3y + 15 = 90$ — m∠ABC = 90°

$y = 25$ — olve for y

Step 2 Solve for x.

$m\angle EBF + m\angle FBD = m\angle EBD$ — um of parts = whole

$7x + 10 + 8x + 5 = 90$ — Substitution

$x = 5$ — Solve for x

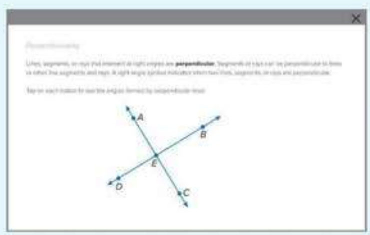
Think About It!

Besides right angles, how else can you describe ∠ABC and ∠EBD?

Sample answer: They are vertical angles.

Lesson 11-2 • Angle Relationships 627

Interactive Presentation



Learn

TAP

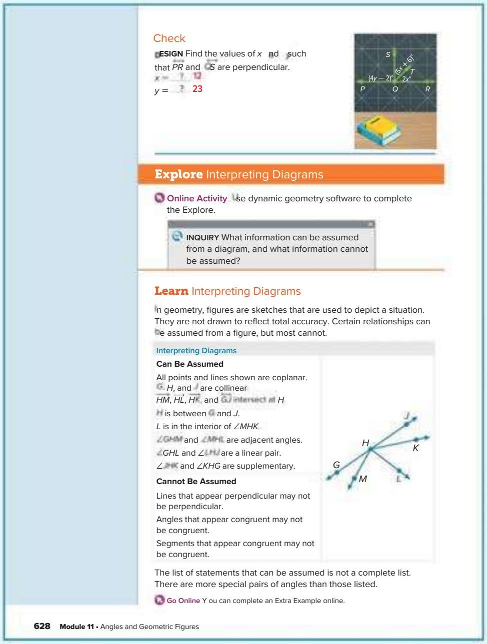
Students tap to see examples of perpendicular lines and right angles.

DIFFERENTIATE

AL ELL

IF students show difficulty determining angle relationships from a diagram,

THEN encourage students to use color-coding to avoid confusion. For example, they can color code vertical angles with red, perpendicular lines with yellow, and so on.



Check

DESIGN Find the values of x and y such that $\overleftrightarrow{PR}$ and $\overleftrightarrow{QS}$ are perpendicular.

x = ? 12

y = ? 23

Explore Interpreting Diagrams

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY What information can be assumed from a diagram, and what information cannot be assumed?

Learn Interpreting Diagrams

In geometry, figures are sketches that are used to depict a situation. They are not drawn to reflect total accuracy. Certain relationships can be assumed from a figure, but most cannot.

Interpreting Diagrams

Can Be Assumed

All points and lines shown are coplanar.

G, H, and J are collinear.

$\overrightarrow{HM}$, $\overrightarrow{HL}$, $\overrightarrow{HK}$, and $\overleftrightarrow{GJ}$ intersect at H.

H is between G and J.

L is in the interior of ∠MHK.

∠GHM and ∠MHL are adjacent angles.

∠GHL and ∠LHJ are a linear pair.

∠JHK and ∠KHG are supplementary.

Cannot Be Assumed

Lines that appear perpendicular may not be perpendicular.

Angles that appear congruent may not be congruent.

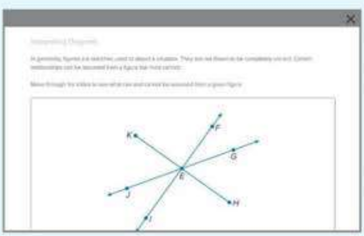
Segments that appear congruent may not be congruent.

The list of statements that can be assumed is not a complete list. There are more special pairs of angles than those listed.

Go Online You can complete an Extra Example online.

628 Module 11 • Angles and Geometric Figures

Interactive Presentation



Learn

TAP

Students move through the slides to learn about interpreting figures.

Learn Interpreting Diagrams

Objective

Students demonstrate understanding of what can and cannot be assumed from a diagram by analyzing line and angle relationships in a given figure.

Teaching the Mathematical Practices

4 Make Assumptions Have students explain an assumption or approximation that was made in this Learn.

Important to Know

Geometry uses figures, which are sketches used to depict various situations. Often sketches are not drawn accurately which means that certain relationships cannot be assumed from figures. Features such as points of intersection can be assumed as can vertical angles and linear pairs, but congruence and perpendicularity cannot be assumed.

Common Misconception

Students tend to make certain assumptions about the geometric relationships of figures in a diagram based on the appearance of the diagram. For example, two angles may look congruent when they are not. Remind students that congruent angles or segments and perpendicular or parallel lines cannot be assumed from a figure.

Essential Question Follow-Up

Students have begun learning about interpreting diagrams.

Ask:

Why should we not assume certain relationships are present based on a diagram? Sample answer: Diagrams are only sketches, which means that they lack accuracy. Assuming that a relationship is present could cause calculations to be incorrect. In the real world, this means construction projects could fail or people could be injured by the lack of precision.

Example 3 Interpreting Diagrams

Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to construct an argument.

Questions for Mathematical Discourse

AL What angle relationships can be assumed from a diagram? Sample answer: linear pairs, vertical angles, and adjacent angles

OL Based on the figure, what statement can be made about $\angle ABG$? Sample answer: It is an obtuse angle.

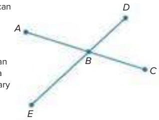
BL To say that $\angle IEB$ and $\angle BEC$ are congruent, what must be given about the figure? $\overline{BE}$ and $\overleftrightarrow{IF}$ are perpendicular.

Common Error

Despite the lesson, students may still assume that angles are congruent or that lines are perpendicular based on the diagram. Remind students that those relationships can never be assumed based on a figure, unless it is stated.

Because points of intersection can be assumed, you can identify vertical angles from the figure.

Because linear pairs can be assumed from the figure, you can apply known characteristics of a linear pair, such as supplementary angles.

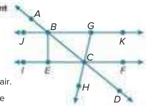


$\angle ABD$ and $\angle CBE$ are vertical angles.

$\angle EBA$ and $\angle ABD$ form a linear pair, so $m\angle EBA + m\angle ABD = 180°$.

Example 3 Interpreting Diagrams

Determine whether each statement can be assumed from the figure. Explain.



a. $\overrightarrow{CE}$ and $\overrightarrow{CF}$ are opposite rays.

Yes; C is a common endpoint.

b. $\angle BGC$ and $\angle KGC$ form a linear pair.

Yes; their noncommon sides are opposite rays.

c. $\angle ABE$ and $\angle CBG$ are vertical angles.

Yes; these angles are nonadjacent and are formed by two intersecting lines.

d. $\angle BCG$ and $\angle DCF$ are congruent.

No; these angles are not vertical angles. There isn't enough information given to determine this.

e. $\overline{BE}$ and $\overleftrightarrow{IF}$ are perpendicular.

No; there isn't enough information given to determine this.

f. $\angle EBC$ and $\angle GBC$ are complementary angles.

No; there isn't any information about perpendicularity or angle measure so this cannot be determined.

g. $\angle ICH$ and $\angle HCD$ are adjacent angles.

Yes; these angles share a common side.

h. $\overline{BC}$ is an angle bisector of $\angle ECG$.

No; there isn't any information about congruent angles so this cannot be determined.

Think About It!

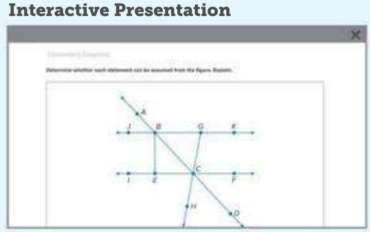
If you are given that $\overline{BE} \perp \overline{IC}$, can you determine whether $\angle BEI \cong \angle BEC$? Explain your solution process.

Yes; sample answer: the segments are perpendicular, then the angles are both right angles. Because they both have the same measure, 90°, they are congruent.

Watch Out!

Congruence and Perpendicularity Remember that congruent angles or segments and perpendicular or parallel lines cannot be assumed from a figure.

Interactive Presentation



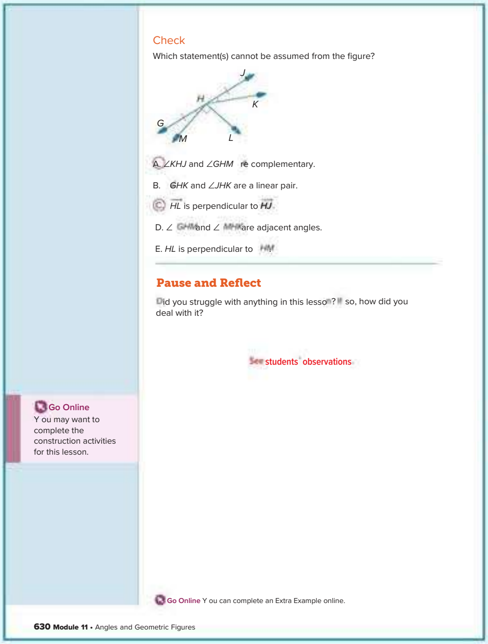
Example 3

TAP

Students move through different statements and determine whether they can be assumed from the figure.

TYPE

Students determine whether two angles are congruent.



Check

Which statement(s) cannot be assumed from the figure?

A. $\angle KHJ$ and $\angle GHM$ are complementary.

B. $\angle GHK$ and $\angle JHK$ are a linear pair.

C. $\overrightarrow{HL}$ is perpendicular to $\overrightarrow{HJ}$.

D. $\angle GHM$ and $\angle MHK$ are adjacent angles.

E. HL is perpendicular to HM.

Pause and Reflect

Did you struggle with anything in this lesson? If so, how did you deal with it?

See students' observations.

Go Online You may want to complete the construction activities for this lesson.

Go Online You can complete an Extra Example online.

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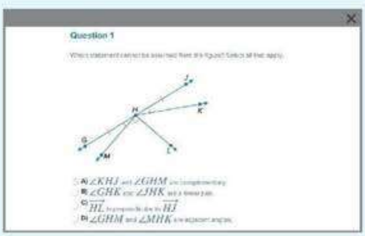
Interactive Presentation



Check

CHECK

Students complete the Check online to determine whether they are ready to move on.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–14
2	exercises that use a variety of skills from this lesson	15–28
3	exercises that emphasize higher-order and critical-thinking skills	29–35

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–28 odd, 29–35
- Extension: Runway Angles
- ALEKS Angles

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–35 odd
- Remediation, Review Resources: Vertical and Adjacent Angles
- Personal Tutors
- Extra Examples 1–3
- ALEKS Angle Relationships

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–13 odd
- Remediation, Review Resources: Vertical and Adjacent Angles
- *Quick Review Math Handbook*: Angle Relationships
- ALEKS Angle Relationships

Practice

Go Online Y ou can complete your homework online.

Example 1

1. Find the measures of two supplementary angles if the difference between the measures of the two angles is 35°. 72.5°, 107.5°

2. $\angle E$ and $\angle F$ are complementary. The measure of $\angle E$ is 54° more than the measure of $\angle F$. Find the measure of each angle. $m\angle F = 18°$; $m\angle E = 72°$

3. The measure of an angle's supplement is 76° less than the measure of the angle. Find the measures of the angle and its supplement. 128°; 52°

4. $\angle Q$ and $\angle R$ are complementary. The measure of $\angle Q$ is 26° less than the measure of $\angle R$. Find the measure of each angle. $m\angle Q = 32°$; $m\angle R = 58°$

5. The measure of the supplement of an angle is three times the measure of the angle. Find the measures of the angle and its supplement. 45°; 135°

6. The bascule bridge shown is opening from its horizontal position to its fully vertical position. So far, the bridge has lifted 35° in 21 seconds. At this rate, how much longer will it take for the bridge to reach its vertical position? 33 s

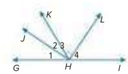
Nick Tzolov/E+/Getty Images

Example 2

7. Rays BA and BC are perpendicular. Point D lies in the interior of $\angle ABC$. If $m\angle ABD = (3r + 5)°$ and $m\angle DBC = (5r - 27)°$, find $m\angle ABD$ and $m\angle DBC$. $m\angle ABD = 47°$; $m\angle DBC = 43°$

8. $\overleftrightarrow{WX}$ and $\overleftrightarrow{YZ}$ intersect at point V. If $m\angle WVY = (4a + 58)°$ and $m\angle XVY = (2b - 18)°$, find the values of a and b such that $\overleftrightarrow{WX}$ is perpendicular to $\overleftrightarrow{YZ}$. $a = 8$; $b = 54$

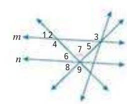
9. Refer to the figure at the right. If $m\angle 2 = (a + 15)°$ and $m\angle 3 = (a + 35)°$, find the value of a such that $\overrightarrow{HL} \perp \overrightarrow{HJ}$. $a = 20$



10. Rays DA and DC are perpendicular. Point B lies in the interior of $\angle ADC$. If $m\angle ADB = (3a + 10)°$ and $m\angle BDC = 13a°$, find a, $m\angle ADB$, and $m\angle BDC$. $a = 5$; $m\angle ADB = 25°$; $m\angle BDC = 65°$

Example 3

Determine whether each statement can be assumed from the given figure. Explain.



11. $\angle 6$ and $\angle 8$ are complementary. Y es; because $\angle 7$ is a right angle, $\angle 6$ and $\angle 8$ must form a right angle.

12. $\angle 7$ and $\angle 8$ form a linear pair. No; the angles do not share a common side.

13. $\angle 2$ and $\angle 4$ are vertical angles. Y es; the angles are nonadjacent and are formed by two intersecting lines.

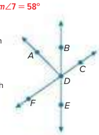
14. $m\angle 9 = m\angle 6 + m\angle 8$ Y es; because $\angle 9$ and $\angle 7$ are vertical angles, $m\angle 9 = 90°$. Because $\angle 6$ and $\angle 8$ are complementary angles, $m\angle 6 + m\angle 8 = 90°$. Thus, $m\angle 9 = m\angle 6 + m\angle 8$.

Mixed Exercises

15. The measure of the supplement of an angle is 60° less than four times the measure of the complement of the angle. Find the measure of the angle. 40°

16. $\angle 6$ and $\angle 7$ form a linear pair . Twice the measure of $\angle 6$ is twelve more than four times the measure of $\angle 7$. Find the measure of each angle. $m\angle 6 = 122°$; $m\angle 7 = 58°$

Refer to the figure at the right.

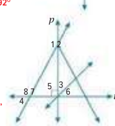


17. If $m\angle ADB = (6x - 4)°$ and $m\angle BDC = (4x + 24)°$, find the value of x such that $\angle ADC$ is a right angle. 7

18. If $m\angle FDE = (3x - 15)°$ and $m\angle FDB = (5x + 59)°$, find the value of x such that $\angle FDE$ and $\angle FDB$ are supplementary. 17

19. If $m\angle BDC = (8x + 12)°$ and $m\angle FDB = (12x - 32)°$, find $m\angle FDE$. 92°

Determine whether each statement can be assumed from the given figure. Explain.



20. $\angle 4$ and $\angle 7$ are vertical angles. Y es; the angles are nonadjacent and are formed by two intersecting lines.

21. $\angle 3 \cong \angle 6$ No; the measures of the angles are unknown.

22. $m\angle 5 = m\angle 3 + m\angle 6$ Y es; $m\angle 5 = 90°$ because it is a right angle, and $m\angle 3 + m\angle 6 = 90°$ because $\angle 3$ and $\angle 6$ are complementary angles.

23. $\angle 5$ and $\angle 7$ form a linear pair. No; the angles are not adjacent.

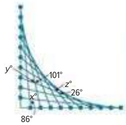
For Exercises 24 and 25, lines p and q intersect to form adjacent angles 1 and 2.

24. If $m\angle 1 = (7x + 6)^\circ$ and $m\angle 2 = (8x - 6)^\circ$, find the value of x such that p is perpendicular to q. $x = 12$

25. If $m\angle 1 = (4x - 3)^\circ$ and $m\angle 2 = (3x + 8)^\circ$, find the value of x such that $\angle 1$ is supplementary to $\angle 2$. $x = 25$

26. COLOR GUARD Shannon is designing a new rectangular flag for the school's color guard and is determining the angles at which to cut the fabric. She wants the measure of $\angle 2$ to be three times as great as the measure of $\angle 1$. She thinks the measures of $\angle 3$ and $\angle 4$ should be equal. Finally, she wants the measure of $\angle 6$ to be half that of $\angle 5$. Determine the measures of the angles. $m\angle 1 = 22.5^\circ$; $m\angle 2 = 67.5^\circ$; $m\angle 3 = 45^\circ$; $m\angle 4 = 45^\circ$; $m\angle 5 = 120^\circ$; $m\angle 6 = 60^\circ$

27. STRING ART String art is created by wrapping string around nails or wires to form patterns. Use the string art pattern below to find the values of x, y, and z.
$x = 94$, $y = 79$, $z = 26$



28. USE TOOLS Draw an acute angle, $\angle ABC$. Let $m\angle ABC = (6x - 1)^\circ$.

a. Use a protractor to determine the measure of $\angle ABC$. Use this measure to determine the value of x. See margin.

b. Explain how you would determine the measure of an angle that is complementary to $\angle ABC$. To find the measure of an angle that is complementary to $\angle ABC$, you would subtract $m\angle ABC$ from 90°.

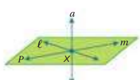
c. Explain how you would determine the measure of an angle that is supplementary to $\angle ABC$. To find the measure of an angle that is supplementary to $\angle ABC$, you would subtract $m\angle ABC$ from 180°.

Higher-Order Thinking Skills

29. ANALYZE Are there angles that do not have a complement? Justify your argument. Yes; sample answer: Angles that are right or obtuse do not have complements because their measures are greater than or equal to 90°.

30. PERSEVERE If a line, line segment, or ray is perpendicular to a plane, then it is perpendicular to every line, line segment, or ray in the plane that intersects it.

a. If a line is perpendicular to each of two intersecting lines at their point of intersection, then the line is perpendicular to the plane determined by them. If line a is perpendicular to line ℓ and line m at point X, what must also be true? Line a is perpendicular to plane P.



b. If a line is perpendicular to a plane, then any line perpendicular to the given line at the point of intersection with the given plane is in the given plane. If line a is perpendicular to plane P and line m and point X, what must also be true? Line m is in plane P.

c. If a line is perpendicular to a plane, then every plane containing the line is perpendicular to the given plane. If line a is perpendicular to plane P, what must also be true? Any plane containing line a is perpendicular to plane P.

31. WRITE Describe three different ways you can determine that an angle is a right angle. Sample answer: You can determine whether an angle is right if it is marked with a right angle symbol, if the angle is a vertical pair with a right angle, or if the angle forms a linear pair with a right angle.

32. FIND THE ERROR Kaila solved the problem, as shown. Is her solution correct? If it is, explain your reasoning. If not, explain Kaila's mistake and correct the work. No; Kaila solved the problem for complementary angles.

$(6x - 9)^\circ + (2x + 13)^\circ = 180^\circ$

$8x + 4 = 180$

$8x = 176$

$x = 22$

If $m\angle F = (6x - 9)^\circ$ and $m\angle G = (2x + 13)^\circ$, find the value of x such that $\angle F$ and $\angle G$ are supplementary.

$(6x - 9)^\circ + (2x + 13)^\circ = 90^\circ$

$8x - 4 = 90$

$8x = 86$

$x = 10.75$

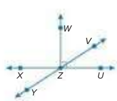
33. CREATE Create $\angle 1$ along with its complement and supplement by drawing only a line and two rays. See margin.

34. WHICH ONE DOESN'T BELONG Three students used the figure to write a statement. Is each statement correct? Justify your conclusion.

Samar: $\angle WZU$ is a right angle.

Jana: $\angle YZU$ and $\angle UZV$ are supplementary.

Antonio: $\angle VZU$ is adjacent to $\angle YZX$. Samar is correct; $\angle WZU$ is marked. Jana is correct; the angles form a linear pair. Antonio is incorrect; the angles do not share a common side.

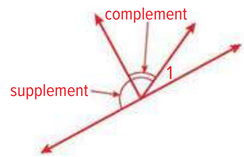


35. ANALYZE Do all angles have a supplement? Explain. No; sample answer: Straight angles or angles that are greater than 180° do not have supplements because their measures are greater than or equal to 180°.

Answers

28a. After drawing an acute angle, label vertex B and point A on one ray and point C on the other ray. Then use a protractor to find the measure of $\angle ABC$. Let the measure of $\angle ABC$ equal $6x - 1$ and solve for x.

33. Sample answer:



Lesson 11-3
Two-Dimensional Figures

G.GPE.7, G.MG.1

LESSON GOAL

Students find measures of two-dimensional figures.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Perimeter, Circumference, and Area
• Find Perimeter, Circumference, and Area

Explore: Modeling Objects by Using Two-Dimensional Figures

Develop:

Modeling with Two-Dimensional Figures
• Modeling with Two-Dimensional Figures
• Using a Two-Dimensional Model

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Angle Relationships	●	●		●
Extension: Pick's Theorem		●	●	●

Language Development Handbook

Assign page 65 of the *Language Development Handbook* to help your students build mathematical language related to finding measures of two-dimensional figures.

ELL You can use the tips and suggestions on page T65 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

G.MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

2 Reason abstractly and quantitatively.

Coherence

Vertical Alignment

Previous
Students understood and used area formulas for two-dimensional figures.
6.G.1, 7.G.4, 7.G.6

Now
Students find measures of two-dimensional objects.
G.GPE.7, G.MG.1

Next
Students will identify transformations and represent reflections, translations, and rotations.
G.CO.2

Rigor

The Three Pillars of Rigor

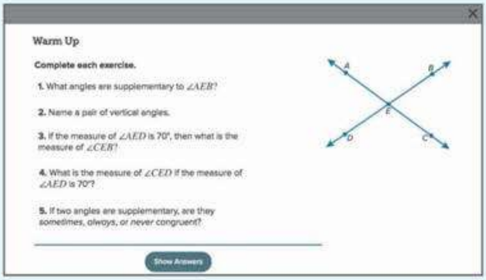
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students draw on their understanding of plane figures to model real-world objects. They build fluency by using coordinates to find perimeters and areas of figures and apply what they know about plane figures to solve real-world problems.

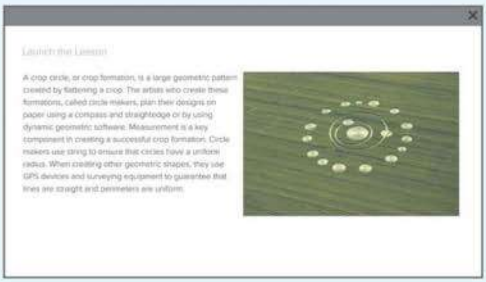
Mathematical Background

A *polygon* is a closed figure formed by a finite number of coplanar segments. The *perimeter* of a polygon is the sum of the lengths of its sides. The *circumference* of a circle is the distance around the circle. The *area* is the number of square units required to cover a surface.

Interactive Presentation



Warm Up



Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- identifying angle pairs

Answers:

1. ∠*BEC* and ∠*AED*
2. ∠*AED*, ∠*CEB* or ∠*AEB*, ∠*CED*
3. 70°
4. 110°
5. sometimes

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of two-dimensional geometric shapes.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Modeling Objects by Using Two-Dimensional Figures

Objective

Students use two-dimensional shapes to model real-world objects and use dynamic geometry software to calculate measures.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this Explore, students will need to use dynamic geometry software. Work with students to explore and deepen their understanding of modeling objects using two-dimensional figures.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

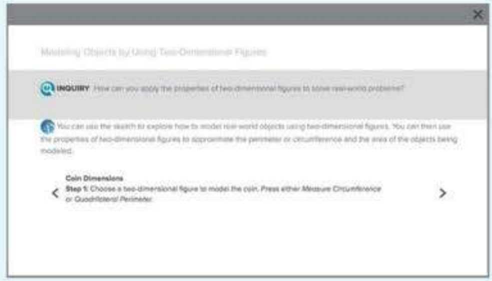
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use a sketch to approximate the shape of a coin and a television to find the circumference or perimeter and the area. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

Explore

WEB SKETCHPAD

Students use a sketch to explore circumference, perimeter, and area.

TYPE

Students type to answer the guiding exercises.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Modeling Objects by Using Two-Dimensional Figures (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What is the first step in finding the perimeter or circumference of an object? Sample answer: Measure the sides or diameter of the object.
- Which area formulas will be useful to know when modeling objects with two-dimensional figures? Sample answer: the area formulas for rectangle, circle, and triangle

Inquiry

How can you apply the properties of two-dimensional figures to solve real-world problems? Sample answer: Two-dimensional figures can be used to model real-world objects such as the coastline of a country or the area of a construction site. Then you can use the known formulas for calculating the perimeter and area of the two-dimensional figures to approximate the perimeter and area of the real-world objects.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Perimeter , Circumference, and Area

Objective

Students find perimeters, circumferences, and areas of two-dimensional geometric shapes by using coordinates and the Distance Formula.

MP Teaching the Mathematical Practices

3 Analyze Cases This Learn guides students to examine cases of perimeter, circumference, and area for various polygons. Encourage students to familiarize themselves with all of the cases.

What Students Are Learning

A figure bounded by three or more straight sides is called a polygon. The perimeter of a polygon is found by adding the lengths of all the sides. The circumference of a circle is the distance around the circle. Area is the number of square units needed to cover a surface. The Distance Formula calculates different characteristics of two-dimensional figures on the coordinate plane, which can be used to determine perimeter, circumference, or area of the figure.

Common Misconception

Students often interchange perimeter and area, believing the two represent the same measurement. Remind students that perimeter is a one-dimensional measurement and that area is a two-dimensional measurement.

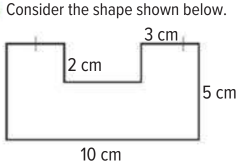
DIFFERENTIATE

AL

IF students have difficulty using or remembering the formulas for perimeter,

THEN have them build their intuition by measuring cutouts of triangles, squares, and rectangles. They can use string to measure circumference of a circle.

BL Consider the shape shown below.



What is the perimeter of the shape?

34 cm

Lesson 11-3

Two-Dimensional Figures

Learn Perimeter , Circumference, and Area

A **polygon** is a closed plane figure with at least three straight sides.

The **perimeter** of a polygon is the sum of the lengths of the sides of the polygon. Some shapes have special formulas for perimeter, but all are derived from the basic definition of perimeter.

The **circumference** of a circle is the distance around the circle.

Area is the number of square units needed to cover a surface.

Perimeter, Circumference, and Area

Triangle	Square
Perimeter $P = b + c + d$ Area $A = \frac{1}{2}bh$	Perimeter $P = s + s + s + s = 4s$ Area $A = s^2$
Rectangle	**Circle**
Perimeter $P = \ell + w + \ell + w = 2\ell + 2w$ Area $A = \ell w$	Circumference $C = 2\pi r$ or $C = \pi d$ Area $A = \pi r^2$

Y ou can use the Distance Formula to find the perimeter and area of a polygon graphed on a coordinate plane. Y ou can also use the Distance Formula to calculate the radius of a circle and then use the appropriate equations for circumference and area.

An **equilateral polygon** has all sides congruent. An **equiangular polygon** has all angles congruent. A **regular polygon** is a convex polygon that is both equilateral and equiangular.

T oday's Goals
- Find perimeters, circumferences, and areas of two-dimensional geometric shapes.
- Calculate the measures of real-world objects.

T oday's Vocabulary
polygon
perimeter
circumference
area
equilateral polygon
equiangular polygon
regular polygon
concave
convex
geometric model

Go Online Y ou can watch a video to see how to find the perimeter and area of a figure on the coordinate plane.

Lesson 11-3 • Two-Dimensional Figures 635

Interactive Presentation

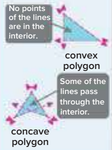
Learn

TAP

Students tap to see the formulas for perimeter and area of various shapes.

Study Tip

Concave and Convex Polygons can be **concave** or **convex**. Suppose the line containing each side is drawn. If any of the lines contain any point in the interior of the polygon, then it is concave. Otherwise, it is convex.



Study Tip

Perimeter vs. Area Because calculating the area of a figure involves multiplying two dimensions, *square units* are used. There is only one dimension used when finding the perimeter, thus, it is given simply in *units*.

Think About It!

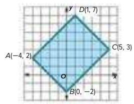
How can you use the coordinate grid to check your answer?

Sample answer: Each square on the grid represents 1 square unit. Y ou could estimate the number of squares within the shape to determine the approximate area.

Example 1 Find Perimeter , Circumference, and Area

Find the perimeter or circumference and area of each figure.

a. Rectangle *ABCD*



First, find the length ℓ of the rectangle by using the Distance Formula.

$\ell = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{[1 - (-4)]^2 + (7 - 2)^2}$	Let $(x_1, y_1) = D(1, 7)$ and $(x_2, y_2) = A(-4, 2)$.
$= \sqrt{5^2 + 5^2}$	Subtract.
$= \sqrt{50}$	Simplify.

Next, find the width w of the rectangle by using the Distance Formula.

$w = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{[0 - (-4)]^2 + [(-2) - 2]^2}$	Let $(x_1, y_1) = A(-4, 2)$ and $(x_2, y_2) = B(0, -2)$.
$= \sqrt{4^2 + (-4^2)}$	Subtract.
$= \sqrt{32}$	Simplify.

Use the length and width that you calculated to find the perimeter and area of the rectangle.

$P = 2\ell + 2w$	Perimeter of a rectangle
$= 2\sqrt{50} + 2\sqrt{32}$	$\ell = \sqrt{50}$ and $w = \sqrt{32}$
≈ 25.5	Simplify.

The perimeter is about 25.5 units.

$A = \ell w$	Area of a rectangle
$= \sqrt{50} \times \sqrt{32}$	$\ell = \sqrt{50}$ and $w = \sqrt{32}$
$= 40$	Simplify.

The area is 40 square units.

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Interactive Presentation

Example 1

TAP Students move through the steps to find the perimeter and circumference of the given figures.

TYPE Students explain how the coordinate grid can be used to check their answer.

CHECK Students complete the Check online to determine whether they are ready to move on.

Example 1 Find Perimeter , Circumference, and Area

MP Teaching the Mathematical Practices

5 Use Estimation Point out that in this example, students can use estimation to check the reasonableness of their answer.

Questions for Mathematical Discourse

AL What other shape names can be used to classify the rectangle? quadrilateral and parallelogram

OL What formulas are used when finding the area of a square, a triangle, or a circle? The formula for the area of a square is $A = s^2$. The formula for the area of a triangle is $A = \frac{1}{2}bh$. The formula for the area of a circle is $A = \pi r^2$.

BL Suppose you forgot the Distance Formula. How else could you determine the lengths of the sides of the rectangle? I could use the Pythagorean Theorem.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Common Error

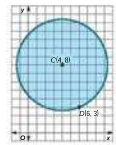
Students often write the wrong units for area. If the problem is measured in inches, then they will write area as inches rather than square inches. Remind students that area is a two-dimensional measurement because area involves multiplying two different dimensions.

DIFFERENTIATE

Language Development Activity ELL

The word *perimeter* comes from the Greek *peri*, which means around and *meter* which means measure. The term is used for the path or for its length. Have students discuss how this can help them remember the definition of *perimeter*.

b. Circle C



Use the Distance Formula to calculate the length of the radius of the circle.

$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{(6 - 4)^2 + (3 - 8)^2}$	$C(4, 8)$ and $D(6, 3)$
$= \sqrt{2^2 + (-5)^2}$	Subtract.
$= \sqrt{29}$	Simplify.

Use the value of *r* to find the circumference and area of the circle.

$C = 2\pi r$	Circumference
$= 2\pi\sqrt{29}$ or about 33.8	$r = \sqrt{29}$

The circumference of the circle is about 33.8 units.

$A = \pi r^2$	Area of a circle
$= \pi(\sqrt{29})^2$	$r = \sqrt{29}$
$= 29\pi$ or about 91.1	Simplify.

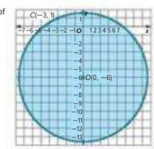
The area of the circle is about 91.1 square units.

Check

Find the circumference and area of the circle. Round to the nearest tenth if necessary.

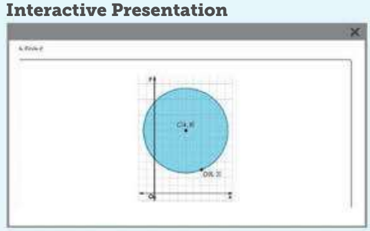
$C \approx$ __47.9__ units

$A \approx$ __182.2__ units^2



Thomas Arbour/istock/Getty Images Plus/Getty Images

Interactive Presentation



Example 1

TAP

Students move through the steps to find the perimeter, circumference, and area of a circle.

Explore Modeling Objects by Using Two-Dimensional Figures

Online Activity Use real-world objects to complete the Explore.

INQUIRY How can you apply the properties of two-dimensional figures to solve real-world problems?

Learn Modeling with Two-Dimensional Figures

A **geometric model** is a geometric figure that represents a real-world object. A good model shows all the important characteristics of the object it represents, although some of the detail may be lost.

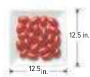
Drafters use two-dimensional geometric models to create technical drawings that communicate an object's function or construction. Scientists may use two-dimensional models to record an object's general shape or mechanics in a field notebook. You can use two-dimensional models to estimate the perimeter, circumference, and area of objects.

Study Tip

Approximations Because the corners of the serving platter are rounded, we can only approximate the perimeter and area of the serving platter using the formulas for a square.

Example 2 Modeling with Two-Dimensional Figures

Use an appropriate two-dimensional model and the dimensions provided in the image to calculate the perimeter and area of the plate.



What two-dimensional figure can be used to model the serving platter? square

What are the perimeter and area of the serving platter? Round to the nearest tenth, if necessary.

Perimeter $= 4s = 4(12.5) = 50$ in.

Area $= s^2 = (12.5)^2 = 156.3$ in^2

Because the platter is a square, the perimeter of the platter is 4 multiplied by the length of the side. The area is the length of the side squared. The perimeter of the platter is 50 inches, and the area of the platter is 156.3 square inches.

Check

Use an appropriate two-dimensional model and the dimensions provided in the image to calculate the perimeter and area of the framed art.

What two-dimensional figure can be used to model the art? rectangle

$P =$? 203.2 cm; $A =$? 2476.6 cm^2

Thomas Arbour/iStock/Getty Images, Mark Steinmetz/McGraw-Hill Education, Sonsedska Yuliia/Shutterstock

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Interactive Presentation

Example 2

SELECT

Students select the two-dimensional figure that can be used to model the serving platter.

TYPE

Students type to complete the perimeter and area formulas.

Learn Modeling with Two-Dimensional Figures

Objective

Students calculate the measures of real-world objects by using two-dimensional geometric shapes and their perimeters, circumferences, areas, and properties to model the objects.

Teaching the Mathematical Practices

7 Use Structure Mathematically proficient students can see real-world objects as being composed of several two-dimensional figures.

About the Key Concept

Geometric models are geometric figures that represent real-life objects. A good model displays all the important characteristics of the object even though some of the detail may be lost. Geometric models are good tools to study objects.

Essential Question Follow-Up

Students have begun learning about geometric models.

Ask:

Why are geometric models a useful tool when dealing with real-world two-dimensional objects? Sample answer: Often real-world objects do not form perfect shapes, so a geometric model can help approximate the shape. Then the perimeter, area, and circumference can be calculated. For example, a building may be constructed and it seems to be circular. Even though the building may not be perfectly circular, we can still use a circle to approximate the characteristics of the building.

Example 2 Modeling with Two-Dimensional Figures

Teaching the Mathematical Practices

4 Make Assumptions Have students explain an assumption or approximation that was made to solve the problem.

Questions for Mathematical Discourse

AL How do you know the figure is a square? Sample answer: All four angles are right angles, and all of the sides are congruent.

OL How many plates could fit length-wise on a table that is six feet long? Explain. 5 plates; Because 6 ft = 72 inches, and each plate is 12.5 inches wide, then $72 \div 12.5 = 5.76$.

BL Suppose the platter is a rectangle and the width is still 12.5 inches. If the length was four more than the width, what is the area of the serving platter rounded to the nearest tenth? $A = 12.5(16.5) = 206.3$ in^2

Common Error

Although a square is a rectangle, students may not pick the most accurate shape for the serving platter. A rectangle is used when opposite sides are not the same length. Because the serving platter has opposite sides of the same length, students should use a square to model the object.

Example 3 Using a Two-Dimensional Model

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about two-dimensional geometric shapes to solving a real-world problem.

Questions for Mathematical Discourse

AL What shapes are present in the diagram? 3 rectangles, 1 triangle

OL How can you find the area of the two tables in the café? Use the Distance Formula to calculate the length and width, then multiply to find the area.

BL Suppose Isaiah decided to add a small outdoor seating space that is roughly circular. If the diameter of the space is 10 feet, how many people can be in the area if there are no tables? $A = 25\pi \approx 78.5$ square feet, which means that a maximum 5 people can be in this area at a time.

Common Error

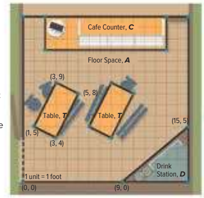
Many students enter the entire Distance Formula into their calculator, including the radical sign. This results in rounding at a very early stage in the problem-solving process. When rounded answers are used in another step and those answers are then rounded, answers can be pretty far off the mark. Remind students to leave answers in radical form until the very end to avoid rounding errors.

(continued on the next page)

Example 3 Using a Two-Dimensional Model

BUSINESS **Isaiah owns a small café.**

Part A A new fire code states that there must be 15 square feet of free space for every customer in the café. How many people can be in the café?



Step 1 Find the amount of free space available.

Find the total area of the café.

Area of the café $= 15 \times 15$ or $225\ \text{ft}^2$

Then, find the area of the counter and the drink station.

$C = 3 \times 11$ or $33\ \text{ft}^2$

$D = \frac{1}{2}(5 \cdot 6)$ or $15\ \text{ft}^2$

Find the areas of the tables by using the Distance Formula.

$\ell = \sqrt{(3-1)^2 + (9-5)^2}$ or $\sqrt{20}$ and $w = \sqrt{(3-1)^2 + (4-5)^2}$ or $\sqrt{5}$

$T = \ell \cdot w = \sqrt{20} \cdot \sqrt{5}$ or $10\ \text{ft}^2$

Find the amount of free space available for Isaiah's customers.

$A =$ area of the café $- C - D - 2T$

$= 225 - 33 - 15 - (2 \times 10)$ or $157\ \text{ft}^2$

Step 2 Find the number of people that can be in the café.

$157\ \text{ft}^2 \cdot \frac{1\ \text{person}}{15\ \text{ft}^2} \approx 10.5$ or 10 people

The café can hold 10 people.

Part B Isaiah wants to hang garland around the tables and the drink station. How much garland does Isaiah need?

Find the sum of the perimeters of the tables and drink station.

length of garland $= 2 \cdot$ perimeter of table $+$ perimeter of drink station

$= 2(2\sqrt{20} + 2\sqrt{5}) + (6 + 5 + \sqrt{(15-9)^2 + (5-0)^2})$

≈ 45.6 feet

Isaiah would need at least 45.6 feet of garland.

Problem-Solving Tip

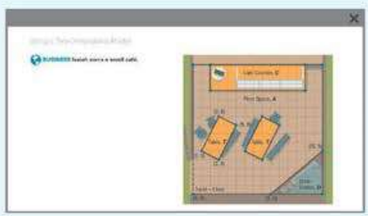
Evaluate Your Answer It canbe tempting to complete the final calculation in a multi-step exercise and conclude that you have arrived at the answer. However, always remember to define appropriate quantities when solving a real-world problem. In this example, it does not make sense to have 10.5 people. Y ou can determine that a correct answer for this exercise must be a whole number.

Study Tip

Radical Form Leave answers in radical form until the last calculation. This will prevent compounding errors caused by rounding throughout steps within a problem.

Lesson 11-3 • Two-Dimensional Figures **639**

Interactive Presentation



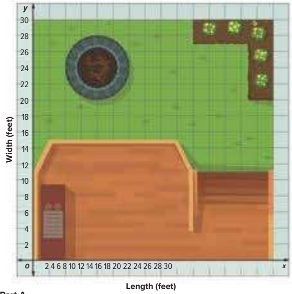
Example 3

TAP Students move through the steps to solve the problem.

SELECT Students select the correct variables and operations to represent the area.

Check

LANDSCAPING Monica is redesigning her backyard. She has created the following blueprint to model her design.



Part A

Monica wants to have at least 300 square feet of grass available in the backyard for her dog. Is there enough space for her dog? If there is, then how much area is available?

A. no

B. yes; 387.7 ft^2

C. yes; 396.7 ft^2

D. yes; 472.7 ft^2

Part B

Monica wants to build a fence in the backyard. She does not want to enclose the edge of the deck that extends from (0, 0) to (30, 0). If Monica wants to enclose the rest of the backyard, including the side edges of the deck and the side edge of the stairs, then how many feet of material are needed to complete the project?

? feet
90

Go Online You can complete an Extra Example online.

640 **Module 11** • Angles and Geometric Figures

Interactive Presentation

Check

CHECK

Students complete the Check online to determine whether they are ready to move on.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–13
2	exercises that use a variety of skills from this lesson	14–23
3	exercises that emphasize higher-order and critical-thinking skills	24–27

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–23 odd, 24–27
- Extension: Pick's Theorem
- ALEKS Introduction to Perimeter and Area

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–23 odd
- Remediation, Review Resources: Angle Relationships
- Personal Tutors
- Extra Examples 1–3
- ALEKS Angle Pairs

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–13 odd
- Remediation, Review Resources: Angle Relationships
- *Quick Review Math Handbook*: Two-Dimensional Figures
- ALEKS Angle Pairs

Answers

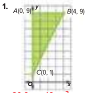
13b. Sample answer: In part **a**, I assumed that there was no space between the field and the first lane of the track. I also assumed that the athlete's body was centered on the border of the track.

Practice

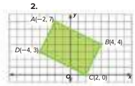
Go Online You can complete your homework online.

Example 1

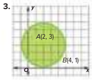
Find the perimeter or circumference and area of each figure if each unit on the graph measures 1 centimeter. Round answers to the nearest tenth, if necessary.

1. 

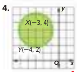
20.9 cm; 16 cm^2

2. 

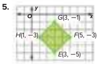
22.4 cm; 30 cm^2

3. 

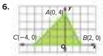
17.8 cm; 25.1 cm^2

4. 

14.0 cm; 15.7 cm^2

5. 

11.3 cm; 8 cm^2

6. 

16.1 cm; 12 cm^2

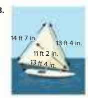
Magnetix/Shutterstock.com

Example 2

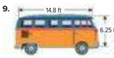
Use a two-dimensional model and the dimensions provided to calculate the perimeter or circumference and area of each object. Round to the nearest tenth, if necessary.

7. 

15.7 in.; 19.6 in^2

8. 

41 ft 3 in.; 81.4 ft^2

9. 

42.1 ft; 92.5 ft^2

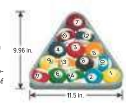
Example 3

10. **DESIGN** Dev is designing a new sign for his art studio. However, he needs to make several improvements to the sign before it is ready to be hung.

a. Dev wants to add a metal trim around the perimeter of the sign. How much trim should Dev purchase? Round answer to the nearest foot. 23 ft
b. The front of the sign also needs to be waterproofed with a protective sealer. How much area needs to be covered by the sealer? Round answer to the nearest square foot. 28 ft^2
c. If a pint of sealer covers an area of 20 square feet, then how many pints of sealer should Dev purchase? 2 pints

11. **WORLD RECORD** The world's largest ice cream cake was created on May 10, 2011, in Toronto, Canada. The cake was 4.45 meters long, 4.06 meters wide, and 1 meter tall. All surfaces of the cake except the bottom were covered with a cookie crumble topping. Use an appropriate two-dimensional model to approximate the area covered by the cookie crumble topping. Round the answer to the nearest tenth of a square meter. 35.1 m^2

12. **POOL** Eight-ball pool is a popular game played on a pool table that has six pockets. In eight-ball pool, there are 7 striped balls, 7 solid-colored balls, and a black eight ball. At the beginning of each game, players position the 15 balls in a rack in preparation for the first shot.



a. Find the area contained by the rack using an appropriate two-dimensional model. Round the answer to the nearest tenth of a square inch. 57.3 in^2
b. Approximate the area covered by a single ball to the nearest tenth of a square inch. 3.8 in^2

13. **TRACK** A 400-meter Olympic-size track can be modeled with a rectangle and two semicircles.

a. If an athlete runs around the track once, then how far has the athlete traveled to the nearest meter? 398 m
b. What assumption can be used to explain the difference between your answer in **part a** and the actual length around the track? See margin.
c. Each lane is 1.22 meters wide. If the athlete runs in the center of the inside lane, then how far has she traveled after a single lap to the nearest meter? 402 m
d. How far inside the track should the athlete be positioned to run exactly 400 meters? Round the answer to the nearest centimeter. 30 cm

Mixed Exercises

Identify the figure with the given vertices. Find the perimeter and area of the figure.

14. $A(3, 5)$, $B(3, 1)$, $C(0, 1)$
triangle; 12 units; 6 units^2

15. $Q(-3, 2)$, $R(1, 2)$, $S(1, -4)$, $T(-3, -4)$
quadrilateral; 20 units; 24 units^2

16. $G(-4, 1)$, $H(4, 1)$, $I(0, -2)$
triangle; 18 units; 12 units^2

17. $K(-1, 1)$, $L(3, 4)$, $M(6, 0)$, $N(2, -3)$
quadrilateral; 20 units; 25 units^2

18. Rectangle *WXYZ* has a length that is 5 more than three times its width.

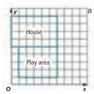
a. D raw and label a figure for rectangle *WXYZ*. See margin.

b. Write an algebraic expression for the perimeter of the rectangle. An expression for the perimeter, where x is the width, would be either $2(3x + 5) + 2x$ or $8x + 10$.

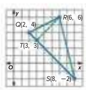
c. Find the width if the perimeter is 58 millimeters. Explain how you can check that your answer is correct. Solving $58 = 8x + 10$ for x, the width is found to be 6 mm. To check that this answer is correct, use the value of the width to determine the length, 23. The sum of all four sides, $23 + 23 + 6 + 6$, should equal 58.

d. Use a ruler to draw and label $\overline{PQ}$, which is congruent to the segment representing the length of rectangle *WXYZ*. What is the measure of $\overline{PQ}$? See margin.

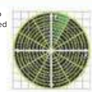
19. FENCING The figure shows Derek's house and his backyard on coordinate grid. Derek is planning to fence in the play area in his backyard. Part of the play area is enclosed by the house and does not need to be fenced. Each unit on the coordinate grid represents 5 feet. The cost for the fencing materials and installation is $10 per foot. How much will it cost Derek to install the fence? Explain. See margin.



20. Explain a method to find the area of $\triangle QRS$ given that $\overline{RT} \perp \overline{QS}$. Then find the area. Show your work. See margin.



21. SONAR Sonar is used by oceanographers to locate marine animals and to map the contours of the ocean floor. Sonar sends out sound pulses, called pings, and receives the returning sound echo. Sonar uses the returning sound echo to detect the location of animals or the distance from a rock formation. If each unit on the coordinate grid measures 1 mile, then what area does the sonar system cover? Round to the nearest tenth.
78.5 square miles



22. Two vertices of square *ABCD* are $C(5, 8)$ and $D(2, 4)$.

a. Do you need to find the coordinates for the other two vertices to find the perimeter and area of the square? Justify your argument. No; sample answer: The perimeter and area of a square can be found using the length of just one side.

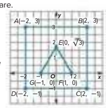
b. Find the perimeter and area of square *ABCD*. Show your work. $CD = 5$; $P = 4(5) = 20$ units; $A = (5)^2 = 25$ square units

23. The coordinate grid shows an equilateral triangle that fits inside a square.

a. Find the area of the square. Show your work.
$s = 4$, so $A = 4^2 = 16$ units^2

b. Find the area of the triangle. Show your work.
$b = 2$, $h = \sqrt{3}$, so $A = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$ units^2

c. Find the area of the square that is not covered by the triangle. Write an exact value and then round to the nearest tenth. Justify your reasoning. 14.3 units^2; sample answer: The area not covered by the triangle is equal to the area of the square minus the area of the triangle. So, $A = (16 - \sqrt{3})$ units^2, 14.3 units^2.



Higher-Order Thinking Skills

24. PERSEVERE The floor plan of a rectangular room has the coordinates (0, 12.5), (20, 12.5), (20, 0), and (0, 0) when it is placed on the coordinate plane. Each unit on the coordinate plane measures 1 foot. How many square tiles will it take to cover the floor of the room if the tiles have a side length of 5 inches? Explain. 1440 square tiles; sample answer: The space is 20 • 12 or 240 inches long and 12.5 • 12 or 150 inches wide. The area of the floor is 240 • 150 or 36,000 square inches. The area of a square tile is 5 • 5 or 25 square inches. So, the number of tiles needed is 36,000 ÷ 25 or 1440 tiles.

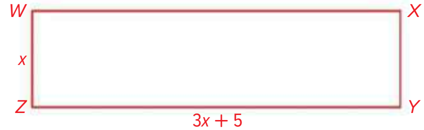
25. PERSEVERE The vertices of a rectangle with side lengths of 10 and 24 units are on a circle of radius 13 units. Find the area between the figures. 290.93 units^2

26. WRITE Give an example of a polygon that is equiangular but not a regular polygon. Explain your reasoning. Sample answer: rectangle; A rectangle has 4 right angles so it is equiangular. The adjacent sides of a rectangle are not always congruent, so a rectangle may not be a regular polygon.

27. ANAL YZE Find the perimeter of equilateral triangle *KLM* given the vertices $K(-2, 1)$ and $M(10, 6)$. Explain your reasoning. Sample answer: An equilateral triangle has congruent side lengths. Use the Distance Formula to find $KM = \sqrt{(10-(-2))^2 + (6-1)^2} = 13$. So, $P = 3(KM) = 3(13)$ or 39 units.

Answers

18a. Sample answer: Let x represent the width of *WXYZ*.



18d. 23 mm; Sample answer:

19. $650; Sample answer: The side of the play area that is adjacent to the house does not need fencing. The remaining three sides of the play area on the grid have lengths of 4, 5, and 4 units. The perimeter of the play area on the grid is $P = 4 + 5 + 4 = 13$ units. Each unit on the grid represents 5 ft, so Derek will need 13(5 ft) or 65 ft of fencing. The cost of the fencing is $10 per foot, so the total cost will be 65($10) = $650.

20. Sample answer: Use the Distance Formula to find the base *QS* and the height *RT*. $QS = \sqrt{72}$, and the height $RT = \sqrt{18}$. Then use the area formula: $A = \frac{1}{2}(\sqrt{72})(\sqrt{18}) = \frac{1}{2}(36) = 18$. So, the area is 18 units^2.

Lesson 11-4

Transformations in the Plane

G.CO.2

LESSON GOAL

Students calculate the coordinates of the vertices of transformed images given the coordinates of the preimages.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Introducing Transformations

Develop:

Identifying Transformations
- Identify Transformations in the Real World
- Identify Transformations on the Coordinate Plane

Representing Reflections
- Reflections in the *x*- or *y*-Axis

Representing Translations
- Translations

Representing Rotations
- Rotations

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Area of Parallelograms	●	●		●
Extension: Compositions of Transformations		●	●	●

Language Development Handbook

Assign page 66 of the *Language Development Handbook* to help your students build mathematical language related to the transformations of figures.

ELL You can use the tips and suggestions on page T66 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1.5 days
45 min	3 days

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

Standards for Mathematical Practice:

4 Model with mathematics.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous
Students described the effect of transformations on two-dimensional figures using coordinates. **8.G.3**

Now
Students identify transformations and represent reflections, translations, and rotations.
G.CO.2

Next
Students will find measures of three-dimensional objects.
G.MG.1, G.GMD.3

Rigor

The Three Pillars of Rigor

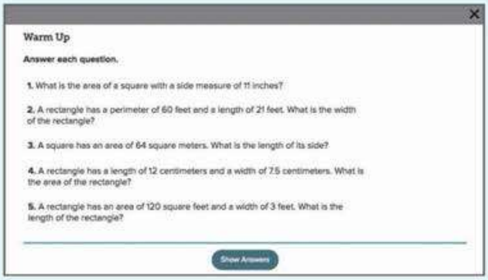
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of transformations in the plane. They apply their understanding by solving real-world problems related to transformations.

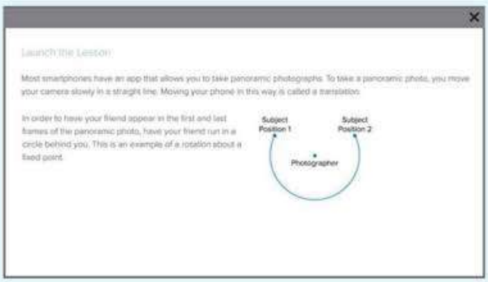
Mathematical Background

A translation is an operation that maps one geometric figure, the pre-image, onto another geometric figure, the image. A rigid motion is one in which the position of the image may differ from the preimage, but the segment and angle measures are preserved. Reflections, translations, and rotations are three types of rigid motions.

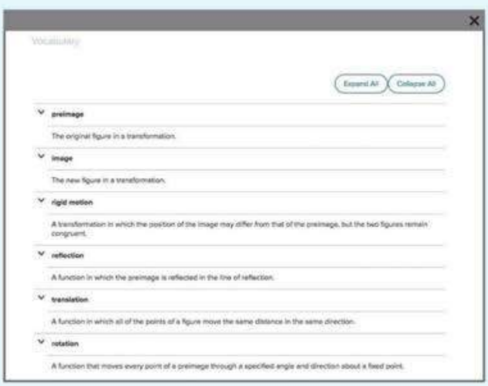
Interactive Presentation



Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- reviewing perimeter and area

Answers:

1. 121 in^2
2. 9 ft
3. 8 m
4. 90 cm^2
5. 40 ft

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of a translation.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?,* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Introducing Transformations

Objective

Students use dynamic geometry software to identify and represent transformations in the plane.

MP Teaching the Mathematical Practices

5 Use Appropriate Tools Strategically Throughout the Explore, encourage students to use the sketch to help identify the different transformations.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

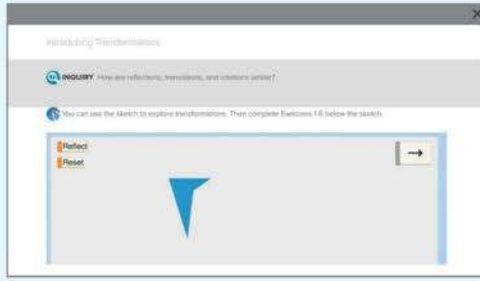
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

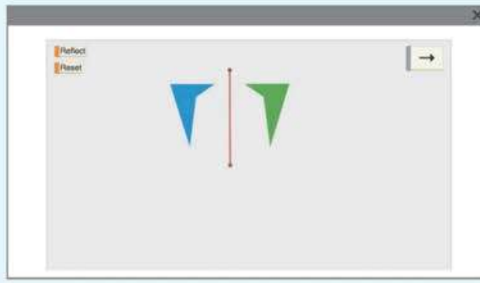
Students will complete guiding exercises throughout the Explore activity. They will use a sketch to investigate the different types of transformations that can be performed on a figure in the plane. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

WEB SKETCHPAD

Students use a sketch to explore transformations in the plane.

TYPE

Students answer guiding exercises about transformations.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Introducing Transformations (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How do the preimage and the image compare? Sample answer: They are congruent.
- How do the distance between the vertices of the preimage and the vertices of the image compare? Sample answer: They are preserved.

Inquiry

How are reflections, translations, and rotations similar? Sample answer: Each transformation results in a figure that is identical to the original figure. Shape and size are preserved. In each of the three transformations, the position of the copy differs from that of the original figure.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Identifying T ransformations

Objective

Students analyze figures to identify the types of rigid motions represented.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Important to Know

Transformations map the preimage onto a new figure, called the image. Transformations can change the position, size, or shape of a figure. Rigid motions are transformations that produce an image that remains congruent to the preimage. Reflections are transformations over a line of reflection. Translations are transformations that move all points of the preimage the same distance and direction. Rotations are transformations about a fixed point, through a specific angle, and in a specific direction.

Common Misconception

Many students confuse rotations with reflections, believing that they are the same transformation. Remind students that reflections occur over a line of reflection whereas rotations occur around a point of rotation.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

DIFFERENTIATE

AL ELL

IF students have difficulty identifying the different rigid motions, **THEN** have students draw a shape on a piece of paper and translate the shape. Next have student reflect the shape, and then rotate the shape. Performing the transformations helps make identifying easier.

AL BL ELL

Have students photograph or draw representations of rigid motions found in nature. Each photo or drawing should include a description of the transformation shown.

Lesson 11-4

Transformations in the Plane

Explore Introducing Transformations

Online Activity Use graphing technology to complete the Explore.

INQUIRY How are reflections, translations, and rotations similar?

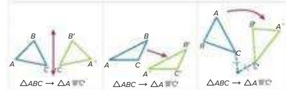
Learn Identifying T ransformations

A **transformation** is a function that takes points in the plane as inputs and gives other points as outputs. In a transformation, the **preimage** is mapped onto the **image**. A **rigid motion**, also called a *congruence transformation* or an *isometry*, is a transformation that preserves distance and angle measure.

The three main types of rigid motions are shown below. The preimage is shown in blue, and the image is shown in green. Prime notation is used to indicate transformations. If *A* is the preimage, then *A′* is the image after one transformation.

Key Concept • Reflections, Translations, and Rotations

A **reflection** or *flip* is a transformation in a line called the **line of reflection**. Each point of the preimage and its image are the same distance from the line of reflection.	A **translation** or *slide* is a transformation that moves all points of the original figure the same distance in the same direction.	A **rotation** or *turn* is a transformation about a fixed point (called the **center of rotation**), through a specific angle, and in a specific direction. Each point of the original figure and its image are the same distance from the center.



Today's Goals

- Analyze figures to identify the types of rigid motions represented.
- Calculate the coordinates of the vertices of images given the coordinates of the preimages.

Today's Vocabulary

transformation
preimage
image
rigid motion
reflection
translation
rotation
line of reflection
center of rotation
translation vector
component form
angle of rotation

Study Tip

Rigid Motion A rigid motion is also called a *rigid transformation*. The two terms can be used interchangeably.

Lesson 11-4 • Transformations in the Plane 645

Interactive Presentation

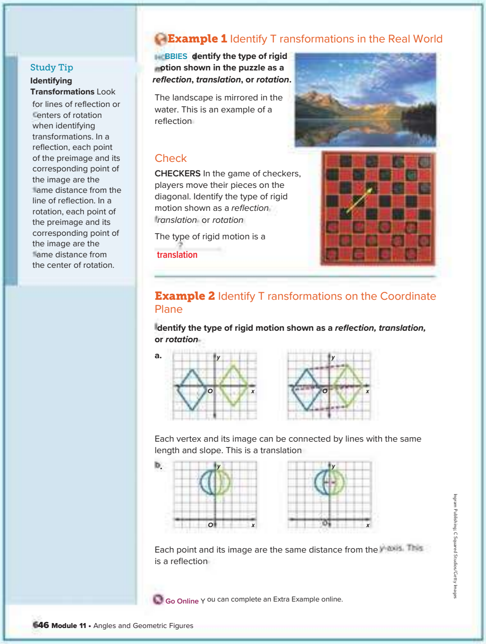
Learn

EXPAND

Students tap to reveal definitions and examples of each type of rigid motion.

TYPE

Students answer a question about transformations.



Study Tip

Identifying Transformations Look for lines of reflection or centers of rotation when identifying transformations. In a reflection, each point of the preimage and its corresponding point of the image are the same distance from the line of reflection. In a rotation, each point of the preimage and its corresponding point of the image are the same distance from the center of rotation.

Example 1 Identify Transformations in the Real World

HOBBIES **Identify the type of rigid motion shown in the puzzle as a *reflection, translation,* or *rotation.***

The landscape is mirrored in the water. This is an example of a reflection.

Check

CHECKERS In the game of checkers, players move their pieces on the diagonal. Identify the type of rigid motion shown as a *reflection, translation,* or *rotation.*

The type of rigid motion is a translation.

Example 2 Identify Transformations on the Coordinate Plane

Identify the type of rigid motion shown as a *reflection, translation,* or *rotation.*

a.

Each vertex and its image can be connected by lines with the same length and slope. This is a translation.

b.

Each point and its image are the same distance from the y-axis. This is a reflection.

Go Online You can complete an Extra Example online.

646 Module 11 • Angles and Geometric Figures

Interactive Presentation

Example 1

TAP

Students move through the slides to identify rigid motions.

Example 1 Identify Transformations in the Real World

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about rigid motions to real-world situations.

Questions for Mathematical Discourse

AL Which transformation moves each point in a figure the same distance in the same direction? translation

OL Over what is a figure reflected about? the line of reflection

BL Suppose the image was placed on a coordinate plane with the line of reflection being the *x*-axis and the mountaintop falling on (4, 25). What are the coordinates of the reflected mountaintop? (4, −25)

Common Error

Students may incorrectly identify transformations in a real-world setting. Encourage students to memorize the definition of each rigid motion and then to look for the indicators in each image.

Example 2 Identify Transformations on the Coordinate Plane

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to identify the transformations in each part, students will need to analyze the relationship between the two figures.

Questions for Mathematical Discourse

AL If the blue triangle is translated five units to the left, what points identify the vertices of the image? (−4, 2), (−3, 5), and (2, 3)

OL Suppose an image of the blue triangle is plotted using the following points: (1, −2), (2, −5), and (7, −3). What transformation was applied? The triangle was reflected in the *x*-axis.

BL Suppose the blue triangle is rotated about the origin in the opposite direction of the green image. What points identify the vertices of the new image? (2, −1), (5, −2), and (3, −7)

Common Error

Students may believe that all transformations preserve congruence, meaning that the shapes will always be the same size and shape. Remind students that translations, reflections, and rotations preserve congruence, and that those are not the only transformations.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Representing Reflections

Objective

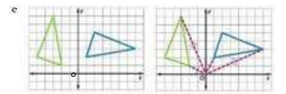
Students calculate the coordinates of the vertices of reflected images given the coordinates of the preimages.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

About the Key Concept

A reflection is a function in which the preimage is reflected in the line of reflection. The preimage and the image are the same distance from the line of reflection. When an image is reflected in the *x*-axis, the *y*-coordinates of the preimage are multiplied by −1. When an image is reflected in the *y*-axis, the *x*-coordinates of the preimage are multiplied by −1.


Each vertex and its image are the same distance from the origin. The angles formed by each pair of corresponding points and the origin are congruent. This is a rotation.

Check

The type of rigid motion shown is a ___?___ . rotation

Learn Representing Reflections

In a reflection, each point of the preimage and its corresponding point on the image are the same distance from the line of reflection.

A reflection can be described as a function in which the preimage is reflected in the line of reflection. The points of the preimage are the input, and the corresponding points on the image are the output.

Key Concept • Reflections in the *x*- or *y*-axis

	Reflections in the *x*-axis	Reflections in the *y*-axis
Words	To reflect a point in the *x*-axis, multiply its *y*-coordinate by −1.	To reflect a point in the *y*-axis, multiply its *x*-coordinate by −1.
Symbols	$(x, y) \to (x, -y)$	$(x, y) \to (-x, y)$
Example	A(4, 1), B(7, 3), A′(4, −1), B′(7, −3)	B(−2, 3), A(2, 3), B′(−6, −4), A(6, −4)

Study Tip

What Is Preserved? Because it is a rigid motion, all lengths and angle measures are preserved in a reflection.

Lesson 11-4 • Transformations in the Plane 647

Interactive Presentation

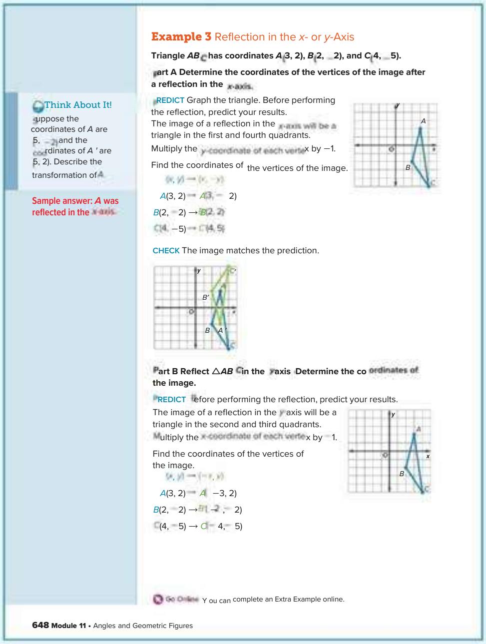
Learn

TAP

Students move through the definition, function, and notation of reflections.

TYPE

Students describe a transformation given the coordinates of a point and its image.



Example 3 Reflection in the x- or y-Axis

Triangle ABC has coordinates $A(3, 2)$, $B(2, -2)$, and $C(4, -5)$.

Part A Determine the coordinates of the vertices of the image after a reflection in the x-axis.

PREDICT Graph the triangle. Before performing the reflection, predict your results.

The image of a reflection in the x-axis will be a triangle in the first and fourth quadrants.

Multiply the y-coordinate of each vertex by -1.

Find the coordinates of the vertices of the image.

$(x, y) \rightarrow (x, -y)$

$A(3, 2) \rightarrow A'(3, -2)$

$B(2, -2) \rightarrow B'(2, 2)$

$C(4, -5) \rightarrow C'(4, 5)$

CHECK The image matches the prediction.

Part B Reflect $\triangle ABC$ in the y-axis. Determine the coordinates of the image.

PREDICT Before performing the reflection, predict your results.

The image of a reflection in the y-axis will be a triangle in the second and third quadrants.

Multiply the x-coordinate of each vertex by -1.

Find the coordinates of the vertices of the image.

$(x, y) \rightarrow (-x, y)$

$A(3, 2) \rightarrow A'(-3, 2)$

$B(2, -2) \rightarrow B'(-2, -2)$

$C(4, -5) \rightarrow C'(-4, -5)$

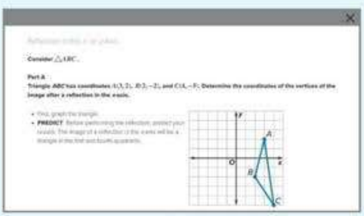
Think About It! Suppose the coordinates of A are $(5, -2)$ and the coordinates of A' are $(5, 2)$. Describe the transformation of A.

Sample answer: A was reflected in the x-axis.

Go Online You can complete an Extra Example online.

648 **Module 11** • Angles and Geometric Figures

Interactive Presentation



Example 3

DRAG & DROP

Students drag coordinates to complete the reflection.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Reflection in the x- or y-Axis

MP Teaching the Mathematical Practices

5 Compare Predictions Point out that in this example, students make a prediction and then check against the prediction at the end

Questions for Mathematical Discourse

AL When a point is reflected in the x-axis, how can you find its coordinates? Sample answer: The x-coordinate remains the same, but the y-coordinate is multiplied by -1.

OL When a point is reflected in the y-axis, how can you find its coordinates? Sample answer: The y-coordinate remains the same, but the x-coordinate is multiplied by -1.

BL Suppose a triangle with the vertices $A(1, 1)$, $B(3, 5)$ and $C(6, 6)$ is reflected in the coordinate plane. If the vertices of the image are $A'(-1, -1)$, $B'(-3, -5)$ and $C'(-6, -6)$, what reflection(s) occurred? The triangle was reflected in the x-axis, and then reflected in the y-axis.

Common Misconception

Students may invert the order of the coordinate pair when they represent a reflection in the x- or y-axis. Have students draw a point on a piece of paper, then fold the paper vertically and horizontally. Have them consider what changed and what did not.

Common Error

When a figure is reflected in the x-axis, many students will multiply the x-coordinates by -1. The same thing occurs when a figure is reflected in the y-axis; they multiply the y-coordinates by -1. Remind students that the axis of reflection is the coordinate that does not change, so reflections in the x-axis have the same x-coordinates, and reflections in the y-axis have the same y-coordinates.

DIFFERENTIATE

BL ELL

Allow the class to discuss examples of reflections in nature and in everyday objects that they use. Students can explain where lines of reflection are in objects. Natural examples could be leaves, flowers, fruits, vegetables, animals, eggs, and so on. Everyday objects could be pencils, paper, cars, clothing, and so on.

Learn Representing T ranslations

Objective

Students calculate the coordinates of the vertices of translated images given the coordinates of the preimages.

Teaching the Mathematical Practices

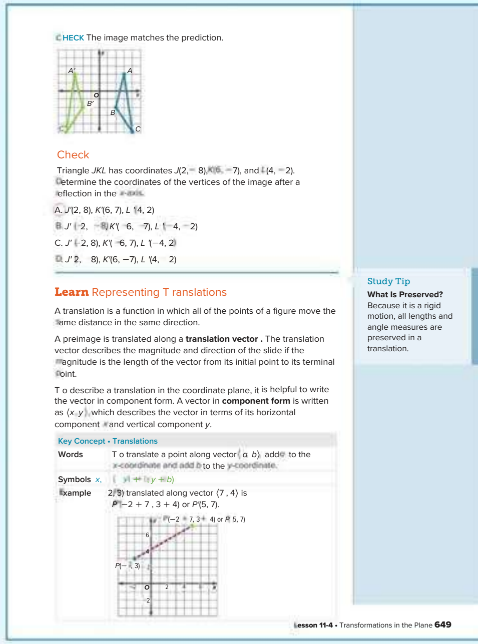
7 Use Structure Help students to explore the structure of translations in this Learn.

About the Key Concept

A translation is a function in which all of the points of a figure move the same distance in the same direction. A preimage is translated along a translation vector, which describes both the magnitude and direction of the slide. A vector in component form is often used to describe a translation, so $\langle x, y\rangle$ would describe the horizontal and vertical shift of the preimage.

Common Misconception

Students may believe that translations change the size of the figure, not just move it a set distance and direction. Remind students that translations are rigid motions and that the size of the image is preserved.



CHECK The image matches the prediction.

Check

Triangle *JKL* has coordinates *J*(2, −8), *K*(6, −7), and *L*(4, −2). Determine the coordinates of the vertices of the image after a reflection in the *x*-axis.

A. *J*′(2, 8), *K*′(6, 7), *L*′(4, 2)

B. *J*′(−2, −8), *K*′(−6, −7), *L*′(−4, −2)

C. *J*′(−2, 8), *K*′(−6, 7), *L*′(−4, 2)

D. *J*′(2, 8), *K*′(6, −7), *L*′(4, 2)

Learn Representing T ranslations

A translation is a function in which all of the points of a figure move the same distance in the same direction.

A preimage is translated along a **translation vector**. The translation vector describes the magnitude and direction of the slide if the magnitude is the length of the vector from its initial point to its terminal point.

T o describe a translation in the coordinate plane, it is helpful to write the vector in component form. A vector in **component form** is written as $\langle x, y\rangle$, which describes the vector in terms of its horizontal component *x* and vertical component *y*.

Key Concept • Translations

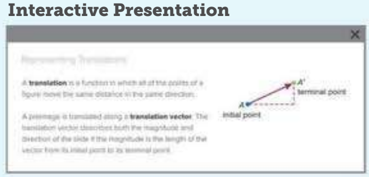
Words	T o translate a point along vector $\langle a, b\rangle$, add *a* to the *x*-coordinate and add *b* to the *y*-coordinate.
Symbols	$(x, y) \to (x + a, y + b)$
Example	$P(-2, 3)$ translated along vector $\langle 7, 4\rangle$ is $P'(-2 + 7, 3 + 4)$ or $P'(5, 7)$.

Study Tip

What Is Preserved? Because it is a rigid motion, all lengths and angle measures are preserved in a translation.

Lesson 11-4 • Transformations in the Plane 649

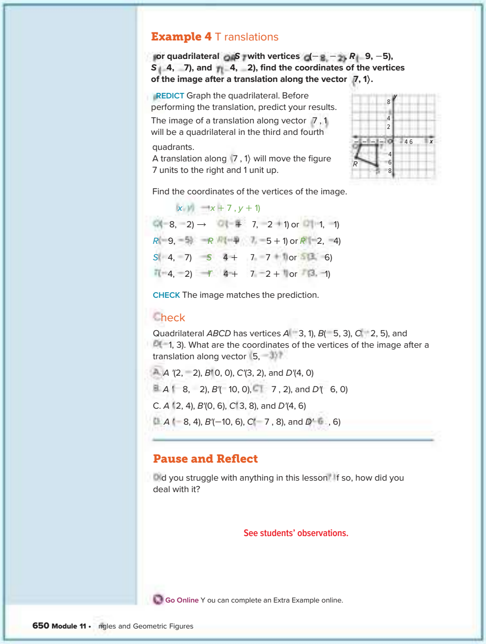
Interactive Presentation



Learn

TAP

Students tap to reveal a Study Tip.



Example 4 Translations

For quadrilateral QRST with vertices Q(−8, −2), R(−9, −5), S(−4, −7), and T(−4, −2), find the coordinates of the vertices of the image after a translation along the vector ⟨7, 1⟩.

PREDICT Graph the quadrilateral. Before performing the translation, predict your results.

The image of a translation along vector ⟨7, 1⟩ will be a quadrilateral in the third and fourth quadrants.

A translation along ⟨7, 1⟩ will move the figure 7 units to the right and 1 unit up.

Find the coordinates of the vertices of the image.

$(x, y) \to (x + 7, y + 1)$

$Q(-8, -2) \to Q'(-8 + 7, -2 + 1)$ or $Q'(-1, -1)$

$R(-9, -5) \to R'(-9 + 7, -5 + 1)$ or $R'(-2, -4)$

$S(-4, -7) \to S'(-4 + 7, -7 + 1)$ or $S'(3, -6)$

$T(-4, -2) \to T'(-4 + 7, -2 + 1)$ or $T'(3, -1)$

CHECK The image matches the prediction.

Check

Quadrilateral *ABCD* has vertices *A*(−3, 1), *B*(−5, 3), *C*(−2, 5), and *D*(−1, 3). What are the coordinates of the vertices of the image after a translation along vector ⟨5, −3⟩?

A. *A*′(2, −2), *B*′(0, 0), *C*′(3, 2), and *D*′(4, 0)

B. *A*′(−8, −2), *B*′(−10, 0), *C*′(−7, 2), and *D*′(−6, 0)

C. *A*′(2, 4), *B*′(0, 6), *C*′(3, 8), and *D*′(4, 6)

D. *A*′(−8, 4), *B*′(−10, 6), *C*′(−7, 8), and *D*′(−6, 6)

Pause and Reflect

Did you struggle with anything in this lesson? If so, how did you deal with it?

See students' observations.

Go Online You can complete an Extra Example online.

650 Module 11 • Angles and Geometric Figures

Interactive Presentation

Example 4

DRAG & DROP

Students drag operations to complete the translation.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 4 Translations

MP Teaching the Mathematical Practices

1 Explain Correspondences Guide students as they use the information in this example to graph data to represent the situation.

Questions for Mathematical Discourse

AL What is the horizontal movement for the translation along ⟨7, 1⟩? 7 units right

OL How will a figure move if it is translated along ⟨7, 1⟩? 7 units right and 1 unit up

BL Suppose the point (2, 3) were translated to (−1, −1). Along what vector would the point be translated? ⟨−3, −4⟩

Common Error

Students tend to reverse the operation based on the sign of the numbers in the vector For example, students may say *subtract 7 and subtract 1* for the vector ⟨7, 1⟩. Remind students that the figure moves in the same direction as the sign of each number in the vector. Positive numbers mean addition, and negative numbers mean subtraction.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are not mastering translations,

THEN create three or four large coordinate grids using poster board. Provide several laminated shapes, such as rectangles, hexagons, pentagons, and trapezoids. Students can practice physically translating shapes on the grids. Students can use examples of translations in the lesson or create their own.

Learn Representing Rotations

Objective

Students calculate the coordinates of the vertices of rotated images given the coordinates of the preimages.

Teaching the Mathematical Practices

3 Construct Arguments In this Learn, students will use stated assumptions, definitions, and previously established results to construct an argument.

About the Key Concept

A rotation is a function that moves every point of a preimage through a specified angle (the angle of rotation) and direction about a fixed point (the center of rotation.) Rotations can either be clockwise or counterclockwise. Rotations of 90˚, 180˚, and 270˚ each have a function rule that can be applied to find the coordinates of the resulting image.

Common Misconception

Students often think that rotations are always about the origin. Remind students that the center of the rotation can be any point. Illustrate using graph paper and varying centers of rotation to help students have a better understanding about rotations.

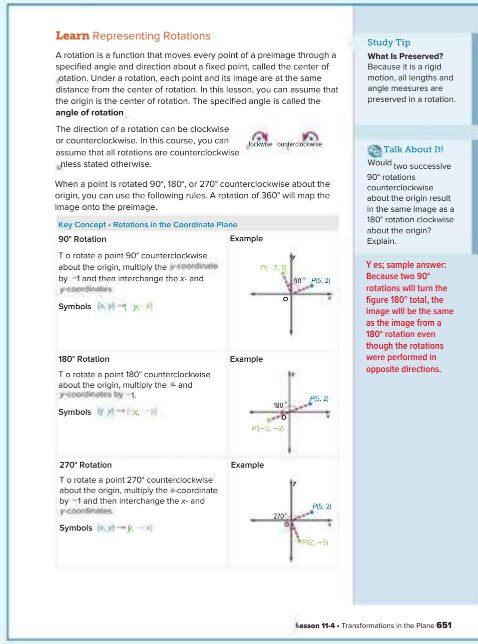
DIFFERENTIATE

Language Development Activity AL ELL

IF students confuse the terms *clockwise* and *counterclockwise*, **THEN** ask students to think about the direction of the hands on a clock. This direction is clockwise.

Enrichment Activity BL

Have students draw a triangle in Quadrant I. Then have students apply each of the three congruent transformations so that Quadrant II, Quadrant III, and Quadrant IV each contain a triangle congruent to the original triangle.



Learn Representing Rotations

A rotation is a function that moves every point of a preimage through a specified angle and direction about a fixed point, called the center of rotation. Under a rotation, each point and its image are at the same distance from the center of rotation. In this lesson, you can assume that the origin is the center of rotation. The specified angle is called the **angle of rotation**

The direction of a rotation can be clockwise or counterclockwise. In this course, you can assume that all rotations are counterclockwise unless stated otherwise.

When a point is rotated 90°, 180°, or 270° counterclockwise about the origin, you can use the following rules. A rotation of 360° will map the image onto the preimage.

Key Concept • Rotations in the Coordinate Plane

90° Rotation

To rotate a point 90° counterclockwise about the origin, multiply the y-coordinate by −1 and then interchange the x- and y-coordinates.

Symbols $(x, y) \to (-y, x)$

180° Rotation

To rotate a point 180° counterclockwise about the origin, multiply the x- and y-coordinates by −1.

Symbols $(x, y) \to (-x, -y)$

270° Rotation

To rotate a point 270° counterclockwise about the origin, multiply the x-coordinate by −1 and then interchange the x- and y-coordinates.

Symbols $(x, y) \to (y, -x)$

Study Tip

What Is Preserved? Because it is a rigid motion, all lengths and angle measures are preserved in a rotation.

Talk About It!

Would two successive 90° rotations counterclockwise about the origin result in the same image as a 180° rotation clockwise about the origin? Explain.

Yes; sample answer: Because two 90° rotations will turn the figure 180° total, the image will be the same as the image from a 180° rotation even though the rotations were performed in opposite directions.

Lesson 11-4 • Transformations in the Plane 651

Interactive Presentation

Learn

TAP

Students move through the slides to learn about rotations.

TYPE

Students answer a question to show they understand rotations.

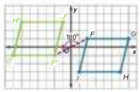
Think About It!
Describe the figure after the rotation.

Sample answer: After the figure is rotated, it is upside down with side $\overline{F'J'}$ on the right. The figure is located mostly in the second quadrant.

Example 5 Rotations

Parallelogram *FGHJ* has vertices *F*(2, 1), *G*(7, 1), *H*(6, −3), and *J*(1, −3). What are the coordinates of the vertices of its image after a rotation of 180° about the origin?

PREDICT Graph parallelogram *FGHJ*.



Before performing the rotation, predict your results.

The image of the parallelogram rotated 180° will be a parallelogram in the second and third quadrants.

To rotate a point 180° counterclockwise about the origin, multiply the *x*- and *y*-coordinates by −1. Find the coordinates of the vertices of the image.

$(x, y) \rightarrow (-x, -y)$
$F(2, 1) \rightarrow F'(-2, -1)$
$G(7, 1) \rightarrow G'(-7, -1)$
$H(6, -3) \rightarrow H'(-6, 3)$
$J(1, -3) \rightarrow J'(-1, 3)$

CHECK The image meets the prediction.

Study Tip
Rotations
Verify the coordinates of the image by graphing the preimage and image on the coordinate grid.

Check

Quadrilateral *JKLM* has coordinates *J*(1, 2), *K*(4, 3), *L*(6, 1), and *M*(3, 1). Determine the coordinates of the vertices of the image after a 270° rotation about the origin.

A. *J′*(2, −1), *K′*(3, −4), *L′*(1, −6), and *M′*(1, −3)
B. *J′*(2, 1), *K′*(3, 4), *L′*(1, 6), and *M′*(1, 3)
C. *J′*(−2, 1), *K′*(−3, 4), *L′*(−1, 6), and *M′*(−1, 3)
D. *J′*(−2, −1), *K′*(−3, −4), *L′*(−1, −6), and *M′*(−1, −3)

Go Online You can complete an Extra Example online.

652 Module 11 • Angles and Geometric Figures

Interactive Presentation

Example 5

DRAG & DROP
Students drag coordinates to complete the rotation.

TYPE
Students describe the figure after a rotation.

CHECK
Students complete the Check online to determine whether they are ready to move on.

Example 5 Rotations

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL Do rotations change the size or shape of an object? no

OL What happens to each coordinate when a 180° counterclockwise rotation about the origin is performed? Sample answer: The *x*- and *y*-coordinates are multiplied by −1.

BL What is the difference between rotating a point 180° clockwise about the origin and rotating a point 180° counterclockwise about the origin? Sample answer: The direction of each rotation is different, but the image is the same.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–21
2	exercises that use a variety of skills from this lesson	22–31
3	exercises that emphasize higher-order and critical-thinking skills	32–38

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: OL BL

- Practice, Exercises 1–31 odd, 32–38
- Extension: Compositions of Transformations

IF students score 66%–89% on the Checks, **THEN** assign: AL OL

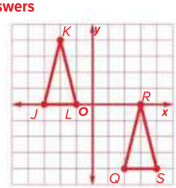
- Practice, Exercises 1–31 odd
- Remediation, Review Resources: Area of Parallelograms
- Personal Tutors
- Extra Examples 1–5
- ALEKS Reviewing Perimeter and Area

IF students score 65% or less on the Checks **THEN** assign: AL

- Practice, Exercises 1–21 odd
- Remediation, Review Resources: Area of Parallelograms
- *Quick Review Math Handbook*: Transformations in the Plane
- ALEKS Reviewing Perimeter and Area

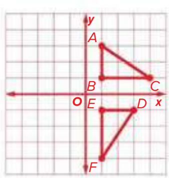
Answers

26.



Q(2, −4), *R*(3, 0), *S*(4, −4)

27.



D(3, −1), *E*(1, −1), *F*(1, −4)

Practice

Go Online You can complete your homework online.

Examples 1 and 2

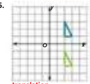
Identify the type of rigid motion shown as a *reflection*, *translation*, or *rotation*.

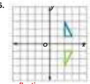
1. 
reflection

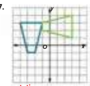
2. 
translation

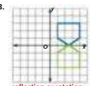
3. 
rotation

4. 
translation

5. 
translation

6. 
reflection

7. 
rotation

8. 
reflection or rotation

Examples 3–5

Triangle *ABC* has coordinates *A*(2, 0), *B*(−1, 5), and *C*(4, 3). Determine the coordinates of the vertices of the image after each transformation.

9. reflection in *x*-axis
A′(2, 0), *B*′(−1, −5), and *C*′(4, −3)

10. reflection in *y*-axis
A′(−2, 0), *B*′(1, 5), and *C*′(−4, 3)

11. translation along the vector ⟨0, 2⟩
A′(2, 2), *B*′(−1, 7), and *C*′(4, 5)

12. translation along the vector ⟨3, −4⟩
A′(5, −4), *B*′(2, 1), and *C*′(7, −1)

13. rotation 180° about the origin
A′(−2, 0), *B*′(1, −5), and *C*′(−4, −3)

14. rotation 90° counterclockwise about the origin *A*′(0, 2), *B*′(−5, −1), and *C*′(−3, 4)

Triangle *DEF* has coordinates *D*(4, −1), *E*(5, 2), and *F*(1, 2). Determine the coordinates of the vertices of the image after each transformation.

15. reflection in *x*-axis
D′(4, 1), *E*′(5, −2), and *F*′(1, −2)

16. reflection in *y*-axis
D′(−4, −1), *E*′(−5, 2), and *F*′(−1, 2)

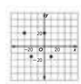
17. translation along the vector ⟨1, 0⟩
D′(5, −1), *E*′(6, 2), and *F*′(2, 2)

18. translation along the vector ⟨−3, 1⟩
D′(1, 0), *E*′(2, 3), and *F*′(−2, 3)

19. rotation 180° about the origin
D′(−4, 1), *E*′(−5, −2), and *F*′(−1, −2)

20. rotation 270° counterclockwise about the origin
D′(−1, −4), *E*′(2, −5), and *F*′(2, −1)

21. AIR SHOW At a flight demonstration, two planes are flying in a synchronized pattern. Describe the transformation that represents the planes' flight pattern to their final destinations at (−30, 20) and (0, 20). translation along vector ⟨−10, 35⟩



Mixed Exercises

22. OFFICE Francesca draws a plan of her office before she rearranges the furniture. She decides to reflect the entire room over a vertical line through the center of the drawing of the room.

Which is the reflected plan? A

A. 

B. 

C. 

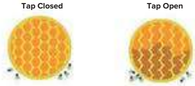
23. BASKETBALL James spins a basketball on his finger and then passes the ball to his friend.

a. What type of transformation is used when James spins the basketball on his finger? rotation

b. What type of transformation is used to pass the basketball? translation

24. COMBINATION LOCKS Benicio locks his safe by setting each of the three dials to 8. To unlock the safe, he turns the left dial 90° counterclockwise, the middle dial 270° clockwise, and the right dial 180° counterclockwise. Which three numbers, in order, unlock the safe? 2-2-4

25. BEEKEEPING A beekeeper uses a frame of partial honeycomb cells that bees fill with honey and complete with wax. When the honey is ready for harvest, the beekeeper turns the tap allowing the honey to flow out of the hive without disturbing the bees. By what transformation are the sides of the partial honeycomb cells related when the tap is closed? when the tap is open? reflection; translation



Find the coordinates of the figure with the given coordinates after the transformation on the plane. Then graph the preimage and image.

26. preimage: $J(-3, 0)$, $K(-2, 4)$, $L(-1, 0)$, image: triangle QRS, translation of JKL along vector ⟨5, −4⟩ See margin.

27. preimage: $A(1, 3)$, $B(1, 1)$, $C(4, 1)$, image: triangle DEF, rotation of ABC 270° counterclockwise about the origin See margin.

28. FIND THE ERROR Saurabh and Elena visit a craft fair and notice a quilt with a pattern. Saurabh claims the pattern is made using translations. Elena believes that the pattern is made using rotations. Who is correct? Justify your argument. Elena; sample answer: The triangles shown in the pattern appear to be made using translations but the outlined hexagonal patterns are created using rotations.

29. The vertices of $\triangle ABC$ are $A(-1, 1)$, $B(4, 2)$, and $C(1, 5)$. The vertices of $\triangle DEF$ are $D(-1, -1)$, $E(4, -2)$, and $F(1, -5)$ such that $\triangle ABC \cong \triangle DEF$. Identify the congruence transformation. reflection in the x-axis

30. STRUCTURE $\overline{XY}$ has endpoints $X(-5, 6)$ and $Y(0, 4)$, the image of $\overline{XY}$ has the endpoints $X'(6, 5)$ and $Y'(4, 0)$, and $\overline{XY} \cong \overline{X'Y'}$. Identify the transformation. rotation of 270° about the origin

31. STRUCTURE The vertices of quadrilateral $FGHJ$ are $F(2, -3)$, $G(-2, -5)$, $H(-3, 6)$, and $J(3, 5)$. The vertices of quadrilateral $KLMN$ are $K(5, -5)$, $L(1, -7)$, $M(0, 4)$, and $N(6, 3)$ such that $FGHJ \cong KLMN$. If quadrilateral $FGHJ$ is the preimage and quadrilateral $KLMN$ is the image, identify the transformation. translation 3 units right and 2 units down

Higher-Order Thinking Skills

32. ANALYZE The image of $\triangle ABC$ reflected in the y-axis is $\triangle A'B'C'$.

a. Describe the result of reflecting $\triangle A'B'C'$ in the y-axis. Explain. See margin.

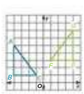
b. Describe the result of reflecting $\triangle A'B'C'$ in the x-axis. Explain. See margin.

33. FIND THE ERROR Antwan and Diamond are finding the coordinates of the image of $P(2, 3)$ after a reflection in the x-axis. Is either of them correct? Explain your reasoning.

Antwan	Diamond
P'(2, −3)	P'(2, −3)

Antwan; sample answer: When you reflect a point across the x-axis, the reflected point is in the same place horizontally, but not vertically. When (2, 3) is reflected across the x-axis, the coordinates of the reflected point are (2, −3) because it is in the same location horizontally, but the other side of the x-axis vertically.

34. WRITE In the diagram, $\triangle DEF$ is called a *glide reflection* of $\triangle ABC$. Based on the diagram, define a glide reflection. Explain your reasoning. Sample answer: A glide reflection is a reflection over a line and then a translation in a direction that is parallel to the line of reflection.



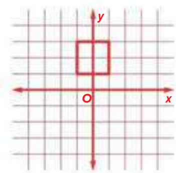
35. CREATE Draw a polygon on the coordinate plane that when reflected in the y-axis looks exactly like the original figure. See margin.

36. ANALYZE Is the reflection of a figure in the x-axis equivalent to the rotation of that same figure 180° about the origin? Explain. See margin.

32a. $\triangle ABC$; Sample answer: To find the coordinates of the vertices of $\triangle A'B'C'$, you would multiply the x-coordinates of the vertices of $\triangle ABC$ by −1. To find the coordinates of the vertices of the reflection of $\triangle A'B'C'$, you would multiply the x-coordinates of the vertices by −1. Because $(-1)(-1) = 1$, the coordinates of the vertices of the image are the same as the coordinates of the vertices of $\triangle ABC$.

32b. This reflection results in a 180° rotation of $\triangle ABC$ about the origin. To find the coordinates of the vertices of $\triangle A'B'C'$, you would multiply x-coordinates of the vertices of $\triangle ABC$ by −1. To find the coordinates of the vertices of the reflection of $\triangle A'B'C'$ in the x-axis, you would multiply y-coordinates of the vertices by −1. If a vertex of $\triangle ABC$ has coordinates (x, y), then the coordinates of the image of that point would have coordinates $(-x, -y)$. Those coordinates describe a 180° rotation about the origin.

35. Sample answer:



36. No; sample answer: When a figure is reflected in the x-axis, the x-coordinates of the transformed figure remain the same, the y-coordinates are negated. When a figure is rotated 180° about the origin, both the x- and y-coordinates are negated. Therefore, the transformations are not equivalent.

Lesson 11-5

Three-Dimensional Figures

G.MG.1, G.GMD.3

LESSON GOAL

Students find measures of three-dimensional figures.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Identifying Three-Dimensional Figures
- Identify Properties of Three-Dimensional Figures
- Model Three-Dimensional Figures

Explore: Measuring Real-World Objects

Develop:

Measuring Three-Dimensional Figures
- Find Measurements of Three-Dimensional Figures
- Calculate Measurements by Using Three-Dimensional Models
- Solve for Unknown Values

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Three-Dimensional Figures	●	●		●
Extension: Cubes		●	●	●

Language Development Handbook

Assign page 67 of the *Language Development Handbook* to help your students build mathematical language related to finding? measures of three-dimensional figures.

ELL You can use the tips and suggestions on page T67 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g. modeling a tree trunk or a human torso as a cylinder).

G.GMD.3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

Standards for Mathematical Practice:

4 Model with mathematics.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students understood and used surface area and volume formulas for two-dimensional figures.
6.G.2, 6.G.4, 7.G.6, 8.G.9

Now
Students find measures of three-dimensional objects.
G.MG.1, G.GMD.3

Next
Students will model three-dimensional figures with two-dimensional representations.
G.MG.1

Rigor

The Three Pillars of Rigor

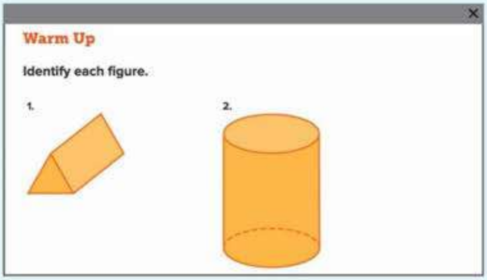
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students draw on their understanding of solid figures to model real-world objects. They build fluency by using coordinates to find volumes of solid figures and apply what they know about solid figures to solve real-world problems.

Mathematical Background

A solid with all flat surfaces that encloses a single region of space is called a *polyhedron*. Each flat surface, or face, is a polygon. A regular polyhedron has all congruent edges and all of its faces are congruent regular polygons. Two common types of polyhedra are prisms and pyramids. A prism has two parallel congruent faces called bases.

Interactive Presentation



Warm Up


Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- identifying three-dimensional figures

Answers:
1. triangular prism
2. cylinder
3. cone
4. sphere
5. triangular pyramid

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of modeling objects using three-dimensional geometric figures.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the question below with the class.

Explore Measuring Real-World Objects

Objective

Students use dynamic geometry software and what they know about area and volume to calculate the surface area and volume of a Platonic solid.

MP Teaching the Mathematical Practices

5 Use appropriate tools strategically Throughout the Explore, encourage students to use the appropriate tools to explore their understanding of area and volume.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use a sketch to explore the surface area and volume of a Platonic solid. They will answer questions regarding surface area and volume of different shapes, and they will consider the units required. Then, students will answer the Inquiry Question.

(*continued on the next page*)

Interactive Presentation

Explore

Explore

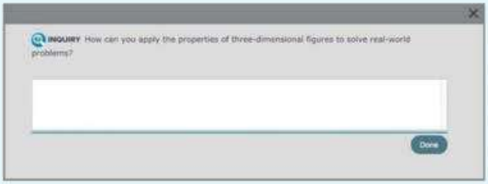
WEB SKETCHPAD

Students use a sketch to explore the surface area and volume of a Platonic solid.

TYPE

Students answer questions about the surface area and volume of the solids.

Interactive Presentation



Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Measuring Real-World Objects *(continued)*

Questions

Have students complete the Explore activity.

Ask:

- What is the formula used to find the area of a triangle? $A = \frac{1}{2}bh$
- Consider a regular die with square faces. How would you find the surface area of this type of die? Sample answer: Find the area of one square face and then multiply by 6.

Inquiry

How can you apply the properties of three-dimensional figures to solve real-world problems? Sample answer: Three-dimensional figures can be used to model real-world objects such a grain silos, water tanks, jewelry, or pottery. Then you can use the known formulas for calculating the surface area and volume of the three-dimensional figures to approximate the amount of material it would take to build an object or how much material an object can hold.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Identifying Three-Dimensional Figures

Objective

Students identify and determine characteristics of three-dimensional figures.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

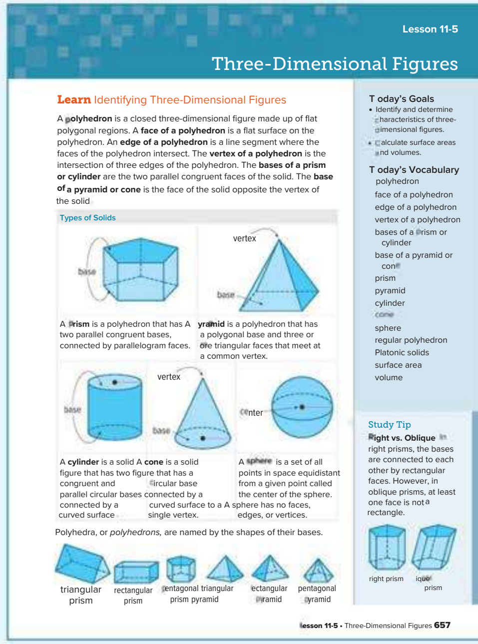
7 Use Structure Help students to explore the structure of polyhedra in this Learn.

Important to Know

Three-dimensional figures are often used as models for real-world objects. Polyhedrons are closed three-dimensional figures made up of flat polygonal regions. The faces are flat surfaces on the polyhedron. The edges are line segments where the faces intersect. The vertex is the intersection of three edges. A prism is a polyhedron with two parallel congruent faces connected by parallelogram faces. A pyramid is also a polyhedron, but it has a polygonal base with three or more triangular faces that meet at the common vertex. Cylinders, cones, and spheres are not polyhedral because they have curved surfaces.

Common Misconception

Students may believe that all three-dimensional figures are polyhedral, including those with curved surfaces. Remind students that a polyhedron must have bases by which it can be named, such as a square, triangle, or rectangle, but it cannot contain curved surfaces.


Lesson 11-5

Three-Dimensional Figures

Learn Identifying Three-Dimensional Figures

A **polyhedron** is a closed three-dimensional figure made up of flat polygonal regions. A **face of a polyhedron** is a flat surface on the polyhedron. An **edge of a polyhedron** is a line segment where the faces of the polyhedron intersect. The **vertex of a polyhedron** is the intersection of three edges of the polyhedron. The **bases of a prism or cylinder** are the two parallel congruent faces of the solid. The **base of a pyramid or cone** is the face of the solid opposite the vertex of the solid.

Types of Solids

A **prism** is a polyhedron that has two parallel congruent bases, connected by parallelogram faces.

A **pyramid** is a polyhedron that has a polygonal base and three or more triangular faces that meet at a common vertex.

A **cylinder** is a solid figure that has two congruent and parallel circular bases connected by a curved surface.

A **cone** is a solid figure that has a circular base connected by a curved surface to a single vertex.

A **sphere** is a set of all points in space equidistant from a given point called the center of the sphere. A sphere has no faces, edges, or vertices.

Polyhedra, or *polyhedrons*, are named by the shapes of their bases.

triangular prism, rectangular prism, pentagonal prism, triangular pyramid, rectangular pyramid, pentagonal pyramid

Today's Goals
- Identify and determine characteristics of three-dimensional figures.
- Calculate surface areas and volumes.

Today's Vocabulary: polyhedron, face of a polyhedron, edge of a polyhedron, vertex of a polyhedron, bases of a prism or cylinder, base of a pyramid or cone, prism, pyramid, cylinder, cone, sphere, regular polyhedron, Platonic solids, surface area, volume

Study Tip

Right vs. Oblique In right prisms, the bases are connected to each other by rectangular faces. However, in oblique prisms, at least one face is not a rectangle.

right prism, oblique prism

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Interactive Presentation


Learn

TAP

Students tap to learn about the characteristics of polyhedra.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students confuse the terms *pyramid* and *prism*,

THEN have students create nets for a rectangular prism and triangular prism along with a rectangular pyramid and a triangular pyramid to make a visual connection to the properties of each figure.

Math History Minute

The Platonic solids are named for the Greek philosopher **Plato** **(c. 428 BCE–c. 348 BCE)**, who theorized in his dialogue, the *Timaeus*, that the classical elements were made of them. Of all the Platonic solids, the Greeks believed that the dodecahedron represented the entire universe.

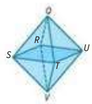
A polyhedron is a **regular polyhedron** if all of its faces are regular congruent polygons and all of the edges are congruent. There are exactly five types of regular polyhedra, called **platonic solids** because Plato used them extensively.

Platonic Solids

Tetrahedron	Hexahedron or Cube	Octahedron	Dodecahedron	Icosahedron
4 equilateral triangular faces	6 square faces	8 equilateral triangular faces	12 regular pentagonal faces	20 equilateral triangular faces

Example 1 Identify Properties of Three-Dimensional Figures

Determine whether each solid is a polyhedron. Then identify the solid. If it is a polyhedron, name the bases, faces, edges, and vertices.



The solid is formed by polygonal faces, so it is a polyhedron. There is no base, but the solid has 8 equilateral triangular faces, so it is an octahedron.

Bases: none

Faces: $\triangle QRU$, $\triangle QRS$, $\triangle QST$, $\triangle QTU$, $\triangle RSV$, $\triangle STV$, $\triangle TUV$, $\triangle RUV$

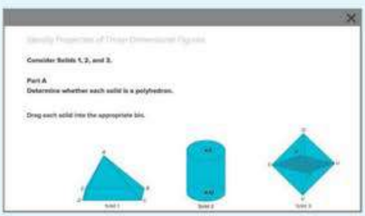
Edges: $\overline{QR}$, $\overline{QS}$, $\overline{QT}$, $\overline{QU}$, $\overline{RV}$, $\overline{UV}$, $\overline{TV}$, $\overline{SV}$, $\overline{RS}$, $\overline{ST}$, $\overline{TU}$, $\overline{RU}$

Vertices: Q, R, S, T, U, V

The solid has a curved surface, so it is not a polyhedron. It has two congruent and parallel circular bases and parallel sides, so it is a cylinder.

Go Online You can complete an Extra Example online.

Interactive Presentation



Example 1

DRAG & DROP

Students drag shapes to classify them.

TAP

Students tap to see the descriptions for various solids.

Example 1 Identify Properties of Three-Dimensional Figures

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of each solid in this example to classify them as polyhedra or not polyhedra.

Questions for Mathematical Discourse

AL What is the difference between shapes that are polyhedral and *not* polyhedral? All polyhedra have polygonal surfaces. Other 3-D shapes with curved surfaces are *not* polyhedra.

OL What 3-D shapes make up the octahedron? two square pyramids

BL What shape has 10 faces, 24 edges and 16 vertices? octagonal prism

Common Error

Students may think that only Platonic solids with a flat base, such as tetrahedrons or cubes, are polyhedra. They may try and classify the octahedron as a nonpolyhedra. Encourage students to consider if the faces are flat or curved, rather than looking only at the base.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 2 Model Three-Dimensional Figures

Teaching the Mathematical Practices

4 Make Assumptions In the Study Tip, have students point out where an assumption or approximation was made in the solution.

Questions for Mathematical Discourse

AL What shape would best model the base of the beverage container? a circle

OL Would the beverage container be considered a polyhedron? Explain. No; sample answer: The beverage container has curved surfaces, so it would not be a polyhedron.

BL Suppose the beverage container was packaged in a box. What three-dimensional figure could model the box? a rectangular prism

Learn Measuring Three-Dimensional Figures

Objective

Students solve for unknown measures of three-dimensional figures by calculating surface areas and volumes.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

What Students Are Learning

Surface area is the sum of the areas of all faces and side surfaces of a three-dimensional figure. Volume is the measure of the amount of space enclosed by a three-dimensional object. Geometric figures are used to model real-world objects in order to estimate those measurements.

Common Misconception

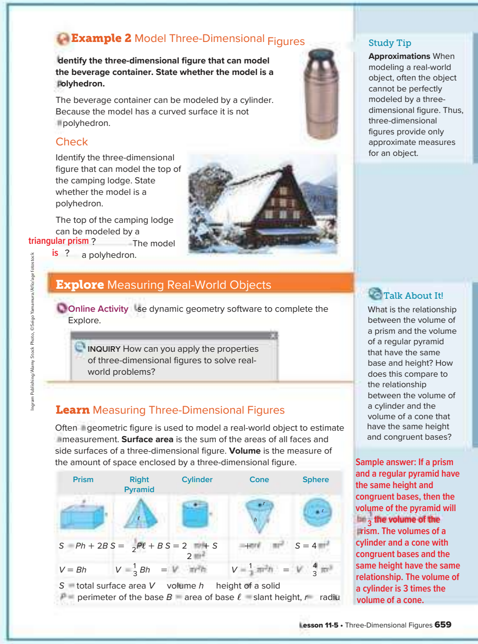
Students may not realize units are an important part of any measurement, including volume and surface area. Remind the students to use square units when dealing with area, use cubic units when dealing with volume.

DIFFERENTIATE

AL ELL

IF students struggle to calculate surface area,

THEN have students create nets for various figures, cut the out nets and put them together to form a solid. Then they can physically see the different sides that generate surface area.



Example 2 Model Three-Dimensional Figures

Identify the three-dimensional figure that can model the beverage container. State whether the model is a polyhedron.

The beverage container can be modeled by a cylinder. Because the model has a curved surface it is not a polyhedron.

Check

Identify the three-dimensional figure that can model the top of the camping lodge. State whether the model is a polyhedron.

The top of the camping lodge can be modeled by a triangular prism ? The model is ? a polyhedron.

Study Tip

Approximations When modeling a real-world object, often the object cannot be perfectly modeled by a three-dimensional figure. Thus, three-dimensional figures provide only approximate measures for an object.

Explore Measuring Real-World Objects

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How can you apply the properties of three-dimensional figures to solve real-world problems?

Talk About It!

What is the relationship between the volume of a prism and the volume of a regular pyramid that have the same base and height? How does this compare to the relationship between the volume of a cylinder and the volume of a cone that have the same height and congruent bases?

Sample answer: If a prism and a regular pyramid have the same height and congruent bases, then the volume of the pyramid will be $\frac{1}{3}$ the volume of the prism. The volumes of a cylinder and a cone with congruent bases and the same height have the same relationship. The volume of a cylinder is 3 times the volume of a cone.

Learn Measuring Three-Dimensional Figures

Often a geometric figure is used to model a real-world object to estimate a measurement. **Surface area** is the sum of the areas of all faces and side surfaces of a three-dimensional figure. **Volume** is the measure of the amount of space enclosed by a three-dimensional figure.

Prism	Right Pyramid	Cylinder	Cone	Sphere
$S = Ph + 2B$	$S = \frac{1}{2}P\ell + B$	$S = 2\pi rh + 2\pi r^2$	$S = \pi r\ell + \pi r^2$	$S = 4\pi r^2$
$V = Bh$	$V = \frac{1}{3}Bh$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$	$V = \frac{4}{3}\pi r^3$

S = total surface area, V = volume, h = height of a solid, P = perimeter of the base, B = area of base, ℓ = slant height, r = radius

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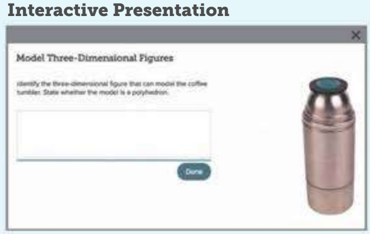
Interactive Presentation



Example 2

TYPE

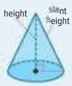
Students describe the three-dimensional figures that can be used to model the beverage container.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Watch Out!

Height vs. Slant Height The height of a pyramid or cone is not the same as its *slant height*. The height of a pyramid or cone is the perpendicular distance from the center of the base to the vertex. The slant height is the height of a lateral side of a pyramid or cone.



Watch Out!

Square and Cubic Units Remember to use square units when measuring surface area and cubic units when measuring volume.

Think About It!

How could you write the surface area of the cone in terms of pi?

480π

Example 3 Find Measurements of Three-Dimensional Figures

Find the surface area and volume of the cone. Round each measure to the nearest tenth, if necessary.

Surface Area

Because the radius of the base is 15 inches and the height of the cone is 8 inches, you can use the Pythagorean Theorem to find the slant height.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$c^2 = 8^2 + 15^2$	$a = 8$ and $b = 15$
$c^2 = 289$	Simplify.
$c = \sqrt{289}$ or 17	Simplify.

The slant height is 17 inches, and the radius is 15 inches. Use the formula for the surface area of a cone.

$S = \pi r\ell + \pi r^2$	Surface area of cone
$= \pi(15)(17) + \pi(15)^2$	$r = 15$ in. and $\ell = 17$ in.
≈ 1508.0	Use a calculator.

The surface area of the cone is 1508.0 square inches.

Volume

Use the formula for the volume of a cone.

$V = \frac{1}{3}\pi r^2 h$	Volume of cone
$= \frac{1}{3}\pi(15)^2(8)$ or about 1885.0	$r = 15$ in. and $h = 8$ in.

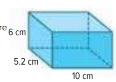
The volume of the cone is about 1885.0 cubic inches.

Check

Find the surface area and volume of the rectangular prism. Round each measure to the nearest tenth, if necessary.

$S =$? cm^2 286.4

$V =$? cm^3 312



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Example 3 Find Measurements of Three-Dimensional Figures

Teaching the Mathematical Practices

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. In this example, notice the relationship between the problem variables and the units involved.

Questions for Mathematical Discourse

AL Using words, how can you describe is the slant height of a pyramid? the height of each lateral face of the pyramid

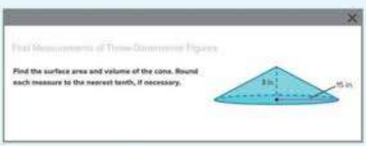
OL In the formula for the surface area of a pyramid, what do the variables P, ℓ, and B represent? P is the perimeter of the base, ℓ is the slant height, and B is the area of the base.

BL What would be the approximate volume of a cylinder with the same height and congruent base as the given cone? 5655 in^3

Common Error

When dealing with volume, many students will use the slant height rather than the height of the cone. Remind students that when calculating area, the height is the perpendicular distance from the center of the base to the vertex of the cone.

Interactive Presentation



Example 3

TYPE

Students complete the calculations to find the slant height, surface area, and volume.

Example 4 Calculate Measurements by Using Three-Dimensional Models

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about surface areas and volumes to solving a real-world problem.

Questions for Mathematical Discourse

AL What is the difference in finding surface area and volume of a sphere? Sample answer: When finding the volume, multiply four times pi times the radius cubed, and divide the product by three. When finding the surface area, multiply four times pi times the radius squared.

OL If the diameter doubled in size, would the surface area also double in size? Explain. No; if the diameter was 24 feet, then the surface area would equal approximately 1809 square feet, which is four times as big.

BL In **Part B**, suppose a second section of the ball must be repaired. If the section is 6 cubic feet, how much does that section weigh? Show all work. $\frac{6 \text{ ft}^3}{1} \times \frac{11{,}875 \text{ lb}}{904.8 \text{ ft}^3} = 78.7$ lbs

Common Error

Students that do well with surface area and volume of shapes may have difficulty when presented with an application problem. They may not know how to use surface area and volume with the given information of lights and weight. Encourage students to write down the given information and then underline the important values for the question asked.

Example 4 Calculate Measurements by Using Three-Dimensional Models

NEW YEAR'S EVE **The New Year's Eve ball is a geodesic sphere that is 12 feet in diameter. It weighs 11,875 pounds, is lit by 32,256 LED lights, and is covered with 2688 crystal triangles.**

Part A How many lights are contained on the ball's surface within an area of 4 square feet?

Step 1 Find the surface area of the ball.

Because the diameter is 12 feet, the radius of the sphere is 6 feet.

$S = 4\pi r^2$ — Surface area of a sphere

$= 4\pi(6)^2$ — $r = 6$

$= 144\pi$ or about 452.4 — Use a calculator

The surface area of the ball is about 452.4 square feet.

Step 2 Determine the number of lights within an area of 4 square feet.

$4 \text{ ft}^2 \times \frac{32{,}256 \text{ lights}}{452.4 \text{ ft}^2} = 285.2$ or 285 lights

There are 285 lights within an area of 4 square feet.

Part B Tony is repairing a section of the ball that has an area of 8 cubic feet. How much does the section weigh?

Step 1 Find the volume of the ball.

$V = \frac{4}{3}\pi r^3$ — Volume of a sphere

$= \frac{4}{3}\pi(6)^3$ — $r = 6$

$= 288\pi$ or about 904.8 — Use a calculator.

Step 2 Determine the weight of the section.

$8 \text{ ft}^3 \times \frac{11{,}875 \text{ lb}}{904.8 \text{ ft}^3} = 105.0$

The section of the ball weighs about 105.0 pounds.

Think About It!
What assumption did you make about the New Year's Eve ball to solve the problem?

Sample answer: I assumed that the ball is a perfect sphere and that the weight of the ball is evenly distributed throughout the object.

Check

POOLS Mateo's family is building a new inground pool. A cross section of the pool is shown.

Part A What is the volume of the pool to the nearest tenth?

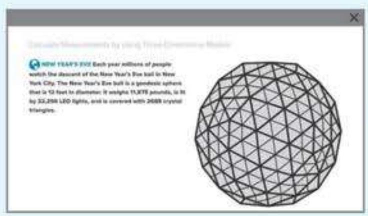
$V =$? ft^3 3142.5

Part B Mateo's family needs to install a protective liner to cover the walls and flat base of the deep end of the pool. How much liner is required to cover the deep end of the pool in square feet?

A. 570 ft^2 B. 750 ft^2 C. 900 ft^2 D. 1800 ft^2

Lesson 11-5 • Three-Dimensional Figures 661

Interactive Presentation



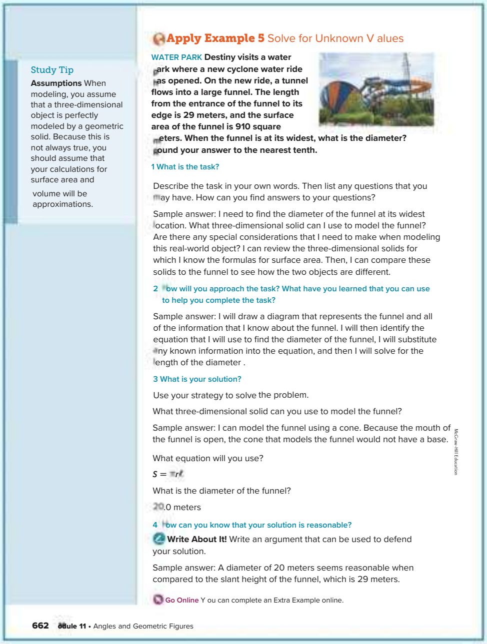
Example 4

TAP

Students move through the steps to solve the problem.

TYPE

Students respond to a question about making assumptions.


Study Tip

Assumptions When modeling, you assume that a three-dimensional object is perfectly modeled by a geometric solid. Because this is not always true, you should assume that your calculations for surface area and volume will be approximations.

Apply Example 5 Solve for Unknown V alues

WATER PARK **Destiny visits a water park where a new cyclone water ride has opened. On the new ride, a tunnel flows into a large funnel. The length from the entrance of the funnel to its edge is 29 meters, and the surface area of the funnel is 910 square meters. When the funnel is at its widest, what is the diameter? Round your answer to the nearest tenth.**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to find the diameter of the funnel at its widest location. What three-dimensional solid can I use to model the funnel? Are there any special considerations that I need to make when modeling this real-world object? I can review the three-dimensional solids for which I know the formulas for surface area. Then, I can compare these solids to the funnel to see how the two objects are different.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will draw a diagram that represents the funnel and all of the information that I know about the funnel. I will then identify the equation that I will use to find the diameter of the funnel. I will substitute any known information into the equation, and then I will solve for the length of the diameter .

3 What is your solution?

Use your strategy to solve the problem.

What three-dimensional solid can you use to model the funnel?

Sample answer: I can model the funnel using a cone. Because the mouth of the funnel is open, the cone that models the funnel would not have a base.

What equation will you use?

$S = \pi r\ell$

What is the diameter of the funnel?

20.0 meters

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: A diameter of 20 meters seems reasonable when compared to the slant height of the funnel, which is 29 meters.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Apply Example 5

TAP

Students tap to view a Study Tip.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Apply Example 5 Solve for Unknown Values

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- W hat three-dimensional object can model the shape of the funnel?
- W hy does the formula for the surface area for the funnel not include the base?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Common Error

The problem asked students to find the diameter at the widest part of the funnel, but when solving the surface area formula, students will calculate the radius. Some students may stop at this answer, but remind students to read the question carefully to ensure that they have found the requested measurement.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–19
2	exercises that use a variety of skills from this lesson	20–24
3	exercises that emphasize higher-order and critical-thinking skills	25–30

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: (OL BL)

- Practice, Exercises 1–24 odd, 25–30
- Extension: Cubes
- ALEKS Solids and Cross Sections

IF students score 66%–89% on the Checks, **THEN** assign: (AL OL)

- Practice, Exercises 1–30 odd
- Remediation, Review Resources: Three-Dimensional Figures
- Personal Tutors
- Extra Examples 1–4
- ALEKS Identifying Three-Dimensional Figures

IF students score 65% or less on the Checks, **THEN** assign: (AL)

- Practice, Exercises 1–19 odd
- Remediation, Review Resources: Three-Dimensional Figures
- *Quick Review Math Handbook*: Three-Dimensional Figures
- ALEKS Identifying Three-Dimensional Figures

Answers

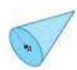
1. not a polyhedron; cone
2. polyhedron; rectangular prism; bases ▭*DEFG*, ▭*HJKL*; faces ▭*DEFG*, ▭*HJKL*, ▭*DEJH*, ▭*EFKJ*, ▭*FKLG*, ▭*LGDH*; edges $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, $\overline{GD}$, $\overline{DH}$, $\overline{EJ}$, $\overline{FK}$, $\overline{GL}$, $\overline{HJ}$, $\overline{JK}$, $\overline{KL}$, $\overline{LH}$; vertices *D, E, F, G, H, J, K, L*
3. polyhedron; rectangular pyramid; base *WXYZ*; faces ▭*WXYZ*, △*VWX*, △*VXY*, △*VYZ*, △*VZW*; edges $\overline{WX}$, $\overline{XY}$, $\overline{YZ}$, $\overline{ZW}$, $\overline{WV}$, $\overline{XV}$, $\overline{YV}$, $\overline{ZV}$; vertices *W, X, Y, Z, V*

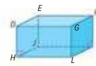
Practice

Go Online You can complete your homework online.

Example 1

Determine whether each solid is a polyhedron. Then identify the solid. If it is a polyhedron, name the bases, faces, edges, and vertices. 1-3. See margin.

1.


2.


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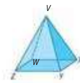


Example 2

Identify the three-dimensional figure that can model each object. State whether the model is or is not a polyhedron.

4.

cone; not a polyhedron

5.

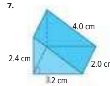
sphere; not a polyhedron

6.

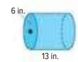
rectangular prism; polyhedron

Example 3

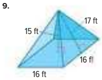
Find the surface area and volume of each solid. Round each measure to the nearest tenth, if necessary.

7.


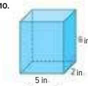
26.9 cm²; 7.7 cm³

8.


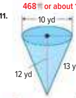
228π or about 716.3 in²; 468π or about 1470.3 in³

9.


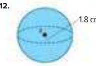
800 ft²; 1280 ft³

10.


104 in²; 60 in³

11.


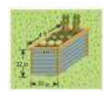
90π or about 282.7 yd²; 100π or about 314.2 yd³

12.


12.96π or about 40.7 cm²; 7.776π or about 24.4 cm³

Example 4

13. GARDENING The plans for constructing a raised vegetable garden use corrugated metal in a wooden frame. The finished garden is 4 feet long, 30 inches wide, and 32 inches tall.



a. The metal is only used on the lateral faces, so how many square feet of metal should be purchased? Round to the nearest square foot. 15 ft²

b. How many bags containing 2 cubic feet of soil will be needed to fill the garden if the soil level is 1 inch below the top of the frame? 13 bags

14. TRASH CANS A cylindrical trash can is 30 inches high and has a base radius of 7 inches. A manufacturer wants to know the surface area of this trash can, including the top of the lid. What is the surface area? Round to the nearest square inch. 1627 in²

15. ALGAE A scientist has a fish tank in the shape of a rectangular prism. The tank is 18 inches high, 14 inches wide, and 30 inches long. After one month, the scientist found that the sides and bottom of his fish tank were covered with algae. The scientist wants to run tests on the algae to help determine why it started to grow. How much algae is there for the scientist to test? 2004 in²

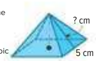
16. GEOLOGY A *tiankeng* is a sinkhole with nearly vertical walls. The Tianpingmiao tiankeng is approximately cylindrical with a diameter of 180 meters and a depth of 420 meters.

a. If the top of the tiankeng is open and plants can grow on the bottom and sides, what is the surface area available for plants? Round to the nearest square meter. 262,951 m²

b. What is the volume of water that could fill the Tianpingmiao tiankeng? 10,687,698 m³

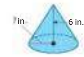
Example 5

17. The model of a roof is in the shape of a square pyramid, as shown. If the surface area of the model is 64 cm², what is the slant height? 3.9 cm



18. A candle is in the shape of a pyramid. The volume of a candle is 27 cubic centimeters and its height is 6 centimeters. Find the area of the base of the candle. 13.5 cm²

19. A disposable cup is in the shape of a cone, as shown. The cup has a volume of about 48.8 in³. What is the radius of the cup to the nearest inch? 3 in.



Mixed Exercises

20. PLANETS For a time, Johannes Kepler thought that the Platonic solids were related to the orbits of the planets. He made models of each of the Platonic solids He made a frame of each of the Platonic solids by fashioning together wooden edges. How many edges did Kepler have to make for the cube? **12**

21. SILO A silo used for storing grain is shaped like a cylinder with a cone on top. The radius of the base of the cylinder and cone is 8 feet. The height of the cylindrical part is 25 feet, and the height of the cone is 6 feet.

a. What is the volume of the cylindrical part of the silo? Round to the nearest cubic foot. **5027 ft^3**

b. What is the volume of the conical part of the silo? Round to the nearest cubic foot. **402 ft^3**

c. What is the volume of the entire silo? Round to the nearest cubic foot. **5429 ft^3**

22. USE A SOURCE Find a real object that can be modeled with one or more three-dimensional figures. Identify the best three-dimensional model and calculate the surface area and volume of the object. **See students' work.**

23. A garden shop sells pyramid-shaped lawn ornaments that each have a base area of 900 square centimeters and a height of 40 centimeters. The lawn ornaments are made of concrete, granite, or marble.

Material	Density (kg/m^3)
Concrete	2371
Granite	2691
Marble	2711

a. What is the volume of one lawn ornament in cubic meters? Explain.
0.012 m^3; $V = \frac{1}{3}Bh$, so $V = \frac{1}{3}(900)(40)$ or 12,000 cm^3. One cubic meter equals 1 million cubic centimeters, so the volume of one pyramid shaped lawn ornament is 0.012 m^3.

b. Find the weights of three of these ornaments that are each made from a different material. Round to the nearest tenth of a kilogram.
concrete: 28.5 kg; granite: 32.3 kg; marble 32.5 kg

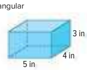
c. What generalization can you make about the relationships among the volume of an ornament, the weight of the lawn ornament, and the density of the material used to make it?
If the volume of the lawn ornament stays the same, then the weight of the ornament increases as the density of the material used to make it increases.

24. REASONING The volume of a new extra large toy tennis ball for pets is about 221 cubic centimeters. If 3 extra large toy tennis balls are packaged and sold in cylindrical package as shown, what is the approximate volume of the cylindrical package? Explain. **See margin.**

Higher-Order Thinking Skills

25. FIND THE ERROR Alex and Sia are calculating the surface area of the rectangular prism shown. Is either of them correct? Explain your reasoning.

Alex	Sia
(5 · 3) · 6 faces	2(5 • 4 • 3)
= 90 in^2	= 120 in^2



Neither; sample answer: The surface area is twice the sum of the areas of the top, front, and left side of the prism or 2(5 • 3 + 5 • 4 + 3 • 4), which is 94 square inches.

26. ANALYZE Consider a pyramid and a prism that have bases that are regular polygons inscribed in a circle. What solid results if the number of sides of the bases is increased infinitely?

The pyramid becomes a cone; the prism becomes a cylinder.

27. WRITE Which solid has a greater volume: cone with a base radius of 7 centimeters and a height of 28 centimeters or a pyramid with base area of 154 square centimeters and height of 28 centimeters? Explain your reasoning.
Sample answer: The cone and pyramid have nearly the same volumes. Cone: The area of the base is approximately 154 square centimeters, so $V = \frac{1}{3}(154)(28)$ or about 1437 cubic centimeters. The volume of the pyramid is greater by such a small amount that we can say that the volumes are approximately equal.

28. CREATE Draw an irregular 14-sided polyhedron that has two congruent bases. **See margin.**

29. PERSEVERE Find the volume of a cube that has a total surface area of 54 square millimeters. **27 mm^3**

30. ANALYZE Is a cube a regular polyhedron? Justify your argument.
Yes; sample answer: All of the faces are regular congruent squares, and all of the edges are congruent.

24. 994 cm^3; substitute 221 for V in the formula $V = \frac{4}{3}\pi r^3$, $221 = \frac{4}{3}\pi r^3$, so $52.8 \approx r^3$ and $r \approx 3.75$. The base of the cylindrical package will have a radius equal to that of the tennis ball, or 3.75 cm. The height of the package will equal the diameter of three tennis balls, or 3[2(3.75)] = 3(7.5) or 22.5 cm. So, the volume of the package is $V = \pi(3.75)^2(22.5)$ or about 994 cubic centimeters.

28.

Lesson 11-6

Two-Dimensional Representations of Three- Dimensional Figures

G.MG.1

LESSON GOAL

Students model three-dimensional figures with two-dimensional representations.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Representing Three-Dimensional Figures

Develop:

Representing Three-Dimensional Figures with Orthographic Drawings

- Make a Model from an Orthographic Drawing
- Make an Orthographic Drawing

Representing Three-Dimensional Figures with Nets

- Use a Net to Find Surface Area
- Identify Platonic Solids
- Draw Nets for Three-Dimensional Figures
- Represent a Real-World Object with Nets

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Three-Dimensional Figures	●	●		●
Extension: Polyhedrons		●	●	●

Language Development Handbook

Assign page 68 of the *Language Development Handbook* to help your students build mathematical language related to modeling three-dimensional figures with two-dimensional representations.

ELL You can use the tips and suggestions on page T68 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.
4 Model with mathematics.
6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students represented three-dimensional figures with nets.
6.G.4

Now
Students model three-dimensional figures with two-dimensional representations.
G.MG.1

Next
Students will identify the shapes of two-dimensional cross sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.
G.GMD.4

Rigor

The Three Pillars of Rigor

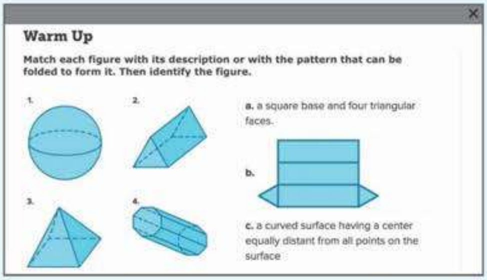
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of solid figures to nets and orthographic drawings, and they apply their understanding to solve real-world problems.

Mathematical Background

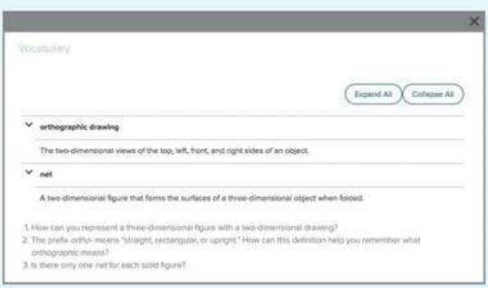
Two-dimensional shapes can be represented using an orthographic drawing or a net. An orthographic drawing shows the top, left, front, and right side of an object. Nets show all surfaces of a three-dimensional figure in one two-dimensional drawing.

Interactive Presentation



Warm Up


Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- identifying three-dimensional figures

Answers:

1. c; sphere
2. b; triangular prism
3. a; square pyramid
4. d; octagonal prism
5. f; pentagonal pyramid
6. e; cube

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of two-dimensional representations of three-dimensional figures.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Representing Three-Dimensional Figures

Objective

Students explore how to represent three-dimensional figures with orthographic drawings.

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this Explore, students will need to use dynamic geometry software. Work with students to explore and deepen their understanding of representing three-dimensional figures with orthographic drawings.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share his or her responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

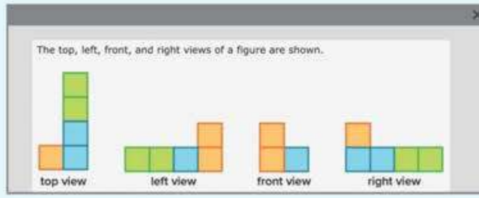
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will explore different viewpoints of a three-dimensional shape as orthographic drawings, and they will use orthographic drawings to identify possible three-dimensional shapes. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore



Explore

TAP

Students tap to explore three-dimensional figures.

MULTIPLE CHOICE

Students select the three-dimensional shape for a given orthographic drawing.

TYPE

Students respond to a question about representing three-dimensional figures.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Representing Three-Dimensional Figures (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why are color-coded blocks easier to visualize when creating a three-dimensional figure from an orthographic drawing? Sample answer: The color-coded blocks make it easier to spot the different views in the three-dimensional object.
- Does a right and a left view need to be given? Explain. No; sample answer: The views are just mirror images of each other, so if one is not given we could create it.

Inquiry

How can you accurately represent a three-dimensional figure with two-dimensional drawings? Sample answer: Visualize the object from various perspectives and draw a sketch of each one.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Representing Three-Dimensional Figures with Orthographic Drawings

Objective

Students identify which orthographic drawings best model three-dimensional geometric figures.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of orthographic drawings in this Learn.

What Students Are Learning

Three-dimensional figures can be represented in two dimensions using orthographic drawings. Two-dimensional views of the top, left, front and right sides of a three-dimensional object are called *orthographic drawings*.

Common Misconception

Students often believe right- and left-side drawings are not both needed even though many objects do not have matching left and right sides. Remind students that orthographic drawings must represent the front, top, left and right sides to be complete.

Example 1 Make a Model from an Orthographic Drawing

Teaching the Mathematical Practices

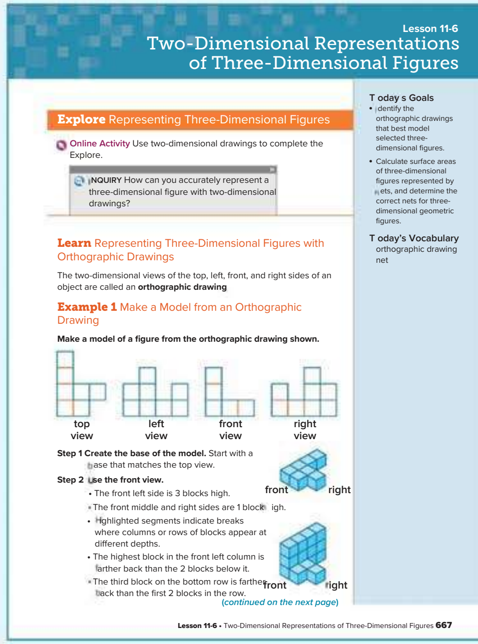
6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL What shape does the front of the three-dimensional figure follow? an L shape

OL How many blocks high is the figure? 3

BL What is the best order in which to consider the two-dimensional views? top, front, left, then right



Lesson 11-6

Two-Dimensional Representations of Three-Dimensional Figures

Explore Representing Three-Dimensional Figures

Online Activity Use two-dimensional drawings to complete the Explore.

INQUIRY How can you accurately represent a three-dimensional figure with two-dimensional drawings?

Learn Representing Three-Dimensional Figures with Orthographic Drawings

The two-dimensional views of the top, left, front, and right sides of an object are called an **orthographic drawing**.

Example 1 Make a Model from an Orthographic Drawing

Make a model of a figure from the orthographic drawing shown.

top view, left view, front view, right view

Step 1 Create the base of the model. Start with a base that matches the top view.

Step 2 Use the front view.

- The front left side is 3 blocks high.
- The front middle and right sides are 1 block high.
- Highlighted segments indicate breaks where columns or rows of blocks appear at different depths.
- The highest block in the front left column is farther back than the 2 blocks below it.
- The third block on the bottom row is farther back than the first 2 blocks in the row.

front, right

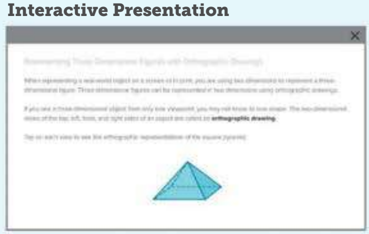
(continued on the next page)

Today's Goals
- Identify the orthographic drawings that best model selected three-dimensional figures.
- Calculate surface areas of three-dimensional figures represented by nets, and determine the correct nets for three-dimensional geometric figures.

Today's Vocabulary
orthographic drawing
net

Lesson 11-6 • Two-Dimensional Representations of Three-Dimensional Figures 667

Interactive Presentation



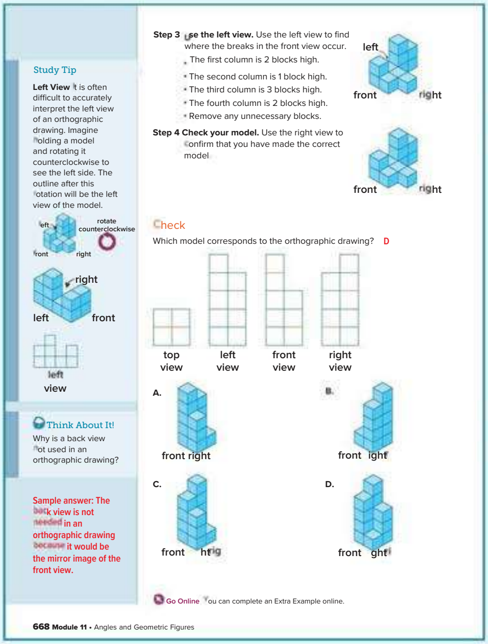
Learn

TAP

Students tap to learn about orthographic drawings.

G.MG.1

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION



Study Tip

Left View It is often difficult to accurately interpret the left view of an orthographic drawing. Imagine holding a model and rotating it counterclockwise to see the left side. The outline after this rotation will be the left view of the model.

Think About It!

Why is a back view not used in an orthographic drawing?

Sample answer: The back view is not needed in an orthographic drawing because it would be the mirror image of the front view.

Step 3 Use the left view. Use the left view to find where the breaks in the front view occur.

- The first column is 2 blocks high.
- The second column is 1 block high.
- The third column is 3 blocks high.
- The fourth column is 2 blocks high.
- Remove any unnecessary blocks.

Step 4 Check your model. Use the right view to confirm that you have made the correct model.

Check

Which model corresponds to the orthographic drawing? D

Go Online You can complete an Extra Example online.

668 Module 11 • Angles and Geometric Figures

Common Error

Students may draw the left side of the three-dimensional figure backwards because they follow the order of the given two-dimensional drawing. Remind students that the left view is reversed and instead of considering the shape from left to right, they should look right to left.

DIFFERENTIATE

AL ELL

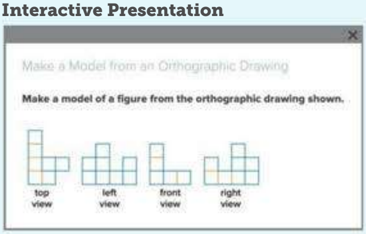
IF students struggle drawing either the orthographic drawing or the three-dimensional object,

THEN have the entire class make the same three-dimensional figure and give students time to draw the four orthographic views. Repeat as many times as needed to help students to master the concept.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Interactive Presentation



Example 1

TAP

Students move through the steps to make a three-dimensional figure from an orthographic drawing.

TYPE

Students explain why a back view is not included in orthographic drawings.

Example 2 Make an Orthographic Drawing

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

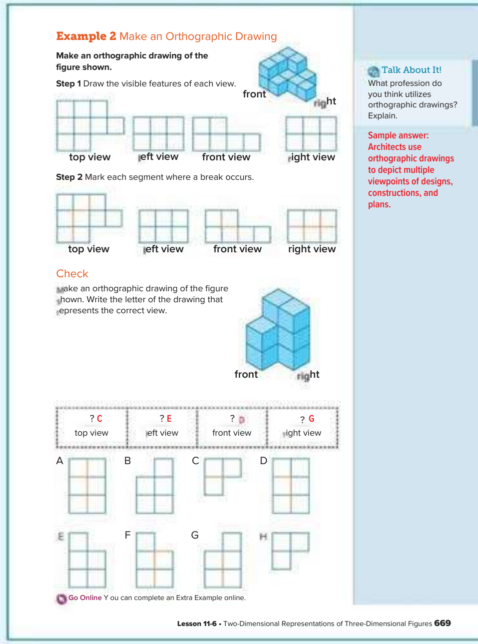
AL What will be the shape of the left and right view of the object? a rectangle

OL How many blocks will be needed for the bottom row of the front view? 4 How many blocks will be needed for the top row of the front view? 2

BL Suppose the figure was made of two identical rows of blocks matching the shape of the current bottom row of the figure. Which view(s) of the orthographic drawing will change? Explain. Front view; sample answer: The location of the orange segments would not be needed in the top view, and they would change in the right view.

Common Error

Students may forget to identify the breaks of the figure on the orthographic drawing. Remind students that after they create the different views, they should identify all breaks.



Example 2 Make an Orthographic Drawing

Make an orthographic drawing of the figure shown.

Step 1 Draw the visible features of each view.

Step 2 Mark each segment where a break occurs.

Check

Make an orthographic drawing of the figure shown. Write the letter of the drawing that represents the correct view.

Go Online You can complete an Extra Example online.

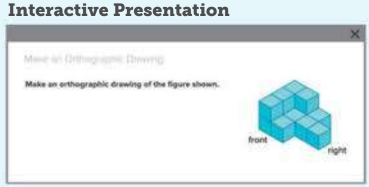
Talk About It!

What profession do you think utilizes orthographic drawings? Explain.

Sample answer: Architects use orthographic drawings to depict multiple viewpoints of designs, constructions, and plans.

Lesson 11-6 • Two-Dimensional Representations of Three-Dimensional Figures 669

Interactive Presentation



Example 2

TAP

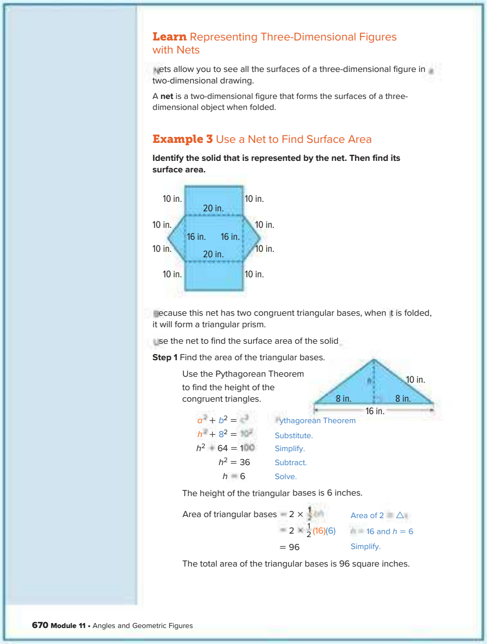
Students move through the steps to make an orthographic drawing.

TYPE

Students answer a question to show that they understand orthographic drawings.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Learn Representing Three-Dimensional Figures with Nets

Nets allow you to see all the surfaces of a three-dimensional figure in a two-dimensional drawing.

A **net** is a two-dimensional figure that forms the surfaces of a three-dimensional object when folded.

Example 3 Use a Net to Find Surface Area

Identify the solid that is represented by the net. Then find its surface area.

Because this net has two congruent triangular bases, when it is folded, it will form a triangular prism.

Use the net to find the surface area of the solid.

Step 1 Find the area of the triangular bases.

Use the Pythagorean Theorem to find the height of the congruent triangles.

$a^2 + b^2 = c^2$ Pythagorean Theorem

$h^2 + 8^2 = 10^2$ Substitute.

$h^2 + 64 = 100$ Simplify.

$h^2 = 36$ Subtract.

$h = 6$ Solve.

The height of the triangular bases is 6 inches.

Area of triangular bases $= 2 \times \frac{1}{2}bh$ Area of 2 $\cong$ △s

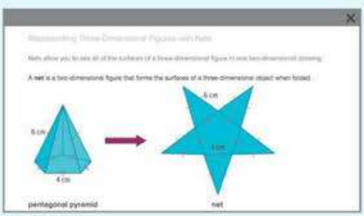
$= 2 \times \frac{1}{2}(16)(6)$ $b = 16$ and $h = 6$

$= 96$ Simplify.

The total area of the triangular bases is 96 square inches.

670 Module 11 • Angles and Geometric Figures

Interactive Presentation



Learn

WEB SKETCHPAD

Students use a sketch to view the net of a prism.

Learn Representing Three-Dimensional Figures with Nets

Objective

Students calculate surface areas of three-dimensional figures represented by nets and determine the correct nets for three-dimensional geometric figures.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that in this Learn, students will need to use a sketch. Work with students to explore and deepen their understanding of representing three-dimensional figures with nets.

What Students Are Learning

Nets show all of the surfaces of a three-dimensional figure in a two-dimensional drawing. If a net is folded, the three-dimensional object is formed.

Common Misconception

Students tend to draw or describe shapes with a lack of precision. If a three-dimensional figure contains congruent sides, then students may not illustrate that relationship when drawing a net. Remind students that nets show the exact relationships of a three-dimensional object as a two-dimensional drawing.

Example 3 Use a Net to Find Surface Area

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL Are the triangles of the triangular prism similar or congruent? congruent

OL How can the net be used to find the surface area of the three-dimensional object? Sample answer: The area of each shape of the net can be found and added together.

BL Check your answer by finding the surface area of the triangular prism. Show all work. $S = Ph + 2B$, the perimeter of the base is 72 inches and the height is 10 inches. The area of each base is $\frac{1}{2}(16)(6) = 48$, so the surface area is $72(10) + 2(48) = 816$ square inches.

DIFFERENTIATE

Enrichment Activity BL

Surface area of a sphere is found by $S = 4\pi r^2$, and the volume of a sphere is found by $V = \frac{4}{3}\pi r^3$. If the measure of a sphere's volume is the same as the measure of its surface area, what is the radius of the sphere? Explain. A sphere with radius 3 has a surface area of 36π square units, and a volume of 36π cubic units.

Common Error

When finding the area of the triangle, students may not find the height of the triangle, or altitude, but instead use the leg length. Remind students that when finding the area of the triangle, the vertical height is a requirement.

Example 4 Identify Platonic Solids

MP Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to construct an argument.

Questions for Mathematical Discourse

AL What shape is each face of the solid? equilateral triangle

OL Suppose the net only had 8 equilateral triangles. What Platonic solid would the net represent? octahedron

BL What shape would comprise the net of a dodecahedron? regular pentagon

Common Error

Students may not know all of the five Platonic solids by name. Encourage students to make a list of the most common solids for easy reference.

Step 2 Find the total surface area of the triangular prism.

$S = 96 + 2(10)(20) + 16(20)$ — Area of triangular bases plus area of three rectangles

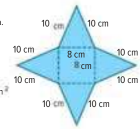
$= 96 + 400 + 320$ or 816 in² — Simplify.

The surface area of the triangular prism is 816 square inches.

Check

Identify the solid that is represented by the net. Then find its surface area.

A. square pyramid; 104 cm²
B. tetrahedron; $64 + 64\sqrt{21}$ cm²
C. tetrahedron; 88 cm²
D. square pyramid; $64 + 32\sqrt{21}$ cm² (circled)

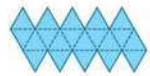


Think About It! If a solid can be represented by more than one net, will the surface area of the solid change? Explain.

No; sample answer: In any net for a given solid, each face of the solid occurs only once. Though the faces may be connected to one another in different ways, they are always connected along the edges of the solid. How the faces are connected in a net does not affect the total surface area.

Example 4 Identify Platonic Solids

Identify the Platonic solid that is represented by the net.

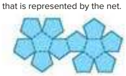


Because this net has 20 equilateral triangles, it represents a net of a(n) icosahedron.

Check

Identify the Platonic solid that is represented by the net.

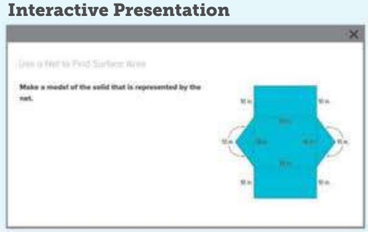
A. octahedron
B. pentagonal prism
C. dodecahedron (circled)
D. icosahedron



Go Online You can complete an Extra Example online.

Lesson 11-6 • Two-Dimensional Representations of Three-Dimensional Figures 671

Interactive Presentation



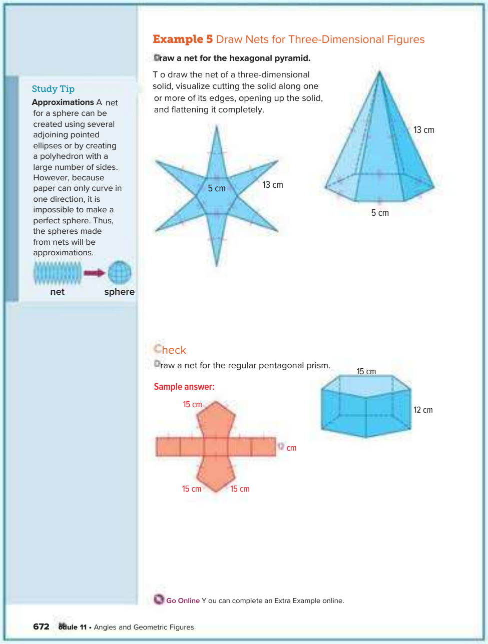
Example 3

TAP

Students move through the slides to see how a net folds into a three-dimensional figure.

TYPE

Students answer a question to show that they understand how to use a net to find surface area.



Study Tip

Approximations A net for a sphere can be created using several adjoining pointed ellipses or by creating a polyhedron with a large number of sides. However, because paper can only curve in one direction, it is impossible to make a perfect sphere. Thus, the spheres made from nets will be approximations.

net sphere

Example 5 Draw Nets for Three-Dimensional Figures

Draw a net for the hexagonal pyramid.

T o draw the net of a three-dimensional solid, visualize cutting the solid along one or more of its edges, opening up the solid, and flattening it completely.

5 cm
13 cm
13 cm
5 cm

Check

Draw a net for the regular pentagonal prism.

Sample answer:

15 cm
12 cm
15 cm
15 cm
15 cm
12 cm

Go Online Y ou can complete an Extra Example online.

672 Module 11 • Angles and Geometric Figures

Interactive Presentation

Example 5

TAP

Students tap to see the pyramid unfold.

Example 5 Draw Nets for Three- Dimensional Figures

Teaching the Mathematical Practices

4 Make Assumptions In the Study Tip, have students point out where an assumption or approximation was made in the solution.

Questions for Mathematical Discourse

AL What shape is the base of the solid? regular hexagon

OL What shape are the faces of the solid? triangles

How many faces does the solid have? 6

BL How would the net change if the solid were a hexagonal prism? Sample answer: The net would have two hexagons and six rectangles.

Common Error

Students may try to place the faces together for the pyramid when drawing the net because of previous examples. Encourage students to watch the illustration of the solid being unfolded to see how the faces are connected to the base.

Example 6 Represent a Real-World Object with a Net

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

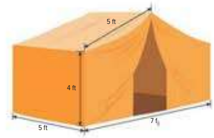
AL What shapes make up the surfaces of the tent? 2 pentagons, 2 squares, and 3 rectangles

OL Which surfaces have the same area? the 2 pentagons (front and back), the 2 squares (roof), and the 2 rectangles (sides)

BL Where else could the two parts of the roof be connected to the net instead of the rectangular sides? Sample answer: Each roof part could be attached to one slanted side of each side pentagon.

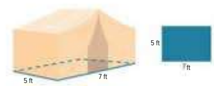
Example 6 Represent a Real-World Object with a Net

TENTS Draw a net to represent the three-dimensional figure that can be used to model the tent.

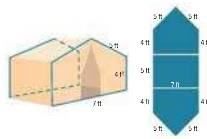


The tent can be modeled by a(n) pentagonal prism.

Step 1 Start by drawing the bottom of the tent.



Step 2 Next, draw the pentagonal bases of the prism in the net. The pentagonal faces will attach to the rectangle at the 7-foot edges.



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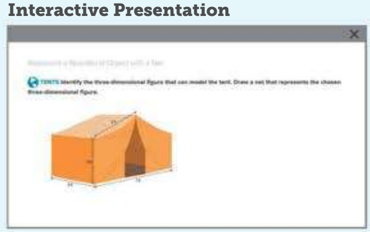
Lesson 11-6 • Two-Dimensional Representations of Three-Dimensional Figures **673**

Interactive Presentation



Example 6

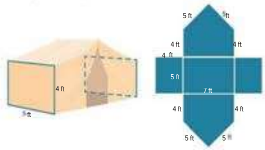
TAP

Students move through the slides to create a net for the prism.

TYPE

Students respond to a question about creating a net.

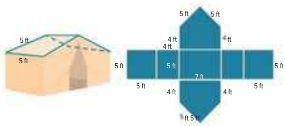
Step 3 Draw the rectangular faces of the prism that represent the side of the tent.



Think About It!
Is there more than one way to correctly draw a net? Explain using the net from Example 6.

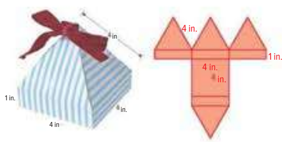
Yes; sample answer: Because the rectangular faces of the tent's roof are also connected to the pentagonal faces, the net can be drawn with these faces being attached to one another.

Step 4 Draw the rectangular faces of the prism that represent the roof of the tent. Compare your net to the original figure to ensure that the dimensions are correct.



Check

GIFT WRAPPING Draw a net to represent the three-dimensional figure that can be used to model the gift box.



Go Online You can complete an Extra Example online.

674 Module 11 • Angles and Geometric Figures

Interactive Presentation

Check

CHECK

Students complete the Check online to determine whether they are ready to move on.

Common Error

Students may not know how to connect the rectangular roof pieces of the tent to the net. Encourage students to visualize taking the tent apart by each shape and seeing what piece it is connected to. The roof shapes have two possible connections.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–22
2	exercises that use a variety of skills from this lesson	23–26
3	exercises that emphasize higher-order and critical-thinking skills	27–32

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

OL BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–25 odd, 27–32
- Extension: Polyhedrons
- ALEKS Solids and Cross Sections

AL OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–31 odd
- Remediation, Review Resources: Three-Dimensional Figures
- Personal Tutors
- Extra Examples 1–6
- ALEKS Three–Dimensional Figures

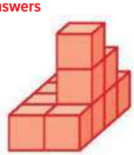
AL

IF students score 65% or less on the Checks,
THEN assign:

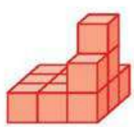
- Practice, Exercises 1–21 odd
- Remediation, Review Resources: Three-Dimensional Figures
- ALEKS Three–Dimensional Figures

Answers

1.



2.



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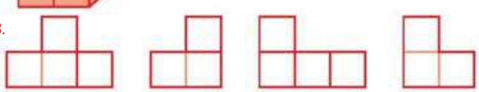


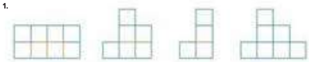
Practice

Go Online Y ou can complete your homework online.

Example 1

Make a model of a figure for each orthographic drawing. 1–2. See margin.

1.



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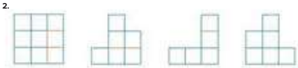


Example 2

Make an orthographic drawing of each figure. 3. See margin. 4–8. See Module 11 Answer Appendix.

3.

4.

5.

6.

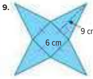
7.

8.

Example 3

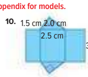
Make a model of the solid that is represented by each net. Then identify the solid and find its surface area. 9–12. See Module 11 Answer Appendix for models.

9.



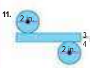
square pyramid, 144 cm^2

10.



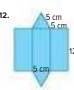
triangular prism; 22.8 cm^2

11.



cylinder; 3.5π in^2 or 11.0 in^2

12.

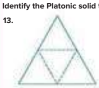


triangular prism, 201.7 cm^2

Example 4

Identify the Platonic solid that is represented by the net.

13.



tetrahedron

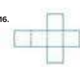
14.

dodecahedron

15.

octahedron

16.

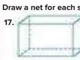


cube or hexahedron

Examples 5 and 6

Draw a net for each solid or object.

17.



See Module 11 Answer Appendix.

18.

See Module 11 Answer Appendix.

Draw a net for each solid or object. 19–22 See margin.

19.


20.

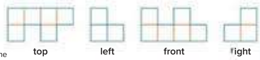

21.


22.


t to b, l to r): Africa Studio/Shutterstock, Nerfy/Shutterstock, Lorie Leigh Lawrence/Alamy Stock Photo, Kelly Redinger/Design Pics

Mixed Exercises

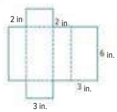
23. GAMING Candela is playing a game that has game pieces. Use the orthographic drawing to make a model of the game piece.
See margin.



24. FURNITURE Make an orthographic drawing to show the top, front, left, and right views of the storage cabinet.
See margin.

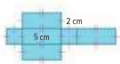
25. MODELING Identify a real-world object that can be represented by the net shown.
Sample answer: ice cream cone

26. GIFT WRAP Olive is wrapping a gift for her sister in a box that has the net shown here. How many square inches of wrapping paper will it take to cover the box?
72 in^2

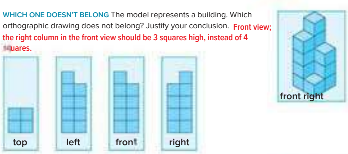


Higher-Order Thinking Skills

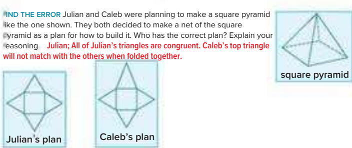
27. ANALYZE Julia knows that a figure has a surface area of 40 square centimeters. The net shown has 5-centimeters and 2-centimeter edges. Could the net represent the figure? Justify your argument.
No; the area of this net is 48 square centimeters.



28. WHICH ONE DOESN'T BELONG The model represents a building. Which orthographic drawing does not belong? Justify your conclusion. Front view; the right column in the front view should be 3 squares high, instead of 4 squares.



29. FIND THE ERROR Julian and Caleb were planning to make a square pyramid like the one shown. They both decided to make a net of the square pyramid as a plan for how to build it. Who has the correct plan? Explain your reasoning. Julian; All of Julian's triangles are congruent. Caleb's top triangle will not match with the others when folded together.

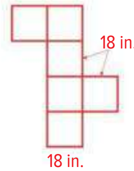


30. CREATE Adriana works for a company called Boxes R Us making different sizes and shapes of boxes for packages. Adriana's boss wants her to sketch nets of one of the boxes. Sketch a possible net that Adriana could have drawn. See margin.

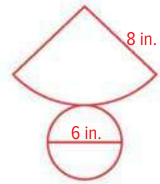
31. WRITE Describe the similarities and differences in orthographic drawings and nets. See margin.

32. PERSEVERE How many Platonic solids are there? Give a description of each solid that includes the number of two-dimensional shapes that meet at each vertex, number of faces, number of vertices, and number of edges. See margin.

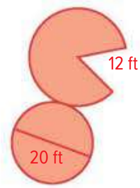
19. Sample answer:



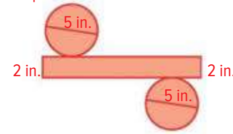
20. Sample answer:



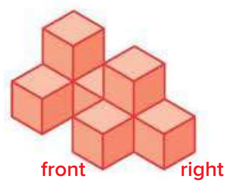
21. Sample answer:



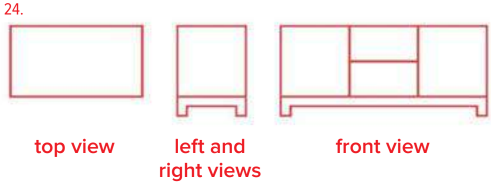
22. Sample answer:



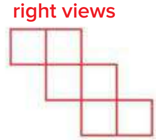
23.



24.



30. Sample answer:



31. Orthographic drawings and nets are both two-dimensional shapes used to describe three-dimensional figures. Orthographic drawings are views of the top, left, front, and right sides of an object, whereas nets can be folded to create a three-dimensional object.

32. There are 5 Platonic solids. A tetrahedron has 3 triangles that meet at each vertex, 4 faces, 4 vertices, and 6 edges. A cube has 3 squares that meet at each vertex, 6 faces, 8 vertices, and 12 edges. An octahedron has 4 triangles that meet at each vertex, 8 faces, 6 vertices, and 12 edges. A dodecahedron has 3 pentagons that meet at each vertex, 12 faces, 20 vertices, and 30 edges. An icosahedron has 5 triangles that meet at each vertex, 20 faces, 12 vertices, and 30 edges.

Lesson 11-7

Precision and Accuracy

N.Q.3

LESSON GOAL

Students apply the definitions of precision, accuracy, and error to measurements and computed values.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Precision and Accuracy in Basketball

Develop:

Precision and Accuracy
- Identify Precision and Accuracy

Approximate Error
- Find Approximate Error

Calculating with Rounded Measurements
- Calculate with Rounded Measurements

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Expressions Involving Absolute Value	●	●		●
Extension: Comparing Precision: Metric and Customary Measurements		●	●	●

Language Development Handbook

Assign page 69 of the *Language Development Handbook* to help your students build mathematical language related to applying the defintions of precisions, accuracy, and error to measurements and computed values.

ELL You can use the tips and suggestions on page T69 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Number and Quantity

Standards for Mathematical Content:

N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students calculated percent error.
7.RP.3

Now
Students apply the definitions of precision, accuracy, and error to measurements and computed values.
N.Q.3

Next
Students will use significant figures in measurements.
N.Q.3

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of precision and accuracy. They apply their understanding by determining the levels of precision and accuracy in real-world scenarios.

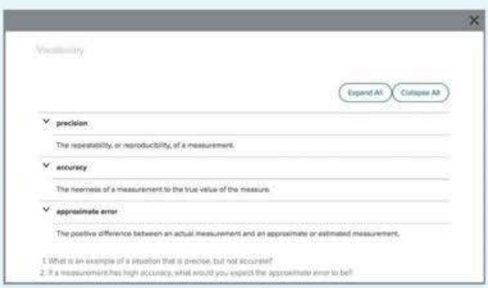
Mathematical Background

All measurements are approximations. Two main factors of approximation are precision and accuracy. Precision refers to the repeatability or reproducibility of a group of measurements. Accuracy refers to the nearness of a measured value to the actual or desired value. The positive difference between an actual measurement and an approximate measurement is called *approximate error*.

Interactive Presentation

Warm Up

Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- evaluating expressions with absolute value

Answers:

1. 8
2. 5
3. $3\frac{2}{5}$
4. 19
5. 2

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of accuracy and precision of measurements

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Precision and Accuracy in Basketball

Objective

Students apply the definitions of precision and accuracy in a real-world setting.

Teaching the Mathematical Practices

5 Use Appropriate Tools Strategically Throughout the Explore, encourage students to use the necessary tools, including the sketch, to explore the concepts of precision and accuracy.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

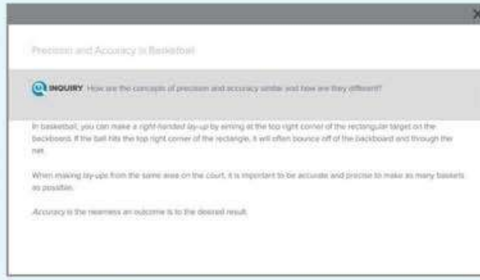
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use different illustrations to understand the concept of precision and accuracy. Students will use a sketch to explore precision and accuracy. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

Explore

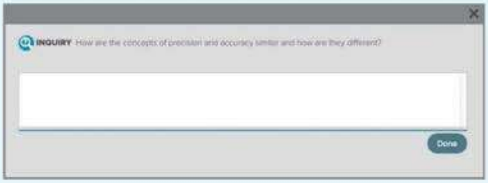
WEB SKETCHPAD

Students use a sketch to explore precision and accuracy.

TYPE

Students answer guiding exercises about precision and accuracy.

Interactive Presentation



Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Precision and Accuracy in Basketball (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- If the player always hits the left corner of the backboard, but misses the shot, does this description relate to accuracy and precision? Sample answer: The player has high precision but low accuracy.

Inquiry

How are the concepts of precision and accuracy similar and how are they different? Sample answer: Precision and accuracy both represent the ability of someone to perform consistently relative to a given goal. Precision describes the repeatability of a result, regardless of how close to the goal someone is. Accuracy describes how close someone is to achieving the desired goal.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Precision and Accuracy

Objective

Students determine the level of accuracy in real-world scenarios.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Important to Know

Accuracy is the nearness of a measurement to the true value of the measure. On a target accuracy is represented by the marks being close to the bulls-eye.

Common Misconception

Students may not believe that there is a difference between precision and accuracy. Remind students that precision is how close the measured values are to each other, and that accuracy is how close measured values are to achieving the desired goal.

DIFFERENTIATE

Language Development AL ELL

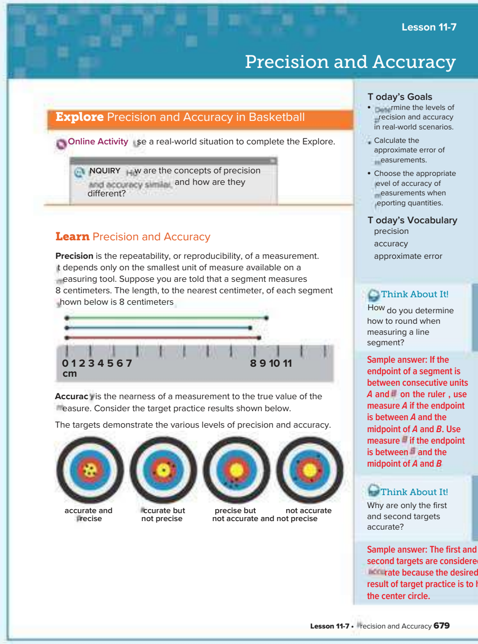
IF students do not know how to distinguish between accuracy and precision, **THEN** give students this easy way to remember the difference between accuracy and precision:

ACCura**C**y is **C**orrect (or **C**lose to the real value)

PRecision is **R**epeating (or **R**epeatable)

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 11-7

Precision and Accuracy

Today's Goals
- Determine the levels of precision and accuracy in real-world scenarios.
- Calculate the approximate error of measurements.
- Choose the appropriate level of accuracy of measurements when reporting quantities.

Today's Vocabulary
precision
accuracy
approximate error

Explore Precision and Accuracy in Basketball

Online Activity Use a real-world situation to complete the Explore.

INQUIRY How are the concepts of precision and accuracy similar and how are they different?

Learn Precision and Accuracy

Precision is the repeatability, or reproducibility, of a measurement. It depends only on the smallest unit of measure available on a measuring tool. Suppose you are told that a segment measures 8 centimeters. The length, to the nearest centimeter, of each segment shown below is 8 centimeters.

Accuracy is the nearness of a measurement to the true value of the measure. Consider the target practice results shown below.

The targets demonstrate the various levels of precision and accuracy.

Think About It! How do you determine how to round when measuring a line segment?

Sample answer: If the endpoint of a segment is between consecutive units A and B on the ruler, use measure A if the endpoint is between A and the midpoint of A and B. Use measure B if the endpoint is between B and the midpoint of A and B.

Think About It! Why are only the first and second targets accurate?

Sample answer: The first and second targets are considered accurate because the desired result of target practice is to hit the center circle.

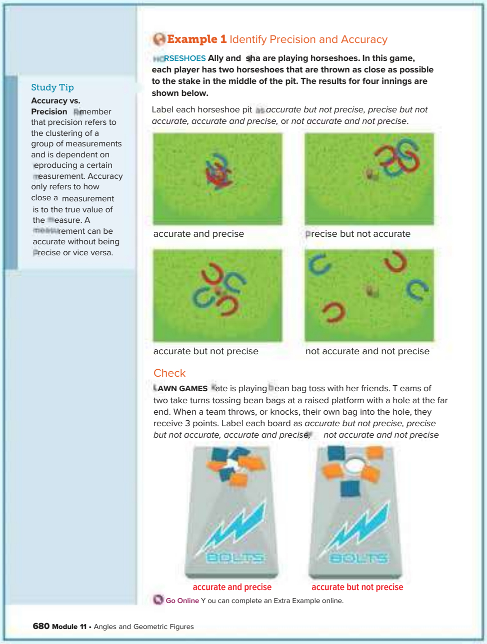
Lesson 11-7 • Precision and Accuracy 679

Interactive Presentation

Learn

TYPE

Students respond to a question about precise measurements.



Study Tip

Accuracy vs. Precision Remember that precision refers to the clustering of a group of measurements and is dependent on reproducing a certain measurement. Accuracy only refers to how close a measurement is to the true value of the measure. A measurement can be accurate without being precise or vice versa.

Example 1 Identify Precision and Accuracy

HORSESHOES **Ally and sha are playing horseshoes. In this game, each player has two horseshoes that are thrown as close as possible to the stake in the middle of the pit. The results for four innings are shown below.**

Label each horseshoe pit as *accurate but not precise, precise but not accurate, accurate and precise,* or *not accurate and not precise*.

accurate and precise

precise but not accurate

accurate but not precise

not accurate and not precise

Check

LAWN GAMES Kate is playing bean bag toss with her friends. T eams of two take turns tossing bean bags at a raised platform with a hole at the far end. When a team throws, or knocks, their own bag into the hole, they receive 3 points. Label each board as *accurate but not precise, precise but not accurate, accurate and precise,* or *not accurate and not precise*

accurate and precise

accurate but not precise

Go Online Y ou can complete an Extra Example online.

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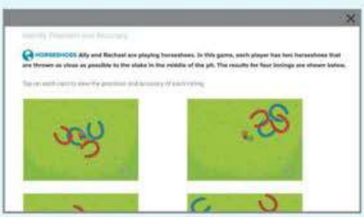
Interactive Presentation



Example 1

FLASHCARDS

Students flip cards to view the precision and accuracy of each horseshoe toss.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 1 Identify Precision and Accuracy

Teaching the Mathematical Practices

2 Different Properties Mathematically proficient students look for different ways to solve problems. Encourage them to work through both ways to solve the problem and to choose the method that works best for them.

Questions for Mathematical Discourse

AL What is the goal when playing horseshoes? Throw the horseshoe closest to the stake in the middle of the pit.

OL When considering how close the horseshoes are to the stake, does that describe accuracy or precision? Explain. Accuracy; sample answer: Accuracy is the nearness to the goal, and because the goal is to hit the stake, this would describe accuracy.

BL If you were playing horseshoes, would you want accuracy, precision, or both? Explain. Both; sample answer: Accuracy will put me close to the stake and precision will put my horseshoes closer together, so both will put my horseshoes close together at the stake.

Common Error

Students may forget the critical step of finding the absolute value of the difference. This could lead to negative answers, which are incorrect. Remind students that approximate error represents the difference in the actual and estimate measurements, regardless of direction.

Common Error

Many students interchange the definition of accuracy with precision. Encourage students to write the definitions down with an example so they can refer back to them when in doubt.

Learn Approximate Error

Objective

Students calculate the approximate error of measurements.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

About the Key Concept

Measurements are always approximations, which means they have error. The approximate error of a measurement helps determine how accurate the calculation will be using the measurement. The formula is the absolute value of the difference between the actual measurement and the estimated measurement.

Common Misconception

Students often believe the measurement with the most decimals is the actual measurement, not the estimate. Remind students that the actual measurement is the true weight, length, height, etc. of an object while the estimate is the measurement taken.

Example 2 Find Approximate Error

Teaching the Mathematical Practices

6 Use Precision Students use approximate error as a means to evaluate the precision of measurements.

Questions for Mathematical Discourse

AL What is the actual weight of the mass? 10 grams

OL Suppose the mass were weighed again on the spring scale, and the approximate error was found to be 0.7 grams. What could be the possible weight reported by the spring scale? Either 10.07 grams or 9.93 grams

BL Suppose a different scale reported the object as 9.7 grams. Why will the approximate error be the same as the food scale? Sample answer: Because both scales are measuring 0.3 grams away from the actual weight, they have the same approximate error. Approximate error is always the absolute value of the difference, so the negative does not matter.

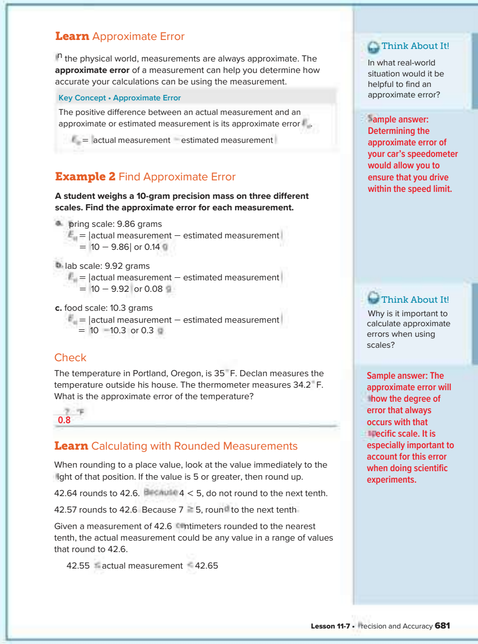
Learn Calculating with Rounded Measurements

Objective

Students choose the appropriate level of accuracy of measurements when reporting quantities.

Teaching the Mathematical Practices

2 Reason Abstractly and Quantitatively Students will make sense of quantities and their relationships in problem situations.
6 Attend to Precision Guide students to express numerical answers with a degree of precision appropriate for the problem context.



Learn Approximate Error

In the physical world, measurements are always approximate. The **approximate error** of a measurement can help you determine how accurate your calculations can be using the measurement.

Key Concept • Approximate Error

The positive difference between an actual measurement and an approximate or estimated measurement is its approximate error E_a.

$E_a = |\text{actual measurement} - \text{estimated measurement}|$

Example 2 Find Approximate Error

A student weighs a 10-gram precision mass on three different scales. Find the approximate error for each measurement.

a. spring scale: 9.86 grams
$E_a = |\text{actual measurement} - \text{estimated measurement}|$
$= |10 - 9.86|$ or 0.14 g

b. lab scale: 9.92 grams
$E_a = |\text{actual measurement} - \text{estimated measurement}|$
$= |10 - 9.92|$ or 0.08 g

c. food scale: 10.3 grams
$E_a = |\text{actual measurement} - \text{estimated measurement}|$
$= |10 - 10.3|$ or 0.3 g

Check

The temperature in Portland, Oregon, is 35°F. Declan measures the temperature outside his house. The thermometer measures 34.2°F. What is the approximate error of the temperature?

? °F
0.8

Learn Calculating with Rounded Measurements

When rounding to a place value, look at the value immediately to the right of that position. If the value is 5 or greater, then round up.

42.64 rounds to 42.6. Because 4 < 5, do not round to the next tenth.

42.57 rounds to 42.6. Because 7 ≥ 5, round to the next tenth.

Given a measurement of 42.6 centimeters rounded to the nearest tenth, the actual measurement could be any value in a range of values that round to 42.6.

$42.55 \le \text{actual measurement} < 42.65$

Think About It!
In what real-world situation would it be helpful to find an approximate error?

Sample answer: Determining the approximate error of your car's speedometer would allow you to ensure that you drive within the speed limit.

Think About It!
Why is it important to calculate approximate errors when using scales?

Sample answer: The approximate error will show the degree of error that always occurs with that specific scale. It is especially important to account for this error when doing scientific experiments.

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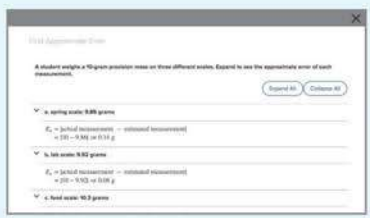
Interactive Presentation



Example 2

EXPAND

Students expand each example to see how approximate error is calculated.

TYPE

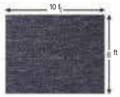
Students explain why approximate error should be calculated when using scales.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Calculate with Rounded Measurements

CARPETING **Alejandro wants to carpet his bedroom. He measures the dimensions of his bedroom and rounds to the nearest foot. The carpet he chose costs $2.63 per square foot.**



Part A What is the possible range for how much it will cost to carpet Alejandro's bedroom?

Step 1 Find the possible range for the area of the room.

$9.5 \text{ feet} \le \text{actual length} < 10.5 \text{ feet}$

$7.5 \text{ feet} \le \text{actual width} < 8.5 \text{ feet}$

least possible area $= 9.5 \cdot 7.5$ or 71.25 ft^2
greatest possible area $= 10.5 \cdot 8.5$ or 89.25 ft^2

The area is at least 71.25 square feet but less than 89.25 square feet.

Step 2 Find the cost to buy carpet for the room.

cost for least possible area: $71.25 \cdot \$2.63 = \$187.39$
cost for greatest possible area: $89.25 \cdot \$2.63 = \$234.73$

The cost would be at least $187.39 but less than $234.73.

Part B Alejandro checks the dimensions of the room and measures it to be 9.8 feet by 8.2 feet to the nearest tenth of a foot. How does this change the range for the cost of the carpeting?

Step 1 Find the possible range for the area of the room.

$9.75 \text{ feet} \le \text{actual length} < 9.85 \text{ feet}$

$8.15 \text{ feet} \le \text{actual width} < 8.25 \text{ feet}$

least possible area $= 9.75 \cdot 8.15$ or 79.4625 ft^2
greatest possible area $= 9.85 \cdot 8.25$ or 81.2625 ft^2

Step 2 Find the cost to buy carpet for the room.

cost for least possible area: $79.4625 \cdot \$2.63 = \$208.99$
cost for greatest possible area: $81.2625 \cdot \$2.63 = \$213.72$

The cost would be at least $208.99 but less than $213.72.

When the measurements were rounded to the nearest foot, the range of costs was more than $45. With the measurements rounded to the nearest tenth, the range of costs is about $4.75. Rounding to the nearest tenth creates a more accurate range for the cost of the carpeting.

Talk About It!
How would a recorded time be affected in a stopwatch rounded to the nearest second versus to the nearest millisecond?

Sample answer: The time would be more accurate in the stopwatch rounded to the nearest millisecond.

lucato/Getty Images

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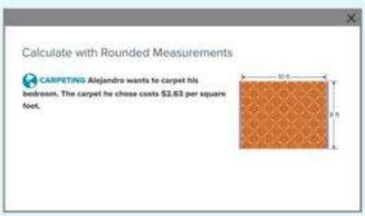
Interactive Presentation



Example 3

TAP

Students tap through the steps to solve the problem.

TYPE

Students respond to a question about rounded measurements of time.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Example 3 Calculate with Rounded Measurements

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about accuracy of measurements to solving a real-world problem.

Questions for Mathematical Discourse

AL How can you find the area of the room? length times width

OL When calculating the area of the room, why do we only find the least and greatest possible area? Sample answer: We only find the least and greatest because all other values will fall in-between those two values. Knowing the least and greatest allows us to find the smallest and largest price.

BL Suppose Alejandro has a friend who works in construction measure his room, and the dimensions were found to be 9.76 feet by 8.16. What is the possible range for the area of the room now? $9.755 \le$ actual length < 9.765 while $8.155 \le$ actual width < 8.165; Least possible area: $9.755(8.155) = 79.552025 \text{ ft}^2$; Greatest possible area: $9.7655(8.165) = 79.7353075 \text{ ft}^2$

Common Error

Students may not understand how to calculate the range of values for the length and width of the room. Encourage students to plot the reported length on a number line scaled with the same precision as the number. Then students can mark half a unit below and half a unit above to see the range.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students have a hard time rounding,
THEN have them plot the number on an appropriately scaled number line. Students can then see to which digit the number is closer. For example, if the number 31.42 is to be rounded to the nearest tenth, scale the number line from 31 to 32 by tenths. Then students can see 31.42 is close to 31.4 than 31.5.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–7
2	exercises that use a variety of skills from this lesson	8–16
3	exercises that emphasize higher-order and critical-thinking skills	17–20

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: OL BL

- Practice, Exercises 1–15 odd, 17–20
- Extension: Comparing Precision: Metric and Customary Measurements

IF students score 66%–89% on the Checks, **THEN** assign: AL OL

- Practice, Exercises 1–19 odd
- Remediation, Review Resources: Expressions Involving Absolute Value
- Personal Tutors
- Extra Examples 1–3
- ALEKS Evaluating Expressions with Absolute Value

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–7 odd
- Remediation, Review Resources: Expressions Involving Absolute Value
- ALEKS Evaluating Expressions with Absolute Value

Practice

Go Online You can complete your homework online.

Example 1

1. PRECISION A manufacturer claims that its rice cakes are packaged with 20 in each package. A sample of 12 packages is counted for accuracy. The sample yields a count of {18, 17, 17, 17, 18, 18, 18, 17, 18, 17, 18, 17} rice cakes. How accurate and precise is the manufacturer's claim? Explain your reasoning. **The sample is precise because there are consistently 17 or 18 rice cakes in each package. The sample indicates an inaccurate claim of 20 rice cakes per package.**

Example 2

2. SCALES A 10-pound weight is weighed on two different scales. Find the approximate error of each weight.
   a. digital bathroom scale: 9.59 pounds **0.41 lb**
   b. food scale: 10.09 pounds **0.09 lb**

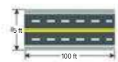
3. PHYSICS A circuit has amperage of 0.01 milliamp. A multimeter measures the amperage of the circuit at 0.06 milliamp. What is the approximate error? **0.05 milliamp**

4. CARPENTRY A door frame is 2.13 meters high. Chandra measures the height of the door frame with a carpenter's rule. She measures 2.22 meters. What is the approximate error of the height? **0.09 m**

5. COOKING Water boils at 212.0°F. Jeremiah uses a kitchen thermometer to measure the temperature of a pot of boiling water. The thermometer measures 213.1°F. What is the approximate error of the temperature? **1.1°F**

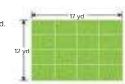
Example 3

6. CONSTRUCTION The public works department is repaving some of the roads in the city. The materials needed to repave this section of the road cost $2.50 per square foot.



   a. What is the possible range for the area of the road? **least possible area = 4427.75 ft²; greatest possible area = 4572.75 ft²**
   b. What is the possible range for the cost, c, of the materials needed to repave this section of the road? **$11,069.38 ≤ c < $11,431.88**

7. GRASS SEED George is buying grass seed for his lawn. Grass seed is sold at $0.40 per square yard.



   a. What is the least value for the length of the lawn? **16.5 yd**
   b. What is the greatest value for the width of the lawn? **12.5 yd**
   c. What is the possible range for the area of the lawn? **least possible area = 189.75 yd²; greatest possible area = 218.75 yd²**
   d. What is the possible range for the cost of the grass seed? **The cost would be at least $75.90 but less than $87.50.**

Mixed Exercises

8. THERMOMETER The thermostat on a heated pool is set at 76.5°F. A thermometer in the pool is shown. What is the approximate error of the temperature? **0.6°F**

9. SANDWICHES A sandwich shops claims to sell foot-long sandwiches. Xiao uses a ruler to measure her sandwich. The ruler measures 11 inches. What is the approximate error of the length? **1 in.**

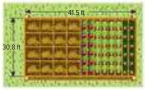
10. SPEED A police officer uses a radar detector to measure the speed of Roya's car. Roya's speedometer reads 55 miles per hour. The radar detector measures her speed at 56.71 miles per hour. What is the approximate error of the speed? **1.71 mph**

11. BICYCLES An assembly line supervisor weighs three 25-pound bicycle frames on a scale. Find the approximate error of each weight.
   a. Bicycle A: 25.11 pounds **0. 11 lb**
   b. Bicycle B: 24.99 pounds **0.01 lb**
   c. Bicycle C: 24.36 pounds **0.64 lb**

12. DELI Josephina works at a deli. She is testing the scales at the deli to make sure they are accurate. She uses a weight that is exactly 1 pound and gets the following results shown in the table. Which scale is the most accurate? **scale 2**

Scale	Weight (lb)
1	1.013
2	1.0[illegible]
3	0.97

13. GARDEN Mr. Granger wants to spread fertilizer on his vegetable garden that has dimensions 41.5 feet by 30.8 feet. The fertilizer he chose costs $0.75 per square foot for adequate coverage. What is the possible range for the cost of the fertilizer that is needed to cover the vegetable garden?
The cost would be at least $955.94 but less than $961.36.



14. HEIGHT Lucas was proud of how much he had grown over the last six months since his grandma had seen him last. He told her that he was 6 feet 3 inches. His grandma didn't believe him, so she measured him again, and he was 6 feet 1 inch. What is the approximate error of Lucas' height? 2 in.

15. PAINT Y ou measure a wall of your room as 8 feet high and 12 feet wide. Y ou want to apply wallpaper to only this wall. The wallpaper is expensive and will cost $1.25 per square foot. What is the possible range for the cost of the wallpaper?
The cost would be at least $107.81 but less than $132.81.

16. Four measurements were taken three different times. The correct measurement is 52.4 cm. Determine whether the set of measurements is *accurate*, *precise*, *both*, or *neither*. Explain your reasoning.

a. 56.1 cm, 48.9 cm, 24.2 cm, 5 cm This set is neither accurate nor precise. The measures are not close to each other, and they are not close to the correct value.

b. 73.1 cm, 74.0 cm, 73.5 cm, 73.7 cm This set is precise. All the measures are very close together. They are not, however, close to the correct value, so the set is not accurate.

c. 52.6 cm, 52.5 cm, 52.2 cm, 52.3 cm This set is both accurate and precise. All the measures are close to each other, and they are close to the correct value.

Higher-Order Thinking Skills

17. WRITE Many people confuse the definitions of accuracy and precision. What is the difference between accuracy and precision? Give an example of a set of four numbers that represents accurate and precise measurements for a cut of meat at a steakhouse that advertises a 16-ounce ribeye steak special on Tuesday nights. Accuracy is how well the information or data matches the true values. Precision is the repeatability of the measurement and level of measurement. Sample answer: 16. 1 oz, 16.3 oz, 15.93 oz, 15.8 oz.

18. WRITE Isabel says that if a set of measurements is accurate, then it is also precise. If you agree, explain your reasoning. If you disagree, provide a counterexample. Sample answer: A set of measurements may be accurate and close to the actual measurement of 0.5 inches, for example. However, the set of measurements may not also be precise if they are written using varying place values.

19. PERSEVERE Jayden measures and labels the dimensions of a box.

a. Calculate the areas of the faces of the box. There are two faces that have an area of 111.4 in^2, two faces that have an area 35.82 in^2, and two faces that have an area 75.3 in^2.

b. Determine the surface area of the box. 445.0 in^2

c. Determine the range of values that should contain the actual (true) measure of the surface area of the box. Explain your reasoning. The calculation of surface area is accurate to the nearest tenth. The true surface area falls between 444.95 in^2 and 445.05 in^2.

d. Suppose that Jayden had incorrectly measured the first dimension as 15.1 inches. Find the surface area of the box using this measure. 440. 1 in^2

20. CREATE A manufacturer claims that its bags of sweetener contain 9.7 ounces in each bag. Create a sample of weights of 10 bags of sweetener such that the sample is precise and accurate. Explain your reasoning. See margin.

Answers

20. Sample answer: {9.8, 9.7, 9.7, 9.6, 9.6, 9.8, 9.7, 9.7, 9.6, 9.8}; The set is precise because there are consistently 9.6, 9.7, or 9.8 ounces of sweetener in each bag. The sample is also accurate to their claim of 9.7 ounces per bag.

Representing Measures

N.Q.3

LESSON GOAL

Students use significant figures in measurements.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Significant Figures

Develop:

Determining Significant Figures
- Determine Significant Figures
- Find Significant Figures by Using Tools

Calculating with Significant Figures
- Calculate with Significant Figures
- Use Significant Figures in the Real World
- Use Tools to Calculate Measurements

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Convert Customary Measurement Units	●	●		●
Extension: While Loops		●	●	●

Language Development Handbook

Assign page 70 of the *Language Development Handbook* to help your students build mathematical language related to significant figures.

ELL You can use the tips and suggestions on page T70 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Number and Quantity

Standards for Mathematical Content:

N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.
5 Use appropriate tools strategically.
6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students applied the definitions of precision, accuracy, and error to measurements and computed values
N.Q.3

Now
Students use significant figures in measurements.
N.Q.3

Next
Students will analyze conjectures by using inductive reasoning and disprove conjectures by using counterexamples.

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

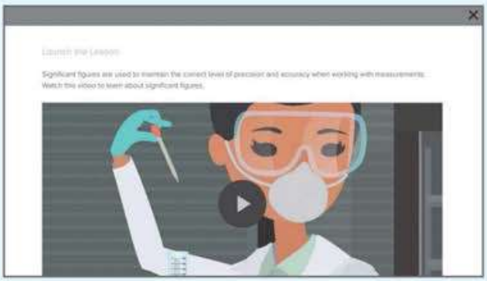
Conceptual Bridge In this lesson, students draw on their understanding of precision and accuracy. They apply their understanding by determining the correct numbers of significant figures in recorded measurements.

Mathematical Background

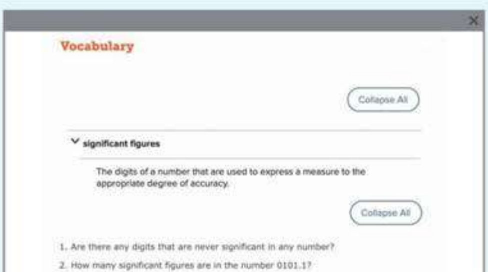
The digits that are used to express a measure to the appropriate degree of accuracy are called *significant figures*. The rules to determine whether digits are considered significant are: nonzero digits are always significant; in whole numbers, zeros are significant if they fall between nonzero digits; in decimal numbers greater than or equal to 1, every digit is significant; in decimal numbers less than 1, the first nonzero digit and every digit to the right are significant.

Interactive Presentation

Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- converting measurements

Answers:

1. 4
2. 5000
3. 0.08
4. 2; 1
5. 13; 1

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see real-world applications of significant figures in measurements.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then, discuss the questions below with the class.

Explore Significant Figures

Objective

Students explain the level of accuracy of measurements and choose the level of accuracy appropriate for measurements.

MP Teaching the Mathematical Practices

1 Understand the Approaches of Others Mathematically proficient students can explain the methods used to solve a problem. This Explore asks students to justify a step.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

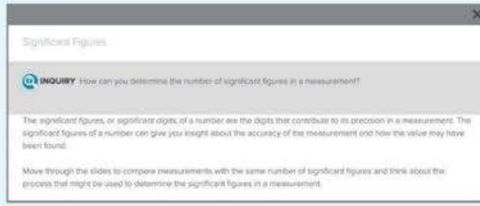
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

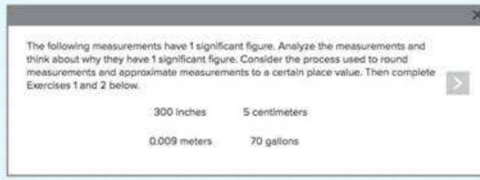
Students will complete guiding exercises throughout the Explore activity. They will investigate different measurements and the number of significant figures in each measurement. Students will answer questions to help find the process that might be used to determine the significant figures in a measurement. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

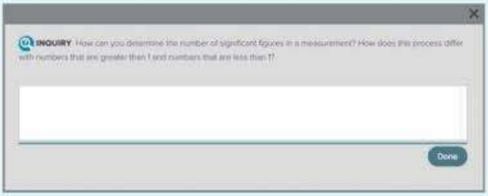
TAP

Students move through the steps to Explore significant figures.

TYPE

Students answer guiding exercises about significant figures.

Interactive Presentation



Explore

TYPE

a|

Students respond to the Inquiry Question and can view a sample answer.

Explore Significant Figures (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Which measurement only has one significant figure: 1.0 centimeter, 0.30 gram, or 0.08 liter? 0.08 liter
- How does the presence of a decimal point affect significant figures? Sample answer: The presence of a decimal point can indicate a greater level of accuracy, and therefore the existance of more significant figures in a measurement.

Inquiry

How can you determine the number of significant figures in a measurement? Sample answer: To determine the number of significant figures in a measurement, you must identify the place value to which the measurement is accurate. When the measurement is greater than 1, count the number of digits including and to the left of the place value you previously identified. When the measurement is less than 1, count the number of digits including and to the left of the place value you identified, but do not count zeros that are to the left of the place value and act as placeholders.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Determining Significant Figures

Objective

Students determine the correct number of significant figures in recorded measurements.

MP Teaching the Mathematical Practices

6 Use Precision In this Learn, students learn how to calculate accurately and efficiently and to express numerical answers with a degree of precision appropriate to the problem context.

About the Key Concept

Significant figures allow us to maintain the correct level of precision when working with measurements. Nonzero digits are always significant. In whole numbers, zeros are significant when they are between nonzero digits. In decimal numbers greater than or equal to 1, every digit is significant. However in decimal numbers less than 1, the first nonzero digit and every digit to the right are significant.

Common Misconception

Students often count zero as a significant digit without considering the value of the number or the presence of a decimal. Remind students that zero is a special digit and that there are only certain times it is considered significant.

Essential Question Follow-Up

Students have begun learning about significant figures.

Ask:

Why are significant figures important in the scientific field? Sample answer: Scientists need a precise way of reporting measurements based on the tool used, so that others know how precise the measurement is.

Example 1 Determine Significant Figures

MP Teaching the Mathematical Practices

3 Compare Arguments Mathematically proficient students can compare arguments, determine which one is flawed, and explain the flaw. In this example, students have to identify the flawed argument and choose the correct one.

Questions for Mathematical Discourse

- **AL** How many significant figures are in 1500 inches? 2
- **OL** How many significant figures are in 1501 inches? 4
- **BL** How many significant figures are in 0.00500? 3

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 11-8

Representing Measurements

Explore Significant Figures

Online Activity Use the guiding exercises to complete the Explore.

INQUIRY How can you determine the number of significant figures in a measurement?

Learn Determining Significant Figures

Using significant figures allows you to maintain the correct level of precision when you are working with measurements. The **significant figures**, or *significant digits*, of a number are the digits that are used to express a measure to the appropriate degree of accuracy.

Key Concept • Significant Figures

Rules	Examples
Nonzero digits are always significant.	214 **3 significant figures: 2, 1, and 4**
In whole numbers, zeros are significant if they fall between nonzero digits.	5078 **4 significant figures: 5, 0, 7, and 8**
In decimal numbers greater than or equal to 1, every digit is significant.	7.60 **3 significant figures: 7, 6, and 0**
In decimal numbers less than 1, the first nonzero digit and every digit to the right are significant.	0.029 **2 significant figures: 2 and 9**

Example 1 Determine Significant Figures

Determine the number of significant figures in each measurement.

0.0320 inches

This is a decimal number less than 1. The first nonzero digit is 3, and there are two digits to the right of 3; 2 and 0. So, this measurement has 3 significant figures.

107,000 centimeters

Because this is a whole number, zeros are only significant if they fall between nonzero digits. There is one zero that falls between 1 and 7. So, this measurement has 3 significant figures.

Today's Goals

- Determine the correct number of significant figures in recorded measurements.
- Round measurements to the correct number of significant figures.

Today's Vocabulary

significant figures

Watch Out!

Look for the Decimal If a number has no decimal place, then zeros are only significant if they fall between nonzero digits. For example, 165,000 has only 3 significant figures because all three zeros fall after the 5. If a number does have a decimal place, then follow the rules listed.

Talk About It!

What is the purpose of using significant figures?

Sample answer: We use significant figures because a calculation can never be more accurate than the measurements that are used in it.

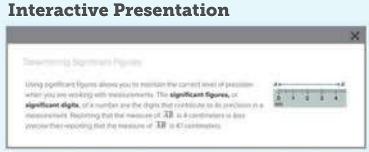
Think About It!

Create three different measurements that each have four significant figures.

Sample answer: 6,891,000 pounds, 12.87 meters, 0.08127 liter

Lesson 11-8 • Representing Measurements 687

Interactive Presentation



Learn

TAP

Students move through the slides to learn about significant figures.

TYPE

Students respond to a question about using significant figures.

Check

Determine the number of significant figures in each measurement.

a. 0.03927 milliliter has ? 4 significant figures

b. 5,134,180 pounds has ? 6 significant figures

Example 2 Find Significant Figures by Using Tools

Find the possible range for the length of the segment using the correct number of significant figures.

The length of the segment is approximately $1\frac{1}{2}$ inches.

This measurement was given to the nearest $\frac{1}{4}$ inch, so the possible range of this measurement is within $\frac{1}{2}\left(\frac{1}{4}\right)$ or $\frac{1}{8}$ inch of the measured length.

The exact measurement is between $1\frac{3}{8}$ and $1\frac{5}{8}$ inches or 1.375 and 1.625 inches.

Due to the precision of the ruler, the length of the segment has 4 significant figures.

Check

Find the possible range for the length of the segment.

A. 2.0 cm to 2.2 cm

B. 2.00 cm to 3.00 cm

C. 2.15 cm to 2.25 cm

D. 2.08 cm to 2.12 cm

Go Online You can complete an Extra Example online.

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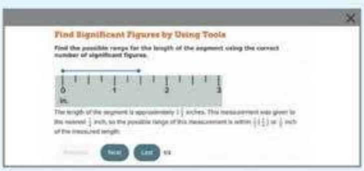
Interactive Presentation



Example 2

TAP

Students move through the steps to determine the correct number of significant figures.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Find Significant Figures by Using Tools

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this example, students will need to use a ruler. Work with students to explore and deepen their understanding of significant figures.

Questions for Mathematical Discourse

AL Suppose a ruler measured to the nearest centimeter. What will the possible range of any measurement be within the measured length? 0.5 cm

OL Suppose a ruler measured to the quarter of an inch. What will the possible range of any measurement be within the measured length? 0.125 of an inch

BL What is the possible range for the length of a segment measured on the same ruler as $2\frac{1}{4}$ inches? $2\frac{1}{8}$ in. to $2\frac{3}{8}$ in.

Common Error

Students may not understand finding the range of possible values using the measurement tool. Remind students that the precision of the answer is based on the reported measurement.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students do not know how to write the correct number of decimal places after adding or subtracting measurements,

THEN remind students to leave the same number of decimal places in the answer as there are in the measurement with the least amount of significant figures.

Learn Calculating with Significant Figures

Objective

Students round measurements to the correct number of significant figures.

Teaching the Mathematical Practices

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. In this Learn, notice the relationship between the problem variables and the units involved.

Important to Know

When calculating with significant figures, the accuracy of the result is limited by the least accurate measurement. When adding or subtracting, the result cannot have more decimal places than either of the original numbers. When multiplying or dividing, the number of significant figures of the result is determined by the original number with the least amount of figures. Significant figures are not affected by conversion factors.

Common Misconception

Students tend to believe the answer should have the same number of significant figures as the original measurements, without regard to the operation or significant figures of the original numbers. Remind students that calculating with significant figures has rules to follow, and answers must be precise based on the situation.

Example 3 Calculate with Significant Figures

Teaching the Mathematical Practices

6 Use Precision In this example, students will calculate accurately and efficiently and to express numerical answers with a degree of precision appropriate to the problem context.

Questions for Mathematical Discourse

AL When adding a measurement with three decimal places to a measurement with one decimal place, how many decimal places will the result have? 1 decimal place

OL When dividing a measurement with three significant figures by a measurement with four significant figures, how many significant figures will the quotient have? 3

BL Give an example of a multiplication of measurements where the product has 2 significant figures.
Sample answer: 4.5 inches × 100.0 inches = 450 square inches

Common Error

Students may assume the product should have the total number of decimal places as the values being multiplied, due to the rules when multiplying decimals. Remind students that when calculating with measurements, we must follow the rules of significant figures.

Learn Calculating with Significant Figures

When you are calculating with significant figures, the accuracy of the result is limited by the least accurate measurement.

Key Concept • Calculations with Significant Figures

Addition and Subtraction	Multiplication and Division
When using addition and subtraction, a calculation cannot have more digits to the right of the decimal point than either of the original numbers.	When using multiplication and division, the number of significant figures in the final product or quotient is determined by the original number that has the *fewest* number of figures.

Numbers that are not measured are not considered when determining significant figures. For example, if you have 5 cereal boxes that weigh 14 ounces each, then the significant figures used in a calculation would be determined from the measurement, 14 ounces, not the quantity.

Significant figures are also not affected by conversion factors. For example, when using the conversion 12 inches = 1 foot, the significant figures are determined by the original measurement being converted.

Talk About It!
Why is it important to have a standard method for calculations with significant figures?

Sample answer: Having a standard method for calculations with significant figures guarantees consistency when performing computations.

Example 3 Calculate with Significant Figures

Find each measurement rounded to the correct number of significant figures.

a. volume of an 837.24-mL sample after 276.516 mL is removed

837.24 has 2 digits after the decimal, and 276.516 has 3. So the result should have 2 digits after the decimal.

Find the difference. Then round to the hundredths place.

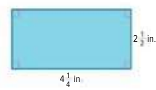
837.24 − 276.516 = 560.724 or 560.72 mL.

b. area of the rectangle

$A = (4.25)(2.5)$

$= 10.625$

Using significant figures, the area is 11 square inches.



Check

A mixing bowl contains 8.5 fluid ounces of water. If 4.25 fluid ounces are removed from the bowl, how many fluid ounces of water remain? Round to the correct number of significant figures.

? fl oz
4.3

Go Online You can complete an Extra Example online.

Lesson 11-8 • Representing Measurements 689

Interactive Presentation

Example 3

TAP

Students tap to reveal a Common Error.

Example 4 Use Significant Figures in the Real World

Think About It!
What assumption did you make while solving this problem?

Sample answer: I assumed that all of the bulbs were distributed evenly throughout the 79 acres.

FLOWER GARDEN **The world's second largest flower garden is Keukenhof Park in Lisse, the Netherlands, which covers 79 acres and contains 7 million flower bulbs. About 30 gardeners work together each year to design the flower formations. How many bulbs do they use in each square yard of the garden? Round to the correct number of significant figures.**

$$\frac{\text{total bulbs}}{\text{total area}} = \frac{7{,}000{,}000 \text{ bulbs}}{79 \text{ acres}}$$

$$= \frac{7{,}000{,}000 \text{ bulbs}}{79 \text{ acres}} \times \frac{1 \text{ acre}}{4840 \text{ yd}^2} \quad 1 \text{ acre} = 4840 \text{ yd}^2$$

$$= 18.30735433 \text{ bulbs/yd}^2 \quad \text{Simplify.}$$

Because the number of bulbs is a quantity, not a measurement, the number of significant figures is determined by the given measurement, which is 79 acres. Because 79 has 2 significant figures, the final product should also have 2 significant figures. Thus, they use 18 bulbs in each square yard of the garden.

Check

SNOW REMOVAL Mark's snow plow truck can clear 1600 tons of snow in an hour. How many pounds can Mark's truck clear in a minute? Round to the correct number of significant figures. (Hint: 1 T = 2000 lb)

A. 20,000 lb
B. 32,000 lb
C. 53,000 lb
D. 53,330 lb
E. 53,400 lb

Go Online You can complete an Extra Example online.

690 Module 11 • Angles and Geometric Figures

Interactive Presentation

Example 4

TAP
Students move through the steps to solve the problem

TYPE
Students respond to a question to show they understand using significant figures.

Example 4 Use Significant Figures in the Real World

Teaching the Mathematical Practices

4 Make Assumptions Have students explain an assumption or approximation that was made to solve the problem.

Questions for Mathematical Discourse

AL Which numerical value is irrelevant to the question? 30 gardeners

OL Regardless of significant figures, why does an answer of 18.3 not make sense? Sample answer: You can't plant 0.3 of a bulb in a square yard. A whole-number answer makes sense.

BL Suppose the Keukenhof Park covers 80 acres. Without recalculating, approximately how many bulbs are planted per square yard? Explain your reasoning. 20; Sample answer: Because 80 acres is very close to 79 acres, I can assume the result will be approximately 18, but because 80 has only 1 significant figure, the answer will only have 1 significant figure.

Common Error

Students may not know the conversion factor for an acre to a square yard. Encourage students to look up the conversion factor when working through the problem.

Example 5 Use Tools to Calculate Measurements

MP Teaching the Mathematical Practices

6 Use Precision In this example, students will calculate accurately and efficiently and to express numerical answers with a degree of precision appropriate to the problem context.

Questions for Mathematical Discourse

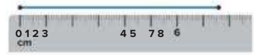
AL To how many centimeters is the measurement given? 0.1 centimeters

OL How do you know the length of the radius has 3 significant figures? The range of the measurement is from 7.55 to 7.65, which both have 3 significant figures.

BL Why do you need to calculate two areas of the circle? Using significant figures tells you the greatest and least length of the radius of the circle. You need to calculate the area for each measure.

Example 5 Use Tools to Calculate Measurements

The radius of a circle has the measurement shown. What is the possible range for the area of the circle? Round to the correct number of significant figures.



Step 1 Find the possible range for the length of the radius.

The approximate length of the segment is 7.6 centimeters. This measurement is given to the nearest 0.1 centimeter, so the approximate error is $\frac{1}{2}(0.1)$ or 0.05 centimeter. Therefore, the exact length is between 7.55 and 7.65 centimeters.

Step 2 Determine the number of significant figures.

Because the range of the length is between 7.55 and 7.65 centimeters, the length has 3 significant figures.

Step 3 Calculate the area of the circle.

The area of a circle is equal to πr^2, where r is the length of the radius. Complete the expressions to calculate the least and greatest possible areas of the circle.

least possible area: $\pi(7.55)^2 = 179.0786352 \text{ cm}^2$

greatest possible area: $\pi(7.65)^2 \approx 183.8538561 \text{ cm}^2$

Using significant figures, the area of the circle is between 179 and 184 square centimeters.

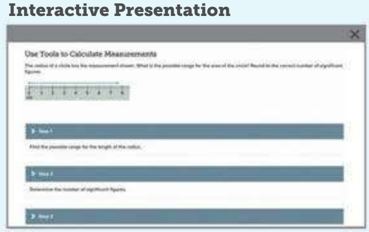
Check

The radius of a circle has the measurement shown. What is the possible range for the area of the circle? Round to the correct number of significant figures.

The area of the circle is between __?__ (86.6) and __?__ (104) square inches.

Lesson 11-8 • Representing Measurements 691

Interactive Presentation



Example 5

TAP

Students move through the steps to find a possible range of areas.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Pause and Reflect

Did you struggle with anything in this lesson? If so, how did you deal with it?

See students' observations.

Practice

Example 1

Determine the number of significant digits in each measurement.

1. 54.023
   5
2. 0.923
   3
3. 0.30
   2
4. 100.58
   5
5. 0.0002
   1
6. 101.01
   5
7. ACADEMICS The students in Miss Li's class are measuring the height of a chalkboard. Miss Li asked the students to write the measurement with 4 significant digits. Which student correctly followed her instructions? Michelle

Student	Measure
Sasha	48.5 centimeters
Michelle	48.53 centimeters
Alwan	49 centimeters
Remmie	48.530 centimeters

Example 2

Find the possible range for each length of the segment using the correct number of significant figures.

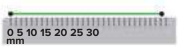
8.

2.85 to 2.95 cm

9.

4.5 to 5.5 in.

10.



28.5 to 29.5 mm

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–26
2	exercises that use a variety of skills from this lesson	27–42
3	exercises that emphasize higher-order and critical-thinking skills	43–48

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–41 odd, 43–48
- Extension: While Loops

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–47 odd
- Remediation, Review Resources: Convert Customary Measurement Units
- Personal Tutors
- Extra Examples 1–5
- ALEKS Converting Measurements

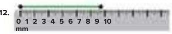
IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–25 odd
- Remediation, Review Resources: Convert Customary Measurement Units
- ALEKS Converting Measurements

Go Online You can complete your homework online.

11. 

2.5 to 3.5 cm

12. 

5.65 to 5.75 mm

13. 

4.375 to 4.625 in.

Example 3

The base of a triangle is fixed at 2.218 millimeters. Determine the number of significant figures of the area of the triangle with each given height.

14. 1.86 mm 3

15. 0.099 mm 2

16. 0.1279 mm 4

17. 2.109 mm 4

18. 11.0 mm 3

19. 1.7 mm 2

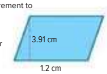
20. Using significant figures, which of the following students wrote a calculation that could have a sum or difference of 51.9? Nobu

Student	Calculation
Juliana	48.222 + 3.769
Lori	48.22 + 3.76
Nobu	48.222 + 3.7
Jerome	48.2 + 3.769

Example 4

21. CHEMISTRY Angel has 8.341 mL of saline. She pours 1.1 mL of saline into another solution. How much saline does Angel have left? Round your measurement to the correct number of significant figures. 7.2 mL

22. A parallelogram with base b and height h has area, A, given by the formula $A = bh$. Find the area of the given parallelogram. Round your measurement to the correct number of significant figures. $4.7\ cm^2$



23. AREA Find the area of a triangle with a height of 4.90 centimeters and a base length of 6.174 centimeters. Round your measurement to the correct number of significant figures. $15.1\ cm^2$

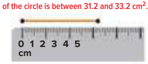
24. DIMENSIONS Rafael is building a horseshoe pit in his backyard. The width of the pit is 29.71 inches, and the length is 30.1 inches.

Part A Estimate the area of Rafael's horseshoe pit. $900\ in^2$

Part B Rafael finds the exact area of the horseshoe pit and rounds his answer to the correct number of significant figures. What area did Rafael find? $894\ in^2$

Example 5

25. AREA The radius of a circle has the measurement shown. What is the possible range for the area of the circle? Round to the correct number of significant figures. The area of the circle is between 31.2 and $33.2\ cm^2$.



26. CIRCUMFERENCE The radius of a circle has the measurement shown. What is the possible range for the circumference of the circle? Round to the correct number of significant figures. The circumference of the circle is between 21.598 and 22.384 inches.

Mixed Exercises

Determine the number of significant digits in each measurement.

27. 53.74 4

28. 0.03298 4

29. 10.500 5

30. 6,102.0 5

31. 7,109,100 5

32. 0.110 3

33. CHEMISTRY A beaker contains a sample of NaCl weighing 49.8767 grams. If the empty beaker weighs 49.214 grams, what is the weight of the NaCl? Round to the correct number of significant figures. 0.663 gram

34. COOKING Jordan makes a sandwich on a paper plate weighing 32.47 grams. The bread weighs 60.13 grams. Jordan adds 12.3 grams of turkey, 2.4 grams of mayonnaise, and 3.0 grams of lettuce. What is the final weight of the plate and sandwich? 110.3 g

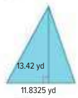
35. HEMISTRY Three chemists weigh an item using different scales. The values they report are shown on the scales. How many significant figures should be used for each measurement? scale 1: 4; scale 2: 3; scale 3: 4

36. SWIMMING POOL A rectangular swimming pool measures 24.2 feet by 76 feet.

a. Find the perimeter of the pool. Round to the correct umber of significant figures. 200 ft

b. Find the area of the pool. Round to the correct umber of significant figures. 1800 ft

37. AREA Find the area of the given triangle. Round your measure to the correct number of significant figures. 79.40 yd



38. MURAL Krista is painting a rectangular wall that has an area of 247 square feet. If she can paint 5.25 square feet in an hour, about how long will it take for Krista to finish the mural on her own? Round to the correct number of significant figures. 47 1

39. MASS Suppose that you measured the volume of a rock to be 2.3 cm and you know the density to be 3.6 g/cm. What is the mass of the rock? Round your measure to the correct number of significant figures. 8.3 g

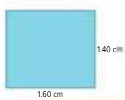
40. DRIVING Keandra is taking a trip to visit her extended family and makes a stop somewhere during her trip. The distance between Cincinnati, where Keandra started, and Dayton, where Keandra made the stop is 54 miles. The distance between Dayton and T oledo, where Keandra's family lives is 150.2 miles. How far did Keandra travel on her trip? Round to the correct number of significant figures. 204 mi

41. VOLUME A rectangular box has a length of 10.876 inches, a width of 4.34 inches, and a height of 13.22 inches. What is the volume of the rectangular prism? Round to the correct number of significant figures. 624 in^3

42. TRAVEL Y ou estimate that your car gets 28 miles per gallon. The cost of gas per gallon is shown. How much does it cost you to travel 455 miles? Round to the correct number of significant figures. $40

Higher-Order Thinking Skills

43. FIND THE ERROR A student found that the dimensions of a rectangle were 1.40 meters and 1.60 meters. She was asked to report the area using the correct number of significant figures. She reported the area as 2.2 square meters. What error did the student make? Explain your reasoning. Sample answer: The 0 in each dimension, 1.40 cm and 1.60 cm, is significant. The answer should be given with 3 significant figures as 2.24 square centimeters.



44. PERSEVERE The Sun is an excellent source of electrical energy. A field of solar panels yields 19.23 Watts per square foot. Determine the amount of electricity produced by a field of solar panels that is 410 feet by 201 yards. 4,800,000 W

45. WRITE When explaining the process of finding the perimeter of a triangle using significant digits, Trinidad claimed that 0.045 inch and 0.0045 inch have the same number of significant figures. Is she correct? Explain your answer. Y es; sample answer: The zeros before and after the decimal are not significant because a nonzero number did not come before them. Therefore, both numbers have two significant figures.

46. WRITE How do you use significant figures to determine how to report a sum or product of two measures? For addition, the sum should be rounded to the same place value of the least-precise measure. For multiplication, the product should be rounded to have the same number of significant figures as the original measure with the fewest significant figures.

47. ANAL YZE Determine whether the following statement is *sometimes, always,* or *never* true. Justify your argument.
Zeros are significant figures.
Sometimes; sample answer: A zero between two nonzero significant figures is always significant, a leading zero is never significant, and a zero at the end of a number is only significant when a decimal point is given in the number.

48. CREATE The swim team measures time to the hundredth of a second. Amanda's time was slower than Jocelyn's time in the 100-meter freestyle. What are possible times for Amanda and Jocelyn if each has times with 4 significant digits? Sample answer: Amanda's time = 72.50 s and Jocelyn's time = 70.91 s

Module 11 • Angles and Geometric Figures

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why are angles important in the real world?
- Why should we not assume certain relationships are present based off a diagram?
- Why are geometric models a useful tool when dealing with real world two-dimensional objects?
- Why are significant figures important in the scientific field?

Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to angles, geometric figures, transformations, accuracy, precision, and significant figures.

LS **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Congruence, Proof, and Constructions** and **Extend to Three Dimensions.**

- Experiment with transformations in the plane
- Make Geometric Constructions
- Visualize the relation between two-dimensional and three-dimensional objects

Module 11 • Angles and Geometric Figures

Review

Essential Question

How are angles and two-dimensional figures used to model the real world?

Two-dimensional figures can be drawn to represent real-world objects. Two-dimensional figures can model three-dimensional figures in the form of nets so that computations about those objects can be made more easily. Angles and sides should be labeled in these representations to help when computations are made.

Module Summary

Lessons 11-1 and 11-2

Angles

- Angles that have the same measure are congruent angles.
- A ray or segment that divides an angle into two congruent parts is an angle bisector.
- Relationships between special angle pairs can be used to find missing measures.
- Complementary angles are two angles with measures that have a sum of 90°. Supplementary angles are two angles that have measures that have a sum of 180°.
- Certain relationships can be assumed from a figure, but most cannot.

Lessons 11-3 and 11-4

Two-Dimensional Figures

- The perimeter of a polygon is the sum of the lengths of the sides of the polygon.
- The circumference of a circle is the distance around the circle.
- Area is the number of square units needed to cover a surface.
- A transformation is a function that takes points in the plane as inputs and gives other points as outputs.
- A rigid motion is a transformation that preserves distance and angle measure.
- The three main types of rigid motions are reflection, translation, and rotation.

Lessons 11-5 and 11-6

Three-Dimensional Figures

- Surface area is the sum of the areas of all faces and side surfaces of a three-dimensional figure.
- Volume is the measure of the amount of space enclosed by a three-dimensional figure.
- A three-dimensional figure can be modeled by an orthographic drawing, which shows its top, left, front, and right views.
- A net is a two-dimensional figure that forms the surfaces of a three-dimensional object when folded.

Lessons 11-7 and 11-8

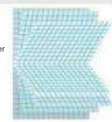
Measurements

- Precision is the repeatability, or reproducibility, of a measurement.
- Accuracy is the nearness of a measurement to the true value of the measure.
- The significant figures, or significant digits, of a number are the digits that contribute to its precision in a measurement.

Study Organizer

Foldables

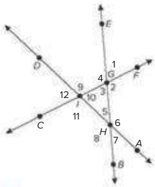
Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.



Module 11 Review • Angles and Geometric Figures **697**

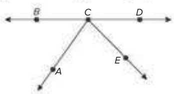
Test Practice

1. **MULTI-SELECT** Select all the angles for which $\overrightarrow{HA}$ and $\overrightarrow{HE}$ are the sides. (Lesson 11-1)



A. ∠AHE
B. ∠AGE
C. ∠EHA
D. ∠EGA
E. ∠AHB

2. **OPEN RESPONSE** In the figure, $\overrightarrow{CD}$ and $\overrightarrow{CB}$ are opposite rays, and $\overrightarrow{CA}$ bisects ∠BCE.



Suppose $m\angle ECA = 14x - 2$ and $m\angle ACB = 12x + 8$. What is $m\angle ECA$? (Lesson 11-1)

58°

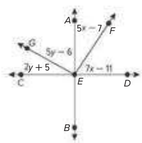
3. **OPEN RESPONSE** Describe how you would construct an angle bisector using paper-folding. (Lesson 11-1)

Sample answer: Use paddy paper or wax paper . Draw an angle on the paper . Fold the paper so that one side of the angle is directly on top of the other side. Draw the angle bisector in the crease of the fold.

4. **MULTIPLE CHOICE** Two angles are supplementary. The measure of the larger angle is 12 less than 3 times the measure of the smaller angle. Find the measure of the larger angle. (Lesson 11-2)

A. 25.5°
B. 48°
C. 64.5°
D. 132°

5. **MULTIPLE CHOICE** Which value of x will make $\overleftrightarrow{AB}$ perpendicular to $\overleftrightarrow{CD}$? (Lesson 11-2)



A. 6
B. 9
C. 11
D. 13

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Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

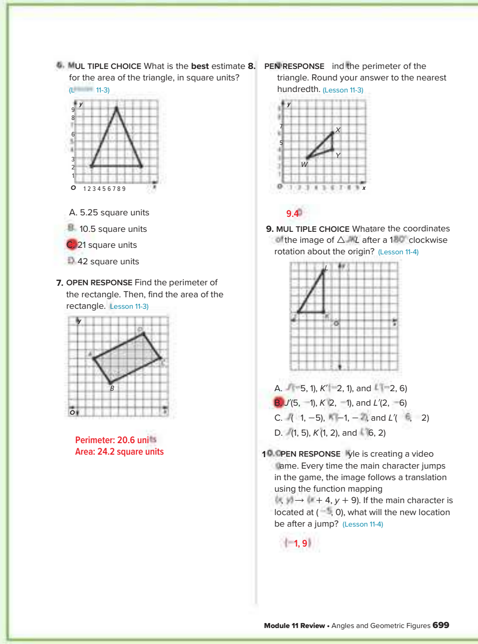
Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–15 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	4–6, 9, 11, 15
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	1
Open Response	Students construct their own response.	2, 3, 7, 8, 10, 12–14

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
G.CO.1	11-1, 11-2	1, 2, 4, 5
G.CO.2	11-4	9–11
G.CO.12	11-1	3
G.GPE.7	11-3	6–8
G.GMD.3	11-5	12, 13
G.MG.1	11-6	14, 15


6. **MULTIPLE CHOICE** What is the **best** estimate for the area of the triangle, in square units? (Lesson 11-3)

A. 5.25 square units
B. 10.5 square units
C. 21 square units
D. 42 square units

7. **OPEN RESPONSE** Find the perimeter of the rectangle. Then, find the area of the rectangle. (Lesson 11-3)

Perimeter: 20.6 units
Area: 24.2 square units

8. **OPEN RESPONSE** Find the perimeter of the triangle. Round your answer to the nearest hundredth. (Lesson 11-3)

9.40

9. **MULTIPLE CHOICE** What are the coordinates of the image of △JKL after a 180° clockwise rotation about the origin? (Lesson 11-4)

A. J′(−5, 1), K′(−2, 1), and L′(−2, 6)
B. J′(5, −1), K′(2, −1), and L′(2, −6)
C. J′(−1, −5), K′(−1, −2), and L′(−6, −2)
D. J′(1, 5), K′(1, 2), and L′(6, 2)

10. **OPEN RESPONSE** Kyle is creating a video game. Every time the main character jumps in the game, the image follows a translation using the function mapping $(x, y) \rightarrow (x + 4, y + 9)$. If the main character is located at (−5, 0), what will the new location be after a jump? (Lesson 11-4)

(−1, 9)

Module 11 Review • Angles and Geometric Figures **699**

11. **MULTIPLE CHOICE** $\triangle KLM$ has coordinates $K(4, -2)$, $L(6, -1)$, and $M(5, 5)$. What would be the coordinates of the vertices of the image after a reflection in the x-axis? (Lesson 11-4)

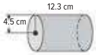
A. $K'(-4, -2)$, $L'(-6, -1)$, and $M'(-5, 5)$

B. $K'(-4, 2)$, $L'(-6, -1)$, and $M'(-5, -5)$

C. $K'(4, 2)$, $L'(6, 1)$, and $M'(5, -5)$

D. $K'(-2, 4)$, $L'(-1, 6)$, and $M'(5, 5)$

12. **OPEN RESPONSE** What is the volume, in cubic centimeters, of the cylinder? (Lesson 11-5)



782.49 cm^3

13. **OPEN RESPONSE** A beach ball has a radius of 8 inches. Find how many cubic inches to the nearest hundredth of air was used to fill the beach ball. (Lesson 11-5)

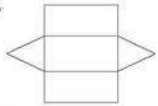
2,144.66 in^3

14. **OPEN RESPONSE** Identify two three-dimensional shapes that are represented by the grain silo. (Lesson 11-6)

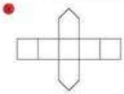
hemisphere and cylinder

15. **MULTIPLE CHOICE** Which net could be used to represent the storage shed? (Lesson 11-6)

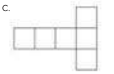
A.



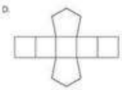
B.



C.

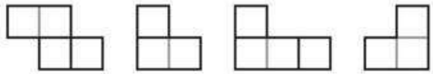


D.



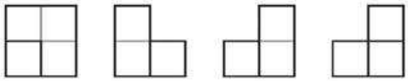
Lesson 11-6

4.



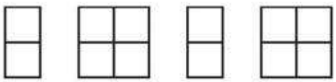
top view left view front view right view

5.



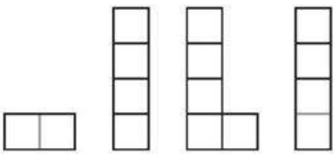
top view left view front view right view

6.



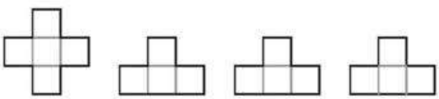
top left front right

7.



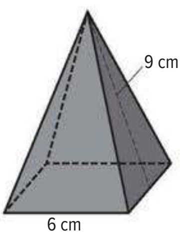
top left front right

8.

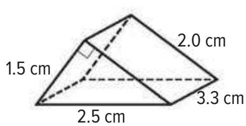


top left front right

9.



10.



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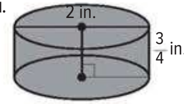
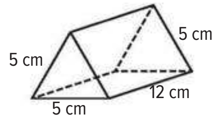
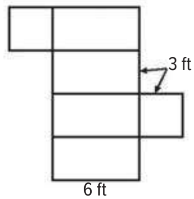


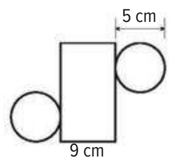
12.



17. Sample answer:



18. Sample answer:



Module 12

Logical Arguments and Line Relationships

Module Goals

- Students look for patterns and write conjectures based on those patterns.
- Students prove conjectures using logical arguments or disprove conjectures using counterexamples.
- Students apply logical arguments to basic line and angle relationships.

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.9 Prove theorems about lines and angles.

G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

Also addresses G.CO.1, G.GPE.5, and *G.MG.3.*

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Be Sure to Cover

To completely cover G.CO.12, go online to assign the following construction activities:

- Construct a Segment Twice as Long as a Given Segment (Lesson 12-5)
- Construct a Line Parallel to a Given Line Through a Given Point (Lesson 12-9)

Coherence

Vertical Alignment

Previous
Students defined and used lines, line segments, angles, and two-dimensional figures.
G.CO.1

Now
Students prove theorems about lines, line segments, and angles.
G.CO.9

Next
Students will prove theorems about triangles.
G.CO.10

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. After they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Independent Practice.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

EXPLORE → LEARN → EXAMPLE & PRACTICE

Suggested Pacing

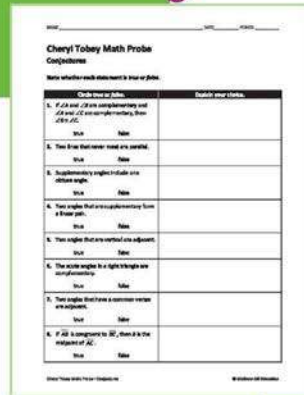
Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
12-1 Conjectures and Counterexamples		1	0.5
12-2 Statements, Conditionals, and Biconditionals		1	0.5
12-3 Deductive Reasoning		1	0.5
Put It All Together: Lessons 1 through 3		1	0.5
12-4 Writing Proofs		3	1.5
12-5 Proving Segment Relationships	G.CO.9, G.CO.12	1	0.5
12-6 Proving Angle Relationships	G.CO.9	2	1
12-7 Parallel Lines and Transversals	G.CO.1, G.CO.9	1	0.5
12-8 Slope and Equations of Lines	G.GPE.5	2	1
12-9 Proving Lines Parallel	G.CO.9, G.CO.12	1	0.5
12-10 Perpendiculars and Distance	G.CO.12, G.MG.3	2	1
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	19	9.5

CHERYL TOBEY MATH PROBES

Formative Assessment Math Probe

Conjectures

Module Resource



Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students determine whether conjectures are true and explain their choices.

Targeted Concepts Understand the basic principles of logical reasoning to determine the truth values of conjectures.

Targeted Misconceptions

- Students have difficulty distinguishing the hypothesis from the conclusion if conjectures are not written in if-then form.
- Students incorrectly believe there are exceptions to a property or theorem.
- Students think that one counterexample is not enough to prove a statement false.
- Students think the converse of a true statement is also always true.

Use the Probe after Lesson 12-1.

Answers:

1. T	**2. F**	**3. F**	**4. F**
5. F	**6. T**	**7. F**	**8. F**

Collect and Assess Student Answers

If the student selects these responses...	**Then** the student likely...
2. true **4.** true **7.** true **8.** true	is basing his or her decision on the converse statements being true. **Example:** For Item 2, the converse is true (parallel lines never meet) but students are not including skew lines to analyze this statement.
2. true **3.** true **4.** true **5.** true **7.** true **8.** true	is not considering exceptions (counterexamples); believes that if there is only one counterexample, then the statement is still true; and/or believes that if the statement is true some of the time, then it is considered true. **Example:** For Item 8, the student does not consider situations where congruent segments *AB* and *BC* are perpendicular, or the student does not know that if point *A*, *B*, and *C* are collinear, the statement is true.
1. false **2.** false	is not considering that an angle can be complementary to more than one angle or does not have a thorough understanding of the term *complementary*.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS Patterns and Inductive Reasoning
- Lesson 12-1, Learn, Examples 1–4

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

IGNITE!

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

What makes a logical argument, and how are logical arguments used in geometry? Sample answer: A logical argument is well organized and has statements that can be justified using postulates, which are assumed to be true, or previously proved statements.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

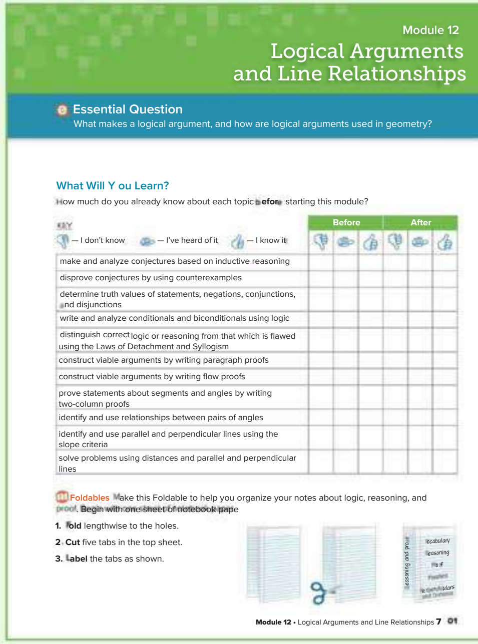
Focus Students write about reasoning and proofs.

Teach Throughout the module, have students take notes under the tabs of their Foldables. Instruct students to take notes while reading each lesson and listening to instruction. They should include definitions of terms and key concepts. Encourage students to record examples of each type of logical reasoning from a lesson on the back of the Foldable.

When to Use It Use the appropriate tabs as students cover each lesson in this module. Students should add to the vocabulary tab during each lesson.

Launch the Module

For this module, the Launch the Module video shows how the process of conducting experiments involves making a conjecture, gathering evidence, and then making a conclusion. Students will learn how proving geometric theorems follows a similar pattern of thinking.



Module 12

Logical Arguments and Line Relationships

Essential Question

What makes a logical argument, and how are logical arguments used in geometry?

What Will Y ou Learn?

How much do you already know about each topic before starting this module?

KEY — I don't know — I've heard of it — I know it

	Before			After		
	I don't know	I've heard of it	I know it	I don't know	I've heard of it	I know it
make and analyze conjectures based on inductive reasoning						
disprove conjectures by using counterexamples						
determine truth values of statements, negations, conjunctions, and disjunctions						
write and analyze conditionals and biconditionals using logic						
distinguish correct logic or reasoning from that which is flawed using the Laws of Detachment and Syllogism						
construct viable arguments by writing paragraph proofs						
construct viable arguments by writing flow proofs						
prove statements about segments and angles by writing two-column proofs						
identify and use relationships between pairs of angles						
identify and use parallel and perpendicular lines using the slope criteria						
solve problems using distances and parallel and perpendicular lines						

Foldables Make this Foldable to help you organize your notes about logic, reasoning, and proof. Begin with one sheet of notebook pape

1. Fold lengthwise to the holes.
2. Cut five tabs in the top sheet.
3. Label the tabs as shown.

Module 12 • Logical Arguments and Line Relationships 701

Interactive Presentation GM03_001SC

What Vocabulary Will Y ou Learn?

- alternate exterior angles
- alternate interior angles
- biconditional statement
- compound statement
- conclusion
- conditional statement
- conjecture
- conjunction
- consecutive interior angles
- contrapositive
- converse
- corresponding angles
- counterexample
- deductive argument
- deductive reasoning
- disjunction
- equidistant
- exterior angles
- flow proof
- hypothesis
- if-then statement
- inductive reasoning
- interior angles
- inverse
- logically equivalent
- negation
- paragraph proof
- parallel lines
- parralel planes
- proof
- skew lines
- slope
- slope criteria
- statement
- transversal
- truth value
- two-column proof
- valid argument

Are Y ou Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review

Example 1

Solve $36x - 14 = 16x + 58$.

$36x - 14 = 16x + 58$ Original equation

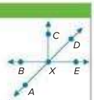
$20x - 14 = 58$ Subtract $16x$ from each side.

$20x = 72$ Add 14 to each side.

$x = 3.6$ Divide each side by 20.

Example 2

If $m\angle BXA = 3x + 5$ and $m\angle DXE = 56$, find x.



$m\angle BXA = m\angle DXE$

Vertical ∠s are ≅.

$3x + 5 = 56$ Substitution.

$3x = 51$ Subtract 5 from each side.

$x = 17$ Divide each side by 3.

Quick Check

Solve each equation.

1. $8x - 10 = 6x$ 5
2. $18 + 7x = 10x + 39$ −7
3. $3(11x - 7) = 13x + 25$ 2.3
4. $3x + 8 = 0.5x + 35$ 10.8

Refer to the figure above.

5. Identify a pair of vertical angles that appear to be obtuse. $\angle BXD$, $\angle AXE$
6. If $m\angle DXB = 116$ and $m\angle EXA = 3x + 2$, find x. 38
7. If $m\angle BXC = 90$, $m\angle CXD = 6x - 13$, and $m\angle DXE = 10x + 7$, find x. 6

How did you do?

Which exercises did you answer correctly in the Quick Check?

702 Module 12 • Logical Arguments and Line Relationships

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define A conditional statement is a compound statement that consists of a premise, or *hypothesis*, and a *conclusion*, which is false only when its premise is true and its conclusion is false.

Example If you finish your homework, then you can go to the movies.

Ask Is this statement in *if-then form*? yes What is the hypothesis? Y ou finish your homework. What is the conclusion? You can go to the movies.

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- using patterns
- analyzing angles
- finding slopes
- finding square roots

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the **Reasoning; Lines** section to ensure student success in this module.

Mindset Matters

Attitude Ownership

Part of developing a growth mindset involves acknowledging progress in growth thinking and sharing it with others. It's important for students to own their mindset, be proud of their growth, and see themselves as someone who has a growth mentality—not just in math, but with learning in general.

How Can I Apply It?

Have students complete a math mindset project to share how they have grown throughout the year. They might choose the delivery method, such as a **poster, blog post, video, or podcast**. Students can share their mindset journey with their classmates as part of a class discussion, or they might post their projects for others to see.

Lesson 12-1

Conjectures and Counterexamples

LESSON GOAL

Students analyze conjectures by using inductive reasoning and disprove conjectures by using counterexamples.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Using Inductive Reasoning to Make Conjectures

Develop:

Inductive Reasoning and Conjecture
- Patterns and Conjectures
- Algebraic Conjectures
- Geometric Conjectures
- Make Conjectures from Data

Counterexamples
- Find Counterexamples

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Powers and Exponents	●	●		●
Extension: Mathematical Induction		●	●	●

Language Development Handbook

Assign page 71 of the *Language Development Handbook* to help your students build mathematical language related to making conjectures and finding counterexamples.

ELL You can use the tips and suggestions on page T71 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
3 Construct viable arguments and critique the reasoning of others.
5 Use appropriate tools strategically.
6 Attend to precision.
7 Look for and make use of structure.
8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Throughout Grades 6-8 and Course 1, students have made conjectures and cited counterexamples.
MP3

Now
Students write and analyze conjectures by using inductive reasoning.

Next
Students will determine the truth values of given statements.

Rigor

The Three Pillars of Rigor

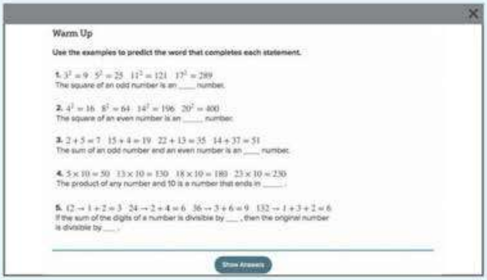
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of conjectures. They write and analyze conjectures by using inductive reasoning and disprove conjectures by using counterexamples.

Mathematical Background

A *conjecture* is an educated guess based on known information. Examining several specific situations to arrive at a conjecture is called *inductive reasoning*. If just one example contradicts the conjecture, then the conjecture is not true. The example that is used to disprove the conjecture is called a *counterexample*.

Interactive Presentation



Warm Up

Warm Up

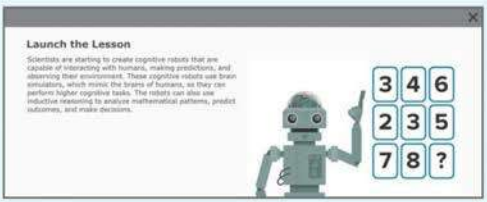
Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

• using patterns

Answers:

1. odd
2. even
3. odd
4. zero
5. 3



Launch the Lesson

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of inductive reasoning.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Vocabulary

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Using Inductive Reasoning to Make Conjectures

Objective

Students use dynamic geometry software and inductive reasoning to make conjectures.

MP Teaching the Mathematical Practices

8 Attend to Details Mathematically proficient students continually ask themselves, “Does this make sense?” to evaluate the reasonableness of their answer.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don’t have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students enter a length for the side of a regular hexagon into the dynamic geometry software. The software computes the perimeter and area of the regular hexagon. Students can press buttons to double the side length, and then double it again, to see what happens with the perimeter and area. Then students record their results in a table and try new numbers for the length. Students then answer guiding exercises that lead them to write conjectures about their observations. Then, students answer the Inquiry Question.

(*continued on the next page*)

Interactive Presentation

Explore

WEB SKETCHPAD

Students use the sketch to complete the activity in which they explore inductive reasoning.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Using Inductive Reasoning to Make Conjectures (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How does the side length of a regular hexagon relate to the perimeter? Because each side length is the same, the perimeter is 6 times the side length, or $P = 6s$.
- Why does it make sense to explore with a regular polygon instead of an irregular polygon? Sample answer: Using a regular polygon makes finding the perimeter and area easier because all information can be found with one side length. By simplifying the calculations, I can focus on the relationships.

Inquiry

How can you use observations and patterns to make predictions? Sample answer: If you use data to quantify your observations, then patterns within the data can help you make predictions about the situation you are observing.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Inductive Reasoning and Conjecture

Objective

Students write and analyze conjectures by using inductive reasoning.

Teaching the Mathematical Practices

7 Look for a Pattern Help students to see the pattern in this Learn.

Important to Know

Tell students to test all fundamental operations, including powers and roots, when they are looking for patterns in a series of numbers. Advise students that sometimes a pattern requires two operations.

Common Misconception

Students have used inductive reasoning to find missing terms in a pattern. They might be good at finding the next term, or the tenth term, but then have trouble finding a generic term or rule to describe the pattern. If the sequence is linear (the difference between terms is constant), then they can use methods that they learned in Algebra for writing the equation of a line.

Example 1 Patterns and Conjectures

Teaching the Mathematical Practices

3 Make Conjectures In this example, students will make conjectures and then build a logical progression of statements to validate the conjectures. Once students have made their conjectures, guide the students to validate them.

Questions for Mathematical Discourse

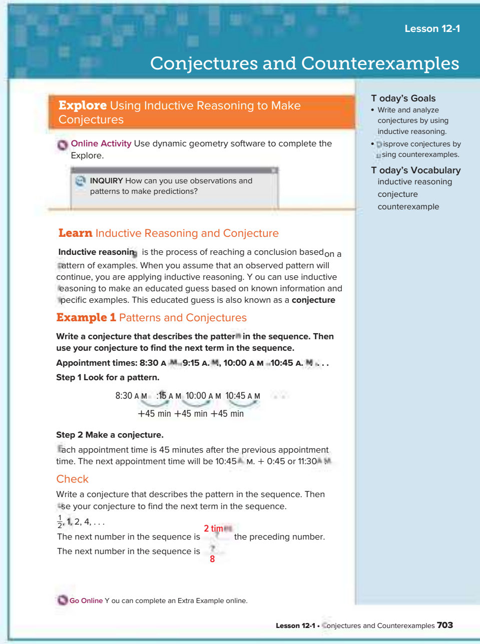
AL In your own words, what is a conjecture? Sample answer: an idea without proof

OL An *arithmetic* sequence is one in which there is a constant difference between each term. Is this an arithmetic sequence? Explain. The sequence is arithmetic, the constant difference is 45 minutes.

BL When using the expression 4^{n-1} to find the nth term in a sequence, what is an example of three consecutive terms in that sequence? Sample answer: 1, 4, 16

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 12-1

Conjectures and Counterexamples

Explore Using Inductive Reasoning to Make Conjectures

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How can you use observations and patterns to make predictions?

Today's Goals
- Write and analyze conjectures by using inductive reasoning.
- Disprove conjectures by using counterexamples.

Today's Vocabulary
inductive reasoning
conjecture
counterexample

Learn Inductive Reasoning and Conjecture

Inductive reasoning is the process of reaching a conclusion based on a pattern of examples. When you assume that an observed pattern will continue, you are applying inductive reasoning. You can use inductive reasoning to make an educated guess based on known information and specific examples. This educated guess is also known as a **conjecture**.

Example 1 Patterns and Conjectures

Write a conjecture that describes the pattern in the sequence. Then use your conjecture to find the next term in the sequence.

Appointment times: 8:30 A.M., 9:15 A.M., 10:00 A.M., 10:45 A.M., . . .

Step 1 Look for a pattern.

8:30 A.M. 9:15 A.M. 10:00 A.M. 10:45 A.M. . . .
+45 min +45 min +45 min

Step 2 Make a conjecture.

Each appointment time is 45 minutes after the previous appointment time. The next appointment time will be 10:45 A.M. + 0:45 or 11:30 A.M.

Check

Write a conjecture that describes the pattern in the sequence. Then use your conjecture to find the next term in the sequence.

$\frac{1}{2}$, 1, 2, 4, . . .

The next number in the sequence is 2 times the preceding number.

The next number in the sequence is 8.

Go Online You can complete an Extra Example online.

Lesson 12-1 • Conjectures and Counterexamples 703

Interactive Presentation

Learn

TAP

Students click through to follow the steps of finding a pattern and writing a conjecture.

Think About It!

It is possible for more than one conjecture to correctly describe a pattern. Each number in the sequence 2, 4, 8, 16, 32, . . . is 2^n where $n \geq 1$. What other conjecture can be made about the values?

Sample answer: Each number in the sequence can be generated by doubling the previous number

Example 2 Algebraic Conjectures

Make a conjecture about the sum of the squares of two consecutive natural numbers. List or draw some examples that support your conjecture.

Step 1 List examples.

$1^2 + 2^2 = 5$ $\quad$ $6^2 + 7^2 = 85$

$2^2 + 3^2 = 13$ $\quad$ $10^2 + 11^2 = 221$

Step 2 Look for a pattern.

Notice that all the sums are odd numbers.

Step 3 Make a conjecture.

The sum of the squares of two consecutive natural numbers is an odd number.

Check

Make a conjecture about the sum of two odd numbers.

The sum of two odd numbers is always a(n) __even__ number.

Example 3 Geometric Conjectures

Make a conjecture about the relationship between the segments joining opposite vertices of isosceles trapezoids.

Step 1 Draw several examples.

An isosceles trapezoid is a trapezoid with two opposite congruent legs.

Step 2 Look for a pattern.

Notice that the segments joining opposite vertices of each isosceles trapezoid appear to have the same measure. Use a ruler or compass to confirm this.

Step 3 Make a conjecture.

The segments joining opposite vertices of an isosceles trapezoid are congruent.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 2

TYPE

Students fill in the blanks to find a pattern and write a conjecture.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Example 2 Algebraic Conjectures

Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to construct an argument.

Questions for Mathematical Discourse

AL If you are having trouble finding a pattern, what steps can you take to help? Sample answer: generate more examples

OL What conjecture can be made about the sum of three odd numbers? Sample answer: The sum of three odd numbers is an odd number.

BL What conjecture can be made about the square of a number? Sample answer: If the number is odd, then the square of that number is odd. If the number is even, then the square of that number is even.

Common Error

Students may write a conjecture based on only a couple of examples. Encourage them to generate multiple examples in a pattern, and to check that their conjecture works with each example.

Essential Question Follow-Up

Students have begun to write conjectures.

Ask:

Why are conjectures important in a logical argument? Sample answer: Conjectures are the statements that logical arguments are trying to prove.

Example 3 Geometric Conjectures

Teaching the Mathematical Practices

3 Make Conjectures In this example, students will make conjectures and then build a logical progression of statements to validate the conjectures. Once students have made their conjectures, guide the students to validate them.

Questions for Mathematical Discourse

AL Measure the legs in **Step 1** with a ruler. What can you infer? Sample answer: The legs in each trapezoid are the same length.

OL What conjecture can be made about the diagonals of a trapezoid? Sample answer: The diagonals are congruent.

BL What conjecture can be made about the diagonals of a square? Sample answer: The diagonals are congruent and perpendicular bisectors of each other.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Example 4 Make Conjectures from Data

Teaching the Mathematical Practices

3 Reason Inductively In this example, students will use inductive reasoning to make plausible arguments.

Questions for Mathematical Discourse

AL What kind of statistical display best shows change over time? Sample answer: a scatter plot or a line graph

OL Which year has shown the greatest increase in gas price? 2011

BL Make a conjecture about the price of gas in 2020. Sample answer: The price of gas will begin to decrease after increasing for a few years.

Common Error

Students may try to prove that their conjecture based on data is correct using a logical argument, but the only way to determine whether such a conjecture is correct is by finding out what the data is for the time of their prediction.

DIFFERENTIATE

Reteaching Activity AL

IF students have trouble recognizing geometric patterns,
THEN have them write down the sequence of numbers that may be contained in the pattern.

Check

Make a conjecture about the relationships between *AD* and *AB*, if *C* is the midpoint of $\overline{AB}$ and *D* is the midpoint of $\overline{AC}$.

AD is ___?___ of *AB*. one fourth

Example 4 Make Conjectures from Data

GAS PRICES **The table shows the average price of gasoline in the United States for the years 2010 through 2018. Make a conjecture about the price of gas in 2019. Explain how this conjecture is supported by the data given.**

Year	Price (dollars per gallon)
2010	2.84
2011	3.58
2012	3.68
2013	3.58
2014	3.44
2015	2.43
2016	2.14
2017	2.42
2018	2.84

Look for patterns in the data.

The price of gasoline increased from 2010 to 2012. From 2012 to 2016, the price of gas decreased, at first at a steady rate, and then more dramatically. Beginning in 2017 , the price of gas began to increase at a steady rate.

The data shows that the price of gas follows an oscillating pattern, increasing in price for several years before decreasing in price for several years.

Conjecture: In 2019, the price of gas will continue to increase.

Check

HEARING LOSS Almost 50% of young adults between the ages of 12 and 35 years old are exposed to damaging levels of sound from the use of personal electronic devices. The intensity of a sound and the time spent listening to a sound highly affects the amount of damage that can be done to someone's hearing. The intensity of a sound to the human ear is measured in A-weighted decibels, or dBA. For every 3 decibels over 85 decibels, the exposure time it takes to cause hearing damage is cut in half. How long does it take to cause hearing damage at 106 decibels? Write your answer as a decimal.

Decibel Exposure Level (dBA)	Time (hours)
85	8
88	4
91	2
94	1
97	$\frac{1}{2}$
100	$\frac{1}{4}$

___?___ minutes 3.75

Go Online Y ou can complete an Extra Example online.

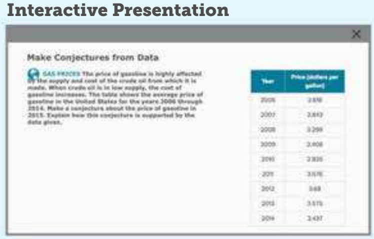
Think About It!
Could the pattern of the data change over time? Explain your reasoning.

Y es; sample answer: If the supply of crude oil decreases, then the price of gasoline will begin to increase.

Use a Source
Find data about the digital music revenue in the United States in recent years. Make a conjecture about the future trends in digital music revenue.

Sample answer: In the future, the revenue for digital music in the United States will continue to increase.

Interactive Presentation



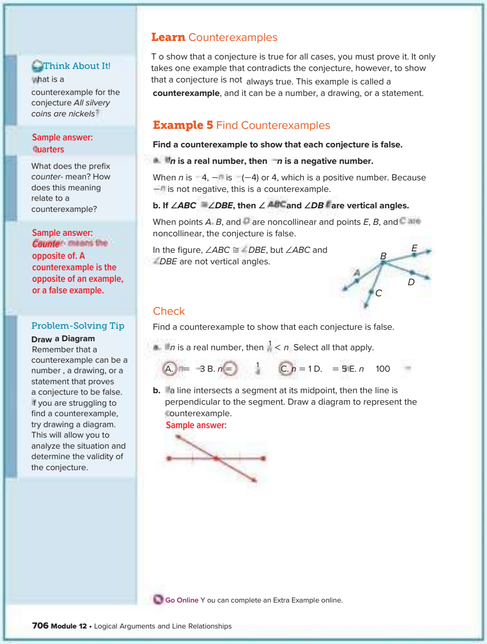
Example 4

TYPE

Students use the table to find a pattern and write a conjecture.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Learn Counterexamples

T o show that a conjecture is true for all cases, you must prove it. It only takes one example that contradicts the conjecture, however, to show that a conjecture is not always true. This example is called a **counterexample**, and it can be a number, a drawing, or a statement.

Example 5 Find Counterexamples

Find a counterexample to show that each conjecture is false.

a. If n is a real number, then $-n$ is a negative number.

When n is -4, $-n$ is $-(-4)$ or 4, which is a positive number. Because $-n$ is not negative, this is a counterexample.

b. If $\angle ABC \cong \angle DBE$, then $\angle ABC$ and $\angle DBE$ are vertical angles.

When points A, B, and D are noncollinear and points E, B, and C are noncollinear, the conjecture is false.

In the figure, $\angle ABC \cong \angle DBE$, but $\angle ABC$ and $\angle DBE$ are not vertical angles.

Check

Find a counterexample to show that each conjecture is false.

a. If n is a real number, then $\frac{1}{n} < n$. Select all that apply.

(A.) $n = -3$ (B.) $n = \frac{1}{4}$ (C.) $n = 1$ D. $n = 5$ E. $n = 100$

b. If a line intersects a segment at its midpoint, then the line is perpendicular to the segment. Draw a diagram to represent the counterexample.

Sample answer:

Think About It!

What is a counterexample for the conjecture *All silvery coins are nickels*?

Sample answer: Quarters

What does the prefix *counter-* mean? How does this meaning relate to a counterexample?

Sample answer: *Counter-* means the opposite of. A counterexample is the opposite of an example, or a false example.

Problem-Solving Tip

Draw a Diagram Remember that a counterexample can be a number , a drawing, or a statement that proves a conjecture to be false. If you are struggling to find a counterexample, try drawing a diagram. This will allow you to analyze the situation and determine the validity of the conjecture.

Go Online Y ou can complete an Extra Example online.

706 Module 12 • Logical Arguments and Line Relationships

Interactive Presentation

Learn

FLASHCARDS

Students use flashcards to see different counterexamples of given conjectures.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Counterexamples

Objective

Students disprove conjectures by using counterexamples.

Teaching the Mathematical Practices

3 Analyze Cases Work with students to look at the Think About It! feature. Have students identify a counterexample that disproves the conjecture.

Important to Know

To help students understand what a counterexample of a particular conjecture will look like, ask them to explain what must be true about a counterexample to prove that the conjecture is false.

Example 5 Find Counterexamples

Teaching the Mathematical Practices

3 Analyze Cases Work with students to look at each conjecture in this example. Ask students to identify a counterexample that disproves each erroneous conjecture.

Questions for Mathematical Discourse

AL In part **a**, what is another counterexample, and what is not a counterexample? Sample answer: $n = -5$ is a counterexample and $n = \frac{1}{4}$ is not a counterexample

OL In part **a**, what other numbers work as a counterexample? Sample answer: any number < 0

BL In general, when would you use a counterexample to prove a statement false? Sample answer: When you can find one instance where the statement is false, you can use it as a counterexample.

Common Error

Students may try to find examples that support the conjecture rather than examples that prove it to be false.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–23
2	exercises that use a variety of skills from this lesson	24–30
3	exercises that emphasize higher-order and critical-thinking skills	31–36

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–29 odd, 31–36
- Extension: Mathematical Induction
- ALEKS Patterns and Inductive Reasoning

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–35 odd
- Remediation, Review Resources: Powers and Exponents
- Personal Tutors
- Extra Examples 1–5
- ALEKS Exponents and Order of Operations

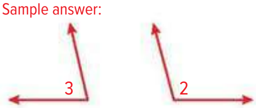
AL

IF students score 65% or less on the Checks,
THEN assign:

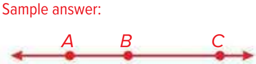
- Practice, Exercises 1–23 odd
- Remediation, Review Resources: Powers and Exponents
- ALEKS Exponents and Order of Operations

Answers

20. Sample answer:



21. Sample answer:



Practice

Go Online Y ou can complete your homework online.

Example 1

Write a conjecture that describes the pattern in each sequence. Then use your conjecture to find the next term in the sequence.

1. 4, 8, 12, 16, 20
Each term in the pattern is four more than the previous term; 24.

2. 2, 22, 222, 2222
Each term has an additional digit two as part of the number; 22222.

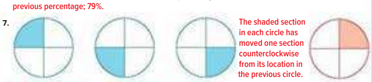
3. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$
Each term is one half the previous term; $\frac{1}{16}$.

4. $6, \frac{11}{2}, 5, \frac{9}{2}, 4$ Each term is one half less than the previous term; $\frac{7}{2}$.

5. A rrival times: 3:00 P.M., 12:30 P.M., 10:00 A.M., . . . Each arrival time is 2 hours and 30 minutes prior to the previous arrival time; 7:30 A.M.

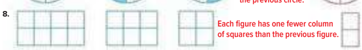
6. P ercent humidity: 100%, 93%, 86%, . . . Each percentage is 7% less than the previous percentage; 79%.

7.



The shaded section in each circle has moved one section counterclockwise from its location in the previous circle.

8.



Each figure has one fewer column of squares than the previous figure.

Examples 2 and 3

Make a conjecture about each value or geometric relationship.

9. the product of two odd numbers The product is an odd number.

10. the product of two and a number, plus one The result is odd.

11. the relationship between a and c if $ab = bc, b \neq 0$ They are equal.

12. the relationship between a and b if $ab = 1$ They are reciprocals.

13. the relationship between two intersecting lines that form four congruent angles
The lines are perpendicular.

14. the relationship between the angles of a triangle with all sides congruent
All the angles are congruent.

15. the relationship between NP and PQ if point P is midpoint of NQ $NP = PQ$

16. the relationship between the volume of a prism and a pyramid with the same base and equal heights The volume of the prism is three times the volume of the pyramid.

Example 4

17. RAMPS Xio is rolling marbles down a ramp. Every second that passes, she measures how far the marbles travel. She records the information in the table shown below.

Second	1st	2nd	3rd	4th	
Distance (cm)	20	60	100	140	

Make a conjecture about how far the marble will roll in the fifth second. 180 cm

Example 5

Determine whether each conjecture is *true* or *false*. Find a counterexample for any false conjecture.

18. I f n is a prime number, then $n + 1$ is not prime. False; sample answer: If $n = 2$, then $n + 1 = 3$, a prime number.

19. I f x is an integer, then $-x$ is positive. False; sample answer: Suppose $x = 2$, then $-x = -2$.

20. If $\angle 2$ and $\angle 3$ are supplementary angles, then $\angle 2$ and $\angle 3$ form a linear pair.
False; see margin for counterexample.

21. If you have three points A, B, and C, then A, B, and C are noncollinear.
False; see margin for counterexample.

22. If in $\triangle ABC$, $(AB)^2 + (BC)^2 = (AC)^2$, then $\triangle ABC$ is a right triangle. true

23. If the area of a rectangle is 20 square meters, then the length is 10 meters and the width is 2 meters. False; sample answer: The length could be 4 m, and the width could be 5 m.

Mixed Exercises

24. REASONING Given: $2a^2 = 72$. Conjecture: $a = 6$. Write a counterexample. $a = -6$

25. CONSTRUCT ARGUMENTS Barbara is in charge of the award medals for a sporting event. She has 31 medals to present to various individuals on 6 competing teams. She asserts that at least one team will end up with more than 5 medals. Do you believe her assertion? Justify your argument. Y es; sample answer: If no team got more than 5 medals, then the total number of medals could not be more than 5×6 or 30 medals.

26. USE TOOLS Miranda is developing a chart that shows her ancestry. She makes the three sketches shown. The first dot represents herself. The second sketch represents herself and her parents. The third sketch represents herself, her parents, and her grandparents. Sketch what you think would be the next figure in the sequence. See margin.

27. REGULARITY The figure shows a sequence of squares each made out of identical square tiles.

a. Starting from zero tiles, how many tiles do you need to make the first square? How many tiles do you have to add to the first square to get the second square? How many tiles do you have to add to the second square to get the third square? **1, 3, 5**

b. Make a conjecture about the list of numbers that you started writing in your answer to **part a**. **Y ou get all the odd numbers.**

c. Make a conjecture about the sum of the first n odd numbers. **Their sum equals n^2.**

28. STRUCTURE Adric made the following pattern by connecting points with line segments.

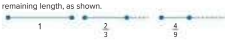
a. Suppose Adric continues the pattern. How many line segments will he need to make 4 triangles? 5 triangles? **4 triangles: 9 line segments; 5 triangles: 11 line segments**

b. Suppose Adric makes n triangles. Make a conjecture about the number of line segments he will need to make the triangles. **He will need $2n + 1$ line segments.**

c. Compare the number of line segments to the number of points in each step of the pattern. How many more line segments than points will there be if Adric continues the pattern to 4 triangles? 5 triangles? Extend the pattern to make a conjecture stating how many more line segments than points are needed to draw n triangles. **See margin.**

29. A prime number is a number, other than 1, that is divisible by only itself and 1. Lucille read that prime numbers are very important in cryptography, so she decided to find a systematic way of producing prime numbers. After some experimenting, she conjectured that $2^n - 1$ is a prime for all whole numbers $n > 1$. Find a counterexample to this conjecture. **Sample answer: When $n = 4$, $2^n - 1 = 15$ and $15 = 3 \times 5$.**

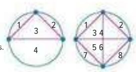
30. A line segment of length 1 is repeatedly shortened by removing one third of its remaining length, as shown.



Find and use a pattern to make a conjecture about the length of the line segment after being shortened n times. **See margin.**

Higher-Order Thinking Skills

31. PERSEVERE If you draw points on a circle and connect every pair of points, then the circle is divided into regions. For example, two points form two regions, three points form four regions, and four points form eight regions.



a. Make a conjecture about the relationship between the number of points on a circle and the number of regions formed in the circle. **Sample answer: The number of regions doubles when you add a point on the circle.**

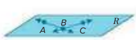
b. Does your conjecture hold true when there are six points? Support your answer with a diagram. **For six points, there should be 32 regions; however only 31 regions are formed. The conjecture is false. See margin for diagram.**

32. CREATE Write a number sequence that can be generated by two different patterns. Explain your patterns. **See margin.**

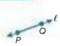
33. ANALYZE Consider the conjecture *If two points are equidistant from a third point, then the three points are collinear*. Is the conjecture *true* or *false*? Justify your argument. If false, give a counterexample. **See margin.**

In Exercises 34 and 35, use undefined terms, definitions, and/or postulates to explain why each conjecture is true.

34. WRITE Wei drew the figure at the right. Then she stated the following conjecture: A plane contains at least two lines. **See margin.**



35. WRITE Andre drew the figure at the right. Then he stated the following conjecture: Every line contains at least one line segment. **See margin.**



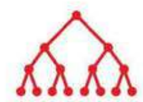
36. ANALYZE Kayla owns a company that makes patios and garden paths out of square tiles. The figures show the patterns used to make paths of different lengths.

a. Kayla would like to find the number of tiles needed to make a path of any length. Look for a pattern and make a conjecture about the number of tiles needed to make a path of length n. **$2n + 2$**

b. Kayla said that one path her company made last week required exactly 103 tiles. Is this possible? Justify your argument. **See margin.**

Answers

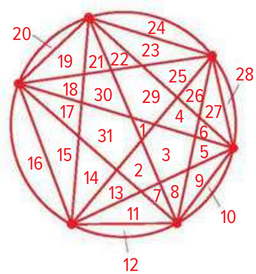
26. Sample answer:



28c. Let L be line segments and P be points. 1 triangle: $L - P = 0$; 2 triangles: $L - P = 1$; 3 triangles: $L - P = 2$; 4 triangles: $L - P = 3$; 5 triangles: $L - P = 4$. To draw n triangles, $L - P = n - 1$. There will be $n - 1$ more line segments than points if n triangles are drawn.

30. The new length of the line segment at each step is $\frac{2}{3}$ the length of the previous line segment. The length of the line segment after being shortened n times is $\left(\frac{2}{3}\right)^n$.

31b. Sample answer:



32. Sample answer: 2, 4, 8, 16, 32, . . . Each number in the sequence can be generated by adding each number to itself to form the next number. Each number in the sequence is 2^n, where $n \geq 1$.

33. False; sample answer: If the two points create a straight angle that includes the third point, then the conjecture is true. If the two points do not create a straight angle with the third point, then the conjecture is false.

34. Sample answer: A postulate states that a plane contains at least three noncollinear points. Another postulate states that if two points lie in a plane, then the line containing the points lies in the plane. Because the points are noncollinear, the points determine at least two distinct lines that lie in the plane.

35. Sample answer: A postulate states that a line contains at least two points. These two points and all the points between them are line segments, by the definition of a line segment.

36b. No; sample answer: The number of tiles is $2n + 2$, where n is the length. For any whole-number value of n, the value of $2n + 2$ is even; however, 103 is odd, so this cannot be the number of tiles.

Lesson 12-2

Statements, Conditionals, and Biconditionals

LESSON GOAL

Students write and analyze compound statements by using logic.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Truth Values

Develop:

Using Logic
- Truth Values of Conjunctions
- Truth Values of Disjunctions

Conditionals
- Identify the Hypothesis and Conclusion
- Write a Conditional in If-Then Form
- Related Conditionals

Biconditionals
- Write Biconditionals
- Determine Truth Values of Biconditionals

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL	
Extension: Sudoku		●	●		●

Language Development Handbook

Assign page 72 of the *Language Development Handbook* to help your students build mathematical language related to writing and analyzing compound statements by using logic.

ELL You can use the tips and suggestions on page T72 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
3 Reason abstractly and quantitatively.
4 Construct viable arguments and critique the reasoning of others.
5 Use appropriate tools strategically.
6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students wrote and analyzed conjectures by using inductive reasoning.

Now
Students write and analyze compound statements by using logic.

Next
Students will use conditional statements to solve math problems.

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

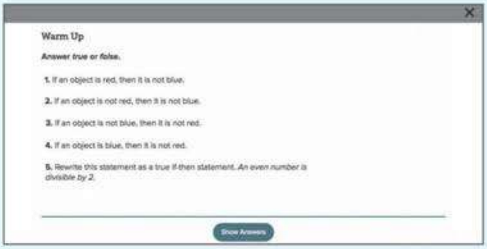
Conceptual Bridge In this lesson, students develop an understanding of statements. They write, analyze, and determine truth values of conditional statements.

Mathematical Background

A *statement* is a sentence that is either true or false, but not both. The truth or falsity of a statement is called its *truth value*. The negation of a statement p is denoted not p or $\sim p$ and has the opposite meaning as well as an opposite truth value.

A conditional statement is a statement that can be written in if-then form: *if p, then q*. A conditional statement is true in all cases except where the hypothesis is true and the conclusion is false. A biconditional statement, or *p if and only if q*, is true when both the conditionals, *if p, then q* and *if q, then p,* are true.

Interactive Presentation



Warm Up

Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- determining truth values of statements

Answers:

1. true
2. false
3. false
4. true
5. If a number is even, then it is divisible by 2. If a number is divisible by 2, then it is even.

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of conditional statements.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Truth Values

Objective

Students watch a video about red pandas and determine the truth value of statements in a series of questions.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

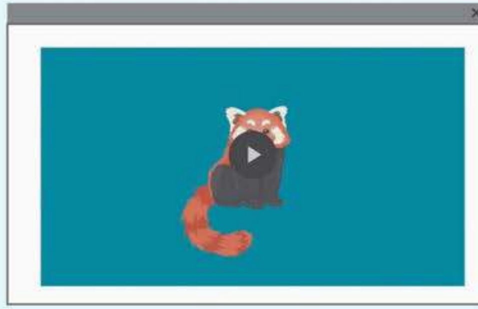
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students watch a video about red pandas. They use the information given in the video to determine the truth value of given statements. Then, students will answer the Inquiry Question.

(*continued on the next page*)

Interactive Presentation



Explore

WATCH

Students can watch a video that provides information about red pandas to explore the truth values of statements.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Truth Values (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why is a statement that uses the word *or* true if only one part of the statement is true? Sample answer: Because the statement uses *or*, if one part of the statement is true or the other part of the statement is true, then the statement is true.
- How else can you complete the following statement so that it is true? If an animal is a red panda, then it___. Sample answer: is an endangered species; is more similar to a racoon than a giant panda; spends most of its time in trees.

Inquiry

How can you determine the truth value of a statement? Sample answer: You can use given information to determine whether all or part of a statement is true. If you believe a statement is false, you can provide a counterexample that proves the statement is false.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Using Logic

Objective

Students write compound statements for conjunctions and disjunctions and determine truth values of statements.

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This Learn asks students to justify their conclusions.

Essential Question Follow-Up

Students have begun to learn to use logic.

Ask:

Why is it important to understand the truth values of combinations of statements? Sample answer: If you know that a statement or a combination of statements is true, then you can use it in a logical argument.

Example 1 Truth Values of Conjunctions

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL In part **a**, why is p false? Sample answer: A trapezoid has exactly one pair of parallel sides, but this shape has two pairs of parallel sides.

OL Using the given statements, write another compound statement. What is its truth value? Sample answer: $q \wedge r$. The figure has four congruent sides, and the figure has four right angles. Both q and r are true, so $q \wedge r$ is true.

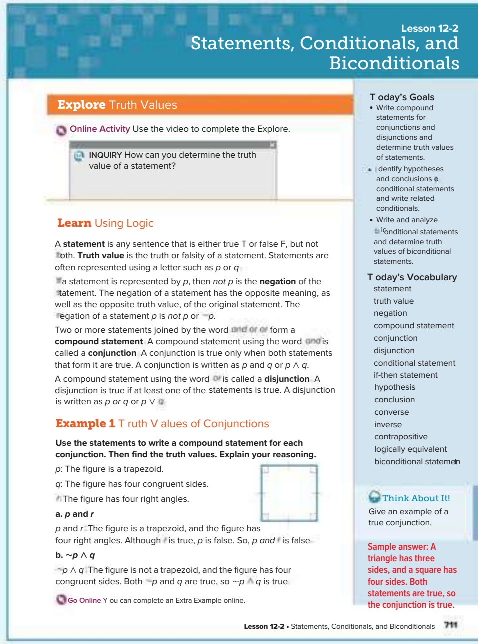
BL Write a compound statement for $p \wedge q \wedge \sim r$. What is its truth value? The figure is a trapezoid and the figure has four sides and the figure does not have four right angles. The compound statement is false.

Common Error

Students may assume that the negation of a statement is always false, when it has the opposite truth value of the original statement.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 12-2
Statements, Conditionals, and Biconditionals

Explore Truth Values

Online Activity Use the video to complete the Explore.

INQUIRY How can you determine the truth value of a statement?

Learn Using Logic

A **statement** is any sentence that is either true T or false F, but not both. **Truth value** is the truth or falsity of a statement. Statements are often represented using a letter such as p or q.

If a statement is represented by p, then *not p* is the **negation** of the statement. The negation of a statement has the opposite meaning, as well as the opposite truth value, of the original statement. The negation of a statement p is *not p* or $\sim p$.

Two or more statements joined by the word *and* or *or* form a **compound statement**. A compound statement using the word *and* is called a **conjunction**. A conjunction is true only when both statements that form it are true. A conjunction is written as p and q or $p \wedge q$.

A compound statement using the word *or* is called a **disjunction**. A disjunction is true if at least one of the statements is true. A disjunction is written as p or q or $p \vee q$.

Example 1 Truth Values of Conjunctions

Use the statements to write a compound statement for each conjunction. Then find the truth values. Explain your reasoning.

p: The figure is a trapezoid.

q: The figure has four congruent sides.

r: The figure has four right angles.

a. p **and** r

p and r: The figure is a trapezoid, and the figure has four right angles. Although r is true, p is false. So, p and r is false.

b. $\sim p \wedge q$

$\sim p \wedge q$: The figure is not a trapezoid, and the figure has four congruent sides. Both $\sim p$ and q are true, so $\sim p \wedge q$ is true.

Go Online You can complete an Extra Example online.

Today's Goals
- Write compound statements for conjunctions and disjunctions and determine truth values of statements.
- Identify hypotheses and conclusions of conditional statements and write related conditionals.
- Write and analyze biconditional statements and determine truth values of biconditional statements.

Today's Vocabulary
statement
truth value
negation
compound statement
conjunction
disjunction
conditional statement
if-then statement
hypothesis
conclusion
converse
inverse
contrapositive
logically equivalent
biconditional statement

Think About It!
Give an example of a true conjunction.

Sample answer: A triangle has three sides, and a square has four sides. Both statements are true, so the conjunction is true.

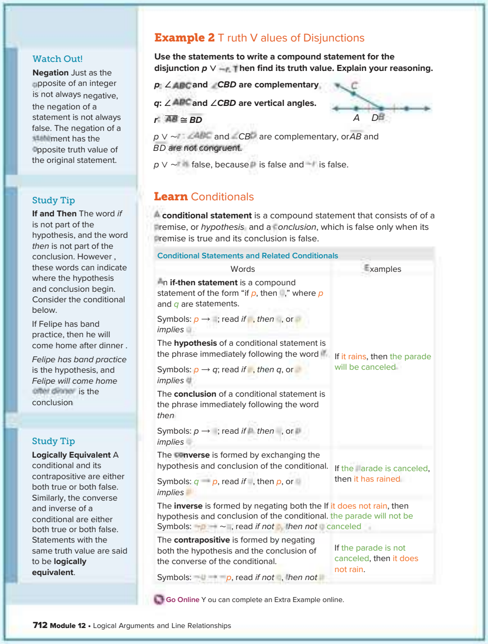
Lesson 12-2 • Statements, Conditionals, and Biconditionals 711

Interactive Presentation

Learn

TAP

Students select *True* or *False* to determine truth values.



Watch Out!

Negation Just as the opposite of an integer is not always negative, the negation of a statement is not always false. The negation of a statement has the opposite truth value of the original statement.

Study Tip

If and Then The word *if* is not part of the hypothesis, and the word *then* is not part of the conclusion. However, these words can indicate where the hypothesis and conclusion begin. Consider the conditional below.

If Felipe has band practice, then he will come home after dinner.

Felipe has band practice is the hypothesis, and *Felipe will come home after dinner* is the conclusion

Study Tip

Logically Equivalent A conditional and its contrapositive are either both true or both false. Similarly, the converse and inverse of a conditional are either both true or both false. Statements with the same truth value are said to be **logically equivalent**.

Example 2 Truth Values of Disjunctions

Use the statements to write a compound statement for the disjunction $p \vee \sim r$. Then find its truth value. Explain your reasoning.

p: $\angle ABC$ and $\angle CBD$ are complementary.

q: $\angle ABC$ and $\angle CBD$ are vertical angles.

r: $\overline{AB} \cong \overline{BD}$

$p \vee \sim r$: $\angle ABC$ and $\angle CBD$ are complementary, or $\overline{AB}$ and $\overline{BD}$ are not congruent.

$p \vee \sim r$ is false, because p is false and $\sim r$ is false.

Learn Conditionals

A **conditional statement** is a compound statement that consists of of a premise, or *hypothesis*, and a *conclusion*, which is false only when its premise is true and its conclusion is false.

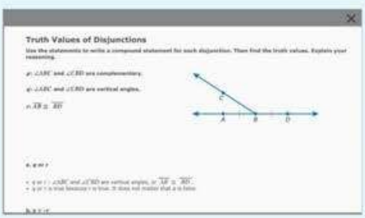
Conditional Statements and Related Conditionals

Words	Examples
An **if-then statement** is a compound statement of the form "if p, then q," where p and q are statements. Symbols: $p \rightarrow q$; read *if* p, *then* q, or p *implies* q	
The **hypothesis** of a conditional statement is the phrase immediately following the word *if*. Symbols: $p \rightarrow q$; read *if* p, *then* q, or p *implies* q	If it rains, then the parade will be canceled.
The **conclusion** of a conditional statement is the phrase immediately following the word *then*. Symbols: $p \rightarrow q$; read *if* p, *then* q, or p *implies* q	
The **converse** is formed by exchanging the hypothesis and conclusion of the conditional. Symbols: $q \rightarrow p$, read if q, then p, or q *implies* p	If the parade is canceled, then it has rained.
The **inverse** is formed by negating both the hypothesis and conclusion of the conditional. Symbols: $\sim p \rightarrow \sim q$, read *if not* p, *then not* q	If it does not rain, then the parade will not be canceled.
The **contrapositive** is formed by negating both the hypothesis and the conclusion of the converse of the conditional. Symbols: $\sim q \rightarrow \sim p$, read *if not* q, *then not* p	If the parade is not canceled, then it does not rain.

Go Online You can complete an Extra Example online.

712 **Module 12** • Logical Arguments and Line Relationships

Interactive Presentation



Example 2

TAP

Students tap to learn about negation.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 2 Truth Values of Disjunctions

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Important to Know

To help students understand when a disjunction is true, list all the possible combinations of the two statements and their negations, and ask students the truth value in each situation.

Questions for Mathematical Discourse

AL What is the difference between a conjunction and a disjunction? Sample answer: If the compound statement uses *and* then it is a conjunction. However, if the compound statement uses *or* then it is a disjunction.

OL Is it possible for a disjunction to be written with more than one true statement? Explain. Yes; sample answer: If three statements are used to write the disjunction, then two of the statements will be true.

BL Write a compound statement for $p \vee q \vee \sim r$. What is its truth value? $\angle ABC$ and $\angle CBD$ are complementary or $\angle ABC$ and $\angle CBD$ are vertical angles or $\overline{AB} \ncong \overline{BD}$. $p \vee q \vee \sim r$ is false, because p is false and q is false and $\sim r$ is false.

Common Error

Students may confuse the symbols for conjunctions and disjunctions. Tell them that the symbol $\wedge$ looks like the *A* in *And*.

Learn Conditionals

Objective

Students identify hypotheses and conclusions of conditional statements, and write related conditionals.

Teaching the Mathematical Practices

2 Represent a Situation Symbolically Guide students to define variables to solve the problem in this Learn. Help students to identify the different variables. Then work with them to find the other relationships in the problem.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Example 3 Identify the Hypothesis and Conclusion

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL How do you know which statement is the hypothesis? Sample answer: The hypothesis follows the word *if*.

OL In part **b**, what is the statement if it is written with the hypothesis before the conclusion? If the first performance is sold out, then another performance will be scheduled.

BL In a conditional statement, does the hypothesis always precede the conclusion? No; sample answer: In part **b**, the conclusion precedes the hypothesis.

Common Error

Students may not recognize conditional statements that are written without the words *if* and *then*. For example, the statement *all squares are rectangles* can be written as a conditional as *if a figure is a square, then it is a rectangle*.

Example 4 Write a Conditional in If-Then Form

Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to construct an argument.

Questions for Mathematical Discourse

AL In a conditional statement, what depends on the hypothesis? the conclusion

OL In part **b**, how do you know that the hypothesis is *Two angles are supplementary*? Sample answer: Having an angle sum of 180° depends on having angles that are supplementary.

BL In part **b**, switch the hypothesis and conclusion. Is this new statement true? Explain. Yes; sample answer: If the sum of the measures of two angles is 180°, then the two angles are supplementary.

Example 3 Identify the Hypothesis and Conclusion

Identify the hypothesis and conclusion of each conditional statement.

a. If a polygon has six sides, then it is a hexagon.

Hypothesis: A polygon has six sides.

Conclusion: The polygon is a hexagon.

b. Another performance will be scheduled if the first one is sold out.

Notice that the word *if* appears in the second portion of the sentence.

Hypothesis: The first performance is sold out.

Conclusion: Another performance will be scheduled.

Check

Identify the hypothesis and conclusion of each conditional statement.

a. If the forecast is rain, then I will take an umbrella.

Hypothesis: ? The forecast is rain.

Conclusion: ? I will take an umbrella.

b. A number is divisible by 10 if its last digit is a 0.

Hypothesis: ? The last digit of a number is 0.

Conclusion: ? A number is divisible by 10.

Think About It!

If a conditional is true, are the converse and inverse *sometimes*, *always*, or *never* true? Support your answer with an example.

Sample answer: Sometimes; for the conditional *if a square has a side length of 4 inches, then it has an area of 16 square inches*, the converse and inverse are true. However, for the conditional *if an insect is a monarch butterfly, then it is has orange wings*, the converse and inverse are false.

Example 4 Write a Conditional in If-Then Form

Identify the hypothesis and conclusion for each conditional statement. Then write the statement in if-then form.

a. Four quarters can be exchanged for a $1 bill.

Hypothesis: You have four quarters.

Conclusion: You can exchange them for a $1 bill.

If-then: If you have four quarters, then you can exchange them for a $1 bill.

b. The sum of the measures of two supplementary angles is 180°.

Hypothesis: Two angles are supplementary.

Conclusion: The sum of their measures is 180°.

If-then: If two angles are supplementary, then the sum of their measures is 180°.

Think About It!

How do you identify the hypothesis and conclusion of a conditional statement when the statement is not in if-then form?

Sample answer: The hypothesis *implies* the conclusion in a conditional statement. To determine the hypothesis and conclusion, identify the cause and the effect. The hypothesis is the cause and the conclusion is the effect.

Go Online You can complete an Extra Example online.

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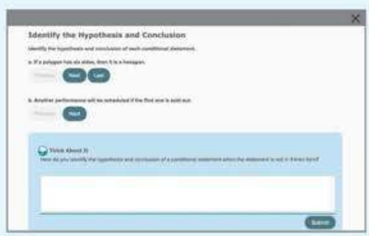
Interactive Presentation



Example 3

TYPE

Students type their answers for the Think About It! question.

Check

Identify the hypothesis and conclusion of the conditional statement *A polygon with two sets of parallel sides is a parallelogram*. Then write the statement in if-then form.

Hypothesis: ? A polygon has two sets of parallel sides.

Conclusion: ? The polygon is a parallelogram.

? If a polygon has two sets of parallel sides, then it is a parallelogram.

Example 5 Related Conditionals

NATURE **The tang is a saltwater fish that inhabits shallow coral reefs in tropical areas. Tangs are a part of the Acanthuridae family along with surgeonfish and unicornfish. All members of the Acanthuridae family are saltwater fish. Write the converse, inverse, and contrapositive of the true conditional statement *Tangs are fish that live in salt water*. Determine whether each related conditional is *true* or *false*. If a statement is false, then find a counterexample.**

Think About It!
Write a conditional statement in which the converse, inverse, and contrapositive are true.

Sample answer: If a number is divisible by two, then the number is even.

Conditional: *If a fish is a tang, then it lives in salt water.*

Converse: *If a fish lives in salt water, then it is a tang.*

Counterexample: A surgeonfish lives in salt water, but it is not a tang.

Therefore, the converse is false.

Inverse: *If a fish is not a tang, then it does not live in salt water.*

Counterexample: A surgeonfish is not a tang, but it does live in salt water. Therefore, the inverse is false.

Contrapositive: *If a fish does not live in salt water, then it is not a tang.*

Based on the information above, this statement is true.

Check

MUSIC Symphony orchestras contain instruments from 4 musical families: strings, woodwinds, brass, and percussion. However, string orchestras only contain string instruments. String instruments include the violin, viola, cello, bass, and harp. Write the converse, inverse, and contrapositive of the true conditional statement *If an orchestra is a string orchestra, then it contains string instruments*. Determine whether each related conditional is *true* or *false*. If the statement is false, find a counterexample.

Converse: ? If an orchestra contains string instruments, then it is a string orchestra. False; symphony orchestras contain string instruments.

Inverse: ? If an orchestra is not a string orchestra, then it does not contain string instruments. False; a symphony orchestra is not a string orchestra, but it does contain string instruments.

Contrapositive: ? If an orchestra does not contain string instruments, then it is not a string orchestra. True.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 5

TAP

Students tap through the parts of the Example.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 5 Related Conditionals

Teaching the Mathematical Practices

1 Check Answers Mathematically proficient students continually ask themselves, "Does this make sense?" Point out that in this example, students need to check their answer. Point out that they should ask themselves whether their answer makes sense and whether they have answered the question.

Questions for Mathematical Discourse

AL What are related conditionals? the converse, the inverse, and the contrapositive that can be formed from the original conditional statement

OL What pairs of statements are logically equivalent? the conditional and the contrapositive; the converse and the inverse

BL How do you form the converse from the original conditional? the inverse? the contrapositive? Exchange the conclusion and the hypothesis; negate the hypothesis and the conclusion; negate the hypothesis and the conclusion, then exchange them.

Common Error

Students may assume that the truth value of the converse is the same as the truth value of the original statement, but this is not always the case.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Biconditionals

Objective

Students write and analyze biconditional statements and determine truth values of biconditional statements.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Common Misconception

Students may assume that a biconditional is true when one of the related conditionals is true, but students must check both of the related conditionals. The biconditional is true only when both of the related conditionals are true.

Example 6 Write Biconditionals

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL What conditional statement applies to the following statement? The intersection of two lines is a point. Sample answer: If two lines intersect, then they intersect at a point.

OL Does every true statement have a true biconditional? Explain. No; sample answer: Some true statements have a false converse, so they have a false biconditional statement.

BL How do you form the biconditional from the original conditional? Sample answer: If the conditional statement and its converse are both true, then delete *if* and *then* from the conditional, and write *if* and *only if* between the hypothesis and conclusion.

Learn Biconditionals

You can use logic and *biconditional statements* to indicate exclusivity in situations. For example, Aarón is applying for admission into culinary school. He must earn a 3.5 GPA or higher this semester to be accepted. You can express this as two if-then statements.

- If he earns a 3.5 GPA or higher this semester, then he will be accepted.
- If Aarón is accepted into culinary school, then he has earned a 3.5 GPA or higher for the semester.

Biconditional Statement	
Words	A **biconditional statement** is the conjunction of a conditional and its converse.
Symbols	$(p \rightarrow q) \wedge (q \rightarrow p) \rightarrow (p \leftrightarrow q)$, read *p if and only if q*

So, the biconditional statement for the example above is *Aarón will be accepted into culinary school if and only if he earns a 3.5 GPA or higher this semester.*

Example 6 Write Biconditionals

Write the conditional and converse for each statement. Determine the truth values of the conditionals and converses. If false, find a counterexample. Write a biconditional statement if possible.

a. Rasha listens to music when she is in study hall.

Conditional: If Rasha is in study hall, then she is listening to music.

Is the conditional statement *true* or *false*? If false, provide a counterexample. true

Converse: If Rasha is listening to music, then she is in study hall.

Is the converse *true* or *false*? If false, provide a counterexample. False; sample answer: Rasha could be listening to music in the cafeteria.

Because the converse is false, a biconditional statement cannot be written.

b. If two lines are parallel, then they have the same slope.

Conditional: If two lines are parallel, then they have the same slope.

Converse: If two lines have the same slope, then they are parallel.

The conditional and the converse are true. So, a biconditional can be written.

Biconditional: Two lines are parallel if and only if they have the same slope.

Go Online You can complete an Extra Example online.

Study Tip
If and Only If The phrase *if and only if* can be abbreviated with *iff*.

Think About It! Compare the mathematical meanings of the symbols $\rightarrow$ and $\leftrightarrow$ in $p \rightarrow q$ and $p \leftrightarrow q$.

Sample answer: In the statement $p \rightarrow q$, the symbol $\rightarrow$ means that p implies q. In the statement, $p \leftrightarrow q$, the symbol $\leftrightarrow$ means that p implies q and q implies p. Because the arrow goes in both directions, both p and q imply the other.

Think About It! If a biconditional is true, what do you know about the conditional and converse? If a biconditional is false, what do you know about the conditional and converse?

Sample answer: If a biconditional is true, both the conditional and converse are true. If a biconditional is false, either the conditional is false or the converse is false or both.

Lesson 12-2 • Statements, Conditionals, and Biconditionals 715

Interactive Presentation

Learn

TYPE

Students compare the mathematical meanings of symbols.

Check

Write the conditional and converse for the statement. Determine the truth values of the conditional and converse. If false, find a counterexample. Write a biconditional statement if possible.

Isosceles triangles have at least two congruent sides.

Conditional: If a triangle is isosceles, then it has at least two congruent sides

Converse: If a triangle has at least two congruent sides, then it is isosceles

The conditional is true, and the converse is true.

Biconditional: A triangle is isosceles iff it has at least two congruent sides

Example 7 Determine T ruth V alues of Biconditionals

Write each biconditional as a conditional and its converse. Then determine whether the biconditional is *true* or *false*. If it is false, give a counterexample.

Two angles are complements if and only if their measures have a sum of 90°

Write the biconditional statement as a conditional.

Sample answer: If two angles are complements, then their measures have a sum of 90°

Write the converse of your conditional statement.

Sample answer: If the measures of two angles have a sum of 90°, then the two angles are complements.

The conditional and the converse are true, so the biconditional is true.

Go Online An alternate method is available for this example.

Check

Write the biconditional as a conditional and its converse. Then, determine whether the biconditional is *true* or *false*. If false, give a counterexample.

$x > -2$, if and only if x is positive.

Conditional: ___?___ If $x > -2$, then x is positive.

Converse: ___?___ If x is positive, then $x > -2$

The biconditional is ___?___ false, because ___?___ $x = -1$ is a counterexample

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 7

TYPE

Students type to complete the example.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Example 7 Determine T ruth Values of Biconditionals

Teaching the Mathematical Practices

1 Understand the Approaches of Others Work with students to look at the Alternate Method. Ask students to compare and contrast the original method and the alternate method.

Questions for Mathematical Discourse

AL When are biconditionals true? Sample answer: Both a conditional and its converse must be true for a biconditional to be true.

OL Does the order of the hypothesis and conclusion of a biconditional affect its truth value? Explain. No; sample answer: Because a conditional and its converse must both be true to write a biconditional, the order of the hypothesis and conclusion does not affect the truth value of the biconditional.

BL If the inverse of a conditional statement is false, can a biconditional statement be written? Explain. No; sample answer: Because the converse and inverse of a conditional statement are logically equivalent, if the inverse is false, then the converse is false. Therefore, a biconditional cannot be written.

DIFFERENTIATE

Language Development Activity AL BL ELL

IF students are having difficulty determining the truth values of the two conditional statements related to a biconditional,
THEN have students write the biconditional on a strip of paper. Cut the paper into pieces, separating the two component statements. Then make a framework on which to place the pieces of paper with the words "If ______ is true, does this mean that ______ must also be true?" Have students place the pieces of paper into the blanks in each order and have them answer the question.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–26
2	exercises that use a variety of skills from this lesson	27–31
2	exercises that extend concepts learned in this lesson to new contexts	32–38
3	exercises that emphasize higher-order and critical-thinking skills	39–50

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–37 odd, 39–50
- Extension: Sudoku
- ALEKS Deductive Reasoning

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–49 odd
- Personal Tutors
- Extra Examples 1–7
- ALEKS

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–25 odd
- ALEKS

Answers

1. $-3 - 2 = -5$, and vertical angles are congruent; p is true, and q is true, so p and q is true.
2. $-3 - 2 = -5$, and $2 + 8 > 10$; p is true, and r is false, so $p \wedge r$ is false.
3. Vertical angles are congruent, or $2 + 8 \leq 10$; q is true, and $\sim r$ is true, so $q \vee \sim r$ is true.
4. $2 + 8 > 10$, or vertical angles are congruent; r is false, and q is true, so $r \vee q$ is true.
5. $-3 - 2 \neq -5$, and not all vertical angles are not congruent; $\sim p$ is false, and $\sim q$ is false, so $\sim p \wedge \sim q$ is false.
6. $2 + 8 \leq 10$ or $-3 - 2 \neq -5$; $\sim r$ is true, and $\sim p$ is false, so $\sim r \vee \sim p$ is true.

12. H: you buy a 1-year membership; C: you get a free water bottle; If you buy a 1-year membership, then you get a free water bottle.
13. H: you were at the party; C: you received a gift; If you were at the party, then you received a gift.

Practice

Go Online You can complete your homework online.

Examples 1 and 2

Use the statements to write a compound statement for each conjunction or disjunction. Then find the truth values. Explain your reasoning.

p: $-3 - 2 = -5$

q: Vertical angles are congruent.

r: $2 + 8 > 10$

1. p and q See margin.
2. $p \wedge r$ See margin.
3. $q \vee \sim r$ See margin.
4. $r \vee q$ See margin.
5. $\sim p \wedge \sim q$ See margin.
6. $\sim r \vee \sim p$ See margin.

Example 3

Identify the hypothesis and conclusion of each conditional statement.

7. "If there is no struggle, there is no progress." (Frederick Douglass). H: there is no struggle; C: there is no progress
8. If two angles are adjacent, then they have a common side. H: two angles are adjacent; C: they have a common side
9. If you lead, then I will follow. H: you lead; C: I will follow
10. If $3x - 4 = 11$, then $x = 5$. H: $3x - 4 = 11$; C: $x = 5$
11. If two angles are vertical, then they are congruent. H: two angles are vertical; C: they are congruent

Example 4

Identify the hypothesis and conclusion for each conditional statement. Then write each statement in if-then form.

12. Get a free water bottle with a one-year membership. See margin.
13. Everybody at the party received a gift. See margin.
14. The intersection of two planes is a line. See margin.
15. The area of a circle is πr^2. See margin.
16. Collinear points lie on the same line. See margin.
17. A right angle measures 90 degrees. See margin.

Example 5

Write the converse, inverse, and contrapositive of each true conditional statement. Determine whether each related conditional is *true* or *false*. If a statement is false, then find a counterexample.

18. AIR TRAVEL Ulma is waiting to board an airplane. Over the speakers she hears a flight attendant say "If you are seated in rows 10 to 20, you may now board." See margin.
19. RAFFLE If you have five dollars, then you can buy five raffle tickets. See margin.
20. GEOMETRY If two angles are complementary, then the angles are acute. See margin.
21. MEDICATION A medicine bottle says "If you will be driving, then you should not take this medicine." See margin.

Lesson 12-2 • Statements, Conditionals, and Biconditionals 717

Example 6

Write the conditional and converse for each statement. Determine the truth values of the conditionals and converses. If false, find a counterexample. Write a biconditional statement if possible.

22. 89 is an even number if it is divisible by 2. See margin.
23. The game will be cancelled if it is raining. See margin.
24. Laura's soccer team plays on Saturdays. See margin.

Example 7

Write each biconditional as a conditional and its converse. Then determine whether the biconditional is *true* or *false*. If it is false, give a counterexample.

25. A polygon is a quadrilateral if and only if it has four sides. See margin.
26. An angle is acute if and only if it has a measure less than 90°. See margin.

Mixed Exercises

27. Find the truth value of $(p \wedge q) \vee r$. true
    p: $(-4)^2 > 0$
    q: An isosceles triangle has at least two congruent sides.
    r: Two angles, whose measure have a sum of 90, are supplements.
28. Suppose p and q are both false. What is the truth value of $(p \wedge \sim q) \vee \sim p$? true
29. What is the truth value of $(\sim p \vee q) \wedge r$ if p is true, q is false, and r is true? false
30. What is the truth value of $(\sim p \wedge q) \vee r$ if p is true, q is false, and r is true? true
31. CHOCOLATE Luca has a bag of miniature chocolate bars that come in two distinct types: dark and milk. Luca picks a chocolate out of the bag. Use the following statements to determine whether the statement $\sim(\sim p \vee \sim q)$ is true. yes
    p: the chocolate bar is dark chocolate
    q: the chocolate bar is milk chocolate

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32. Clark says that a parallelogram is a quadrilateral with equal opposite angles. Write his statement in if-then form.
If a figure is a parallelogram, then it is a quadrilateral with equal opposite angles.

33. REASONING Kala asked Elijah whether his hockey team won the game last night and whether he scored a goal. Elijah said "yes." Kalia then asked Goldi whether she or Elijah scored a goal at the game. Goldi said "yes." What can you conclude about whether or not Goldi scored? nothing

34. PRECISION If I roll two 6-sided dice and the sum of the numbers is 11, then one die must be a 5. Write the converse, inverse, and contrapositive of the true conditional statement. Determine whether each related conditional is *true* or *false*. If a statement is false, then find a counterexample. See margin.

For Exercises 35 and 36, use the following statement.

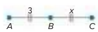
If a ray bisects an angle, then it divides the angle into two congruent angles.

35. Write the inverse of the given statement. If a ray does not bisect an angle, then it does not divide the angle into two congruent angles.

36. Write the contrapositive of the given statement. If a ray does not divide an angle into two congruent angles, then it does not bisect the angle.

37. Write the statement *All right angles are congruent* in if-then form. If two angles are right angles, then they are congruent.

38. Use the segment to write a statement that has the same truth value as $3 = 5$. Sample answer: $BC = 3 + x$



Higher-Order Thinking Skills

39. CREATE Consider a situation that can be represented with an if-then statement.

a. Write a true if-then statement for which the converse is false. Sample answer: If you are in Houston, then you are in Texas.

b. Write the converse, inverse, and contrapositive of your sentence. Sample answer: Converse: If you are in Texas, then you are in Houston. Inverse: If you are not in Houston, then you are not in Texas. Contrapositive: If you are not in Texas, then you are not in Houston.

c. Give the truth value of each statement you wrote for part **b.**
Converse: false; Inverse: false; Contrapositive: true

40. ANAL YZE Y ou are evaluating a conditional statement in which the hypothesis is true, but the conclusion is false. Is the inverse of the statement true or false? Justify your argument. True; sample answer: Because the conclusion is false, the converse of the statement must be true. The converse and inverse are logically equivalent, so the inverse is also true.

PERSEVERE T o negate a statement containing the words *all* or *for every*, you can use the phrase *at least one* or *there exists*. T o negate a statement containing the phrase *there exists*, use the phrase *for all* or *for every*.

p: All polygons are convex. ~*p*: *At least one* polygon is *not* convex.

q: *There exists* a problem that has no solution. ~*q*: *For every* problem, there is a solution.

Sometimes there are phrases that may be implied. For example, *The square of a real number is nonnegative* implies the following conditional and its negation.

p: *For every* real number x, $x^2 \geq 0$.

~*p*: *There exists* a real number x, such that $x^2 < 0$.

Use the information above to write the negation of each statement.

41. Every student at Hammond High School has a locker.
There exists at least one student at Hammond High school that does not have a locker.

42. All squares are rectangles. There exists at least one square that is not a rectangle.

43. There exists a real number x, such that $x^2 = x$. For every real number x, $x^2 \neq x$.

44. There exists a student who has at least one class in the C-Wing.
No students have classes in the C-wing.

45. Every real number has a real square root.
There exists a real number that does not have a real square root.

46. There exists a segment that has no midpoint. Every segment has a midpoint.

47. CREATE Research truth tables online. Then make a truth table to prove that an if-then statement is equivalent to its contrapositive and its inverse is equivalent to its converse. See Mod. 12 Answer Appendix.

48. WRITE Describe the relationship among a conditional, its converse, its inverse, and its contrapositive. See Mod. 12 Answer Appendix.

49. FIND THE ERROR Nicole and Kiri are evaluating the conditional *If 15 is prime, then 20 is divisible by 4*. Both think that the conditional is true, but their reasoning differs. Is either of them correct? Explain your reasoning. Kiri; sample answer: When the hypothesis of a conditional is false, the conditional is always true.

Nicole	Kiri
The conclusion is true because 20 is divisible by 4. So, the conditional is true.	The hypothesis is false because 15 is not prime. So, the conditional is true.

50. CREATE Write a conditional statement for which the converse, inverse, and contrapositive are all true. Explain your reasoning. See Mod. 12 Answer Appendix.

Answers

14. H: two planes intersect; C: the intersection is a line; If two planes intersect, then the intersection is a line.

15. H: a figure is a circle; C: the area is πr^2; If a figure is a circle, then the area is πr^2.

16. H: points are collinear; C: they lie on the same line; If points are collinear, then they lie on the same line.

17. H: an angle is right; C: the angle measures 90°; If an angle is right, then the angle measures 90°.

18. Converse: If you may board now, then you are seated in rows 10 to 20. The converse is true. Inverse: If you are not seated in rows 10 to 20, then you may not board now. The inverse is true. Contrapositive: If you are not allowed to board now, then you are not seated in rows 10 to 20. The contrapositive is true.

19. Converse: If you can buy five raffle tickets, then you have five dollars. The converse is true. Inverse: If you do not have five dollars, then you cannot buy five raffle tickets. The inverse is true. Contrapositive: If you cannot buy five raffle tickets, then you do not have five dollars. The contrapositive is true.

20. Converse: If you have two acute angles, then the angles are complementary. Counterexample: You have two acute angles, and the sum of the measures of the angles is less than 90°. The converse is false. Inverse: If two angles are not complementary, then the angles are not acute. Counterexample: You have two acute angles, and the sum of the measures of the angles is not 90°. The inverse is false. Contrapositive: If you have two angles that are not acute, then the angles are not complementary. The contrapositive is true.

21. Converse: If you do not take this medicine, then you can drive. The converse is true. Inverse: If you are not driving, then you can take this medicine. The inverse is true. Contrapositive: If you take this medicine, then you are not driving. The contrapositive is true.

22. Conditional: If 89 is divisible by 2, then it is an even number. The conditional is true. Converse: If 89 is an even number, then it is divisible by 2. The converse is true. Biconditional: 89 is divisible by 2 if and only if 89 is an even number.

23. Conditional: If it is raining, then the game will be cancelled. The conditional is true. Converse: If the game is cancelled, then it is raining. Counterexample: The game could be cancelled, and it is not raining. The converse is false. Because the converse is false, a biconditional statement cannot be written.

24. Conditional: If it is Saturday, then Laura's soccer team is playing. The conditional is true. Converse: If Laura's soccer team is playing, then it is Saturday. Counterexample: Laura's soccer team could be playing on Thursday. The converse is false. Because the converse is false, a biconditional statement cannot be written.

25. Conditional: If a polygon has four sides, then it is a quadrilateral. Converse: If a polygon is a quadrilateral, then it has four sides. The conditional and the converse are true, so the biconditional is true.

26. Conditional: If an angle is acute, then it measures less than 90°. Converse: If an angle measures less than 90°, then it is acute. The conditional and the converse are true, so the biconditional is true.

34. Converse: If I roll two 6-sided dice and one die is a 5, then the sum of the number is 11. Counterexample: I roll two 6-sided dice and one die is a 5, then the sum of the numbers is 9. The converse is false. Inverse: If I roll two 6-sided dice and the sum of the numbers is not 11, then one die is not a 5. Counterexample: I roll two 6-sided dice and the sum of the numbers is not 11, but one die is a 5. The inverse is false. Contrapositive: If I roll two 6-sided dice and one of the die is not a 5, then the sum is not 11. The contrapositive is true.

Lesson 12-3

Deductive Reasoning

LESSON GOAL

Students apply the Laws of Detachment and Syllogism.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Applying Laws of Deductive Reasoning by Using Venn Diagrams

Develop:

The Law of Detachment
- Inductive and Deductive Reasoning
- The Law of Detachment

The Law of Syllogism
- The Law of Syllogism

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Conditionals	●	●		●
Extension: Necessary and Sufficient Conditions		●	●	●

Language Development Handbook

Assign page 73 of the *Language Development Handbook* to help your students build mathematical language related to applying the Laws of Detachment and Syllogism.

ELL You can use the tips and suggestions on page T73 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
5 Use appropriate tools strategically.
6 Attend to precision.
7 Look for and make use of structure.
8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students explored the concept of conditional statements.

Now
Students apply the Laws of Detachment and Syllogism in deductive reasoning.

Next
Students will write proofs using various arguments.

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of deductive reasoning. They apply the Law of Detachment, determine the validity of conclusions, and they apply the Law of Syllogism to draw valid conclusions.

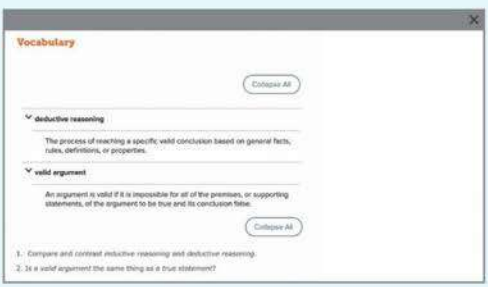
Mathematical Background

Deductive reasoning uses facts, rules, definitions, and properties to reach logical conclusions. A form of deductive reasoning that is used to draw conclusions from true conditional statements is called the Law of Detachment. This law states that if $p \longrightarrow q$ is true and p is true, then q is also true. The Law of Syllogism is another law of logic. It states that if $p \longrightarrow q$ and $q \longrightarrow r$ are true, then $p \longrightarrow r$ is also true.

Interactive Presentation

Warm Up

Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- determining the truth values of conditional statements

Answers:

1. true
2. false
3. false
4. true
5. false

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of deductive reasoning.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Applying Laws of Deductive Reasoning by Using Venn Diagrams

Objective

Students use dynamic geometry software to create Venn diagrams to determine the truth value of a statement.

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this Explore, students will need to use dynamic geometry software. Work with students to explore and deepen their understanding of Venn diagrams.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students are given three true conditional statements. They must then use these statements to create a Venn diagram that represents the statements. The guiding exercises relate students' knowledge to the information in the diagram. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore

Explore

WEB SKETCHPAD

Students use the sketch to explore Venn Diagrams.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Applying Laws of Deductive Reasoning by Using Venn Diagrams (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Which statement gives the most specific location? Which gives the most generic location? Sample answer: The first statement is the most specific because it gives the city of Tucson. The last statement is the most generic because it gives the continent.
- How does looking for generic or specific information help you draw the Venn diagram? Sample answer: Generic information will be the bigger or outer-most circles in your diagram. You know that more specific information will belong inside these circles.

Inquiry

How can you use Venn Diagrams to determine the truth value of a statement? Sample answer: Y ou can use the relationships between or among the circles of a Venn diagram to determine whether the conclusion of a statement is true based on whether the hypothesis is true.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn The Law of Detachment

Objective

Students apply the Law of Detachment to determine the validity of conclusions.

Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in this Learn.

Common Misconception

Students may think that the Law of Detachment also implies that if *p implies q* is true and *q* is true, then *p* must be true. Help them understand that the Law of Detachment only works if you know that the conditional and its hypothesis are true.

Example 1 Inductive and Deductive Reasoning

Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to construct an argument.

Questions for Mathematical Discourse

AL What's the difference between using inductive reasoning and deductive reasoning? Sample answer: Inductive reasoning uses examples or observations to make a conjecture while deductive reasoning uses facts and rules.

OL In part **a**, are facts or examples used to make the conjecture? On which type of reasoning is it based? facts; deductive

BL Do science experiments use inductive or deductive reasoning? Explain. Sample answer: Some experiments use inductive reasoning to determine if a pattern or relationship exists.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 12-3

Deductive Reasoning

Explore Applying Laws of Deductive Reasoning by Using Venn Diagrams

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How can you use Venn diagrams to determine the truth value of a statement?

Today's Goals
- Apply the Law of Detachment to determine the validity of conclusions.
- Apply the Law of Syllogism to make valid conclusions from given statements

Today's Vocabulary
deductive reasoning
valid argument

Learn The Law of Detachment

Unlike inductive reasoning, which uses a specific pattern of examples or observations to make a general conclusion, **deductive reasoning** uses general facts, rules, definitions, or properties to reach specific valid conclusions from given statements. An argument is **valid** if it is impossible for all the premises, or supporting statements, of the argument to be true and for its conclusion to be false. One law related to deductive reasoning is the Law of Detachment.

Key Concept • Law of Detachment	
Words	If $p \rightarrow q$ is a true statement and p is true, then q is true.
Example	*Given*: If a car is out of gas, then it will not start. Sarah's car is out of gas. *Valid Conclusion*: Sarah's car will not start.

Example 1 Inductive and Deductive Reasoning

Determine whether each conclusion is based on *inductive* or *deductive* reasoning.

a. If a student is late returning a library book, then he or she will be charged a $2 late fee. Chang returned a library book late, so he concludes that he will be charged a $2 late fee.

Chang is basing his conclusion on the library's policies, so he is using deductive reasoning.

b. Every time Tamika has worn her favorite jersey to a football game, her school's team has won the game. Tamika is wearing her favorite jersey to the football game tonight, so she concludes that her school's team will win the game.

Tamika is basing her conclusion on a specific pattern of observations, so she is using inductive reasoning.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Learn

MULTI-SELECT

Students choose whether each conclusion is based on inductive or deductive reasoning.

Check

Determine whether each conclusion is based on *inductive* or *deductive* reasoning.

a. Newton's first law of motion states that an object at rest will remain at rest unless acted on by an unbalanced force. Elisa watches a soccer ball roll across the field. She concludes that an unbalanced force has acted upon the soccer ball ___?___ deductive

b. Mrs. Jackson notices that her family's data usage is increasing by approximately 2500 megabytes of data every month. So, she concludes that her family's data usage next month will be 2500 megabytes greater than this month's data usage ___?___ inductive

Study Tip

Given Information All information provided after the *Given* can be assumed to be true. The *Conclusion* cannot be assumed to be true.

Example 2 The Law of Detachment

Determine whether each conclusion is valid based on the given information. Write *valid* or *invalid*. Explain your reasoning.

a. Given: To go on the field trip, a student must turn in a permission slip. Mariana turned in her permission slip.

Conclusion: Mariana can go on the field trip.

Step 1 Identify the hypothesis and conclusion.

Because a student must turn in a permission slip to go on the field trip, the phrase *a student must turn in a permission slip* is the hypothesis of the conditional statement.

p: A student turns in a permission slip.

q: The student can go on the field trip.

Step 2 Analyze the conclusion.

The given statement *Mariana turned in her permission slip* satisfies the hypothesis, so p is true. By the Law of Detachment, *Mariana can go on the field trip,* which matches q, is a true or valid conclusion.

b. Given: If a figure is a square, then it is a polygon.

Figure *A* is a polygon.

Conclusion: Figure *A* is a square.

Step 1 Identify the hypothesis and conclusion.

p: A figure is a square.

q: It is a polygon.

Step 2 Analyze the conclusion.

The given statement *Figure A is a polygon* satisfies the conclusion q of a true conditional. However, knowing that a conditional statement and its conclusion are true does not make the hypothesis true. Figure *A* could be a triangle. The conclusion is *invalid*.

Study Tip

Arguments An argument consists of reasons, proof, or evidence to support a position. A *logical argument*, such as the one shown in part **a**, is supported by the rules of logic. These rules include the Law of Detachment. This is different from a *statistical argument*, which is supported by examples or data.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 2

TYPE

Students answer a question to see if they recognize the common error.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Example 2 The Law of Detachment

Teaching the Mathematical Practices

1 Special Cases Work with students to evaluate the two methods shown. Encourage students to familiarize themselves with both methods, and to know the best time to use each one.

Questions for Mathematical Discourse

AL How do you know if a conclusion is valid or invalid? Sample answer: If the information presents a logical argument, then the conclusion is valid. If not, then the conclusion is invalid.

OL The conclusions are valid if the conditional statement is true. When is a conditional false? when the hypothesis is true and the conclusion is false

BL In part **b**, the conclusion of the conditional is true. Why can't you conclude that the hypothesis is also true? When the conclusion of a conditional is true, the entire conditional is true whether the hypothesis is true or false.

Common Error

The Law of Detachment can be applied only when the hypothesis of a conditional statement is true. If only the conclusion of a conditional is true, then the Law of Detachment cannot be used to make a conclusion about the situation.

Learn The Law of Syllogism

Objective

Students apply the Law of Syllogism to make valid conclusions from given statements.

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This Learn asks students to justify their conclusions.

Common Misconception

Students cannot just match any two components of the two given conditionals. The conclusion of one conditional must match the hypothesis of the other conditional for the Law of Syllogism to work.

Essential Question Follow-Up

Students learn the two main laws of deductive reasoning.

Ask:

Why is it important to understand the laws of Detachment and Syllogism for understanding logical arguments? Sample answer: These two laws are important tools for writing valid logical arguments.

DIFFERENTIATE

Enrichment Activity BL ELL

Write an example to illustrate the correct use of the Law of Syllogism.

Sample answer:

1. Students need to be organized.
2. If you are organized, then you have good study habits.
3. If you have good study habits, then you get good grades.
4. Students who are organized get good grades.

Check

Determine whether the conclusion is valid based on the given information. Select the correct answer and justification.

a. Given: If three points are noncollinear, then they determine a plane. Points A, B, and C lie in plane G.

Conclusion: Points A, B, and C are noncollinear.

A. Valid; points A, B, and C determine plane G. Therefore, they are noncollinear.

B. Valid; because points A, B, and C are noncollinear, they determine plane G.

C. Invalid; points A, B, and C determine plane G. Therefore, they are noncollinear.

(D.) Invalid; points A, B, and C can be collinear and lie in plane G.

b. Given: If Dakota goes to the video game store, then he will buy a new game. Dakota went to the video game store this afternoon.

Conclusion: Dakota bought a new game.

A. Invalid; because the statement *Dakota bought a new game* does not satisfy the hypothesis of the conditional statement, the conclusion is not true.

B. Valid; because the statement *Dakota went to the video game store this afternoon* satisfies the conclusion of the conditional statement, the hypothesis of the conditional is true.

(C.) Valid; because the statement *Dakota went to the video game store this afternoon* satisfies the hypothesis of the conditional statement, the conclusion is true.

D. Invalid; because the statement *Dakota went to the video game store this afternoon* satisfies only the hypothesis, the conclusion is not true.

Learn The Law of Syllogism

One law that is related to deductive reasoning is the Law of Syllogism. This law allows you to draw conclusions from two true conditional statements when the conclusion of one statement is the hypothesis of the other.

Key Concept • Law of Syllogism	
Words	If $p \rightarrow q$ and $q \rightarrow r$ are true statements, then $p \rightarrow r$ is a true statement.
Example	Given: If you get a job, then you will earn money. If you earn money, then you will buy a car. Valid Conclusion: If you get a job, then you will buy a car.

Talk About It!

Do you think that the order of the given statements is important when applying the Law of Syllogism? Justify your argument.

No; sample answer: As long as the conclusion of one statement is the hypothesis of the other statement, the Law of Syllogism can be applied.

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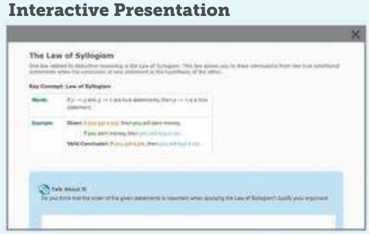
Interactive Presentation



Learn

TYPE

Students answer a question to show they understand the Law of Syllogism.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 The Law of Syllogism

SLEEP Scientists have found that the quality and amount of sleep greatly impact learning and memory. Lack of sleep causes students to have trouble focusing and receiving new information. Sleep deprivation also makes it difficult to retrieve previously-learned information. Draw a valid conclusion from the given statements, if possible.

Given: If you are tired, then you will not do well on your test.
If you do not get enough sleep, then you will be tired.

Step 1 Identify the hypothesis and conclusion that are the same.

Determine whether the conclusion of one statement is the hypothesis of the other statement.

Given: If you are tired, then you will not do well on your test.
If you do not get enough sleep, then you will be tired.

Reorder the given statements so the conclusion of the first statement is the hypothesis of the second statement. This will allow you to make a valid conclusion using the Law of Syllogism.

Given: If you do not get enough sleep,	p: You do not get enough sleep.
then you will be tired	q: You will be tired.
If you are tired, then you will not do well on your test.	r: You will not do well on your test.

Step 2 Represent the statements with symbols.

Let p, q, and r represent the parts of the given conditional statements. Analyze the logic of the given conditional statements using symbols.

Statement 1: $p \rightarrow q$

Statement 2: $q \rightarrow r$

Because both statements are true and the conclusion of the first statement is the hypothesis of the second statement, $p \rightarrow r$ by the Law of Syllogism. A valid conclusion is *If you do not get enough sleep, then you will not do well on your test.*

Check

GRAND CANYON The Grand Canyon covers an area of 1900 square miles and contains 277 miles of the Colorado River . Since the Grand Canyon became a national park in 1919, over 193 million people have visited.

Draw a valid conclusion from the given statements, if possible.

Given: If Ebony takes a vacation, then she will go to the Grand Canyon. If Ebony goes to the Grand Canyon, then she will hike to the Colorado River.

If Ebony takes a vacation, then she will hike to the Colorado River.

Go Online You can complete an Extra Example online.

Study Tip

True vs. Valid Conclusions A true conclusion is not the same as a valid conclusion. True conclusions reached using invalid reasoning are still invalid.

Think About It!

Can the Law of Syllogism be applied if the two given statements have the same conclusion? Justify your argument.

No; sample answer: Even though both statements have the same conclusion, the statements are not connected to each other unless there is a hypothesis and conclusion that are the same. Therefore, the Law of Syllogism cannot be applied.

Go Online to practice what you've learned about deductive reasoning in the Put It All Together over Lessons 12-1 through 12-3.

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Interactive Presentation

Example 3

TAP

Students tap to move through the example.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Law of Syllogism

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL In the first statement, what is the hypothesis? conclusion? The hypothesis is *you are tired*, and the conclusion is *you will not do well on your test*. In the second statement, what is the hypothesis? conclusion? The hypothesis is *you do not get enough sleep* and the conclusion is *you will be tired*.

OL How is the Law of Syllogism related to deductive reasoning? Sample answer: Both allow you to reach valid conclusions based on properties and facts.

BL Use the inverses of the original conditionals. Can you write a statement that is valid based on the Law of Syllogism? Show your work. Sample answer: Inverses: If you get enough sleep, then you won't be tired. If you are not tired, then you will do well on your test. So, if you get enough sleep, then you will do well on your test.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–17
2	exercises that use a variety of skills from this lesson	18–23
2	exercises that extend concepts learned in this lesson to new contexts	24–31
3	exercises that emphasize higherorder and critical-thinking skills	32–39

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–31 odd, 32–39
- Extension: Necessary and Sufficient Conditions
- ALEKS Deductive Reasoning

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–39 odd
- Remediation, Review Resources: Statements, Conditionals, and Biconditionals
- Personal Tutors
- Extra Examples 1–3
- ALEKS Deductive Reasoning

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Statements, Conditionals, and Biconditionals
- ALEKS Deductive Reasoning

Practice

Go Online Y ou can complete your homework online.

Example 1

Determine whether each conclusion is based on *inductive* or *deductive* reasoning.

1. At Fumio's school, if a student is late five times, then the student will receive a detention. Fumio has been late to school five times. Therefore, he will receive a detention. deductive
2. A dental assistant notices that a patient has never been on time for an appointment. She concludes that the patient will be late for her next appointment. inductive
3. A person must have a membership to work out at a gym. Jessie is working out at that gym. Jessie has a membership to that gym. deductive
4. If Emilio decides to go to a concert tonight, then he will miss football practice. Tonight, Emilio went to a concert. Emilio missed football practice. deductive
5. Every Wednesday, Jacy's mother calls. Today is Wednesday, so Jacy concludes that her mother will call. inductive
6. Whenever Juanita has attended a tutoring session, she notices that her grades have improved. Juanita attends a tutoring session, and she concludes her grades will improve. inductive

Example 2

Determine whether each conclusion is valid based on the given information. Write *valid* or *invalid*. Explain your reasoning.

7. **Given:** Right angles are congruent. $\angle 1$ and $\angle 2$ are right angles.
   Conclusion: $\angle 1 \cong \angle 2$ valid; Law of Detachment
8. **Given:** If a figure is a square, then it has four right angles. Figure *ABCD* has four right angles.
   Conclusion: Figure *ABCD* is a square. Invalid; the figure could be a rectangle.
9. **Given:** If you leave your lights on while your car is off, then your battery will die. Your battery is dead.
   Conclusion: You left your lights on while your car was off. Invalid; your battery could be dead because it was old.
10. **Given:** If Dennis gets a part-time job, then he can afford a car payment. Dennis can afford a car payment.
    Conclusion: Dennis got a part-time job. Invalid; Dennis could afford a car payment because he paid off his other bills.
11. **Given:** If 75% of the prom tickets are sold, then the prom will be held at the country club. 75% of the prom tickets were sold.
    Conclusion: The prom will be held at the country club. valid; Law of Detachment

Example 3

Use the Law of Syllogism to draw a valid conclusion from each set of given statements, if possible. If no valid conclusion can be drawn, write *no valid conclusion* and explain your reasoning.

12. If you interview for a job, then you wear a suit.
    If you interview for a job, then you will update your resume.
    No valid conclusion; the conclusion of statement (1) is not the hypothesis of statement (2).
13. If Tina has a grade point average of 3.0 or greater, she will be on the honor role.
    If Tina is on the honor role, then she will have her name in the school paper.
    If Tina has a grade point average of 3.0 or greater, then she will have her name in the school paper.
14. If two lines are perpendicular, then they intersect to form right angles.
    Lines *s* and *r* form right angles.
    No valid conclusion; the conclusion of statement (1) is not the hypothesis of statement (2).
15. If the measure of an angle is between 90° and 180°, then it is obtuse.
    If an angle is obtuse, then it is not acute.
    If the measure of an angle is between 90° and 180°, then it is not acute.
16. If two lines in a plane are not parallel, then they intersect.
    If two lines intersect, then they intersect in a point.
    If two lines in a plane are not parallel, then they intersect in a point.
17. If a number ends in 0, then it is divisible by 2.
    If a number ends in 4, then it is divisible by 2.
    No valid conclusion; the conclusion of statement (1) is not the hypothesis of statement (2).

Mixed Exercises

CONSTRUCT ARGUMENTS **Draw a valid conclusion from the given statements, if possible. Then state whether your conclusion was drawn using the Law of Detachment or the Law of Syllogism. If no valid conclusion can be drawn, write *no valid conclusion*. Justify your argument.**

18. **Given:** If a figure is a square, then all the sides are congruent. Figure *ABCD* is a square. Figure *ABCD* has all sides congruent; Law of Detachment.
19. **Given:** If two angles are complementary, the sum of the measures of the angles is 90°. $\angle 1$ and $\angle 2$ are complementary angles. The sum of the measures of $\angle 1$ and $\angle 2$ is 90°; Law of Detachment.
20. **Given:** Ballet dancers like classical music. If you like classical music, then you enjoy the opera. If you are a ballet dancer, then you enjoy the opera; Law of Syllogism.
21. **Given:** If you are an athlete, then you enjoy sports. If you are competitive, then you enjoy sports. No valid conclusion; the conclusion of statement (1) is not the hypothesis of statement (2).
22. **Given:** If a polygon is regular, then all of its sides are congruent. All of the sides of polygon *WXYZ* are congruent. No valid conclusion; knowing a conclusion is true does not imply the hypothesis will be true.
23. **Given:** If Terryl completes a course with a grade of C, then he will not receive credit. If Terryl does not receive credit, he will have to take the course again. If Terryl completes a course with a grade of C, then he will have to take the course again; Law of Syllogism.

USE TOOLS Determine whether each conclusion is valid based on the given information. Write *valid* or *invalid*. Explain your reasoning using a Venn diagram.

24. **Given:** If the temperature drops below 32°F, it may snow. The temperature did not drop below 32°F on Monday.
    Conclusion: It did not snow on Monday. See margin.

25. **Given:** All vegetarians do not eat meat. Theo is a vegetarian.
    Conclusion: Theo does not eat meat. See margin.

26. TUTORING Marla sometimes stays after school to tutor classmates. If it is Tuesday, then Marla tutors chemistry. If Marla tutors chemistry, then she arrives home at 4 P.M. Today Marla arrived home at 4 P.M. Can it be concluded that today is Tuesday? Explain your reasoning. **No; sample answer: Today could be Thursday, and Marla could have arrived home at 4 P.M. due to softball practice.**

27. MUSIC Composer Ludwig van Beethoven wrote 9 symphonies and 5 piano concertos. If you lived in Vienna in the early 1800s, then you could attend a concert conducted by Beethoven himself. Write a valid conclusion to the hypothesis: *If Mozart could not attend a concert conducted by Beethoven, . . .* **then Mozart did not live in Vienna in the early 1800s.**

28. DIRECTIONS Paolo has an appointment to see a financial advisor on the fifteenth floor of an office building. When he gets to the building, the people at the front desk tell him that if he wants to go to the fifteenth floor, then he must take the red elevator. While looking for the red elevator, a guard informs him that if he wants to find the red elevator, then he must find the replica of Michelangelo's David. When he finally got to the fifteenth floor, his financial advisor greeted him asking, "What did you think of the Michelangelo?" How did Paolo's financial advisor conclude that Paolo must have seen the Michelangelo statue? **He used the Law of Syllogism to conclude that if Paolo made it to the fifteenth floor, then he must have found the statue and then used the Law of Detachment to conclude that Paolo did find the statue**

29. SIGNS Two signs are posted outside a trampoline park. Inside the trampoline park, you see a child with a parent. Write a valid conclusion based on the given information about the age of the child.
    The child is at least 5 years old.

30. LOGIC As Maite's mother left for work, she quickly gave Maite some instructions. "If you need me, call my cell phone. If I do not answer, then it means I'm in a meeting. The meeting will not last more than 30 minutes, and I will call you back when the meeting is over." Later that day, Maite tried to call her mother's cell phone, but her mother was in a meeting and could not answer the phone. Maite concludes that she will have to wait no more than 30 minutes before she gets a call back from her mother. What law of logic did Maite use to draw this conclusion? **Law of Detachment**

Lesson 12-3 • Deductive Reasoning **727**

31. ENERGY Use deductive reasoning to draw a valid conclusion from the following statements: If a heat wave occurs, then air conditioning will be used more frequently; if air conditioning is used more frequently, then energy costs will be higher; there is a heat wave in Florida. If no valid conclusion can be drawn, then write *no valid conclusion* and explain your reasoning. **Energy costs will be higher in Florida.**

Higher-Order Thinking Skills

32. WRITE Explain why the Law of Syllogism cannot be used to draw a conclusion from these conditionals.
    If you wear winter gloves, then you will have warm hands.
    If you do not have warm hands, then your gloves are too thin.
    See margin.

33. PERSEVERE Use symbols for *conjunction*, *disjunction*, and *implies* to represent the Law of Detachment and the Law of Syllogism symbolically. Let p represent the hypothesis, and let q represent the conclusion. **Law of Detachment: $[(p \to q) \wedge p] \to q$ Law of Syllogism: $[(p \to q) \wedge (q \to r)] \to (p \to r)$**

34. CREATE Write a pair of statements in which the Law of Syllogism can be used to reach a valid conclusion. Specify the conclusion that can be reached. **See margin.**

35. ANALYZE Students in Mr. Kendrick's class are divided into two groups for an activity. Students in Group A must always tell the truth. Students in Group B must always lie. Jonah and Janeka are in Mr. Kendrick's class. When asked whether he and Janeka are in group A or B, Jonah says, "We are both in Group B." To which group does each student belong? Justify your argument. **See margin.**

36. WRITE Compare and contrast inductive and deductive reasoning when making conclusions and proving conjectures. **See margin.**

37. CREATE Write three statements that illustrate the Law of Syllogism. **See margin.**

38. CREATE Write three statements that illustrate the Law of Detachment. **See margin.**

39. WHICH ONE DOESN'T BELONG? Use statements (1) and (2). Determine which statement does not belong. Justify your conclusion.
    (1) *If a triangle is equilateral, then it has three congruent sides.*
    (2) *If all the sides of a triangle are congruent, then each angle measures 60°.*
    A. If a triangle is not equilateral, then it cannot have congruent angles.
    B. A figure with three congruent sides is always an equilateral triangle.
    C. If a triangle is not equilateral, then none of the angles measures 60°.
    D. If a triangle is equilateral, then each of its angles measures 60°.
    See margin.

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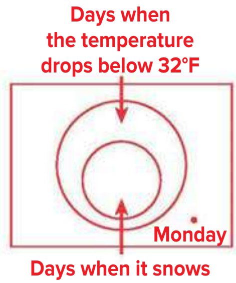
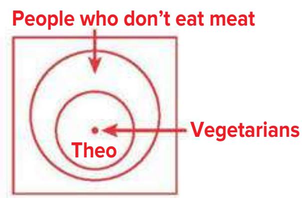
Answers

24. Valid; Monday is outside of the days when the temperature drops below 32°F, so it cannot be inside the days when it snows circle either. Thus, the conclusion is valid.



25. Valid; Theo is inside the small and large circles, so the conclusion is valid.



32. Sample answer: The Law of Syllogism cannot be used, because the hypothesis of the second conditional is the negation of the conclusion of the first conditional. To use the Law of Syllogism, the conclusion of one conditional must be the hypothesis of the other conditional.

34. Sample answer: (1) If a student earns 40 credits, then he or she will graduate from high school. (2) If a student graduates from high school, then he or she will receive a diploma. Conclusion: If a student earns 40 credits, then he or she will receive a diploma.

35. Sample answer: Jonah's statement can be restated as, "Jonah is in Group B, and Janeka is in Group B." For this compound statement to be true, both parts of the statement must be true. If Jonah was in Group A, he would not be able to say that he is in Group B, because students in Group A must always tell the truth. Therefore, the statement that Jonah is in Group B is true. For the compound statement to be false, the statement that Janeka is in Group B must be false, Therefore, Jonah is in Group B, and Janeka is in Group A.

36. Sample answer: Inductive reasoning uses several specific examples to reach a conclusion, and deductive reasoning relies on established facts, rules, definitions, and/or properties to reach a conclusion. One counterexample is enough to disprove a conjecture reached using inductive or deductive reasoning.

37. Sample answer: Given: If you are at the Willis Tower, then you are in Chicago. If you are in Chicago, then you are in Illinois. Conclusion: Therefore, if you are at the Willis Tower, then you are in Illinois.

38. Sample answer: Given: If two numbers are even, then their sum is even. The numbers 4 and 6 are even. Conclusion: The sum of 4 and 6 is even.

39. D; Statement D follows logically from statements (1) and (2). Statements A, B, and C do not follow logically from statements (1) and (2).

Lesson 12-4

Writing Proofs

LESSON GOAL

Students analyze and construct viable arguments.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Algebraic Proof

Develop:

Postulates About Points, Lines, and Planes
- Identify Postulates
- Use Postulates

Two-Column Proofs
- Two-Column Proof

Flow Proofs
- Flow Proofs

Paragraph Proofs
- Paragraph Proof

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Biconditoinals	●	●		●
Extension: Even and Odd		●	●	●

Language Development Handbook

Assign page 74 of the *Language Development Handbook* to help your students build mathematical language related to analyzing and constructing viable arguments.

ELL You can use the tips and suggestions on page T74 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1.5 days
45 min	3 days

Focus

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

3 Construct viable arguments and critique the reasoning of others.

Coherence

Vertical Alignment

Previous
Students developed an understanding of the validity of arguments.

Now
Students analyze and construct viable arguments.

Next
Students will construct arguments to prove geometric relationships.
G.CO.9

Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of the process of writing proofs, and they begin to build fluency in writing proofs.

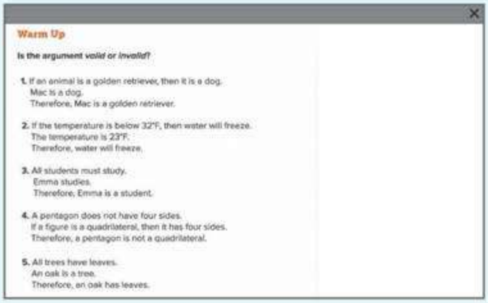
Mathematical Background

In geometry, a *postulate* is a statement that describes a fundamental relationship between the basic terms of geometry. Postulates are accepted as true without proof.

After a statement or conjecture is proved to be true, it is called a *theorem*. A theorem can be used like a definition or postulate to justify whether other statements are true.

A *proof* is a logical argument in which each statement you make is supported by a statement that is accepted as true. A proof states the hypotheses and conclusion and develops a system of deductive reasoning to prove the conclusion, assuming that the hypotheses are true.

Interactive Presentation



Warm Up

Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

• determining the validity of arguments

Answers:

1. invalid
2. valid
3. invalid
4. valid
5. valid

Launch the Lesson

Teaching the Mathematical Practices

3 Construct Arguments In this Launch the Lesson, students see a historical example of mathematical proof.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Algebraic Proof

Objective

Students apply the properties of real numbers to algebraic proofs.

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This Explore asks students to justify their conclusions.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

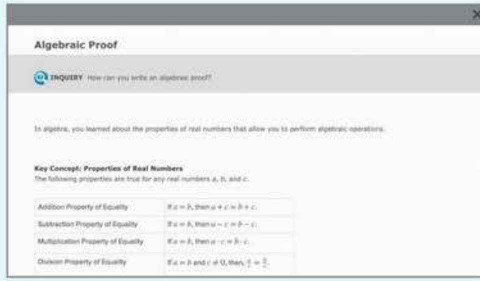
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

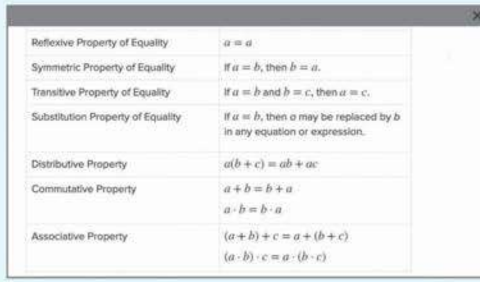
Students will read properties of real numbers. Then students will complete four guiding exercises where they determine which property is being used or complete a statement based on a property. Then students use the properties of real numbers to complete two proofs. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

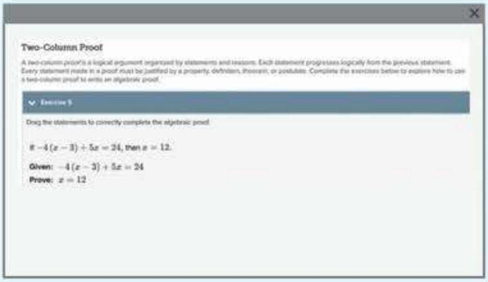


Explore



Explore

Interactive Presentation



Explore

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Algebraic Proof (*continued*)

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions

Have students complete the Explore activity.

Ask:

- Why is it important to justify or explain each step when solving an algebraic problem? Sample answer: If you can justify each step, then there is a mathematical reason for it and it must be true.
- How is writing an algebraic proof similar to solving a problem? Sample answer: I follow the same steps as I would to solve, but make sure to justify each step and use the correct properties.

Inquiry

How can you write an algebraic proof? Sample answer: Write each step in solving an algebraic equation in the Statements column of a two-column proof, and then write the corresponding property of real numbers in the Reasons column for each corresponding step.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Postulates About Points, Lines, and Planes

Objective

Students analyze figures to identify and use postulates about points, lines, and planes.

Teaching the Mathematical Practices

3 Analyze Cases This Learn guides students to examine cases of postulates on points, lines, and planes. Encourage students to familiarize themselves with all of the cases.

Common Misconception

Students occasionally think that postulates must be proved. However, postulates are accepted as true and are used to prove conjectures and theorems.

Example 1 Identify Postulates

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about postulates to solving a real-world problem.

Questions for Mathematical Discourse

AL Why are postulates helpful? Sample answer: Postulates are used to create mathematical proofs.

OL Which postulates relate to lines? 3.1, 3.3, 3.5, 3.6, 3.7

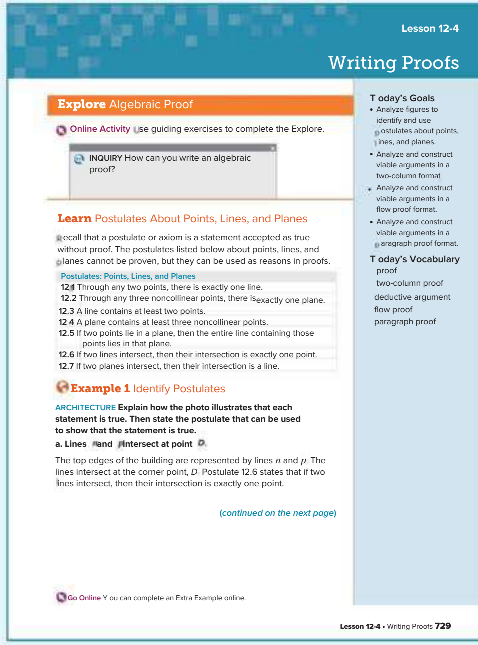
BL In the photo, what is an example of Postulate 3.4? Sample answer: Points *A*, *E*, and *D* lie in the same plane.

Common Error

Students may incorrectly match geometric objects with hypotheses of postulates. Remind them that they can rewrite postulates in if-then form to more easily identify the hypotheses.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.


Lesson 12-4

Writing Proofs

Explore Algebraic Proof

Online Activity Use guiding exercises to complete the Explore.

INQUIRY How can you write an algebraic proof?

Learn Postulates About Points, Lines, and Planes

Recall that a postulate or axiom is a statement accepted as true without proof. The postulates listed below about points, lines, and planes cannot be proven, but they can be used as reasons in proofs.

Postulates: Points, Lines, and Planes

12.1 Through any two points, there is exactly one line.
12.2 Through any three noncollinear points, there is exactly one plane.
12.3 A line contains at least two points.
12.4 A plane contains at least three noncollinear points.
12.5 If two points lie in a plane, then the entire line containing those points lies in that plane.
12.6 If two lines intersect, then their intersection is exactly one point.
12.7 If two planes intersect, then their intersection is a line.

Example 1 Identify Postulates

ARCHITECTURE **Explain how the photo illustrates that each statement is true. Then state the postulate that can be used to show that the statement is true.**

a. Lines n and p intersect at point D.

The top edges of the building are represented by lines n and p. The lines intersect at the corner point, D. Postulate 12.6 states that if two lines intersect, then their intersection is exactly one point.

(continued on the next page)

Go Online You can complete an Extra Example online.

Today's Goals
- Analyze figures to identify and use postulates about points, lines, and planes.
- Analyze and construct viable arguments in a two-column format.
- Analyze and construct viable arguments in a flow proof format.
- Analyze and construct viable arguments in a paragraph proof format.

Today's Vocabulary
proof
two-column proof
deductive argument
flow proof
paragraph proof

Lesson 12-4 • Writing Proofs 729

Interactive Presentation

Learn

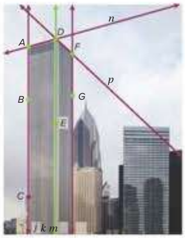
EXPAND

Students tap to see a diagram of each postulate.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

b. Points *A*, *B*, and *D* determine a plane.

Points *A*, *B*, and *D* are three noncollinear points on the front face of the building. By Postulate 12.2, through any three noncollinear points, there is exactly one plane.



c. The plane that contains *A*, *D*, and *E* intersects the plane that contains *F*, *G*, and *E* in line *k*.

The front face of the building can be represented by the plane that contains *A*, *D*, and *E*. The side face of the building can be represented by the plane that contains *F*, *G*, and *E*. These planes intersect at the corner of the building represented by line *k*. Postulate 12.7 states that if two planes intersect, then their intersection is a line.

Check

ANCIENT MONUMENTS The image illustrates the statement $\overleftrightarrow{AB}$ *is the only line through A and B.* Which postulate proves that this statement is true?

Postulate 12. 1:

Through any two points, there is exactly one line.

Interactive Presentation

Example 1

TAP

Students tap to view answers to the exercise.

Example 2 Use Postulates

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This example asks students to respond to the arguments of others.

Questions for Mathematical Discourse

AL In part **a**, there are three planes. Which postulates relate to planes? 3.2, 3.4, 3.5, 3.7

OL In part **a**, what counterexample shows that the statement is only sometimes true? The intersection of three planes could be a point.

BL Determine whether the following statement is *sometimes*, *always*, or *never* true. Plane T and Plane S intersect at a single point P. Never; sample answer: Postulate 3.7 states that if two planes intersect, then their intersection is a line.

Learn Two-Column Proofs

Objective

Students analyze and construct viable arguments in a two-column format.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of two-column proofs in the Learn to use them to write logical arguments.

About the Key Concept

Using a two-column proof style can be very useful in making sure that students understand that there must be a reason for each step in a proof.

Example 2 Use Postulates

Determine whether each statement is *always, sometimes,* or *never* true. Justify your argument.

a. The intersection of three planes is a line.

Sometimes; if three planes intersect, then their intersection could be a line or a point.

b. Line r contains only point P.

Never; Postulate 12.3 states that a line contains at least two points.

c. Through points H and K, there is exactly one line.

Always; Postulate 12.1 states that through any two points, there is exactly one line.

Check

Determine whether the statement is *always, sometimes,* or *never* true. Justify your argument.

Two intersecting lines determine a plane. **Always; sample answer: Between any two intersecting lines there are always at least three noncollinear points, and Postulate 2.2 states that through any three noncollinear points there is exactly one plane.**

Think About It! Martin claims that this is a true statement. *Through any three points, there is exactly one plane.* Do you agree? Explain.

No; sample answer: This statement is sometimes true. If the three points were noncollinear, there would be exactly one plane per Postulate 12.2. If the points were collinear, then there would be infinitely many planes.

Learn Two-Column Proofs

A **proof** is a logical argument in which each statement is supported by a statement that is accepted as true. These supporting statements can include definitions, postulates, and theorems. A **two-column proof** is a proof that contains statements and reasons that are organized in a two-column format. You can develop a **deductive argument** to prove a statement by building a logical chain of statements and reasons.

Key Concept • How to Write a Proof

Step 1 List the given information. Draw a diagram if needed.

Step 2 Create a deductive argument that links the given information to the statement that you are proving.

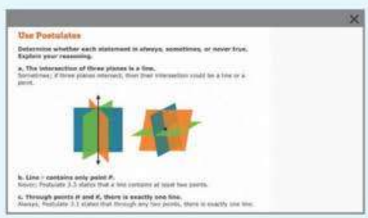
Step 3 Justify each statement with a reason. Reasons include definitions, postulates, theorems, and algebraic properties.

Step 4 State what it is that you have proven.

Go Online You can complete an Extra Example online.

Lesson 12-4 • Writing Proofs 731

Interactive Presentation



Example 2

TYPE

Students answer a question about using postulates.

CHECK

Students complete the Check online to determine whether they are ready to move on.

STATEMENTS/REASONS:
Definition of midpoint
Definition of congruence
$PQ = QR$
Betweenness of points
Q is between P and R

Study Tip
Midpoint Theorem and Definition The definition of *midpoint* is in terms of equality, and the Midpoint Theorem is in terms of congruence.

STATEMENTS/REASONS:
$AB = BC, DC = CE$
Substitution
$BC = CE$
Transitive Property
$AC = DE$
$AB = DC, BC = CE$

Example 3 Two-Column Proof

Complete the two-column proof by selecting the correct statements and reasons.

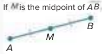
Given: Q is the midpoint of $\overline{PR}$

Prove: $\overline{PQ} \cong \overline{QR}$

Statements	Reasons
1. Q is the midpoint of $\overline{PR}$	1. Given
2. $PQ = QR$	2. Definition of midpoint
3. $\overline{PQ} \cong \overline{QR}$	3. Definition of congruence

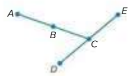
Once a conjecture has been proven true, it can be used as a reason in other proofs. The conjecture proven above is known as the Midpoint Theorem.

Theorem 12.1: Midpoint Theorem
If M is the midpoint of $\overline{AB}$, then $\overline{AM} \cong \overline{MB}$.



Check

Copy and complete the two-column proof by selecting the correct statements and reasons.



Given: B is the midpoint of $\overline{AC}$. C is the midpoint of $\overline{DE}$. $AB = CE$

Prove: $BC = DC$

Statements	Reasons
1. B is the midpoint of $\overline{AC}$. C is the midpoint of $\overline{DE}$	1. Given
2. ? $AB = BC, DC = CE$	2. Definition of midpoint
3. $AB = CE$	3. Given
4. ? $BC = CE$	4. ? Substitution
5. $BC = DC$	5. Substitution

Learn Flow Proofs

A **flow proof** uses boxes and arrows to show the logical progression of an argument. The statement is in the box, and the reason is below it. Arrows indicate the order of the steps.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 3

DRAG & DROP

Students drag the statements and reasons to complete the proof.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 3 Two-Column Proof

Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to complete the given proof.

Questions for Mathematical Discourse

AL How are postulates used to write a proof? Sample answer: Postulates can be used to support a statement in a proof.

OL What is the difference between reasons 2 and 3? Sample answer: The definition of midpoint refers to equality while the definition of congruence refers to equal distances.

BL If $PQ = 3x$ and $QR = \frac{1}{4}x + 11$, what is the value of x? $x = 4$

Common Error

Students may confuse distance, from the definition of midpoint, with congruence, from the conclusion of the Midpoint Theorem. The two are related but not the same thing.

Learn Flow Proofs

Objective

Students analyze and construct viable arguments in a flow proof format.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of flow proofs in the Learn so they can use them to write logical arguments.

Important to Know

The advantage of flow proofs in helping students understand logical reasoning is that they show how one step leads to another. This is not always obvious in two-column format, especially in proofs where some steps could occur in a different order. In this case, use a flow proof to show students why the order of some steps may not matter.

Example 4 Flow Proofs

Teaching the Mathematical Practices

1 Analyze Givens and Constraints In this example, guide students through the meaning of the problem and look for entry points to its solution.

Questions for Mathematical Discourse

AL Why is the first item in the flow proof the statement *P is the midpoint of* $\overline{JK}$? Sample answer: It is the given information in the proof.

OL What does the Midpoint Theorem state? If a point is the midpoint of a segment, then it divides the segment into two congruent segments.

BL If $JP = 7$ and $PK = 2y - 3$, what is y? $y = 5$

Essential Question Follow-Up

Students learn proof methods such as the two-column proof.

Ask:

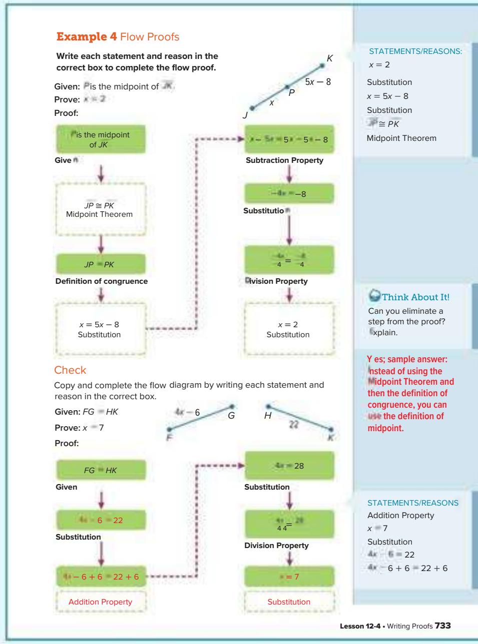
Why is it important to learn different proof methods? Sample answer: So that you can better write logical arguments for geometric facts and theorems.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are having difficulty knowing where to start writing a paragraph proof, or they do not know how to correctly order the steps of the proof,

THEN have them outline the proof beforehand, or write the proof using the two-column proof or flow proof method first, and then write the full proof in paragraph form.



Example 4 Flow Proofs

Write each statement and reason in the correct box to complete the flow proof.

Given: P is the midpoint of $\overline{JK}$.

Prove: $x = 2$

Proof:

P is the midpoint of $\overline{JK}$ — Given

$\overline{JP} \cong \overline{PK}$ — Midpoint Theorem

$JP = PK$ — Definition of congruence

$x = 5x - 8$ — Substitution

$x - 5x = 5x - 5x - 8$ — Subtraction Property

$-4x = -8$ — Substitution

$\frac{-4x}{-4} = \frac{-8}{-4}$ — Division Property

$x = 2$ — Substitution

STATEMENTS/REASONS: $x = 2$; Substitution; $x = 5x - 8$; Substitution; $\overline{JP} \cong \overline{PK}$; Midpoint Theorem

Think About It! Can you eliminate a step from the proof? Explain.

Yes; sample answer: Instead of using the Midpoint Theorem and then the definition of congruence, you can use the definition of midpoint.

Check

Copy and complete the flow diagram by writing each statement and reason in the correct box.

Given: $FG = HK$

Prove: $x = 7$

Proof:

$FG = HK$ — Given

$4x - 6 = 22$ — Substitution

$4x - 6 + 6 = 22 + 6$ — Addition Property

$4x = 28$ — Substitution

$\frac{4x}{4} = \frac{28}{4}$ — Division Property

$x = 7$ — Substitution

STATEMENTS/REASONS: Addition Property; $x = 7$; Substitution; $4x - 6 = 22$; $4x - 6 + 6 = 22 + 6$

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Interactive Presentation

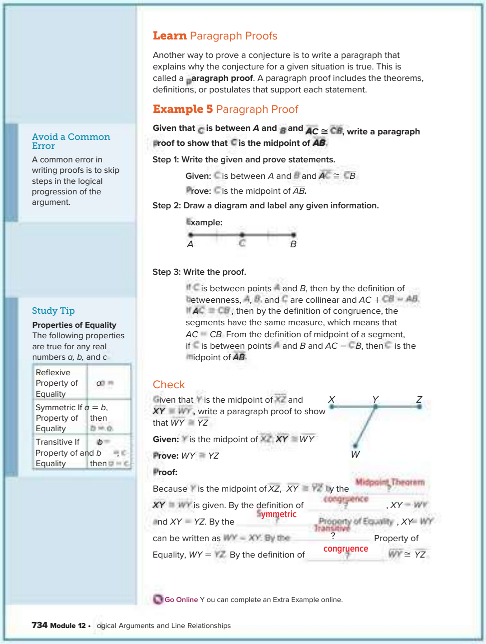
Example 4

DRAG & DROP

Students drag and drop the steps of the proof.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Avoid a Common Error

A common error in writing proofs is to skip steps in the logical progression of the argument.

Study Tip

Properties of Equality The following properties are true for any real numbers a, b, and c

Reflexive Property of Equality	$a = a$
Symmetric Property of Equality	If $a = b$, then $b = a$.
Transitive Property of Equality	If $a = b$ and $b = c$, then $a = c$.

Learn Paragraph Proofs

Another way to prove a conjecture is to write a paragraph that explains why the conjecture for a given situation is true. This is called a **paragraph proof**. A paragraph proof includes the theorems, definitions, or postulates that support each statement.

Example 5 Paragraph Proof

Given that C is between A and B and $\overline{AC} \cong \overline{CB}$, write a paragraph proof to show that C is the midpoint of $\overline{AB}$.

Step 1: Write the given and prove statements.

Given: C is between A and B and $\overline{AC} \cong \overline{CB}$

Prove: C is the midpoint of $\overline{AB}$.

Step 2: Draw a diagram and label any given information.

Example:

Step 3: Write the proof.

If C is between points A and B, then by the definition of betweenness, A, B, and C are collinear and $AC + CB = AB$. If $\overline{AC} \cong \overline{CB}$, then by the definition of congruence, the segments have the same measure, which means that $AC = CB$. From the definition of midpoint of a segment, if C is between points A and B and $AC = CB$, then C is the midpoint of $\overline{AB}$.

Check

Given that Y is the midpoint of $\overline{XZ}$ and $\overline{XY} \cong \overline{WY}$, write a paragraph proof to show that $\overline{WY} \cong \overline{YZ}$.

Given: Y is the midpoint of $\overline{XZ}$, $\overline{XY} \cong \overline{WY}$

Prove: $\overline{WY} \cong \overline{YZ}$

Proof:

Because Y is the midpoint of $\overline{XZ}$, $\overline{XY} \cong \overline{YZ}$ by the Midpoint Theorem. $\overline{XY} \cong \overline{WY}$ is given. By the definition of congruence, $XY = WY$ and $XY = YZ$. By the Symmetric Property of Equality, $XY = WY$ can be written as $WY = XY$. By the Transitive Property of Equality, $WY = YZ$. By the definition of congruence, $\overline{WY} \cong \overline{YZ}$.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 5

TAP

Students tap through the parts of the Example.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Paragraph Proofs

Objective

Students analyze and construct viable arguments in a paragraph proof format.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

What Students Are Learning

Writing a geometric proof in paragraph form may be preferable to some students. Explain that the steps for each type of proof are the same, but that the execution varies. Encourage students to first plan the steps of their proof out entirely before starting to write the paragraph.

Example 5 Paragraph Proof

MP Teaching the Mathematical Practices

3 Compare Arguments Mathematically proficient students can compare arguments, determine which one is flawed, and explain the flaw. In this example, students have to identify the flawed argument and choose the correct one.

Questions for Mathematical Discourse

AL What is a paragraph proof? Sample answer: It is a proof where the givens, statements, and conclusion are written in sentences and paragraphs.

OL How should you start a paragraph proof? Sample answer: Write the *Given* and *Prove* statements, then draw a diagram and label any given information.

BL How do you determine if a paragraph proof is correct? Sample answer: Determine whether each statement is logically true and whether the statements progress from the given information to the conclusion without skipping any steps.

Common Error

A common error in writing paragraph proofs is to skip steps in the logical progression of the argument. Outlining the proof beforehand may help avoid this.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–18
2	exercises that use a variety of skills from this lesson	19–20
2	exercises that extend concepts learned in this lesson to new contexts	21–24
3	exercises that emphasize higher-order and critical-thinking skills	25–30

ASSESS AND DIFFERENTIATE

Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–23 odd, 25–30
- Extension: Even and Odd
- ALEKS Proofs Involving Segments and Angles

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–29 odd
- Remediation, Review Resources: Angle Relationships
- Personal Tutors
- Extra Examples 1–5
- ALEKS Angles

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Angle Relationships
- *Quick Review Math Handbook*: Postulates and Paragraph Proofs
- ALEKS Angles

Answers

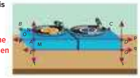
5. Always; Postulate 3.2 states that through any three noncollinear points, there is exactly one plane.
6. Never; Postulate 3.1 states that through any two points, there is exactly one line.
7. Sometimes; the points do not have to be collinear to lie in a plane.
8. Always; Postulate 3.5 states that if two points lie in a plane, then the entire line containing those points lies in that plane.

Practice

Go Online Y ou can complete your homework online.

Example 1

MUSIC **Explain how the figure illustrates that each statement is true. Then state the postulate that can be used to show that each statement is true.**



1. Planes O and M intersect in line t. The two planes meet at the edge, which lies on line t. Postulate: If two planes intersect, then their intersection is a line.

2. Line p lies in plane N. Points A and D both lie on line p and in plane N. Postulate: If two points lie in a plane, then the entire line containing those points lies in that plane.

SIGNS **In the figure, $\overleftrightarrow{DG}$ and $\overrightarrow{DP}$ are in plane J and H lies on $\overleftrightarrow{DG}$. State the postulate that can be used to show that each statement is true.**

3. Points G and P are colli near. Postulate 12. 1; Through any two points, there is exactly one line.

4. Points D, H, and P are coplanar. Postulate 12.2; Through any three noncollinear points, there is exactly one plane.

Example 2

CONSTRUCT ARGUMENTS **Determine whether each statement is *always*, *sometimes*, or *never* true. Justify your argument.** 5–10. See margin.

5. There is exactly one plane that contains noncollinear points A, B, and C.
6. There are at least three lines through points J and K.
7. If points M, N, and P lie in plane X, then they are collinear.
8. Points X and Y are in plane Z. Any point collinear with points X and Y is in plane Z.
9. The intersection of two planes can be a point.
10. Points A, B, and C determine a plane.

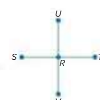
Example 3

11. PROOF Point Y is the midpoint of $\overline{XZ}$. Point W is collinear with X, Y, and Z. Z is the midpoint of $\overline{YW}$. Write a two-column proof to prove that $\overline{XY} \cong \overline{ZW}$. See margin.

12. PROOF Write a two-column proof to prove that $w = 3.5$.

Given: $\overline{JK} \cong \overline{LM}$
Prove: $w = 3.5$ See margin.

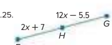
13. PROOF Copy and complete the two-column proof.



Given: $SR = RT$, $SR = UR$, and $RT = RV$
Prove: R is the midpoint of $\overline{ST}$. R is the midpoint of $\overline{UV}$.

Statements	Reasons
1. $SR = RT$	1. ? Given
2. ? R is the midpoint of $\overline{ST}$.	2. Definition of midpoint
3. $SR = UR$ and $RT = RV$	3. ? Given
4. ? $SR = RT$, so $UR = RV$.	4. Substitution
5. ? R is the midpoint of $\overline{UV}$.	5. ? Definition of midpoint

14. PROOF Copy and complete the two-column proof to prove that $x = 1.25$.



Given: H is the midpoint of $\overline{FG}$.
Prove: $x = 1.25$

Statements	Reasons
1. H is the midpoint of $\overline{FG}$.	1. ? Given
2. ? $\overline{FH} \cong \overline{HG}$	2. Definition of midpoint
3. ? $FH = HG$	3. Congruent segments have equal lengths.
4. $2x + 7 = 12x - 5.5$	4. ? Substitution Property of Equality
5. ? $2x + 12.5 = 12x$	5. Addition Property of Equality
6. ? $12.5 = 10x$	6. ? Subtraction Property of Equality
7. ? $1.25 = x$	7. Division Property of Equality
8. $x = 1.25$	8. ? Symmetric Property of Equality

Example 4

15. PROOF Point L is the mid point of $\overline{JK}$. $\overline{JK}$ intersects $\overline{MK}$ at K. If $\overline{MK} \cong \overline{JL}$, write a flow proof to prove that $\overline{LK} \cong \overline{MK}$. See margin.

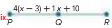
16. PROOF Copy and complete the flow proof to prove that if $\overline{MN} \cong \overline{PQ}$, $MN = 5x - 10$, and $PQ = 4x + 10$, then $MN = 90$.

$MN \cong PQ$, $MN = 5x - 10$, and $PQ = 4x + 10$ (Given) → $MN = PQ$, $5x - 10 = 4x + 10$ (Congruent segments have equal lengths; Substitution Property of Equality) → $x - 10 = 10$ (Subtraction Property of Equality) → $x = 20$ (Addition Property of Equality) → $MN = 5(20) - 10 = 90$ (Substitution Property of Equality)

Example 5

17. PROOF In the figure at the right, point B is the midpoint of $\overline{AC}$ and point C is the midpoint of $\overline{BD}$. Write a paragraph proof to prove that $AB = CD$. See margin.

18. PROOF Write a paragraph proof to prove that if $PQ = 4(x - 3) + 1$, $QR = x + 10$, and $x = 7$, then $\overline{PQ} \cong \overline{QR}$. See Mod. 12 Answer Appendix.

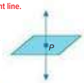


Mixed Exercises

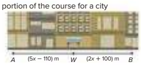
19. What postulate can be used to show the following statement is true?
Line m contains points A and F. Postulate 3.3 A line contains at least two points.

20. ROOFING Fai and Max are building a new roof. They wanted a roof with two sloping planes that intersect in a curved arch. Is this possible?
Such a roof is impossible because two intersecting planes intersect in a straight line.

21. Carson claims that a line will always intersect a plane at only one point, and he draws this picture to show his reasoning. Iza thinks it is possible for a line to intersect a plane at more than one point. Who is correct? Explain.
Iza is because a line can lie in a plane and intersect it in infinite points.



22. REASONING The figure shows a straight portion of the course for a city marathon. The water station W is located at the midpoint of $\overline{AB}$.



a. What is the length of the course from point A to point W? 240 m

b. Write a paragraph proof for your answer to part **a**.
Because W is the midpoint of $\overline{AB}$, $\overline{AW} \cong \overline{WB}$ and $AW = WB$. This means $5x - 110 = 2x + 100$, so $3x - 110 = 100$ by the Subtraction Property of Equality and $3x = 210$ by the Addition Property of Equality. Therefore, $x = 70$ by the Division Property of Equality. By the Substitution Property of Equality, $AW = 5(70) - 110 = 240$ meters.

c. Explain how you used a definition in your paragraph proof.
The definition of *midpoint* allows you to conclude that $AW = WB$.

23. AIRLINES An airline company wants to provide service to San Francisco, Los Angeles, Chicago, Dallas, Washington D.C., and New Y ork City. The company's president draws lines between each pair of cities in the list on a map. No three of the cities are collinear. How many lines did the president draw? 15

24. SMALL BUSINESSES A small company has 16 employees. The owner placed 16 points on a sheet of paper in such a way that no 3 were collinear. Each point represented a different employee. He then connected two points with a line segment if they represented coworkers in the same department.

a. What is the maximum number of line segments that can be drawn between pairs among the 16 points? 120

b. When the owner finished the diagram, he found that his company was split into two groups, one with 10 people and the other with 6. All the people within a group were in the same department, but nobody from one group was from the other group. How many line segments were there? 60

Higher-Order Thinking Skills

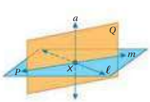
25. FIND THE ERROR Omair and Ana were working on a paragraph proof to prove that if $\overline{AB}$ is congruent to $\overline{BD}$ and A, B, and D are collinear, then B is the midpoint of $\overline{AD}$. Each student started his or her proof in a different way. Is either of them correct? Explain your reasoning. See Mod. 12 Answer Appendix.

Omair	Ana
If B is the midpoint of $\overline{AB}$, then B divides $\overline{AD}$ into two congruent segments.	$\overline{AB}$ is congruent to $\overline{BD}$, and A, B, and D are collinear.

26. CREATE Draw a figure that satisfies five of the seven postulates you have learned. Explain which postulates you chose and how your figure satisfies each postulate. See Mod. 12 Answer Appendix.

27. PERSEVERE Use the following true statements and the definitions and postulates you have learned to answer each question.

Two planes are perpendicular if and only if one plane contains a line perpendicular to the second plane.



a. Through a given point, there passes one and only one plane perpendicular to a given line. If plane Q is perpendicular to line ℓ at point X and line ℓ lies in plane P, what must also be true? Plane Q is perpendicular to plane P.

b. Through a given point, there passes one and only one line perpendicular to a given plane. If plane Q is perpendicular to plane P at point X and line a lies in plane Q, what must also be true? Line a is perpendicular to plane P.

28. WRITE How does writing a proof require logical thinking? See Mod. 12 Answer Appendix.

ANAL YZE **Determine whether each statement is *sometimes*, *always*, or *never* true. Justify your argument.**

29. Through any three points, there is exactly one plane. See Mod. 12 Answer Appendix.

30. A plane contains at least two distinct lines. See Mod. 12 Answer Appendix.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Answers

9. Never; Postulate 3.7 states that if two planes intersect, then their intersection is a line.

10. Sometimes; the points must be noncollinear.

11. Proof:

Statements (Reasons)

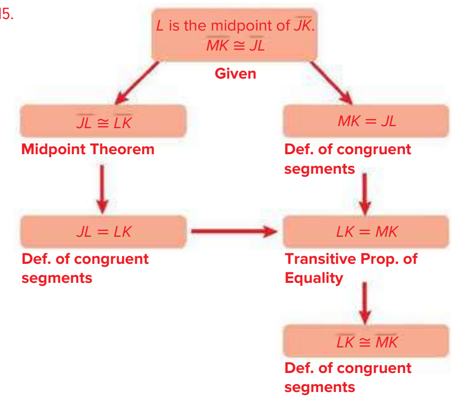
1. Y is the midpoint of $\overline{XZ}$. W is collinear with X, Y, and Z. Z is the midpoint of $\overline{YW}$. (Given)
2. $\overline{XY} \cong \overline{YZ}$ and $\overline{YZ} \cong \overline{ZW}$ (Midpoint Theorem)
3. $XY = YZ$ and $YZ = ZW$ (Definition of congruent segments)
4. $XY = ZW$ (Transitive Property of Equality)
5. $\overline{XY} \cong \overline{ZW}$ (Definition of congruent segments)

12. Proof:

Statements (Reasons)

1. $\overline{JK} \cong \overline{LM}$ (Given)
2. $JK = LM$ (Definition of congruent segments)
3. $4w + 1 = 6w - 6$ (Substitution Property of Equality)
4. $4w + 7 = 6w$ (Addition Property of Equality)
5. $7 = 2w$ (Subtraction Property of Equality)
6. $3.5 = w$ (Division Property of Equality)
7. $w = 3.5$ (Symmetric Property of Equality)

15.



17. Given: B is the midpoint of $\overline{AC}$. C is the midpoint of $\overline{BD}$.
Prove: $AB = CD$
Proof: Because B is the midpoint of $\overline{AC}$ and C is the midpoint of $\overline{BD}$, we know by the Midpoint Theorem that $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{CD}$. Because congruent segments have equal measures, $AB = BC$ and $BC = CD$. Thus, by the Transitive Property of Equality, $AB = CD$.

Lesson 12-5

Proving Segment Relationships

G.CO.9, G.CO.12

LESSON GOAL

Students prove theorems about line segments.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Segment Relationships

Develop:

Segment Addition
- Segment Addition Postulate

Segment Congruence
- Prove Segment Congruence
- Determine Congruence

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Deductive Reasoning	● ●			●
Extension: Axioms and Propositions		● ●		●

Language Development Handbook

Assign page 75 of the *Language Development Handbook* to help your students build mathematical language related to proving relationships about line segments.

ELL You can use the tips and suggestions on page T75 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.9 Prove theorems about lines and angles.

G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students wrote proofs in two-column, flow, and paragraph styles.

Now
Students prove theorems about line segments.
G.CO.9

Next
Students will write proofs of theorems about angles.
G.CO.9

Rigor

The Three Pillars of Rigor

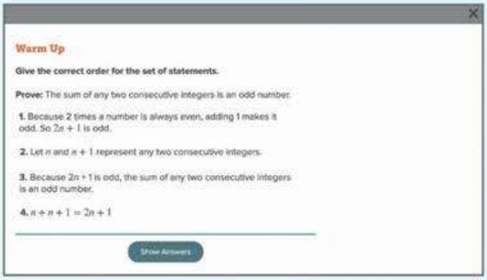
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of proofs, and they build fluency by proving theorems about line segment relationships.

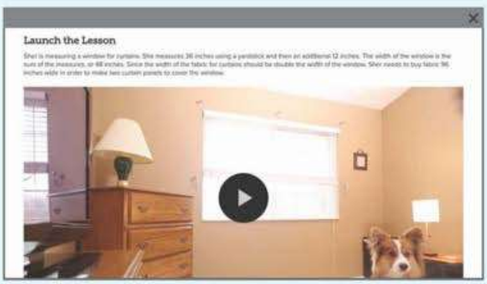
Mathematical Background

A segment can be measured, and measures can be used in calculations because they are real numbers. The Ruler Postulate states that the points on any line or line segment can be paired with real numbers such that, given any two points A and B on a line, A corresponds to 0, and B lies between points A and C on the same line, $AB + BC = AC$. The Reflexive, Symmetric, and Transitive Properties of Equality can be used to write proofs about segment congruence.

Interactive Presentation



Warm Up



Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

• making a valid argument about algebra

Answer:

2., 4., 1., 3.

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of segment relationships.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Explore Segment Relationships

Objective

Students use dynamic geometry software to prove theorems about line segments.

Teaching the Mathematical Practices

3 Make Conjectures In this explore, students will make conjectures and then build a logical progression of statements to validate the conjectures. Once students have made their conjectures, guide the students to validate them.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

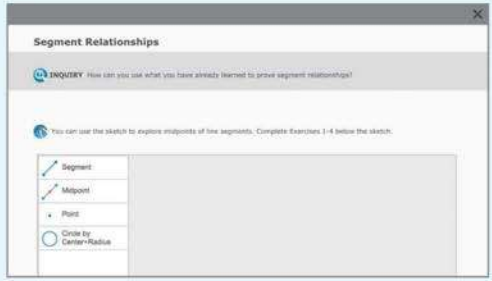
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

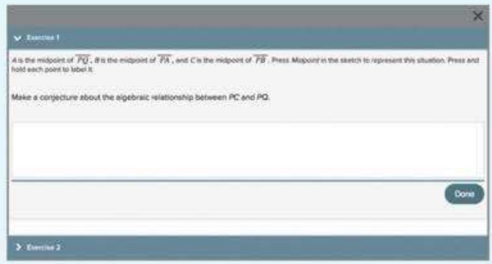
Students make a series of constructions involving midpoints of segments. Then students make a conjecture about lengths of related segments. Students then make some constructions designed to show how to prove the conjecture. Then students fill in the missing parts of a proof of their conjecture. Finally, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

WEB SKETCHPAD

Students use the sketch to explore segment length relationships.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Segment Relationships (*continued*)

Question

Have students complete the Explore activity.

Ask:

- How is finding the midpoint related to the segment length? Sample answer: The midpoint divides the segment into two congruent segments. So, the length of each segment is half of the original.
- Describe what happens each time you find the midpoint in this activity. Sample answer: Each midpoint is dividing a segment in half. First I found one-half, then one-fourth, and finally one-eight of the original segment length.

Inquiry

How can you use what you have already learned to prove segment relationships? Sample answer: Y ou can use properties of real numbers to help prove relationships between lengths of segments.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Segment Addition

Objective

Students prove theorems about line segments by using the Segment Addition Postulate.

MP Teaching the Mathematical Practices

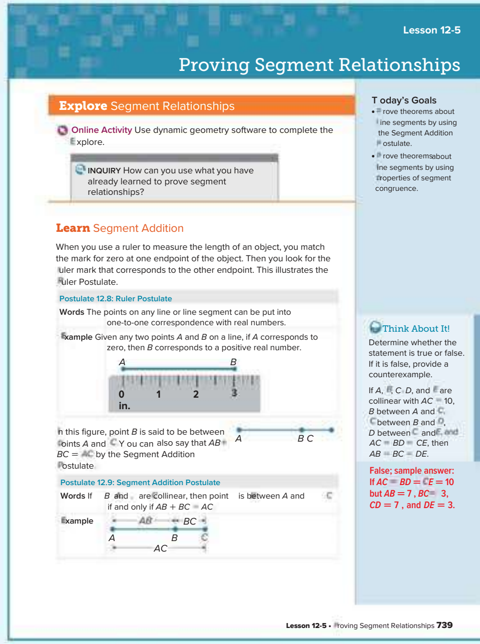
3 Analyze Cases Work with students to look at the Think About It! feature. Ask students to determine whether the statement is true or false. If false, have students identify a counterexample that disproves the claim.

About the Key Concept

The Ruler Postulate and Segment Addition Postulate are important because they give us a way to measure the lengths of line segments using real numbers. This is needed to be able to define congruence of segments as segments with the same length.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 12-5

Proving Segment Relationships

Explore Segment Relationships

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How can you use what you have already learned to prove segment relationships?

Today's Goals

- Prove theorems about line segments by using the Segment Addition Postulate.
- Prove theorems about line segments by using properties of segment congruence.

Learn Segment Addition

When you use a ruler to measure the length of an object, you match the mark for zero at one endpoint of the object. Then you look for the ruler mark that corresponds to the other endpoint. This illustrates the Ruler Postulate.

Postulate 12.8: Ruler Postulate

Words The points on any line or line segment can be put into one-to-one correspondence with real numbers.

Example Given any two points A and B on a line, if A corresponds to zero, then B corresponds to a positive real number.

In this figure, point B is said to be between points A and C. You can also say that $AB + BC = AC$ by the Segment Addition Postulate.

Postulate 12.9: Segment Addition Postulate

Words If A, B, and C are collinear, then point B is between A and C if and only if $AB + BC = AC$.

Example

Think About It!

Determine whether the statement is true or false. If it is false, provide a counterexample.

If A, B, C, D, and E are collinear with $AC = 10$, B between A and C, C between B and D, D between C and E, and $AC = BD = CE$, then $AB = BC = DE$.

False; sample answer: If $AC = BD = CE = 10$ but $AB = 7$, $BC = 3$, $CD = 7$, and $DE = 3$.

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Interactive Presentation

Learn

TYPE

Students answer a question about segment addition.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

STATEMENTS/REASONS

Segment Addition Postulate

$QT = RV$

Substitution Property of Equality

$QR + RT = RT + TV$

Example 1 Segment Addition Postulate

Write the correct statements and reasons to complete the two-column proof.

Given: $\overline{QT} \cong \overline{RV}$

Prove: $QR \cong \overline{TV}$

Proof:

Statements	Reasons
1. $\overline{QT} \cong \overline{RV}$	1. Given
2. $QT = RV$	2. Definition of congruence
3. $QR + RT = QT$; $RT + TV = RV$	3. Segment Addition Postulate
4. $QR + RT = RT + TV$	4. Substitution Property of Equality
5. $QR + RT - RT = RT + TV - RT$	5. Subtraction Property of Equality
6. $QR = TV$	6. Substitution Property of Equality
7. $\overline{QR} \cong \overline{TV}$	7. Definition of congruence

Check

Copy and complete the two-column proof by writing the correct statement and reason.

REASONS

Definition of congruence

Given

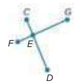
Segment Addition Postulate

Substitution Property of Equality

Transitive Property of Equality

Given: $\overline{CE} \cong \overline{FE}$; $\overline{ED} \cong \overline{EG}$

Prove: $CD \cong FG$



Proof:

Statements	Reasons
1. $\overline{CE} \cong \overline{FE}$; $\overline{ED} \cong \overline{EG}$	1. ? Given
2. $CE = FE$; $ED = EG$	2. ? Definition of congruence
3. $CE + ED = CD$	3. ? Segment Addition Postulate
4. $FE + EG = CD$	4. ? Substitution Property of Equality
5. $FE + EG = FG$	5. ? Segment Addition Postulate
6. $CD = FG$	6. ? Transitive Property of Equality
7. $\overline{CD} \cong \overline{FG}$	7. ? Definition of congruence

Go Online You can complete an Extra Example online.

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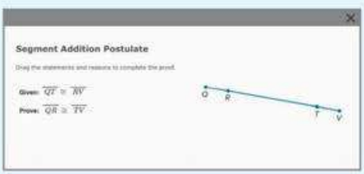
Interactive Presentation



Example 1

DRAG & DROP

Students drag statements and reasons to complete a two-column proof.

CHECK

Complete the Check exercise online to determine whether students are ready to move on.

Example 1 Segment Addition Postulate

Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to complete the given proof.

Questions for Mathematical Discourse

AL What is the Segment Addition Postulate in your own words? Sample answer: If three points are on the same line, then point B is between A and C if $AB + BC = AC$.

OL How does the definition of congruence help you in the final step of the proof? If you know that two segments have the same length, then you can say that they are congruent.

BL In Step 6, how are you using substitution to say that $QR = TV$? In Step 5, you can substitute QR for the left side of the equation, and substitute TV for the right side.

Common Error

Students will often assume that line segments that look congruent in a figure are congruent. Remind students to check this against the given information in the proof.

DIFFERENTIATE

Reteaching Activity **AL**

IF students have difficulty identifying the given information implicit in a given figure,
THEN encourage students to read through the given information, identifying each point and line segment in the figure. Have students mark the figures so they can easily refer to the relationships while writing their proofs.

Learn Segment Congruence

Objective

Students prove theorems about line segments by using properties of segment congruence.

Teaching the Mathematical Practices

3 Analyze Cases This Learn guides students to examine cases of the properties of segment congruence. Encourage students to familiarize themselves with all of the cases.

Common Misconception

Students may assume that all relations have the reflexive, symmetric, and transitive properties. For a counterexample, remind them that the relation "<" is not reflexive or symmetric.

Example 2 Prove Segment Congruence

Teaching the Mathematical Practices

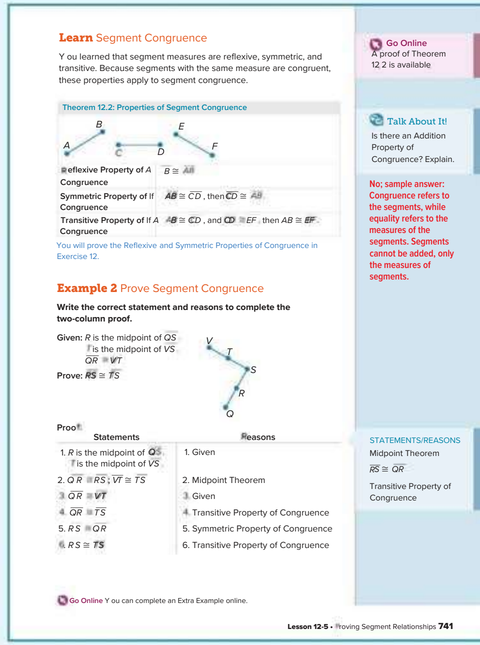
3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to complete the given proof.

Questions for Mathematical Discourse

AL How is what we have in Step 2 from the Midpoint Theorem different from what the definition of midpoint tells us? Sample answer: The definition tells us that the segments are the same length, and the theorem tells us that they are congruent.

OL In Step 4, what are the two individual congruence statements that allow you to state that $\overline{QR} \cong \overline{TS}$? $\overline{QR} \cong \overline{VT}$ and $\overline{VT} \cong \overline{TS}$

BL In Step 5, why do you need to use the Symmetric Property to change $\overline{QR} \cong \overline{RS}$ to $\overline{RS} \cong \overline{QR}$? Sample answer: The congruence must be in the correct order, then you can use the Transitive Property in Step 6.



Learn Segment Congruence

You learned that segment measures are reflexive, symmetric, and transitive. Because segments with the same measure are congruent, these properties apply to segment congruence.

Theorem 12.2: Properties of Segment Congruence	
Reflexive Property of Congruence	$\overline{AB} \cong \overline{AB}$
Symmetric Property of Congruence	If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.
Transitive Property of Congruence	If $\overline{AB} \cong \overline{CD}$, and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.

You will prove the Reflexive and Symmetric Properties of Congruence in Exercise 12.

Example 2 Prove Segment Congruence

Write the correct statement and reasons to complete the two-column proof.

Given: R is the midpoint of $\overline{QS}$. T is the midpoint of $\overline{VS}$. $\overline{QR} \cong \overline{VT}$

Prove: $\overline{RS} \cong \overline{TS}$

Proof:

Statements	Reasons
1. R is the midpoint of $\overline{QS}$. T is the midpoint of $\overline{VS}$.	1. Given
2. $\overline{QR} \cong \overline{RS}$; $\overline{VT} \cong \overline{TS}$	2. Midpoint Theorem
3. $\overline{QR} \cong \overline{VT}$	3. Given
4. $\overline{QR} \cong \overline{TS}$	4. Transitive Property of Congruence
5. $\overline{RS} \cong \overline{QR}$	5. Symmetric Property of Congruence
6. $\overline{RS} \cong \overline{TS}$	6. Transitive Property of Congruence

Go Online You can complete an Extra Example online.

Go Online A proof of Theorem 12.2 is available.

Talk About It! Is there an Addition Property of Congruence? Explain.

No; sample answer: Congruence refers to the segments, while equality refers to the measures of the segments. Segments cannot be added, only the measures of segments.

STATEMENTS/REASONS
Midpoint Theorem
$\overline{RS} \cong \overline{QR}$
Transitive Property of Congruence

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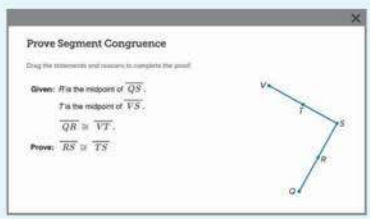
Interactive Presentation



Example 2

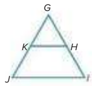
DRAG & DROP

Students drag statements and reasons to complete a two-column proof.

REASONS:
Add.
Definition of congruence
Divide by 2.
Given
Midpoint Theorem
Segment Addition Postulate
Substitution Property of Equality

Check

Copy and complete the two-column proof by writing the correct reasons.



Given: $\overline{GJ} \cong \overline{GI}$
K is the midpoint of $\overline{GJ}$.
H is the midpoint of $\overline{GI}$.

Prove: $\overline{GK} \cong \overline{GH}$

Proof:

Statements	Reasons
1. K is the midpoint of $\overline{GJ}$. H is the midpoint of $\overline{GI}$.	1. ? Given
2. $\overline{GK} \cong \overline{KJ}$; $\overline{GH} \cong \overline{HI}$	2. ? Midpoint Theorem
3. $GK = KJ$; $GH = HI$	3. ? Definition of congruence
4. $\overline{GJ} \cong \overline{GI}$	4. ? Given
5. $GJ = GI$	5. ? Definition of congruence
6. $GJ = GK + KJ$; $GI = GH + HI$	6. ? Segment Addition Postulate
7. $GK + KJ = GH + HI$	7. ? Substitution Property of Equality
8. $GK + GK = GH + GH$	8. ? Substitution Property of Equality
9. $2GK = 2GH$	9. ? Add.
10. $GK = GH$	10. ? Divide by 2.
11. $\overline{GK} \cong \overline{GH}$	11. ? Definition of congruence

Example 3 Determine Congruence

CITY PLANNING **Marcellus is planning a birthday party. He measures a length of ribbon for a balloon, and then uses this ribbon to measure and cut a second ribbon. He continues this pattern of using the last ribbon that he cut to measure the next ribbon until 10 ribbons have been cut for balloons. Is the last ribbon that he cut the same length as the first ribbon? Justify your argument.**

Y es; because the first ribbon is the same length as the second ribbon and the third ribbon is the same length as the second ribbon, the first ribbon is the same length as the third ribbon by the Transitive Property of Congruence.

This logic can be applied until the last ribbon is shown to be the same length as the first ribbon.

Go Online Y ou may want to complete the construction activities for this lesson.

Check

CITY PLANNING A city council plans to convert a section of a city block into green space for the community. The north sidewalk is congruent to the south sidewalk, which is congruent to the west sidewalk. Therefore, the north sidewalk is congruent to the west sidewalk. What theorem, postulate, or property justifies this statement?
Transitive Property of Congruence

Go Online Y ou can complete an Extra Example online.

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Interactive Presentation

Example 3

TAP

Students select the correct term to complete the argument.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Determine Congruence

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about segment congruence to solving a real-world problem.

Questions for Mathematical Discourse

AL Which property of congruence requires knowing the relationships between two pairs of objects? Explain. The Transitive Property requires knowing that one pair of objects is congruent and that another pair of objects is congruent.

OL State the Transitive Property of Congruence in your own words. If one object is congruent to two other objects, then those two objects are congruent to each other.

BL What would the Reflexive Property tell you about the 1st balloon string? It is the same length as itself.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–7
2	exercises that use a variety of skills from this lesson	8–10
2	exercises that extend concepts learned in this lesson to new contexts	11–12
3	exercises that emphasize higher-order and critical-thinking skills	13–19

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–11 odd, 13–19
- Extension: Axioms and Propositions
- ALEKS Proofs Involving Segments and Angles

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–19 odd
- Remediation, Review Resources: Deductive Reasoning
- Personal Tutors
- Extra Examples 1–3
- ALEKS Conditional Statements and Deductive Reasoning

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–7 odd
- Remediation, Review Resources: Deductive Reasoning
- *Quick Review Math Handbook*: Proving Segment Relationships
- ALEKS Conditional Statements and Deductive Reasoning

Practice

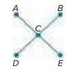
Go Online You can complete your homework online.

Example 1

1. PROOF Write the correct statements and reasons to complete the two-column proof.

Given: C is the midpoint of $\overline{AE}$.
C is the midpoint of $\overline{BD}$.
$\overline{AE} \cong \overline{BD}$

Prove: $\overline{AC} \cong \overline{CD}$



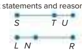
Proof:

Statements	Reasons
1. ? C is the midpoint of $\overline{AE}$. ? C is the midpoint of $\overline{BD}$. ? $\overline{AE} \cong \overline{BD}$	1. Given
2. $AC = CE$, $BC = CD$	2. ? Definition of midpoint
3. $AE = BD$	3. ? Definition of ≅ segments
4. ? $AE = AC + CE$? $BD = BC + CD$	4. Segment Addition Property
5. $AC + CE = BC + CD$	5. ? Substitution Property
6. $AC + AC = CD + CD$	6. ? Substitution Property
7. ? $2AC = 2CD$	7. Simplify.
8. ? $AC = CD$	8. Division Property
9. $\overline{AC} \cong \overline{CD}$	9. ? Definition of ≅ segments

2. PROOF Write the correct statements and reasons to complete the two-column proof.

Given: $\overline{SU} \cong \overline{LR}$
$\overline{TU} \cong \overline{LN}$

Prove: $\overline{ST} \cong \overline{NR}$



Proof:

Statements	Reasons
1. $\overline{SU} \cong \overline{LR}$, $\overline{TU} \cong \overline{LN}$	1. ? Given
2. ? $SU = LR$, $TU = LN$	2. Definition of ≅ segments
3. $SU = ST + TU$, $LR = LN + NR$	3. ? Segment Addition Postulate
4. $ST + TU = LN + NR$	4. ? Substitution Property
5. $ST + LN = LN + NR$	5. ? Substitution Property
6. $ST + LN - LN = LN + NR - LN$	6. ? Subtraction Property
7. ? $ST = NR$	7. Substitution Property
8. $\overline{ST} \cong \overline{NR}$	8. ? Definition of ≅ segments

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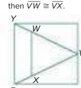
Example 2

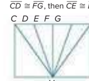
PROOF Write a two-column proof to prove each geometric relationship.

3. If $\overline{VZ} \cong \overline{VY}$ and $\overline{WY} \cong \overline{XZ}$, then $\overline{VW} \cong \overline{VX}$.



See margin.

4. If E is the midpoint of $\overline{DF}$ and $\overline{CD} \cong \overline{FG}$, then $\overline{CE} \cong \overline{EG}$.



See margin.

Example 3

5. FAMILY Maria is 11 inches shorter than her sister Clara. Luna is 11 inches shorter than her brother Chad. If Maria is shorter than Luna, how do the heights of Clara and Chad compare? What else can be concluded if Maria and Luna are the same height? Clara is shorter than Chad when Maria is shorter than Luna; Clara and Chad are the same height when Maria is the same height as Luna.

6. LUMBER Byron works in a lumberyard. His boss just cut a dozen planks and asked Byron to double check that they are all the same length. The planks were numbered 1 through 12. Byron took out plank number 1 and checked that the other planks are all the same length as plank 1. He concluded that they must all be the same length. Explain how you know that plank 7 and plank 10 are the same length even though they were never directly compared to each other. Plank 7 is the same length as plank 1, and plank 1 is the same length as plank 10. By the Transitive Property, plank 7 must be the same length as plank 10.

7. NEIGHBORHOODS Karla, Lola, and Mandy live in three houses that are on the same line. Lola lives between Karla and Mandy. Karla and Mandy live a mile apart. Is it possible for Lola's house to be a mile from both Karla's and Mandy's houses? No, it's not possible. Lola's house must be less than a mile from each house because she lives between them.

Mixed Exercises

8. PROOF Five lights, A, B, C, D, and E, are aligned in a row. The middle light is the midpoint of the segment between the second and fourth lights and also the midpoint of the segment between the first and last lights.

a. Draw a figure to illustrate the situation. See margin.

b. Complete this proof.

Given: C is the midpoint of $\overline{BD}$ and $\overline{AE}$.

Prove: $AB = DE$

Statement	Reason
1. C is the midpoint of $\overline{BD}$ and $\overline{AE}$.	1. Given
2. $BC = CD$ and ? $AC = CE$	2. ? Def. of mdpt.
3. $AC = AB + BC$, $CE = CD + DE$	3. ? Seg. Add. Post.
4. $AC - BC = AB$	4. ? Subtr. Prop.
5. ? $CE - CD = AB$	5. Substitution Property
6. $CE - CD = DE$	6. ? Subtr. Prop.
7. $AB = CE - CD$	7. Symmetric Property of Equality
8. ? $AB = DE$	8. ? Trans. Prop.

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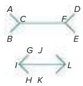
9. PROOF $\overline{AC} \cong \overline{GI}$, $\overline{FE} \cong \overline{LK}$, and $AC + CF + FE = GI + IL + LK$. Prove that $\overline{CF} \cong \overline{IL}$. See margin.

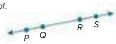


10. PROOF Consider $\overleftrightarrow{PS}$.

a. Complete the two-column proof.

Given: $\overline{PQ} \cong \overline{RS}$

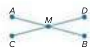
Prove: $\overline{PR} \cong \overline{QS}$



Statement	Reason
1. $\overline{PQ} \cong \overline{RS}$	1. ___?___ Given
2. ___?___ $PQ = RS$	2. Congruent segments have equal lengths.
3. $PQ + QR = PR$ and $QR + RS = QS$	3. ___?___ Segment Addition Property
4. $RS + QR = PR$	4. ___?___ Substitution Property of Equality
5. $QR + RS = PR$	5. ___?___ Commutative Property of Addition
6. $QS = PR$	6. ___?___ Substitution Property of Equality
7. $PR = QS$	7. Symmetric Property of Equality
8. ___?___ $\overline{PR} \cong \overline{QS}$	8. Segments with equal lengths are congruent.

b. Can it also be proved that $\overline{PQ} \cong \overline{RS}$ if $\overline{PR} \cong \overline{QS}$? Explain. See margin.

11. PROOF A city planner is designing a new park. The park has two straight paths, $\overline{AB}$ and $\overline{CD}$, which are the same length. A monument, M, is located at the midpoint of both paths.



a. The city planner thinks that the length of $\overline{AM}$ will be the same as the length of $\overline{CM}$. Explain why this makes sense. See margin.

b. Complete the two-column proof.

Given: $\overline{AB} \cong \overline{CD}$; M is the midpoint of $\overline{AB}$ and $\overline{CD}$.

Prove: $\overline{AM} \cong \overline{CM}$

Statement	Reason
1. ___?___ $\overline{AB} \cong \overline{CD}$; M is the midpoint of $\overline{AB}$ and $\overline{CD}$.	1. Given
2. $AB = CD$	2. ___?___ Congruent segments have equal lengths.
3. $\overline{AM} \cong \overline{MB}$; $\overline{CM} \cong \overline{MD}$	3. ___?___ Definition of midpoint
4. $AM = MB$; $CM = MD$	4. Congruent segments have equal lengths.
5. $AM + MB = AB$; $CM + MD = CD$	5. ___?___ Segment Addition Postulate
6. $AM + MB = CM + MD$	6. ___?___ Substitution Property of Equality
7. $AM + AM = CM + CM$	7. Substitution Property of Equality
8. $2AM = 2CM$	8. ___?___ Substitutive Property of Equality
9. ___?___ $AM = CM$	9. Division Property of Equality
10. ___?___ $\overline{AM} \cong \overline{CM}$	10. Segments with equal lengths are congruent.

12. PROOF Write a paragraph proof for each property of segment congruence.

a. Reflexive Property of Segment Congruence

Given: $\overline{XY}$

Prove: $\overline{XY} \cong \overline{XY}$

Proof:
It is given that $\overline{XY}$ is a segment. By the Reflexive Property of Equality, $XY = XY$. Thus, $\overline{XY} \cong \overline{XY}$ by the definition of congruent segments.

b. Symmetric Property of Segment Congruence

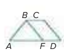
Given: $\overline{AB} \cong \overline{CD}$

Prove: $\overline{CD} \cong \overline{AB}$

Proof:
It is given that $\overline{AB} \cong \overline{CD}$. By the definition of congruent segments, $AB = CD$. By the Symmetric Property of Equality, $CD = AB$. So, by the definition of congruent segments, $\overline{CD} \cong \overline{AB}$.

Higher-Order Thinking Skills

13. FIND THE ERROR In the diagram, $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{BF}$. Examine the conclusions made by Leslie and Shantice. Is either of them correct? Explain your reasoning.
Neither; because $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{BF}$, then $\overline{AB} \cong \overline{BF}$ by the Transitive Property of Congruence.



Leslie	Shantice
Because $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{BF}$, $\overline{AB} \cong \overline{AF}$ by the Transitive Property of Congruence.	Because $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{BF}$, $\overline{AB} \cong \overline{BF}$ by the Reflexive Property of Congruence.

14. PROOF $ABCD$ is a square. Prove that $\overline{AC} \cong \overline{BD}$. See Mod. 12 Answer Appendix.

15. CREATE Draw a representation of the Segment Addition Postulate in which the segment is two inches long, contains four collinear points, and contains no congruent segments. See Mod. 12 Answer Appendix.

16. CREATE Write an example of the Transitive Property and the Substitution Property that illustrates the difference between them. See Mod. 12 Answer Appendix.

17. FIND THE ERROR Justin knows that point R is the midpoint of $\overline{QS}$, and he knows that this means that $QR = RS$. He says that $PR = PQ + QR$ by the Segment Addition Postulate. So, $PR = PQ + RS$ by substitution. Do you agree with Justin's reasoning? Explain your reasoning. See Mod. 12 Answer Appendix.

18. WRITE Compare and contrast paragraph proofs and two-column proofs. See Mod. 12 Answer Appendix.

19. PROOF Write a paragraph proof to prove that if P, Q, R, and S are collinear, $\overline{PQ} \cong \overline{RS}$, and Q is the midpoint of $\overline{PR}$, then R is the midpoint of $\overline{QS}$. See Mod. 12 Answer Appendix.

Answers

3. Given: $\overline{VZ} \cong \overline{VY}$ and $\overline{WY} \cong \overline{XZ}$
   Prove: $\overline{VW} \cong \overline{VX}$
   Proof
   Statements (Reasons)
   1. $\overline{VZ} \cong \overline{VY}$ and $\overline{WY} \cong \overline{XZ}$ (Given)
   2. $VZ = VY$ and $WY = XZ$ (Definition of ≅ segments)
   3. $VZ = VX + XZ$ and $VY = VW + WY$ (Segment Addition Postulate)
   4. $VX + XZ = VW + WY$ (Substitution Property)
   5. $VX + WY = VW + WY$ (Substitution Property)
   6. $VX = VW$ (Subtraction Property of Equality)
   7. $VW = VX$ (Symmetric Property)
   8. $\overline{VW} \cong \overline{VX}$ (Definition of ≅ segments)

4. Given: E is the midpoint of $\overline{DF}$ and $\overline{CD} \cong \overline{FG}$.
   Prove: $\overline{CE} \cong \overline{EG}$
   Proof
   Statements (Reasons)
   1. E is the midpoint of $\overline{DF}$ and $\overline{CD} \cong \overline{FG}$. (Given)
   2. $DE = EF$ (Definition of midpoint)
   3. $CD = FG$ (Definition of ≅ segments)
   4. $CD + DE = EF + FG$ (Addition Property of Equality)
   5. $CE = CD + DE$ and $EG = EF + FG$ (Segment Addition Postulate)
   6. $CE = EG$ (Substitution Property)
   7. $\overline{CE} \cong \overline{EG}$ (Definition of ≅ segments)

8a. Sample answer:

9. Given: $\overline{AC} \cong \overline{GI}$, $\overline{FE} \cong \overline{LK}$, $AC + CF + FE = GI + IL + LK$
   Prove: $\overline{CF} \cong \overline{IL}$
   Proof
   Statements (Reasons)
   1. $\overline{AC} \cong \overline{GI}$, $\overline{FE} \cong \overline{LK}$, $AC + CF + FE = GI + IL + LK$ (Given)
   2. $AC = GI$ and $FE = LK$ (Definition of ≅ segments)
   3. $AC + CF + FE = AC + IL + LK$ (Substitution Property)
   4. $AC - AC + CF + FE = AC - AC + IL + LK$ (Subtraction Property of Equality)
   5. $CF + FE = IL + LK$ (Substitution Property)
   6. $CF + FE = IL + FE$ (Substitution Property)
   7. $CF + FE - FE = IL + FE - FE$ (Subtraction Property of Equality)
   8. $CF = IL$ (Substitution Property)
   9. $\overline{CF} \cong \overline{IL}$ (Definition of ≅ segments)

10b. Yes; the Segment Addition Postulate can be used to show that $PR = PQ + QR$ and $QS = QR + RS$. Both equations can be solved for QR, and substituting PR for QS will lead to $\overline{PQ} \cong \overline{RS}$.

11a. Both segments are half the length of two congruent segments, so the lengths of the shorter segments must be the same.

Proving Angle Relationships

G.CO.9

LESSON GOAL

Students prove theorems about angles.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Angle Relationships

Develop:

Angle Addition
- Angle Addition Postulate
- Complement and Supplement Theorems

Congruent Angles
- Congruent Supplements and Complements
- Vertical Angles

Right Angle Theorems
- Right Angle Theorems in Proofs

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports on student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Deductive Reasoning	●	●		●
Extension: Symmetric, Reflexive, and Transitive Properties		●	●	●

Language Development Handbook

Assign page 76 of the *Language Development Handbook* to help your students build mathematical language related to proving relationships about angles.

ELL You can use the tips and suggestions on page T76 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.9 Prove theorems about lines and angles.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students proved theorems about line segments.
G.CO.9

Now
Students prove theorems about angles using the Angle Addition Postulate.
G.CO.9

Next
Students will identify special angle pairs, parallel lines, and transversals.
G.CO.1

Rigor

The Three Pillars of Rigor

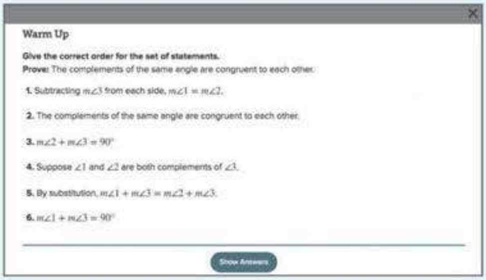
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of proofs, and they build fluency by proving theorems about angle relationships.

Mathematical Background

This lesson introduces postulates and theorems about angle relationships. The Protractor Postulate and the Angle Addition Postulate can be used to prove theorems about angle relationships.

Interactive Presentation



Warm Up

Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- making a valid argument about geometry

Answers:

1. 5
2. 6
3. 3
4. 1
5. 4
6. 2

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of angle relationships.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Explore Angle Relationships

Objective

Students use dynamic geometry software to explore angle relationships.

MP Teaching the Mathematical Practices

1 Monitor and Evaluate Point out that in this Explore, students must stop and evaluate their progress and change course to find the ultimate solution.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students perform a number of guided steps with dynamic geometry software, interspersed with guiding exercises intended to guide students through proving that the complements of an angle are congruent. Then, students answer the Inquiry Question.

(continued on the next page)

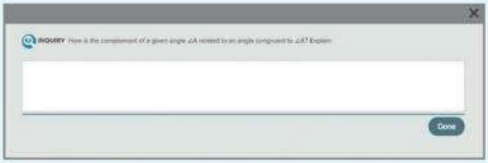
Interactive Presentation

Explore

WEB SKETCHPAD

Students use a sketch to explore angle relationships.

Interactive Presentation



Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Angle Relationships (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why does it matter that point D is in the interior of $\angle ABC$? Sample answer: You know that angle ABC is a right angle, so its measure is 90°. If you place point D in the interior, you also know that the two angles are complementary, because they have to add to 90°.
- What would the relationship be if you constructed $\angle JKL$ congruent to $\angle DBC$? Sample answer: Then $\angle JKL$ would be complementary to $\angle ABD$, because $\angle ABD$ is complementary to $\angle DBC$.

Inquiry

How is the complement of a given angle A related to an angle congruent to $\angle A$? Sample answer: It is the complement of the congruent angle. By the definition of congruence, the measures of two congruent angles are equal. By the Transitive Property, you can substitute the measure of one angle for the measure of a congruent angle.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Angle Addition

Objective

Students prove theorems about angles by using the Angle Addition Postulate.

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This Learn asks students to respond to the arguments of others.

About the Key Concept

The Protractor Postulate and Angle Addition Postulate perform the same function for angles as the Ruler Postulate and Segment Addition Postulate do for line segments.

Example 1 Angle Addition Postulate

Teaching the Mathematical Practices

1 Understand the Approaches of Others Mathematically proficient students can explain the methods used to solve a problem. This example asks students to justify reasoning.

Questions for Mathematical Discourse

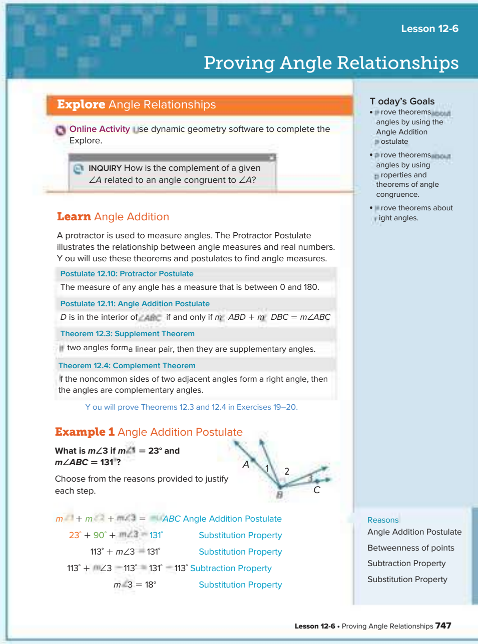
AL What other postulate is similar to the Angle Addition Postulate? Segment Addition Postulate

OL What Property could be used as justification in Step 2 of the solution? Substitution Property of Equality

BL Suppose $m\angle ABC = 145°$, $m\angle 1 = 2x$, and $m\angle 2 = x + 10$. What are the measures of $\angle 1$ and $\angle 2$? 90° and 55°

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 12-6

Proving Angle Relationships

Explore Angle Relationships

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How is the complement of a given $\angle A$ related to an angle congruent to $\angle A$?

Today's Goals
- Prove theorems about angles by using the Angle Addition Postulate
- Prove theorems about angles by using properties and theorems of angle congruence.
- Prove theorems about right angles.

Learn Angle Addition

A protractor is used to measure angles. The Protractor Postulate illustrates the relationship between angle measures and real numbers. You will use these theorems and postulates to find angle measures.

Postulate 12.10: Protractor Postulate
The measure of any angle has a measure that is between 0 and 180.

Postulate 12.11: Angle Addition Postulate
D is in the interior of $\angle ABC$ if and only if $m\angle ABD + m\angle DBC = m\angle ABC$

Theorem 12.3: Supplement Theorem
If two angles form a linear pair, then they are supplementary angles.

Theorem 12.4: Complement Theorem
If the noncommon sides of two adjacent angles form a right angle, then the angles are complementary angles.

You will prove Theorems 12.3 and 12.4 in Exercises 19–20.

Example 1 Angle Addition Postulate

What is $m\angle 3$ if $m\angle 1 = 23°$ and $m\angle ABC = 131°$?

Choose from the reasons provided to justify each step.

$m\angle 1 + m\angle 2 + m\angle 3 = m\angle ABC$	Angle Addition Postulate
$23° + 90° + m\angle 3 = 131°$	Substitution Property
$113° + m\angle 3 = 131°$	Substitution Property
$113° + m\angle 3 - 113° = 131° - 113°$	Subtraction Property
$m\angle 3 = 18°$	Substitution Property

Reasons
Angle Addition Postulate
Betweenness of points
Subtraction Property
Substitution Property

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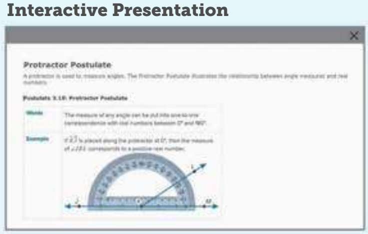
Interactive Presentation



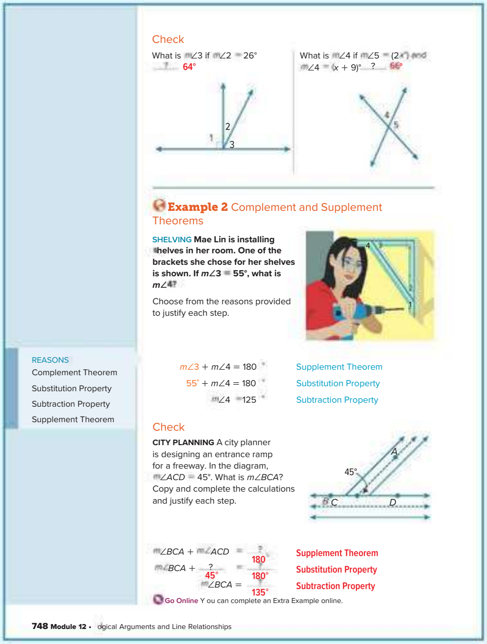
Learn

TYPE

Students answer a question to show that they understand angle addition.

G.CO.9

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**



Check

What is $m\angle 3$ if $m\angle 2 = 26°$? **64°**

What is $m\angle 4$ if $m\angle 5 = (2x)°$ and $m\angle 4 = (x + 9)°$? **66°**

Example 2 Complement and Supplement Theorems

SHELVING **Mae Lin is installing shelves in her room. One of the brackets she chose for her shelves is shown. If $m\angle 3 = 55°$, what is $m\angle 4$?**

Choose from the reasons provided to justify each step.

REASONS
Complement Theorem
Substitution Property
Subtraction Property
Supplement Theorem

$m\angle 3 + m\angle 4 = 180°$	Supplement Theorem
$55° + m\angle 4 = 180°$	Substitution Property
$m\angle 4 = 125°$	Subtraction Property

Check

CITY PLANNING A city planner is designing an entrance ramp for a freeway. In the diagram, $m\angle ACD = 45°$. What is $m\angle BCA$? Copy and complete the calculations and justify each step.

$m\angle BCA + m\angle ACD = 180$	Supplement Theorem
$m\angle BCA + 45° = 180°$	Substitution Property
$m\angle BCA = 135°$	Subtraction Property

Go Online You can complete an Extra Example online.

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Example 2 Complement and Supplement Theorems

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about the Angle Addition Postulate to solving a real-world problem.

Questions for Mathematical Discourse

AL Based on the diagram, what is another pair of supplementary angles? Sample answer: ∠1 and ∠2

OL Which of the angles in the diagram are not needed to answer the question? the right angle, ∠1, ∠2

BL Suppose ∠2 and ∠3 are congruent. What is the measure of ∠1? 135°

Common Error

Students may confuse complementary angles and supplementary angles. One way to remember them is that the name that comes earlier in the alphabet, complementary, coincides with the smaller angle sum, 90°.

Interactive Presentation

Example 2

DRAG & DROP

Students drag statements and reasons to complete a proof.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Learn Congruent Angles

Objective

Students prove theorems about angles by using properties and theorems of angle congruence.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Learn Congruent Angles

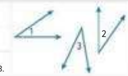
The properties of algebra that apply to the congruence of segments and the equality of their measures also hold true for the congruence of angles and the equality of their measures.

Theorem 12.5: Properties of Angle Congruence

Reflexive Property of Congruence
$\angle 1 \cong \angle 1$

Symmetric Property of Congruence
If $\angle 1 \cong \angle 2$, then $\angle 2 \cong \angle 1$.

Transitive Property of Congruence
If $\angle 1 \cong \angle 2$ and $\angle 2 \cong \angle 3$, then $\angle 1 \cong \angle 3$.



Go Online Proofs of the Reflexive Property of Congruence and the Transitive Property of Congruence are available.

Proof: Symmetric Property of Congruence

Given: $\angle J \cong \angle K$

Prove: $\angle K \cong \angle J$

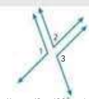
Paragraph Proof:
We are given that $\angle J \cong \angle K$. By the definition of congruent angles, $m\angle J = m\angle K$. Using the Symmetric Property of Equality, $m\angle K = m\angle J$. Thus, $\angle K \cong \angle J$ by the definition of congruent angles.

Theorems

Theorem 12.6: Congruent Supplements Theorem

Angles supplementary to the same angle or to congruent angles are congruent.

Abbreviation ∠s suppl. to same ∠ or ≅ ∠s are ≅.

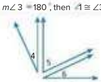


If $m\angle 1 + m\angle 2 = 180°$ and $m\angle 2 + m\angle 3 = 180°$, then $\angle 1 \cong \angle 3$.

Theorem 12.7: Congruent Complements Theorem

Angles complementary to the same angle or to congruent angles are congruent.

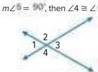
Abbreviation ∠s compl. to same ∠ or ≅ ∠s are ≅.



If $m\angle 4 + m\angle 5 = 90°$ and $m\angle 5 + m\angle 6 = 90°$, then $\angle 4 \cong \angle 6$.

Theorem 12.8: Vertical Angles Theorem

If two angles are vertical angles, then they are congruent.



$\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$

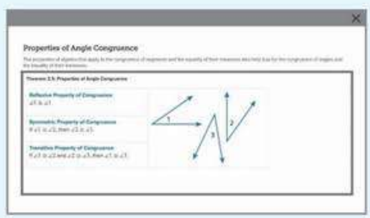
Talk About It! Explain the difference between the Complement Theorem and the Congruent Complements Theorem.

Sample answer: The Complement Theorem states that if the noncommon sides of two adjacent angles form a right angle, then the angles are complementary angles. The Congruent Complements Theorem states that angles complementary to the same angle or to congruent angles are congruent.

You will prove one case of Theorems 12.6 and 12.7 in Exercises 21-22. You will prove the second case of each theorem in Exercise 31.

Lesson 12-6 • Proving Angle Relationships **749**

Interactive Presentation



Learn

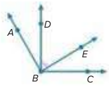
TYPE

Students answer a question to show they understand congruent angles.

REASONS:
Complement Theorem
Congruent Complements Theorem
Definition of complementary angles
Definition of congruence

Example 3 Congruent Supplements and Complements

In the figure, ∠ABE and ∠DBC are right angles.



Select from the reasons provided to complete the proof.

Given: ∠ABE and ∠DBC are right angles.

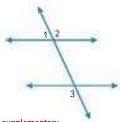
Prove: ∠ABD ≅ ∠EBC

Proof:

Statements	Reasons
1. ∠ABE and ∠DBC are right angles.	1. Given
2. m∠ABE = 90°; m∠DBC = 90°	2. Definition of right angle
3. ∠ABD and ∠DBE are complementary; ∠DBE and ∠EBC are complementary.	3. Complement Theorem
4. ∠ABD ≅ ∠EBC	4. Congruent Complements Theorem

Check

Copy and complete the proof.



Given: ∠1 and ∠3 are supplementary.

Prove: ∠2 ≅ ∠3

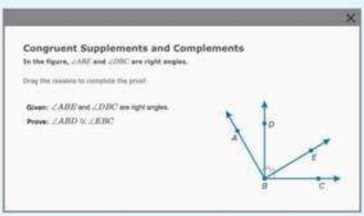
Proof:

It is given that ∠1 and ∠3 are supplementary. By the definition of linear pair, ∠1 and ∠2 are a linear pair. So, by the Supplement Theorem, ∠1 and ∠2 are supplementary. Thus, ∠2 ≅ ∠3 by the Congruent Supplements Theorem.

Go Online You can complete an Extra Example online.

750 Module 12 • Logical Arguments and Line Relationships

Interactive Presentation



Example 3

DRAG & DROP

Students drag statements and reasons to complete a proof.

Example 3 Congruent Supplements and Complements

MP Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This example asks students to respond to the arguments of others.

Questions for Mathematical Discourse

AL Do you need to know $m\angle ABD$ and $m\angle EBC$ to prove that they are congruent? Explain. No; sample answer: You can use definitions and theorems to prove that the angles are congruent without knowing the measures.

OL What do you know about a pair of angles that comprise a right angle? They are complementary angles.

BL If you extend $\overrightarrow{BA}$ past point B to a new point F, what angle can you prove is congruent to $\angle CBF$? $\angle DBE$

Common Error

Students may think that complements are congruent to each other, rather than two complements of the same angle are congruent to each other.

Example 4 Vertical Angles

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This example asks students to respond to the arguments of others.

Questions for Mathematical Discourse

AL What are vertical angles? a pair of nonadjacent angles formed when two lines intersect

OL What does the Vertical Angles Theorem say about a pair of nonadjacent angles formed when two lines intersect? Vertical angles are congruent.

BL If the Given and Prove statements were switched, would the reasons remain the same? Explain. Yes; sample answer: The thought process would remain the same even though the angles would be different.

Learn Right Angle Theorems

Objective

Students prove theorems about right angles.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of right angle theorems to understand and prove theorems about right angles.

Common Misconception

Students may forget that the only time supplementary angles are congruent to each other is when they are right angles.

Essential Question Follow-Up

Students learn to use angle congruence theorems.

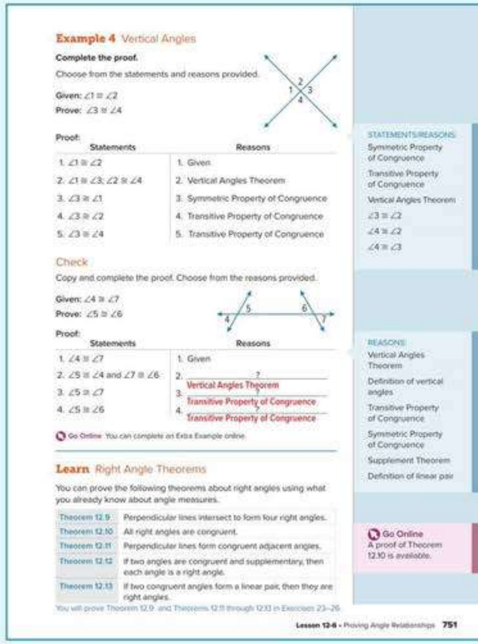
Ask:

Why is it important to know how to use right angle theorems? Sample answer: These theorems are useful for writing logical arguments in geometry.

DIFFERENTIATE

Language Development Activity AL ELL

IF students have difficulty remembering the difference between complementary and supplementary angles,
THEN have them write a short poem or rhyme to help them remember the definitions.



Example 4 Vertical Angles

Complete the proof.

Choose from the statements and reasons provided.

Given: $\angle 1 \cong \angle 2$
Prove: $\angle 3 \cong \angle 4$

Proof:

Statements	Reasons
1. $\angle 1 \cong \angle 2$	1. Given
2. $\angle 1 \cong \angle 3$; $\angle 2 \cong \angle 4$	2. Vertical Angles Theorem
3. $\angle 3 \cong \angle 1$	3. Symmetric Property of Congruence
4. $\angle 3 \cong \angle 2$	4. Transitive Property of Congruence
5. $\angle 3 \cong \angle 4$	5. Transitive Property of Congruence

STATEMENTS/REASONS: Symmetric Property of Congruence; Transitive Property of Congruence; Vertical Angles Theorem; $\angle 3 \cong \angle 2$; $\angle 4 \cong \angle 2$; $\angle 4 \cong \angle 3$

Check

Copy and complete the proof. Choose from the reasons provided.

Given: $\angle 4 \cong \angle 7$
Prove: $\angle 5 \cong \angle 6$

Proof:

Statements	Reasons
1. $\angle 4 \cong \angle 7$	1. Given
2. $\angle 5 \cong \angle 4$ and $\angle 7 \cong \angle 6$	2. ? Vertical Angles Theorem
3. $\angle 5 \cong \angle 7$	3. ? Transitive Property of Congruence
4. $\angle 5 \cong \angle 6$	4. ? Transitive Property of Congruence

REASONS: Vertical Angles Theorem; Definition of vertical angles; Transitive Property of Congruence; Symmetric Property of Congruence; Supplement Theorem; Definition of linear pair

Go Online You can complete an Extra Example online.

Learn Right Angle Theorems

You can prove the following theorems about right angles using what you already know about angle measures.

Theorem 12.9	Perpendicular lines intersect to form four right angles.
Theorem 12.10	All right angles are congruent.
Theorem 12.11	Perpendicular lines form congruent adjacent angles.
Theorem 12.12	If two angles are congruent and supplementary, then each angle is a right angle.
Theorem 12.13	If two congruent angles form a linear pair, then they are right angles.

You will prove Theorem 12.9 and Theorems 12.11 through 12.13 in Exercises 23–26.

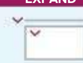
Go Online A proof of Theorem 12.10 is available.

Lesson 12-6 • Proving Angle Relationships 751

Interactive Presentation

Learn

EXPAND



Students tap to see various right angle theorems.

STATEMENTS/REASONS:

Reflexive Property of Congruence

Symmetric Property of Congruence

Transitive Property of Congruence

Vertical Angles Theorem

∠1 and ∠2 are right angles. ∠1 ≅ ∠2

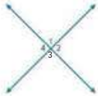
Example 5 Right Angle Theorems in Proofs

Complete the proof.

Choose from the statements and reasons provided.

Given: ∠1 ≅ ∠4

Prove: ∠1 and ∠2 are right angles.



Proof:

Statements	Reasons
1. ∠1 ≅ ∠4	1. Given
2. ∠2 ≅ ∠4	2. Vertical Angles Theorem
3. ∠4 ≅ ∠2	3. Symmetric Property of Congruence
4. ∠1 ≅ ∠2	4. Transitive Property of Congruence
5. ∠1 and ∠2 are right angles.	5. If two angles are congruent and supplementary, then each angle is a right angle.

REASONS

Perpendicular lines intersect to form four right angles.

All right angles are congruent.

Perpendicular lines form congruent adjacent angles.

If two angles are congruent and supplementary, then each angle is a right angle.

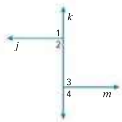
If two congruent angles form a linear pair, then they are right angles.

Check

Copy and complete the proof. Choose from the reasons provided.

Given: ∠1 ≅ ∠4
Lines *j* and *k* are perpendicular.

Prove: ∠2 ≅ ∠4



Proof:

Statements	Reasons
1. Lines *j* and *k* are perpendicular.	1. Given
2. ∠2 ≅ ∠1	2. Perpendicular lines form congruent adjacent angles.
3. ∠1 ≅ ∠4	3. Given
4. ∠2 ≅ ∠4	4. Transitive Property of Congruence

Go Online You can complete an Extra Example online.

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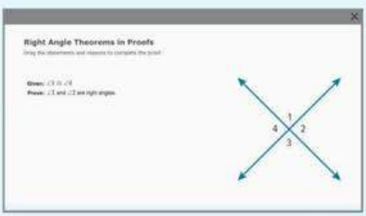
Interactive Presentation



Example 5

DRAG

Students drag statements and reasons to complete a two-column proof.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Right Angle Theorems in Proofs

Teaching the Mathematical Practices

1 Understand the Approaches of Others Work with students to look at the Alternate Method. Ask students to compare and contrast the original method and the alternate method.

Questions for Mathematical Discourse

AL This proof has a right angle in its conclusion. Which right angle theorems have a right angle in their conclusion? Theorem 3.12 and Theorem 3.13

OL Why can't we use Theorem 3.10 to write this proof? Sample answer: Being a right angle is part of its givens, not its conclusion.

BL Which angles form linear pairs in the diagram? ∠1 and ∠2, ∠2 and ∠3, ∠3 and ∠4, ∠4 and ∠1

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

The Practice pages are meant to be used as a homework assignment. You will also find these questions online in the Practice Bank for customization, digital assignment, and auto-scoring.

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–10
2	exercises that use a variety of skills from this lesson	11–18
2	exercises that extend concepts learned in this lesson to new contexts	19–27
3	exercises that emphasize higher-order and critical-thinking skills	28–31

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–27 odd, 28–31
- Extension: Symmetric, Reflexive, and Transitive Properties
- ALEKS Proofs Involving Segments and Angles

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–31 odd
- Remediation, Review Resources: Writing Proofs
- Personal Tutors
- Extra Examples 1–5
- ALEKS Proofs Involving Segments and Angles

AL

IF students score 65% or less on the Checks,
THEN assign:

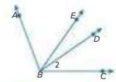
- Practice, Exercises 1–9 odd
- Remediation, Review Resources: Writing Proofs
- *Quick Review Math Handbook*: Proving Angle Relationships
- ALEKS Proofs Involving Segments and Angles

Practice

Go Online Y ou can complete your homework online.

Example 1

Find the measure of each angle.



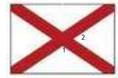
1. Find $m\angle ABC$ if $m\angle ABD = 70°$ and $m\angle DBC = 43°$. 113°

2. If $m\angle EBC = 55°$ and $m\angle EBD = 20°$, find $m\angle 2$. 35°

3. Find $m\angle ABD$ if $m\angle ABC = 110°$ and $m\angle 2 = 36°$. 74°

Example 2

4. FLAGS The Alabama state flag is white and has two diagonal red stripes. If the $m\angle 1 = 112°$, what is $m\angle 2$? 68°

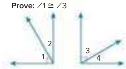


5. CONSTRUCTION Aaron has installed a new window above the entrance of an office building. If $m\angle 2 = 44°$, what is $m\angle 1$? 46°

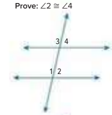
Example 3

PROOF **Write a two-column proof.** 6 –7. See margin.

6. Given: $\angle 2 \cong \angle 4$
Prove: $\angle 1 \cong \angle 3$



7. Given: $\angle 1 \cong \angle 3$
Prove: $\angle 2 \cong \angle 4$

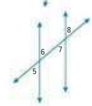


Example 4

PROOF **Write a two-column proof.**

8. Given: $\angle 5 \cong \angle 7$
Prove: $\angle 5 \cong \angle 8$

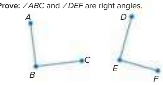
Statements (Reasons)
1. $\angle 5 \cong \angle 7$ (Given)
2. $\angle 7 \cong \angle 8$ (Vertical Angles Theorem)
3. $\angle 5 \cong \angle 8$ (Transitive Prop. of Congruence)



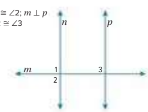
Example 5

PROOF **Write a two-column proof.** 9–10. See margin.

9. Given: $m\angle ABC = m\angle DEF$
$\angle ABC$ and $\angle DEF$ are supplementary.
Prove: $\angle ABC$ and $\angle DEF$ are right angles.

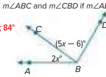


10. Given: $\angle 1 \cong \angle 2$; $m \perp p$
Prove: $\angle 2 \cong \angle 3$

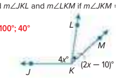


Mixed Exercises

11. Find $m\angle ABC$ and $m\angle CBD$ if $m\angle ABD = 120°$. 36°; 84°

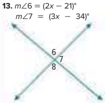


12. Find $m\angle JKL$ and $m\angle LKM$ if $m\angle JKM = 140°$. 100°; 40°



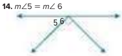
Find the measure of each numbered angle and name the theorems that you used to justify your work.

13. $m\angle 6 = (2x - 21)°$
$m\angle 7 = (3x - 34)°$



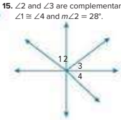
$m\angle 6 = m\angle 8 = 73°$, $m\angle 7 = 107°$ (≅ Supp. Thm. and Vert. ∠s Thm.)

14. $m\angle 5 = m\angle 6$



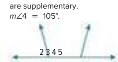
$m\angle 5 = m\angle 6 = 45°$ (≅ Supp. Thm.)

15. $\angle 2$ and $\angle 3$ are complementary.
$\angle 1 \cong \angle 4$ and $m\angle 2 = 28°$.



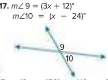
$m\angle 3 = 62°$, $m\angle 1 = m\angle 4 = 45°$ (≅ Comp. and Supp. Thm.)

16. $\angle 2$ and $\angle 4$ and $\angle 4$ and $\angle 5$ are supplementary.
$m\angle 4 = 105°$.



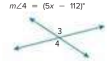
$m\angle 2 = 75°$, $m\angle 3 = 105°$, $m\angle 5 = 75°$ (≅ Supp. Thm.)

17. $m\angle 9 = (3x + 12)°$
$m\angle 10 = (x - 24)°$



$m\angle 9 = 156°$, $m\angle 10 = 24°$ (≅ Supp. Thm.)

18. $m\angle 3 = (2x + 23)°$
$m\angle 4 = (5x - 112)°$

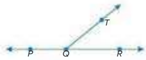


$m\angle 3 = m\angle 4 = 113°$ (Vert. ∠s Thm.)

PROOF **Write a two-column proof for each theorem.**

19. Supplement Theorem

Given: $\angle PQT$ and $\angle TQR$ form a linear pair.
Prove: $\angle PQT$ and $\angle TQR$ are supplementary.

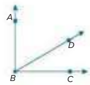


Statements (Reasons)

1. $\angle PQT$ and $\angle TQR$ form a linear pair. (Given)
2. $\angle PQR$ is a straight angle. (Given from figure)
3. $m\angle PQR = 180°$ (Def. of straight angle)
4. $m\angle PQT + m\angle TQR = m\angle PQR$ (Angle Add. Post.)
5. $m\angle PQT + m\angle TQR = 180°$ (Substitution)
6. $\angle PQT$ and $\angle TQR$ are supplementary. (Def. of supp. angles)

20. Complement Theorem

Given: $\angle ABC$ is a right angle.
Prove: $\angle ABD$ and $\angle CBD$ are complementary.

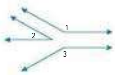


Statements (Reasons)

1. $\angle ABC$ is a right angle. (Given)
2. $m\angle ABC = 90°$ (Def. of rt. angle)
3. $m\angle ABC = m\angle ABD + m\angle CBD$ (Angle Add. Post.)
4. $m\angle ABD + m\angle CBD = 90°$ (Substitution)
5. $\angle ABD$ and $\angle CBD$ are complementary. (Def. of comp. angles)

21. Congruent Supplements Theorem (Case 1)

Given: $\angle 1$ and $\angle 2$ are supplementary.
$\angle 2$ and $\angle 3$ are supplementary.
Prove: $\angle 1 \cong \angle 3$

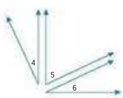


Statements (Reasons)

1. $\angle 1$ and $\angle 2$ are supplementary.
   $\angle 2$ and $\angle 3$ are supplementary. (Given)
2. $m\angle 1 + m\angle 2 = 180°$
   $m\angle 2 + m\angle 3 = 180°$ (Def. of supp. angles)
3. $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$ (Substitution)
4. $m\angle 1 = m\angle 3$ (Subtraction Prop. of Equality)
5. $\angle 1 \cong \angle 3$ (Def. of congruent angles)

22. Congruent Complements Theorem (Case 1)

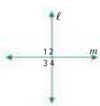
Given: $\angle 4$ and $\angle 5$ are complementary.
$\angle 5$ and $\angle 6$ are complementary.
Prove: $\angle 4 \cong \angle 6$



Statements (Reasons)

1. $\angle 4$ and $\angle 5$ are complementary.
   $\angle 5$ and $\angle 6$ are complementary. (Given)
2. $m\angle 4 + m\angle 5 = 90°$
   $m\angle 5 + m\angle 6 = 90°$ (Def. of comp. angles)
3. $m\angle 4 + m\angle 5 = m\angle 5 + m\angle 6$ (Substitution)
4. $m\angle 4 = m\angle 6$ (Subtraction Prop. of Equality)
5. $\angle 4 \cong \angle 6$ (Def. of congruent angles)

PROOF **Use the figure to write a proof of each theorem.**
23–26. See Mod. 12 Answer Appendix.



23. Perpendicular lines intersect to form four right angles. (Theorem 12.9)

24. Perpendicular lines form congruent adjacent angles. (Theorem 12.11)

25. If two angles are congruent and supplementary, then each angle is a right angle. (Theorem 12.12)

26. If two congruent angles form a linear pair, then they are right angles. (Theorem 12.13)

27. CONSTRUCT ARGUMENTS For a school project, students are making a giant icosahedron, which is a large solid with twenty identical triangular faces. John is in charge of quality control. He must make sure that the measures of all the angles in all the triangles are the same. He does this by using a precut template and comparing the corner angles of every triangle to the template. How does this assure that the angles in all the triangles will be congruent to each other?
By the Transitive Property, if any two angles are equal to the angle of the template, then they must be equal to each other.

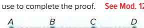
Higher-Order Thinking Skills

28. ANAL YZE Find $m\angle C$ if $\angle C \cong \angle A$, $m\angle A = 3x°$, $m\angle B = (x + 20)°$, and $\angle A$ and $\angle B$ are supplementary. Justify your argument. See Mod. 12 Answer Appendix.

29. CREATE Draw $\angle WXZ$ such that $m\angle WXZ = 45°$. Construct $\angle YXZ \cong \angle WXZ$. Make a conjecture about the measure of $\angle WXY$, and then prove your conjecture.
See Mod. 12 Answer Appendix.

30. WRITE Write the steps that you would use to complete the proof. See Mod. 12 Answer Appendix.

Given: $\overline{BC} \cong \overline{CD}$, $AB = \frac{1}{2}BD$
Prove: $\overline{AB} \cong \overline{CD}$



31. PERSEVERE In Exercises 21 and 22, you proved one case of the Congruent Supplements Theorem and one case of the Congruent Complements Theorem. Explain why there is another case for each of these theorems. Then write a proof of this second case for each theorem. See Mod. 12 Answer Appendix.

Answers

6. Proof:

Statements (Reasons)

1. $\angle 1$ and $\angle 2$ form a right angle.
   $\angle 3$ and $\angle 4$ form a right angle. (Given)
2. $\angle 1$ and $\angle 2$ are complementary.
   $\angle 3$ and $\angle 4$ are complementary. (Complement Thm.)
3. $\angle 2 \cong \angle 4$ (Given)
4. $\angle 1 \cong \angle 3$ (Congruent Complements Thm.)

7. Proof:

Statements (Reasons)

1. $\angle 1$ and $\angle 2$ form a linear pair.
   $\angle 3$ and $\angle 4$ form a linear pair. (Def. of linear pair)
2. $\angle 1$ and $\angle 2$ are supplementary.
   $\angle 3$ and $\angle 4$ are supplementary. (Supp. Thm)
3. $\angle 1 \cong \angle 3$ (Given)
4. $\angle 2 \cong \angle 4$ (≅ Supp. Thm)

9. Proof:

Statements (Reasons)

1. $m\angle ABC = m\angle DEF$ (Given)
2. $\angle ABC \cong \angle DEF$ (Def. of ≅ angles)
3. $\angle ABC$ and $\angle DEF$ are supplementary. (Given)
4. $\angle ABC$ and $\angle DEF$ are rt. angles. (If two ∠s are ≅ and supp., then each ∠ is a rt. ∠.)

10. Proof:

Statements (Reasons)

1. $\angle 1 \cong \angle 2$; $m \perp p$ (Given)
2. $\angle 1$ and $\angle 2$ form a linear pair. (Def. of linear pair)
3. $\angle 1$ and $\angle 2$ are right angles. (If 2 ≅ ∠s form a linear pair, they are rt. ∠s.)
4. $\angle 3$ is a right angle. (⊥ lines form 4 rt. angles.)
5. $\angle 2 \cong \angle 3$ (All rt. ∠s are congruent.)

Lesson 12-7

G.CO.1, G.CO.9

Parallel Lines and Transversals

LESSON GOAL

Students identify and use relationships between parallel lines and transversals.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Parallel Lines and Transversals
- Identify Parallel and Skew Relationships
- Classify Angle Pair Relationships
- Identify Transversals and Classify Angle Pairs

Explore: Relationships Between Angles and Parallel Lines

Develop:

Angles and Parallel Lines
- Use Theorems About Parallel Lines
- Find Values of Variables

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports on student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Angle Relationships and Parallel Lines	●	●		●
Extension: Parallelism in Space		●	●	●

Language Development Handbook

Assign page 77 of the *Language Development Handbook* to help your students build mathematical language related to relationships between parallel lines and angles.

ELL You can use the tips and suggestions on page T77 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G.CO.9 Prove theorems about lines and angles.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

3 Construct viable arguments and critique the reasoning of others.

6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students analyzed angle relationships.
8.G.5, G.CO.9

Now
Students identify and use relationships between parallel lines and transversals.
G.CO.1

Next
Students will classify lines as parallel, perpendicular, or neither by using the slope criteria.
G.GPE.5

Rigor

The Three Pillars of Rigor

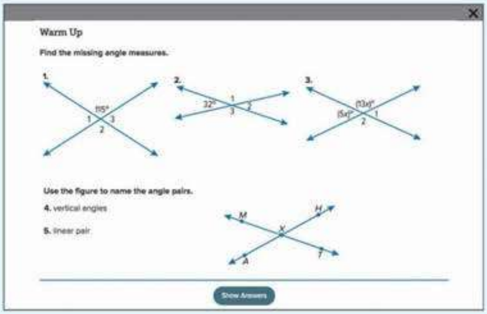
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of parallel line relationships and build fluency by proving theorems related to parallel lines. They apply their understanding by solving real-world problems related to parallel lines and transversals.

Mathematical Background

The angles created by parallel lines and transversals have properties that can be used to make conjectures and to determine the validity of the conjectures.

Interactive Presentation



Warm Up

Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skills for this lesson:

- finding missing angle measures using angle relationships
- analyzing angles and parallel lines

Answers:

1. $m\angle 1 = m\angle 3 = 65°$, $m\angle 2 = 115°$
2. $m\angle 1 = m\angle 3 = 148°$, $m\angle 2 = 32°$
3. $x = 10$; $m\angle 1 = 5x = 50°$, $m\angle 2 = 13x = 130°$
4. $\angle MXH$, $\angle AXT$; $\angle MXA$, $\angle HXT$
5. $\angle MXH$, $\angle HXT$; $\angle HXT$, $\angle TXA$; $\angle TXA$, $\angle AXM$; $\angle AXM$, $\angle MXH$

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of parallel lines, transversals, and angles.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards*? and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Relationships Between Angles and Parallel Lines

Objective

Students use dynamic geometry software to determine the relationships between special angle pairs and parallel lines.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

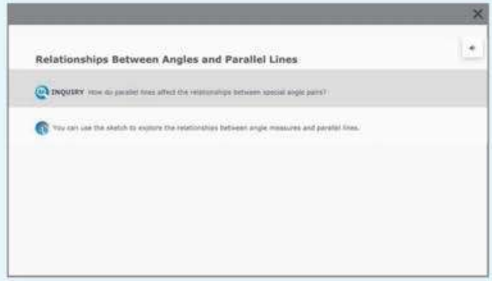
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students use dynamic geometry software to explore the relationships between angles formed when parallel lines are cut by a transversal. They record their observations and make conjectures about the relationships they find between various types of angles. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

WEB SKETCHPAD

Students use a sketch to explore angle relationships.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Relationships Between Angles and Parallel Lines (*continued*)

Question

Have students complete the Explore activity.

Ask:

- What do you know about the angles formed by $\overleftrightarrow{AB}$ and $\overleftrightarrow{FG}$? Sample answer: Several angles are formed by the intersection of these two lines. I know that vertical angles are congruent and linear pairs are supplementary.
- Why does it matter that $\overleftrightarrow{FG}$ and $\overleftrightarrow{JK}$ are parallel? Sample answer: The parallel lines intersect with the transversal in the same way, so the angles have special relationships.

Inquiry

How do parallel lines affect the relationships between special angle pairs? Sample answer: Parallel lines make special angle pairs that are either congruent or supplementary.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Parallel Lines and Transversals

Objective

Students identify special angle pairs, parallel and skew lines, and transversals.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Example 1 Identify Parallel and Skew Relationships

Teaching the Mathematical Practices

3 Find the Error This example requires students to read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Questions for Mathematical Discourse

AL Are $\overline{AD}$ and $\overline{BC}$ coplanar? $\overline{DE}$ and $\overline{BC}$? yes; no

OL How are skew lines different from parallel lines? Parallel lines are coplanar, and skew lines are not.

BL Are lines in parallel planes *always, sometimes,* or *never* parallel? Explain. Sometimes; sample answer: If there is a plane that can be drawn that will contain both lines, then they are parallel. If there is no plane that can contain both lines, then they are skew.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 12-7

Parallel Lines and Transversals

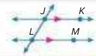
Learn Parallel Lines and Transversals

If two lines do not intersect, then they are either parallel or skew.

Parallel and Skew

Parallel Lines

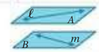
Parallel lines are coplanar lines that do not intersect.



Example $\overleftrightarrow{JK} \parallel \overleftrightarrow{LM}$

Skew Lines

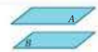
Skew lines are lines that do not intersect and are not coplanar.



Example Lines ℓ and m are skew.

Parallel Planes

Parallel planes are planes that do not intersect.

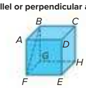


Example Planes A and B are parallel.

If segments or rays are contained within lines that are parallel or skew, then the segments or rays are parallel or skew.

Example 1 Identify Parallel and Skew Relationships

Identify each of the following using the cube shown. Assume lines and planes that appear to be parallel or perpendicular are parallel or perpendicular, respectively.



a. all lines skew to $\overleftrightarrow{BC}$

$\overleftrightarrow{AF}$, $\overleftrightarrow{DE}$, $\overleftrightarrow{FG}$, and $\overleftrightarrow{HE}$

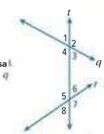
b. all lines parallel to $\overleftrightarrow{EH}$

$\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, or $\overleftrightarrow{FG}$

c. all planes parallel to plane *DCH*

Plane *ABG* is the only plane parallel to plane *DCH*.

A line that intersects two or more lines in a plane at different points is called a **transversal**. In the diagram, line t is a transversal of lines q and r. Notice that line t forms a total of eight angles with lines q and r. These angles and specific pairings of these angles are given special names.



Today's Goals

- Identify special angle pairs, parallel and skew lines, and transversals.
- Find values by applying theorems about parallel lines and transversals.

Today's Vocabulary

parallel lines
skew lines
parallel planes
transversal
interior angles
exterior angles
consecutive interior angles
alternate interior angles
alternate exterior angles
corresponding angles

Study Tip

Parallel Lines The statement $\overleftrightarrow{JK} \parallel \overleftrightarrow{LM}$ is read as *line JK is parallel to line LM*. In a figure, arrowheads are used to indicate that lines are parallel.

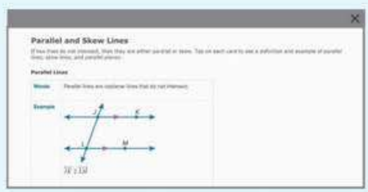
Talk About It!

Can a two-dimensional figure contain skew lines? Justify your argument.

No; sample answer: Skew lines are noncoplanar. Because a two-dimensional figure is contained within a plane, any lines that form the figure cannot be skew.

Lesson 12-7 • Parallel Lines and Transversals 757

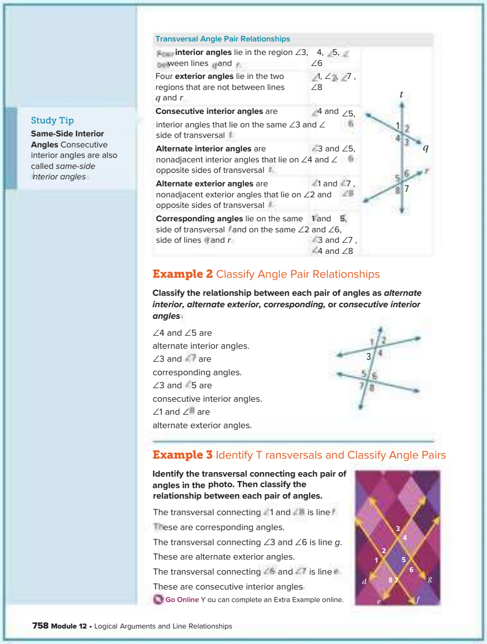
Interactive Presentation



Learn

TAP

Students tap to see line and angle definitions.



Study Tip
Same-Side Interior Angles Consecutive interior angles are also called *same-side interior angles*.

Transversal Angle Pair Relationships	
Four **interior angles** lie in the region between lines q and r.	∠3, ∠4, ∠5, ∠6
Four **exterior angles** lie in the two regions that are not between lines *q* and *r*.	∠1, ∠2, ∠7, ∠8
Consecutive interior angles are interior angles that lie on the same side of transversal t.	∠4 and ∠5, ∠3 and ∠6
Alternate interior angles are nonadjacent interior angles that lie on opposite sides of transversal t.	∠3 and ∠5, ∠4 and ∠6
Alternate exterior angles are nonadjacent exterior angles that lie on opposite sides of transversal t.	∠1 and ∠7, ∠2 and ∠8
Corresponding angles lie on the same side of transversal t and on the same side of lines q and *r*.	∠1 and ∠5, ∠2 and ∠6, ∠3 and ∠7, ∠4 and ∠8

Example 2 Classify Angle Pair Relationships

Classify the relationship between each pair of angles as *alternate interior, alternate exterior, corresponding, or consecutive interior angles*.

∠4 and ∠5 are alternate interior angles.
∠3 and ∠7 are corresponding angles.
∠3 and ∠5 are consecutive interior angles.
∠1 and ∠8 are alternate exterior angles.

Example 3 Identify T ransversals and Classify Angle Pairs

Identify the transversal connecting each pair of angles in the photo. Then classify the relationship between each pair of angles.

The transversal connecting ∠1 and ∠8 is line f. These are corresponding angles.
The transversal connecting ∠3 and ∠6 is line *g*. These are alternate exterior angles.
The transversal connecting ∠6 and ∠7 is line e. These are consecutive interior angles.

Go Online Y ou can complete an Extra Example online.

758 Module 12 • Logical Arguments and Line Relationships

Example 2 Classify Angle Pair Relationships

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to solve the problem in this example, students will need to analyze the mathematical relationships in the problem to draw a conclusion.

Questions for Mathematical Discourse

AL What does it mean for an angle to be an interior angle? Sample answer: The angle lies in the region bounded by the two lines that are cut by the transversal.

OL Are angles 4 and 5 interior or exterior angles? Are they on the same or alternate sides of the transversal? interior; alternate

BL Name a pair of alternate interior angles. angles 4 and 5 or angles 3 and 6

Common Error

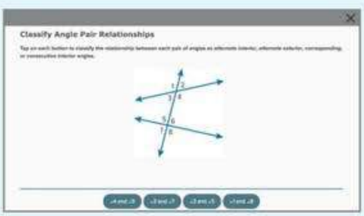
Students tend to confuse the angle pair relationships. Return to this example as needed to reinforce the correct definitions.

DIFFERENTIATE

Language Development Activity AL BL ELL

Kinesthetic Learners Use masking tape to mark two parallel lines and a transversal on the floor. Have pairs of students stand in angles that are congruent or supplementary, and have them explain whether their angles are alternate interior, alternate exterior, corresponding, or consecutive interior angles.

Interactive Presentation



Example 2

TAP

Students tap to see angle pair relationships.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Identify T ransversals and Classify Angle Pairs

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to solve the problem in this example, students will need to analyze the mathematical relationships in the problem to draw a conclusion.

Questions for Mathematical Discourse

AL What line connects the vertices of angles 1 and 8? line f

OL Are angles 6 and 7 on the same side or different sides of the transversal? What angle pairs have that relationship? same; consecutive interior angles and corresponding angles

BL Name a pair of corresponding angles with line *d* as a transversal connecting them. angles 2 and 3

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Angles and Parallel Lines

Objective

Students find values by applying theorems about parallel lines and transversals.

Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in this Learn.

Common Misconception

Students may assume that special angle pairs like corresponding angles are always congruent, but they only are if the two lines cut by a transversal are parallel.

Essential Question Follow-Up

Students learn theorems about parallel lines that are cut by transversals.

Ask:

Why is it important to understand and use theorems about parallel lines? Sample answer: These theorems are very useful in writing logical arguments about geometry.

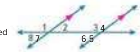
Explore Relationships Between Angles and Parallel Lines

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How do parallel lines affect the relationships between special angle pairs?

Learn Angles and Parallel Lines

If two lines are parallel and cut by a transversal, then there are special relationships in the angle pairs formed by the lines.



Theorem 12.14: Corresponding Angles Theorem

If two parallel lines are cut by a transversal, then each pair of corresponding angles is congruent. ∠1 ≅ ∠3, ∠2 ≅ ∠4, ∠5 ≅ ∠7, ∠6 ≅ ∠8

Theorem 12.15: Alternate Interior Angles Theorem

If two parallel lines are cut by a transversal, then each pair of alternate interior angles is congruent. ∠2 ≅ ∠6, ∠3 ≅ ∠7

Theorem 12.16: Consecutive Interior Angles Theorem

If two parallel lines are cut by a transversal, then each pair of consecutive interior angles is supplementary. ∠2 and ∠3, ∠6 and ∠7

Theorem 12.17: Alternate Exterior Angles Theorem

If two parallel lines are cut by a transversal, then each pair of alternate exterior angles is congruent. ∠5 ≅ ∠1, ∠4 ≅ ∠8

You will prove Theorems 12.16 and 12.17 in Exercises 47 and 48.

A special relationship also exists when the transversal of two parallel lines is a perpendicular line.

Theorem 12.18: Perpendicular Transversal Theorem

In a plane, if a line is perpendicular to one of two parallel lines, then it is perpendicular to the other.

You will prove Theorem 12.18 in Exercise 48.

Study Tip

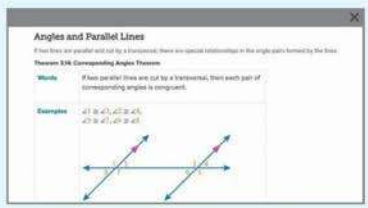
Angle Relationships Theorems 12.15–12.17 generalize the relationships between specific pairs of angles. If you get confused about the relationships, you can verify them using only corresponding angles, vertical angles, and linear pairs.

Go Online Proofs of Theorems 12.14 and 12.15 are available.

Go Online You can complete an Extra Example online.

Lesson 12-7 • Parallel Lines and Transversals 759

Interactive Presentation



Learn

EXPAND

Students tap to see a proof of the Alternate Interior Angles Theorem.

Go Online An alternate method is available for this example.

Example 4 Use Theorems About Parallel Lines

RAILROADS Crossties *j* and *k* are parallel. Both crossties are intersected by crosstie *h*. If $m\angle 1 = 42°$, find $m\angle 7$.

$\angle 7 \cong \angle 1$ Alternate Exterior Angles Theorem

$m\angle 7 = m\angle 1$ Definition of congruent angles

$m\angle 7 = 42°$ Substitution

The measure of ∠7 is 42°.

Check

COMMUNITY PLANNING Dennis Avenue and State Road are parallel streets that intersect Newport Lane along the south side of Oak Creek Park. If $m\angle 3 = 62°$, find $m\angle 4$.

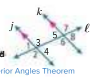
___?___ 62°

Study Tip

Precision Theorems 12.14–12.17 only apply to parallel lines cut by a transversal. You should assume that lines are parallel only if the information is given or the lines are marked with parallel arrows.

Example 5 Find Values of Variables

Use the figure to find the value of the indicated variable. Justify your reasoning.



a. If $m\angle 3 = (4x + 7)°$ and $m\angle 6 = (5x - 13)°$, find *x*.

$\angle 3 \cong \angle 6$ Alternate Interior Angles Theorem

$m\angle 3 = m\angle 6$ Definition of congruent angles

$4x + 7 = 5x - 13$ Substitution

$x = 20$ Simplify.

b. Find *y* if $m\angle 8 = 68°$ and $m\angle 3 = (3y - 2)°$.

$\angle 5 \cong \angle 8$ Vertical Angles Theorem

$m\angle 5 = m\angle 8$ Definition of congruent angles

$m\angle 5 = 68°$ Substitution

Because lines *j* and *k* are parallel, ∠5 and ∠3 are supplementary by the Consecutive Interior Angles Theorem.

$m\angle 3 + m\angle 5 = 180°$ Definition of supplementary angles

$3y - 2 + 68 = 180$ Substitution

$3y + 66 = 180$ Simplify.

$y = 38$ Simplify.

Go Online You can complete an Extra Example online.

sundown001/iStockphoto.com

Interactive Presentation

Example 4

EXPAND

Students tap to see an alternate method.

TYPE

Students type to complete the solution.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 4 Use Theorems About Parallel Lines

Teaching the Mathematical Practices

1 Understand the Approaches of Others Work with students to look at the Alternate Method. Ask students to compare and contrast the original method and the alternate method.

Questions for Mathematical Discourse

AL What is one pair of alternate exterior angles created by transversal *k*? Sample answer: ∠12 and ∠6

OL If line *h* is perpendicular to lines *i* and *k*, what is true about angles 1–8? All angles are right angles.

BL If $m\angle 1 = 83°$, what is $m\angle 5$? $m\angle 5 = 83°$

Example 5 Find Values of Variables

Teaching the Mathematical Practices

2 Create Representations Guide students to write an equation that models the situation in this example. Then use the equation to solve the problem.

Questions for Mathematical Discourse

AL Name the relationship between ∠3 and ∠5; ∠5 and ∠6. consecutive interior; linear pair

OL In part **b**, what angle relates to both ∠8 and ∠3? Possible answers: ∠1, ∠4, ∠5, ∠6, ∠7

BL In part **b**, what alternative path could be taken to solve the problem? Sample answer: $\angle 8 \cong \angle 4$, so $m\angle 4 = 68°$; ∠3 and ∠4 are supplementary, so their measures total 180°; so, $(3y - 2) + 68 = 180$; $y = 38$.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–31
2	exercises that use a variety of skills from this lesson	32–42
2	exercises that extend concepts learned in this lesson to new contexts	43–48
3	exercises that emphasize higher-order and critical-thinking skills	49–55

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:
- Practice, Exercises 1–47 odd, 49–55
- Extension: Parallelism in Space
- ALEKS Parallel Lines and Transversals

IF students score 66%–89% on the Checks, **OL**
THEN assign:
- Practice, Exercises 1–55 odd
- Remediation, Review Resources: Angle Relationships and Parallel Lines
- Personal Tutors
- Extra Examples 1–5
- ALEKS Parallel Lines

IF students score 65% or less on the Checks, **AL**
THEN assign:
- Practice, Exercises 1–31 odd
- Remediation, Review Resources: Angle Relationships and Parallel Lines
- *Quick Review Math Handbook*: Parallel Lines and Transversals
- ALEKS Parallel Lines

Answers

29. $x = 28$, $y = 47$; Use the supplementary angles to find x. Then use alternate exterior angles to find y.

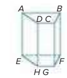
30. $x = 10$, $y = 15$; Use alternate interior angles to find x. Then use supplementary angles to find y.

Practice

Go Online Y ou can complete your homework online.

Example 1

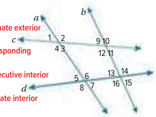
Identify each of the following using the figure shown. Assume lines and planes that appear to be parallel or perpendicular are parallel or perpendicular, respectively.



1. three segments parallel to $\overline{AE}$ $\overline{BF}$, $\overline{CG}$, and $\overline{DH}$
2. a segment skew to $\overline{AB}$ Sample answer: $\overline{EH}$
3. a pair of parallel planes *ABCD* and *EFGH* or *ABFE* and *CDHG*
4. a segment parallel to $\overline{AD}$ $\overline{EH}$
5. three segments parallel to $\overline{HG}$ $\overline{EF}$, $\overline{AB}$,and $\overline{DC}$
6. five segments skew to $\overline{BC}$ $\overline{AE}$, $\overline{EH}$, $\overline{HG}$,$\overline{EF}$, and $\overline{DH}$
7. How could you characterize the relationship between faces *ABCD* and *DCGH*? Explain. Sample answer: *ABCD* and *DCGH* could be characterized as perpendiculars, because *DCGH* contains segment $\overline{CG}$, which is perpendicular to *ABCD*.

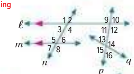
Examples 2 and 3

Identify the transversal connecting each pair of angles. Then classify the relationship between each pair of angles as *alternate interior*, *alternate exterior*, *corresponding*, or *consecutive interior* angles.

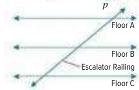


8. ∠4 and ∠5 line *a*; consecutive interior
9. ∠5 and ∠15 line *d*; alternate exterior
10. ∠12 and ∠14 line *b*; alternate interior
11. ∠7 and ∠15 line *d*; corresponding
12. ∠2 and ∠12 line *c*; alternate interior
13. ∠3 and ∠6 line *a*; consecutive interior
14. ∠1 and ∠9 line *c*; corresponding
15. ∠3 and ∠9 line *c*; alternate interior
16. ∠10 and ∠16 line *b*; alternate exterior
17. ∠5 and ∠13 line *d*; corresponding

For Exercises 18 and 19, use the figure.



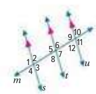
18. What type of angles are ∠3 and ∠10? alternate exterior angles
19. State the transversal that connects ∠11 and ∠13. line *p*
20. ESCALATORS An escalator at a shopping mall runs up several levels. The escalator railing can be modeled by a straight line running past horizontal lines that represent the floors. Describe the relationships of these lines. The lines representing floors A, B, and C are parallel, and the escalator railing is a transversal intersecting each of the three lines.



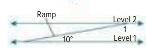
Lesson 12-7 • Parallel Lines and Transversals 761

Example 4

In the figure, $m\angle 7 = 100°$. Find the measure of each angle.



21. ∠9 100°
22. ∠6 80°
23. ∠8 80°
24. ∠2 80°
25. ∠5 100°
26. ∠11 100°
27. RAMPS A parking garage ramp rises to connect two horizontal levels of a parking lot. The ramp makes a 10° angle with the horizontal. What is the measure of angle 1 in the figure? 170°

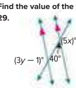


28. CITY ENGINEERING Seventh Avenue runs perpendicular to 1st and 2nd Streets, which are parallel. However, Maple Avenue makes a 115° angle with 2nd Street. What is the measure of angle 1? 65°

Example 5

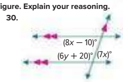
Find the value of the variables in each figure. Explain your reasoning.

29.



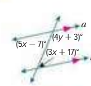
See margin.

30.



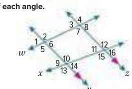
See margin.

31. Find the value of the variables in the figure. $x = 12, y = 31$



Mixed Exercises

In the figure, $m\angle 3 = 75$ and $m\angle 10 = 105°$. Find the measure of each angle.



32. ∠2 105°
33. ∠5 105°
34. ∠7 105°
35. ∠15 105°
36. ∠14 75°
37. ∠9 75°

762 Module 12 • Logical Arguments and Line Relationships

USE A MODEL Lines a and b are parallel and are cut by transversal t to form interior angles ∠7, ∠8, ∠9, and ∠10. ∠7 and ∠8 are consecutive interior angles, and $m∠7 = 94°$. ∠8 and ∠10 are alternate interior angles. Find the measure of each angle.

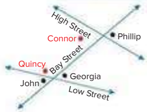
38. ∠10 **86°** **39.** ∠9 **94°** **40.** ∠8 **86°**

41. CARPENTRY A carpenter is building a podium. The side panel of the podium is cut from a rectangular piece of wood. The rectangle must be sawed along the dashed line in the figure. What is the measure of ∠1? Explain your reasoning. See margin.

42. MAPPING Copy the figure.

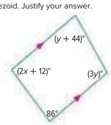
a. Connor lives at the angle that forms an alternate interior angle with Georgia's residence. Label the location of Connor's home on the map.

b. Quincy lives at the angle that forms a consecutive interior angle with Connor's residence. Label the location of Quincy's home on the map.



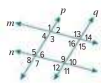
43. USE A SOURCE Research the flag for the Solomon Islands. Sketch the flag. Label angles formed by the yellow stripe, or transversal. Describe the relationship between the angles you labeled on the flag. See margin.

44. PRECISION Find the values of x and y in the trapezoid. Justify your answer.



See margin.

45. PROOF In the figure, lines m and n are parallel and lines p and q are parallel. Write a paragraph proof to prove that if $m∠1 - m∠4 = 25°$, then $m∠9 - m∠12 = 25°$.

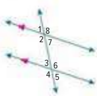


See margin.

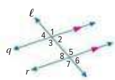
46. In the figure, $m∠4 = 118°$. Find each angle measure. Justify each step.

a. $m∠8$ ∠4 ≅ ∠8 by Alt. Ext. ∠s Thm. $m∠4 = m∠8$ by the def. of cong. ∠s, so $m∠8 = 118°$ by substitution.

b. $m∠7$ See margin.



47. PROOF Write a paragraph proof of the Alternate Exterior Angles Theorem. Given: $q \parallel r$; Prove: ∠1 ≅ ∠7. See margin.

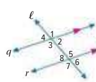


48. PROOF **Write a two-column proof to prove each theorem.**

a. Consecutive Interior Angles Theorem

Given: $q \parallel r$

Prove: ∠2 and ∠5 are supplementary



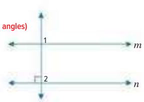
Statements (Reasons)

1. $q \parallel r$ (Given)
2. ∠1 ≅ ∠5 (Corresponding Angles Thm)
3. ∠1 and ∠2 are a linear pair. (Def. of linear pair)
4. ∠1 and ∠2 are supplementary. (Supplement Thm)
5. $m∠1 + m∠2 = 180°$ (Def. of supp. angles)
6. $m∠5 + m∠2 = 180°$ (Substitution)
7. ∠2 and ∠5 are supplementary. (Def. of supp. angles)

b. Perpendicular Transversal Theorem.

Given: $m \parallel n$; $p \perp n$

Prove: $p \perp m$



Statements (Reasons)

1. $m \parallel n$; $p \perp n$ (Given)
2. ∠2 is a right angle. (Def. of perpendicular)
3. $m∠2 = 90°$ (Def. of right angle)
4. ∠1 ≅ ∠2 (Corresponding Angles Theorem)
5. $m∠1 = m∠2$ (Def. of congruent angles)
6. $m∠1 = 90°$ (Substitution)
7. ∠1 is a right angle. (Def. of right angle)
8. $p \perp m$ (Def. of perpendicular)

Higher-Order Thinking Skills

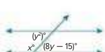
49. CREATE Plane P contains lines a and b. Line c intersects plane P at point J. Lines a and b are parallel lines, lines a and c are skew, and lines b and c are not skew. Draw a figure based upon this description. See margin.

ANALYZE **Plane X and plane Y are parallel and plane Z intersects plane X. Line $\overleftrightarrow{AB}$ is in plane X, line $\overleftrightarrow{CD}$ is in plane Y, and line $\overleftrightarrow{EF}$ is in plane Z. Determine whether each statement is *always, sometimes,* or *never* true. Justify your argument.**

50. $\overleftrightarrow{AB}$ is skew to $\overleftrightarrow{CD}$. See margin. **51.** $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{EF}$. See margin.

52. WRITE Compare and contrast the Alternate Interior Angles Theorem and the Consecutive Interior Angles Theorem. See margin.

53. PERSEVERE Find the values of x and y.
$x = 171$ or $x = 155$; $y = 3$ or $y = 5$



54. ANALYZE Determine the minimum number of angle measures you would have to know to find the measures of all the angles formed by two parallel lines cut by a transversal. Justify your argument. See margin.

55. WRITE Can a pair of planes be described as skew? Explain. See margin.

Answers

41. 64°; Sample answer: Opposite sides of a rectangle are parallel. So, the top and bottom lines on the side panel are parallel and cut by a transversal, which is the dashed line. Therefore, ∠1 and the 116°-angle are consecutive interior angles, so their sum is 180°. $m∠1 + 116° = 180°$, so $m∠1 = 64°$.

43.

Sample answer: ∠1 and ∠4 are alternate interior angles. ∠2 and ∠3 are alternate interior angles. ∠1 and ∠2 are complementary angles, and ∠3 and ∠4 are complementary angles.

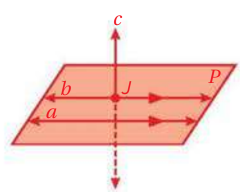
44. $(2x + 12)° + 86° = 180°$ (Consecutive Interior Angles Theorem and definition of supplementary angles); $2x + 98° = 180°$; $x = 41$; $(y + 44)° + (3y)° = 180°$ (Consecutive Interior Angles Theorem and the definition of supplementary angles); $4y + 44° = 180°$; $y = 34$.

45. By the Corresponding Angles Postulate, ∠1 ≅ ∠13 and ∠13 ≅ ∠9. By the Transitive Property, ∠1 ≅ ∠9. So, $m∠1 = m∠9$. By the Corresponding Angles Postulate, ∠4 ≅ ∠8 and ∠8 ≅ ∠12. By the Transitive Property, ∠4 ≅ ∠12. So, $m∠4 = m∠12$. It is given that $m∠1 - m∠4 = 25°$. By the Substitution Property, $m∠9 - m∠12 = 25°$.

46b. Sample answer: ∠4 ≅ ∠6 by Vert. ∠s Thm., so $m∠6 = 118°$ (def. of cong. ∠s). ∠6 and ∠7 are supplementary angles by Cons. Int. ∠s Thm., so $m∠6 + m∠7 = 180°$. By substitution, $118° + m∠7 = 180°$, and by subtraction, $m∠7 = 62°$.

47. By the Vertical Angles Theorem, ∠7 ≅ ∠5. By the Corresponding Angles Theorem ∠5 ≅ ∠1. By the Transitive Property, ∠1 ≅ ∠7.

49.



50. Sometimes; sample answer: $\overleftrightarrow{AB}$ is either skew or parallel to $\overleftrightarrow{CD}$ because the lines will never intersect and are not parallel.

51. Sometimes; sample answer: $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{EF}$ depending on where the planes intersect.

52. Sample answer: In both theorems, a pair of angles is formed when two parallel lines are cut by a transversal. However, in the Alternate Interior Angles Theorem, each pair of alternate interior angles that are formed is congruent, whereas in the Consecutive Interior Angles Theorem, each pair of angles that are formed is supplementary.

54. One; sample answer: When the measures of one angle is known, the rest of the angles are congruent or supplementary to the given angle.

55. No; sample answer: From the definition of skew lines, the lines must not intersect and cannot be coplanar. Different planes cannot be coplanar, but they are always parallel or intersecting. Therefore, planes cannot be skew.

Lesson 12-8

Slope and Equations of Lines

G.GPE.5

LESSON GOAL

Students classify lines as parallel, perpendicular, or neither by using the slope criteria.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Slope Criteria for Parallel and Perpendicular Lines
- Determine Line Relationships When Given Points
- Determine Line Relationships When Given Graphs

Explore: Equations of Lines

Develop:

Equations of Lines
- Determine Line Relationships When Given Equations
- Use Slope to Graph a Line
- Write Equations of Parallel and Perpendicular Lines

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Parallel Lines and Transversals	●	●		●
Extension: Polygons on a Coordinate Plane		●	●	●

Language Development Handbook

Assign page 78 of the *Language Development Handbook* to help your students build mathematical language related to using slope criteria to classify lines as parallel or perpendicular.

ELL You can use the tips and suggestions on page T78 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Geometry

Standards for Mathematical Content:

G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point.).

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
3 Construct viable arguments and critique the reasoning of others.
8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students analyzed linear equations in slope-intercept form to determine if two lines are parallel.
8.EE.7a

Now
Students classify lines as parallel, perpendicular, or neither by using the slope criteria.
G.GPE.5

Next
Students will identify and use parallel lines using angle relationships.
G.CO.9

Rigor

The Three Pillars of Rigor

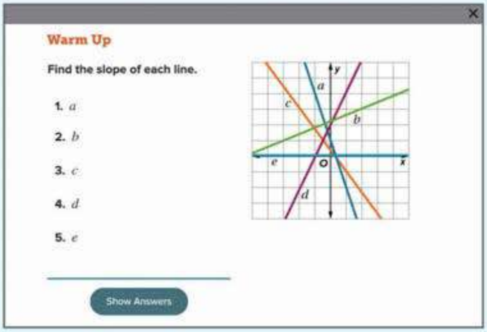
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of parallel line relationships to the coordinate plane. They build fluency and apply their understanding by solving real-world problems related to parallel and perpendicular lines.

Mathematical Background

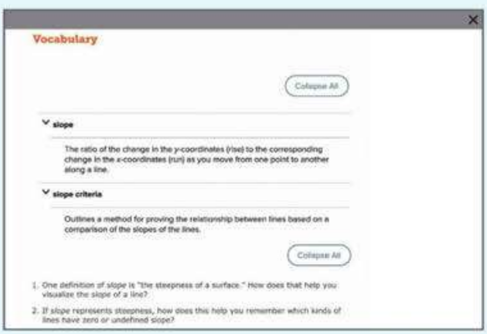
The slope of a line is the ratio of its vertical rise to its horizontal run. The slope of a vertical line is undefined, and the slope of a horizontal line is zero. Two nonvertical lines have the same slope if and only if they are parallel. Two nonverticall lines are perpendicular if and only if the product of their slopes is −1. This means that you can use slope to identify parallel and perpendicular lines. You can also use slope to graph parallel and perpendicular lines.

Interactive Presentation



Warm Up

Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- classifying lines as parallel, perpendicular, or neither

Answers:

1. -3
2. $\frac{2}{5}$
3. $-\frac{4}{3}$
4. 2
5. 0

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of slope.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the (standards).

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Equations of Lines

Objective

Students use dynamic geometry software to make conjectures about whether lines are parallel or perpendicular.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this Explore, students will need to use dynamic geometry software. Work with students to explore and deepen their understanding of parallel and perpendicular lines.

Ideas for Use

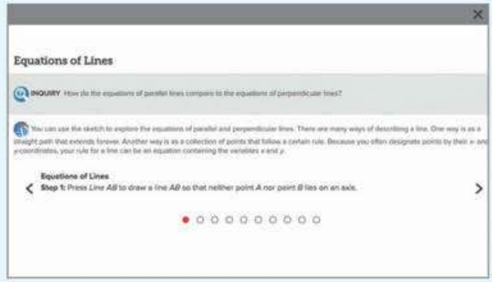
Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

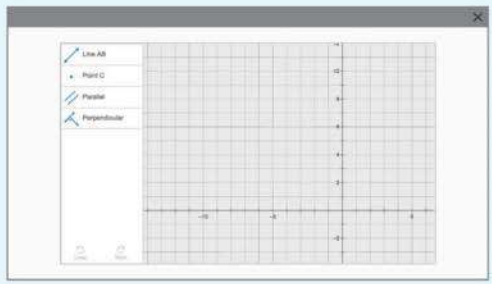
Summary of the Activity

Students will construct parallel and perpendicular lines using dynamic geometry software. Students will observe relationships between slopes of parallel and perpendicular lines. Then students will write conjectures about the relationships of the slopes of parallel and perpendicular lines. Then, students will answer the Inquiry Question.

(continued on the next page)



Explore



Explore

WEB SKETCHPAD

Students use the sketch to explore equations of lines.

Explore

TYPE

Students respond to the Inquiry Question and view a sample answer.

Explore Equations of Lines (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- If the slope of the original line is positive, what is the slope of a line parallel to the original line? Of a line perpendicular? Sample answer: The parallel line should have a positive slope because they should be going in the same direction. The perpendicular line would have a negative slope because the slopes are negative reciprocals.
- Given a line $y = -3x + 2$, what is the slope of a line perpendicular? Sample answer: Because the slopes of perpendicular lines are negative reciprocals, the slope of a perpendicular line would be $\frac{1}{3}$.

Inquiry

How do the equations of parallel lines compare to the equations of perpendicular lines? Sample answer: The slopes of parallel lines are the same, and the slopes of perpendicular lines are negative reciprocals.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Slope Criteria for Parallel and Perpendicular Lines

Objective

Students classify lines as parallel, perpendicular, or neither by comparing the slopes of the lines.

MP Teaching the Mathematical Practices

3 Analyze Cases The Concept Check guides students to examine the cases of vertical and horizontal lines being parallel or perpendicular. Encourage students to familiarize themselves with all of the cases.

Important to Know

Students may be curious why slopes of nonvertical perpendicular lines are negative reciprocals. To explain why, sketch a graph of two such lines. Note that one line must be increasing, or going up, from left to right, and that the other must be decreasing, so the signs of the slopes must be different. Note that the rise and run of one line are interchanged in the slope of the other line and that the slope have opposite signs.

Example 1 Determine Line Relationships When Given Points

MP Teaching the Mathematical Practices

8 Attend to Details Mathematically proficient students continually ask themselves, "Does this make sense?" Point out that in this example, students will evaluate the reasonableness of their answer.

Questions for Mathematical Discourse

AL What do you know about the slopes of parallel lines? They are equal. What do you know about the slopes of perpendicular lines? The product of the slopes is equal to -1.

OL Suppose the slope of a line is $\frac{5}{4}$. What is the slope of a parallel line? $\frac{5}{4}$ What is the slope of a perpendicular line? $-\frac{4}{5}$

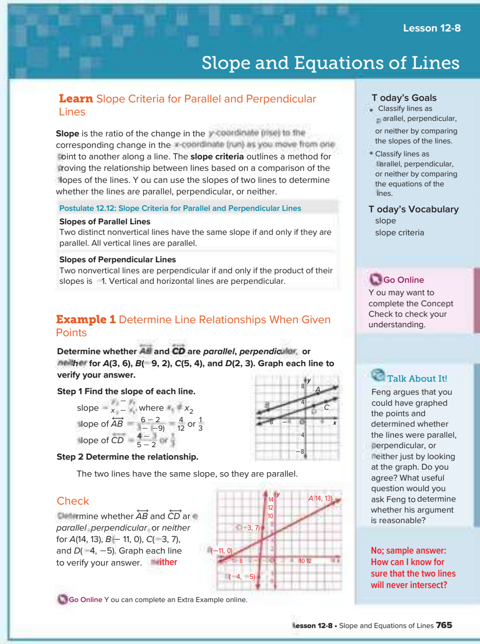
BL Choose a point F such that $\overleftrightarrow{CF}$ is perpendicular to $\overleftrightarrow{AB}$. Sample answer: (6, 1)

Common Error

Students may incorrectly compute slopes. Remind them that slope $= \frac{\text{rise}}{\text{run}}$, so the rise, or change in y values, is divided by the run, or change in x values.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 12-8

Slope and Equations of Lines

Learn Slope Criteria for Parallel and Perpendicular Lines

Slope is the ratio of the change in the y-coordinate (rise) to the corresponding change in the x-coordinate (run) as you move from one point to another along a line. The **slope criteria** outlines a method for proving the relationship between lines based on a comparison of the slopes of the lines. You can use the slopes of two lines to determine whether the lines are parallel, perpendicular, or neither.

Postulate 12.12: Slope Criteria for Parallel and Perpendicular Lines

Slopes of Parallel Lines
Two distinct nonvertical lines have the same slope if and only if they are parallel. All vertical lines are parallel.

Slopes of Perpendicular Lines
Two nonvertical lines are perpendicular if and only if the product of their slopes is −1. Vertical and horizontal lines are perpendicular.

Example 1 Determine Line Relationships When Given Points

Determine whether $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are *parallel*, *perpendicular*, or *neither* for $A(3, 6)$, $B(-9, 2)$, $C(5, 4)$, and $D(2, 3)$. Graph each line to verify your answer.

Step 1 Find the slope of each line.

slope $= \frac{y_2 - y_1}{x_2 - x_1}$ where $x_1 \neq x_2$

slope of $\overleftrightarrow{AB} = \frac{6-2}{3-(-9)} = \frac{4}{12}$ or $\frac{1}{3}$

slope of $\overleftrightarrow{CD} = \frac{4-3}{5-2}$ or $\frac{1}{3}$

Step 2 Determine the relationship.

The two lines have the same slope, so they are parallel.

Check

Determine whether $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are *parallel*, *perpendicular*, or *neither* for $A(14, 13)$, $B(-11, 0)$, $C(-3, 7)$, and $D(-4, -5)$. Graph each line to verify your answer. neither

Go Online You can complete an Extra Example online.

Today's Goals
- Classify lines as parallel, perpendicular, or neither by comparing the slopes of the lines.
- Classify lines as parallel, perpendicular, or neither by comparing the equations of the lines.

Today's Vocabulary
slope
slope criteria

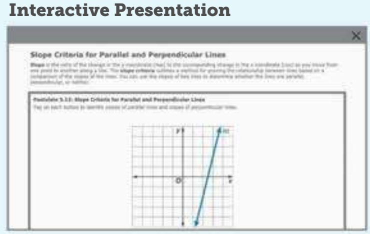
Go Online
You may want to complete the Concept Check to check your understanding.

Talk About It!
Feng argues that you could have graphed the points and determined whether the lines were parallel, perpendicular, or neither just by looking at the graph. Do you agree? What useful question would you ask Feng to determine whether his argument is reasonable?

No; sample answer: How can I know for sure that the two lines will never intersect?

Lesson 12-8 • Slope and Equations of Lines 765

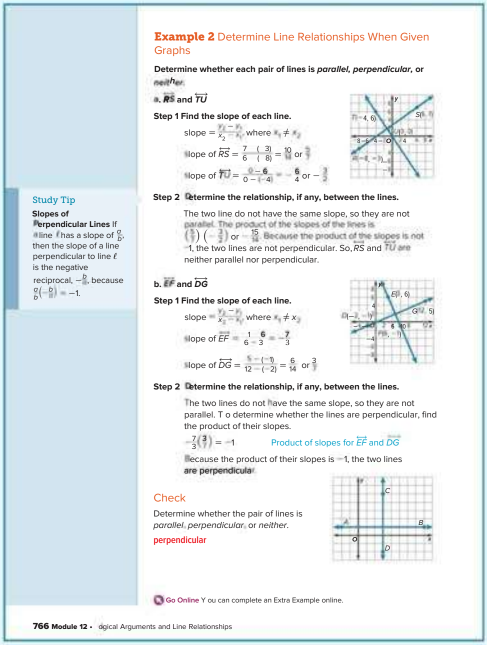
Interactive Presentation



Learn

TAP

Students tap on each button to see slope information about line relationships.



Example 2 Determine Line Relationships When Given Graphs

Determine whether each pair of lines is *parallel*, *perpendicular*, or *neither*.

a. $\overleftrightarrow{RS}$ and $\overleftrightarrow{TU}$

Step 1 Find the slope of each line.

slope $= \frac{y_2 - y_1}{x_2 - x_1}$ where $x_1 \neq x_2$

slope of $\overleftrightarrow{RS} = \frac{7-(-3)}{6-(-8)} = \frac{10}{14}$ or $\frac{5}{7}$

slope of $\overleftrightarrow{TU} = \frac{0-6}{0-(-4)} = -\frac{6}{4}$ or $-\frac{3}{2}$

Step 2 Determine the relationship, if any, between the lines.

The two lines do not have the same slope, so they are not parallel. The product of the slopes of the lines is $\left(\frac{5}{7}\right)\left(-\frac{3}{2}\right)$ or $-\frac{15}{14}$. Because the product of the slopes is not -1, the two lines are not perpendicular. So, $\overleftrightarrow{RS}$ and $\overleftrightarrow{TU}$ are neither parallel nor perpendicular.

b. $\overleftrightarrow{EF}$ and $\overleftrightarrow{DG}$

Step 1 Find the slope of each line.

slope $= \frac{y_2 - y_1}{x_2 - x_1}$ where $x_1 \neq x_2$

slope of $\overleftrightarrow{EF} = \frac{-1-6}{6-3} = -\frac{7}{3}$

slope of $\overleftrightarrow{DG} = \frac{5-(-1)}{12-(-2)} = \frac{6}{14}$ or $\frac{3}{7}$

Step 2 Determine the relationship, if any, between the lines.

The two lines do not have the same slope, so they are not parallel. To determine whether the lines are perpendicular, find the product of their slopes.

$-\frac{7}{3}\left(\frac{3}{7}\right) = -1$ Product of slopes for $\overleftrightarrow{EF}$ and $\overleftrightarrow{DG}$

Because the product of their slopes is -1, the two lines are perpendicular.

Study Tip
Slopes of Perpendicular Lines If a line ℓ has a slope of $\frac{a}{b}$, then the slope of a line perpendicular to line ℓ is the negative reciprocal, $-\frac{b}{a}$, because $\frac{a}{b}\left(-\frac{b}{a}\right) = -1$.

Check

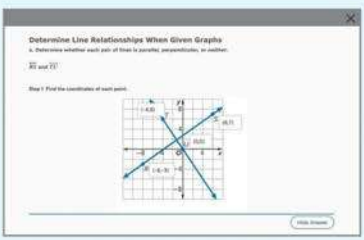
Determine whether the pair of lines is *parallel*, *perpendicular*, or *neither*.

perpendicular

Go Online You can complete an Extra Example online.

766 Module 12 • Logical Arguments and Line Relationships

Interactive Presentation



Example 2

TYPE

Students enter coordinates to determine information about line relationships.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Determine Line Relationships When Given Graphs

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL What is another method for finding the slope of $\overleftrightarrow{EF}$? Sample answer: Start at point (3, 6) and count down, then over, to point (6, −1). The slope of the line is equal to $-\frac{7}{3}$.

OL The line containing (−2, 1) and (x, −2) has a slope of $\frac{3}{7}$. What is the value of x? $x = -9$

BL Choose a point B so that $\overleftrightarrow{BF}$ is parallel to $\overleftrightarrow{DG}$? Sample answer: (−1, −4)

Common Error

Students may confuse x-coordinates and y-coordinates. Encourage them to slow down and check their work in finding coordinates on a graph.

Learn Equations of Lines

Objective

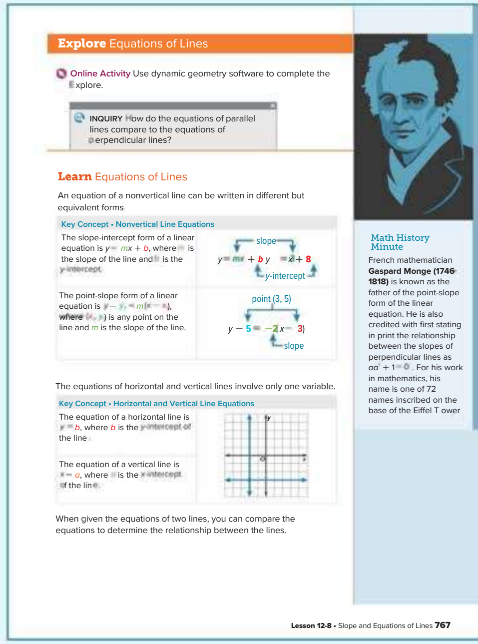
Students classify lines as parallel, perpendicular, or neither by comparing the equations of the lines.

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the equations of a line used in this Learn.

Common Misconception

Students often neglect horizontal and vertical lines when they think about or discuss equations of lines and their relationships. Make sure you remind them about these possibilities when you are discussing these topics.



Explore Equations of Lines

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How do the equations of parallel lines compare to the equations of perpendicular lines?

Learn Equations of Lines

An equation of a nonvertical line can be written in different but equivalent forms.

Key Concept • Nonvertical Line Equations

The slope-intercept form of a linear equation is $y = mx + b$, where m is the slope of the line and b is the y-intercept.	slope; $y = mx + b$; $y = x + 8$; y-intercept
The point-slope form of a linear equation is $y - y_1 = m(x - x_1)$, where (x_1, y_1) is any point on the line and m is the slope of the line.	point (3, 5); $y - 5 = -2(x - 3)$; slope

The equations of horizontal and vertical lines involve only one variable.

Key Concept • Horizontal and Vertical Line Equations

The equation of a horizontal line is $y = b$, where b is the y-intercept of the line.	
The equation of a vertical line is $x = a$, where a is the x-intercept of the line.	

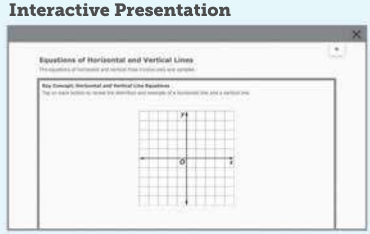
When given the equations of two lines, you can compare the equations to determine the relationship between the lines.

Math History Minute

French mathematician **Gaspard Monge (1746–1818)** is known as the father of the point-slope form of the linear equation. He is also credited with first stating in print the relationship between the slopes of perpendicular lines as $aa' + 1 = 0$. For his work in mathematics, his name is one of 72 names inscribed on the base of the Eiffel Tower.

Lesson 12-8 • Slope and Equations of Lines 767

Interactive Presentation



Learn

TAP

Tap to reveal definitions and examples.

Example 3 Determine Line Relationships When Given Equations

Determine whether each pair of lines is *parallel, perpendicular*, or *neither*.

a. $y = 3x - 2$; $y - 0 = -\frac{1}{3}(x - 2)$

slope-intercept form: $y = 3x - 2$ point-slope form: $y - 0 = -\frac{1}{3}(x - 2)$

slope

The two lines do not have the same slope, so the lines are not parallel. To determine whether the lines are perpendicular, find the product of the slopes.

$3\left(-\frac{1}{3}\right) = -1$ Product of slopes

Because the product of their slopes is −1, the two lines are perpendicular.

b. $y = 3$; $x = 1$

$y = 3$ horizontal line, slope of 0 $x = 1$ vertical line, undefined slope

Vertical and horizontal lines are always perpendicular.

c. $y - 5 = -\frac{3}{4}(x + 2)$; $y = -\frac{3}{4}x + 2$

point-slope form: $y - 5 = -\frac{3}{4}(x + 2)$ slope-intercept form: $y = -\frac{3}{4}x + 2$

slope

Because the slopes of both lines are $-\frac{3}{4}$, the lines are parallel.

d. $y = 2x + 3$; $y - 1 = \frac{1}{2}(x + 2)$

slope-intercept form: $y = 2x + 3$ point-slope form: $y - 1 = \frac{1}{2}(x + 2)$

slope

The two lines do not have the same slope, so the lines are not parallel. To determine whether the lines are perpendicular, find the product of the slopes.

$2\left(\frac{1}{2}\right) = 1$ Product of slopes

Because the product of the slopes is not −1, the two lines are not perpendicular. So, the two lines are neither parallel nor perpendicular.

e. $x = -2$; $x = 4$

Both lines are vertical with undefined slope. Vertical lines are always parallel.

Watch Out!

Vertical Lines If you calculate the slope of the line $x = 1$ using the slope formula, you get $m = \frac{-1-4}{1-1} = -\frac{5}{0}$ or an undefined slope. You cannot find the product of the slope of $x = 1$ and $y = 3$. However, vertical and horizontal lines are always perpendicular.

Study Tip

Zero and Undefined Slope If the change in *y* values is 0, then the line is horizontal. If the change in *x* values is 0, then the line is vertical.

Go Online You can complete an Extra Example online.

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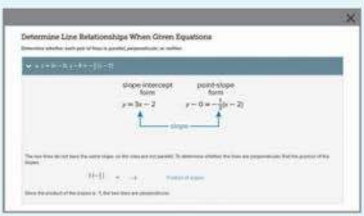
Interactive Presentation



Example 3

TAP

Students tap to reveal a Study Tip.

Example 3 Determine Line Relationships When Given Equations

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the equations of a line used in this example.

Questions for Mathematical Discourse

AL What is the difference between slopes of horizontal and vertical lines? Sample answer: Horizontal lines have a slope of zero while vertical lines have an undefined slope.

OL In part **c**, what is the *y*-intercept of $y = -\frac{3}{4}(x + 2)$? $-\frac{3}{2}$

BL What is the slope of a line that is perpendicular to a vertical line? 0

Common Error

Students may think that they are unable to solve a problem like Example 3 when they cannot find a slope of the line. This usually occurs when the lines are horizontal or vertical, so ask them whether they are sure that the line is nonvertical.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 4 Use Slope to Graph a Line

Teaching the Mathematical Practices

5 Decide When to Use Tools Mathematically proficient students can make sound decisions about when to use mathematical tools such as dynamic geometry software. Help them see why using this tool will help them solve problems and what the limitations are of using the tool.

Questions for Mathematical Discourse

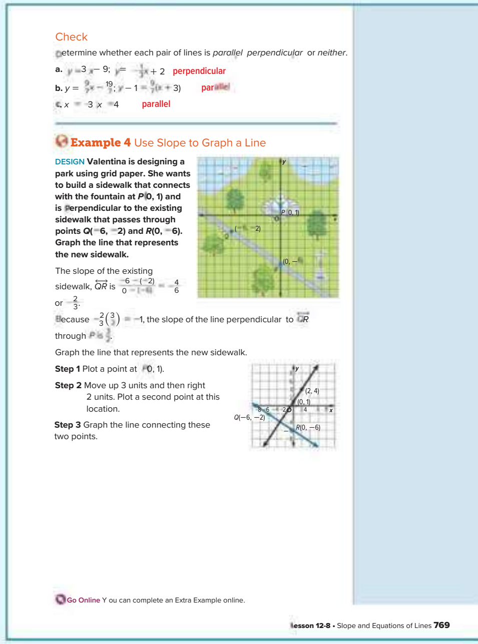
AL Will the slope of $\overleftrightarrow{QR}$ be positive or negative? Explain. Negative; sample answer: The line goes down from left to right.

OL What is the slope of $\overleftrightarrow{QR}$? What is the slope of a line perpendicular to $\overleftrightarrow{QR}$? $-\frac{2}{3}$; $\frac{3}{2}$

BL Write the equation of the perpendicular line in slope-intercept form. $y = \frac{3}{2}x + 1$

Common Error

Students may try to graph perpendicular lines, so that the lines intersect at a point with whole number coordinates. This is not always the case, as can be seen in Example 4.


Check

Determine whether each pair of lines is *parallel*, *perpendicular*, or *neither*.

a. $y = 3x - 9$; $y = -\frac{1}{3}x + 2$ perpendicular

b. $y = \frac{9}{7}x - \frac{19}{7}$; $y - 1 = \frac{9}{7}(x + 3)$ parallel

c. $x = -3$; $x = 4$ parallel

Example 4 Use Slope to Graph a Line

DESIGN **Valentina is designing a park using grid paper. She wants to build a sidewalk that connects with the fountain at $P(0, 1)$ and is perpendicular to the existing sidewalk that passes through points $Q(-6, -2)$ and $R(0, -6)$. Graph the line that represents the new sidewalk.**

The slope of the existing sidewalk, $\overleftrightarrow{QR}$ is $\frac{-6-(-2)}{0-(-6)} = -\frac{4}{6}$ or $-\frac{2}{3}$.

Because $-\frac{2}{3}\left(\frac{3}{2}\right) = -1$, the slope of the line perpendicular to $\overleftrightarrow{QR}$ through P is $\frac{3}{2}$.

Graph the line that represents the new sidewalk.

Step 1 Plot a point at $P(0, 1)$.

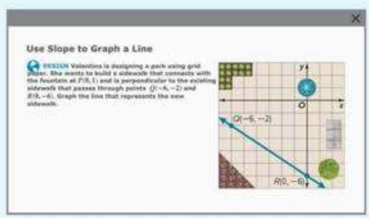
Step 2 Move up 3 units and then right 2 units. Plot a second point at this location.

Step 3 Graph the line connecting these two points.

Go Online You can complete an Extra Example online.

Lesson 12-8 • Slope and Equations of Lines 769

Interactive Presentation



Example 4

WEB SKETCHPAD

Students use the sketch to graph parallel and perpendicular lines.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

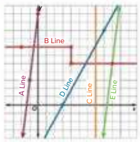
Think About It!
Kennedy suggests that there is another line parallel to $y = -\frac{3}{4}x + 3$ that contains the point $(-3, 6)$. She says that the equation of the line is $y - 6 = -\frac{3}{4}(x + 3)$. Do you agree? Explain your reasoning.

No; sample answer: The equations $y = -\frac{3}{4}x + \frac{15}{4}$ and $y - 6 = -\frac{3}{4}(x + 3)$ are equations for the same line. One equation is presented in slope-intercept form, and the other equation is presented in point-slope form.

Go Online
An alternate method is available for this example.

Check

MAPS Isabella is creating a map of her town's metro lines. She knows that the A Line and the E Line are parallel. On her map, the equation that represents the A Line is $y = 8x + 11$ and the E Line passes through (9, 5). Write the equation in slope-intercept form that represents the E Line. $y = 8x - 67$



Example 5 Write Equations of Parallel and Perpendicular Lines

Write an equation in slope-intercept form for the line parallel to $y = -\frac{3}{4}x + 3$ containing $(-3, 6)$.

The slope of $y = -\frac{3}{4}x + 3$ is $-\frac{3}{4}$, so the slope of the line parallel to it is $-\frac{3}{4}$.

$y = mx + b$	Slope-intercept form
$6 = -\frac{3}{4}(-3) + b$	$m = -\frac{3}{4}$ and $(x, y) = (-3, 6)$
$6 = \frac{9}{4} + b$	Simplify.
$\frac{15}{4} = b$	Subtract $\frac{9}{4}$ from each side.

So, the equation is $y = -\frac{3}{4}x + \frac{15}{4}$.

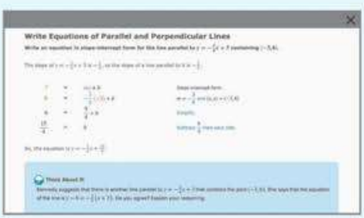
Check

Write an equation in slope-intercept form for the line parallel to $y = \frac{1}{2}x + \frac{5}{2}$ containing $\left(\frac{3}{2}, 1\right)$.

$y = \frac{1}{2}x + \frac{1}{4}$

Go Online Y ou can complete an Extra Example online.

Interactive Presentation



Example 5

TAP

Students tap to view an alternate method.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Write Equations of Parallel and Perpendicular Lines

Teaching the Mathematical Practices

3 Find the Error This example requires students to read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Questions for Mathematical Discourse

AL What is the slope of a line that is perpendicular to the given line? $m = \frac{4}{3}$

OL What equation represents a line that is parallel to the given line? Write the equation in slope-intercept form. Sample answer: $y = -\frac{3}{4}x + 21$

BL Write the equation of the line that is perpendicular to the line $y = -5$ containing $(-6, -3)$. $x = -6$

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–25
2	exercises that use a variety of skills from this lesson	26–35
2	exercises that extend concepts learned in this lesson to new contexts	36–40
3	exercises that emphasize higher-order and critical-thinking skills	41–46

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–39 odd, 41–46
- Extension: Polygons on a Coordinate Plane
- ALEKS Slopes of Lines, Equations of Lines

IF students score 66%–89% on the Checks, **THEN** assign: AL OL

- Practice, Exercises 1–45 odd
- Remediation, Review Resources: Parallel Lines and Transversals
- Personal Tutors
- Extra Examples 1–5
- ALEKS parallel, perpendicular, or oblique

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–25 odd
- Remediation, Review Resources: Parallel Lines and Transversals
- *Quick Review Math Handbook*: Slopes of Lines
- ALEKS parallel, perpendicular, or oblique

Practice

Go Online You can complete your homework online.

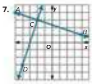
Example 1

Determine whether $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are *parallel, perpendicular*, or *neither*. Graph each line to verify your answer. See margin for graphs.

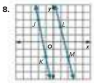
1. $A(1, 5)$, $B(4, 4)$, $C(9, -10)$, $D(-6, -5)$ parallel
2. $A(-6, -9)$, $B(8, 19)$, $C(0, -4)$, $D(2, 0)$ parallel
3. $A(4, 2)$, $B(-3, 1)$, $C(6, 0)$, $D(-10, 8)$ neither
4. $A(8, -2)$, $B(4, -1)$, $C(3, 11)$, $D(-2, -9)$ perpendicular
5. $A(8, 4)$, $B(4, 3)$, $C(4, -9)$, $D(2, -1)$ perpendicular
6. $A(4, -2)$, $B(-2, -8)$, $C(4, 6)$, $D(8, 5)$ neither

Example 2

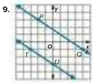
Determine whether each pair of lines is *parallel*, *perpendicular*, or *neither*.

7. 

perpendicular

8. 

neither

9. 

parallel

Example 3

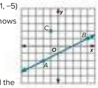
Determine whether each pair of lines is *parallel*, *perpendicular*, or *neither*.

10. $y = 2x + 4$, $y = 2x - 10$ parallel
11. $y = -\frac{1}{2}x - 12$, $y - 3 = 2(x + 2)$ perpendicular
12. $y - 4 = 3(x + 5)$, $y + 3 = -\frac{1}{3}(x + 1)$ perpendicular
13. $y - 3 = 6(x + 2)$, $y + 3 = -\frac{1}{3}(x - 4)$ neither
14. $x = -2$, $y = 10$ perpendicular
15. $y = 5$, $y = -3$ parallel

Example 4

Graph the line that satisfies each condition. 16–19. See margin.

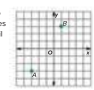
16. passes through $A(2, -5)$, parallel to $\overleftrightarrow{BC}$ with $B(1, 3)$ and $C(4, 5)$
17. passes through $X(1, -4)$, parallel to $\overleftrightarrow{YZ}$ with $Y(5, 2)$ and $Z(-3, -5)$
18. passes through $K(3, 7)$, perpendicular to $\overleftrightarrow{LM}$ with $L(-1, -2)$ and $M(-4, 8)$
19. passes through $D(-5, -6)$, perpendicular to $\overleftrightarrow{FG}$ with $F(-2, -9)$ and $G(1, -5)$
20. SKIING Gavin is working on an animated film about skiing. The figure shows a ski slope, represented by $\overleftrightarrow{AB}$, and one of the chairs on the chair lift, represented by point C.



   a. The chair needs to move along a straight line that is parallel to $\overleftrightarrow{AB}$. What is the equation of this line? $y = \frac{1}{2}x + \frac{5}{2}$

   b. The top of the chair lift occurs at $y = 20$. Explain how Gavin can find the coordinates of the chair when it reaches the top of the chair lift. See margin.

21. REASONING The director of a marching band uses a coordinate plane to design the band's formations. During one formation, a drummer marches from point A to point B and then turns 90° to her right and marches until she reaches the x-axis.



   a. When the drummer marches from point B to the x-axis, what is the equation of the line that she marches along? $y = -\frac{2}{3}x + \frac{11}{3}$

   b. The director wants to know whether the drummer will cross the x-axis at a point where the x-coordinate is greater than or less than 5. Explain how the director can answer this question. When the drummer crosses the x-axis, the y-coordinate will be 0, so solve $0 = -\frac{2}{3}x + \frac{11}{3}$ to find the x-coordinate; solving shows that $x = \frac{11}{2}$ or $5\frac{1}{2}$ so the x-coordinate will be greater than 5.

Example 5

Write an equation in slope-intercept form for each line described.

22. passes through $(-7, -4)$, perpendicular to $y = \frac{1}{2}x + 9$ $y = -2x - 18$
23. passes through $(-1, -10)$, parallel to $y = 7$ $y = -10$
24. passes through $(6, 2)$, parallel to $y = -\frac{2}{3}x + 1$ $y = -\frac{2}{3}x + 6$
25. passes through $(-2, 2)$, perpendicular to $y = -5x - 8$ $y = \frac{1}{5}x + \frac{12}{5}$

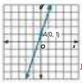
Mixed Exercises

Find the value of x or y that satisfies the given conditions. Then graph the line. 26–29. See Mod. 12 Answers Appendix.

26. The line containing $(4, -2)$ and $(x, -6)$ is perpendicular to the line containing $(-2, -9)$ and $(3, -4)$.
27. The line containing $(-4, 9)$ and $(4, 3)$ is parallel to the line containing $(-8, 1)$ and $(4, y)$.
28. The line containing $(8, 7)$ and $(7, -6)$ is perpendicular to the line containing $(2, 4)$ and $(x, 3)$.
29. The line containing $(1, -3)$ and $(3, y)$ is parallel to the line containing $(5, -6)$ and $(9, y)$.

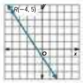
Write equations in slope-intercept form for a line that is *parallel* and a line that is *perpendicular* to the given line and that passes through the given point.

30. passes through $B(6, 3)$



$y = 3x - 15$; $y = -\frac{1}{3}x + 5$

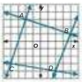
31. passes through $S(-2, -4)$



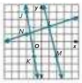
$y = -\frac{3}{2}x - 7$; $y = \frac{2}{3}x - \frac{8}{3}$

PRECISION **Determine whether any of the lines in each figure are parallel or perpendicular. Justify your answers.** 32–34. See Mod. 12 Answer Appendix.

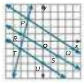
32.



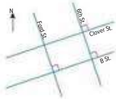
33.



34.



35. CITY BLOCKS The figure shows a map of part of a city consisting of two pairs of parallel roads. If a coordinate grid is applied to this map, Ford Street would have a slope of −3.



a. The intersection of B Street and Ford Street is 150 yards east of the intersection of Ford Street and Clover Street. How many yards south is it? 450 yd

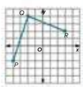
b. What is the slope of 6th Street? Explain. −3; Ford Street and 6th Street are parallel, so they have the same slope.

c. What are the slopes of Clover and B Streets? Explain. See Mod. 12 Answer Appendix.

d. The intersection of B Street and 6th Street is 600 yards east of the intersection of B Street and Ford Street. How many yards north is it? 200 yd

36. REASONING $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$. The coordinates of A, B, and C are $A(-3, 1)$, $B(6, 4)$, and $C(1, -1)$. What is a possible set of coordinates for point D? Describe the reasoning you used to find the coordinates. See Mod. 12 Answer Appendix.

37. USE A MODEL A video game designer is using a coordinate plane to plan the path of a helicopter. She has already determined that the helicopter will move along straight segments from P to Q to R. The designer wants the next part of the path, $\overline{RS}$, to be perpendicular to $\overline{QR}$, and she wants point S to lie on the y-axis. What should the coordinates of point S be? Justify your answer. See Mod. 12 Answer Appendix.



38. Line p passes through (1, 3) and (4, 7), and line q passes through (0, −2) and (a, b).

a. Find the slopes of lines p and q. slope of p: $\frac{4}{3}$; slope of q: $\frac{b+2}{a}$

b. Find possible values of a and b if $p \parallel q$. Sample answer: $a = 3$ and $b = 2$

39. STRUCTURE Let a and b be nonzero real numbers. Line p has the equation $y = ax + b$.

a. Find the equation of the line through (5, 1) that is parallel to line p. Write the equation in point-slope form. Explain your reasoning. $y - 1 = a(x - 5)$; the line must have slope a to be parallel to line p.

b. Find the equation of the line through (2, 3) that is perpendicular to line p. Write the equation in slope-intercept form. Explain your reasoning. $y = -\frac{1}{a}x + \frac{2}{a} + 3$; the line must have slope $-\frac{1}{a}$ to be perpendicular to line p.

40. CONSTRUCT ARGUMENTS The equation of line ℓ is $3y - 2x = 6$.

a. Line m is perpendicular to line ℓ and passes through the point $P(6, -2)$. Find the equation of line m. $y = -\frac{3}{2}x + 7$

b. Line n is parallel to line m. Is it possible to write the equation of line n in the form $2x + 3y = k$ for some constant k? Justify your argument. See Mod. 12 Answer Appendix.

Higher-Order Thinking Skills

41. ANALYZE Draw a square $ABCD$ with opposite vertices at $A(2, -4)$, and $C(10, 4)$.

a. Find the other two vertices of the square and label them B and D. $B(2, 4)$ and $D(10, -4)$

b. Show that $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$. See Mod. 12 Answer Appendix.

c. Show that the measure of each angle inside the square is equal to 90°. See Mod. 12 Answer Appendix.

42. PERSEVERE Find the value of n so that the line perpendicular to the line with the equation $-2y + 4 = 6x + 8$ passes through the points $(n, -4)$ and $(2, -8)$. 14

43. ANALYZE Determine whether the points at (−2, 2), (2, 5), and (6, 8) are collinear. Justify your argument. See Mod. 12 Answer Appendix.

44. CREATE Write equations for a pair of perpendicular lines that intersect at the point at (−3, −7). Sample answer: $y = 2x - 1$, $y = -\frac{1}{2}x - \frac{17}{2}$

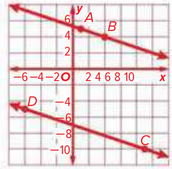
45. WRITE Write biconditionals to determine whether lines are parallel or perpendicular using slopes. See Mod. 12 Answer Appendix.

46. FIND THE ERROR A student was asked to find the equation of the line perpendicular to $\overleftrightarrow{AB}$ that passes through point P, given that A, B, and P have coordinates $A(0, 3)$, $B(2, 2)$, and $P(1, 4)$. The student's work is shown at the right. Do you agree with the student's solution? Explain your reasoning. See Mod. 12 Answer Appendix.

> Slope of $\overleftrightarrow{AB} = \frac{2-3}{2-0} = -\frac{1}{2}$.
>
> So, the slope of the required line is 2. The equation of this line is $y = 2x + b$. The line passes through P(1,4).
>
> To find b: $1 = 2(4) + b$
>
> $1 = 8 + b$
>
> $-7 = b$
>
> So, the equation is $y = 2x - 7$.

Answers

1.



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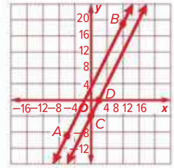

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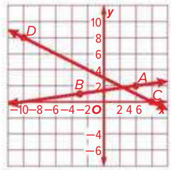

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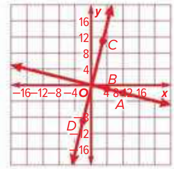

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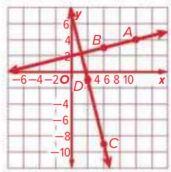

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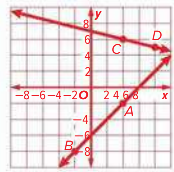

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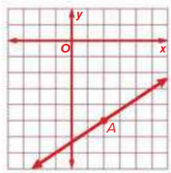

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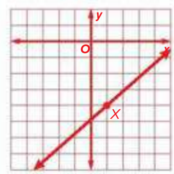
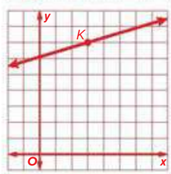
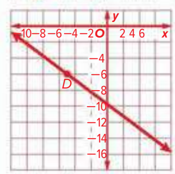


18.



19.



20b. The y-coordinate will be 20, so let the coordinates of the point be $(x, 20)$. Then $20 = \frac{1}{2}x + \frac{5}{2}$. Solving for x shows that $x = 35$. The coordinates of the chair at the top of the chair lift are (35, 20).

Lesson 12-9

Proving Lines Parallel

G.CO.9, G.CO.12

LESSON GOAL

Students identify and use parallel lines by using angle relationships.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Intersecting Lines

Develop:

Identifying Parallel Lines
- Identify Parallel Lines
- Use Angle Relationships
- Prove Lines Parallel

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Rate of Change and Slope	●	●		●
Extension: Eratosthenes		●	●	●

Language Development Handbook

Assign page 79 of the *Language Development Handbook* to help your students build mathematical language related to parallel lines and angle relationships.

ELL You can use the tips and suggestions on page T79 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.9 Prove theorems about lines and angles.

G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper foldering, dynamic geometry software, etc.).

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

3 Construct viable arguments and critique the reasoning of others.

4 Model with mathematics.

Coherence

Vertical Alignment

Previous
Students classified lines as parallel, perpendicular, or neither by using the slope criteria.
G.GPE.5

Now
Students identify and use parallel lines by using angle relationships.
G.CO.9

Next
Students will use perpendicular lines to find distance between a point and a line.
G.CO.12

Rigor

The Three Pillars of Rigor

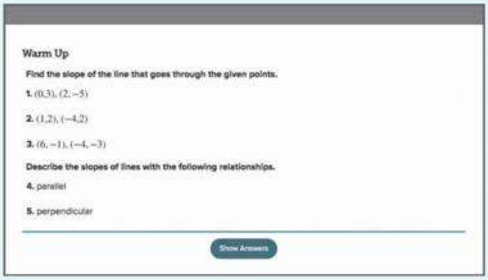
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of parallel lines, and they build fluency by proving theorems about parallel lines. They apply their understanding by solving real-world problems related to parallel lines.

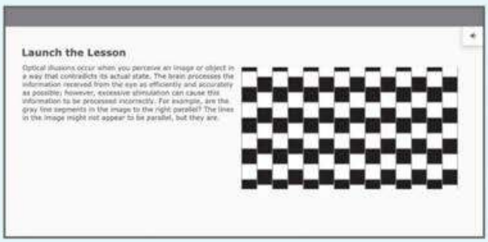
Mathematical Background

If two lines in a plane are cut by a transversal so that corresponding angles are congruent, then the lines are parallel. This postulate justifies the construction of parallel lines. A transversal is drawn through a given point to intersect a given line. The given point becomes the vertex for constructing an angle congruent to the one formed by the line and the transversal. The result is a pair of parallel lines cut by a transversal. This construction leads to the Parallel Postulate: If given a line and a point not on the line, then there exists exactly one line through the point that is parallel to the given line.

Interactive Presentation



Warm Up



Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skills for this lesson:

- finding slopes
- describing slopes of parallel and perpendicular lines

Answers:

1. -4
2. 0
3. $\frac{1}{5}$
4. equal slopes
5. The product of the slopes is -1.

Launch the Lesson

Teaching the Mathematical Practices

3 Construct Arguments Students will use stated assumptions, definitions, and previously established results to prove that lines are parallel.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Explore Intersecting Lines

Objective

Students use dynamic geometry software to analyze the relationship between pairs of related angles and parallel lines.

Teaching the Mathematical Practices

5 Decide When to Use Tools Mathematically proficient students can make sound decisions about when to use mathematical tools such as dynamic geometry software. Help them see why using these tools will help to solve problems and what the limitations are of using the tools.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students use dynamic geometry software to construct alternate exterior and alternate interior angles that are congruent, and measure the slopes of the lines to see that they are parallel. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore

Explore

WEB SKETCHPAD

Students use a sketch to explore angle relationships.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Intersecting Lines (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Describe where alternate exterior angles lie in relation to the transversal. Sample answer: By definition, alternate exterior angles lie outside a pair of lines and on opposite (or alternate) sides of the transversal.
- Why does stating alternate exterior or interior angles congruent prove that the lines cut by the transversal are parallel? Explain using your knowledge of parallel lines. Sample answer: We found that when parallel lines are cut by a transversal, then the alternate exterior angles are congruent and the alternate interior angles are congruent. If two lines are cut by a transversal, and the alternate exterior or interior angles are congruent, then the angle relationships are true and the lines must be parallel.

Inquiry

If a pair of alternate exterior or alternate interior angles is congruent, what relationship is formed? Sample answer: The two lines being cut by the transversal are parallel.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Identifying Parallel Lines

Objective

Students apply angle relationship theorems to identify parallel lines and find missing values.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of parallel lines in the Learn to be able to apply them later.

Common Misconception

Students may think that these new theorems are not necessary, because they are converses of previous theorems. Remind them that the converse of a true conditional statement is not necessarily also true, but these particular statements are special because the converses are also true.

Essential Question Follow-Up

Students learn the theorems used to prove that lines are parallel.

Ask:

Why is it important to know how to prove that lines are parallel using angles? Sample answer: This is useful for writing logical arguments to prove geometry theorems.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 12-9

Proving Lines Parallel

Explore Intersecting Lines

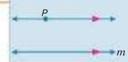
Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY If a pair of alternate exterior or alternate interior angles is congruent, what relationship is formed?

T oday's Goals
- Apply angle relationship theorems to identify parallel lines and find missing values.

Learn Identifying Parallel Lines

Corresponding angles are congruent when the lines cut by the transversal are parallel. The converse of this relationship is also true.



Theorem 12.19: Converse of Corresponding Angles Theorem
If two lines are cut by a transversal so that corresponding angles are congruent, then the lines are parallel.

Postulate 12.13: Parallel Postulate
If given a line and a point not on the line, then there exists exactly one line through the point that is parallel to the given line.

Parallel lines that are cut by a transversal create several pairs of congruent angles. These special angle pairs can be used to prove that a pair of lines is parallel.

Theorem 12.20: Alternate Exterior Angles Converse
If two lines in a plane are cut by a transversal so that a pair of alternate exterior angles is congruent, then the lines are parallel. If ≅ ∠5, then a ∥ b.

Theorem 12.21: Consecutive Interior Angles Converse
If two lines in a plane are cut by a transversal so that a pair of consecutive interior angles is supplementary, then the lines are parallel. If m + m∠6 180°, then b ∥

Theorem 12.22: Alternate Interior Angles Converse
If two lines in a plane are cut by a transversal so that a pair of alternate interior angles is congruent, then the lines are parallel. If ≅ ∠3, then a ∥ b.

Theorem 12.23: Perpendicular Transversal Converse
If two lines in a plane are perpendicular to the same line, then the lines are parallel.

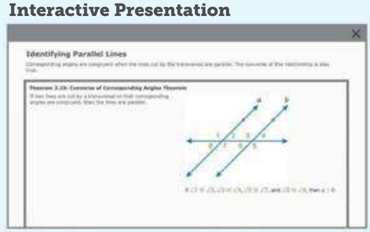
Y ou will prove Theorems 12.20, 12.22, and 12.23 in Exercises 20, 19, and 18, respectively.

Study Tip
Euclid's Postulates The father of modern geometry, Euclid (c. 300 B.C.), realized that only a few postulates were needed to prove the theorems in his day. The Parallel Postulate is one of Euclid's five original postulates.

Go Online Proofs of Theorems 12.19 and 12.21 are available.

Lesson 12-9 • Proving Lines Parallel 775

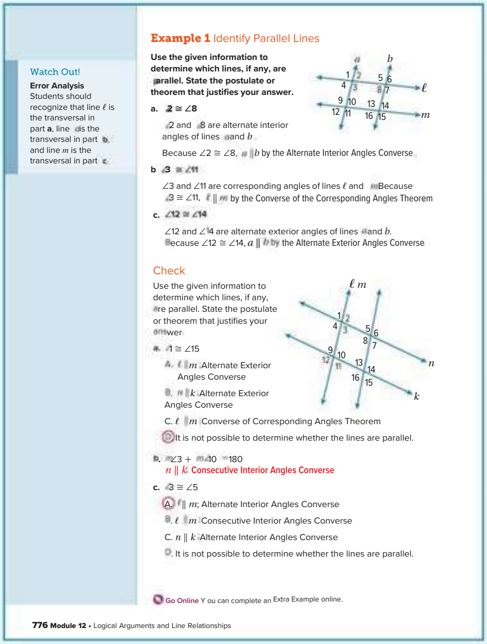
Interactive Presentation



Learn

TAP

Students tap to reveal a Study Tip.


Watch Out!

Error Analysis Students should recognize that line ℓ is the transversal in part **a**, line n is the transversal in part **b**, and line m is the transversal in part **c**.

Example 1 Identify Parallel Lines

Use the given information to determine which lines, if any, are parallel. State the postulate or theorem that justifies your answer.

a. $\angle 2 \cong \angle 8$

$\angle 2$ and $\angle 8$ are alternate interior angles of lines a and b.

Because $\angle 2 \cong \angle 8$, $a \parallel b$ by the Alternate Interior Angles Converse.

b. $\angle 3 \cong \angle 11$

$\angle 3$ and $\angle 11$ are corresponding angles of lines ℓ and m. Because $\angle 3 \cong \angle 11$, $\ell \parallel m$ by the Converse of the Corresponding Angles Theorem.

c. $\angle 12 \cong \angle 14$

$\angle 12$ and $\angle 14$ are alternate exterior angles of lines a and b. Because $\angle 12 \cong \angle 14$, $a \parallel b$ by the Alternate Exterior Angles Converse.

Check

Use the given information to determine which lines, if any, are parallel. State the postulate or theorem that justifies your answer.

a. $\angle 1 \cong \angle 15$

A. $\ell \parallel m$; Alternate Exterior Angles Converse

B. $n \parallel k$; Alternate Exterior Angles Converse

C. $\ell \parallel m$; Converse of Corresponding Angles Theorem

(D.) It is not possible to determine whether the lines are parallel.

b. $m\angle 3 + m\angle 10 = 180$

$n \parallel k$; Consecutive Interior Angles Converse

c. $\angle 3 \cong \angle 5$

(A.) $\ell \parallel m$; Alternate Interior Angles Converse

B. $\ell \parallel m$; Consecutive Interior Angles Converse

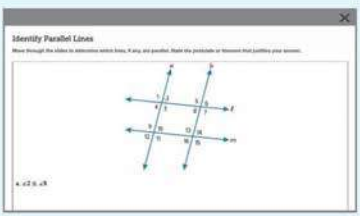
C. $n \parallel k$; Alternate Interior Angles Converse

D. It is not possible to determine whether the lines are parallel.

Go Online You can complete an Extra Example online.

776 Module 12 • Logical Arguments and Line Relationships

Interactive Presentation



Example 1

TAP

Students tap to move through the steps.

Example 1 Identify Parallel Lines

Teaching the Mathematical Practices

2 Different Properties Mathematically proficient students look for different ways to solve problems. Encourage them to consider the different theorems to solve the problem and to choose the method that works best for them.

Questions for Mathematical Discourse

AL In part **a,** what is the relationship between $\angle 2$ and $\angle 8$? They are alternate interior angles of lines a and b.

OL In part **a,** what theorem will you use to show that $a \parallel b$? Theorem 3.22: Alternate Interior Angles Converse

BL If you know that $\angle 1$ and $\angle 10$ are supplementary, determine a set of lines, if any, that are parallel. Justify your answer. Sample answer: $\ell \parallel m$; $\angle 10$ and $\angle 9$ form a linear pair; $m\angle 9 + m\angle 10 = 180°$ and $m\angle 1 + m\angle 10 = 180°$; $m\angle 9 + m\angle 10 = m\angle 1 + m\angle 10$; $m\angle 9 = m\angle 1$. By Theorem 3.19, $\ell \parallel m$.

Common Error

Students may incorrectly think that the transversal is one of the lines they can prove is parallel to another because the transversal passes through both vertices of any angle pair that can be used to prove lines parallel. Remind students that this line is the transversal, and the parallel lines cross it but not each other.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Example 2 Use Angle Relationships

Teaching the Mathematical Practices

2 Create Representations Guide students to write an equation that models the situation in Example 2. Then use the equation to solve the problem.

Questions for Mathematical Discourse

AL What do you know about the three angles formed by the intersection of lines d and f that are not marked? Sample answer: The three angles are right angles.

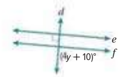
OL What do you need to show about the angle marked $(4y + 10)°$ to show that $e \parallel f$? that the angle marked $(4y + 10)°$ is a right angle

BL Suppose the marked right angle was instead adjacent to the angle marked $(4y + 10)°$. Could you still prove that $e \parallel f$? Explain. No; sample answer: In that case all the angles you could use have vertices on line f and none on line e.

Example 2 Use Angle Relationships

Find the value of y so that $e \parallel f$.

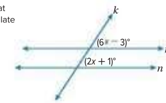
From the figure, you know that line d is perpendicular to line e. For lines e and f to be parallel, line f must also be perpendicular to line d. If line f is perpendicular to line d, then $(4y + 10) = 90°$. Solve for y.



$90 = 4y + 10$	Definition of perpendicular
$80 = 4y$	Subtract 10 from each side
$20 = y$	Divide each side by 4.

Check

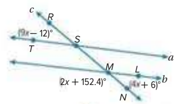
a. Find the value of x so that $m \parallel n$. Identify the postulate or theorem you used.



$x =$? ; ?

1 Converse of Corresponding Angles Theorem

b. Find $m\angle LMN$ so that $a \parallel b$.



A 3.6° B 20.4° C 100° D 159.6°

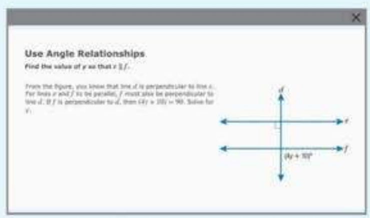
Think About It! Lakeisha argues that we do not have enough information to determine the correct value of y. She says that the two angles are not corresponding angles; therefore, it does not help to assume that the two angles are congruent. What theorems can you use to prove that line e is parallel to line f?

Sample answer: You can use Theorem 3.9 to show that because d and e are perpendicular, all of the angles formed by the intersection are right angles. You can then use the Converse of the Corresponding Angles Theorem to show that $(4y + 10)° = 90°$

Go Online You can complete an Extra Example online.

Lesson 12-9 • Proving Lines Parallel 777

Interactive Presentation



Example 2

TYPE

Students answer a question to show they understand how to use angle relationships.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Study Tip

Assumptions When applying geometric theorems to real-world objects, we often make assumptions about the relationships between the objects being represented. In Example 3, the boat is the transversal and we are trying to determine whether the oars are parallel lines.

Talk About It!

What other information can be used to show that the oars are not parallel?

Sample answer: The consecutive interior angles are not supplementary, so the lines are not parallel.

Study Tip

Proving Lines Parallel When two parallel lines are cut by a transversal, the angle pairs formed are either congruent or supplementary. When a pair of lines forms angles that do not meet this criterion, the lines cannot be parallel.

Go Online

You may want to complete the construction activities for this lesson.

The angle pair relationships formed by a transversal can be used to prove that two lines are parallel.

Example 3 Prove Lines Parallel

ROWING **To move in a straight line with maximum efficiency, rowers' oars should be parallel. Refer to the photo at the right. Is it possible to prove that any of the oars are parallel? Justify your answer.**

The angle that forms a linear pair with the 50° angle has a measure of 180° − 50° or 130°.

The angle measuring 130° is the corresponding angle to the 124° angle. Because the corresponding angles are not congruent, the lines are not parallel.

Therefore, it is not possible to prove that the oars are parallel.

Check

ANTENNAS Is it possible to prove that the support poles of the antenna complex are parallel? Justify your answer.

A. No; because the consecutive interior angles are not supplementary, the support poles cannot be parallel.

B. No; because the alternate interior angles are not supplementary, the support poles cannot be parallel.

C. Yes; because the alternate interior angles are supplementary, the support poles are parallel.

D. Yes; because the consecutive interior angles are congruent, the support poles are parallel.

Go Online You can complete an Extra Example online.

Ivan Smuk/Shutterstock; shakzu/Shutterstock

778 Module 12 • Logical Arguments and Line Relationships

Interactive Presentation

Example 3

TYPE

Students type to complete the solution.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Prove Lines Parallel

Teaching the Mathematical Practices

4 Apply Mathematics In Example 3, students apply what they have learned about proving lines parallel to solving a real-world problem.

Questions for Mathematical Discourse

AL What kinds of angle relationships do we need to see in a special angle pair to prove that two lines are parallel? congruent or supplementary

OL In the diagram, what are the parallel lines and what is the transversal? The oars are the parallel lines and the side of the boat is the transversal.

BL What angle would the bottom pair of oars need to make with the side of the boat to ensure that both pairs of oars are parallel? 56°

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–17
2	exercises that use a variety of skills from this lesson	18–22
2	exercises that extend concepts learned in this lesson to new contexts	23–24
3	exercises that emphasize higher-order and critical-thinking skills	25–30

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–23 odd, 25–30
- Extension: Eratosthenes
- ALEKS Proofs Involving Parallel Lines, Parallel and Perpendicular Lines

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–29 odd
- Remediation, Review Resources: Rate of Change and Slope
- Personal Tutors
- Extra Examples 1–3
- ALEKS finding slopes

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Rate of Change and Slope
- *Quick Review Math Handbook:* Proving Lines Parallel
- ALEKS finding slopes

Important to Know

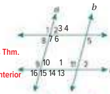
Digital Exercise Alert Exercises 21-22 and 27 require constructions and are not available online. To fully address G.CO.12, have students complete these exercises using their books.

Practice

Go Online You can complete your homework online.

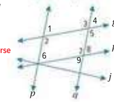
Example 1

Use the given information to determine which lines, if any, are parallel. State the postulate or theorem that justifies your answer.



1. ∠3 ≅ ∠7
   $a \parallel b$; Alternate Interior Angles Converse
2. ∠9 ≅ ∠11
   $a \parallel b$; Converse of Corresponding Angles Thm.
3. ∠2 ≅ ∠16
   $\ell \parallel m$; Alternate Exterior Angles Converse
4. $m\angle 5 + m\angle 12 = 180°$
   $\ell \parallel m$; Consecutive Interior Angles Converse

Given the following information, determine which lines, if any, are parallel. State the theorem that justifies your answer.

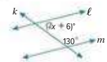


5. ∠1 ≅ ∠6
   $g \parallel h$; Converse of Corresponding Angles Thm.
6. $m\angle 7 + m\angle 6 = 180°$
   $p \parallel q$; Consecutive Interior Angles Converse

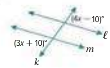
Example 2

Find the value of x so that $\ell \parallel m$.

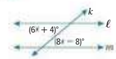
7. 22



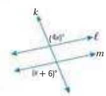
8. 20



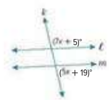
9. 6



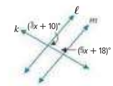
10. 2



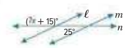
11. 13



12. 19

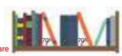


13. Find the value of x so that $\ell \parallel m$. 20



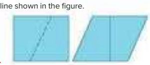
Example 3

14. **BOOKS** Each orange book on the bookshelf makes a 70° angle with the base of the shelf. What more can you say about these two orange books? Explain.



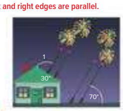
Sample answer: They are parallel. Because corresponding angles are congruent, the books are parallel.

15. **PATTERNS** A rectangle is cut along the slanted, dashed line shown in the figure. The two pieces are rearranged to form another figure. Describe as precisely as you can the shape of the new figure. Explain.



Parallelogram; sample answer: The top edges are perpendicular to the vertical line, so they are a single line. The bottom edge is also a single line and perpendicular to the same line as the top, so it is parallel to the top. The top edge is a transversal to the left and right slanted edges, and the angles are supplementary. So, the left and right edges are parallel.

16. **FIREWORKS** The designers of a fireworks display want to have four fireworks travel along parallel trajectories. They decide to place two launchers on a dock and two launchers on the roof of a building. To make this display work correctly, what should the measure of angle 1 be? Explain. See margin.



17. **REASONING** Chaska is making a giant letter A to put on the rooftop of the A Is for Apple Orchard Store. The figure shows a sketch of the design.

a. What should the measures of angles 1 and 2 be so the horizontal part of the A is truly horizontal? Explain. See margin.

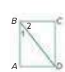
b. When building the A, Chaska makes sure that angle 1 is correct, but when he measures angle 2, it is not correct. What does this imply about the A? Sample answer: One side of the A is longer than the other.

Mixed Exercises

18. **PROOF** Provide a reason for each statement in the proof of the Perpendicular Transversal Converse.

Given: ∠1 and ∠2 are complementary; $\overline{BC} \perp \overline{CD}$

Prove: $\overline{BA} \parallel \overline{CD}$



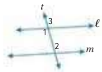
Proof:

Statements	Reasons
1. $\overline{BC} \perp \overline{CD}$	1. ___?___ Given
2. $m\angle ABC = m\angle 1 + m\angle 2$	2. ___?___ Angle Addition Postulate
3. ∠1 and ∠2 are complementary.	3. ___?___ Given
4. $m\angle 1 + m\angle 2 = 90$	4. ___?___ Definition of complementary angles
5. $m\angle ABC = 90$	5. ___?___ Transitive Property of Equality
6. $BA \perp \overline{BC}$	6. ___?___ Definition of perpendicular
7. $\overline{BA} \parallel \overline{CD}$	7. ___?___ If 2 lines are ⊥ to the same line, the lines are ∥.

19. PROOF Write a paragraph proof to prove the Alternate Interior Angles Converse.

Given: $\angle 1 \cong \angle 2$

Prove: $\ell \parallel m$

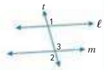


Sample answer: It is given that $\angle 1 \cong \angle 2$. Also, $\angle 1 \cong \angle 3$, because these are vertical angles. Therefore, $\angle 2 \cong \angle 3$ by the Transitive Property of Congruence. This shows that $\ell \parallel m$ by the Converse of Corresponding Angles Theorem.

20. PROOF Write a paragraph proof to prove the Alternate Exterior Angles Converse.

Given: $\angle 1 \cong \angle 2$

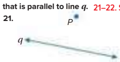
Prove: $\ell \parallel m$



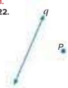
Sample answer: It is given that $\angle 1 \cong \angle 2$. Because vertical angles are congruent, $\angle 2 \cong \angle 3$. By the Transitive Property of Congruence, $\angle 1 \cong \angle 3$. Thus, $\ell \parallel m$ by the Converse of Corresponding Angles Theorem.

USE TOOLS **Use a compass and straightedge to construct the line through point *P* that is parallel to line *q*.** 21–22. See margin.

21.



22.



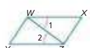
23. PICTURE FRAMES Lindy is making a wooden picture frame. She cuts the top and bottom pieces at a 45° angle. If the corners are right angles, explain how Lindy knows that each pair of opposite sides is parallel.

Sample answer: Because the corners are right angles, each pair of opposite sides is perpendicular to the same line. Therefore, each pair of opposite sides is parallel.

24. REASONING Jim made a frame for a painting. He wants to check to make sure that opposite sides are parallel by measuring the angles at the corners and seeing whether they are right angles. How many corners must he check to be sure that the opposite sides are parallel? 3

Higher Order Thinking Skills

25. FIND THE ERROR Sean and Daniela are determining which lines are parallel in the figure at the right. Sean says that because $\angle 1 \cong \angle 2$, $\overline{WY} \parallel \overline{XZ}$. Daniela disagrees and says that because $\angle 1 \cong \angle 2$, $\overline{WX} \parallel \overline{YZ}$. Is either of them correct? Explain your reasoning.



Daniela is correct. $\angle 1$ and $\angle 2$ are alternate interior angles for $\overline{WX}$ and $\overline{YZ}$. So, if alternate interior angles are congruent, then the lines are parallel.

26. ANALYZE Is Theorem 3.23 still true if the two lines are not coplanar? Draw a figure to justify your argument. See margin.

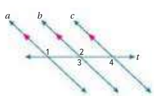
27. CREATE Draw a triangle *ABC*.

a. Construct the line parallel to $\overline{BC}$ through point *A*. See margin.

b. Use measurements to justify that the line you constructed is parallel to $\overline{BC}$. Sample answer: Using a straightedge, the lines are equidistant. So, they are parallel.

c. Justify the construction. See margin.

28. PROOF Use the figure at the right to complete the two-column proof to prove that two lines parallel to a third line are parallel to each other.



Given: $a \parallel b$ and $b \parallel c$

Prove: $a \parallel c$

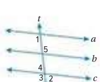
Proof:

Statements	Reasons
1. $a \parallel b$ and $b \parallel c$	1. ? Given
2. $\angle 1 \cong \angle 3$	2. ? Alternate Interior Angles Theorem
3. ? $\angle 3 \cong \angle 2$	3. ? Vertical angles are congruent.
4. ? $\angle 2 \cong \angle 4$	4. ? Alternate Interior Angles Theorem
5. ? $\angle 1 \cong \angle 4$	5. ? Transitive Property
6. $a \parallel c$	6. ? Alternate Interior Angles Converse

29. WRITE Can a pair of angles be supplementary and congruent? Explain your reasoning.

Y es; sample answer: A pair of angles can be supplementary and congruent if the measure of both angles is 90°, because the sum of the angle measures would be 180°.

30. PROOF Refer to the figure at the right.



a. If $m\angle 1 + m\angle 2 = 180°$, prove that $a \parallel c$. See margin.

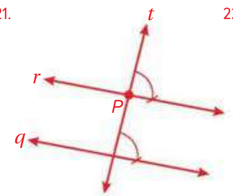
b. Given that $a \parallel c$, if $m\angle 1 + m\angle 3 = 180°$, prove that $t \perp c$. See margin.

Answers

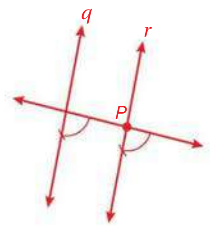
16. 80°; The supplementary angle to the 30°-angle and $\angle 1$ is a corresponding angle to the 70°-angle. This means that the supplementary angle to the 30°-angle and $\angle 1$ is a 70°-angle. Therefore, $30° + m\angle 1 + 70° = 180°$. So, $m\angle 1 = 80°$.

17a. 108°; Sample answer: To ensure that the horizontal part of the A is truly horizontal, it should be parallel to the dashed line. Therefore, $\angle 2$ and the 108°-angle are alternate interior angles, and $m\angle 2 = 108°$. $\angle 1$ and $\angle 2$ are congruent angles, so $m\angle 1 = 108°$.

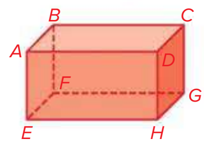
21.



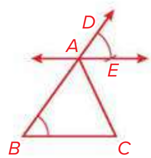
22.



26. No; sample answer: In the figure shown, $\overline{AB} \perp \overline{BC}$ and $\overline{GC} \perp \overline{BC}$, but $\overline{AB} \not\perp \overline{GC}$



27a.



27c. Sample answer: $\angle ABC$ was copied to construct $\angle DAE$. So, $\angle ABC \cong \angle DAE$. $\angle ABC$ and $\angle DAE$ are corresponding angles, so by the Converse of the Corresponding Angles Theorem, $\overleftrightarrow{AE} \parallel \overleftrightarrow{BC}$.

30a. We know that $m\angle 1 + m\angle 2 = 180°$. Because $\angle 2$ and $\angle 3$ are a linear pair, $m\angle 2 + m\angle 3 = 180°$. By substitution, $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$. By subtracting $m\angle 2$ from both sides, we get $m\angle 1 = m\angle 3$. $m\angle 1 \cong m\angle 3$, by the definition of congruent angles. Therefore, $a \parallel c$ because the corresponding angles are congruent.

30b. We know that $a \parallel c$ and $m\angle 1 + m\angle 3 = 180°$. Because $\angle 1$ and $\angle 3$ are corresponding angles, they are congruent and their measures are equal. By substitution, $m\angle 3 + m\angle 3 = 180°$ or $2m\angle 3 = 180°$. By dividing both sides by 2, we get $m\angle 3 = 90°$. Therefore, $t \perp c$ because they form a right angle.

Perpendiculars and Distance

G.CO.12, G.MG.3

LESSON GOAL

Students use perpendicular lines to find distance.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Distance from a Point to a Line

Develop:

Distance Between a Point and a Line
- Distance from a Point to a Line on the Coordinate Plane
- Solve a Design Problem by Using Distance

Distance Between Parallel Lines
- Distance Between Parallel Lines

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Roots	●	●		●
Extension: Perpendicular Lines in Spherical Geometry		●	●	●

Language Development Handbook

Assign page 80 of the *Language Development Handbook* to help your students build mathematical language related to using perpendicular lines to find distance.

ELL You can use the tips and suggestions on page T80 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

G.MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
4 Model with mathematics.
5 Use appropriate tools strategically.
6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students identified and used parallel lines by using angle relationships.
G.CO.9

Now
Students use perpendicular lines to find distance.
G.CO.12

Next
Students will study perpendicular bisectors of triangles.
G.CO.9 (Course 2)

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

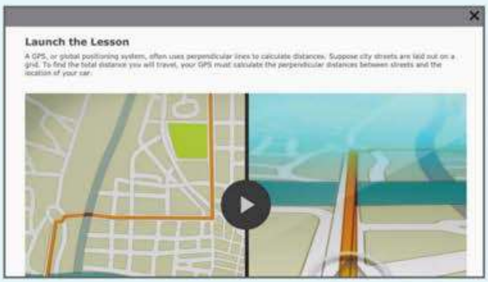
Conceptual Bridge In this lesson, students expand on their understanding of perpendicular lines, and they build fluency by making constructions related to perpendicular lines. They apply their understanding by solving real-world problems about distances between points and lines and between parallel lines.

Mathematical Background

The distance from a line to a point not on the line is the length of the segment perpendicular to the line from the point. This is the shortest distance from the point to the line. Distance can be used to determine parallel lines. Two lines in a plane are parallel if they are equidistant everywhere. To find the distance between two parallel lines, measure the length of a perpendicular segment whose endpoints lie on the two lines.

Interactive Presentation

Warm Up



Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

• finding square roots

Answers:

1. 2.24
2. 6.71
3. 9.27
4. 11.49
5. 16

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of perpendicular lines.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then, discuss the questions below with the class.

Explore Distance from a Point to a Line

Objective

Students use dynamic geometry software to determine how to measure the shortest distance between a point and a line.

MP Teaching the Mathematical Practices

5 Decide When to Use Tools Mathematically proficient students can make sound decisions about when to use mathematical tools such as dynamic geometry software. Help them see why using these tools will help to solve problems and what the limitations are of using the tools.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

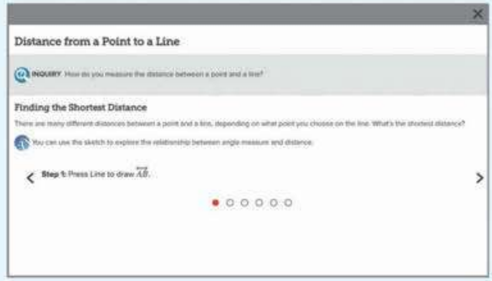
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use dynamic geometry software to explore the distance between a line and a point not on the line. In doing so, students will discover that the shortest distance between the point and the line is along the perpendicular to the line through the point. Students apply this knowledge to a real-world situation. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



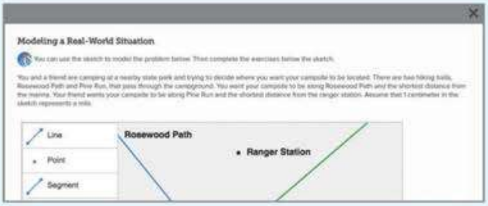
Explore

Explore

WEB SKETCHPAD

Students use a sketch to explore distance between two lines.

Interactive Presentation



Explore

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Distance from a Point to a Line (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How is finding the distance between a line and a point not on the line related to finding the distance between two points? Sample answer: A line is a collection of points that follow a certain rule. To find the distance between a line and a point not on the line, you have to first find the point on the line that's closest to the point not on the line.
- Think about crossing the street. Does it make sense that the shortest distance from your location to the other side of the street is along a perpendicular line? Sample answer: My location is the point not on the line and the other side of the street is the line. When I want to use the shortest distance, it makes most sense to travel in a path perpendicular to the sidewalk and not at a different angle.

Inquiry

How do you measure the distance between a point and a line? Sample answer: You must first find a line that is perpendicular to the given line and passes through the given point. Then you must calculate the distance between the given point and the intersection point of the perpendicular line and the given line.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Distance Between a Point and a Line

Objective

Students use perpendicular lines to find the distance between a point and a line.

Teaching the Mathematical Practices

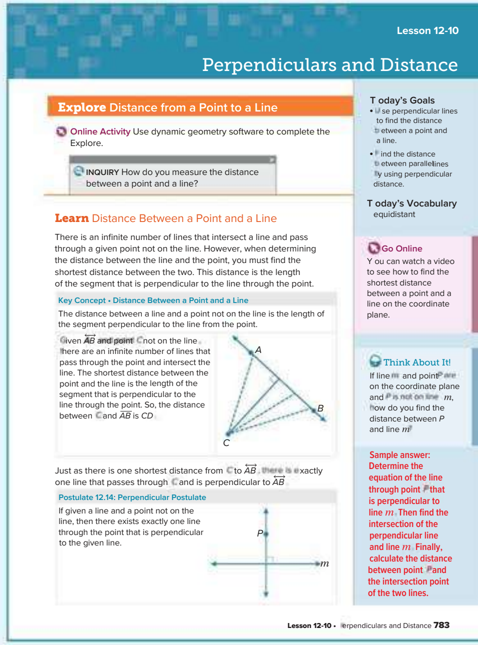
1 Explain Correspondences Encourage students to explain the relationships between the point and the line used in this Learn.

Common Misconception

Students may not understand why going through the thinking behind this concept is important. However, looking at the other lines between the point and the line shows visually why the perpendicular is the shortest, even though students do not yet have the tools to prove it.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.


Lesson 12-10

Perpendiculars and Distance

Explore Distance from a Point to a Line

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How do you measure the distance between a point and a line?

Learn Distance Between a Point and a Line

There is an infinite number of lines that intersect a line and pass through a given point not on the line. However, when determining the distance between the line and the point, you must find the shortest distance between the two. This distance is the length of the segment that is perpendicular to the line through the point.

Key Concept • Distance Between a Point and a Line

The distance between a line and a point not on the line is the length of the segment perpendicular to the line from the point.

Given $\overleftrightarrow{AB}$ and point C not on the line, there are an infinite number of lines that pass through the point and intersect the line. The shortest distance between the point and the line is the length of the segment that is perpendicular to the line through the point. So, the distance between C and $\overleftrightarrow{AB}$ is CD.

Just as there is one shortest distance from C to $\overleftrightarrow{AB}$, there is exactly one line that passes through C and is perpendicular to $\overleftrightarrow{AB}$.

Postulate 12.14: Perpendicular Postulate

If given a line and a point not on the line, then there exists exactly one line through the point that is perpendicular to the given line.

Today's Goals
- Use perpendicular lines to find the distance between a point and a line.
- Find the distance between parallel lines by using perpendicular distance.

Today's Vocabulary
equidistant

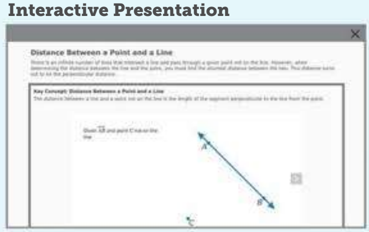
Go Online
You can watch a video to see how to find the shortest distance between a point and a line on the coordinate plane.

Think About It!
If line m and point P are on the coordinate plane and P is not on line m, how do you find the distance between P and line m?

Sample answer: Determine the equation of the line through point P that is perpendicular to line m. Then find the intersection of the perpendicular line and line m. Finally, calculate the distance between point P and the intersection point of the two lines.

Lesson 12-10 • Perpendiculars and Distance 783

Interactive Presentation



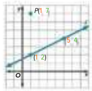
Learn

TAP

Students tap to see the steps behind the reasoning.

Example 1 Distance from a Point to a Line on the Coordinate Plane

Line ℓ contains points (1, 2) and (5, 4). Find the distance between line ℓ and the point P(1, 7).



Step 1 Find the equation of line ℓ.

Begin by finding the slope of the line through points (1, 2) and (5, 4).

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{5 - 1} = \frac{2}{4}$ or $\frac{1}{2}$

Then write the equation of the line using the point (1, 2).

$y = mx + b$	Slope-intercept form
$2 = \frac{1}{2}(1) + b$	$m = \frac{1}{2}$ and $(x, y) = (1, 2)$
$2 = \frac{1}{2} + b$	Simplify.
$\frac{3}{2} = b$	Subtract $\frac{1}{2}$ from each side.

The equation of line ℓ is $y = \frac{1}{2}x + \frac{3}{2}$.

Step 2 Find the equation of the line perpendicular to line ℓ.

Write the equation of line w that is perpendicular to line ℓ and contains P(1, 7). Because the slope of line ℓ is $\frac{1}{2}$, the slope of line w is -2. Write the equation of line w through P(1, 7) with slope -2.

$y = mx + b$	Slope-intercept form
$7 = -2(1) + b$	$m = -2$, $(x, y) = (1, 7)$
$7 = -2 + b$	Simplify.
$9 = b$	Add 2 to each side.

The equation of the line is $y = -2x + 9$.

Step 3 Solve the system of equations.

Find the point of intersection of lines ℓ and w.

Solve the system of equations to determine the point of intersection.

$y = \frac{1}{2}x + \frac{3}{2}$	Equation of line ℓ
$y = -2x + 9$	Equation of line w
$y = 3$	Solve for y.

Solve for x.

$y = -2x + 9$	Equation of line w
$3 = -2x + 9$	Substitute 3 for y.
$-6 = -2x$	Subtract 9 from each side.
$3 = x$	Divide each side by -2.

The point of intersection is (3, 3). Let this be point Q.

Talk About It!

Why do you use the perpendicular distance to find the distance between a line and a point not on the line?

Sample answer: The perpendicular distance will always be the shortest distance between a line and a point not on the line.

Study Tip

Solving Systems of Equations Systems of equations can be solved by graphing, substitution, or elimination. Keep this in mind when you are finding the intersection point of perpendicular lines.

Go Online You can complete an Extra Example online.

784 Module 12 • Logical Arguments and Line Relationships

Interactive Presentation

Example 1

TAP

Students tap to see the steps in the problem, enter solutions, and choose correct answers.

Example 1 Distance from a Point to a Line on the Coordinate Plane

Teaching the Mathematical Practices

2 Create Representations Guide students to write an equation that models the situation in Example 1. Then use the equation to solve the problem.

Questions for Mathematical Discourse

AL How do you find the distance between two points on a coordinate plane? Use the Distance Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

OL In Step 1, why do you need to find the equation of line ℓ? Sample answer: You need to find the equation of line ℓ to write the equation of the perpendicular line that passes through point P.

BL Why can't you determine the coordinates of Q by looking at the diagram? Sample answer: The diagram is not always an accurate representation. To find an exact answer, you need to calculate the exact coordinates.

Common Error

Students may find the information needed for one of the intermediate steps in the problem and stop. Remind them that this problem has multiple steps.

Apply Example 2 Solve a Design Problem by Using Distance

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- H ow can you restate the problem outside of the real-world context?
- W hat information can you use to write an equation of a line in slope-intercept form?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

(*continued on the next page*)

Step 4 Calculate the distance between P and Q.

Use the Distance Formula to determine the distance between P(1, 7) and Q(3, 3).

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$= \sqrt{(3 - 1)^2 + (3 - 7)^2}$ $x_2 = 3$, $x_1 = 1$ and $y_2 = 3$, and $y_1 = 7$

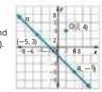
$= \sqrt{20}$ Simplify.

The distance between point P and line ℓ is $\sqrt{20}$ or about 4.47 units.

Check

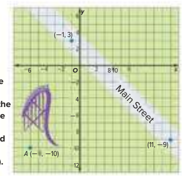
Line n contains points (−5, 3) and (4, −6). Find the distance between line n and point Q(2, 4). Round to the nearest tenth, if necessary.

? 5.7 units



Apply Example 2 Solve a Design Problem by Using Distance

AMUSEMENT PARK **The developers of an amusement park want to build a new attraction. According to park regulations, the entrance to each attraction must be at least 10 yards from the center of Main Street. In the design plans, the entrance to the new attraction is located at A(−6, −10), and Main Street contains the points (−1, 3) and (11, −9). If each unit represents 1 yard, will the new attraction comply with park regulations?**



1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to determine whether the entrance to the new attraction is at least 10 yards from the center of Main Street. How can I represent Main Street as a linear equation? How can I find the perpendicular distance from the entrance to the center of Main Street? I can review using points to write the equation of a line, and I can review finding the perpendicular distance between a point and a line.

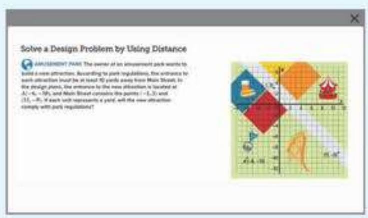
(*continued on the next page*)

Lesson 12-10 • Perpendiculars and Distance 785

Study Tip

Units of Measure When finding the distance between a point and a line on the coordinate plane, your final measurement should be labeled with *units* unless the problem is set in a real-world context.

Interactive Presentation



Example 2

TYPE

Students type to explain their solution process.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Study Tip

Estimation Y ou can also use the horizontal distance between a line and a point not on the line to estimate the distance between the point and the line. When you graph the horizontal and perpendicular lines that contain the point and intersect the given line, a right triangle is created. The horizontal distance between the given point and line is the same as the length of the hypotenuse of the right triangle. So, you know the perpendicular distance, or the length of the right triangle's leg, must be less than the horizontal distance between the point and the given line.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will use the given points to write an equation in slope-intercept form that can be used to represent Main Street. Then, I will find the equation of a line that is perpendicular to Main Street that also passes through point *A*. I will find the point of intersection of Main Street and the perpendicular line that I found. Finally, I will use the Distance Formula, the point of intersection, and point *A* to calculate the distance between the entrance of the new attraction and the center of Main Street. I have learned how to calculate the slope of a line using two points and how to use a slope and a given point to calculate the *y*-intercept. I have learned how to find the equation of a line perpendicular to a given line and through a point not on the given line. I have learned how to find the point of intersection of two lines. I have learned how to use the Distance Formula to find the distance between two points.

3 What is your solution?

Use your strategy to solve the problem.

What is the equation of the line in slope-intercept form that represents Main Street?

$y = -x + 2$

What is the equation of the line in slope-intercept form that is perpendicular to Main Street and passes through point *A*?

$y = x - 4$

What is the point of intersection of these two lines?

$(3, -1)$

What is the distance between the entrance to the new attraction and the center of Main Street? Will the new attraction be located far enough away from the center of Main Street to comply with park regulations?

The new attraction will be located about 12.7 yards, so it will be located far enough away from the center of Main Street.

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: I can use the coordinate grid to estimate the distance between the entrance of the new attraction and the center of Main Street. After sketching a line that appears to be perpendicular to Main Street through point *A*, I estimate the number of units between point *A* and Main Street. My estimation supports my solution.

Go Online Y ou can complete an Extra Example online.

786 **Module 12** • Logical Arguments and Line Relationships

DIFFERENTIATE

Language Development Activity AL ELL

Kinesthetic Learners Identify examples of lines in the classroom, like the grout lines of the tile floor or the frame of the chalkboard. Have students work in pairs to measure the distance of various points along one line to a fixed point. Have students discuss their findings. Facilitate the discussions so that students are able to see the relationships of the segments and distances between a point and a line.

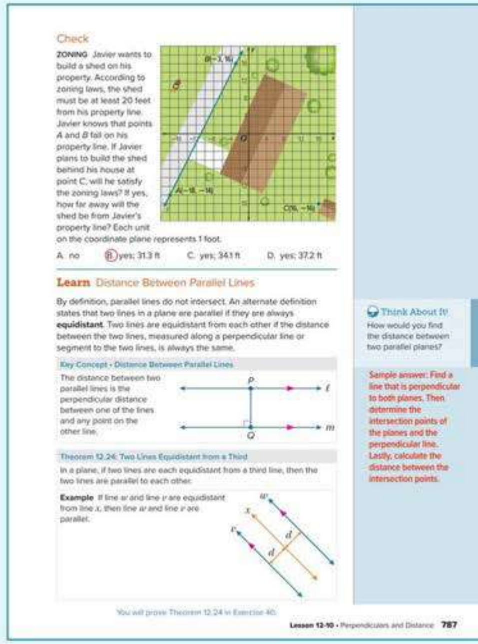
Learn Distance Between Parallel Lines

Objective

Students find the distance between parallel lines by using perpendicular distance.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of parallel lines to understand how to find the distance between them.


Check

ZONING Javier wants to build a shed on his property. According to zoning laws, the shed must be at least 20 feet from his property line. Javier knows that points A and B fall on his property line. If Javier plans to build the shed behind his house at point C, will he satisfy the zoning laws? If yes, how far away will the shed be from Javier's property line? Each unit on the coordinate plane represents 1 foot.

A. no B. yes; 31.3 ft C. yes; 34.1 ft D. yes; 37.2 ft

Learn Distance Between Parallel Lines

By definition, parallel lines do not intersect. An alternate definition states that two lines in a plane are parallel if they are always **equidistant**. Two lines are equidistant from each other if the distance between the two lines, measured along a perpendicular line or segment to the two lines, is always the same.

Key Concept • Distance Between Parallel Lines

The distance between two parallel lines is the perpendicular distance between one of the lines and any point on the other line.

Theorem 12.24: Two Lines Equidistant from a Third

In a plane, if two lines are each equidistant from a third line, then the two lines are parallel to each other.

Example If line w and line v are equidistant from line x, then line w and line v are parallel.

You will prove Theorem 12.24 in Exercise 40.

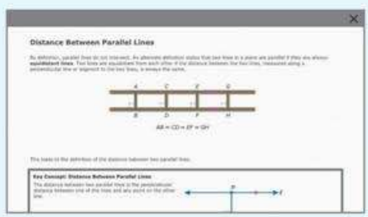
Think About It!

How would you find the distance between two parallel planes?

Sample answer: Find a line that is perpendicular to both planes. Then determine the intersection points of the planes and the perpendicular line. Lastly, calculate the distance between the intersection points.

Lesson 12-10 • Perpendiculars and Distance 787

Interactive Presentation



Learn

TYPE

Students answer a question to show they understand distance between parallel lines.

Study Tip

Perpendicular Lines When you are finding the equation of a line perpendicular to a pair of parallel lines, using the y-intercept of one of the parallel lines as a point contained by the perpendicular line will allow for easier calculations. The y-intercept is also used because it can easily be determined when the equation of a line is given in slope-intercept form.

Think About It!

Compare and contrast the processes for finding the distance between a point and a line and for finding the distance between parallel lines.

Sample answer: Both methods use perpendicular lines to find distances. When finding the distance between a point and a line, you find the equation of the line perpendicular to the given line that passes through the given point. When finding the distance between parallel lines, you must find the equation of a line that is perpendicular to the parallel lines. Then calculate the distance between the intersection points of the perpendicular and parallel lines.

Example 3 Distance Between Parallel Lines

Find the distance between the parallel lines r and t with equations $y = -3x - 5$ and $y = -3x + 6$, respectively.

You need to solve a system of equations to find the endpoints of a segment perpendicular to lines r and t. Lines r and t have slope −3.

Step 1 Write an equation of line q.

The slope of q is the opposite reciprocal of −3, or $\frac{1}{3}$. Use the y-intercept of line r, (0, −5), as a point through which line q will pass.

$y - y_1 = m(x - x_1)$ Point-slope form

$[y - (-5)] = \frac{1}{3}(x - 0)$ $x_1 = 0$, $y_1 = -5$, and $m = \frac{1}{3}$

$y = \frac{1}{3}x - 5$ Solve

Step 2 Solve the system of equations.

Determine the point of intersection of lines t and q.

$t: y = -3x + 6$ $q: y = \frac{1}{3}x - 5$

$-3x + 6 = \frac{1}{3}x - 5$ Substitute.

$6 + 5 = \frac{1}{3}x + 3x$ Group like terms.

$\frac{33}{10} = x$ Solve

Solve for y when $x = \frac{33}{10}$.

$y = \frac{1}{3}\left(\frac{33}{10}\right) - 5$ Substitute $\frac{33}{10}$ for x in the equation for q.

$y = -\frac{39}{10}$ Simplify.

The point of intersection is $\left(\frac{33}{10}, -\frac{39}{10}\right)$ or (3.3, −3.9).

Step 3 Calculate the distance between lines r and t.

Use the Distance Formula to determine the distance between (0, −5) and (3.3, −3.9).

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$= \sqrt{(3.3 - 0)^2 + [-3.9 - (-5)]^2}$ $x_2 = 3.3$, $x_1 = 0$, $y_2 = -3.9$, and $y_1 = -5$

≈ 3.5 Use a calculator.

The distance between the lines is about 3.5 units.

Check

Find the distance between parallel lines a and b with equations $x + 3y = 6$ and $x + 3y = -14$, respectively. Round to the nearest hundredth, if necessary.

? 6.32 units

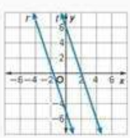
Go Online You can complete an Extra Example online.

788 Module 12 • Logical Arguments and Line Relationships

Interactive Presentation

Distance Between Parallel Lines

Find the distance between the parallel lines r and t with equations $y = -3x - 5$ and $y = -3x + 6$, respectively.



Example 3

TAP

Students tap to move through the steps of the solution.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Distance Between Parallel Lines

Teaching the Mathematical Practices

2 Create Representations Guide students to write an equation that models the situation in Example 3. Then use the equation to solve the problem.

Questions for Mathematical Discourse

AL What do you know about the distance between two parallel lines? Sample answer: Parallel lines are everywhere equidistant.

OL What is the slope of a line perpendicular to the parallel lines? $\frac{1}{3}$

BL Is the point used in the problem the only point through which to draw the perpendicular? No; sample answer: You could use any point because the distance between the lines is always the same.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–17
2	exercises that use a variety of skills from this lesson	18–24
2	exercises that extend concepts learned in this lesson to new contexts	25–28
3	exercises that emphasize higher-order and critical-thinking skills	29–34

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–27 odd, 29–34
- Extension: Perpendicular Lines in Spherical Geometry
- ALEKS® Parallel and Perpendicular Lines

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–33 odd
- Remediation, Review Resources: Roots
- Personal Tutors
- Extra Examples 1–3
- ALEKS® Radicals

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Roots
- *Quick Review Math Handbook:* Perpendiculars and Distance
- ALEKS® Radicals

Important to Know

Digital Exercise Alert Exercises 23-24 and 32 require constructions and are not available online. To fully address G.CO.12, have students complete these exercises using their books.

Practice

Go Online You can complete your homework online.

Examples 1 and 2

Find the distance between point P and line ℓ.

1. Line ℓ contains points (0, −3) and (7, 4). Point P has coordinates (4, 3). $\sqrt{2}$ or about 1.41 units

2. Line ℓ contains points (11, −1) and (−3, −11). Point P has coordinates (−1, 1). $\sqrt{74}$ or about 8.60 units

3. Line ℓ contains points (−2, 1) and (4, 1). Point P has coordinates (5, 7). 6 units

4. Line ℓ contains points (4, −1) and (4, 9). Point P has coordinates (1, 6). 3 units

5. Line ℓ contains points (1, 5) and (4, −4). Point P has coordinates (−1, 1). $\sqrt{10}$ or about 3. 16 units

6. Line ℓ contains points (−8, 1) and (3, 1). Point P has coordinates (−2, 4). 3 units

7. DESIGN Dante is designing a poster for prom using a design program with a coordinate grid. He starts by creating a geometric border. Dante wants the text on the poster to be at least 3 inches away from the top left-hand corner of the border. The border contains the points (0, 7) and (7, 14). If Dante places the text at (7, 8), is the text at least 3 inches away from the border? If yes, how far away is the text from the border? Let every unit represent an inch. Round your answer to the nearest hundredth, if needed. yes; 4.24 in.

8. PHYSICS Mrs. Holmes's physics class is using 3D-printing software to create miniature bridges that can hold at least 5 pounds. Teams will print multiple parts of the bridges and then assemble the parts. One team wants there to be at least 6 inches between the upper rail and the lower rail of the bridge. The lower rail of the bridge contains the points (2, 10) and (16, 3). If the upper rail contains the point (5, 1), will the bridge meet the team's specifications? If yes, how far apart are the rails? Let every unit represent an inch. Round your answer to the nearest hundredth, if needed. yes; 6.71 in.

Example 3

Find the distance between each pair of parallel lines with the given equations.

9. $y = 7$
$y = -1$
8 units

10. $x = -6$
$x = 5$
11 units

11. $y = 3x$
$y = 3x + 10$
$\sqrt{10}$ or about 3. 16 units

12. $y = -5x$
$y = -5x + 26$
$\sqrt{26}$ or about 5. 10 units

13. $y = x + 9$
$y = x + 3$
$3\sqrt{2}$ or about 4.24 units

14. $y = -2x + 5$
$y = -2x - 5$
$2\sqrt{5}$ or about 4.47 units

15. $y = \frac{1}{4}x + 2$
$4y - x = -60$
$4\sqrt{17}$ or about 16.49 units

16. $3x + y = 3$
$y + 17 = -3x$
$2\sqrt{10}$ or about 6.32 units

17. $y = -\frac{5}{4}x + 3.5$
$4y + 10.6 = -5x$
$\sqrt{14.76}$ or about 3.84 units

Mixed Exercises

Find the distance from the line to the given point.

18. $y = -3$; (5, 2)
5 units

19. $y = \frac{1}{6}x + 6$; (−6, 5)
0 units

20. $x = 4$; (−2, 5)
6 units

21. TELEPHONE WIRES Isaiah works for a telephone company. He rewired some telephone wires on a pole. How can Isaiah use perpendicular distances to confirm that the wires are parallel? Sample answer: Isaiah can measure the perpendicular distance between the wires in two different places. If the distances are equal, then the wires are parallel.

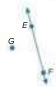
lkpro/Shutterstock

22. STATE YOUR ASSUMPTION A city planner is designing a new park using a map of the city on a coordinate plane. The planner wants the entrance of the park to be at least 4 meters away from Washington Avenue. On the map, Washington Avenue contains the points (2, −4) and (11, −1).

a. If the city planner wants to build the entrance of the park at (3, 3), will the entrance be at least 4 meters away from Washington Avenue? If yes, how far away will the entrance be from the street? Let every unit represent 1 meter. Round your answer to the nearest hundredth, if needed. **yes; 6.32 m**

b. What assumption did you make while solving this problem?
Sample answer: I assumed that the section of Washington Avenue between the given points was straight.

23. Construct the line through G perpendicular to $\overleftrightarrow{EF}$.



See margin.

24. Construct the line through P perpendicular to m.

See margin.

25. REASONING The diagram at the right shows the path that Mark walked from the tee box to where his ball landed on the green. Is the path the shortest possible one from the tee box to the golf ball? Explain why or why not.
No; sample answer: A path that is perpendicular to the tee box would be the shortest. The angle that the tee box makes with the path that Mark walked is less than 90°, so it is not the shortest possible path.

26. $\overline{AB}$ has a slope of 2 and midpoint $M(3, 2)$. A segment perpendicular to $\overline{AB}$ has midpoint $P(4, -1)$ and shares endpoint B with $\overline{AB}$.

a. Graph the segments. **See margin.**

b. Find the coordinates of A and B. **$A(4, 4)$, $B(2, 0)$**

27. What does it mean if the distance between a point P and a line ℓ is zero? If the distance between two lines is zero?
Sample answer: The point is on the line. The two lines are the same line.

28. PROOF Copy and complete the two-column proof of Theorem 12.24.
Given: ℓ is equidistant to m, and n is equidistant to m.
Prove: $\ell \parallel n$
Proof:

Statements	Reasons
1. ℓ is equidistant to m, and n is equidistant to m.	1. ? **Given**
2. ? **$\ell \parallel m$ and $m \parallel n$**	2. Definition of equidistant
3. ? **slope of ℓ = slope of m, slope of m = slope of n**	3. Definition of parallel lines
4. ? **slope of ℓ = slope of n**	4. Transititve Property of Equality
5. $\ell \parallel n$	5. ? **Definition of parallel lines**

Higher-Order Thinking Skills

29. WRITE Summarize the steps that are necessary to find the distance between a pair of parallel lines given the equations of the two lines.
Sample answer: First, a point on one of the parallel lines is found. Then the line that is perpendicular to the line through the point is found. Then the point of intersection is found between the perpendicular line and the other line that is not used in the first step. Last, the Distance Formula is used to determine the distance between the pair of intersection points. This value is the distance between the pair of parallel lines.

30. PERSEVERE Suppose a line perpendicular to a pair of parallel lines intersects the lines at the points $(a, 4)$ and $(0, 6)$. If the distance between the parallel lines is $\sqrt{5}$, find the value of a and the equations of the parallel lines.
$a = \pm 1$; $y = \frac{1}{2}x + 6$ and $y = \frac{1}{2}x + \frac{7}{2}$ or $y = -\frac{1}{2}x + 6$ and $y = -\frac{1}{2}x + \frac{7}{2}$

31. ANALYZE Determine whether the following statement is *sometimes, always,* or *never* true. Justify your argument.

The distance between a line and a plane can be found.
Sometimes; sample answer: The distance can only be found if the line is parallel to the plane.

32. CREATE Draw an irregular convex pentagon using a straightedge.

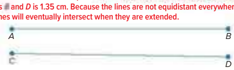
a. Use a compass and straightedge to construct a perpendicular line between one vertex and a side opposite the vertex. **See margin.**

b. Use measurement to justify that the constructed line is perpendicular to the side chosen. **See margin.**

c. Use mathematics to justify this conclusion. **See margin.**

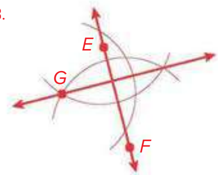
33. WRITE Rewrite Theorem 12.24 in terms of two planes that are equidistant from a third plane. Sketch an example. **See margin.**

34. FIND THE ERROR Harold draws the segments AB and CD shown below using a straightedge. He claims that these two lines, if extended in both directions, will never intersect. Olga claims that the lines will eventually intersect. Who is correct? Explain your reasoning.
Olga; the distance between points A and C is 1.2 cm. The distance between points B and D is 1.35 cm. Because the lines are not equidistant everywhere, the lines will eventually intersect when they are extended.

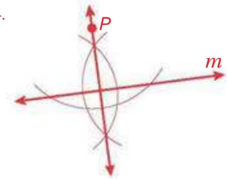


Answers

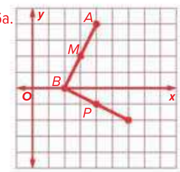
23.



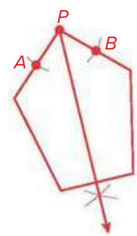
24.



26a.



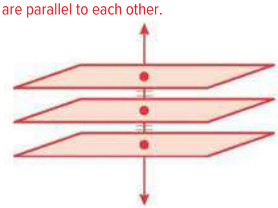
32a. Sample answer:



32b. Sample answer: Using a protractor, the measurement of the constructed angle is equal to 90°. So, the line constructed from vertex P is perpendicular to the nonadjacent chosen side.

32c. Sample answer: The same compass setting was used to construct points A and B. Then the same compass setting was used to construct the perpendicular line to the chosen side. Because the compass setting was equidistant in both steps, a perpendicular line was constructed.

33. If two planes are each equidistant from a third plane, then the two planes are parallel to each other.



Review

Rate Yourself

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why are conjectures important in a logical argument?
- Why is it important to understand the truth values of combinations of statements?
- Why is it important to understand the laws of Detachment and Syllogism for understanding logical arguments?
- Why is it important to learn different proof styles?
- Why is it important to know how to use right angle theorems?
- Why is it important to understand and use theorems about parallel lines?
- Why is it important to know how to prove that lines are parallel using angles?

Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

A completed Foldable for this module should include the Key Concepts related to reasoning and proof.

LearnSmart Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on this topic: **Congruence, Proof, and Constructions.**

- Prove Geometric Theorems

Module 12 • Logical Arguments and Line Relationships

Review

Essential Question

What makes a logical argument, and how are logical arguments used in geomtetry?

A logical argument is well organized and has statements that can be justified using postulates, theorems, and definitions.

Module Summary

Lessons 12-1 and 12-2

Conjectures and Logical Statements

- To show that a conjecture is not true for all cases, find a counterexample.
- An if-then statement is a compound statement of the form "if *p*, then *q*," where *p* and *q* are statements.
- The converse of a conditional statement is formed by exchanging the hypothesis and conclusion of the conditional statement. The inverse is formed by negating the hypothesis and the conclusion of the conditional statement. The contrapositive is formed by negating the hypothesis and the conclusion of the converse of the conditional statement

Lessons 12-3 and 12-4

Reasoning and Proof

- If $p \rightarrow q$ is a true statement and *p* is true, then *q* is true
- If $p \rightarrow q$ and $q \rightarrow r$ are true statements, then $p \rightarrow r$ is a true statement.
- A postulate or axiom is a statement that is accepted as true without proof.
- A proof contains statements and reasons that are organized to show progression from given information to a conclusion. Proofs can be in a two-column format, a flow format (using boxes and arrows), or in a paragraph format.

Lessons 12-5 and 12-6

Proving Segment and Angle Relationships

- The Angle Addition Postulate can be used with other angle relationships to prove theorems about supplementary and complementary angles.
- The properties of algebra that apply to the congruence of segments and the equality of their measures also hold true for the congruence of angles and the equality of their measures.

Lessons 12-7 through 12-10

Relationships Among Angles and Lines

- When two parallel lines are cut by a transversal, there are relationships between specific pairs of angles.
- If two lines are cut by a transversal so corresponding angles are congruent, then the lines are parallel.
- The distance between a line and a point not on the line is the length of the segment perpendicular to the line from the point.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

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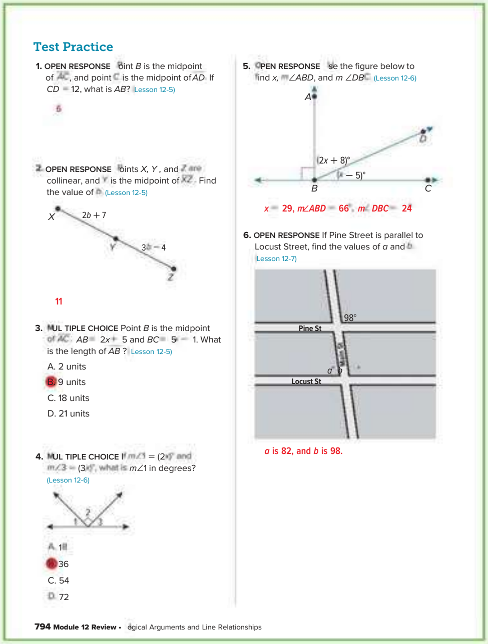

Test Practice

1. **OPEN RESPONSE** Point B is the midpoint of $\overline{AC}$, and point C is the midpoint of $\overline{AD}$. If $CD = 12$, what is AB? (Lesson 12-5)

   6

2. **OPEN RESPONSE** Points X, Y, and Z are collinear, and Y is the midpoint of $\overline{XZ}$. Find the value of b. (Lesson 12-5)

   11

3. **MULTIPLE CHOICE** Point B is the midpoint of $\overline{AC}$. $AB = 2x + 5$ and $BC = 5x - 1$. What is the length of AB? (Lesson 12-5)

   A. 2 units
   B. 9 units
   C. 18 units
   D. 21 units

4. **MULTIPLE CHOICE** If $m\angle 1 = (2x)°$ and $m\angle 3 = (3x)°$, what is $m\angle 1$ in degrees? (Lesson 12-6)

   A. 18
   B. 36
   C. 54
   D. 72

5. **OPEN RESPONSE** Use the figure below to find x, $m\angle ABD$, and $m\angle DBC$. (Lesson 12-6)

   $x = 29$, $m\angle ABD = 66°$, $m\angle DBC = 24°$

6. **OPEN RESPONSE** If Pine Street is parallel to Locust Street, find the values of a and b. (Lesson 12-7)

   a is 82, and b is 98.

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Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Put It All Together: Lessons 12-1 through 12-3
Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–15 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	3, 4, 12, 15
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	7, 13
Table Item	Students complete a table by entering in the correct values.	9
Open Response	Students construct their own response.	1, 2, 5, 6, 8, 10, 11, 14

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
G.CO.1	12-7	7
G.CO.9	12-5, 12-6, 12-7, 12-9	1–6, 11–13
G.CO.12	12-10	15
G.GPE.5	12-8	8–10
G.MG.3	12-10	14

7. **MULTI-SELECT** Select all the statements that describe parallel lines. (Lesson 12-7)

A. If lines are parallel, then they are coplanar.

B. If lines are parallel, then they are not coplanar.

C. If lines are parallel, then they intersect.

D. If lines are parallel, then they do not intersect.

E. If lines are parallel, then they are not skew.

8. **OPEN RESPONSE** A line passes through points at (9, 5) and (4, 3). What is the slope of the line perpendicular to this line? (Lesson 12-8)

$-\frac{5}{2}$

9. **OPEN RESONSE** Three lines have these equations:

Line m: $y = \frac{2}{3}x - 7$

Line n: $y = -\frac{2}{3}(x + 1)$

Line p: $y = -\frac{3}{2}x + 4$

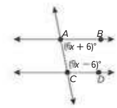
Identify the lines that have a perpendicular relationship. (Lesson 12-8)

Lines m and p

10. **OPEN RESPONSE** Write an equation in slope-intercept form for the line that passes through (−3, 2), perpendicular to $y = \frac{1}{2}x + 9$. (Lesson 12-8)

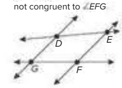
$y = -2x - 4$

11. **OPEN RESPONSE** $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, what is $m\angle ACD$? (Lesson 12-9)



102°

12. **MULTIPLE CHOICE** In the diagram, $\angle GDE$ and $\angle DEF$ are supplementary, but $\angle GDE$ is not congruent to $\angle EFG$.



Which lines are parallel? (Lesson 12-9)

A. $\overleftrightarrow{DE} \parallel \overleftrightarrow{GF}$

B. $\overleftrightarrow{DG} \parallel \overleftrightarrow{EF}$

C. $\overleftrightarrow{DE} \parallel \overleftrightarrow{GF}$ and $\overleftrightarrow{DG} \parallel \overleftrightarrow{EF}$

D. Neither pair of lines is parallel.

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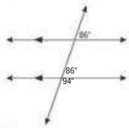
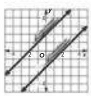
13. **MULTI-SELECT** Using the given figure, which theorem(s) could be used to prove the lines are parallel? Select all that apply. (Lesson 12-9)



A. Alternate Exterior Angles Converse
B. Alternate Interior Angles Converse
C. Consecutive Interior Angles Converse
D. Corresponding Angles Converse
E. Perpendicular Transversal Converse
F. None of the above

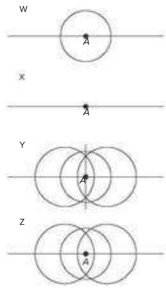
14. **OPEN RESPONSE** Two ships follow the parallel paths shown on the map.



If one unit is 1 nautical mile, what is the shortest distance between the two paths? Round your answer to the nearest tenth. (Lesson 12-10)

2.8 nautical mi

15. **MULTIPLE CHOICE** Which indicates the correct order of steps for the construction of a perpendicular line through a point on the line using dynamic software? (Lesson 12-10)



A. X, W, Y, Z
B. X, W, Z, Y
C. W, X, Y, Z
D. W, X, Z, Y

Lesson 12-2

47.

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Sample answer: Because column 5 is the same as column 8, the conditional is equivalent to its contrapositive. Because column 6 is the same as column 7, the converse and the inverse are equivalent.

48. Sample answer: Because they are logically equivalent, a conditional and its contrapositive always have the same truth value. The inverse and converse of a conditional are also logically equivalent and have the same truth value. The conditional and its contrapositive can have the same truth value as its inverse and converse, or it can have the opposite truth value of its inverse and converse.

50. Sample answer: If four is divisible by two, then birds have feathers. For the converse, inverse, and contrapositive to be true, the hypothesis and the conclusion must both be either true or false.

Lesson 12-4

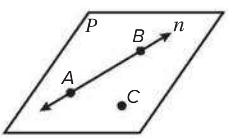
18. Given: $PQ = 4(x - 3) + 1$, $QR = x + 10$, and $x = 7$

Prove: $\overline{PQ} \cong \overline{QR}$

Proof: From the given, $PQ = 4(x - 3) + 1$ and $QR = x + 10$. Because $x = 7$, $PQ = 4(7 - 3) + 1 = 17$ and $QR = 7 + 10 = 17$ by the Substitution Property of Equality. By substitution $PQ = QR$. Any two line segments that have the same length are congruent, so $\overline{PQ} \cong \overline{QR}$.

25. Ana; sample answer: The proof should begin with the given, which is that $\overline{AB}$ is congruent to $\overline{BD}$ and A, B, and D are collinear. Therefore, Ana began the proof correctly.

26. Sample figure shown. Sample answer: It satisfies Postulates 3.1 and 3.3 because points A and B are on line n. It satisfies Postulates 3.2 and 3.4 because 3 points lie in the plane. It satisfies Postulate 3.5 because points A and B lie in plane P, so line n also lies in plane P.



28. Sample answer: When writing a proof, you start with something that you know is true (the given), and then use logic to develop a series of steps that connect the given information to what you are trying to prove.

29. Sometimes; sample answer: If the points were noncollinear, then there would be exactly one plane by Postulate 3.2 shown by Figure 1. If the points were collinear, then there would be infinitely many planes. Figure 2 shows what two planes through collinear points would look like. More planes would rotate around the three points.

Figure 1

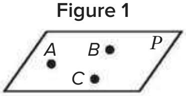
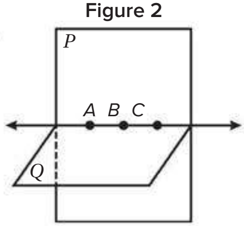
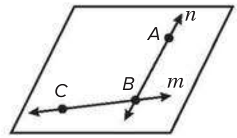


Figure 2



30. Always; sample answer: Because a plane contains at least three noncollinear points and there is exactly one line through any two points, there must be at least two distinct lines in plane.



Lesson 12-5

14. Given: $ABCD$ is a square.

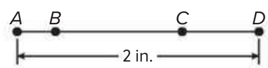
Prove: $\overline{AC} \cong \overline{BD}$

Proof:

Statements (Reasons)

1. $ABCD$ is a square. (Given)
2. $AB = BC = CD = DA$ (Definition of a square)
3. $(AC)^2 = (AB)^2 + (BC)^2$, $(BD)^2 = (AB)^2 + (AD)^2$ (Pythagorean Theorem)
4. $(BD)^2 = (AB)^2 + (BC)^2$ (Substitution Property)
5. $(AC)^2 = (BD)^2$ (Transitive Property of Equality)
6. $AC = \pm\sqrt{(BD)^2}$ (Square Root Property)
7. $AC = BD$ (Definition of square root)
8. $\overline{AC} \cong \overline{BD}$ (Definition of $\cong$ segments)

15. Sample answer:



16. Sample answer: An example of the Transitive Property could be *If $AB = BC$ and $BC = EF$, then $AB = EF$*. However, an example that illustrates the Substitution Property, and cannot be justified using the Transitive Property, would be *If $AB = BC$ and $AB + EF = GH$, then $BC + EF = GH$*.

17. No; sample answer: The Segment Addition Postulate only applies to points that are collinear, but points P, Q, and R are not collinear.

18. Sample answer: Paragraph proofs and two-column proofs both use deductive reasoning presented in a logical order along with the postulates, theorems, and definitions used to support the steps of the proofs. Paragraph proofs are written as a paragraph with the reasons for each step incorporated into the sentences. Two-column proofs are numbered and itemized. Each step of the proof is provided on a separate line with the support for that step in the column beside the step.

19. Because $\overline{PQ} \cong \overline{RS}$ and congruent segments have equal lengths, $PQ = RS$. Because Q is the midpoint of $\overline{PR}$, $PQ = QR$. By the Substitution Property of Equality, $QR = RS$ so R is the midpoint of $\overline{QS}$.

Lesson 12-6

23. Statements (Reasons)

1. $\ell \perp m$ (Given)
2. $\angle 1$ is a right angle. (Def. of $\perp$)
3. $m\angle 1 = 90°$ (Def. of rt. angles)
4. $\angle 1 \cong \angle 4$ (Vert. Angles Thm)
5. $m\angle 1 = m\angle 4$ (Def. of $\cong$ angles)

6. $m\angle 4 = 90°$ (Subs.)

7. $\angle 1$ and $\angle 2$ form a linear pair.

 $\angle 3$ and $\angle 4$ form a linear pair. (Def. of linear pairs)

8. $m\angle 1 + m\angle 2 = 180°$, $m\angle 4 + m\angle 3 = 180°$ (Linear pairs are supp.)

9. $90° + m\angle 2 = 180°$, $90° + m\angle 3 = 180°$ (Subs.)

10. $m\angle 2 = 90°$, $m\angle 3 = 90°$ (Subtraction)

11. $\angle 2$, $\angle 3$, and $\angle 4$ are rt. angles. (Def. of rt. angles (Steps 6, 10))

24. Statements (Reasons)

1. $\ell \perp m$ (Given)

2. $\angle 1$ and $\angle 2$ are rt. angles ($\perp$ lines intersect to form 4 rt. angles.)

3. $\angle 1 \cong \angle 2$ (All rt. angles $\cong$.)

25. Statements (Reasons)

1. $\angle 1 \cong \angle 2$, $\angle 1$ and $\angle 2$ are supplementary. (Given)

2. $m\angle 1 + m\angle 2 = 180°$ (Def. of supp. angles)

3. $m\angle 1 = m\angle 2$ (Def. of $\cong$ angles)

4. $m\angle 1 + m\angle 1 = 180°$ (Subs.)

5. $2(m\angle 1) = 180°$ (Subs.)

6. $m\angle 1 = 90°$ (Div. Prop.)

7. $m\angle 2 = 90°$ (Subs. (steps 3, 6))

8. $\angle 1$ and $\angle 2$ are rt. angles. (Def. of rt. angles)

26. Statements (Reasons)

1. $\angle 1 \cong \angle 2$ (Given)

2. $\angle 1$ and $\angle 2$ form a linear pair. (Given)

3. $\angle 1$ and $\angle 2$ are supplementary. (Linear pairs are suppl.)

4. $\angle 1$ and $\angle 2$ are rt. angles. (If angles $\cong$ and suppl., they are rt. angles.)

28. 120°

$m\angle A + m\angle B = 180°$

$3x + (x + 20) = 180$

$4x + 20 = 180$

$4x = 160$

$x = 40$

$m\angle A = 3x = 3(40°) = 120°$

Because $\angle C \cong \angle A$, $m\angle C = m\angle A$ so $m\angle C = 120°$.

29. Sample answer: $m\angle WXY = 90°$

Given: $m\angle WXZ = 45°$, $\angle WXZ \cong \angle YXZ$

Prove: $m\angle WXY = 90°$

Proof:

Statements (Reasons)

1. $m\angle WXZ = 45°$, $\angle WXZ \cong \angle YXZ$ (Given)

2. $m\angle WXZ = m\angle YXZ$ (Def. of $\cong$ $\angle s$)

3. $m\angle YXZ = 45°$ (Substitution)

4. $m\angle WXY = m\angle WXZ + m\angle YXZ$ (Angle Add. Post.)

5. $m\angle WXY = 45° + 45°$ (Substitution)

6. $m\angle WXY = 90°$ (Substitution)

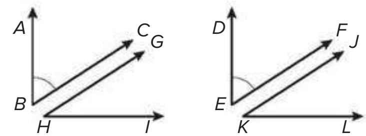
30. Sample answer: First, show that $BC = CD$ and $BC + CD = BD$. Then use substitution to show that $CD + CD = BD$ and $2CD = BD$. Divide to show that $CD = \frac{1}{2}BD$, so $AB = CD$. That means that $AB \cong CD$

31. Each of these theorems uses the words *or to congruent angles* indicating that this case of the theorem must also be proved true. The first proof of each theorem only addressed the *to the same angle* case of the theorem.

Proof of the Congruent Complements Theorem (Case 2: Congruent Angles)

Given: $\angle ABC \cong \angle DEF$, $\angle GHI$ is complementary to $\angle ABC$, $\angle JKL$ is complementary to $\angle DEF$.

Prove: $\angle GHI \cong \angle JKL$



Proof:

Statements (Reasons)

1. $\angle ABC \cong \angle DEF$, $\angle GHI$ is complementary to $\angle ABC$, $\angle JKL$ is complementary to $\angle DEF$. (Given)

2. $m\angle ABC + m\angle GHI = 90°$, $\angle DEF + \angle JKL = 90°$ (Def. of compl. angles)

3. $m\angle ABC = m\angle DEF$ (Def. of cong. angles)

4. $m\angle ABC + m\angle JKL = 90°$ (Substitution)

5. $90° = m\angle ABC + m\angle JKL$ (Symmetric Property of Equality)

6. $m\angle ABC + m\angle GHI = m\angle ABC + m\angle JKL$ (Transitive Property of Equality)

7. $m\angle ABC - m\angle ABC + m\angle GHI = m\angle ABC - m\angle ABC + m\angle JKL$ (Subtraction Property)

8. $m\angle GHI = m\angle JKL$ (Substitution)

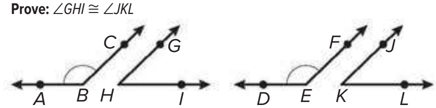
9. $\angle GHI \cong \angle JKL$ (Def. of $\cong$ angles)

Proof of the Congruent Supplements Theorem (Case 2: Congruent Angles)

Given: $\angle ABC \cong \angle DEF$, $\angle GHI$ is supplementary to $\angle ABC$,

$\angle JKL$ is supplementary to $\angle DEF$.

Prove: $\angle GHI \cong \angle JKL$



Proof:

Statements (Reasons)

1. $\angle ABC \cong \angle DEF$, $\angle GHI$ is supplementary to $\angle ABC$, $\angle JKL$ is supplementary to $\angle DEF$. (Given)

2. $m\angle ABC + m\angle GHI = 180°$, $m\angle DEF + m\angle JKL = 180°$ (Def. of suppl. angles)

3. $m\angle ABC = m\angle DEF$ (Def. of cong. angles)

4. $m\angle ABC + m\angle JKL = 180°$ (Substitution)

5. $180° = m\angle ABC + m\angle JKL$ (Symmetric Property of Equality)

6. $m\angle ABC + m\angle GHI = m\angle ABC + m\angle JKL$ (Transitive Property of Equality)

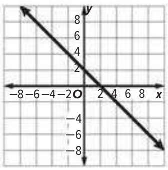
7. $m\angle ABC - m\angle ABC + m\angle GHI = m\angle ABC - m\angle ABC + m\angle JKL$ (Subtraction Property)

8. $m\angle GHI = m\angle JKL$ (Substitution)

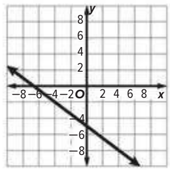
9. $\angle GHI \cong \angle JKL$ (Def. of $\cong$ angles)

Lesson 12-8

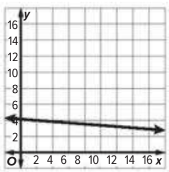
26. $x = 8$



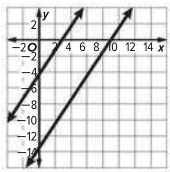
27. $y = -8$



28. $x = 15$



29. $y = 0$



32. Yes, $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$ because the slope of $\overleftrightarrow{AB}$ is $-\frac{1}{3}$, the slope of $\overleftrightarrow{AC}$ is 3, and $-\frac{1}{3} \cdot 3 = -1$.

33. No; none of the slopes are equal, and no two of the slopes have a product of -1.

34. Yes; $\overleftrightarrow{PQ} \parallel \overleftrightarrow{TU}$ because both lines have a slope of $-\frac{2}{3}$.

35c. Both have a slope of $\frac{1}{3}$ because both are perpendicular to Ford and 6th, and the slope of a perpendicular is given by the negative reciprocal.

36. Sample answer: $D(4, 0)$. The slope of $\overleftrightarrow{AB}$ is $\frac{4-1}{6-(-3)} = \frac{1}{3}$; $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, so the slope of $\overleftrightarrow{CD}$ must also be $\frac{1}{3}$. To find the possible coordinates for D, start at $C(1, -1)$ and move 3 units right and 1 unit up. The point is $D(4, 0)$.

37. $S\left(0, -5\frac{1}{2}\right)$; The slope of $\overline{QR}$ is $\frac{2-4}{3-(-2)} = -\frac{2}{5}$ so the slope of $\overline{RS}$ is $\frac{5}{2}$. Let the coordinates of S be $(0, y)$ because S must be on the y-axis. Solve $\frac{5}{2} = \frac{y-2}{0-3}$ for y. $y = -5\frac{1}{2}$, so the coordinates of S are $\left(0, -5\frac{1}{2}\right)$.

40b. No; sample answer: Because line n is parallel to line m, its slope must be $-\frac{3}{2}$, but any line with the equation $2x + 3y = k$ would have a slope of $-\frac{2}{3}$ because $2x + 3y = k$ can be rewritten in slope-intercept form as $y = -\frac{2}{3}x + \frac{k}{3}$.

41b. Sample answer: The slopes of $\overline{AB}$ and $\overline{DC}$ are undefined, so they are parallel to each other. The slopes of $\overline{AD}$ and $\overline{BC}$ are 0, so they are parallel to each other.

41c. Sample answer: Because the slope of $\overline{AB}$ is undefined and the slope of $\overline{BC}$ is zero, the lines are perpendicular to each other. Therefore, they form a right angle, which measures 90°. The same logic applies to all of the sides.

43. Y es; the slope of the line through the points $(-2, 2)$ and $(2, 5)$ is $\frac{3}{4}$. The slope of the line through the points $(2, 5)$ and $(6, 8)$ is $\frac{3}{4}$. Because these lines have the same slope and have a point in common, their equations would be the same. Therefore, all the points are on the same line, and all the points are collinear.

45. Two nonvertical lines are parallel if and only if they have the same slope. Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

46. Disagree; sample answer: The student calculated the value of b incorrectly. The student should have substituted $x = 1$ and $y = 4$ and written $4 = 2(1) + b$, which means $b = 2$. So the correct equation of the line is $y = 2x + 2$.

Module 13

Transformations and Symmetry

Module Goals

- Students perform and use rigid motions including rotations, translations, and reflections.
- Students perform and use compositions of transformations.
- Students explore symmetry using transformations.

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Also addresses G.CO.3, G.CO.4, and G.CO.6.

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Be Sure to Cover

To completely cover G.CO.5, go online to assign the following activities:

- Reflect a Figure in a Line (Construction, Lesson 13-1)
- Determining Congruence with Reflections (Tracing Activity, Lesson 13-1)
- Representing Reflections (Tracing Activity, Lesson 13-1)
- Determining Congruence with Translations (Tracing Activity, Lesson 13-2)
- Representing Translations (Tracing Activity, Lesson 13-2)
- Determining Congruence with Rotations (Tracing Activity, Lesson 13-3)
- Rotating About a Point That is Not the Origin (Tracing Activity, Lesson 13-3)
- Representing Compositions of Transformations (Tracing Activity, Lesson 13-4)

Coherence

Vertical Alignment

Previous
Students described the effect of transformations on two-dimensional figures using coordinates.
8.G.3

Now
Students perform transformations on two-dimensional figures.
G.CO.3

Next
Students will use the definition of congruence in terms of rigid motions to show that two triangles are congruent and use the congruence criteria to solve problems and prove relationships.
G.CO.7, G.CO.8, G.SRT.5

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Practice.

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION
EXPLORE	LEARN	EXAMPLE & PRACTICE

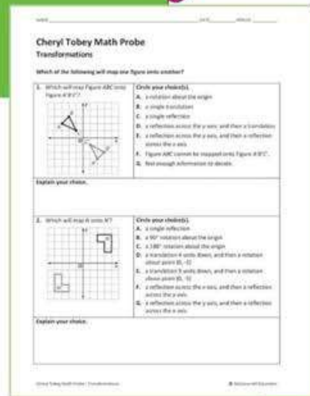
Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
13-1 Reflections	G.CO.4, G.CO.5, G.CO.6	1	0.5
13-2 Translations	G.CO.4, G.CO.5, G.CO.6	1	0.5
13-3 Rotations	G.CO.4, G.CO.5, G.CO.6	1	0.5
13-4 Compositions of Transformations	G.CO.5, G.CO.6	2	1
13-5 Tessellations	G.CO.4, G.CO.5	1	0.5
13-6 Symmetry	G.CO.3, G.CO.5	1	0.5
Put It All Together: Lessons 13-1 through 13-6		1	0.5
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	11	5.5

CHERYL TOBEY MATH PROBES

Formative Assessment Math Probe
Transformations

Module Resource



Answers: 1. C; 2. C, E, F, and G

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will determine which transformations map one figure onto another and explain their choices.

Targeted Concepts Understand how transformations map preimages onto images.

Targeted Misconceptions

- Students are not able to visualize reflections across $y = x$ and $y = -x$ and/or confuse them with reflecting across one of the axes.
- Students do not check to see whether each point of the preimage is mapped onto corresponding points of the image.
- Students may recognize only one transformation as being correct.
- Students have difficulty visualizing compositions of transformations.
- Students have difficulty with angles of rotation and/or rotating around a point.

Use the Probe after Lesson 13-4.

Collect and Assess Student Answers

If the student selects these responses...	**Then** the student likely...
1. A	is not checking to see whether a rotation will map point B onto B' and point A onto A'. (A rotation in this case will map A onto B' and B onto A'.)
1. D, E	is having difficulty visualizing compositions of transformations.
1. F, G	is having difficulty understanding transformations.
Not choosing 1C **2.** A	is having difficulty visualizing a reflection across the lines $y = x$ and $y = -x$. This often happens when the preimage has horizontal and/or vertical lines.
2. B, D	is having difficulty recognizing the angle of rotation or rotating an image about a point other than the origin. With choice D, they may not be able to visualize a composition of transformations.
Not choosing all of the correct transformations for Item 2.	is unaware that a preimage can be mapped onto an image using various and/or multiple transformations. By looking at the student's explanations, the difficulty can be narrowed down.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS Reflections, Translations, or Rotations
- Lesson 13-4, all Learns, all Examples

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

![Ignite!]

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How are rigid motions used to show geometric relationships?
Sample answer: Rigid motions are used to show that figures are congruent. If no series of rigid motions exists from one figure to another, then the figures are not congruent.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

Focus Students read about transformations and symmetry.

Teach Throughout the module, have students take notes under the tabs of their Foldables while working through each lesson. They should include definitions, terms, and key concepts. Encourage students to record examples of each type of transformation from a lesson on the back of their Foldable.

When to Use It Use the appropriate tabs as students cover each lesson in this module. Students should add to the vocabulary tab during each lesson.

Launch the Module

For this module, the Launch the Module video uses a photograph of a reflection in water to describe rigid motions. Students learn about using transformations in photography, beekeeping, and dance.

Module 13

Transformations and Symmetry

Essential Question
How are rigid motions used to show geometric relationships?

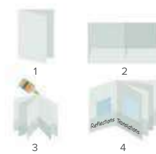
What Will Y ou Learn?
How much do you already know about each topic **before** starting this module?

KEY
— I don't know. — I've heard of it. — I know it!

	Before			After		
	I don't know.	I've heard of it.	I know it!	I don't know.	I've heard of it.	I know it!
define congruence in terms of rigid motions						
reflect figures						
draw and analyze reflected figures						
translate figures						
draw and analyze translated figures						
rotate figures						
draw and analyze rotated figures						
draw and analyze figures under multiple transformations						
identify tessellations						
identify line symmetries in two-dimensional figures						
identify rotational symmetries in two-dimensional figures						

Foldables Make this Foldable to help you organize your notes about transformations and symmetry. Begin with two sheets of paper.

1. **Fold** each sheet of paper in half.
2. **Open** the folded papers and fold each paper lengthwise two inches, to form a pocket.
3. **Glue** the sheets side-by-side to create a booklet.
4. **Label** each of the pockets as shown.



Interactive Presentation

What Vocabulary Will Y ou Learn?

- center of symmetry
- composition of transformations
- glide reflection
- line of symmetry
- line symmetry
- magnitude
- magnitude of symmetry
- order of symmetry
- point of symmetry
- point symmetry
- regular tessellation
- rotational symmetry
- semiregular tessellation
- symmetry
- tessellation
- uniform tessellation

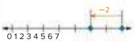
Are Y ou Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review

Example 1

Find the sum of $7 + (-2)$.



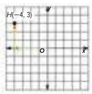
$7 + (-2) = 5$

Example 2

Identify the ordered pair for *H*.

The point is 4 units left and 3 units up.

H is located at $(-4, 3)$.



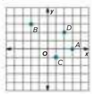
Quick Check

Find each sum.

1. $-9 + (-5)$ -14
2. $6 + (-4)$ 2
3. $1 + (-3)$ -2
4. $-1 + (-7)$ -8

Identify each ordered pair.

5. *A* $(3, 0)$
6. *B* $(-2, 3)$
7. *C* $(1, -1)$
8. *D* $(2, 2)$



How Did Y ou Do?

Which exercises did you answer correctly in the Quick Check?

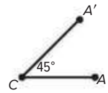
798 Module 13 • Transformations and Symmetry

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define The center of rotation is the fixed point about which an angle of $x°$ maps a point to its image.

Example



Ask What point is the center of rotation? In what direction is the rotation? point *C*; counterclockwise

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- graphing ordered pairs and slope
- graphing ordered pairs and changing coordinates
- identifying translations, rotations, and reflections
- identifying angles formed by parallel lines cut by a transversal

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the **Transformations** section to ensure student success in this module.

Mindset Matters

Growth Mindset vs. Fixed Mindset

Everyone has a core belief or mindset about how they learn. People with a growth mindset believe that hard work will make them smarter. Those who have a fixed mindset believe that they can learn new things, but can't become smarter. A student who changes his or her mindset is more likely to work through challenging problems, to learn from mistakes, and ultimately to learn more deeply.

How Can I Apply It?

Assign students tasks, celebrate mistakes, and provide opportunities for critique, revision, and reflection. The **Explore** activities and discussion prompts are a great tool to begin this journey.

Lesson 13-1

Reflections

G.CO.5, G.CO.6

LESSON GOAL

Students use rigid motions to reflect figures on the coordinate plane.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Developing the Definition of a Reflection

Develop:

Reflections

- Reflection in a Horizontal or Vertical Line
- Reflection in the Line $y = x$

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Proportional Relationships and Slopes	●	●		●
Extension: Reflections in the Coordinate Plane		●	●	●

Language Development Handbook

Assign page 81 of the *Language Development Handbook* to help your students build mathematical language related to reflecting figures on the coordinate plane.

ELL You can use the tips and suggestions on page T81 of the handbook to support students who are building English proficiency.

Mathematical Background

A reflection is a transformation representing a flip of a figure. Reflections can occur in the coordinate plane, allowing you to assign coordinates to each point in the image and preimage.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

G.CO.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
5 Use appropriate tools strategically.
6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students described the effect of reflections on two-dimensional figures using coordinates.
8.G.3

Now
Students use rigid motions to reflect figures on the coordinate plane.
G.CO.5, G.CO.6

Next
Students will use rigid motions to translate figures on the coordinate plane.
G.CO.5, G.CO.6

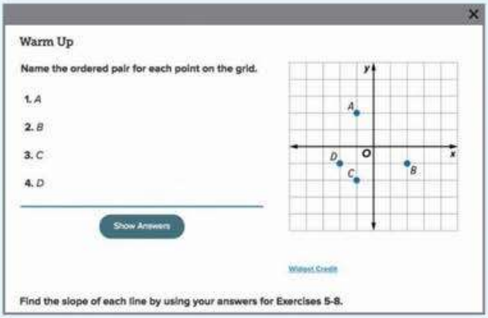
Rigor

The Three Pillars of Rigor

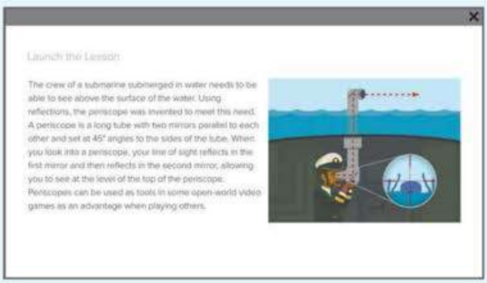
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of transformations in the plane to reflections on the coordinate plane. They build fluency by reflecting figures, and they apply their understanding by solving real-world problems related to reflections.

Interactive Presentation



Warm Up



Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- graphing ordered pairs and slope

Answers:

1. (−1, 2)
2. (2, −1)
3. (−1, −2)
4. (−2, −1)
5. −1
6. undefined
7. 0
8. 3

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of reflections.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Explore Developing the Definition of a Reflection

Objective

Students use dynamic geometry software to explore reflections.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this Explore, students will need to use dynamic geometry software. Work with students to explore and deepen their understanding of reflections.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on his or her device. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

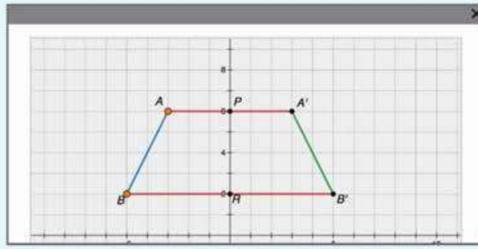
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students use the graph of a reflection of a line segment to answer guiding exercises designed to lead them to writing a definition of a reflection. Students then use their definition to study reflections in the rest of the lesson. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Explore



Explore

TYPE

Students type answers to the guiding exercises.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Developing the Definition of a Reflection (*continued*)

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their mathematical thinking. Point out that they should use clear definitions when they discuss their ideas with others.

Questions

Have students complete the Explore activity.

Ask:

- Over what line is $\overline{AB}$ reflected? *y*-axis
- What does *preserved* mean? Sample answer: *Preserved* means that a property of an object remains unchanged after a transformation.

Inquiry

How can you define a reflection? Sample answer: If $\overline{AB}$ is reflected in the a line, then point A maps to point A' such that $\overline{AA'}$ is perpendicular to the line and the distance from A to the line is the same as the distance from A' to the line. Likewise, $\overline{BB'}$ is perpendicular to the line and point B maps to point B' such that the distance from B to the line is the same as the distance from B' to the line.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Reflections

Objective

Students reflect figures on the coordinate plane and describe the effects of the reflections.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of reflections to develop rules to use when reflecting in specific types of lines.

Things to Know

While the examples presented in this module occur on the coordinate plane, remind students that the properties of rigid motions also apply to figures off the coordinate plane.

Example 1 Reflection in a Horizontal or Vertical Line

Teaching the Mathematical Practices

8 Attend to Details Mathematically proficient students continually ask themselves, "Does this make sense?" Point out that in this example, students will evaluate the reasonableness of their answer.

Questions for Mathematical Discourse

AL In part **a**, how far is point S from the line of reflection? 5 units

OL Which vertical line contains point R? $x = 2$ Which vertical line contains point R'? $x = 2$

BL How far is point R from point R'? 4 units How far is point S from point S'? 10 units In general, how far are points from their images? twice as far as the points are from the line of reflection

Common Error

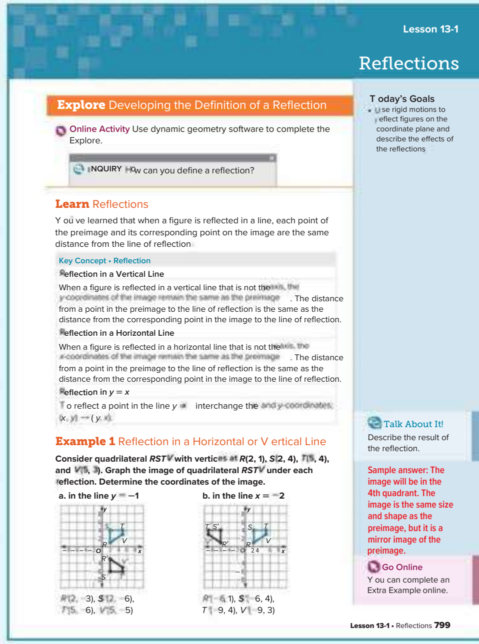
Students may try to reflect the quadrilateral in the x-axis rather than the line $y = -1$. Make sure they remember which is the correct line of reflection.

DIFFERENTIATE

Language Development Activity ELL

Beginning Define the vocabulary words in the module in English and provide examples and explanations. Say the terms aloud and have students repeat the words. Then have students write the word in their notes.

Advanced/Advanced High Allow students to use a search engine to find images for each vocabulary term in the module. Have pairs of students choose a representative image for each term to share with the class. Ask them to explain why their image represents the term.



Lesson 13-1

Reflections

Explore Developing the Definition of a Reflection

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How can you define a reflection?

Learn Reflections

You've learned that when a figure is reflected in a line, each point of the preimage and its corresponding point on the image are the same distance from the line of reflection.

Key Concept • Reflection

Reflection in a Vertical Line

When a figure is reflected in a vertical line that is not the y-axis, the y-coordinates of the image remain the same as the preimage. The distance from a point in the preimage to the line of reflection is the same as the distance from the corresponding point in the image to the line of reflection.

Reflection in a Horizontal Line

When a figure is reflected in a horizontal line that is not the x-axis, the x-coordinates of the image remain the same as the preimage. The distance from a point in the preimage to the line of reflection is the same as the distance from the corresponding point in the image to the line of reflection.

Reflection in $y = x$

To reflect a point in the line $y = x$, interchange the x- and y-coordinates. $(x, y) \rightarrow (y, x)$

Example 1 Reflection in a Horizontal or Vertical Line

Consider quadrilateral $RSTV$ with vertices at $R(2, 1)$, $S(2, 4)$, $T(5, 4)$, and $V(5, 3)$. Graph the image of quadrilateral $RSTV$ under each reflection. Determine the coordinates of the image.

a. in the line $y = -1$

$R'(2, -3)$, $S'(2, -6)$, $T'(5, -6)$, $V'(5, -5)$

b. in the line $x = -2$

$R'(-6, 1)$, $S'(-6, 4)$, $T'(-9, 4)$, $V'(-9, 3)$

Today's Goals
- Use rigid motions to reflect figures on the coordinate plane and describe the effects of the reflections

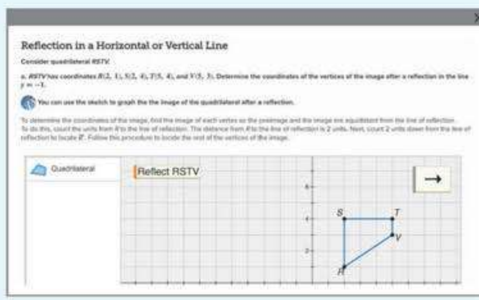
Talk About It!
Describe the result of the reflection.

Sample answer: The image will be in the 4th quadrant. The image is the same size and shape as the preimage, but it is a mirror image of the preimage.

Go Online
You can complete an Extra Example online.

Lesson 13-1 • Reflections 799

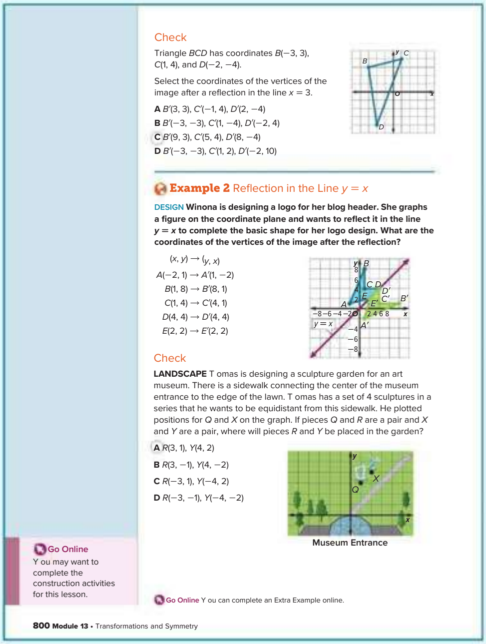
Interactive Presentation



Example 1

TAP

Students tap to reveal steps in locating points in a reflection.



Check

Triangle BCD has coordinates $B(-3, 3)$, $C(1, 4)$, and $D(-2, -4)$.

Select the coordinates of the vertices of the image after a reflection in the line $x = 3$.

A $B'(3, 3)$, $C'(-1, 4)$, $D'(2, -4)$
B $B'(-3, -3)$, $C'(1, -4)$, $D'(-2, 4)$
C $B'(9, 3)$, $C'(5, 4)$, $D'(8, -4)$
D $B'(-3, -3)$, $C'(1, 2)$, $D'(-2, 10)$

Example 2 Reflection in the Line $y = x$

DESIGN **Winona is designing a logo for her blog header. She graphs a figure on the coordinate plane and wants to reflect it in the line $y = x$ to complete the basic shape for her logo design. What are the coordinates of the vertices of the image after the reflection?**

$(x, y) \to (y, x)$
$A(-2, 1) \to A'(1, -2)$
$B(1, 8) \to B'(8, 1)$
$C(1, 4) \to C'(4, 1)$
$D(4, 4) \to D'(4, 4)$
$E(2, 2) \to E'(2, 2)$

Check

LANDSCAPE T omas is designing a sculpture garden for an art museum. There is a sidewalk connecting the center of the museum entrance to the edge of the lawn. T omas has a set of 4 sculptures in a series that he wants to be equidistant from this sidewalk. He plotted positions for Q and X on the graph. If pieces Q and R are a pair and X and Y are a pair, where will pieces R and Y be placed in the garden?

A $R(3, 1)$, $Y(4, 2)$
B $R(3, -1)$, $Y(4, -2)$
C $R(-3, 1)$, $Y(-4, 2)$
D $R(-3, -1)$, $Y(-4, -2)$

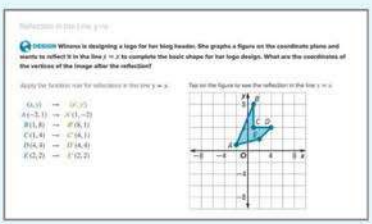
Museum Entrance

Go Online Y ou may want to complete the construction activities for this lesson.

Go Online Y ou can complete an Extra Example online.

800 Module 13 • Transformations and Symmetry

Interactive Presentation



Example 2

TAP

Students tap to reveal steps in a problem or to choose an answer.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Reflection in the Line $y = x$

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about reflections to solving a real-world problem.

Questions for Mathematical Discourse

AL How does reflecting an object in the line $y = x$ affect the coordinates? Sample answer: The x- and y-coordinates are interchanged.

OL How will the size and shape of the image compare to the size and shape of the preimage? Sample answer: Reflections are rigid motions and do not change the lengths of line segments or measures of angles.

BL If you interchange the x- and y-coordinates of the image, what will be the result? the preimage

Common Misconception

When reflecting over an axis, some students think they should multiply by −1 the coordinate that matches the name of the axis. Remind students that reflecting in the x-axis means that the x-coordinate stays the same but the y-coordinate changes. Reflecting in the y-axis means that the y-coordinate stays the same but the x-coordinate changes.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

DIFFERENTIATE

Language Development Activity AL ELL

Allow the class to discuss examples of reflections in nature and in everyday objects that they use. Students can explain where lines of reflection are in objects. Examples from nature could be leaves, flowers, fruits, vegetables, animals, eggs, etc. Everyday objects could be pencils, paper, cars, compact discs, clothing, and so on.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–8
2	exercises that use a variety of skills from this lesson	9–15
3	exercises that emphasize higher-order and critical-thinking skills	16–22

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–15 odd, 16–22
- Extension: Reflections in the Coordinate Plane
- ALEKS Reflections

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–21 odd
- Remediation, Review Resources: Slope of a Line
- Personal Tutors
- Extra Examples 1, 2
- ALEKS Graphing Ordered Pairs and Slope

AL

IF students score 65% or less on the Checks,
THEN assign:

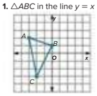
- Practice, Exercises 1–7 odd
- Remediation, Review Resources: Slope of a Line
- *Quick Review Math Handbook*: Reflections
- ALEKS Graphing Ordered Pairs and Slope

Practice

Examples 1 and 2 1–4. See margin for graphs.

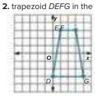
Graph the image of each figure under the given reflection. Determine the coordinates of the image.

1. △*ABC* in the line $y = x$



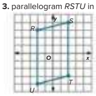
$A'(2, -3)$, $B'(1, 0)$, $C'(-3, -2)$

2. trapezoid *DEFG* in the line $x = -1$



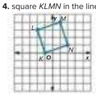
$D'(-2, -3)$, $E'(-3, 3)$, $F'(-5, 3)$, $G'(-6, -3)$

3. parallelogram *RSTU* in the line $y = x$



$R'(3, -2)$, $S'(4, 2)$, $T'(-3, 2)$, $U'(-4, -2)$

4. square *KLMN* in the line $y = -2$



$K'(-1, -4)$, $L'(-2, -7)$, $M'(1, -8)$, $N'(2, -5)$

5. Determine the coordinates of $S(-7, 1)$ after a reflection in the line $y = 3$. $S'(-7, 5)$

6. Determine the coordinates of $Q(6, -4)$ after a reflection in the line $x = 2$. $Q'(-2, -4)$

7. BANNERS Fiona is making a banner in the shape of a triangle for a school project. She graphs the banner on a coordinate plane with vertices at $P(0, 4)$, $Q(2, 8)$, and $R(-3, 6)$. She wants to reflect the banner over the line $x = 1$. Draw the image of the banner reflected in the line $x = 1$. See margin.

8. SANDBOX Aliyah is drawing the top view of a square sandbox on a coordinate plane with vertices at $D(1, 1)$, $E(1, 6)$, $F(6, 6)$, and $G(6, 1)$. She wants to change the location of the sandbox so that it is in the shade. She reflects the sandbox in the line $x = 1$. Find the coordinates of the image of the sandbox. $D'(1, 1)$, $E'(1, 6)$, $F'(-4, 6)$, $G'(-4, 1)$

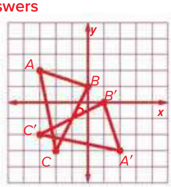
Mixed Exercises

9. Determine the coordinates of $W(-7, 4)$ after a reflection in the line $y = 9$. $W'(-7, 14)$

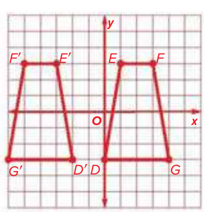
Lesson 13-1 • Reflections 801

Answers

1.



2.



Graph each figure and its image under the given reflection. 10–13. See margin.

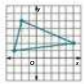
10. rectangle *ABCD* with vertices $A(-5, 2)$, $B(1, 2)$, $C(1, -1)$, and $D(-5, -1)$ in the line $y = -2$

11. $\triangle FGH$ with vertices $F(-3, 2)$, $G(-4, -1)$, and $H(-6, -1)$ in the line $y = x$

12. $\triangle STU$ with vertices $S(-3, -2)$, $T(-2, 3)$, and $U(2, 2)$ in the line $y = x$

13. $\triangle CDE$ with vertices $C(-3, 6)$, $D(-1, 1)$, and $E(3, 5)$ in the line $y = 3x$

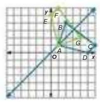
14. Naveen plotted his triangular garden on a coordinate plane. What are the vertices of the image of his garden if it is reflected in the line $y = x$? (2, 5), (5, −2), (1, −3)



15. The image of $A(-1, 1)$ after a reflection is $A'(-1, -3)$. Which reflection produces the image of *A*? reflection in the line $y = -1$

Higher-Order Thinking Skills

16. FIND THE ERROR For the graph at the right, Evelyn maintains that *AEFG* is a reflection of *ABCD* because it fits the definition of a reflection in the line $y = x$. She reasons that *A* is the same point in each figure because it is on the line of reflection and the remaining vertices are equidistant from that line. Do you agree with Evelyn's analysis? Explain your reasoning. See margin.



17. CREATE Create five points on the coordinate plane to form the letter M. Find their image under a reflection in the line $y = x$.
Sample answer: The M can be represented with the points (0, 0), (0, 3), (1, 1), (2, 0), and (2, 3). Reflecting in the line $y = x$ gives (0, 0), (3, 0), (1, 1), (0, 2), and (3, 2).

18. WRITE Describe how to reflect a figure not on the coordinate plane in a line. See margin.

19. PERSEVERE A point in the second quadrant with coordinates $(-a, b)$ is reflected in the line $y = -x$. What are the coordinates of the image? $(-b, -a)$

20. ANAL YZE Is the image of a point reflected in a line *sometimes*, *always*, or *never* located on the other side of the line of reflection? Justify your argument. Sometimes; sample answer: If the point is located on the line of reflection, then the point will remain in its same location.

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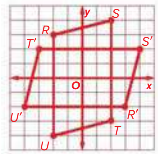
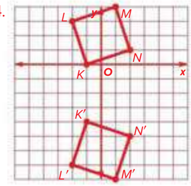
Answers

3.



4.



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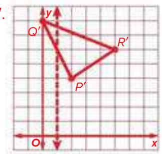
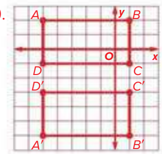
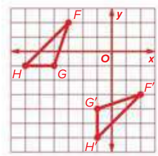


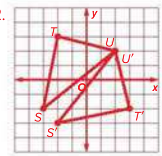
10.



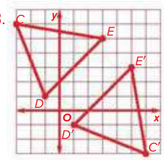
11.



12.



13.



16. No; sample answer: Evelyn is correct that the point on the line of reflection will stay on the line when it is reflected, and she is correct that there are three other pairs of points equidistant to the line of reflection. However, they are not corresponding points. We cannot say that *AEFG* is a reflection of *ABCD*, but we could say that *AGFE* is a reflection of *ABCD*.

18. Sample answer: Draw a line through each vertex of the image that is perpendicular to the line of reflection. Next, measure the distance from each vertex to the line of reflection. Locate each vertex the same perpendicular distance on the perpendicular from the opposite side of the line. Connect each of the vertices to form the reflected image.

Lesson 13-2

Translations

G.CO.5, G.CO.6

LESSON GOAL

Students use rigid motions to translate figures on the coordinate plane.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Developing the Definition of a Translation

Develop:

Translations
- Determine a Translation Vector
- Translations on the Coordinate Plane

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL OL BL ELL			
Remediation: Graph Translations	● ●			●
Extension: Reflections over Parallel Lines		● ●		●

Language Development Handbook

Assign page 82 of the *Language Development Handbook* to help your students build mathematical language related to translating figures on the coordinate plane.

ELL You can use the tips and suggestions on page T82 of the handbook to support students who are building English proficiency.

Mathematical Background

A translation is a transformation that moves all points of a figure the same distance in the same direction. Translations on the coordinate plane can be drawn if you know the direction and how far the figure is moving horizontally and/or vertically. One way to translate a figure in the coordinate plane is to count units on the x-axis and on the y-axis, similar to counting for slope.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.5 Given a geometric figure and a rotation, reflection or translation, draw the transformed figure using: e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

G.CO.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
5 Use appropriate tools strategically.
6 Attend to precision.
8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students described the effect of translations on two-dimensional figures using coordinates.
8.G.3

Now
Students translate figures on the coordinate plane.
G.CO.5, G.CO.6

Next
Students will rotate figures around points on the coordinate plane.
G.CO.5, G.CO.6

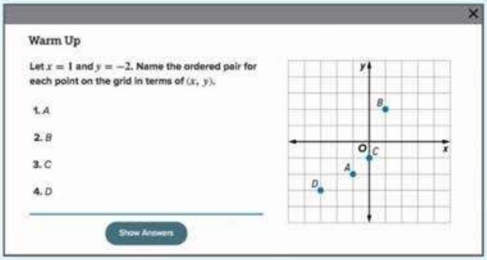
Rigor

The Three Pillars of Rigor

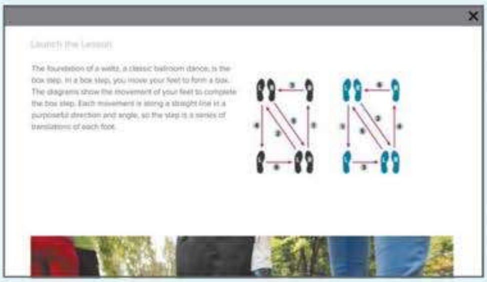
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of transformations in the plane to translations on the coordinate plane. They build fluency by translating figures, and they apply their understanding by solving real-world problems related to translations.

Interactive Presentation



Warm Up



Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- graphing ordered pairs and changing coordinates

Answers:

1. $(-x, y)$
2. $(x, -y)$
3. $(x - 1, y + 1)$
4. $(x - 4, y - 1)$
5. right
6. up
7. left

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of translations.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then discuss the questions below with the class.

Explore Developing the Definition of a Translation

Objective

Students use dynamic geometry software to explore translations.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

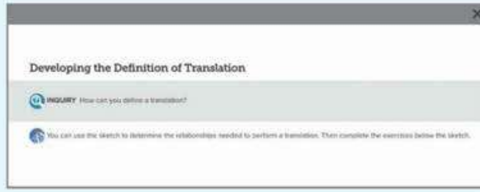
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use the graph of a translation of a line segment to answer guiding exercises designed to lead them to writing a definition of a translation. Students will then use their definition to study translations in the rest of the lesson. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

WEB SKETCHPAD

Students use a sketch to explore the definition of a translation.

TYPE

Students type answers to the guiding exercises.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Developing the Definition of a Translation (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How can you verify that $\overline{AB}$ and $\overline{A'B'}$ are congruent? Sample answer: You can compute their lengths and see that they are the same.
- How do the slopes of $\overline{AB}$ and $\overline{A'B'}$ compare? Sample answer: The slopes are the same.

Inquiry

How can you define a translation if $\overline{AB}$ is translated along a vector? Sample answer: If $\overline{AB}$ is translated along a vector, then point A maps to point A' such that the distance from A to A' is the same as the distance from B to B', both of which are the same length as the magnitude of the vector. The vector is parallel to both $\overline{AB}$ and $\overline{A'B'}$.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Translations

Objective
Students determine the translation vector.

Teaching the Mathematical Practices
7 Use Structure Help students to explore the structure of translations in this Learn to determine if figures were translated.

Common Misconception
Students may confuse the terms *translation* and *transformation* as the words are similar. Work with students to understand the relationship between the terms and the differences between the terms.

Example 1 Determine a Translation Vector

Teaching the Mathematical Practices
8 Look for a Pattern Help students to see the pattern in Example 1.

Questions for Mathematical Discourse

AL What is a vector? a quantity that has both magnitude and direction

OL Does it matter which vertex you choose to check first? Explain. Yes; sample answer: If you check the vertex with a different length to its image first, then you will only need to check one other vertex.

BL What would happen if you translated the image along $\langle 4, 2 \rangle$? Sample answer: The translation of the image would be in the same place as the preimage.

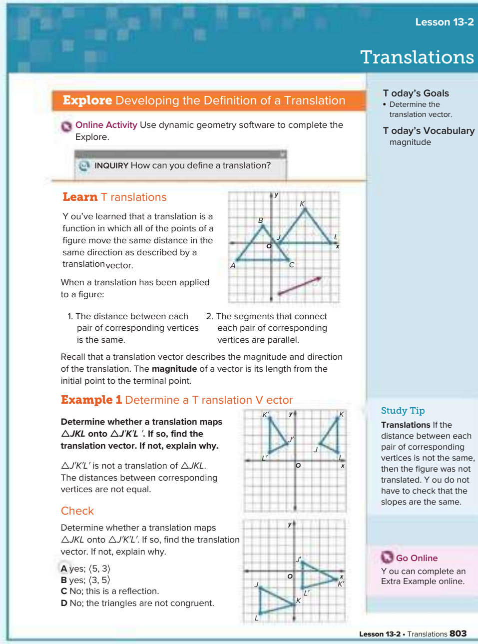
Common Error
Students may try to compute the translation vector starting from the image rather than the original point. Make sure that students are considering the points in the correct order.

DIFFERENTIATE

Reteaching Activity AL
Have students graph images and have them translate them on the coordinate plane.

Reteaching Activity AL ELL
Kinesthetic Learners Create three or four large coordinate grids using poster board. Provide several laminated shapes, such as rectangles, hexagons, pentagons, and trapezoids. Students can practice physically translating shapes on the grids. Students can use examples of translations in the lesson or create their own.

Go Online
- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 13-2

Translations

Explore Developing the Definition of a Translation

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How can you define a translation?

Learn Translations

You've learned that a translation is a function in which all of the points of a figure move the same distance in the same direction as described by a translation vector.

When a translation has been applied to a figure:

1. The distance between each pair of corresponding vertices is the same.
2. The segments that connect each pair of corresponding vertices are parallel.

Recall that a translation vector describes the magnitude and direction of the translation. The **magnitude** of a vector is its length from the initial point to the terminal point.

Example 1 Determine a Translation Vector

Determine whether a translation maps $\triangle JKL$ onto $\triangle J'K'L'$. If so, find the translation vector. If not, explain why.

$\triangle J'K'L'$ is not a translation of $\triangle JKL$. The distances between corresponding vertices are not equal.

Check

Determine whether a translation maps $\triangle JKL$ onto $\triangle J'K'L'$. If so, find the translation vector. If not, explain why.

A yes; $\langle 5, 3 \rangle$
B yes; $\langle 3, 5 \rangle$
C No; this is a reflection.
D No; the triangles are not congruent.

Today's Goals
- Determine the translation vector.

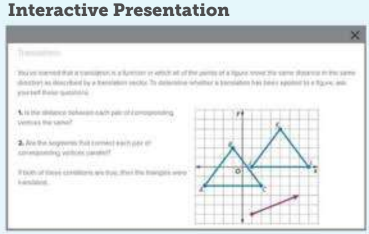
Today's Vocabulary
magnitude

Study Tip
Translations If the distance between each pair of corresponding vertices is not the same, then the figure was not translated. You do not have to check that the slopes are the same.

Go Online
You can complete an Extra Example online.

Lesson 13-2 • Translations 803

Interactive Presentation



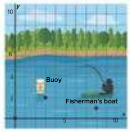
Learn

TAP

Students tap through the slides to determine whether a translation occurs.

Apply Example 2 T ranslations on the Coordinate Plane

SCAVENGER HUNT **Travis is on a scavenger hunt at the lake. He needs to swim to the nearest buoy, pick up a card, and then swim until he reaches a fisherman who will give him the next clue in exchange for the card. Describe the translation from the buoy to the fisherman's boat by using a translation vector.**



Talk About It!
Is the shortest distance between the buoy and the boat the same length as the translation vector? Justify your argument.

Y es; sample answer: By definition, the length of the translation vector is the same as the length of the translation, which was Travis's swim from the buoy to the boat.

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to determine the translation vector that describes Travis's route from the buoy to the fisherman's boat. What are the locations of the buoy and the fisherman's boat? How can I determine the magnitude and direction of the translation vector? I can reread the problem and review the definition of vector to find the magnitude and direction of the translation vector.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will find the coordinates of the buoy and the fisherman's boat. I will determine the length in the x- and y-directions and the general direction of Travis's route. Then, I will determine the translation vector that describes that route. I have learned how to understand points on the coordinate plane, and I have learned how to describe translations using translation vectors.

3 What is your solution?

Use your strategy to solve the problem.

Describe the location of the buoy on the coordinate plane. (3, 2)

Describe the location of the fisherman's boat on the coordinate plane. (8, 1)

Describe the direction of Travis's route. Sample answer: from the buoy toward the fisherman's boat

What is the translation vector that can be used to describe Travis's route? $\langle 5, -1 \rangle$

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: When I apply the translation vector to Travis's position when he is at the buoy, Travis will arrive at the fisherman's boat. Therefore, the translation vector describes Travis's route from the buoy to the fisherman's boat.

Go Online Y ou can complete an Extra Example online.

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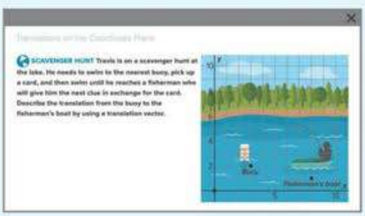
Interactive Presentation



Apply Example 2

TAP

Students tap to reveal steps in a problem or to choose an answer.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Apply Example 2 T ranslations on the Coordinate Plane

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- How far along the x-axis is the boat from the buoy?
- W hat does the direction the boat has to move say about the sign of the sign of the values in the translation vector?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–4
2	exercises that use a variety of skills from this lesson	5–14
3	exercises that emphasize higher-order and critical-thinking skills	15–18

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–14 odd, 15–18
- Extension: Reflections Over Parallel Lines
- ALEKS Translations

IF students score 66%–89% on the Checks, **THEN** assign: OL

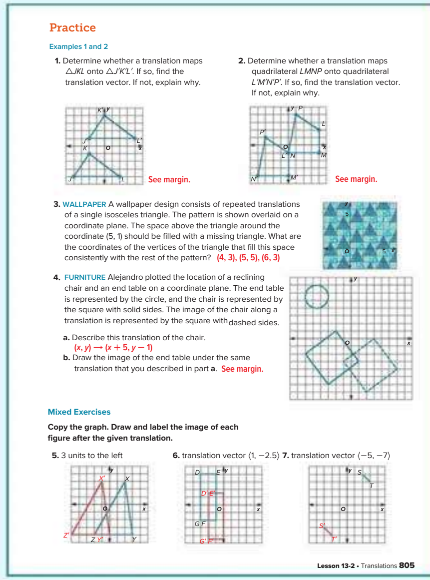
- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Translations
- Personal Tutors
- Extra Examples 1, 2
- ALEKS Translations

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–3 odd
- Remediation, Review Resources: Translations
- *Quick Review Math Handbook*: Translations
- ALEKS Translations

Answers

1. $\triangle J'K'L'$ is a translation of $\triangle JKL$. This translation vector can be represented as $\langle 2, 5 \rangle$.
2. Quadrilateral *LMNP* is a translation of quadrilateral *L'M'N'P'*. This translation vector can be represented as $\langle -4, -3 \rangle$.



Practice

Examples 1 and 2

1. Determine whether a translation maps $\triangle JKL$ onto $\triangle J'K'L'$. If so, find the translation vector. If not, explain why. See margin.

2. Determine whether a translation maps quadrilateral *LMNP* onto quadrilateral *L'M'N'P'*. If so, find the translation vector. If not, explain why. See margin.

3. WALLPAPER A wallpaper design consists of repeated translations of a single isosceles triangle. The pattern is shown overlaid on a coordinate plane. The space above the triangle around the coordinate (5, 1) should be filled with a missing triangle. What are the coordinates of the vertices of the triangle that fill this space consistently with the rest of the pattern? (4, 3), (5, 5), (6, 3)

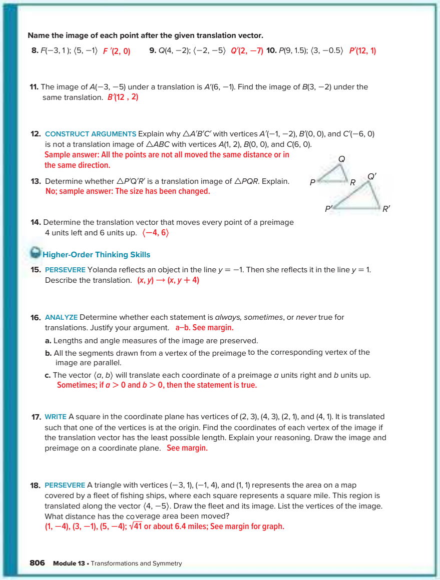
4. FURNITURE Alejandro plotted the location of a reclining chair and an end table on a coordinate plane. The end table is represented by the circle, and the chair is represented by the square with solid sides. The image of the chair along a translation is represented by the square with dashed sides.
   a. Describe this translation of the chair. $(x, y) \rightarrow (x + 5, y - 1)$
   b. Draw the image of the end table under the same translation that you described in part **a**. See margin.

Mixed Exercises

Copy the graph. Draw and label the image of each figure after the given translation.

5. 3 units to the left
6. translation vector $\langle 1, -2.5 \rangle$
7. translation vector $\langle -5, -7 \rangle$

Lesson 13-2 • Translations 805



Name the image of each point after the given translation vector.

8. $F(-3, 1)$; $\langle 5, -1\rangle$ **$F'(2, 0)$** **9.** $Q(4, -2)$; $\langle -2, -5\rangle$ **$Q'(2, -7)$** **10.** $P(9, 1.5)$; $\langle 3, -0.5\rangle$ **$P'(12, 1)$**

11. The image of $A(-3, -5)$ under a translation is $A'(6, -1)$. Find the image of $B(3, -2)$ under the same translation. **$B'(12, 2)$**

12. CONSTRUCT ARGUMENTS Explain why $\triangle A'B'C'$ with vertices $A'(-1, -2)$, $B'(0, 0)$, and $C'(-6, 0)$ is not a translation image of $\triangle ABC$ with vertices $A(1, 2)$, $B(0, 0)$, and $C(6, 0)$. **Sample answer: All the points are not all moved the same distance or in the same direction.**

13. Determine whether $\triangle P'Q'R'$ is a translation image of $\triangle PQR$. Explain. **No; sample answer: The size has been changed.**

14. Determine the translation vector that moves every point of a preimage 4 units left and 6 units up. **$\langle -4, 6\rangle$**

Higher-Order Thinking Skills

15. PERSEVERE Yolanda reflects an object in the line $y = -1$. Then she reflects it in the line $y = 1$. Describe the translation. **$(x, y) \rightarrow (x, y + 4)$**

16. ANALYZE Determine whether each statement is *always*, *sometimes*, or *never* true for translations. Justify your argument. **a–b. See margin.**

a. Lengths and angle measures of the image are preserved.

b. All the segments drawn from a vertex of the preimage to the corresponding vertex of the image are parallel.

c. The vector $\langle a, b\rangle$ will translate each coordinate of a preimage a units right and b units up. **Sometimes; if $a > 0$ and $b > 0$, then the statement is true.**

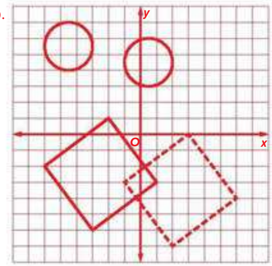
17. WRITE A square in the coordinate plane has vertices of (2, 3), (4, 3), (2, 1), and (4, 1). It is translated such that one of the vertices is at the origin. Find the coordinates of each vertex of the image if the translation vector has the least possible length. Explain your reasoning. Draw the image and preimage on a coordinate plane. **See margin.**

18. PERSEVERE A triangle with vertices (−3, 1), (−1, 4), and (1, 1) represents the area on a map covered by a fleet of fishing ships, where each square represents a square mile. This region is translated along the vector $\langle 4, -5\rangle$. Draw the fleet and its image. List the vertices of the image. What distance has the coverage area been moved?
(1, −4), (3, −1), (5, −4); $\sqrt{41}$ or about 6.4 miles; See margin for graph.

806 Module 13 • Transformations and Symmetry

Answers

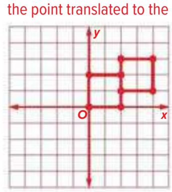
4b.



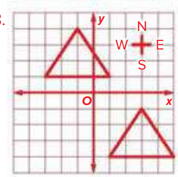
16a. Always; sample answer: Translations move each point of a figure along a vector, the same distance in the same direction, so the figure itself looks the same.

16b. Always; sample answer: Because all the points of the preimage all slide along the same vector that represents the translation, all these lines are parallel.

17. (0, 2), (2, 2), (0, 0), (2, 0); Sample answer: To minimize the length of the vector, I used the vertex closest to the origin, (2, 1), as the preimage for the point translated to the origin.



18.



Lesson 13-3
Rotations

G.CO.5, G.CO.6

LESSON GOAL

Students use rigid motions to rotate figures about points on the coordinate plane.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Developing the Definition of a Rotation

Develop:

Rotations About Points That Are Not the Origin
- Rotation About a Point That Is Not the Origin
- Describe the Effect of a Rotation

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Translations	● ●			●
Extension: Reflections over Intersecting Lines		● ●		●

Language Development Handbook

Assign page 83 of the *Language Development Handbook* to help your students build mathematical language related to rotating figures on the coordinate plane.

ELL You can use the tips and suggestions on page T83 of the handbook to support students who are building English proficiency.

Mathematical Background

A rotation is a transformation that turns every point of a preimage through a specified angle and direction about a fixed point. The fixed point is called the center of rotation. The angle of rotation is the angle formed by a point on the preimage, the center of rotation, and the corresponding point on the rotated image. A rotation exhibits all the properties of isometries, including preservation of distance and angle measure.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.5 Given a geometric figure and a rotation, reflection or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

G.CO.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
5 Use appropriate tools strategically.
6 Attend to precision.
7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students described the effect of rotations on two-dimensional figures using coordinates.
8.G.3

Now
Students use rigid motions to rotate figures about points on a coordinate plane.
G.CO.5, G.CO.6

Next
Students will learn about compositions of transformations and use two or more transformations on the coordinate plane.
G.CO.5, G.CO.6

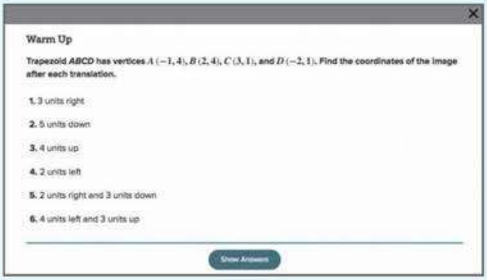
Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of transformations in the plane to rotations on the coordinate plane. They build fluency by rotating figures, and they apply their understanding by solving real-world problems related to rotations.

Interactive Presentation



Warm Up

Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- identifying translations

Answers:

1. $A'(2, 4)$, $B'(5, 4)$, $C'(6, 1)$, and $D'(1, 1)$
2. $A'(-1, -1)$, $B'(2, -1)$, $C'(3, -4)$, and $D'(-2, -4)$
3. $A'(-1, 8)$, $B'(2, 8)$, $C'(3, 5)$, and $D'(-2, 5)$
4. $A'(-3, 4)$, $B'(0, 4)$, $C'(1, 1)$, and $D'(-4, 1)$
5. $A'(1, 1)$, $B'(4, 1)$, $C'(5, -2)$, and $D'(0, -2)$
6. $A'(-5, 7)$, $B'(-2, 7)$, $C'(-1, 4)$, and $D'(-6, 4)$

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of rotations.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Explore Developing the Definition of a Rotation

Objective

Students use dynamic geometry software to explore rotations.

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to solve the problem in this Explore, students will need to analyze the mathematical relationships in the problem and draw a conclusion.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

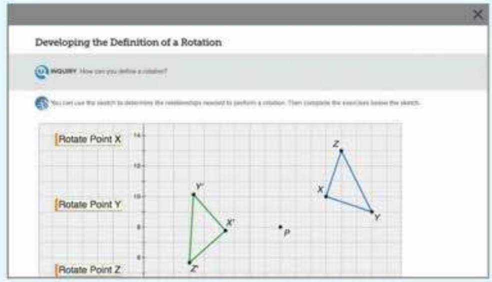
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use the graph of a rotation of a line segment to answer guiding exercises designed to lead them to writing a definition of a rotation. Students will then use their definition to study rotations in the rest of the lesson. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

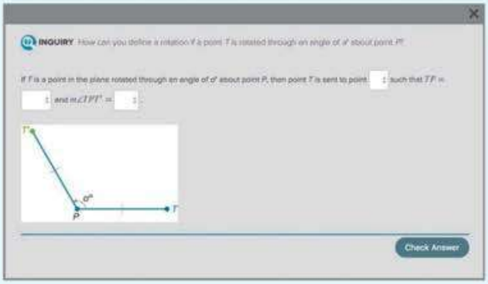
WEB SKETCHPAD

Students use a sketch to reveal information about rotations.

TYPE

Students type answers to the guiding exercises.

Interactive Presentation



Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Developing the Definition of a Rotation (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How does $\triangle XYZ$ compare to $\triangle X'Y'Z'$? Sample answer: The image is turned almost but not quite completely upside down compared to the original triangle.
- How do the slopes of the line segments compare to their images? Sample answer: The slopes are different.

Inquiry

How can you define a rotation if a point T is rotated through an angle of $a°$ about point P? Sample answer: If T is a point in the plane rotated through an angle of $a°$ about point P, then point T is sent to point T' such that $TP = T'P$ and $m\angle TPT' = a°$.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Rotations About Points that Are Not the Origin

Objective

Students use rigid motions to rotate figures about points that are not the origin and describe the effects of the rotations.

MP Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of rotations to develop rules to use when rotating about a point.

Important to Know

Notice that rotations do not change the lengths of line segments or the measures of angles. Like reflections and translations, rotations are rigid motions. So they do not change the size or shape of objects. This will be important later for showing that an image and its preimage in a rotation are congruent.

Example 1 Rotate About a Point That Is Not the Origin

MP Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in Example 1.

Questions for Mathematical Discourse

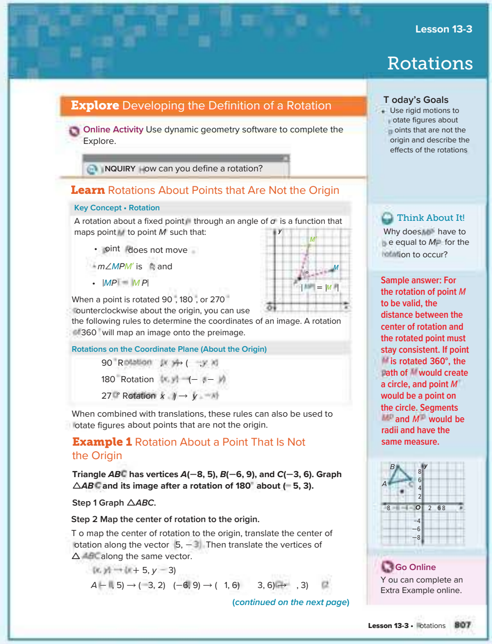
AL How are rotations different from translations? Sample answer: Instead of transforming a figure along a vector, a rotation involves angle rotation around a point.

OL Does it matter which point you choose to rotate first? Explain. No; sample answer: Because the center of rotation is a fixed point, you can rotate the vertices of a figure in any order.

BL What degree of rotation would result in the same image as the preimage? 360°

Common Error

Students may rotate the figure in the wrong direction. Make sure that students are considering the angle in the counterclockwise direction unless specified otherwise.


Lesson 13-3

Rotations

Explore Developing the Definition of a Rotation

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How can you define a rotation?

Learn Rotations About Points that Are Not the Origin

Key Concept • Rotation

A rotation about a fixed point through an angle of α is a function that maps point M to point M' such that:

- point does not move
- $m\angle MPM'$ is and
- $|MP| = |M'P|$

When a point is rotated 90°, 180°, or 270° counterclockwise about the origin, you can use the following rules to determine the coordinates of an image. A rotation of 360° will map an image onto the preimage.

Rotations on the Coordinate Plane (About the Origin)

90° Rotation $(x, y) \to (-y, x)$

180° Rotation $(x, y) \to (-x, -y)$

270° Rotation $(x, y) \to (y, -x)$

When combined with translations, these rules can also be used to rotate figures about points that are not the origin.

Example 1 Rotation About a Point That Is Not the Origin

Triangle ABC has vertices $A(-8, 5)$, $B(-6, 9)$, and $C(-3, 6)$. Graph $\triangle ABC$ and its image after a rotation of 180° about $(-5, 3)$.

Step 1 Graph $\triangle ABC$.

Step 2 Map the center of rotation to the origin.

To map the center of rotation to the origin, translate the center of rotation along the vector $\langle 5, -3 \rangle$. Then translate the vertices of $\triangle ABC$ along the same vector.

$(x, y) \to (x + 5, y - 3)$

$A(-8, 5) \to (-3, 2)$ $(-6, 9) \to (-1, 6)$ $(-3, 6) \to (2, 3)$

(continued on the next page)

Today's Goals
- Use rigid motions to rotate figures about points that are not the origin and describe the effects of the rotations

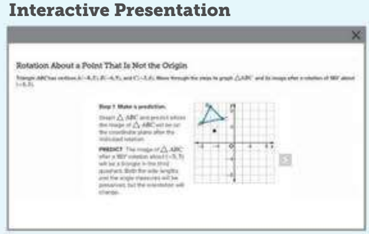
Think About It!
Why does MP have to be equal to $M'P$ for the rotation to occur?

Sample answer: For the rotation of point M to be valid, the distance between the center of rotation and the rotated point must stay consistent. If point M is rotated 360°, the path of M would create a circle, and point M' would be a point on the circle. Segments MP and $M'P$ would be radii and have the same measure.

Go Online
You can complete an Extra Example online.

Lesson 13-3 • Rotations 807

Interactive Presentation



Example 1

TAP

Students move through the steps to graph a rotation.

Step 3 Rotate 180° about the origin.

$(x, y) \rightarrow (-x, -y)$

$A(-3, 2) \rightarrow (3, -2)$

$B(-1, 6) \rightarrow (1, -6)$

$C(2, 3) \rightarrow (-2, -3)$

Step 4 Map the center of rotation to its original position.

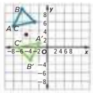
To map the center of rotation to its original position, translate the center of rotation along the vector $\langle -5, 3\rangle$. Then translate the vertices of the rotated triangle along the same vector.

$(x, y) \rightarrow (x - 5, y + 3)$

$A(3, -2) \rightarrow A'(-2, 1)$

$B(1, -6) \rightarrow B'(-4, -3)$

$C(-2, -3) \rightarrow C'(-7, 0)$

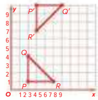


Study Tip

Approximations When you are describing the effects of a rotation, you can approximate the location of the rotated figure without making any calculations. You will know the shape and size of the image because angle measures and lengths are preserved.

Check

Triangle PQR has vertices $P(2, 1)$, $Q(2, 4)$, and $R(5, 1)$. Graph $\triangle PQR$ and its image after a rotation 270° counterclockwise about (7, 5).



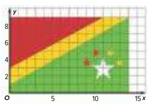
Use a Source

Find an example of a flag that has elements that can be created using a rotation. Describe the center and angle of rotation.

Sample answer: The stars on Venezuela's flag can be created by rotating the bottom left star in the arc 22.5° clockwise. Then repeat the same rotation with each image until there are 8 stars total. The center of rotation would be located at the bottom of the blue stripe and centered horizontally.

Example 2 Describe the Effect of a Rotation

FLAGS Kendrick is working with a team in his social studies class to create a new country and its government. Kendrick is responsible for creating the country's flag. He is using geometry software to design the flag on the coordinate plane. Describe how the two yellow stars would be affected if they were rotated 90° counterclockwise about the center of the white star.

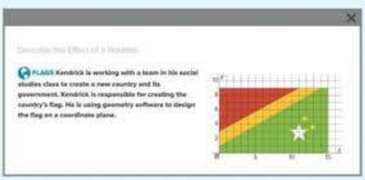


If the stars were rotated, they would curve around the top-left sides of the white star. Together the preimage and the image would create a semicircle of yellow stars above the white star.

Go Online You can complete an Extra Example online.

808 Module 13 • Transformations and Symmetry

Interactive Presentation



Example 2

TAP

Students tap to type answers and to reveal images of the answer.

CHECK

Students complete the Check online to determine whether they are ready to move on.

DIFFERENTIATE

Reteaching Activity AL ELL

Tell students to develop a system for rotating images. First, they should read the problem and locate or plot the figure for visual recognition. They should also carefully note the specifications, especially the direction of the rotation. Finally, they can apply the rotation. Students can use a system similar to this, or they can create their own.

Example 2 Describe the Effect of a Rotation

Teaching the Mathematical Practices

5 Use a Source Guide students to find external information to answer the questions posed in the Use a Source feature.

Questions for Mathematical Discourse

AL How would you describe the counterclockwise direction? Sample answer: A counterclockwise direction is the reverse (or counter) of the hands on a clock.

OL If the two yellow stars were directly above the white star, how would they be affected by this rotation? Sample answer: The yellow stars would be directly to the left of the white star.

BL How would the two yellow stars be affected if they were rotated 90° clockwise around the center of the white star? Sample answer: The yellow stars would be below and to the right of the white star.

DIFFERENTIATE

Enrichment Activity BL

Have students list real-world examples of objects that rotate and discuss the rotational aspect of those objects.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–5
2	exercises that use a variety of skills from this lesson	6–12
3	exercises that emphasize higher-order and critical-thinking skills	13–19

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–11 odd, 13–19
- Extension: Reflections Over Intersecting Lines
- ALEKS Rotations

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–19 odd
- Remediation, Review Resources: Translations
- Personal Tutors
- Extra Examples 1, 2
- ALEKS Translations

AL

IF students score 65% or less on the Checks,
THEN assign:

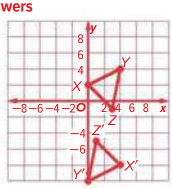
- Practice, Exercises 1–5 odd
- Remediation, Review Resources: Translations
- *Quick Review Math Handbook*: Rotations
- ALEKS Translations

Important to Know

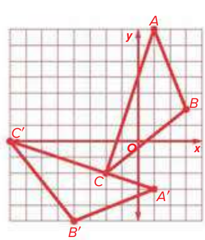
Digital Exercise Alert Exercise 8 requires drawing a transformed figure and is not available online. To fully address G.CO.5, have students complete this exercise using their books.

Answers

1.



2.



Practice

Go Online Y ou can complete your homework online.

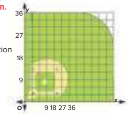
Examples 1 and 2

1. Triangle *XYZ* has vertices *X*(0, 2), *Y*(4, 4), and *Z*(3, −1). Graph △*XYZ* and its image after a rotation of 180° about (2, −3). See margin.

2. Triangle *ABC* has vertices *A*(1, 7), *B*(3, 2), and *C*(−2, −2). Graph △*ABC* and its image after a rotation of 270° counterclockwise about (−4, 2). See margin.

3. Triangle *FGH* has vertices *F*(−3, 4), *G*(2, 0), and *H*(−1, − 2). Graph △*FGH* and its image after a rotation of 180° about (−3, −6). See margin.

4. Quadrilateral *ABCD* has vertices *A*(−2, 4), *B*(1, 3), *C*(2, −3), and *D*(−3, −1). Graph quadrilateral *ABCD* and its image after a rotation of 90° counterclockwise about (−1, 2). See margin.



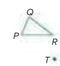
5. BASEBALL A scale drawing of a baseball field is shown on the coordinate plane, where home plate is at (3, 3), first base is at (13, 3), second base is at (13, 13), and third base is at (3, 13). Suppose the baseball field is rotated 270° counterclockwise about second base, what are the coordinates of each base? home plate: (3, 23), first base: (3, 13), second base: (13, 13), third base: (13, 23)

Mixed Exercises

6. Point *Q* with coordinates (4, − 7) is rotated 270° clockwise about (5, 1). What are the coordinates of its image? *Q*′(13, 0)

7. Parallelogram *JKLM* has vertices *J*(2, 1), *K*(7 , 1), *L*(6, −3), and *M*(1, −3). What are the coordinates of the image of *K* if the parallelogram is rotated 270° counterclockwise about (−2, −1)? *K*′(0, −10)

8. USE TOOLS Use a protractor and ruler to draw a rotation of △*PQR* 210° about *T*. See margin.



9. The line segment *XY* with endpoints *X*(3, 1) and *Y*(2, −2) is rotated 90° counterclockwise about (−6, 4). What are the endpoints of $\overline{X'Y'}$? *X*′(−3, 13), *Y*′(0, 12)

10. HIKING A damaged compass points northwest instead of north. If you travel west by the compass, what is your angle of rotation to true north? 45° clockwise or 315° counterclockwise

Lesson 13-3 • Rotations 809

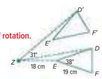
11. A circular dial with the digits 0 through 9 evenly spaced around its edge is rotated clockwise 36°. How many times would you have to perform this rotation in order to bring the dial back to its original position? **10 times**

12. Under a rotation about the origin, the point $A(5, -1)$ is mapped to the point $A'(1, 5)$. What is the image of the point $B(-4, 6)$ under this rotation? Explain. **$B'(-6, -4)$; The rotation maps (x, y) to $(-y, x)$, so it is a 90° rotation. Therefore, the image of $(-4, 6)$ is $(-6, -4)$.**

Higher-Order Thinking Skills

13. CREATE Draw a right triangle ABC and point P not on the triangle.
    a. Rotate triangle ABC about point P 90° counterclockwise. **See margin.**
    b. Name a clockwise rotation that would map triangle ABC onto triangle $A'B'C'$. **Sample answer: 270° clockwise rotation about point P**

14. In the figure, $\triangle D'E'F'$ is the image of $\triangle DEF$ after a rotation about point Z.
    a. What is the distance from E' to Z? Justify your reasoning. **18 cm; A point and its image are the same distance from the center of rotation.**
    b. What is $m\angle FZF'$? Justify your reasoning. **See margin.**

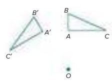


15. ANAL YZE What is the result of a rotation followed by another rotation about the same point? Give an example. **See margin.**

16. FIND THE ERROR Thomas claims that a reflection in the x-axis followed by a reflection in the y-axis is the same thing as a rotation. Is Thomas correct? Explain your reasoning. **See margin.**

17. WRITE Which properties of a figure are preserved under a rotation from the preimage to the image? Explain. **See margin.**

18. FIND THE ERROR Shanice is looking at the figure shown, which shows two congruent triangles. She measures the angle that rotates A to A' about O and finds it to be 30°. She measures the angle that rotates B to B' about O and also finds it to be 30°. She then claims that because the two triangles are congruent, a 30° rotation has occurred about point O. Is Shanice correct? Explain your reasoning. **See margin.**



19. WRITE Are collinearity and betweenness of points maintained under rotations? Explain. **See margin.**

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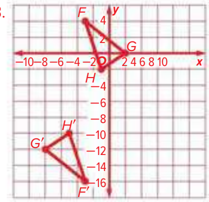
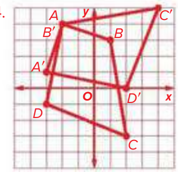
Answers

3.



4.



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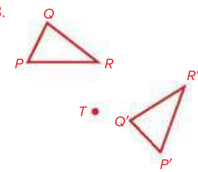
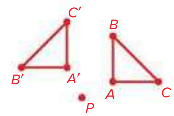


13a. Sample answer:



14b. 31°; Sample answer: The measure of the angle formed by a point, the center of rotation, and the point's image is equal to the angle of rotation, which is 31°.

15. Sample answer: A rotation followed by another rotation is still a rotation. For example, a rotation of 30° clockwise followed by a rotation of 20° counterclockwise is the same as a rotation of 10° clockwise, or 350° counterclockwise. A rotation of 30° counterclockwise followed by a rotation of 15° counterclockwise is the same as a rotation 45° counterclockwise.

16. Yes; sample answer: A reflection in the x-axis followed by a reflection in the y-axis is the same as a rotation of 180° about the origin. This can be seen with the mapping functions. The point (x, y) maps to $(x, -y)$ when reflected in the x-axis. The point $(x, -y)$ maps to $(-x, -y)$ when reflected in the y-axis. So, (x, y) maps to $(-x, -y)$, which is the same as a 180° rotation.

17. Sample answer: Distance is preserved because the lengths of segments remain the same measure. Angle measures are preserved because angle measures remain the same measure. Parallelism is preserved because parallel lines remain parallel. Collinearity is preserved because points remain on the same lines.

18. No; sample answer: Point C has not been rotated 30° around O. It appears that a reflection has occurred in addition to a rotation. The two triangles are congruent, but it is not a rotation that only maps one triangle to the other.

19. Yes; sample answer: A rotation is a transformation that maintains congruence of the original figure and its image. So, the preimage can be mapped onto the image, and corresponding segments will be congruent. Therefore, collinearity and betweenness of points are maintained in rotations.

Lesson 13-4

Compositions of Transformations

G.CO.5, G.CO.6

LESSON GOAL

Students use two or more rigid motions to transform figures on the coordinate plane.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Reflections in Two Lines

Develop:

Compositions of Transformations
- Glide Reflection
- Composition of Isometries

Compositions of Two Reflections
- Reflect a Figure in Two Lines
- Determine Congruence
- Describe Transformations

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Rotations	●	●		●
Extension: Composition of a Translation and a Reflection in a Perpendicular Line		●	●	●

Language Development Handbook

Assign page 84 of the *Language Development Handbook* to help your students build mathematical language related to using two or more rigid motions to transform figures on the coordinate plane.

ELL You can use the tips and suggestions on page T84 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.5 Given a geometric figure and a rotation, reflection or translation, draw the transformed figure using: e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

G.CO.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
2 Reason abstractly and quantitatively.
3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
5 Use appropriate tools strategically.
6 Attend to precision.
7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students reflected, translated, and rotated figures on the coordinate plane.
8.G.3, G.CO.5, G.CO.6

Now
Students determine the image of a figure after two or more transformations have occurred.
G.CO.5, G.CO.6

Next
Students will identify tessellations and transformations in tessellations.
G.CO.4, G.CO.5

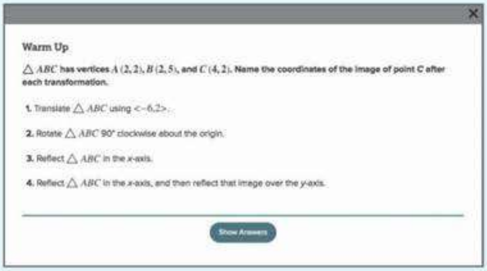
Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

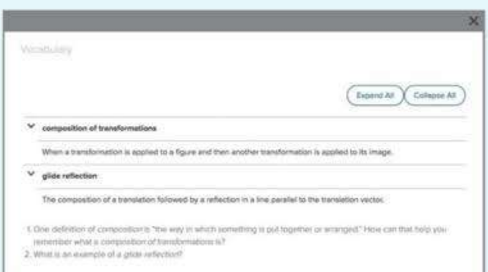
Conceptual Bridge In this lesson, students extend their understanding of transformations on the coordinate plane to compositions of transformations. They build fluency by completing compositions, and they apply their understanding by solving real-world problems related to compositions of transformations.

Interactive Presentation



Warm Up

Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- identifying translations, rotations, and reflections

Answers:

1. (−2, 4)
2. (2, −4)
3. (4, −2)
4. (−4, −2)

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of multiple transformations.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Mathematical Background

When a transformation is applied to a figure, and then another transformation is applied to its image, the resulting transformation is called a composition of transformations. A glide translation is a translation followed by a reflection in a line that is parallel to the translation vector.

Explore Reflections in Two Lines

Objective

Students use dynamic geometry software to explore reflections in two lines.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this Explore, students will need to use dynamic geometry software. Work with students to explore and deepen their understanding of reflections in two lines.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

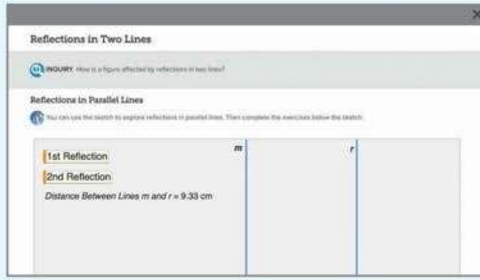
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

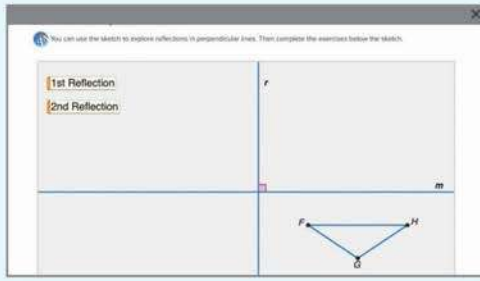
Students will complete guiding exercises throughout the Explore activity. Students first explore what happens to a triangle when it is reflected in one line and then a second line parallel to the first. The second image turns out to be a translation of the original triangle, and the translation vector is perpendicular to the two parallel lines and is twice as long as the distance between them. Students then explore what happens to a triangle when it is reflected in one line and then a second line that is perpendicular to the first. The second image turns out to be a rotation of the original triangle, where the point of rotation is the intersection of the two lines, and the angle of rotation has measure twice the measure of the angle between the two lines where the first image falls. Students then complete the Exercises to help them discover these facts. Then, students will answer the Inquiry Question.

(*continued on the next page*)

Interactive Presentation



Explore



Explore

WEB SKETCHPAD

Students use the sketch to explore the reflections of triangles in two lines.

TYPE

Students type to complete the guiding exercises.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Reflections in Two Lines (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How does the translation vector relate to the two parallel lines? Sample answer: The translation vector is perpendicular to the two parallel lines and is twice as long as the distance between them.
- Where is the rotation point? at the intersection of the two perpendicular lines.

Inquiry

How is a figure affected by reflections in two lines? Sample answer: When a figure is reflected in two parallel lines, the image is the same as the image created by a translation. When a figure is reflected in two intersecting lines, the image is the same as the image created by a rotation.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Compositions of T ransformations

Objective

Students determine the image of a figure after a composition of transformations by analyzing vertices.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Common Misconception

Some students rush to the last transformation. Each transformation needs to be looked at individually as well as part of the whole.

Example 1 Glide Reflection

Teaching the Mathematical Practices

6 Use Quantities Use the Study Tip to encourage students to clarify their use of quantities in this example. Ensure that they carefully compute coordinates used in the problem and label axes appropriately.

Questions for Mathematical Discourse

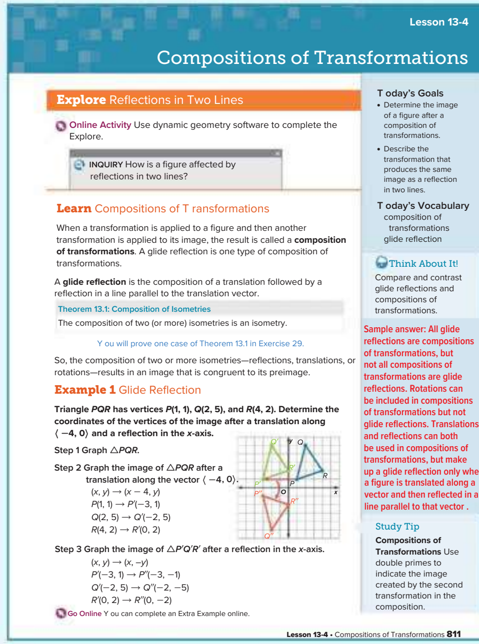
AL What does the translation vector of a glide reflection tell us? how far to translate a figure in the first step of the glide reflection

OL What are the coordinates of the transformed image after a translation along $\langle -5, 4\rangle$ and a reflection in the *x*-axis? $P'''(-8,-3)$, $Q'''(-7, 1)$, $R'''(-5,-2)$

BL A glide reflection is defined as a translation followed by a reflection. Does the order matter to the final image? Explain. No; sample answer: Because the translation vector is parallel to the axis of reflection for a glide reflection, the order of transformations does not matter.

Go Online

- F ind additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.


Lesson 13-4

Compositions of Transformations

Explore Reflections in Two Lines

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How is a figure affected by reflections in two lines?

Learn Compositions of T ransformations

When a transformation is applied to a figure and then another transformation is applied to its image, the result is called a **composition of transformations**. A glide reflection is one type of composition of transformations.

A **glide reflection** is the composition of a translation followed by a reflection in a line parallel to the translation vector.

Theorem 13.1: Composition of Isometries

The composition of two (or more) isometries is an isometry.

Y ou will prove one case of Theorem 13.1 in Exercise 29.

So, the composition of two or more isometries—reflections, translations, or rotations—results in an image that is congruent to its preimage.

Example 1 Glide Reflection

Triangle *PQR* has vertices *P*(1, 1), *Q*(2, 5), and *R*(4, 2). Determine the coordinates of the vertices of the image after a translation along $\langle -4, 0\rangle$ and a reflection in the *x*-axis.

Step 1 Graph $\triangle PQR$.

Step 2 Graph the image of $\triangle PQR$ after a translation along the vector $\langle -4, 0\rangle$.

$(x, y) \to (x - 4, y)$
$P(1, 1) \to P'(-3, 1)$
$Q(2, 5) \to Q'(-2, 5)$
$R(4, 2) \to R'(0, 2)$

Step 3 Graph the image of $\triangle P'Q'R'$ after a reflection in the *x*-axis.

$(x, y) \to (x, -y)$
$P'(-3, 1) \to P''(-3, -1)$
$Q'(-2, 5) \to Q''(-2, -5)$
$R'(0, 2) \to R''(0, -2)$

Go Online Y ou can complete an Extra Example online.

T oday's Goals
- Determine the image of a figure after a composition of transformations.
- Describe the transformation that produces the same image as a reflection in two lines.

T oday's Vocabulary
composition of transformations
glide reflection

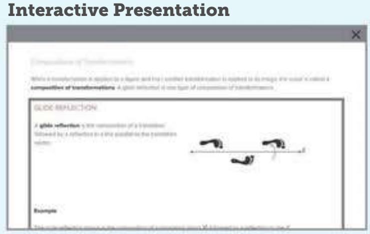
Think About It!
Compare and contrast glide reflections and compositions of transformations.

Sample answer: All glide reflections are compositions of transformations, but not all compositions of transformations are glide reflections. Rotations can be included in compositions of transformations but not glide reflections. Translations and reflections can both be used in compositions of transformations, but make up a glide reflection only when a figure is translated along a vector and then reflected in a line parallel to that vector .

Study Tip
Compositions of Transformations Use double primes to indicate the image created by the second transformation in the composition.

Lesson 13-4 • Compositions of Transformations 811

Interactive Presentation



Learn

TAP

Students tap to reveal tips and additional instruction.

Check

T riangle JKL has vertices $J(6, -1)$, $K(10, -2)$, and $L(5, -3)$. Determine the coordinates of the vertices of the image after a translation along $\langle 0, 4\rangle$ and a reflection in the y-axis.

J''(? , ?), K''(? , ?), L''(? , ?)
−6, 3 −10, 2 −5, 1

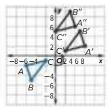
Talk About It!
Do any points remain invariant, or unchanged, under glide reflections? under compositions of transformations? Explain.

No; sample answer: There are no invariant points in a glide reflection because all of the points are translated along a vector . For compositions of transformations, there may be invariant points when a figure is rotated and reflected, rotated twice, or reflected twice.

Example 2 Composition of Isometries

Triangle ABC has vertices $A(-6, -2)$, $B(-5, -5)$, and $C(-2, -1)$. Graph $\triangle ABC$ and its image after a rotation 180° about the origin and a translation along $\langle -2, 4\rangle$.

Step 1 Graph $\triangle ABC$.



Step 2 Graph the image of $\triangle ABC$ after rotation 180° about the origin.

$(x, y) \rightarrow (-x, -y)$

$A(-6, -2) \rightarrow A'(6, 2)$

$B(-5, -5) \rightarrow B'(5, 5)$

$C(-2, -1) \rightarrow C'(2, 1)$

Step 3 Graph the image of $\triangle A'B'C'$ after a translation along $\langle -2, 4\rangle$.

$(x, y) \rightarrow (x - 2, y + 4)$

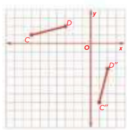
$A'(6, 2) \rightarrow A''(4, 6)$

$B'(5, 5) \rightarrow B''(3, 9)$

$C'(2, 1) \rightarrow C''(0, 5)$

Check

The endpoints of $\overline{CD}$ are $C(-7 , 1)$ and $D(-3, 2)$. Graph $\overline{CD}$ and its image after a reflection in the x-axis and a rotation 90° about the origin.



Go Online Y ou can complete an Extra Example online.

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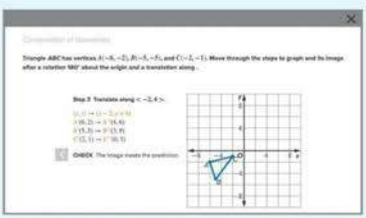
Interactive Presentation



Example 2

TAP

Students tap to reveal steps in a solution.

CHECK

Students complete the Check online to determine whether they are ready to move on.

DIFFERENTIATE

Enrichment Activity BL

Have students identify transformations that occur in the real world.

Example 2 Composition of Isometries

Teaching the Mathematical Practices

5 Use Estimation Guide students to appro ximate the location of the final image and encourage them to check against their approximation when they find the final coordinates of the image.

Questions for Mathematical Discourse

AL Is this a composition of isometries? Explain. Y es; sample answer: This is a composition of a rotation and a translation, both of which are isometries.

OL What are the coordinates of the image if you rotated the triangle 90° counterclockwise about the origin instead of 180° and then translated along the same vector? $A(0, -2)$, $B(3, -1)$, $C(-1, 2)$

BL What single isometry could you use instead of the two in the problem? a 180° rotation about (−1, 2)

Common Error

When predicting the image of a composition of transformations, students may think of the transformations in the wrong order. For some compositions, this will matter for the final image, so remind students to be careful about the order of an unfamiliar composition.

DIFFERENTIATE

Reteaching Activity AL ELL

Have students connect the beauty of art with geometry by designing a figure and then applying transformations, including compositions of transformations, to the figure over a large sheet of paper. Then, have students complete the art project by adding color and decoration as they choose.

Learn Compositions of Two Reflections

Objective

Students describe the transformation that produces the same image as a reflection in two lines by analyzing a given figure.

Teaching the Mathematical Practices

3 Construct Arguments In this Learn, students will use stated assumptions, definitions, and previously established results to construct an argument.

Common Misconception

Students may not understand that a composition of two transformations could be the same as a different single transformation.

Example 3 Reflect a Figure in Two Lines

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In this example, guide students to see what information they can gather about the transformation just from looking at it.

Questions for Mathematical Discourse

AL If a figure is reflected in two intersecting lines that form an angle of 45°, what angle of rotation of the figure will result in the same image? 90°

OL After two reflections in parallel lines, an image and its preimage are 75 centimeters apart. What is the distance between the lines? 37.5 cm

BL If a figure is reflected in two intersecting lines with preimage point $A(2, 6)$ and image point $A'(-2, -6)$, what is the relationship between the lines? They are perpendicular.

Example 4 Determine Congruence

Teaching the Mathematical Practices

1 Understand the Approaches of Others Work with students to look at the Alternate Method. Ask students to compare and contrast the original method and the alternate method.

Questions for Mathematical Discourse

AL What steps do you follow to find the preimage of a transformed image? perform the transformations on the image in reverse order

OL A preimage is reflected in the line $y = x$ and translated along $\langle 1, -3\rangle$. The vertex A'' is located at $(-2, -3)$. What are the coordinates of A? $(0, -3)$

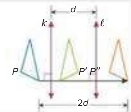
BL Suppose a figure is reflected in the x-axis and then in the y-axis. What single transformation will result in the same image? 180° clockwise rotation about the origin.

Learn Compositions of Two Reflections

The composition of two reflections can result in the same image as a translation or rotation.

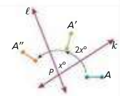
Theorem 13.2: Reflections in Parallel Lines

The composition of two reflections in parallel lines can be described by a translation vector that is
- perpendicular to the two lines and
- twice the distance between the two lines.



Theorem 13.3: Reflections in Intersecting Lines

The composition of two reflections in intersecting lines can be described by a rotation that is
- about the point where the lines intersect and
- through an angle that is twice the measure of the acute or right angle formed by the lines.



You will prove Theorem 13.2 in Exercise 30.

Think About It! In what type of intersecting lines will a composition of two reflections form a 180° rotation?

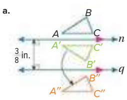
perpendicular lines

Go Online A proof of Theorem 13.3 is available.

Example 3 Reflect a Figure in Two Lines

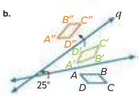
Reflect each figure in line n and then line q. Then describe a single transformation that maps the preimage onto the final image.

a.



By Theorem 13.2, the composition of two reflections in parallel horizontal lines n and q is equivalent to a vertical translation down $2 \cdot \frac{3}{8}$ or $\frac{3}{4}$ inches.

b.



By Theorem 13.3, the composition of two reflections in intersecting lines n and q is equivalent to a $2 \cdot 25°$ or 50° counterclockwise rotation about the intersection point of lines n and q.

Go Online You can complete an Extra Example online.

Lesson 13-4 • Compositions of Transformations 813

Interactive Presentation

Learn

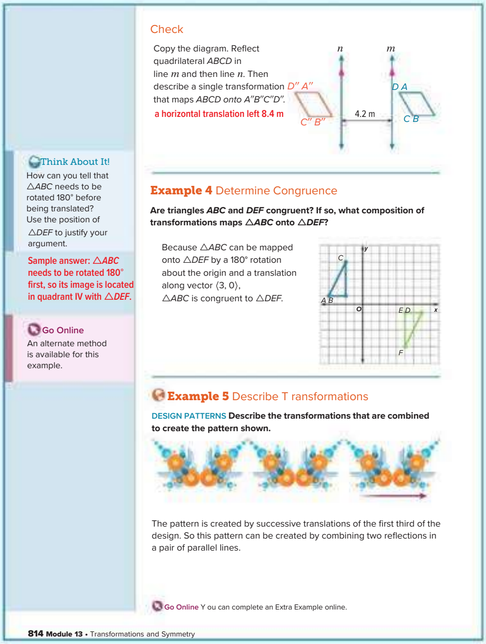
SELECT

Students select to choose a correct answer.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**



Think About It!

How can you tell that $\triangle ABC$ needs to be rotated 180° before being translated? Use the position of $\triangle DEF$ to justify your argument.

Sample answer: $\triangle ABC$ needs to be rotated 180° first, so its image is located in quadrant IV with $\triangle DEF$.

Go Online

An alternate method is available for this example.

Check

Copy the diagram. Reflect quadrilateral *ABCD* in line *m* and then line *n*. Then describe a single transformation that maps *ABCD* onto *A″B″C″D″*.

a horizontal translation left 8.4 m

Example 4 Determine Congruence

Are triangles *ABC* and *DEF* congruent? If so, what composition of transformations maps $\triangle ABC$ onto $\triangle DEF$?

Because $\triangle ABC$ can be mapped onto $\triangle DEF$ by a 180° rotation about the origin and a translation along vector ⟨3, 0⟩, $\triangle ABC$ is congruent to $\triangle DEF$.

Example 5 Describe Transformations

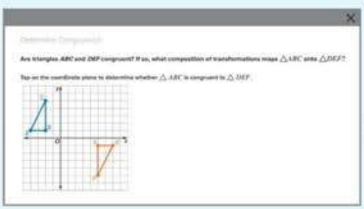
DESIGN PATTERNS **Describe the transformations that are combined to create the pattern shown.**

The pattern is created by successive translations of the first third of the design. So this pattern can be created by combining two reflections in a pair of parallel lines.

Go Online You can complete an Extra Example online.

814 Module 13 • Transformations and Symmetry

Interactive Presentation



Example 4

TAP

Students tap to see an alternate solution method and to choose an answer.

Example 5 Describe Transformations

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about transformations to solving a real-world problem.

Questions for Mathematical Discourse

AL How do you know the pattern isn't created using rotations? Sample answer: The designs are positioned in the same way.

OL How does the length of the translation vector affect the design? Sample answer: The vector is just a little longer than the basic design so there is little space between the repeated parts of the pattern.

BL Can a design that is created using reflections always also be created using translations? Explain. No; sample answer: If the basic design is not symmetric, the reflections and the translations will not produce the same image.

Essential Question Follow-Up

Students study compositions of rigid motions to make new types of rigid motions.

Ask:

Why are compositions of rigid motions important? Sample answer: They can be used to model rigid motions other than reflections, translations, and rotations, such as glide reflections.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–18
2	exercises that use a variety of skills from this lesson	19–30
3	exercises that emphasize higher-order and critical-thinking skills	31–34

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–29 odd, 31–34
- Extension: Composition of a Translation and a Reflection in a Perpendicular Line

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–33 odd
- Remediation, Review Resources: Rotations
- Personal Tutors
- Extra Examples 1–4
- ALEKS Translations, Rotations, Reflections

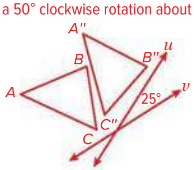
AL

IF students score 65% or less on the Checks,
THEN assign:

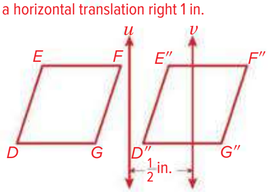
- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Rotations
- *Quick Review Math Handbook*: Compositions of Transformations
- ALEKS Translations, Rotations, Reflections

Answers

9. a 50° clockwise rotation about the point where line u and v intersect



10. a horizontal translation right 1 in.



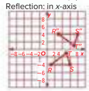
Practice

Go Online You can complete your homework online.

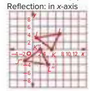
Example 1

Graph each figure with the given vertices and its image after the indicated glide reflection.

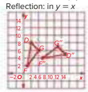
1. $\triangle RST$: $R(1, -4)$, $S(6, -4)$, $T(5, -1)$
   Translation: along $\langle 2, 0\rangle$
   Reflection: in x-axis



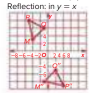
2. $\triangle JKL$: $J(1, 3)$, $K(5, 0)$, $L(7, 4)$
   Translation: along $\langle -3, 0\rangle$
   Reflection: in x-axis



3. $\triangle DFG$: $D(2, 8)$, $F(1, 2)$, $G(4, 6)$
   Translation: along $\langle 3, 3\rangle$
   Reflection: in $y = x$



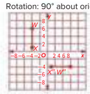
4. $\triangle MPQ$: $M(-4, 3)$, $P(-5, 8)$, $Q(-1, 6)$
   Translation: along $\langle -4, -4\rangle$
   Reflection: in $y = x$



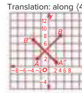
Example 2

Graph each figure with the given vertices and its image after the indicated composition of transformations.

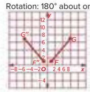
5. $\overline{WX}$: $W(-4, 6)$ and $X(-4, 1)$
   Reflection: in x-axis
   Rotation: 90° about origin



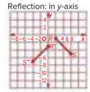
6. $\overline{AB}$: $A(-3, 2)$ and $B(3, 8)$
   Rotation: 90° about origin
   Translation: along $\langle 4, 4\rangle$



7. $\overline{FG}$: $F(1, 1)$ and $G(6, 7)$
   Reflection: in x-axis
   Rotation: 180° about origin



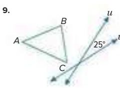
8. $\overline{RS}$: $R(2, -1)$ and $S(6, -5)$
   Translation: along $\langle -2, -2\rangle$
   Reflection: in y-axis



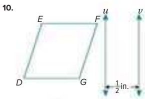
Example 3

Copy and reflect each figure in line u and then line v. Then describe a single transformation that maps the preimage onto the image. 9–12. See margin.

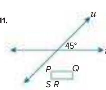
9.



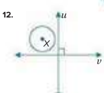
10.



11.



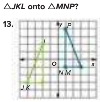
12.



Example 4

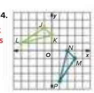
Is $\triangle JKL$ congruent to $\triangle MNP$? If so, what composition of transformations maps $\triangle JKL$ onto $\triangle MNP$?

13.



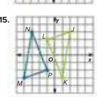
$\triangle JKL \cong \triangle MNP$; reflection in y-axis followed by translation along $\langle -1, 2\rangle$

14.



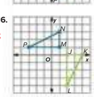
$\triangle JKL \cong \triangle MNP$; reflection in x-axis followed by 90° rotation about the origin

15.



$\triangle JKL \cong \triangle MNP$; translation along $\langle 2, 0\rangle$ followed by 180° rotation about the origin

16.



$\triangle JKL \cong \triangle MNP$; translation along $\langle -1, 1\rangle$ followed by reflection in the line $y = x$

Example 5

17. Describe the transformations that are combined to create the border .

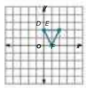
Sample answer: Reflect the first two shapes in a horizontal line through the midpoints of the vertical segments, and then translate to the right and repeat.

18. Describe the transformations that are combined to create the pattern.
Sample answer: Rotate about a point on the corner, and then translate to the right and repeat.

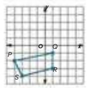
Mixed Exercises

Draw and label the image of each figure after the given composition of transformations. 19–20. See margin.

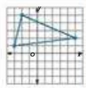
19. 270° rotation about the origin followed by translation along ⟨2, −2⟩



20. reflection in the y-axis followed by 180° rotation about the origin



Determine the coordinates of the preimage given the image and composition of transformations.



21. reflection in the x-axis, reflection in the y-axis (3, −1), (2, −5), (−5, −2)

22. rotation 180° about origin, translation 3 units up (3, 2), (2, −2), (−5, 1)

23. reflection in the y-axis, translation 2 units left (1, 1), (0, 5), (−7, 2)

24. Point K is reflected over line p and then over line d. If lines p and d are parallel and 2.8 feet apart, what single translation maps K onto K''? translation 5.6 ft

Determine whether each statement is *always*, *sometimes*, or *never* true. Justify your argument.

25. A composition of two reflections is a rotation. Sometimes; sample answer: If the lines of reflection intersect, then the composition is a rotation.

26. A composition of two translations is a rotation. Never; sample answer: A composition of two translations is always another translation.

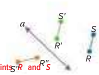
27. A reflection in the x-axis followed by a reflection in the y-axis leaves a point in its original location. Sometimes; sample answer: This is true if the point is the origin.

28. A translation along ⟨a, b⟩ followed by the translation along ⟨c, d⟩ is the translation along ⟨$a + c$, $b + d$⟩. Always; sample answer: The first translation maps (x, y) to $(x + a, y + b)$, and the second translation maps this image to $(x + a + c, y + b + d)$, which is equivalent to the translation along ⟨$a + c$, $b + d$⟩.

29. PROOF Write a paragraph proof for one case of the Composition of Isometries Theorem.

Given: A translation along ⟨a, b⟩ maps R to R' and S to S'.
A reflection in a maps R' to R'' and S' to S''.

Prove: $\overline{RS} \cong \overline{R''S''}$

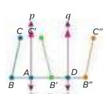


Sample answer: Proof: It is given that a translation along ⟨a, b⟩ maps R to R' and S to S'. Using the definition of a translation, points R and S move the same distance in the same direction, therefore $\overline{RS} \cong \overline{R'S'}$. It is also given that a reflection in a maps R' to R'' and S' to S''. Using the definition of a reflection, points R and S are the same distance from line a, so $\overline{R'S'} \cong \overline{R''S''}$. By the Transitive Property of Congruence, $\overline{RS} \cong \overline{R''S''}$.

30. PROOF Write a two-column proof of Theorem 13.2

Given: A reflection in line p maps $\overline{BC}$ to $\overline{B'C'}$.
A reflection in line q maps $\overline{B'C'}$ to $\overline{B''C''}$.
$p \parallel q$, $AD = x$

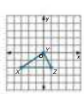
Prove: $\overline{BB''} \perp p$, $\overline{BB''} \perp q$: $BB'' = 2x$ See margin.



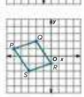
Higher-Order Thinking Skills

31. ANAL YZE When a rotation and a reflection are performed as a composition of transformations on a figure, does the order of the transformations *sometimes*, *always*, or *never* affect the location of the final image? Justify your argument. Sometimes; sample answer: The order of rotating by 180° about the origin and reflecting in the line $y = x$ does not change the location of the final image.

32. FIND THE ERROR Daniel and Lolita are translating △XYZ along ⟨2, 2⟩ and reflecting it in the line $y = 2$. Daniel says that the transformation is a glide reflection. Lolita disagrees and says that the transformation is a composition of transformations. Is either of them correct? Explain your reasoning. See margin.



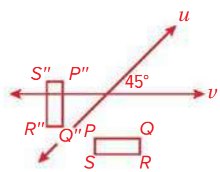
33. PERSEVERE If $PQRS$ is translated along ⟨3, −2⟩, reflected in $y = -1$, and rotated 90° about the origin, what are the coordinates of P''' Q''' R''' S'''?
$P'''(1, -2)$, $Q'''(2, 1)$, $R'''(-1, 3)$, $S'''(-2, 0)$



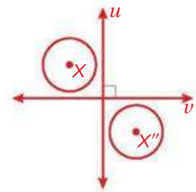
34. ANAL YZE If an image will be reflected in the line $y = x$ and the x-axis, does the order of reflections affect the final image? Explain. See margin.

Answers

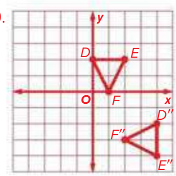
11. a 90° clockwise rotation about the point where line u and v intersect



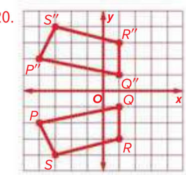
12. 180° rotation about the point where lines u and v intersect followed by a reflection in the v-axis



19.



20.



30. Proof:

Statements (Reasons):

1. A reflection in line p maps $\overline{BC}$ to $\overline{B'C'}$; a reflection in line q maps $\overline{B'C'}$ to $\overline{B''C''}$.; $p \parallel q$; x is the distance between p and q. (Given)
2. p is the perpendicular bisector of $\overline{BB'}$, and q is the perpendicular bisector of $\overline{B'B''}$. (Def. of ⊥ bisector)
3. $BB' + B'B'' = BB''$ (Seg. Add. Post.)
4. $\overline{BB''} \perp p$, $\overline{BB''} \perp q$ (A line perpendicular to a portion of a segment is perpendicular to the whole segment.)
5. $\overline{BA} \cong \overline{AB'}$; $\overline{B'D} \cong \overline{DB''}$ (Def. of refl.)
6. $BA = AB'$; $B'D = DB''$ (Def. of ≅)
7. $BA + AB' + B'D + DB' = BB''$ (Seg. Add. Post.)
8. $AB' + AB' + B'D + B'D = BB''$ (Subs.)
9. $2AB' + 2B'D = BB''$ (Add. Prop.)
10. $2(AB' + B'D) = BB''$ (Dist. Prop.)
11. $AB' + B'D = AD$ (Seg. Add. Post.)
12. $2AD = BB''$ (Subs.)
13. $2x = BB''$ (Subs.)

32. Lolita; sample answer: Because the line $y = 2$ is not parallel to the vector x, the transformation cannot be a glide reflection. It is a composition of a translation and a reflection, so it is a composition of transformations.

34. Yes; sample answer: If a segment with endpoints (a, b) and (c, d) is to be reflected about the x-axis, the coordinates of the endpoints of the reflected image are $(a, -b)$ and $(c, -d)$. If the segment is then reflected about the line $y = x$, the coordinates of the endpoints of the final image are $(-b, a)$ and $(-d, c)$. If the original image is first reflected about $y = x$, the coordinates of the endpoints of the reflected image are (b, a) and (d, c). If the segment is then reflected about the x-axis, the coordinates of the endpoints of the final image are $(b, -a)$ and $(d, -c)$.

Lesson 13-5

Tessellations

G.CO.4, G.CO.5

LESSON GOAL

Students identify figures that tessellate the plane and create tessellations by using rigid transformations.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Creating Tessellations

Develop:

Types of Tessellations
- Regular Tessellation
- Semiregular Tessellation
- Classify a Tessellation

Transformations in Tessellations
- Identify Transformations in a Tessellation

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Parallel Lines and Transversals	●	●		●
Extension: Creating Tessellations		●	●	●

Language Development Handbook

Assign page 85 of the *Language Development Handbook* to help your students build mathematical language related to using rigid motions to tessellate the plane.

ELL You can use the tips and suggestions on page T85 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.5 Given a geometric figure and a rotation, reflection or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

G.CO.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

5 Use appropriate tools strategically.

7 Look for and make use of structure.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students determined the image of a figure after a transformation has occurred.
8.G.3, G.CO.5, G.CO.6

Now
Students identify tessellations and transformations in tessellations.
G.CO.4, G.CO.5

Next
Students will identify line and rotational symmetries in two-dimensional and three-dimensional figures.
G.CO.3, G.CO.5

Rigor

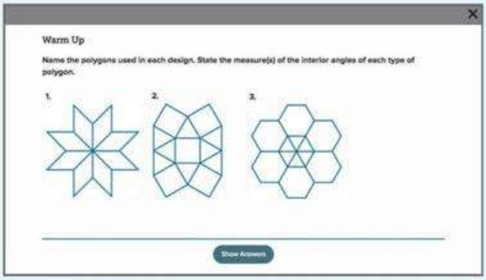
1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of transformations to create tessellations. They build fluency by drawing transformed figures, and they apply their understanding by solving real-world problems related to tessellations.

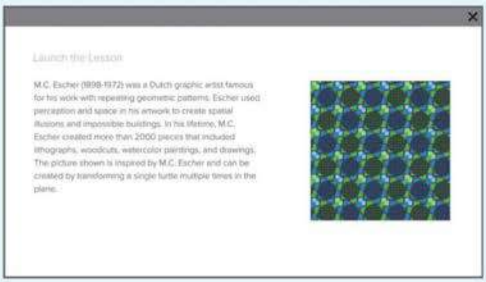
Mathematical Background

A figure has symmetry if there is a rigid motion—reflection, translation, rotation, or glide reflection—that maps the figure onto itself. A figure has line symmetry if it can be mapped onto itself by a reflection in a line. A figure has rotational symmetry if it can be mapped onto itself by a rotation between 0° to 360° about the center of the figure.

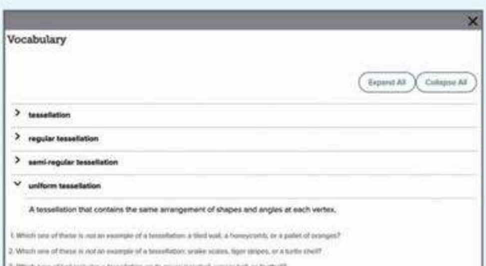
Interactive Presentation



Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- identify types of polygons and angle measures

Answers:

1. rhombus; 45° and 135°
2. square, 90°; equilateral triangle, 60°
3. regular hexagon, 120°; equilateral triangle, 60°

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of tessellations.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Creating Tessellations

Objective

Students use translations to create a tessellation and determine the characteristics needed for a polygon to tessellate the plane.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this Explore, students will need to use dynamic geometry software. Work with students to explore and deepen their understanding of tessellations.

3 Make Conjectures In this Explore, students will make conjectures and then build a logical progression of statements to validate the conjectures. Once students have made their conjectures, guide the students to validate them.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

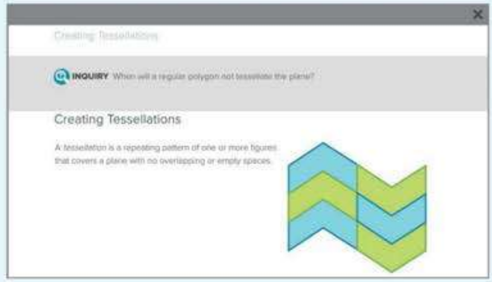
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

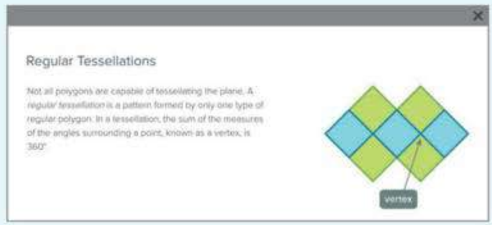
Students will complete guiding exercises throughout the Explore activity. Students read the definition of tessellation. Then they use dynamic geometry software to determine the translation vectors needed to tesselate a geometric figure. Next students use dynamic geometry software to measure angles in regular polygons and to see whether these polygons can tesselate the plane. Students then complete guiding exercises linking the angle measures to the tessellations. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

WEB SKETCHPAD

Students use a sketch to explore tessellations.

TYPE

Students type to complete the guiding exercises.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Creating Tessellations (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What happens when you rotate the first figure at the midpoint of each of its sides? Sample answer: The figure can form a tessellation using those rotations.
- What is the least number of regular polygons that can meet at a vertex? 3

Inquiry

When will a regular polygon not tessellate the plane? Justify your reasoning. Sample answer: A regular polygon will not tessellate the plane when the measure of one of its interior angles is not a factor of 360°. Because the sum of the measures of the angles surrounding a vertex must be 360° and all the interior angles of a regular polygon are all congruent, the measure of each interior angle must be a factor of 360° or else the pattern will have overlapping polygons or empty spaces.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Types of Tessellations

Objective

Students use transformations to classify tessellations and identify figures that tessellate the plane.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of tessellations in this Learn.

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

What Students Are Learning

Tessellations, unlike nonrepeated figures, can have translation symmetry. Encourage students to look for transformations that will not change the appearance of a tessellation, including reflections, rotations, and translations.

Common Misconception

Students may confuse a semiregular tessellation with a tessellation that has only one nonregular polygon. Make sure that students understand that in a semiregular tessellation, the polygons are still regular, but there are more of them than one.

Example 1 Regular Tessellation

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

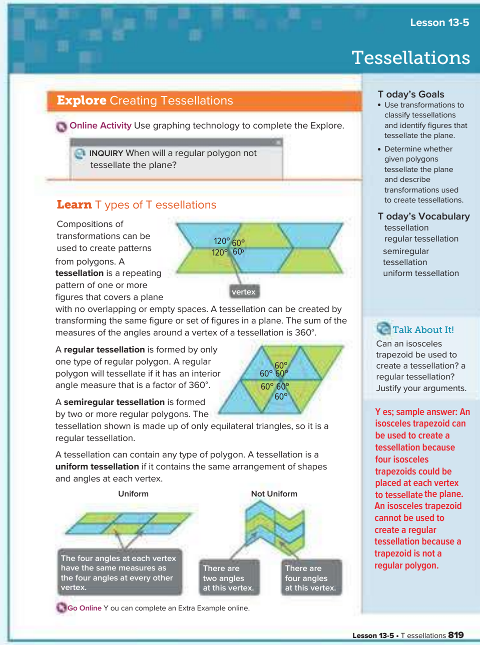
AL What does n represent in the formula? the number of sides of the polygon

OL Why do we need to know the measure of an interior angle? Sample answer: For a regular polygon to tessellate the plane, its interior angle measure must be a factor of 360°.

BL For which values of n will a regular n-gon tessellate the plane? 3, 4, and 6

Common Error

Students may try to cancel the values of n in the formula before multiplying in the numerator. Remind them that they can only cancel expressions that are factors of the numerator and denominator, and that n is not a factor of the numerator.



Lesson 13-5

Tessellations

Explore Creating Tessellations

Online Activity Use graphing technology to complete the Explore.

INQUIRY When will a regular polygon not tessellate the plane?

Learn Types of Tessellations

Compositions of transformations can be used to create patterns from polygons. A **tessellation** is a repeating pattern of one or more figures that covers a plane with no overlapping or empty spaces. A tessellation can be created by transforming the same figure or set of figures in a plane. The sum of the measures of the angles around a vertex of a tessellation is 360°.

A **regular tessellation** is formed by only one type of regular polygon. A regular polygon will tessellate if it has an interior angle measure that is a factor of 360°.

A **semiregular tessellation** is formed by two or more regular polygons. The tessellation shown is made up of only equilateral triangles, so it is a regular tessellation.

A tessellation can contain any type of polygon. A tessellation is a **uniform tessellation** if it contains the same arrangement of shapes and angles at each vertex.

Go Online You can complete an Extra Example online.

Today's Goals
- Use transformations to classify tessellations and identify figures that tessellate the plane.
- Determine whether given polygons tessellate the plane and describe transformations used to create tessellations.

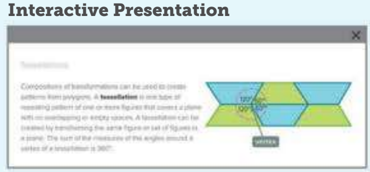
Today's Vocabulary
tessellation
regular tessellation
semiregular tessellation
uniform tessellation

Talk About It!
Can an isosceles trapezoid be used to create a tessellation? a regular tessellation? Justify your arguments.

Yes; sample answer: An isosceles trapezoid can be used to create a tessellation because four isosceles trapezoids could be placed at each vertex to tessellate the plane. An isosceles trapezoid cannot be used to create a regular tessellation because a trapezoid is not a regular polygon.

Lesson 13-5 • Tessellations 819

Interactive Presentation



Learn

TAP

Students tap to view the Math History Minute.

Math History Minute

Although she had only a high-school education, **Marjorie Rice (1923–2017)** devoted her life to finding ways to tessellate a plane with pentagons. She eventually discovered four new types of tessellating pentagons and more than 60 distinct tessellations by pentagons.

Go Online
You can complete an Extra Example online.

Example 1 Regular Tessellation

Determine whether a regular 16-gon will tessellate the plane. Explain.

Let x represent the measure of an interior angle of a regular 16-gon.

$$x = \frac{180(n-2)}{n} = \frac{180(16-2)}{16} = 157.5^\circ$$

Because 157.5° is not a factor of 360°, a regular 16-gon will not tessellate the plane.

Check

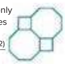
Determine whether a regular decagon will tessellate the plane. Explain.

Because 144° is not a factor of 360°, a regular decagon will not tessellate the plane.

Example 2 Semiregular Tessellation

Determine whether a semiregular tessellation can be created from regular octagons and squares that all have sides 1 unit long. If so, how many regular octagons and squares are needed at each vertex to create the tessellation.

Try to draw a pattern that has no empty spaces using only regular octagons and squares. In the pattern, the vertices are formed by two regular octagons and one square.



Each interior angle of a regular octagon measures $\frac{180(8-2)}{8}$ or 135°. Each interior angle of a square measures 90°.

The sum of the measures of the angles around a vertex of a tessellation is 360°. If there are x regular octagons and y squares at a vertex, then the equation $135x + 90y = 360$ can be used to verify that if there are two regular octagons at a vertex, then there is also a square at the vertex.

Let $x = 2$.	$135x + 90y = 360$	Original equation
	$135(2) + 90y = 360$	Substitution
	$270 + 90y = 360$	Simplify.
	$y = 1$	Solve for y.

So, a semiregular tessellation can be created from two regular octagons and one square.

Check

Determine whether a semiregular tessellation can be created from squares and equilateral triangles that all have sides 1 unit long. If so, how many squares and equilateral triangles are needed at each vertex to create the tessellation? yes; 2 squares and 3 equilateral triangles

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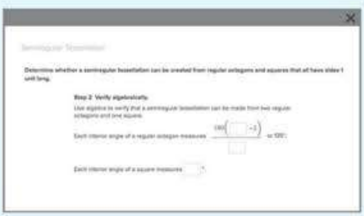
Interactive Presentation



Example 2

TYPE

Students type to verify a tessellation algebraically.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Semiregular Tessellation

Teaching the Mathematical Practices

2 Create Representations Guide students to write an equation that models the situation in Example 2. Then use the equation to solve the problem.

Questions for Mathematical Discourse

AL Why is a tessellation of regular octagons and squares semiregular? Sample answer: Because it is made of more than one shape, but both the shapes are regular.

OL Why do you need to check the angle sum? Sample answer: Sketching by hand is not always reliable; checking that the angle sum is 360° will tell us how reliable the sketch is.

BL Is it possible to have a semiregular tessellation with a regular octagon and a regular hexagon? Explain. No; sample answer: 360° − 135° − 120° = 105°. 105° does not equal the interior angle measure of any regular polygon, and it is too small to be the sum of any two interior angle measures of two regular polygons.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Essential Question Follow-Up

Students learn about tessellations

Ask:

How are tessellations related to transformations? Why might this be useful? Sample answer: Some transformations do not change the way tessellations look. So a tessellation might be useful when repeated in a manufacturing process.

DIFFERENTIATE

Reteaching Activity AL ELL

Have students look for tessellations in the real world, for example in tile patterns or brick sidewalks. Students should classify the tessellations they find using the definitions they learn throughout the lesson.

Enrichment Activity BL

Have students cut a square out of stiff paper, and have them cut a shape out of one side of the paper and tape it to the opposite side. Have students use the new shape to tessellate the plane using translations, and decorate their tessellation with different colors. Have students come up with other ideas to make tessellations using rotations and reflections.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 3 Classify a T essellation

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about tessellations to solving a real-world problem.

Questions for Mathematical Discourse

AL Do the colors of the rectangles used make a difference in determining whether this is a tessellation? Explain. No; sample answer: The only characteristics that determine a tessellation are whether the shapes used overlap and whether the shapes have gaps between them.

OL Is it necessary to use rotations to make this tessellation from one rectangle? Explain. Y es; sample answer: The positions of the rectangles can only occur with rotations.

BL Is there a way to draw a tessellation with this rectangle that is uniform? Explain. Y es; sample answer: Align the rectangles up without rotating them.

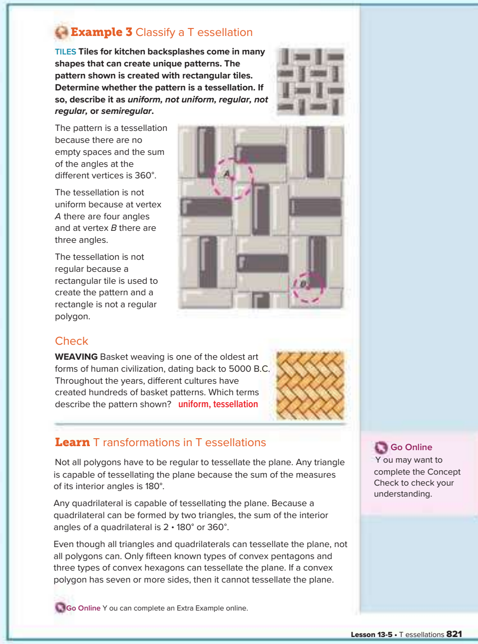
Learn T ransformations in Tessellations

Objective

Students determine whether given polygons tessellate the plane and describe transformations that are used to create tessellations.

Teaching the Mathematical Practices

3 Analyze Cases The Concept Check guides students to examine the cases of different polygons and whether they can tesselate the plane. Encourage students to familiarize themselves with all of the cases.



Example 3 Classify a T essellation

TILES **Tiles for kitchen backsplashes come in many shapes that can create unique patterns. The pattern shown is created with rectangular tiles. Determine whether the pattern is a tessellation. If so, describe it as *uniform, not uniform, regular, not regular,* or *semiregular*.**

The pattern is a tessellation because there are no empty spaces and the sum of the angles at the different vertices is 360°.

The tessellation is not uniform because at vertex *A* there are four angles and at vertex *B* there are three angles.

The tessellation is not regular because a rectangular tile is used to create the pattern and a rectangle is not a regular polygon.

Check

WEAVING Basket weaving is one of the oldest art forms of human civilization, dating back to 5000 B.C. Throughout the years, different cultures have created hundreds of basket patterns. Which terms describe the pattern shown? uniform, tessellation

Learn T ransformations in T essellations

Not all polygons have to be regular to tessellate the plane. Any triangle is capable of tessellating the plane because the sum of the measures of its interior angles is 180°.

Any quadrilateral is capable of tessellating the plane. Because a quadrilateral can be formed by two triangles, the sum of the interior angles of a quadrilateral is 2 • 180° or 360°.

Even though all triangles and quadrilaterals can tessellate the plane, not all polygons can. Only fifteen known types of convex pentagons and three types of convex hexagons can tessellate the plane. If a convex polygon has seven or more sides, then it cannot tessellate the plane.

Go Online Y ou can complete an Extra Example online.

Go Online Y ou may want to complete the Concept Check to check your understanding.

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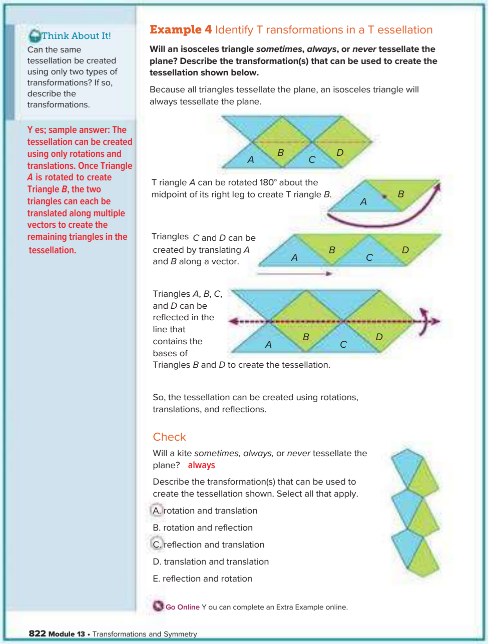
Interactive Presentation

Learn

TAP

Students tap to see a triangle tessellate the plane.


Think About It!
Can the same tessellation be created using only two types of transformations? If so, describe the transformations.

Y es; sample answer: The tessellation can be created using only rotations and translations. Once Triangle *A* is rotated to create Triangle *B*, the two triangles can each be translated along multiple vectors to create the remaining triangles in the tessellation.

Example 4 Identify T ransformations in a T essellation

Will an isosceles triangle *sometimes*, *always*, or *never* tessellate the plane? Describe the transformation(s) that can be used to create the tessellation shown below.

Because all triangles tessellate the plane, an isosceles triangle will always tessellate the plane.

T riangle *A* can be rotated 180° about the midpoint of its right leg to create T riangle *B*.

Triangles *C* and *D* can be created by translating *A* and *B* along a vector.

Triangles *A*, *B*, *C*, and *D* can be reflected in the line that contains the bases of Triangles *B* and *D* to create the tessellation.

So, the tessellation can be created using rotations, translations, and reflections.

Check

Will a kite *sometimes, always*, or *never* tessellate the plane? always

Describe the transformation(s) that can be used to create the tessellation shown. Select all that apply.

A. rotation and translation
B. rotation and reflection
C. reflection and translation
D. translation and translation
E. reflection and rotation

Go Online Y ou can complete an Extra Example online.

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Example 4 Identify T ransformations in a Tessellation

Teaching the Mathematical Practices

1 Understand the Approaches of Others Mathematically proficient students can explain the methods used to solve a problem. The Think About It! feature asks students to justify their reasoning.

Questions for Mathematical Discourse

AL Could you perform the transformation of triangles *A*, *B*, *C*, and *D* using a translation? Explain. Yes; sample answer: You could translate triangle *A* to the green triangle above triangle *B*.

OL Is there another way to transform triangles *A*, *B*, *C*, and *D*? Explain. Y es; sample answer: You could rotate the triangles 180° around the point at the peak of triangle *C*.

BL If these were scalene triangles, could you still use the same set of transformations? Explain. Y es; sample answer: The rotations and translations would be the same. The reflection would look slightly different but still tessellate.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–14
2	exercises that use a variety of skills from this lesson	15–19
3	exercises that emphasize higher-order and critical-thinking skills	20–22

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

- Practice, Exercises 1–19 odd, 20–22
- Extension: Creating New Tessellations

IF students score 66%–89% on the Checks, **OL**
THEN assign:

- Practice, Exercises 1–21 odd
- Remediation, Review Resources: Parallel Lines and Transversals
- Personal Tutors
- Extra Examples 1–4
- ALEKS Parallel Lines and Transversals

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice, Exercises 1–13 odd
- Remediation, Review Resources: Parallel Lines and Transversals
- ALEKS Parallel Lines and Transversals

Important to Know

Digital Exercise Alert Exercise 21 requires drawing transformed figures and is not available online. To fully address G.CO.5, have students complete this exercise using their books.

Answers

Let x represent the measure of an interior angle of each regular polygon.

1. $x = \frac{180(5-2)}{5} = 108°$; Because 108° is not a factor of 360°, a regular pentagon will not tessellate the plane.
2. $x = \frac{180(6-2)}{6} = 120°$; Because 120° is a factor of 360°, a regular hexagon will tessellate the plane.
3. $x = \frac{180(9-2)}{9} = 140°$; Because 140° is not a factor of 360°, a regular 9-gon will not tessellate the plane.

Practice

Go Online Y ou can complete your homework online.

Example 1

Determine whether each regular polygon will tessellate the plane. Explain. 1–3. See margin.

1. pentagon **2.** hexagon **3.** 9-gon

Example 2

Determine whether a semiregular uniform tessellation can be created from the given shapes, assuming that all sides are 1 unit long. If so, determine the number of each shape needed at each vertex to create the tessellation.

4. regular pentagons and squares no

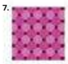
5. regular hexagons and equilateral triangles yes; 2 regular hexagons, 2 equilateral triangles

Example 3

Determine whether the pattern is a tessellation. If so, describe it as *uniform, not uniform, regular, not regular, or semiregular.*

6.

tessellation; uniform, regular

7.

tessellation; uniform, semi-regular

8.

not a tessellation

Example 4

Determine whether a tessellation can be created from each figure. If so, describe the transformation(s) that can be used to create the tessellation and draw a picture to support your reasoning. 9–10. See margin.

9. scalene triangle **10.** rhombus

11. Determine whether a tessellation can be created from a regular dodecagon. If so, describe the transformation(s) that can be used to create the tessellation. Will a regular dodecagon *sometimes, always,* or *never* tessellate the plane? Justify your argument. See margin.

12. Sketch a tessellation that can be created from an isosceles trapezoid. Describe the transformation(s) that can be used to create the tessellation. See margin.

13. Will a regular 15-gon *sometimes, always,* or *never* tessellate the plane? Justify your argument. See margin.

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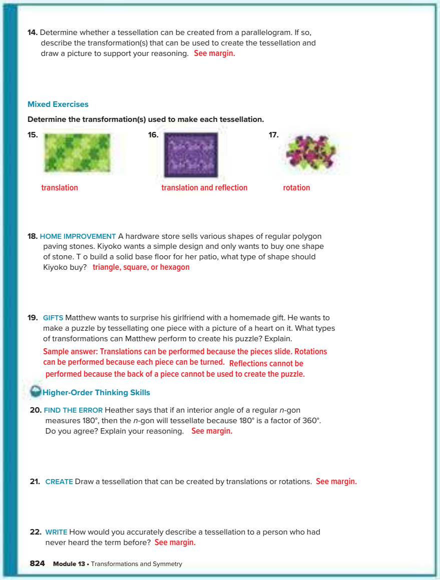

14. Determine whether a tessellation can be created from a parallelogram. If so, describe the transformation(s) that can be used to create the tessellation and draw a picture to support your reasoning. See margin.

Mixed Exercises

Determine the transformation(s) used to make each tessellation.

15. translation

16. translation and reflection

17. rotation

18. HOME IMPROVEMENT A hardware store sells various shapes of regular polygon paving stones. Kiyoko wants a simple design and only wants to buy one shape of stone. To build a solid base floor for her patio, what type of shape should Kiyoko buy? triangle, square, or hexagon

19. GIFTS Matthew wants to surprise his girlfriend with a homemade gift. He wants to make a puzzle by tessellating one piece with a picture of a heart on it. What types of transformations can Matthew perform to create his puzzle? Explain.
Sample answer: Translations can be performed because the pieces slide. Rotations can be performed because each piece can be turned. Reflections cannot be performed because the back of a piece cannot be used to create the puzzle.

Higher-Order Thinking Skills

20. FIND THE ERROR Heather says that if an interior angle of a regular *n*-gon measures 180°, then the *n*-gon will tessellate because 180° is a factor of 360°. Do you agree? Explain your reasoning. See margin.

21. CREATE Draw a tessellation that can be created by translations or rotations. See margin.

22. WRITE How would you accurately describe a tessellation to a person who had never heard the term before? See margin.

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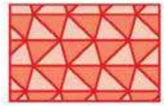
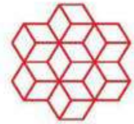
9. yes; Sample answer: reflection, rotation, translation

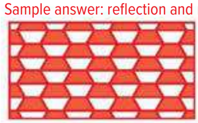


10. yes; Sample answer: rotation, translation



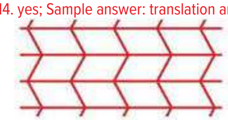
11. Never; sample answer: Each interior angle of a regular dodecagon is $\frac{180°(12-2)}{12} = 150°$. Because 150° is not a factor of 360°, a regular dodecagon will not tessellate the plane.

12. Sample answer: reflection and rotation



13. Never; sample answer: Each interior angle of a regular 15-gon is $\frac{180°(15-2)}{15} = 156°$. Because 156° is not a factor of 360°, a regular 15-gon will not tessellate the plane.

14. yes; Sample answer: translation and reflection



20. No; sample answer: Let n represent the number of interior angles of an n-gon: $180 = \frac{180(n-2)}{n} \rightarrow 180n = 180(n-2) \rightarrow 180n = 180n - 360 \rightarrow 0 \neq -360$; Although 180° is a factor of 360°, because 0 does not equal −360, there is not an n-gon with an interior angle that measures 180°.

21. Sample answer:

22. Sample answer: A tessellation is a pattern that completely covers a surface and is created by using the same or different shapes. To be a tessellation, the shapes in the pattern cannot overlap, and there cannot be any gaps between the shapes.

Lesson 13-6

Symmetry

G.CO.3, G.CO.5

LESSON GOAL

Students use symmetry to describe the transformations that carry a figure onto itself.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Symmetry in Figures

Develop:

Line Symmetry

- Identify Line Symmetry

Rotational Symmetry

- Identify Rotational Symmetry
- Determine Order and Magnitude of Symmetry

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Rotations	●	●		●
Extension: Symmetry in Design		●	●	●

Language Development Handbook

Assign page 86 of the *Language Development Handbook* to help your students build mathematical language related to using symmetry to describe the transformations that carry a figure onto itself.

ELL You can use the tips and suggestions on page T86 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.3 Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

7 Look for and make use of structure.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students identified tessellations and transformations in tessellations.
G.CO.4, G.CO.5

Now
Students identify line and rotational symmetries in two-dimensional and three-dimensional figures.
G.CO.3, G.CO.5

Next
Students will understand and describe any symmetry displayed in graphs of functions.
F.IF.4 (Course 2)

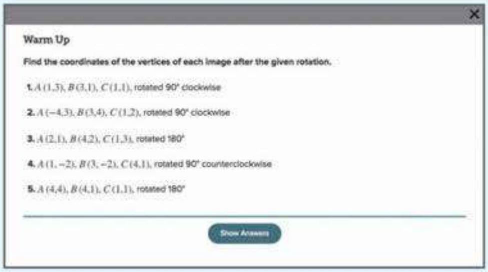
Rigor

The Three Pillars of Rigor

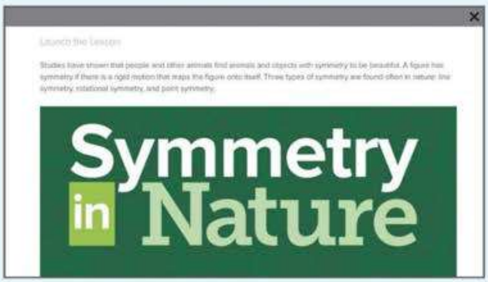
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of and fluency with symmetry (first studied in Grade 4) to connect symmetry to transformations. They apply their understanding by solving real-world problems related to symmetry.

Interactive Presentation



Warm Up



Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- determine coordinates of transformations

Answers:

1. $A'(3, -1)$, $B'(1, -3)$, $C'(1, -1)$
2. $A'(3, 4)$, $B'(4, -3)$, $C'(2, -1)$
3. $A'(-2, -1)$, $B'(-4, -2)$, $C'(-1, -3)$
4. $A'(2, 1)$, $B'(2, 3)$, $C'(-1, 4)$
5. $A'(-4, -4)$, $B'(-4, -1)$, $C'(-1, -1)$

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see real-world applications of the various types of symmetry.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Mathematical Background

A figure has symmetry if there is a rigid motion–reflection, translation, rotation, or glide reflection—that maps the figure onto itself. A figure has line symmetry if it can be mapped onto itself by a reflection in a line. A figure has rotational symmetry if it can be mapped onto itself by a rotation between 0° to 360° about the center of the figure. Similarly, three-dimensional figures can have plane or axis symmetry.

Explore Symmetry in Figures

Objective

Students use dynamic geometry software to explore symmetry in two-dimensional figures.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

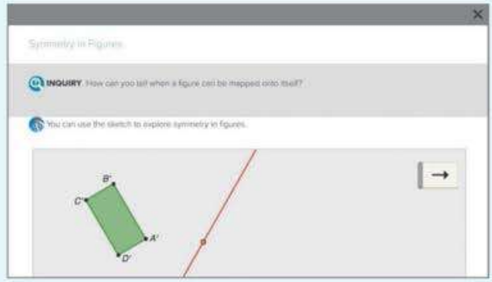
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students use dynamic geometry software to move a line of reflection so that a figure reflected in it does not change. Then students complete guiding exercises about line symmetry. Next students use dynamic geometry software to determine an angle of rotation so that a figure rotated in that angle does not change. Then students complete guiding exercises about point symmetry. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

WEB SKETCHPAD

Students use a sketch to explore symmetry.

TYPE

Students type to complete the guiding exercises.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Symmetry in Figures (*continued*)

Teaching the Mathematical Practices

3 Construct Arguments In this Explore, students will use stated assumptions, definitions, and previously established results to construct an argument.

Questions

Have students complete the Explore activity.

Ask:

- Does the triangle shown have line symmetry? If so, how many lines does it have? yes; 3
- Does the rectangle have point symmetry? If so, what is the smallest angle measure for a rotation where the image is the same as the original rectangle? yes; 180°

Inquiry

How can you tell when a figure can be mapped onto itself? Sample answer: A figure can be mapped onto itself when the figure is regular or when it can be bisected by a line.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Line Symmetry

Objective

Students use line symmetry to describe the reflections that carry a figure onto itself.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

About the Key Concept

Symmetry connects transformations to single figures. Because a single figure cannot be translated onto itself, the only transformations that can show symmetry in a figure are rotations and reflections.

Common Misconception

Students will sometimes find more lines of symmetry than actually exist. Simple or convenient definitions that lines of symmetry cut shapes in half do not imply that these lines must also create two halves that are exact mirror images of each other.

Example 1 Identify Line Symmetry

MP Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This example asks students to respond to the arguments of others.

Questions for Mathematical Discourse

AL Describe reflectional symmetry in your own words. Sample answer: a line along which an object can be folded onto itself and both parts look exactly the same

OL Why does the triangle not have any lines of symmetry? Sample answer: There is no way that this figure can be reflected upon itself.

BL What figure has infinitely many lines of symmetry? a circle

---

Lesson 13-6

Symmetry

Explore Symmetry in Figures

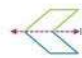
Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY How can you tell when a figure can be mapped onto itself?

Learn Line Symmetry

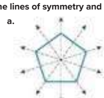
A figure has **symmetry** if there exists a rigid motion—reflection, translation, rotation, or glide reflection—that maps the figure onto itself. Figures that have symmetry are self-congruent. One type of symmetry is *line symmetry*.

A figure in the plane has **line symmetry** (or *reflectional symmetry*) if each half of the figure matches the other side exactly. When a figure has line symmetry, the figure can be mapped onto itself by a reflection in a line, called the **line of symmetry** (or *axis of symmetry*).

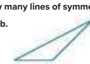


Example 1 Identify Line Symmetry

Determine whether each figure has a line of symmetry. If so, draw the lines of symmetry and state how many lines of symmetry it has.

a. 

5 lines of symmetry

b. 

0 lines of symmetry

Go Online You can complete an Extra Example online.

Today's Goal
- Use line symmetry to describe the reflections that carry a figure onto itself.
- Use rotational symmetry to describe the rotations that carry a figure onto itself.

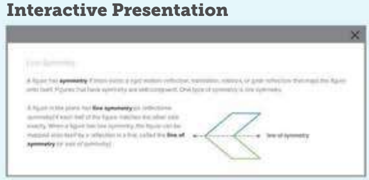
Today's Vocabulary
symmetry
line symmetry
line of symmetry
rotational symmetry
center of symmetry
order of symmetry
magnitude of symmetry
point symmetry
point of symmetry

Talk About It!
Do you think that a figure can have multiple lines of symmetry? Justify your argument.

Yes; sample answer: If a figure can be reflected onto itself in more than one way, then it can have multiple lines of symmetry.

Lesson 13-6 • Symmetry 825

Interactive Presentation



Learn

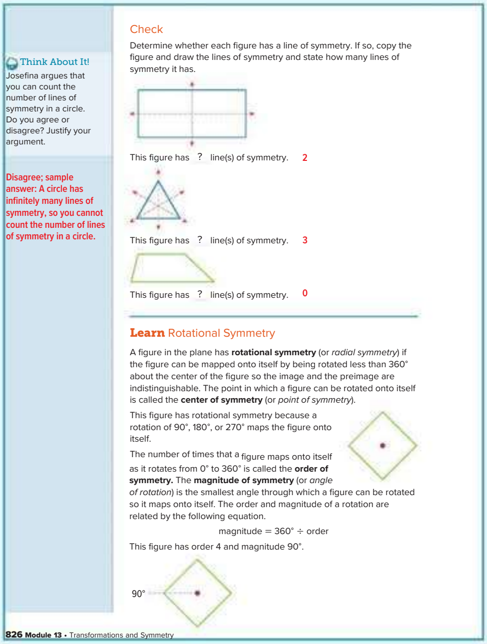
TYPE

Students answer a question to show that they understand line symmetry.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION



Think About It!
Josefina argues that you can count the number of lines of symmetry in a circle. Do you agree or disagree? Justify your argument.

Disagree; sample answer: A circle has infinitely many lines of symmetry, so you cannot count the number of lines of symmetry in a circle.

Check
Determine whether each figure has a line of symmetry. If so, copy the figure and draw the lines of symmetry and state how many lines of symmetry it has.

This figure has ? line(s) of symmetry. 2

This figure has ? line(s) of symmetry. 3

This figure has ? line(s) of symmetry. 0

Learn Rotational Symmetry

A figure in the plane has **rotational symmetry** (or *radial symmetry*) if the figure can be mapped onto itself by being rotated less than 360° about the center of the figure so the image and the preimage are indistinguishable. The point in which a figure can be rotated onto itself is called the **center of symmetry** (or *point of symmetry*).

This figure has rotational symmetry because a rotation of 90°, 180°, or 270° maps the figure onto itself.

The number of times that a figure maps onto itself as it rotates from 0° to 360° is called the **order of symmetry.** The **magnitude of symmetry** (or *angle of rotation*) is the smallest angle through which a figure can be rotated so it maps onto itself. The order and magnitude of a rotation are related by the following equation.

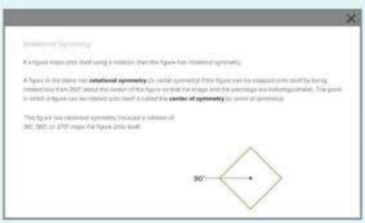
magnitude = 360° ÷ order

This figure has order 4 and magnitude 90°.

90°

826 Module 13 • Transformations and Symmetry

Interactive Presentation



Learn

DRAG & DROP

Students drag answers to the correct figures.

Common Error

Students may have difficulty finding the lines of symmetry that pass through the vertices of polygons that have an even number of sides.

Essential Question Follow-Up

Students learn about different types of symmetry.

Ask:

Why is symmetry important in the real world? Sample answer: You can use symmetry to construct the second part of a symmetric figure from the first part.

Learn Rotational Symmetry

Objective

Students use rotational symmetry to describe the rotations that carry a figure onto itself.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of rotational symmetry in this Learn.

Things to Remember

Some, but not all, figures that have rotational symmetry also have line symmetry. Some, but not all, figures that have line symmetry have rotational symmetry.

Common Misconception

Students sometimes have difficulty with rotational symmetry. The best way to understand rotational symmetry is to visualize it.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 2 Identify Rotational Symmetry

Teaching the Mathematical Practices

4 Apply Mathematics In this example, students apply what they have learned about rotational symmetry to solving a real-world problem.

Questions for Mathematical Discourse

AL If figures have line symmetry, do they also have rotational symmetry? Explain. No; sample answer: Some figures that have line symmetry have rotational symmetry, but not all.

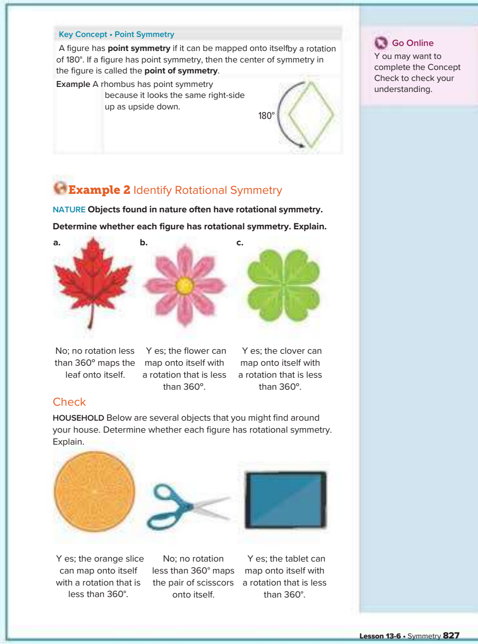
OL What part of the leaf tells you whether it has rotational symmetry? Explain. the stem; Sample answer: No other part of the leaf is as thin, so it cannot have rotational symmetry.

BL What could you do to one of the figures to make it no longer have rotational symmetry? Sample answer: Add a stem to the clover.

DIFFERENTIATE

Enrichment Activity BL

Have the students identify some objects that have rotational symmetry. Sample answer: baseball field, helicopter blades



Key Concept • Point Symmetry

A figure has **point symmetry** if it can be mapped onto itselfby a rotation of 180°. If a figure has point symmetry, then the center of symmetry in the figure is called the **point of symmetry**.

Example A rhombus has point symmetry because it looks the same right-side up as upside down.

180°

Go Online Y ou may want to complete the Concept Check to check your understanding.

Example 2 Identify Rotational Symmetry

NATURE **Objects found in nature often have rotational symmetry.**

Determine whether each figure has rotational symmetry. Explain.

a. No; no rotation less than 360° maps the leaf onto itself.

b. Y es; the flower can map onto itself with a rotation that is less than 360°.

c. Y es; the clover can map onto itself with a rotation that is less than 360°.

Check

HOUSEHOLD Below are several objects that you might find around your house. Determine whether each figure has rotational symmetry. Explain.

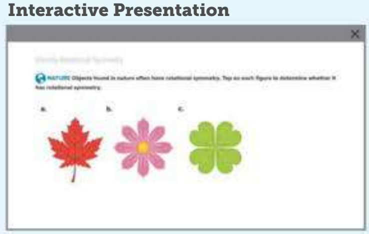
Y es; the orange slice can map onto itself with a rotation that is less than 360°.

No; no rotation less than 360° maps the pair of sciscors onto itself.

Y es; the tablet can map onto itself with a rotation that is less than 360°.

Lesson 13-6 • Symmetry 827

Interactive Presentation



Example 2

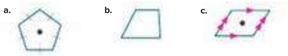
TAP Students tap to reveal a solution.

CHECK Students complete the Check online to determine whether they are ready to move on.

Example 3 Determine Order and Magnitude of Symmetry

Part A State the order and magnitude of symmetry.

Determine whether each figure has rotational symmetry. If so, locate the center of symmetry and state the order and magnitude of symmetry.



a. rotational symmetry: yes; order = 5; magnitude = 72°

b. rotational symmetry: no; order = none; magnitude = none

c. rotational symmetry: yes; order = 2; magnitude = 180°

Part B Identify point symmetry.

Which figure(s) in Part A has point symmetry? Justify your reasoning.

The parallelogram has point symmetry because it can be rotated 180° about its center so it maps onto itself.

Check

Determine whether each figure has rotational symmetry. If so, copy the figure and locate the center of symmetry and state the order and magnitude of symmetry.

a. rotational symmetry: ? yes; order = ? 2; magnitude = ? 180°

b. rotational symmetry: ? no; order = ? none; magnitude = ? none

c. rotational symmetry: ? yes; order = ? 3; magnitude = ? 120°

Which figure(s) has point symmetry? Justify your answer.

The rectangle has point symmetry because it can be rotated 180° about its center so it maps onto itself.

Think About It! Is the following statement *sometimes*, *always*, or *never* true? Explain. *A polygon with order 4 rotational symmetry has point symmetry.*

Always; sample answer: If a polygon has order 4 rotational symmetry, then the magnitude of symmetry is 90°. Because 180 is a multiple of 90, a polygon with order 4 must have point symmetry.

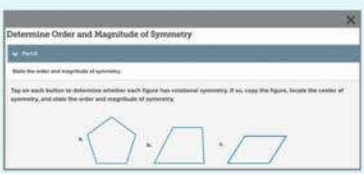
Go Online to practice what you've learned about transformations and symmetry in the Put It All Together over Lessons 13-1 through 13-6.

Go Online You may want to complete the construction activities for this lesson.

Go Online You can complete an Extra Example online.

828 **Module 13** • Transformations and Symmetry

Interactive Presentation



Example 3

TAP Students tap to reveal steps in a solution.

CHECK Students complete the Check online to determine whether they are ready to move on.

Example 3 Determine Order and Magnitude of Symmetry

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL Do all regular polygons have rotational symmetry? Explain. Yes; sample answer: Because there exists an angle through which you can rotate a regular polygon onto itself, regular polygons have rotational symmetry.

OL Do all regular polygons have point symmetry? Explain No; sample answer: Regular polygons with odd numbers of sides cannot map onto themselves with rotations of 180°.

BL What will the order of symmetry for a regular n-gon be? n

Common Error

Students may confuse the order and the magnitude of symmetry. Work with them until they understand that the two terms are related but different.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–14
2	exercises that use a variety of skills from this lesson	15–23
3	exercises that emphasize higher-order and critical-thinking skills	24–31

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–23 odd, 24–31
- Extension: Symmetry in Design
- ALEKS Symmetry, Congruence Transformations

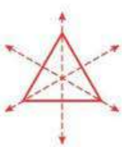
IF students score 66%–89% on the Checks, **THEN** assign: OL

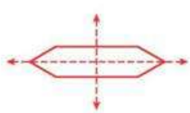
- Practice, Exercises 1–31 odd
- Remediation, Review Resources: Rotations
- Personal Tutors
- Extra Examples 1–3
- ALEKS Rotations

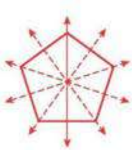
IF students score 65% or less on the Checks, **THEN** assign: AL

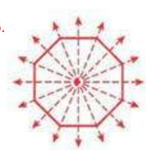
- Practice, Exercises 1–13 odd
- Remediation, Review Resources: Rotations
- *Quick Review Math Handbook*: Symmetry
- ALEKS Rotations

Answers

1. 

2. 

5. 

6. 

Practice

Go Online Y ou can complete your homework online.

Example 1

Determine whether each figure has a line of symmetry. If so, draw the lines of symmetry and state how many lines of symmetry it has. 1–2, 5–6. See margin for graphs.

1. yes; 3

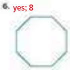
2. yes; 2

3. no

4. no

5. yes; 5

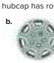
6. yes; 8

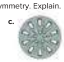


Example 2

7. CARS Steve found the hubcaps shown below at his local junkyard. Determine whether each hubcap has rotational symmetry. Explain.

a. 
Y es; the hubcap can map onto itself with a rotation that is less than 360°.

b. 
Y es; the hubcap can map onto itself with a rotation that is less than 360°.

c. 
Y es; the hubcap can map onto itself with a rotation that is less than 360°.

8. FLAGS The figure shows the Union Jack, which is the flag of the United Kingdom. Does the flag have rotational symmetry? Explain.
Y es; the flag can map onto itself with a rotation that is less than 360°

Stefan Ataman/Shutterstock

9. RECYCLING A waste management company offers recycling programs for its clients. Recycling is denoted by the symbol shown. Does the recycling symbol have rotational symmetry? Explain. Y es; the symbol can map onto itself with a rotation that is less than 360°.

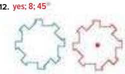
10. VACATION Annabel and her family went to a beach for vacation. While she was on the beach, Annabel collected seashells. Does the seashell shown have rotational symmetry? Explain.
No; no rotation less than 360° maps the seashell onto itself.

Example 3

Determine whether each figure has rotational symmetry. If so, locate the center of symmetry, and state the order and magnitude of symmetry.

11. yes; 3; 120°

12. yes; 8; 45°



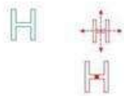
13. yes; 2; 180°

14. no

Frazer Cunningham/Image Source

Mixed Exercises

Refer to the figure at the right.



15. Draw the line(s) of symmetry in the figure.

16. Locate the center of symmetry for the figure.

17. What is the order and magnitude of symmetry for the figure? 2; 180°

18. LETTERS Examine each capital letter in the alphabet. Determine which letters have 180° rotational symmetry about a point in the center of the letter. H, I, N, O, S, X, Z

19. STRUCTURE A regular polygon has rotational symmetry with an order of 5 and a magnitude of 72°. What is the figure? pentagon

20. CONSTRUCT ARGUMENTS Consider the symmetry of a circle.

a. How many lines of symmetry does a circle have? Justify your argument. Infinitely many; every line through the center of the circle is a line of symmetry, and there are infinitely many such lines.

b. What is the order of rotation for a circle? Justify your argument. The order of rotation is infinite; as the circle rotates all the way around, every angle measure, no matter how small, maps the circle onto itself.

State whether each figure has rotational symmetry. If so, describe the rotations that map the figure onto itself by giving the order of symmetry and magnitude of symmetry.

21. equilateral triangle yes; order of symmetry: 3; magnitude of symmetry: 120°

22. scalene triangle no rotational symmetry

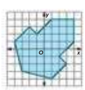
23. regular hexagon yes; order of symmetry: 6; magnitude of symmetry: 60°

Higher-Order Thinking Skills

24. PERSEVERE Draw a three-dimensional object that has a base with line symmetry. See margin.

25. CREATE Draw an object that has at least one line of symmetry. Describe the lines of symmetry in this object. Sample answer: A rectangular mirror with two lines of symmetry, one vertical and one horizontal, through the middle or a spoon with one line of symmetry down the middle.

26. ANALYZE The figure shows the floor plan for a new gallery in an art museum. Describe every reflection or rotation that maps the gallery onto itself. The only transformation that maps the gallery onto itself is a reflection in the line $y = x$.



27. WRITE A regular polygon has magnitude of symmetry 15°. How many sides does the polygon have? Explain. 24; $360° \div 15° = 24$, so the order of symmetry is 24. This means there are 24 sides.

28. FIND THE ERROR Jaime says that Figure A has only line symmetry, and Jewel says that Figure A has only rotational symmetry. Is either of them correct? Explain your reasoning. Neither; Figure A has both rotational and line symmetry.

Figure A

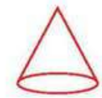
29. PERSEVERE A quadrilateral in the coordinate plane has exactly two lines of symmetry, $y = x - 1$ and $y = -x + 2$. Find possible vertices for the figure. Graph the figure and the lines of symmetry. Sample answer: (−1, 0), (2, 3), (4, 1), and (1, −2); See margin for graph.

30. CREATE Draw a figure that has line symmetry but not rotational symmetry. Explain. See margin.

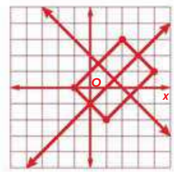
31. WRITE How are line symmetry and rotational symmetry related? See margin.

Answers

24. Sample answer:



29.



30. Sample answer: An isosceles triangle has line symmetry from the vertex angle to the base of the triangle, but it does not have rotational symmetry because it cannot be rotated to map onto itself.

31. Sample answer: In both rotational and line symmetry a figure is mapped onto itself. However, in line symmetry a figure is mapped onto itself by a reflection, and in rotational symmetry a figure is mapped onto itself by a rotation. A figure can have line symmetry and rotational symmetry.

Review

Rate Yourself

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why are compositions of rigid motions important?
- How are tessellations related to transformations? Why might this be useful?
- Why is symmetry important in the real world?

Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to rigid motions and symmetry.

LearnSmart Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Congruence, Proof, and Constructions.**

- Experiment with transformations in the plane

Module 13 • Transformations and Symmetry

Review

Essential Question

How are rigid motions used to show geometric relationships?

Rigid motions are used to show that figures are congruent. If no series of rigid motions exists from one figure to another, then the figures are not congruent.

Module Summary

Lessons 13-1 through 13-3

Reflections, Translations, and Rotations

- When a figure is reflected in a line, each point of the preimage and its corresponding point on the image are the same distance from the line of reflection.
- A translation is a function in which all the points of a figure move the same distance in the same direction as described by a translation vector.
- A translation vector describes the magnitude and direction of the translation. The magnitude of a vector is its length from the initial point to the terminal point.
- A rotation about a fixed point P through an angle of $a°$ is a function that maps point M to point M' such that point P does not move, $m\angle MPM' = a°$, and $MP = MP'$.

Lesson 13-4

Compositions of Transformations

- When a transformation is applied to a figure and then another transformation is applied to its image, the result is called a composition of transformations.
- A glide reflection is the composition of a translation followed by a reflection in a line parallel to the translation vector.
- The composition of two reflections can result in the same image as a translation or rotation.

Lessons 13-5 and 13-6

Tessellations and Symmetry

- A regular polygon will tessellate if it has an interior angle measure that is a factor of 360°.
- A semiregular tessellation is formed by two or more regular polygons.
- A figure has symmetry if there exists a rigid motion—reflection, translation, rotation, or glide reflection—that maps the figure onto itself.
- A figure in the plane has line symmetry (or reflectional symmetry) if each half of the figure matches the other side exactly.
- A figure in the plane has rotational symmetry (or radial symmetry) if the figure can be mapped onto itself by being rotated less than 360° about the center of the figure so the image and the preimage are indistinguishable.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

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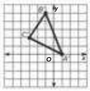
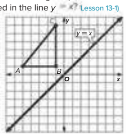
1. **GRAPH** Graph the image of $\triangle ABC$ with vertices at $A(-5, 0)$, $B(-3, 5)$, $C(-1, 2)$ after a reflection in the line $x = -2$. (Lesson 13-1)



2. **MULTIPLE CHOICE** Which best describes a possible step that is used to determine the location of the image of point B when it is reflected in the line $y = x$? (Lesson 13-1)



A. Move down one and right one from (0, 0).
B. Move down one and right one from (−1, 1).
C. Move right two from (1, 1).
D. Move down two from (−1, −1).

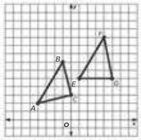
3. **OPEN RESPONSE** When point F is reflected in the line $y = x$, the image is located at $F'(6, -9)$. Find the coordinates of point F. (Lesson 13-1)

(−9, 6)

4. **MULTIPLE CHOICE** Find the vector that translates $A(-2, 7)$ to $A'(6, 4)$. (Lesson 13-2)

A. ⟨−8, 3⟩
B. ⟨−3, 8⟩
C. ⟨3, 8⟩
D. ⟨8, −3⟩

5. **OPEN RESPONSE** Refer to the graph.



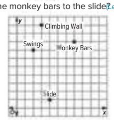
Explain why a translation does not map $\triangle ABC$ to $\triangle EFG$. (Lesson 13-2)

Sample answer: The length of $\overline{AC}$ is not the same as the length of $\overline{EG}$.

6. **MULTIPLE CHOICE** Which is the image of $P(-5, 11)$ along the vector ⟨3, −8⟩? (Lesson 13-2)

A. $P'(-8, 19)$
B. $P'(-8, 3)$
C. $P'(-2, -3)$
D. $P'(-2, 3)$

7. **MULTIPLE CHOICE** Juan is designing a new playground for the elementary school. He needs to determine the shortest distance from the monkey bars to the slide to create a path. Which statement best describes the translation from the monkey bars to the slide? (Lesson 13-2)



A. a translation right 11 units and up 9 units
B. a translation right 3 units and up 7 units
C. a translation left 3 units and down 7 units
D. a translation left 11 units and down 9 units

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Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Put It All Together: Lessons 13-1 through 13-6
Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

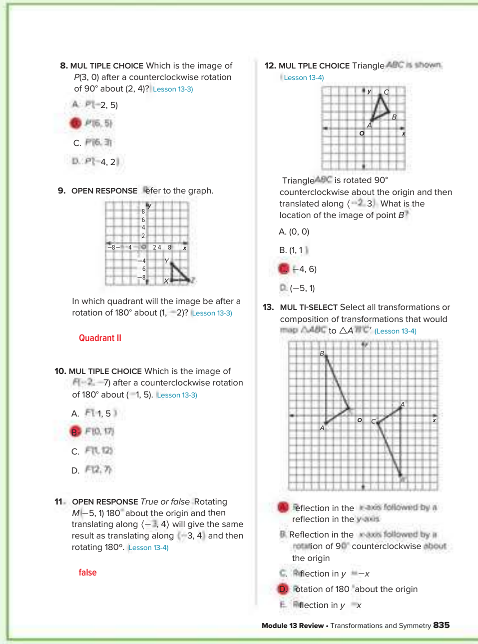
Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–17 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	2, 4, 6–8, 10, 12
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	13
Table Item	Students complete a table by entering in the correct values.	15
Graph	Students create a graph on an online coordinate plane.	1, 14
Open Response	Students construct their own response.	3, 5, 9, 11, 16, 17

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
G.CO.4	13-2, 13-5	5, 15–16
G.CO.5	13-1 through 13-6	1, 4, 7–9, 13, 14, 16
G.CO.6	13-1 through 13-4	1–4, 6, 8–11, 14


8. **MULTIPLE CHOICE** Which is the image of $P(3, 0)$ after a counterclockwise rotation of 90° about (2, 4)? (Lesson 13-3)

A. $P'(-2, 5)$

B. $P'(6, 5)$

C. $P'(6, 3)$

D. $P'(-4, 2)$

9. **OPEN RESPONSE** Refer to the graph.

In which quadrant will the image be after a rotation of 180° about (1, −2)? (Lesson 13-3)

Quadrant II

10. **MULTIPLE CHOICE** Which is the image of $F(-2, -7)$ after a counterclockwise rotation of 180° about (−1, 5). (Lesson 13-3)

A. $F'(1, 5)$

B. $F'(0, 17)$

C. $F'(1, 12)$

D. $F'(2, 7)$

11. **OPEN RESPONSE** *True or false:* Rotating $M(-5, 1)$ 180° about the origin and then translating along ⟨−3, 4⟩ will give the same result as translating along ⟨−3, 4⟩ and then rotating 180°. (Lesson 13-4)

false

12. **MULTIPLE CHOICE** Triangle ABC is shown. (Lesson 13-4)

Triangle ABC is rotated 90° counterclockwise about the origin and then translated along ⟨−2, 3⟩. What is the location of the image of point B?

A. (0, 0)

B. (1, 1)

C. (−4, 6)

D. (−5, 1)

13. **MULTI-SELECT** Select all transformations or composition of transformations that would map $\triangle ABC$ to $\triangle A'B'C'$. (Lesson 13-4)

A. Reflection in the x-axis followed by a reflection in the y-axis

B. Reflection in the x-axis followed by a rotation of 90° counterclockwise about the origin

C. Reflection in $y = -x$

D. Rotation of 180° about the origin

E. Reflection in $y = x$

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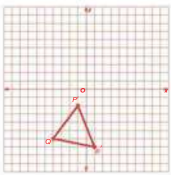
14. GRAPH Graph the image of $\triangle PQR$ with vertices at $P(4, 7)$, $Q(7, 3)$, and $R(2, 2)$ after a translation along $\langle -, -9 \rangle$ and a reflection in $x = 2$. (Lesson 13-4)



15. MULTIPLE CHOICE Which figure has 3 lines of symmetry? (Lesson 13-6)

A. 

B. 

C. 

D. 

16. OPEN RESPONSE How many lines of symmetry does this figure have? Describe the reflections, if any, that map the figure onto itself. (Lesson 13-6)

8; A reflection in any of the 8 lines of symmetry maps the figure onto itself.

17. Open Response State the order and magnitude of symmetry for the object below. (Lesson 13-6)

order = 24; magnitude = 15°

Module 14

Triangles and Congruence

Module Goals

- Students use triangle sum theorems to solve problems.
- Students prove triangles congruent using different congruence criteria.
- Students use congruent triangles to solve problems.

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.10 Prove theorems about triangles.

G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Also addresses G.CO.7, G.CO.8, and G.GPE.4.

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Be Sure to Cover

To completely cover G.CO.12, go online to assign the following constructions:

- Construct a Congruent Triangle (Lessons 14-3 and 14-4)
- Construct an Equilateral Triangle (Lesson 14-6)
- Construct an Isosceles Right Triangle (Lesson 14-6)

Coherence

Vertical Alignment

Previous

Students used transformations to determine congruence between two-dimensional figures.

8.G.2

Now

Students use the definition of congruence in terms of rigid motions to show that two triangles are congruent and use the congruence criteria to solve problems and prove relationships.

G.CO.7, G.CO.8, G.SRT.5

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Practice.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

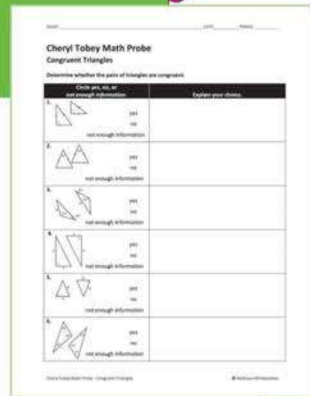
EXPLORE → LEARN → EXAMPLE & PRACTICE

Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
14-1 Angles of Triangles	G.CO.10	2	1
14-2 Congruent Triangles	G.CO.7, G.SRT.5	1	0.5
14-3 Proving Triangles Congruent: SSS, SAS	G.CO.8, G.SRT.5	1	0.5
14-4 Proving Triangles Congruent: ASA, AAS	G.CO.8, G.CO.10, G.SRT.5	1	0.5
Put It All Together: Lessons 14-3 through 14-4		1	0.5
14-5 Proving Right Triangles Congruent	G.CO.8, G.CO.10, G.SRT.5	1	0.5
14-6 Isosceles and Equilateral Triangles	G.CO.10, G.SRT.5	1	0.5
14-7 Triangles and Coordinate Proof	G.CO.10, G.GPE.4	1	0.5
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	12	6

Formative Assessment Math Probe
Congruent Triangles

Module Resource



Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will determine whether pairs of triangles are congruent and explain.

Targeted Concepts Understand the information needed to prove triangle congruence.

Targeted Misconceptions

- Students struggle to connect corresponding sides of triangles with different orientations.
- When using the SAS Postulate, students do not check to make sure that the angle is the included angle.
- When using AAS, students do not check that the congruent sides are corresponding.
- Students sometimes see congruence marks, but do not consider whether the segments of one triangle are congruent to segments of another triangle.
- Students transfer the AA Similarity Postulate to congruent triangles.
- Students only consider using the HL Theorem and not the SAS Postulate in right triangles and/or do not consider using HL as it follows SSA in nonright triangles.

Use the Probe after Lesson 14-6.

Answers: 1. yes; 2. not enough information; 3. no; 4. yes; 5. not enough information; 6. not enough information

Collect and Assess Student Answers

If the student selects these responses...	Then the student likely...
1. no or not enough information **4.** no or not enough information	does not recognize that HL and/or SAS can be used with right triangles. In item 1, students overgeneralize that SSA cannot be used with nonright triangles and only look for HL in Item 4.
2. yes or no	does not recognize that the congruency marks compare segments of individual triangles.
3. yes or not enough information	does not notice that the congruent sides are not corresponding sides.
5. yes	does not recognize that both angles are not included angles.
5. no	is not considering that the other missing side could also be 15.
6. yes	is confusing similarity with congruence.
6. no	is not considering that the corresponding sides could be congruent.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS® Isosceles and Equilateral Triangles
- Lesson 14-6, Learn, Examples 1 and 2

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

IGNITE!

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How can you prove congruence and use congruent figures in real-world situations? Sample answer: Showing combinations of angles and sides in two triangles congruent to one another results in the potential to show two triangle congruent. These congruent triangles can be used to represent objects used in the construction of buildings or mechanical objects.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

Focus Students read about triangle congruence.

Teach Throughout the module, have students take notes under the tabs of their Foldables while working through each lesson. They should include definitions, terms, and key concepts. Encourage students to record examples of each set of triangle congruence criteria from a lesson on the back of their Foldable.

When to Use It Use the appropriate tabs as students cover each lesson in this module. Students should add to the vocabulary tab during each lesson.

Launch the Module

For this module, the Launch the Module video uses viewing artwork to demonstrate the usefulness of congruent triangles. Students learn about using congruent triangles to draw the viewer's eye in a piece of artwork.

Module 14

Triangles and Congruence

Essential Question
How can you prove congruence and use congruent figures in real-world situations?

What Will Y ou Learn?
How much do you already know about each topic before starting this module?

KEY
— I don't know — I've heard of it — I know it!

	Before			After		
	I don't know	I've heard of it	I know it!	I don't know	I've heard of it	I know it!
solve problems using the Triangle Angle-Sum Theorem						
solve problems using the Exterior Angle Theorem						
show that triangles are congruent						
identify corresponding parts of congruent triangles						
solve problems using the SSS Congruence Postulate						
solve problems using the SAS Congruence Postulate						
solve problems using the ASA Congruence Postulate						
solve problems using the AAS Congruence Theorem						
construct congruent triangles						
solve problems using the LL, HA, LA and HL Theorems						
solve problems involving isosceles and equilateral triangles using theorems of triangle congruence						
write coordinate proofs						

Foldables Make this Foldable to help you organize your notes about triangles and congruence. Begin with one sheet of paper.

1. **Fold** a sheet of paper as shown, cutting off the excess paper strip to form a taco.
2. **Open** the fold and refold the square the opposite way to form another taco and an X-fold pattern.
3. **Open** and fold the corners toward the center point of the X, forming a small square.
4. **Label** the flaps as shown.

Module 14 • Triangles and Congruence 837

Interactive Presentation

What Vocabulary Will Y ou Learn?

- auxiliary line
- base angles of an isosceles triangle
- congruent polygons
- coordinate proofs
- corollary
- corresponding parts
- exterior angle of a triangle
- included angle
- included side
- interior angle of a triangle
- isosceles triangle
- legs of an isosceles triangle
- principle of superposition
- remote interior angles
- vertex angle of an isosceles triangle

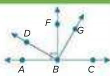
Are Y ou Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review

Example 1

Classify each angle as *right*, *acute*, or *obtuse*.



a. $\angle ABG$

Point G on $\angle ABG$ lies on the exterior of right angle $\angle ABF$, so $\angle ABG$ is an obtuse angle.

b. $\angle DBA$

Point D on $\angle DBA$ lies on the interior of right angle $\angle FBA$, so $\angle DBA$ is an acute angle.

Example 2

Find the distance between $J(5, 2)$ and $K(11, -7)$.

$JK = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$= \sqrt{(11 - 5)^2 + [(-7) - 2]^2}$ Substitute.

$= \sqrt{6^2 + (-9)^2}$ Subtract.

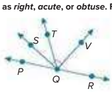
$= \sqrt{36 + 81}$ Simplify.

$= \sqrt{117}$ or about 10.8 Add.

Quick Check

Classify each angle as *right*, *acute*, or *obtuse*.

1. $\angle VQS$ right
2. $\angle TQV$ acute
3. $\angle PQV$ obtuse
4. $\angle SQR$ obtuse



Find the distance between each pair of points. Round to the nearest tenth.

5. $F(3, 6)$ and $G(7, -4)$ 10.8
6. $X(-2, 5)$ and $Y(1, 11)$ 6.7
7. $R(8, 0)$ and $S(-9, 6)$ 18.0
8. $A(14, -3)$ and $B(9, -9)$ 7 .8

How did you do?

Which exercises did you answer correctly in the Quick Check?

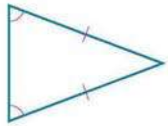
838 **Module 14** • Triangles and Congruence

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define An isosceles triangle is a triangle that has at least two congruent sides.

Example



Ask Do you think the third angle is always the smallest? No; sample answer: The sum of the measures of the angles opposite the congruent sides could be less than 90°, making the measure of the third angle obtuse and the largest angle in the triangle.

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- classifying angles
- analyzing angle relationships in triangles
- analyzing congruent angles and segments
- identifying isosceles and equilateral triangles

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the **Triangles** section to ensure student success in this module.

Mindset Matters

View Challenges as Opportunities

Part of cultivating a growth mindset in math involves viewing challenging problems or tasks as an opportunity to learn and make new connections in your brain.

How Can I Apply It?

Encourage students to embrace challenges by trying problems that are thought provoking, such as the **Higher-Order Thinking Problems** in the practice section of each lesson. Remember to regularly remind students that each new challenge is an opportunity to grow.

Angles of Triangles

G.CO.10

LESSON GOAL

Students solve problems using the Triangle Angle-Sum and Exterior Angle Theorems.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Triangle Angle Sums

Develop:

Interior Angles of Triangles
- Use the Triangle Angle-Sum Theorem

Exterior Angles of Triangles
- Use the Exterior Angle Theorem

Triangle Angle-Sum Corollaries
- Find Angle Measures in Right Triangles

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Complementary and Supplementary Angles	●	●		●
Extension: Stars		●	●	●

Language Development Handbook

Assign page 87 of the *Language Development Handbook* to help your students build mathematical language related to the Triangle Angle-Sum and Exterior Angle Theorems.

ELL You can use the tips and suggestions on page T87 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.10 Prove theorems about triangles.

Standards for Mathematical Practice:

6 Attend to precision.
7 Look for and make use of structure.
8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students used informal arguments to establish facts about the angle sum and exterior angles of triangles.
8.G.5

Now
Students solve problems using Triangle Angle-Sum and Exterior Angle Theorems.
G.CO.10

Next
Students will prove that triangles are congruent.
G.SRT.5

Rigor

The Three Pillars of Rigor

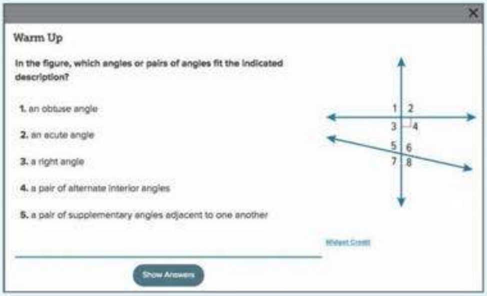
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of angle relationships in triangles and build fluency by proving theorems related to angles of triangles. They apply their understanding by solving real-world problems related to interior and exterior angles of triangles.

Mathematical Background

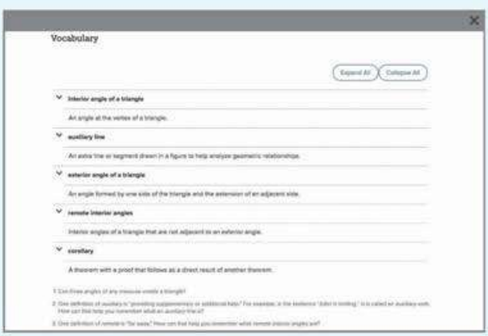
The Triangle Angle-Sum Theorem states that the sum of the measures of the interior angles of a triangle is always 180°. The Triangle Angle-Sum Theorem is used to prove other theorems about angle relationships. Each angle of a triangle has an exterior angle, which is formed by one side of the triangle and the extension of another side.

Interactive Presentation



Warm Up


Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

• classifying angles

Answers:
1. ∠6, ∠7
2. ∠5, ∠8
3. ∠1, ∠2, ∠3, ∠4
4. ∠3, ∠6; ∠5, ∠4
5. ∠1, ∠2; ∠1, ∠3; ∠2, ∠4; ∠3, ∠4; ∠5, ∠6; ∠5, ∠7; ∠6, ∠8; ∠7, ∠8

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of interior and exterior angles of a triangle.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Triangle Angle Sums

Objective

Students use dynamic geometry software to make conjectures about the interior angles of triangles.

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in this Explore, students will need to use dynamic geometry software. Work with students to explore and deepen their understanding of triangle angle sums.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students complete guiding exercises throughout the Explore activity. Students use a sketch to complete the guiding exercises in the Explore. First, students graph a triangle and measure its angles. Then students move the triangle around to observe what happens to the angle measurements. Next students compute the sum of the angle measurements and observe what happens to the sum when they change the triangle. Students write conjectures based on these observations. Then students sketch congruent copies of the triangle in such a way that it leads them to a proof of their conjecture on triangle angle sums. Then, students will answer the Inquiry Question.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Interactive Presentation

Explore

WEB SKETCHPAD

Students use a sketch to complete the activity in the Explore.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and view a sample answer.

Explore Triangle Angle Sums

Questions

Have students complete the Explore activity.

Ask:

- What happens to the shape of the angles as you drag one vertex? Sample answer: Some angles get bigger while others get smaller. This makes sense because the sum is staying equal to 180°.
- What angle relationship could be used with the line parallel to $\overline{AC}$ that goes through *B*? Sample answer: We know that alternate interior angles are congruent, so we could also state that $\angle BAC \cong \angle C'BA$ and $\angle BCA \cong \angle A'BC$.

Inquiry

Is there a relationship associated with the interior angles of a triangle? If so, how do we prove that this relationship is always true? Yes; sample answer: The sum of the measures of the interior angles of a triangle is 180°. By sketching any triangle, we can show that the interior angles can be transformed to create a straight line at one of the vertices of the triangle.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Interior Angles of Triangles

Objective

Students prove the Triangle Angle-Sum Theorem and apply the theorem to solve problems.

MP Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of the Triangle Angle-Sum Theorem in this Learn.

About the Key Concept

The proof of the Triangle Angle-Sum Theorem requires the use of the Parallel Postulate. Because of this, the Triangle Angle-Sum Theorem is only true in Euclidean geometry and not necessarily true in other geometries.

Apply Example 1 Use the Triangle Angle-Sum Theorem

MP Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

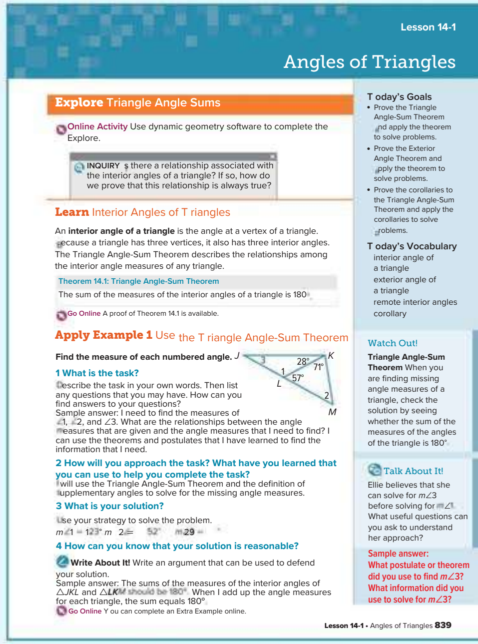
Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- What is the relationship between the angles in $\triangle JKL$?
- What is the relationship between $\triangle JLK$ and $\triangle KLM$?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.



Lesson 14-1

Angles of Triangles

Explore Triangle Angle Sums

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY Is there a relationship associated with the interior angles of a triangle? If so, how do we prove that this relationship is always true?

Learn Interior Angles of Triangles

An **interior angle of a triangle** is the angle at a vertex of a triangle. Because a triangle has three vertices, it also has three interior angles. The Triangle Angle-Sum Theorem describes the relationships among the interior angle measures of any triangle.

Theorem 14.1: Triangle Angle-Sum Theorem

The sum of the measures of the interior angles of a triangle is 180°.

Go Online A proof of Theorem 14.1 is available.

Apply Example 1 Use the Triangle Angle-Sum Theorem

Find the measure of each numbered angle.

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to find the measures of ∠1, ∠2, and ∠3. What are the relationships between the angle measures that are given and the angle measures that I need to find? I can use the theorems and postulates that I have learned to find the information that I need.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

I will use the Triangle Angle-Sum Theorem and the definition of supplementary angles to solve for the missing angle measures.

3 What is your solution?

Use your strategy to solve the problem.

$m\angle 1 = 123^\circ$ $m\angle 2 = 52^\circ$ $m\angle 3 = 29^\circ$

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: The sums of the measures of the interior angles of $\triangle JKL$ and $\triangle LKM$ should be 180°. When I add up the angle measures for each triangle, the sum equals 180°.

Go Online You can complete an Extra Example online.

Today's Goals
- Prove the Triangle Angle-Sum Theorem and apply the theorem to solve problems.
- Prove the Exterior Angle Theorem and apply the theorem to solve problems.
- Prove the corollaries to the Triangle Angle-Sum Theorem and apply the corollaries to solve problems.

Today's Vocabulary
interior angle of a triangle
exterior angle of a triangle
remote interior angles
corollary

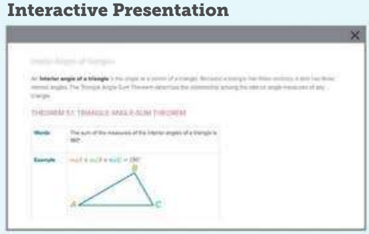
Watch Out!
Triangle Angle-Sum Theorem When you are finding missing angle measures of a triangle, check the solution by seeing whether the sum of the measures of the angles of the triangle is 180°.

Talk About It!
Ellie believes that she can solve for $m\angle 3$ before solving for $m\angle 1$. What useful questions can you ask to understand her approach?

Sample answer: What postulate or theorem did you use to find $m\angle 3$? What information did you use to solve for $m\angle 3$?

Lesson 14-1 • Angles of Triangles **839**

Interactive Presentation



Learn

TAP

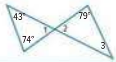
Students tap to see steps in a proof.

CHECK

Students complete the Check online to determine whether they are ready to move on.

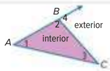
Check

Find the measure of each numbered angle.



$m\angle 1 = 63°$, $m\angle 2 = 63°$, $m\angle 3 = 38$

Learn Exterior Angles of Triangles



interior angles	The sum of the measures of the interior angles of a triangle is 180°.
exterior angles	An **exterior angle of a triangle** is formed by one side of the triangle and the extension of an adjacent side. A triangle has three exterior angles.
remote interior angles	Each exterior angle of a triangle has two **remote interior angles** that are not adjacent to the exterior angle.

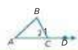
Theorem 14.2 Exterior Angle Theorem

The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

Given: $\triangle ABC$

Prove: $m\angle A + m\angle B = m\angle 1$

Proof:



- $\triangle ABC$ — Given
- $\angle 2$ and $\angle 1$ form a linear pair. — Definition of a linear pair
- $\angle 2$ and $\angle 1$ are supplementary. — If 2 $\angle$s form a linear pair, they are supplementary.
- $m\angle A + m\angle B + m\angle 2 = 180°$ — Triangle Angle-Sum Theorem
- $m\angle 2 + m\angle 1 = 180$ — Definition of supplementary
- $m\angle A + m\angle B + m\angle 2 = m\angle 2 + m\angle 1$ — Substitution
- $m\angle A + m\angle B = m\angle 1$ — Subtraction Property of Equality

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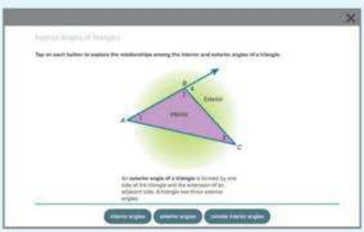
Interactive Presentation



Learn

TAP

Students tap to view examples of definitions.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Learn Exterior Angles of Triangles

Objective

Students prove the Exterior Angle Theorem and apply the theorem to solve problems.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of the Exterior Angle Theorem in this Learn.

Common Misconception

Students frequently have trouble keeping interior and exterior angles straight. Encourage the students to look carefully at the angles and to use the definitions to determine which angles they are.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

DIFFERENTIATE

Language Development Activity AL ELL

Have students draw a copy of the figure for a problem dealing with exterior and interior angles. Then have them color code the angles as exterior or interior, or color code them as an exterior angle matching its remote interior angles, as needed.

DIFFERENTIATE

Reteaching Activity AL

Have students find the sum of the three exterior angles of a triangle and write a proof of their result.

Example 2 Use the Exterior Angle Theorem

Teaching the Mathematical Practices

4 Make Assumptions Have students explain an assumption or approximation that was made to solve the problem.

Questions for Mathematical Discourse

AL What kind of angle is $\angle DAB$? exterior

OL How can you identify the two remote interior angles? They are the two angles that are not adjacent to the exterior angle.

BL What is the measure of $\angle CAB$? 65°

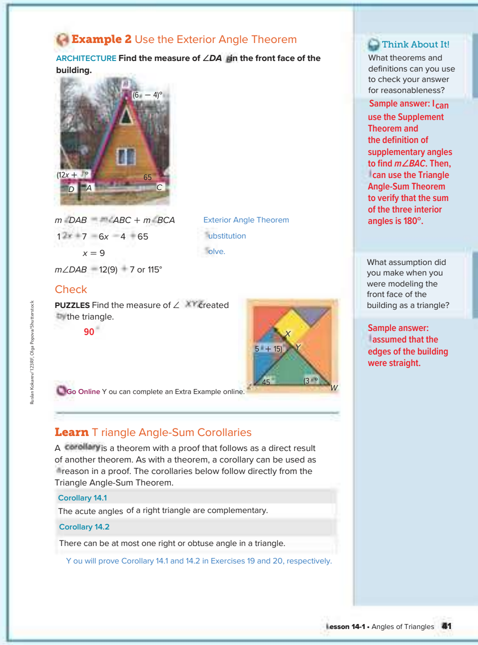
Learn Triangle Angle-Sum Corollaries

Objective

Students prove the corollaries to the Triangle Angle-Sum Theorem and apply the corollaries to solve problems.

Teaching the Mathematical Practices

7 Use Structure Help students use the structure of the Triangle Angle-Sum Theorem to understand the corollaries to the theorem.



Example 2 Use the Exterior Angle Theorem

ARCHITECTURE **Find the measure of ∠DA_ in the front face of the building.**

$m\angle DAB = m\angle ABC + m\angle BCA$ — Exterior Angle Theorem

$12x + 7 = 6x - 4 + 65$ — Substitution

$x = 9$ — Solve.

$m\angle DAB = 12(9) + 7$ or 115°

Check

PUZZLES Find the measure of ∠XYZ created by the triangle.

90°

Go Online You can complete an Extra Example online.

Learn Triangle Angle-Sum Corollaries

A **corollary** is a theorem with a proof that follows as a direct result of another theorem. As with a theorem, a corollary can be used as a reason in a proof. The corollaries below follow directly from the Triangle Angle-Sum Theorem.

Corollary 14.1
The acute angles of a right triangle are complementary.

Corollary 14.2
There can be at most one right or obtuse angle in a triangle.

You will prove Corollary 14.1 and 14.2 in Exercises 19 and 20, respectively.

Think About It!
What theorems and definitions can you use to check your answer for reasonableness?

Sample answer: I can use the Supplement Theorem and the definition of supplementary angles to find $m\angle BAC$. Then, I can use the Triangle Angle-Sum Theorem to verify that the sum of the three interior angles is 180°.

What assumption did you make when you were modeling the front face of the building as a triangle?

Sample answer: I assumed that the edges of the building were straight.

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Interactive Presentation

Learn

CHECK

Students complete the Check online to determine whether they are ready to move on.

Study Tip

Check for Reasonableness When you are solving for the measure of one or more angles of a triangle, always check to make sure that the sum of the angle measures is 180°.

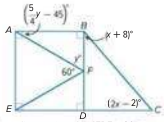
Think About It!

Do you have enough information to solve for $m\angle FED$? If so, explain your solution process.

Yes; sample answer: Because $\triangle FED$ is a right triangle, $m\angle FED + m\angle EFD = 90°$. $\angle AFB$, $\angle AFE$, and $\angle EFD$ form a straight angle, so the sum of the measures of the three angles is 180°. You can find the measure of $\angle EFD$ by subtraction ($180° - 120° = 60°$) and the measure of $\angle FED$ by subtraction ($180° - 150° = 30°$).

Example 3 Find Angle Measures in Right Triangles

Find each measure.



a. $m\angle BCD$

Because $\angle BDC$ and $\angle EDF$ form a linear pair and $m\angle EDF = 90°$, $m\angle BDC = 90°$ by the Supplement Theorem. Therefore, $m\angle BCD + m\angle DBC = 90°$ because the acute angles of a right triangle are complementary.

$m\angle BCD + m\angle DBC = 90°$ Corollary 14.1

$(x + 8) + (2x - 2) = 90°$ Substitution

$x = 28$ Solve

$m\angle BCD = (2(28) - 2)°$ or 54°

b. $m\angle BAF$

Because the acute angles of a right triangle are complementary, $m\angle BAF + m\angle AFB = 90°$.

$90° = m\angle BAF + m\angle AFB$ Corollary 14.1

$= \left(\frac{5}{4}y - 45\right) + y$ Substitution

$60 = y$ Solve

$m\angle BAF = \frac{5}{4}(60) - 45$ or 30°

Check

Find each measure.

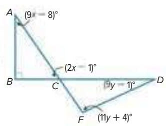
$m\angle BAC$ = ? 73°

$m\angle BCA$ = ? 17°

$m\angle DCF$ = ? 17°

$m\angle CDF$ = ? 71°

$m\angle CFD$ = ? 92°



Go Online You can complete an Extra Example online.

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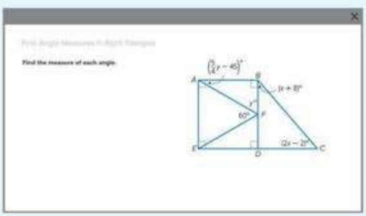
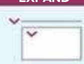
Interactive Presentation



Example 3

EXPAND



Students can tap to see a solution to the problem.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Find Angle Measures in Right Triangles

Teaching the Mathematical Practices

4 Interpret Mathematical Results In this example, point out that to solve the problem, students should interpret their mathematical results in the context of the problem.

Questions for Mathematical Discourse

AL What do you know about $m\angle BDC + m\angle DBC + m\angle C$? The sum is 180°.

OL What kind of angles are $\angle BAF$ and $\angle EAF$? complementary angles

BL Can you find $m\angle EFD$ before you find $m\angle AFB$? Explain. No; sample answer: You don't have enough information to find $m\angle EFD$ until you find $m\angle AFB$.

Common Error

Students may incorrectly set up their equations in Example 3. Help them to use Corollary 5.1 to relate the angle measures correctly.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–14
2	exercises that use a variety of skills from this lesson	15–32
3	exercises that emphasize higher-order and critical-thinking skills	33–36

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

OL BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–31 odd, 33–36
- Extension: Stars
- ALEKS Angles of Triangles

AL OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–35 odd
- Remediation, Review Resources: Complementary and Supplementary Angles
- Personal Tutors
- Extra Examples 1–3
- ALEKS Angle Relationships

AL

IF students score 65% or less on the Checks,
THEN assign:

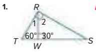
- Practice, Exercises 1–13 odd
- Remediation, Review Resources: Complementary and Supplementary Angles
- *Quick Review Math Handbook*: Angles of Triangles
- ALEKS Angle Relationships

Practice

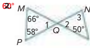
Go Online You can complete your homework online.

Example 1

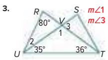
Find the measure of each numbered angle.

1. 

$m\angle 1 = 30°$, $m\angle 2 = 60°$

2. 

$m\angle 1 = 56°$, $m\angle 2 = 56°$, $m\angle 3 = 74°$

3. 

$m\angle 1 = 109°$, $m\angle 2 = 29°$, $m\angle 3 = 71°$

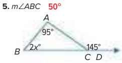
4. 

$m\angle 1 = m\angle 2 = 17°$

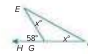
Example 2

Find each measure.

5. $m\angle ABC$ 50°



6. $m\angle F$ 29°



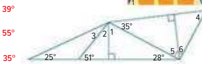
7. TOWERS A lookout tower sits on a network of struts and posts. Leslie measured two angles on the tower. If $m\angle 1 = (7x - 7)°$, $m\angle 2 = (4x + 2)°$, and $m\angle 3 = (2x + 6)°$, what is $m\angle 1$? 98°

8. GARDENING A gardener uses a grow light to grow vegetables indoors. If $m\angle 1 = (8x)°$ and $m\angle 2 = (7x - 4)°$, what is $m\angle 1$? 64°

Example 3

Find each measure.

9. $m\angle 1$ 62°
10. $m\angle 2$ 39°
11. $m\angle 3$ 26°
12. $m\angle 4$ 55°
13. $m\angle 5$ 55°
14. $m\angle 6$ 35°



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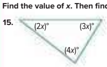
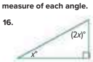
Mixed Exercises

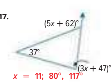
Find the value of *x*. Then find the measure of each angle.

15. 

$x = 20$; 40°, 60°, 80°

16. 

$x = 30$; 30°, 60°

17. 

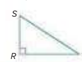
$x = 11$; 80°, 117°

18. CONSTRUCT ARGUMENTS Determine whether the following statement is *true* or *false*. If false, give a counterexample. If true, give an argument to support your conclusion. See margin.

If the sum of two acute angles of a triangle is greater than 90°, then the triangle is acute.

PROOF **Write the specified type of proof for each corollary.**

19. flow proof of Corollary 14.1
Given: $\angle R$ is a right angle.
Prove: $\angle S$ and $\angle T$ are complementary.



Proof:
$\angle R$ is a rt. $\angle$. (Given) → $m\angle R = 90°$ (Def. of rt. $\angle$) → $90 + m\angle S + m\angle T = 180°$ (Substitution) → $m\angle S + m\angle T = 90°$ (Substraction Prop.) → $\angle S$ and $\angle T$ are complementary (Def. of comp. angles)
$m\angle R + m\angle S + m\angle T = 180°$ (Triangle Angle-Sum Thm.) → Substitution

20. paragraph proof of Corollary 14.2

a. Case 1
Given: $\triangle MNO$; $\angle M$ is a right angle.
Prove: There can be at most one right angle in a triangle.

Proof: It is given that in $\triangle MNO$, $\angle M$ is a right angle. $m\angle M + m\angle N + m\angle O = 180°$ by the Triangle Angle-Sum Theorem. $m\angle M = 90°$ by the definition of a right angle. So, $90° + m\angle N + m\angle O = 180°$, and $m\angle N + m\angle O = 90°$ by the Subtraction Property of Equality. Because $\angle N$ and $\angle O$ are angles in a triangle, $m\angle N > 0°$ and $m\angle O > 0°$. Therefore, by the Substitution Property of Equality, $m\angle N \neq 90°$ and $m\angle O \neq 90°$. So, there cannot be two right angles in a triangle.

b. Case 2
Given: $\triangle PQR$; $\angle P$ is an obtuse angle.
Prove: There can be at most one obtuse angle in a triangle.

Proof: It is given that in $\triangle PQR$ $\angle P$ is an obtuse angle. By the Triangle Angle-Sum Theorem, $m\angle P + m\angle Q + m\angle R = 180°$. By the definition of an obtuse angle, $m\angle P > 90°$. So, $m\angle Q + m\angle R < 90$. Therefore, $\angle Q$ and $\angle R$ must be acute angles.

REASONING **Solve each problem.**

21. In triangle DEF, $m\angle E$ is three times $m\angle D$, and $m\angle F$ is 9° less than $m\angle E$. What is the measure of each angle? $m\angle D = 27°$, $m\angle E = 81°$, $m\angle F = 72°$

22. In triangle RST, $m\angle T$ is 5° more than $m\angle R$, and $m\angle S$ is 10° less than $m\angle T$. What is the measure of each angle? $m\angle R = 60°$, $m\angle S = 55°$, $m\angle T = 65°$

23. In triangle JKL, $m\angle K$ is four times $m\angle J$, and $m\angle L$ is five times $m\angle J$. What is the measure of each angle?

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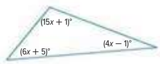
24. Classify the triangle shown by its angles. Justify your reasoning. See margin.



25. In $\triangle XYZ$, $m\angle X = 157°$, $m\angle Y = y°$, and $m\angle Z = z°$. Write an inequality to describe the possible measures of $\angle Z$. Justify your reasoning. See margin.

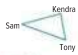
26. AUTOMOBILES Refer to the image at the right. b–c. See margin.

a. Find $m\angle 1$ and $m\angle 2$. $m\angle 1 = 149°$; $m\angle 2 = 31°$

b. If the brown hood prop rod were shorter than the one shown, how would $m\angle 1$ change? Explain.

c. If the brown hood prop rod were shorter than the one shown, how would $m\angle 2$ change? Explain.

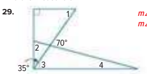
27. BASKETBALL Sam, Kendra, and T ony are passing a basketball. If Sam is looking at Kendra, then he needs to turn 40° to pass to T ony. If T ony is looking at Sam, then he needs to turn 50° to pass to Kendra. How many degrees would Kendra have to turn her head to look at T ony if she is looking at Sam? 90°



28. CONSTRUCTION The diagram shows an example of the Pratt Truss used in bridge construction. Find $m\angle 1$. 55°

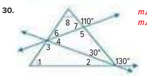
Find the measure of each numbered angle.

29.



$m\angle 1 = 55°$, $m\angle 2 = 75°$, $m\angle 3 = 55°$, $m\angle 4 = 15°$

30.



$m\angle 1 = 65°$, $m\angle 2 = 20°$, $m\angle 3 = 95°$, $m\angle 4 = 40°$, $m\angle 5 = 110°$, $m\angle 6 = 45°$, $m\angle 7 = 70°$, $m\angle 8 = 65°$

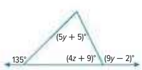
31. USE TOOLS Use tracing paper to verify the Triangle Angle-Sum Theorem. Describe your method and include a sketch. See margin.

32. ANALYZE In $\triangle ABC$, if an exterior angle adjacent to $\angle A$ is acute, is the triangle acute, right, or obtuse, or can its classification not be determined? Explain your reasoning.
Obtuse; because the exterior angle is acute, the sum of the remote interior angles must be acute, which means that the third angle must be obtuse. Therefore, the triangle must be obtuse.

Higher-Order Thinking Skills

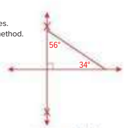
33. WRITE Explain why a triangle cannot have an obtuse, acute, and a right exterior angle.
Sample answer: Because an exterior angle is acute, the adjacent angle must be obtuse. Because another exterior angle is right, the adjacent angle must be right. A triangle cannot contain a right angle and an obtuse angle because the sum would be greater than 180°. Therefore, a triangle cannot have an obtuse, an acute, and a right exterior angle.

34. PERSEVERE Find the values of y and z in the figure at the right. $y = 13$, $z = 14$



35. CREATE Construct a right triangle and measure one of the acute angles. Calculate the measure of the second acute angle and explain your method. Confirm your result using a protractor.

Sample answer: I found the measure of the second angle by subtracting the first angle from 90° because the acute angles of a right triangle are complementary.



36. PERSEVERE The Flatiron Building in New Y ork City is one of America's oldest skyscrapers, completed in 1902. Its floor plan is approximately a right triangle. As shown in the figure, 5th Avenue is perpendicular to East 22nd Street, and $m\angle B$ is 10 less than 3 times $m\angle C$. See margin.

a. Find the angle measures in the floor plan.

b. Find $m\angle BCD$ in two ways. Explain each method.

Answers

18. True; sample answer: Because the sum of the two acute angles is greater than 90°, the measure of the third angle is a number greater than 90° subtracted from 180°, which must be less than 90°. Therefore, the triangle has three acute angles and is acute.

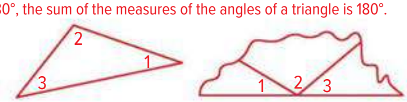
24. Obtuse; the sum of the measures of the three angles of a triangle is 180°. So, $(15x + 1) + (6x + 5) + (4x - 1) = 180°$ and $x = 7$. Substituting 7 into the expressions for each angle, the angle measures are 106°, 47°, and 27°. Because the triangle has an obtuse angle, it is obtuse.

25. $m\angle Z < 23°$; Sample answer: Because the sum of the measures of the angles of a triangle is 180° and $m\angle X = 157°$, $157° + m\angle Y + m\angle Z = 180°$, so $m\angle Y + m\angle Z = 23°$. If $m\angle Y$ was 0°, then $m\angle Z$ would equal 23°. But because an angle must have a measure greater than 0°, $m\angle Z$ must be less than 23°, so $m\angle Z < 23$.

26b. Sample answer: The measure of $\angle 1$ would get larger if the support were shorter because the hood would be closer to the leg of the triangle that is along the engine of the car.

26c. Sample answer: The measure of $\angle 2$ would get smaller if the support were shorter because $\angle 1$ would get larger and they are a linear pair.

31. Sample answer: Draw a triangle and then tear the corners off the triangle. Arrange the three corners so the angles are adjacent. The angles now form a straight angle. Because a straight angle measures 180°, the sum of the measures of the angles of a triangle is 180°.



36a. $m\angle A = 90°$, $m\angle B = 65°$, $m\angle C = 25°$.

36b. $m\angle BCD = m\angle A + m\angle B = 90° + 65°$ or 155° (Exterior Angle Theorem)

$m\angle BCD = 180° - m\angle C = 180° - 25°$ or 155° (Supplement Theorem)

Lesson 14-2

Congruent Triangles

G.CO.7, G.SRT.5

LESSON GOAL

Students prove that triangles are congruent and use congruence statements to solve problems.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Relationships in Congruent Triangles

Develop:

Congruent Triangles
- Identify Corresponding Congruent Parts
- Use Corresponding Parts of Congruent Triangles

Third Angles Theorem and Triangle Congruence
- Use the Third Angles Theorem
- Prove that Two Triangles Are Congruent

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	BL	ELL	
Remediation: Angle Relationships and Triangles	●	●		●
Extension: Overlapping Triangles		●	●	●

Language Development Handbook

Assign page 88 of the *Language Development Handbook* to help your students build mathematical language related to proving that triangles are congruent and using congruence statements.

ELL You can use the tips and suggestions on page T88 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.7 Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

3 Construct viable arguments and critique the reasoning of others.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous
Students solved problems using Triangle Angle-Sum and Exterior Angle Theorems.
G.CO.10

Now
Students prove that triangles are congruent.
G.SRT.5

Next
Students will prove congruent triangles using the SSS and SAS Theorems.
G.SRT.5

Rigor

The Three Pillars of Rigor

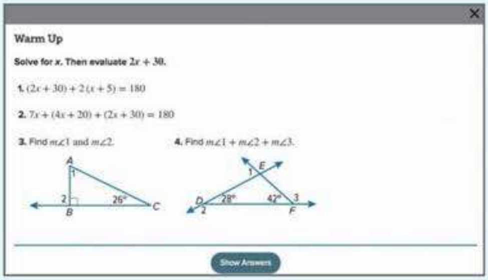
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students use rigid motions to develop an understanding of congruence. They build fluency and apply their understanding by solving real-world problems related to congruent triangles.

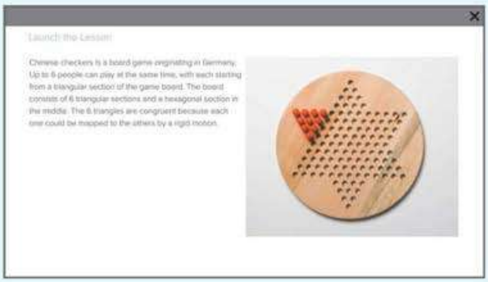
Mathematical Background

Two triangles are congruent if and only if their corresponding parts are congruent. Certain transformations do not affect congruence. These transformations are called rigid motions. Congruence of triangles, like that of angles and segments, is reflexive, symmetric, and transitive.

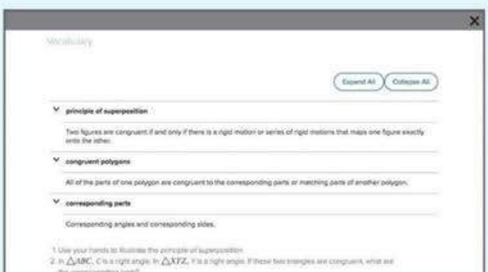
Interactive Presentation



Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- analyzing angle relationships in triangles

Answers:

1. 35; 100
2. 10; 50
3. 64°; 90°
4. 360°

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of congruent triangles.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Relationships in Congruent Triangles

Objective

Students use dynamic geometry software to make conjectures about the relationships between corresponding sides and angles in congruent triangles.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

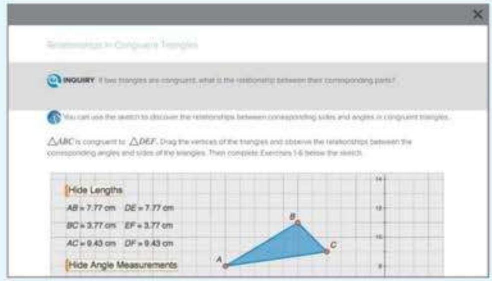
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students use a sketch to explore the relationships between corresponding parts of two congruent triangles. First students discover that corresponding sides are congruent and then that corresponding angles are congruent. Lastly, students are led to discover that all pairs of corresponding parts of congruent triangles are congruent. Then, students will answer the Inquiry Question.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

(*continued on the next page*)

Interactive Presentation



Explore

WEB SKETCHPAD

Students use the sketch to complete the activity in the Explore.

TYPE

Students type to complete the guiding exercises.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and view a sample answer.

Explore Relationships in Congruent Triangles (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What does it mean for two shapes to be congruent? Sample answer: They have the same shape and size, but not necessarily the same orientation.
- If you know $m\angle A = 58°$, what is true about $\triangle DEF$? Sample answer: Because $\angle A \cong \angle D$, you also know that their measures are equal and $m\angle D = 58°$.

Inquiry

If two triangles are congruent, what is the relationship between their corresponding parts? Sample answer: The corresponding sides are congruent, and the corresponding angles are congruent.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Congruent T riangles

Objective

Students use congruence criterion of corresponding congruent parts of triangles to solve problems.

MP Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of triangle congruence in this Learn to determine whether two triangles are congruent.

Common Misconception

After students learn simpler congruence criteria in later lessons, they might forget that the full definition of triangle congruence requires that all corresponding parts are congruent. Remind them to return to the definition throughout the module to reinforce that other triangle congruence criteria are shortcuts, not the full definition of congruence.

Essential Question Follow-Up

Students use triangle congruence to solve problems.

Ask:

Why is it useful to know when two triangles are congruent? Sample answer: When two triangles are congruent, you know that their corresponding parts are congruent.

Example 1 Identify Corresponding Congruent Parts

MP Teaching the Mathematical Practices

3 Find the Error This example requires students to read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Questions for Mathematical Discourse

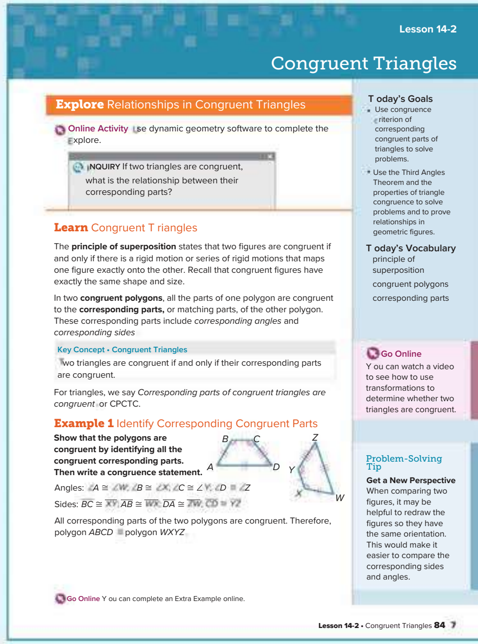
AL What do you need to know to show that two polygons are congruent? Sample answer: Y ou need to know that corresponding angles are congruent and their corresponding sides are congruent.

OL What do the tick marks on the sides mean? the arc marks on the angles? Sides with the same number of tick marks are congruent; angles with the same number of arc marks are congruent.

BL Is it correct to say polygon *ABCD* is congruent to polygon *ZWXY*? Explain. No; sample answer: When you name congruent polygons, you must list the corresponding vertices in order.

Common Error

Students may confuse which of two endpoints of a segment correspond to a particular endpoint of the corresponding segment. Help them determine which one is the corresponding endpoint by looking at the relationships to the other parts of the figures.



Lesson 14-2

Congruent Triangles

Explore Relationships in Congruent Triangles

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY If two triangles are congruent, what is the relationship between their corresponding parts?

Learn Congruent T riangles

The **principle of superposition** states that two figures are congruent if and only if there is a rigid motion or series of rigid motions that maps one figure exactly onto the other. Recall that congruent figures have exactly the same shape and size.

In two **congruent polygons**, all the parts of one polygon are congruent to the **corresponding parts**, or matching parts, of the other polygon. These corresponding parts include *corresponding angles* and *corresponding sides*

Key Concept • Congruent Triangles

Two triangles are congruent if and only if their corresponding parts are congruent.

For triangles, we say *Corresponding parts of congruent triangles are congruent*, or CPCTC.

Example 1 Identify Corresponding Congruent Parts

Show that the polygons are congruent by identifying all the congruent corresponding parts. Then write a congruence statement.

Angles: ∠A ≅ ∠W, ∠B ≅ ∠X, ∠C ≅ ∠Y, ∠D ≅ ∠Z

Sides: $\overline{BC} \cong \overline{XY}$, $\overline{AB} \cong \overline{WX}$, $\overline{DA} \cong \overline{ZW}$, $\overline{CD} \cong \overline{YZ}$

All corresponding parts of the two polygons are congruent. Therefore, polygon *ABCD* ≅ polygon *WXYZ*

Go Online You can complete an Extra Example online.

Today's Goals
- Use congruence criterion of corresponding congruent parts of triangles to solve problems.
- Use the Third Angles Theorem and the properties of triangle congruence to solve problems and to prove relationships in geometric figures.

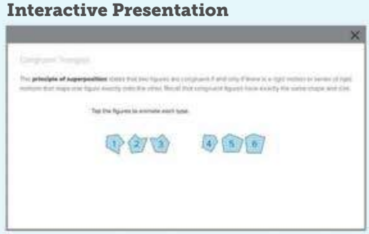
Today's Vocabulary
principle of superposition
congruent polygons
corresponding parts

Go Online
You can watch a video to see how to use transformations to determine whether two triangles are congruent.

Problem-Solving Tip
Get a New Perspective When comparing two figures, it may be helpful to redraw the figures so they have the same orientation. This would make it easier to compare the corresponding sides and angles.

Lesson 14-2 • Congruent Triangles 847

Interactive Presentation



Learn

TAP

Students tap to animate illustrations of congruent figures.

Study Tip

Congruence Statements Valid congruence statements for congruent polygons list corresponding vertices in the same order.

Check

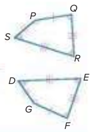
Show that the polygons are congruent by identifying all the congruent corresponding parts. Then write a congruence statement.

$\angle P \cong$? $\angle G$; $\angle Q \cong$? $\angle F$; $\angle R \cong$? $\angle E$; $\angle S \cong$? $\angle D$

$\overline{PQ} \cong$? $\overline{GF}$; $\overline{QR} \cong$? $\overline{FE}$; $\overline{RS} \cong$? $\overline{ED}$; $\overline{SP} \cong$? $\overline{DG}$

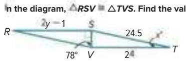
Complete the congruence statement.

Polygon $PQRS \cong$ Polygon ? $GFED$



Example 2 Use Corresponding Parts of Congruent Triangles

In the diagram, $\triangle RSV \cong \triangle TVS$. Find the values of x and y.



Study Tip

Use a Congruence Statement You can use a congruence statement to help you correctly identify corresponding sides.
$\triangle RSV \cong \triangle TVS$
$\overline{RS} \cong \overline{TV}$

Part A Find the value of x.

$\angle T \cong \angle R$	CPCTC
$m\angle T = m\angle R$	Definition of congruence
$= 180° - 90° - 78°$	Triangle Angle-Sum Theorem
$= 12°$	Solve.

The value of x is 12.

Part B Find the value of y.

$\overline{RS} \cong \overline{TV}$	CPCTC
$RS = TV$	Definition of congruence
$2y - 1 = 24$	Substitution
$y = 12.5$	Solve.

The value of y is 12.5.

Talk About It!

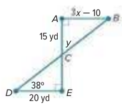
Suppose the congruence statement $\triangle RSV \cong \triangle TVS$ was *not* given. Would you be able to solve this problem? Explain.

No; sample answer: Without knowing which parts are corresponding parts, you would not be able to write the equations to solve.

Check

In the diagram, $\triangle ABC \cong \triangle EDC$. Find the values of x and y.

$x =$? 10; $y =$? 52



Go Online You can complete an Extra Example online.

848 Module 14 • Triangles and Congruence

DIFFERENTIATE

Language Development Activity ELL

Explain to students that congruency can be modeled with drum beats. Point out that to model two congruent equilateral triangles, they could use three equally spaced drum beats for the first and then repeat the exact same rhythm for the second. An isosceles beat could consist of two quick beats and one slow beat or vice versa. Tell students that often in music, a "congruent" rhythm is used throughout a song.

Example 2 Use Corresponding Parts of Congruent Triangles

MP Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to construct an argument.

Questions for Mathematical Discourse

AL What does *CPCTC* mean? Corresponding parts of congruent triangles are congruent.

OL What angle corresponds to $\angle RSV$? $\angle TVS$ What side corresponds to $\overline{ST}$? $\overline{RV}$

BL What is $m\angle RST$? What angle is congruent to $\angle RST$? $m\angle RST = 168°$, and $\angle RST \cong \angle RVT$

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Interactive Presentation

Example 2

TAP

Students tap to reveal parts of solutions and to choose answers.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Third Angles Theorem and T riangle Congruence

Objective

Students use the Third Angles Theorem to solve problems and to prove relationships in geometric figures.

MP Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of the Third Angles Theorem in this Learn.

Example 3 Use the Third Angles Theorem

MP Teaching the Mathematical Practices

3 Find the Error This example requires students to read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Questions for Mathematical Discourse

AL What is the measure of $\angle ABC$? 62°

OL How can symbols be used to represent the congruent sides and angles? Use 1, 2, and 3 tick marks on the three pairs of corresponding sides. Use 1 and 2 arcs in the two pairs of congruent, acute angles.

BL How are the three triangles classified? $\triangle ABC$ is isosceles acute, and $\triangle ABD$ and $\triangle DBC$ are right scalene

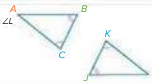
Common Error

Students might confuse the third angles of being congruent with the acute angles of right triangles being complementary. Help them to remember the similarities and differences between the two theorems.

Learn Third Angles Theorem and T riangle Congruence

Theorem 14.3: Third Angles Theorem

Words	If two angles of one triangle are congruent to two angles of a second triangle, then the third angles of the triangles are congruent.
Example	If $\angle C \cong \angle K$ and $\angle B \cong \angle J$, then $\angle A \cong \angle L$.



Y ou will prove Theorem 14.3 in Exercise 25.

Like congruence of segments and angles, congruence of triangles is reflexive, symmetric, and transitive.

Theorem 14.4: Properties of Triangle Congruence

Reflexive Property of Triangle Congruence
$\triangle ABC \cong \triangle ABC$

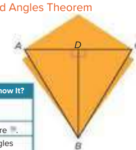
Symmetric Property of Triangle Congruence
If $\triangle ABC \cong \triangle EFG$, then $\triangle EFG \cong \triangle ABC$

Transitive Property of Triangle Congruence
If $\triangle ABC \cong \triangle EFG$ and $\triangle EFG \cong \triangle JKL$, then $\triangle ABC \cong \triangle JKL$

Go Online Proofs of Theorem 14.4 are available.

Example 3 Use the Third Angles Theorem

ORIGAMI Aika is folding origami dragons for a party she is hosting. If $\angle ABD \cong \angle CBD$ and $m\angle BAD = 58°$, find $m\angle CBD$.



What Do You Know?	How Do You Know It?
$\angle ABD \cong \angle CBD$, $m\angle BAD = 58°$	Given
$\angle BDC \cong \angle BDA$	All rt. $\angle$s are $\cong$.
$\angle BAD \cong \angle BCD$	Third Angles Theorem
$m\angle CBD + m\angle BCD = 90°$	The acute $\angle$s of a rt. $\triangle$ are compl.
$m\angle BCD = m\angle BAD$	Def. of congruence

$m\angle BCD = 58°$ Transitive Property

$m\angle CBD + 58° = 90°$ Substitute.

$m\angle CBD = 32°$ Solve

The measure of $\angle CBD$ is 32°.

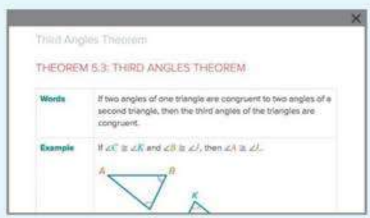
Go Online Y ou can complete an Extra Example online.

Think About It! How could you find the $m\angle CBD$ in a different way?

Sample answer: I can use the Triangle Angle-Sum Theorem to find $m\angle ABD$. Then $m\angle ABD = m\angle CBD$ by the definition of congruent angles.

Lesson 14-2 • Congruent Triangles **849**

Interactive Presentation



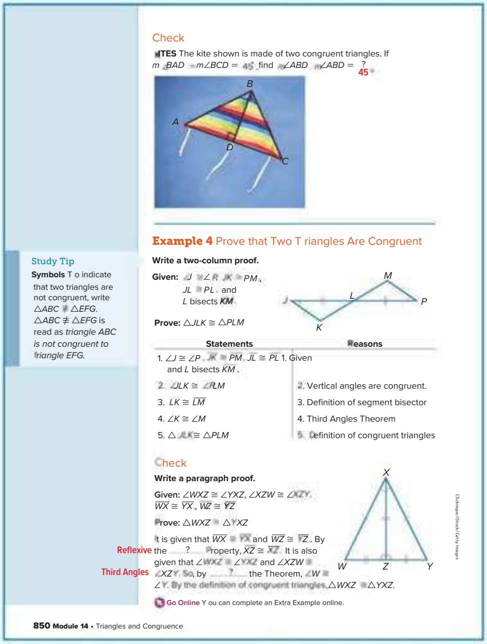
Learn

TAP

Students tap to see animations of triangle congruence properties.

DRAG & DROP

Students drag statements and reasons to their correct locations.



Check

TES The kite shown is made of two congruent triangles. If $m\angle BAD = m\angle BCD = 45^\circ$, find $m\angle ABD$. $m\angle ABD = ?$ 45°

Example 4 Prove that Two Triangles Are Congruent

Write a two-column proof.

Given: $\angle J \cong \angle P$, $\overline{JK} \cong \overline{PM}$, $\overline{JL} \cong \overline{PL}$, and L bisects $\overline{KM}$

Prove: $\triangle JLK \cong \triangle PLM$

Statements	Reasons
1. $\angle J \cong \angle P$, $\overline{JK} \cong \overline{PM}$, $\overline{JL} \cong \overline{PL}$, and L bisects $\overline{KM}$.	1. Given
2. $\angle JLK \cong \angle PLM$	2. Vertical angles are congruent.
3. $\overline{LK} \cong \overline{LM}$	3. Definition of segment bisector
4. $\angle K \cong \angle M$	4. Third Angles Theorem
5. $\triangle JLK \cong \triangle PLM$	5. Definition of congruent triangles

Study Tip

Symbols To indicate that two triangles are not congruent, write $\triangle ABC \ncong \triangle EFG$. $\triangle ABC \ncong \triangle EFG$ is read as *triangle ABC is not congruent to triangle EFG.*

Check

Write a paragraph proof.

Given: $\angle WXZ \cong \angle YXZ$, $\angle XZW \cong \angle XZY$, $\overline{WX} \cong \overline{YX}$, $\overline{WZ} \cong \overline{YZ}$

Prove: $\triangle WXZ \cong \triangle YXZ$

It is given that $\overline{WX} \cong \overline{YX}$ and $\overline{WZ} \cong \overline{YZ}$. By the ? Property, $\overline{XZ} \cong \overline{XZ}$. It is also given that $\angle WXZ \cong \angle YXZ$ and $\angle XZW \cong \angle XZY$. So, by the ? Theorem, $\angle W \cong \angle Y$. By the definition of congruent triangles, $\triangle WXZ \cong \triangle YXZ$.

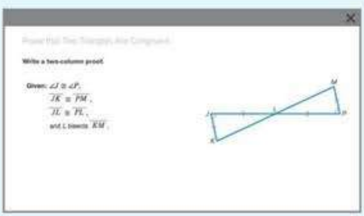
Reflexive
Third Angles

Clicknique/iStock/Getty Images

Go Online You can complete an Extra Example online.

850 Module 14 • Triangles and Congruence

Interactive Presentation



Example 4

TAP

Students tap to reveal parts of solutions and to choose answers.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 4 Prove that Two Triangles Are Congruent

MP Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to complete the given proof.

Questions for Mathematical Discourse

AL What is the relationship between $\angle JLK$ and $\angle PLM$? They are vertical angles.

OL Which corresponding parts of the triangles are not shown to be congruent in the given information? $\angle JLK$ and $\angle PLM$, $\angle K$ and $\angle M$, and $\overline{LK}$ and $\overline{LM}$

BL What piece of given information could have been replaced by the information that L bisects $\overline{JP}$? $\overline{JL} \cong \overline{PL}$

DIFFERENTIATE

Enrichment Activity BL

Have students write their own questions in which two triangles are proved to be congruent. Some corresponding parts should be given as congruent, but other parts must be shown to be congruent from other information. Have students solve each other's questions.

Exit Ticket

Recommended Use

At the end of class, have students respond to the Exit Ticket prompt using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, have students respond to the Exit Ticket prompt verbally or by using a mini-whiteboard. Have students hold up their whiteboards so you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–17
2	exercises that use a variety of skills from this lesson	18–25
3	exercises that emphasize higher-order and critical-thinking skills	26–30

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–25 odd, 26–30
- Extension: Overlapping Triangles
- ALEKS Congruent Triangles

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–29 odd
- Remediation, Review Resources: Congruent Triangles
- Personal Tutors
- Extra Examples 1–4
- ALEKS Angles of Triangles

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Congruent Triangles
- *Quick Review Math Handbook*: Congruent Triangles
- ALEKS Angles of Triangles

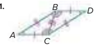
Practice

Go Online You can complete your homework online.

Example 1

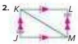
Show that the polygons are congruent by identifying all the congruent corresponding parts. Then write a congruence statement.

1.



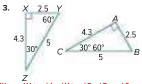
$\angle A \cong \angle D$; $\angle ABC \cong \angle DCB$; $\angle ACB \cong \angle DBC$; $\overline{AC} \cong \overline{DB}$; $\overline{AB} \cong \overline{DC}$; $\triangle ABC \cong \triangle DCB$

2.



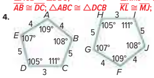
$\angle J \cong \angle L$; $\angle JKM \cong \angle LMK$; $\angle KMJ \cong \angle MKL$; $\overline{KJ} \cong \overline{ML}$; $\overline{KL} \cong \overline{MJ}$; $\triangle JKM \cong \triangle LMK$

3.



$\angle X \cong \angle A$; $\angle Y \cong \angle B$; $\angle Z \cong \angle C$; $\overline{XY} \cong \overline{AB}$; $\overline{XZ} \cong \overline{AC}$; $\overline{YZ} \cong \overline{BC}$; $\triangle XYZ \cong \triangle ABC$

4.



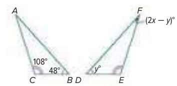
$\angle A \cong \angle F$; $\angle B \cong \angle J$; $\angle C \cong \angle I$; $\angle D \cong \angle H$; $\angle E \cong \angle G$; $\overline{AB} \cong \overline{FJ}$; $\overline{BC} \cong \overline{JI}$; $\overline{CD} \cong \overline{IH}$; $\overline{DE} \cong \overline{HG}$; $\overline{AE} \cong \overline{FG}$; polygon $ABCDE \cong$ polygon $FJIHG$

5.

$\angle R \cong \angle J$; $\angle T \cong \angle K$; $\angle S \cong \angle L$; $\overline{RT} \cong \overline{JK}$; $\overline{TS} \cong \overline{KL}$; $\overline{RS} \cong \overline{JL}$; $\triangle RTS \cong \triangle JKL$

Example 2

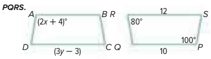
In the diagram, $\triangle ABC \cong \triangle FDE$.



6. Find the value of x. 36

7. Find the value of y. 48

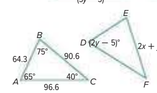
In the diagram, polygon $ABCD \cong$ polygon $PQRS$.



8. Find the value of x. 48

9. Find the value of y. 5

In the diagram, $\triangle ABC \cong \triangle DEF$.

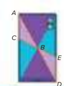


10. Find the value of x. 27.8

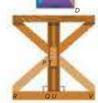
11. Find the value of y. 35

Example 3

12. DESIGN Camila is designing a new image for her cell phone case. If $m\angle ABC = 35°$, $m\angle BAC = 29°$, and $\angle ACB \cong \angle DEB$, what is $m\angle DEB$? 116°



13. CARPENTRY Mr. Lewis is building a rustic dining table. Instead of having four legs, the table has a set of supports at each end. If $\angle PRQ \cong \angle TVU$ and $m\angle RPQ = 49°$, what is $m\angle TVU$? 41°

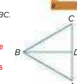


Example 4

PROOF **For 14–16, write a two-column proof.**

14. Given: $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{CD}$, $\angle BAD \cong \angle BCD$, $\overline{BD}$ bisects $\angle ABC$.

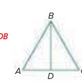
Prove: $\triangle ABD \cong \triangle CBD$



Proof:
Statements (Reasons)
1. $\overline{AB} \cong \overline{CB}$; $\overline{AD} \cong \overline{CD}$ (Given)
2. $\overline{BD} \cong \overline{BD}$ (Reflexive Prop. of Congruence)
3. $\angle BAD \cong \angle BCD$ (Given)
4. $\overline{BD}$ bisects $\angle ABC$. (Given)
5. $\angle ABD \cong \angle CBD$ (Def. of angle bisector)
6. $\angle BDA \cong \angle BDC$ (Third Angles Theorem)
7. $\triangle ABD \cong \triangle CBD$ (Def. of congruent triangles)

15. Given: $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{CD}$, $\angle ABD \cong \angle CBD$, $\angle ADB \cong \angle CDB$

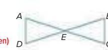
Prove: $\triangle ABD \cong \triangle CBD$



Proof:
Statements (Reasons)
1. $\overline{AB} \cong \overline{CB}$; $\overline{AD} \cong \overline{CD}$ (Given)
2. $\overline{BD} \cong \overline{BD}$ (Reflexive Prop. of Congruence)
3. $\angle ABD \cong \angle CBD$, $\angle ADB \cong \angle CDB$ (Given)
4. $\angle A \cong \angle C$ (Third Angles Theorem)
5. $\triangle ABD \cong \triangle CBD$ (Def. of congruent triangles)

16. Given: $\angle A \cong \angle C$, $\angle D \cong \angle B$, $\overline{AD} \cong \overline{CB}$, $\overline{AE} \cong \overline{CE}$, $\overline{AC}$ bisects $\overline{BD}$.

Prove: $\triangle AED \cong \triangle CEB$

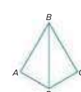


Proof:
Statements (Reasons)
1. $\angle A \cong \angle C$, $\angle D \cong \angle B$ (Given)
2. $\angle AED \cong \angle CEB$ (Vertical angles are ≅.)
3. $\overline{AD} \cong \overline{CB}$; $\overline{AE} \cong \overline{CE}$ (Given)
4. $\overline{AC}$ bisects $\overline{BD}$. (Given)
5. $\overline{DE} \cong \overline{BE}$ (Def. of segment bisector)
6. $\triangle AED \cong \triangle CEB$ (Def. of congruent triangles)

PROOF **Write a paragraph proof.**

17. Given: $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$, $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{CD}$

Prove: $\triangle ABD \cong \triangle CBD$



Proof:
It is given that $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$. Therefore, $\angle ABD \cong \angle CBD$ and $\angle ADB \cong \angle CDB$ by the definition of angle bisector. By the Third Angles Theorem, $\angle A \cong \angle C$. It is given that $\overline{AB} \cong \overline{CB}$ and $\overline{AD} \cong \overline{CD}$. By the Reflexive Property of Congruence, $\overline{BD} \cong \overline{BD}$. Therefore, $\triangle ABD \cong \triangle CBD$ by the definition of congruent triangles.

Mixed Exercises

18. PRECISION Beverly is using loyalty cards at her coffee shop. When a customer purchases a cup of coffee, he or she can present a loyalty card to be stamped with a star-shaped stamp that Beverly purchased specifically for this use. When the customer collects nine stamps, they receive their tenth cup of coffee for free. What property guarantees that the stamped designs are congruent?

Sample answer: Both stamped stars are congruent to the star on the stamper because it was used to create the images. According to the Transitive Property of Polygon Congruence, the two stamped images are congruent to each other because both are congruent to the star on the stamper.

Draw and label a figure to represent the congruent triangles. Then find the values of *x* and *y*.
19–21. See margin.

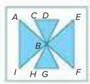
19. $\triangle ABC \cong \triangle DEF$, $AB = 7$, $BC = 9$, $AC = 11 + x$, $DF = 3x - 13$, and $DE = 2y - 5$

20. $\triangle LMN \cong \triangle RST$, $m\angle L = 49°$, $m\angle M = 10y°$, $m\angle S = 70°$, and $m\angle T = (4x + 9)°$

21. $\triangle JKL \cong \triangle MNP$, $JK = 12$, $LJ = 5$, $PM = 2x - 3$, $m\angle L = 67°$, $m\angle K = (y + 4)°$ and $m\angle N = (2y - 15)°$

22. SIERPINSKI TRIANGLE The figure shown is a portion of the Sierpinski triangle. The triangle has the property that any triangle made from any combination of edges is equilateral. How many triangles in this portion are congruent to the black triangle at the bottom? 11

23. LOGO DESIGNS Refer to the design shown.



a. Indicate the triangles that appear to be congruent.
$\triangle ABI \cong \triangle EBF$, $\triangle CBD \cong \triangle HBG$

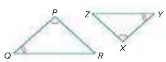
b. Name the congruent angles and congruent sides of a pair of congruent triangles. Sample answer: $\angle A \cong \angle E$, $\angle ABI \cong \angle EBF$, $\angle I \cong \angle F$; $\overline{AB} \cong \overline{EB}$, $\overline{BI} \cong \overline{BF}$, $\overline{AI} \cong \overline{EF}$

24. REASONING Igor noticed on a map that the triangle with vertices that are at the supermarket, the library, and the post office ($\triangle SLP$) is congruent to the triangle with vertices that are at Igor's home, Jasen's home, and Daran's home ($\triangle IJD$). That is, $\triangle SLP \cong \triangle IJD$.

a. The distance between the supermarket and the post office is 1 mile. Which path along the triangle $\triangle IJD$ is congruent to this?
the path between Igor's home and Daran's home

b. The measure of $\angle LPS$ is 40°. Identify the angle that is congruent to this angle in $\triangle IJD$. $\angle JDI$

25. PROOF Copy and complete the two-column proof of the Third Angles Theorem by providing the reason for each statement.



Given: $\angle P \cong \angle X$ and $\angle Q \cong \angle Y$

Prove: $\angle R \cong \angle Z$

Statements	Reasons
1. $\angle P \cong \angle X$, $\angle Q \cong \angle Y$	1. ? Given
2. $m\angle P = m\angle X$, $m\angle Q = m\angle Y$	2. ? Def. of congruent angles
3. $m\angle P + m\angle Q + m\angle R = 180$ $180 = m\angle X + m\angle Y + m\angle Z$	3. ? Triangle Angle-Sum Thm.
4. $m\angle P + m\angle Q + m\angle R = m\angle X + m\angle Y + m\angle Z$	4. ? Transitive Property
5. $m\angle X + m\angle Y + m\angle R = m\angle X + m\angle Y + m\angle Z$	5. ? Substitution Property
6. $m\angle R = m\angle Z$	6. ? Subtraction Prop. of Eq.
7. $\angle R \cong \angle Z$	7. ? Def. of congruent angles

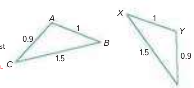
Higher-Order Thinking Skills

26. ANALYZE Determine whether the following statement is *sometimes*, *always*, or *never* true. Justify your argument. See margin.

Equilateral triangles are congruent.

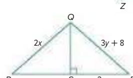
27. CREATE A classmate is using the Third Angles Theorem to show that if two corresponding pairs of the angles of two triangles are congruent, then the third pair is also congruent. Write a question to help him decide whether he can use the same strategy for quadrilaterals.
Sample answer: Do you think that the sum of the angles of a quadrilateral is constant? If so, do you think that the final pair of corresponding angles will be congruent if three other pairs of corresponding angles are congruent for a pair of quadrilaterals?

28. FIND THE ERROR Jasmine and West are evaluating the congruent figures at right. Jasmine says that $\triangle CAB \cong \triangle ZYX$, and West says that $\triangle ABC \cong \triangle YXZ$. Is either of them correct? Explain your reasoning. See margin.



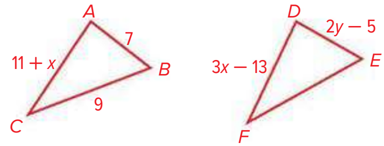
29. WRITE Justify why the order of the vertices is important when naming congruent triangles. Give an example to support your argument. See margin.

30. PERSEVERE Find the values of x and y if $\triangle PQS \cong \triangle RQS$.
$x = 16$, $y = 8$



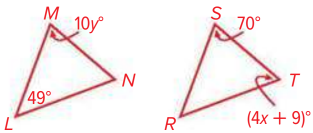
Answers

19.



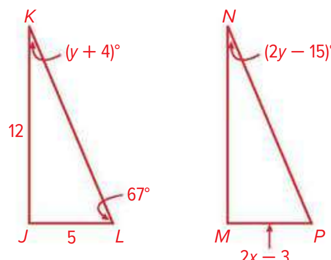
$x = 12$; $y = 6$

20.



$x = 13$; $y = 7$

21.



$x = 4$; $y = 19$

26. Sometimes; sample answer: Equilateral triangles will be congruent if one pair of corresponding sides is congruent.

28. Both; sample answer: $\angle A$ corresponds with $\angle Y$, $\angle B$ corresponds with $\angle X$, and $\angle C$ corresponds with $\angle Z$. $\triangle CAB$ is the same triangle as $\triangle ZYX$, and $\triangle ABC$ is the same triangle as $\triangle YXZ$.

29. Sample answer: When naming congruent triangles, it is important that the corresponding vertices be in the same location for both triangles because the location indicates congruence. For example if $\triangle ABC$ is congruent to $\triangle DEF$, then $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$.

Lesson 14-3

Proving Triangles Congruent: SSS, SAS

G.CO.8, G.SRT.5

LESSON GOAL

Students solve problems using SSS and SAS Congruence Postulates.

1 LAUNCH

Launch the lesson with a Warm Up and an introduction.

2 EXPLORE AND DEVELOP

Explore: Conditions That Prove Triangles Congruent

Develop:

Proving Triangles Congruent: SSS
- Use SSS to Prove Triangles Congruent
- Use SSS on the Coordinate Plane

Proving Triangles Congruent: SAS
- Use SAS to Prove Triangles Congruent

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	BL	ELL	
Remediation: Congruence and Corresponding Parts	●	●		●
Extension: Congruent Triangles in the Coordinate Plane		●	●	●

Language Development Handbook

Assign page 89 of the *Language Development Handbook* to help your students build mathematical language related to solving problems using SSS and SAS Congruence Postulates.

ELL You can use the tips and suggestions on page T89 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.8 Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
2 Reason abstractly and quantitatively.
4 Model with mathematics.

Coherence

Vertical Alignment

Previous
Students used corresponding parts to prove congruent triangles.
G.SRT.5

Now
Students prove that triangles are congruent using the SSS and SAS Postulates.
G.SRT.5

Next
Students will prove that triangles are congruent using the ASA Postulate or AAS Theorem.
G.SRT.5

Rigor

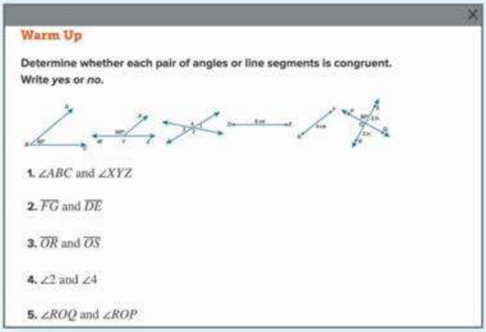
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students show that they understand how the criteria for triangle congruence follow from the definition of congruence in terms of rigid motion. They build fluency by using triangle congruence postulates, and they apply their understanding by solving real-world problems.

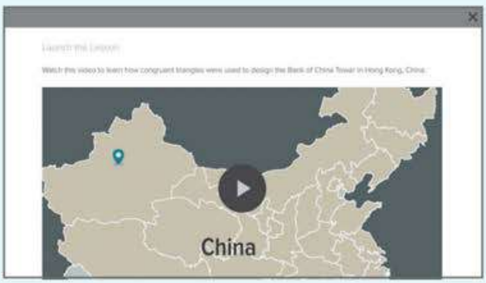
Mathematical Background

The Side-Side-Side Postulate, also written SSS, and the Side-Angle-Side Postulate, also written SAS, are used to prove that two or more triangles are congruent.

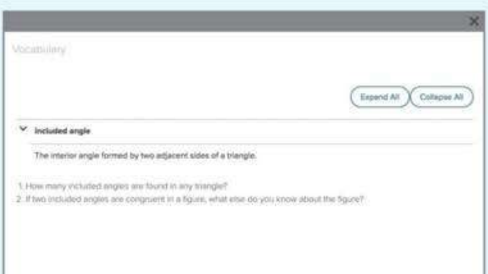
Interactive Presentation



Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- identifying congruent angles and line segments

Answers:

1. yes
2. no
3. yes
4. no
5. yes
6. yes

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of congruent triangles.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then, discuss the questions below with the class.

Explore Conditions That Prove T riangles Congruent

Objective

Students explain how the criteria for triangle congruence follow from the definition of congruence in terms of rigid motion.

Teaching the Mathematical Practices

1 Monitor and Evaluate Point out that in this Explore, students must stop and evaluate their progress and change course to find the ultimate solution.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

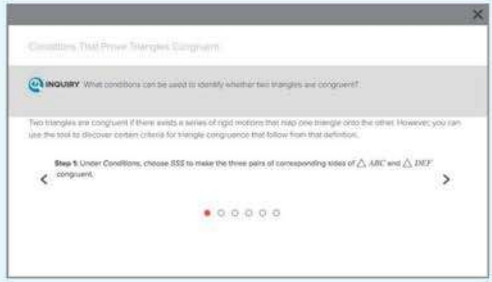
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use a sketch of two triangles to see which sets of corresponding congruent parts of the triangles create congruent triangles. They will answer questions about their findings and write a conjecture on which sets of congruent corresponding parts can be used to identify whether two triangles are congruent without performing a series of rigid motions. Then, students answer the Inquiry Question.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

(continued on the next page)

Interactive Presentation



Explore

WEB SKETCHPAD

Students use a sketch to explore triangle congruence criteria.

Select

Students select answer choices that match their observations of triangle congruence.

Interactive Presentation

Explore

TYPE

a|

Students respond to the Inquiry Question and view a sample answer.

Explore Conditions That Prove T riangles Congruent (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Are all of the corresponding parts of $\triangle ABC$ and $\triangle DEF$ always congruent? SSS: yes; SAS: yes; ASA: yes; AAS: yes; SSA: no
- Can you move the vertices of $\triangle ABC$ so that the two triangles are not congruent? SSS: no; SAS: no; ASA: no; AAS: no; SSA: yes
- Does there exist a rigid motion that will map $\triangle DEF$ onto $\triangle ABC$? SSS: yes; SAS: yes; ASA: yes; AAS: yes; SSA: no
- Must the triangles be congruent? SSS: yes; SAS: yes; ASA: yes; AAS: yes; SSA: no

Inquiry

What conditions can be used to identify whether two triangles are congruent without performing a series of rigid motions? Sample answer: Two triangles can be identified as congruent if they have three pairs of sides (SSS), two pairs of sides and a pair of included angles (SAS), two pairs of angles and a pair of included sides (ASA), or two pairs of angles and a pair of nonincluded sides that are congruent (AAS). Two triangles cannot be identified as congruent without the use of rigid motion if the triangles have two pairs of congruent sides and a pair of congruent nonincluded angles.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Proving T riangles Congruent: SSS

Objective

Students use the SSS Congruence criterion for triangles to solve problems and to prove relationships in geometric figures.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

About the Key Concept

The SSS Congruence postulate and the other triangle congruence criteria can be proved from the other postulates that are assumed to be true in this course. However, the difficulty level of those proofs means that most high school geometry courses will assume at least one of these criteria is true as a postulate. You can engage students who are beyond level by having them read one of those proofs or by trying to write one themselves.

Example 1 Use SSS to Prove T riangles Congruent

MP Teaching the Mathematical Practices

2 Different Properties Mathematically proficient students look for different ways to solve problems. Encourage them to work through both ways to solve the problem and to choose the method that works best for them.

Questions for Mathematical Discourse

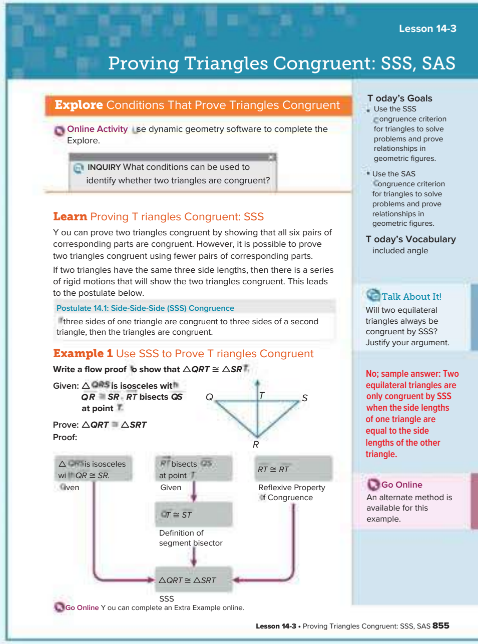
AL How is the flow proof similar to a two-column proof? How is it different? Sample answer: They both use the same statements and reasons. A flow proof uses arrows and boxes to show the logical progression of the proof. A two-column proof shows the logical progression by a list of statements and reasons.

OL What information do you need to know to use the SSS Postulate? Three sides of one triangle are congruent to the corresponding three sides of another triangle.

BL What information tells you that $\overline{QT} \cong \overline{ST}$? $\overline{RT}$ bisects $\overline{QS}$ at point T.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 14-3

Proving Triangles Congruent: SSS, SAS

Explore Conditions That Prove Triangles Congruent

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY What conditions can be used to identify whether two triangles are congruent?

Learn Proving T riangles Congruent: SSS

Y ou can prove two triangles congruent by showing that all six pairs of corresponding parts are congruent. However, it is possible to prove two triangles congruent using fewer pairs of corresponding parts.

If two triangles have the same three side lengths, then there is a series of rigid motions that will show the two triangles congruent. This leads to the postulate below.

Postulate 14.1: Side-Side-Side (SSS) Congruence

If three sides of one triangle are congruent to three sides of a second triangle, then the triangles are congruent.

Example 1 Use SSS to Prove T riangles Congruent

Write a flow proof to show that $\triangle QRT \cong \triangle SRT$.

Given: $\triangle QRS$ is isosceles with $\overline{QR} \cong \overline{SR}$; $\overline{RT}$ bisects $\overline{QS}$ at point T.

Prove: $\triangle QRT \cong \triangle SRT$

Proof:

Go Online Y ou can complete an Extra Example online.

T oday's Goals
- Use the SSS Congruence criterion for triangles to solve problems and prove relationships in geometric figures.
- Use the SAS Congruence criterion for triangles to solve problems and prove relationships in geometric figures.

T oday's Vocabulary
included angle

Talk About It! Will two equilateral triangles always be congruent by SSS? Justify your argument.

No; sample answer: Two equilateral triangles are only congruent by SSS when the side lengths of one triangle are equal to the side lengths of the other triangle.

Go Online An alternate method is available for this example.

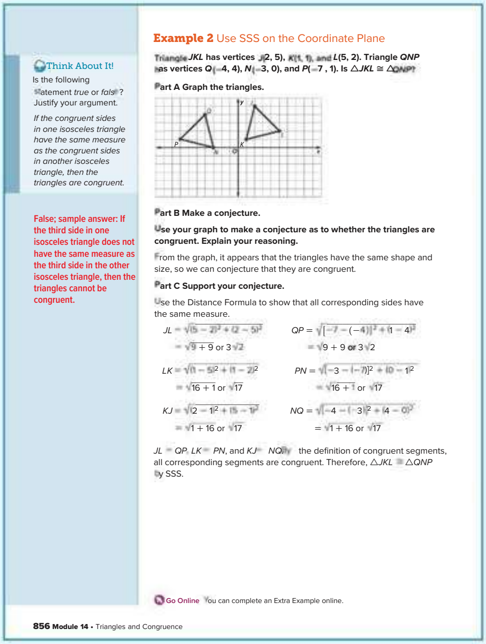
Lesson 14-3 • Proving Triangles Congruent: SSS, SAS 855

Interactive Presentation

Learn

Watch

Students watch an animation about the SSS congruence criterion.



Think About It!

Is the following statement *true* or *false*? Justify your argument.

If the congruent sides in one isosceles triangle have the same measure as the congruent sides in another isosceles triangle, then the triangles are congruent.

False; sample answer: If the third side in one isosceles triangle does not have the same measure as the third side in the other isosceles triangle, then the triangles cannot be congruent.

Example 2 Use SSS on the Coordinate Plane

Triangle *JKL* has vertices $J(2, 5)$, $K(1, 1)$, and $L(5, 2)$. Triangle *QNP* has vertices $Q(-4, 4)$, $N(-3, 0)$, and $P(-7, 1)$. Is $\triangle JKL \cong \triangle QNP$?

Part A Graph the triangles.

Part B Make a conjecture.

Use your graph to make a conjecture as to whether the triangles are congruent. Explain your reasoning.

From the graph, it appears that the triangles have the same shape and size, so we can conjecture that they are congruent.

Part C Support your conjecture.

Use the Distance Formula to show that all corresponding sides have the same measure.

$JL = \sqrt{(5-2)^2 + (2-5)^2} = \sqrt{9+9}$ or $3\sqrt{2}$

$QP = \sqrt{[-7-(-4)]^2 + (1-4)^2} = \sqrt{9+9}$ or $3\sqrt{2}$

$LK = \sqrt{(1-5)^2 + (1-2)^2} = \sqrt{16+1}$ or $\sqrt{17}$

$PN = \sqrt{[-3-(-7)]^2 + (0-1)^2} = \sqrt{16+1}$ or $\sqrt{17}$

$KJ = \sqrt{(2-1)^2 + (5-1)^2} = \sqrt{1+16}$ or $\sqrt{17}$

$NQ = \sqrt{[-4-(-3)]^2 + (4-0)^2} = \sqrt{1+16}$ or $\sqrt{17}$

$JL \cong QP$, $LK \cong PN$, and $KJ \cong NQ$. By the definition of congruent segments, all corresponding segments are congruent. Therefore, $\triangle JKL \cong \triangle QNP$ by SSS.

Go Online You can complete an Extra Example online.

856 Module 14 • Triangles and Congruence

Interactive Presentation

Example 2

TAP

Students tap to see the Alternate Method.

WEB SKETCHPAD

Students use a sketch to apply the SSS congruence criterion to the coordinate plane.

CHECK

Complete the Check exercise online to determine whether students are ready to move on.

DIFFERENTIATE

Language Development Activity BL ELL

Beginning Ask questions about the lesson content to elicit yes/no answers: "Look at Example 1. Are $\overline{SR}$ and $\overline{QR}$ congruent?" yes "Are $\overline{QT}$ and $\overline{ST}$ congruent?" yes

Intermediate/Advanced Ask questions about the lesson content to elicit short answers: "Look at the given information in Example 1. How many pairs of sides do we know are congruent?" two "What else do we need to show to use SSS to prove the triangles congruent?" $\overline{QT} \cong \overline{ST}$

Advanced High Ask questions about lesson content to elicit complete sentences: "How does the SAS acronym represent the postulate?" The SS represents two sides, the A between them represents the angle between them. "How would you choose whether to use SSS or SAS to prove two triangles congruent?" If you can show all three pairs of sides congruent, use SSS. If you can show two sides and the included angle, then use SAS.

Example 2 Use SSS on the Coordinate Plane

MP Teaching the Mathematical Practices

3 Make Conjectures In this example, students will make conjectures and then build a logical progression of statements to validate the conjectures. Once students have made their conjectures, guide the students to validate them.

Questions for Mathematical Discourse

AL How can you find the length of the sides of the triangles on the coordinate plane? Use the Distance Formula.

OL How can you tell that you can use SSS to show that the triangles are congruent? The shapes of the triangles look the same.

BL Can you just say by appearance that the triangles are not congruent? Explain. No; sample answer: Y ou can only make a conjecture from a drawing.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Proving Triangles Congruent: SAS

Objective

Students use the SAS congruence criterion for triangles to solve problems and to prove relationships in geometric figures.

Teaching the Mathematical Practices

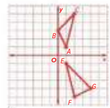
3 Construct Arguments In this Learn, students will use stated assumptions, definitions, and previously established results to construct an argument.

Check

Triangle ABC has vertices $A(1, 1)$, $B(0, 3)$, and $C(2, 5)$. Triangle EFG has vertices $E(1, -1)$, $F(2, -5)$, and $G(4, -4)$. Is $\triangle ABC \cong \triangle EFG$?

Part A

Graph $\triangle ABC$ and $\triangle EFG$ on the same coordinate plane.



Part B

Find the side lengths of each triangle.

$AB = \underline{?}\ \sqrt{5}$; $BC = \underline{?}\ 2\sqrt{2}$; $AC = \underline{?}\ \sqrt{17}$; $EF = \underline{?}\ \sqrt{17}$; $FG = \underline{?}\ \sqrt{5}$; $EG = \underline{?}\ 3\sqrt{2}$

Part C

Is triangle ABC congruent to triangle EFG? Justify your argument.

A. No; $AC \neq FG$ so SSS congruence is not met.

B. No; $BC \neq FG$ so SSS congruence is not met.

C. Yes; all corresponding sides have the same measure, so SSS congruence is met.

D. Yes; all corresponding sides have the same measure, so by the definition of congruent figures, $\triangle ABC \cong \triangle EFG$

Learn Proving Triangles Congruent: SAS

The interior angle formed by two adjacent sides of a triangle is called an **included angle**.

If two triangles are formed using the same side lengths and included angle measure, then there is a series of rigid motions that will show that the two triangles are congruent. This leads to the postulate below.

Postulate 14.2: Side-Angle-Side (SAS) Congruence

If two sides and the included angle of one triangle are congruent to two sides and the included angle of a second triangle, then the triangles are congruent.

Think About It!
Both legs of one right triangle are congruent to the legs of another right triangle. Are the triangles congruent? Justify your argument.

Yes; sample answer: The included angles formed by the legs of the triangles are both right angles. So the included angles are congruent, and by SAS, the triangles are congruent.

Go Online You may want to complete the construction activities for this lesson.

Go Online You can complete an Extra Example online.

Lesson 14-3 • Proving Triangles Congruent: SSS, SAS 857

Interactive Presentation

Learn

TAP

Students tap to see congruent triangles.

Watch

Students watch an animation of the SAS congruence criterion.

DIFFERENTIATE

Reteaching Activity AL ELL

Students can use a systematic approach to write the proofs for problems and examples in this lesson. Have students start by looking for possible methods of proof using SSS or SAS. They should examine the problem to determine how much necessary information is given and how they can find any other information that they need for the proof. Finally, they can draw on prior knowledge of midpoints, distances, angle relationships, and so on, to extract other necessary information and to compile the facts for the final proof.

Study Tip

Assumptions Even though objects in the real world are not two-dimensional, you can still make some assumptions about their geometric relationships. For example, in three-dimensional objects you can apply the Reflexive Property of Congruence and the Vertical Angles Theorem.

Statements/Reasons:
Given
Definition of congruent segments
SAS
SSS
Definition of angle bisector
Reflexive Property of Congruence
$\overline{JL} \cong \overline{JL}$
$\overline{JK} \cong \overline{JM}$
$\angle KJL \cong \angle MJL$

Statements/Reasons:
Given
Definition of congruent segments
SAS
SSS
Definition of angle bisector
Reflexive Property of Congruence
$\overline{DE} \cong \overline{FE}$
$\overline{EG} \cong \overline{EG}$
$\overline{DG} \cong \overline{FG}$
$\angle DEG \cong \angle FEG$

Example 3 Use SAS to Prove Triangles Congruent

PLAYGROUND **The playground equipment shown appears to be made of congruent triangles. If $\overline{KL} \cong \overline{LM}$ and $\angle JLK \cong \angle JLM$, write a two-column proof to prove that $\triangle JLK \cong \triangle JLM$. Complete the two-column proof by selecting the correct statements and reasons.**

Statements	Reasons
1. $\overline{KL} \cong \overline{LM}$	1. Given
2. $\angle JLK \cong \angle JLM$	2. Given
3. $\overline{JL} \cong \overline{JL}$	3. Reflexive Property of Congruence
4. $\triangle JLK \cong \triangle JLM$	4. SAS

Check

KITES The kite shown appears to be made up of congruent triangles. If $\overline{DE} \cong \overline{FE}$ and $\overline{EG}$ bisects $\angle DEF$, prove that $\triangle DEG \cong \triangle FEG$. Copy and complete the two-column proof by selecting the correct statements and reasons.

Given: $\overline{DE} \cong \overline{FE}$, $\overline{EG}$ bisects $\angle DEF$
Prove: $\triangle DEG \cong \triangle FEG$
Proof:

Statements	Reasons
1. ? $\overline{DE} \cong \overline{FE}$	1. Given
2. $\overline{EG}$ bisects $\angle DEF$	2. ? Given
3. ? $\angle DEG \cong \angle FEG$	3. Definition of angle bisector
4. ? $\overline{EG} \cong \overline{EG}$	4. ? Reflexive Property of Congruence
5. $\triangle DEG \cong \triangle FEG$	5. ? SAS

Go Online You can complete an Extra Example online.

858 Module 14 • Triangles and Congruence

Interactive Presentation

Example 3

DRAG & DROP

Students drag statements and reasons to complete the proof.

TAP

Students tap to see a Study Tip and a Common Error.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Use SAS to Prove Triangles Congruent

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL What information is given in the problem? $\overline{KL} \cong \overline{LM}$ and $\angle JLK \cong \angle JLM$

OL What other corresponding parts do you need to prove congruent in order to use SAS? $\overline{JL}$ and itself

BL Now that you have proved that the triangles are congruent, what do you know about $\angle LJK$ and $\angle LJM$? Explain. They are congruent because they are corresponding parts of congruent triangles.

Common Error

Students may attempt to prove triangles congruent using SAS where the pairs of congruent corresponding parts have two sides and a nonincluded angle.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–16
2	exercises that use a variety of skills from this lesson	17–26
3	exercises that emphasize higher-order and critical-thinking skills	27–32

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: OL BL

- Practice, Exercises 1–25 odd, 27–32
- Extension: Congruent Triangles in the Coordinate Plane
- ALEKS Proving Triangle Congruence

IF students score 66%–89% on the Checks, **THEN** assign: AL OL

- Practice, Exercises 1–31 odd
- Remediation, Review Resources: Congruence and Corresponding Parts
- Personal Tutors
- Extra Examples 1–3
- ALEKS Congruence and Similarity

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–15 odd
- Remediation, Review Resources: Congruence and Corresponding Parts
- *Quick Review Math Handbook*: Proving Triangles Congruent (SSS, SAS)
- ALEKS Congruence and Similarity

Important to Know

Digital Exercise Alert Exercise 30 requires a construction. Students will need to complete the construction by using a compass and straightedge.

Practice

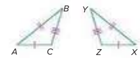
Go Online You can complete your homework online.

Example 1

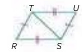
PROOF **Write the specified type of proof.**

1–2. See margin. 3–6. See Mod. 14. Answer Appendix.

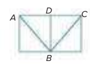
1. two-column proof
   Given: $\overline{AB} \cong \overline{XY}, \overline{AC} \cong \overline{XZ}, \overline{BC} \cong \overline{YZ}$
   Prove: $\triangle ABC \cong \triangle XYZ$



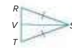
2. flow proof
   Given: $\overline{RS} \cong \overline{UT}, \overline{RT} \cong \overline{US}$
   Prove: $\triangle RST \cong \triangle UTS$



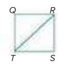
3. two-column proof
   Given: $\overline{AB} \cong \overline{CB}$, D is the midpoint of $\overline{AC}$.
   Prove: $\triangle ABD \cong \triangle CBD$



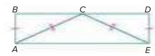
4. flow proof
   Given: $\overline{RS} \cong \overline{TS}$, V is the midpoint of $\overline{RT}$.
   Prove: $\triangle RSV \cong \triangle TSV$



5. paragraph proof
   Given: $\overline{QR} \cong \overline{SR}, \overline{ST} \cong \overline{QT}$
   Prove: $\triangle QRT \cong \triangle SRT$



6. two-column proof
   Given: $\overline{AB} \cong \overline{ED}, \overline{CA} \cong \overline{CE}$, $\overline{AC}$ bisects $\overline{BD}$
   Prove: $\triangle ABC \cong \triangle EDC$



Example 2

REGULARITY **Determine whether $\triangle DEF \cong \triangle PQR$. Explain.**

7. $D(-6, 1), E(1, 2), F(-1, -4), P(0, 5), Q(7, 6), R(5, 0)$ $DE = 5\sqrt{2}, PQ = 5\sqrt{2}, EF = 2\sqrt{10}, QR = 2\sqrt{10}, DF = 5\sqrt{2}, PR = 5\sqrt{2}$; $\triangle DEF \cong \triangle PQR$ by SSS because corresponding sides have the same measure and are congruent.
8. $D(-7, -3), E(-4, -1), F(-2, -5), P(2, -2), Q(5, -4), R(0, -5)$ $DE = \sqrt{13}, PQ = \sqrt{13}, EF = 2\sqrt{5}, QR = \sqrt{26}, DF = \sqrt{29}, PR = \sqrt{13}$. Corresponding sides are not congruent, so $\triangle DEF$ is not congruent to $\triangle PQR$. There are no rigid motions that map one triangle onto the other.

Determine whether $\triangle ABC \cong \triangle KLM$. Explain.

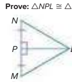
9. $A(-3, 3), B(-1, 3), C(-3, 1), K(1, 4), L(3, 4), M(1, 6)$ $AB = 2, KL = 2, BC = 2\sqrt{2}, LM = 2\sqrt{2}, AC = 2, KM = 2$; The corresponding sides have the same measure and are congruent, so $\triangle ABC \cong \triangle KLM$ by SSS.
10. $A(-4, -2), B(-4, 1), C(-1, -1), K(0, -2), L(0, 1), M(4, 1)$ $AB = 3, KL = 3, BC = \sqrt{13}, LM = 4, AC = \sqrt{10}, KM = 5$. The corresponding sides are not congruent, so $\triangle ABC$ is not congruent to $\triangle KLM$.

Lesson 14-3 • Proving Triangles Congruent: SSS, SAS 859

Example 3

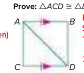
PROOF **Write the specified type of proof.**

11. two-column proof
    Given: $NP = PM, \overline{NP} \perp \overline{PL}$
    Prove: $\triangle NPL \cong \triangle MPL$



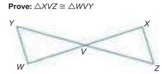
Proof:
Statements (Reasons)
1. $NP = PM$, $\overline{NP} \perp \overline{PL}$ (Given)
2. $\overline{NP} \cong \overline{MP}$ (Def. of congruence)
3. $\angle MPL$ and $\angle NPL$ are rt. angles. ($\perp$ lines form rt. angles.)
4. $\angle MPL \cong \angle NPL$ (All right angles are congruent.)
5. $\overline{PL} \cong \overline{PL}$ (Reflexive Property of $\cong$)
6. $\triangle NPL \cong \triangle MPL$ (SAS)

12. two-column proof
    Given: $AB = CD, \overline{AB} \parallel \overline{CD}$
    Prove: $\triangle ACD \cong \triangle DBA$



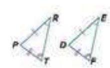
Proof:
Statements (Reasons)
1. $AB = CD$, $\overline{AB} \parallel \overline{CD}$ (Given)
2. $\angle BAD \cong \angle CDA$ (Alternate Interior Angles Thm.)
3. $\overline{AD} \cong \overline{AD}$ (Reflexive Property)
4. $\overline{AB} \cong \overline{CD}$ (Def. of congruent segments)
5. $\triangle ACD \cong \triangle DBA$ (SAS)

13. paragraph proof
    Given: V is the midpoint of $\overline{WX}$ and $\overline{YZ}$.
    Prove: $\triangle XVZ \cong \triangle WVY$



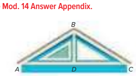
See Mod. 14 Answer Appendix.

14. flow proof
    Given: $\overline{PR} \cong \overline{DE}, \overline{PT} \cong \overline{DF}, \angle R \cong \angle E, \angle T \cong \angle F$
    Prove: $\triangle PRT \cong \triangle DEF$



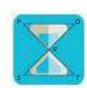
See Mod. 14 Answer Appendix.

15. GAMING Devontae is building a house in a simulation video game. He wants the roof of the house and the main support beam to create congruent triangles. If $\overline{BD} \perp \overline{AC}$ and $\overline{BD}$ bisects $\overline{AC}$, write a two-column proof to prove $\triangle ABD \cong \triangle CBD$.



Proof:
Statements (Reasons)
1. $\overline{BD} \perp \overline{AC}$; $\overline{BD}$ bisects $\overline{AC}$. (Given)
2. $\angle BDA$ and $\angle BDC$ are rt. angles. ($\perp$ lines form rt. angles.)
3. $\angle BDA \cong \angle BDC$ (All right angles are congruent.)
4. $\overline{AD} \cong \overline{DC}$ (Def. of segment bisector)
5. $\overline{BD} \cong \overline{BD}$ (Reflexive Property of $\cong$)
6. $\triangle ABD \cong \triangle CBD$ (SAS)

16. TECHNOLOGY Nevaeh has developed a new timer app. The icon for the app contains an hourglass that can be modeled by two triangles. If R is the midpoint of $\overline{QS}$ and $\overline{PT}$, write a paragraph proof to prove $\triangle PRQ \cong \triangle TRS$. See Mod. 14 Answer Appendix.

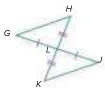


860 Module 14 • Triangles and Congruence

Mixed Exercises

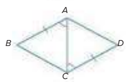
Explain whether there is enough information given in each figure to prove that the triangles are congruent using SSS or SAS. 17–20. See margin.

17.



18.

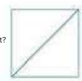
19.



20.

21. REASONING Tyson had three sticks of lengths 24 inches, 28 inches, and 30 inches. Is it possible to make two non-congruent triangles using the same three sticks? Explain. No; sample answer: The sticks do not change size, so any arrangement will yield a congruent triangle.

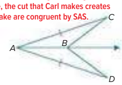
22. BAKERY Sonia made a sheet of baklava. She has markings on her pan so that she can cut them into large squares. After she cuts the pastry in squares, she cuts them diagonally to form two congruent triangles, as shown. Which postulate could you use to prove the two triangles congruent? either SSS or SAS



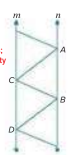
23. TILES Tammy installs bathroom tiles. Her current job requires tiles that are equilateral triangles, and the tiles have to be congruent to each other. She has a sack of tiles that are in the shape of equilateral triangles. She knows that all the tiles are equilateral, but she is not sure whether they are the same size. What must she measure on each tile to be sure that they are congruent? Sample answer: She needs to measure one side of each tile because all the tiles are equilateral triangles.

24. CAKE Carl had a piece of cake in the shape of an isosceles triangle with angles measuring 26°, 77°, and 77°. He wanted to divide it into two equal parts, so he cut it through the middle of the 26°-angle to the midpoint of the opposite side. He claims that the two pieces are congruent. Do you agree? Explain. Yes; sample answer: We are given that the piece of cake is an isosceles triangle, so we know that the two sides opposite the 77°-angles are congruent. Also, Carl cuts the piece of cake through the middle of the 26°-angle creating two congruent angles. Finally, by the Reflexive Property of Congruence, the cut that Carl makes creates two congruent corresponding sides. Therefore, the two pieces of cake are congruent by SAS.

25. In the figure, $\overline{AC} \cong \overline{AD}$. Suppose you know $\angle C \cong \angle D$. Can you prove that $\triangle ABC \cong \triangle ABD$? Why or why not? No; sample answer: Y ou cannot use SAS because the angle congruence that we are given is not an included angle between two sides that are known to be congruent, and SSS cannot be used because only 2 sides of each triangle are known to be congruent.



26. USE A SOURCE An engineer is designing a new cell phone tower. Part of the tower is shown in the figure. The engineer makes sure that line m is parallel to line n and that $\overline{AB} \cong \overline{CD}$.



a. Can the engineer prove that $\triangle ABC \cong \triangle DCB$? Explain why or why not. Y es; sample answer: It is given that $\overline{AB} \cong \overline{CD}$, and $\overline{BC} \cong \overline{BC}$ by the Reflexive Property of Congruence. $\angle ABC \cong \angle DCB$ because they are alternate interior angles and line $m \parallel$ line n. Therefore, $\triangle ABC \cong \triangle DCB$ by SAS.

b. Go online to find an image of a bridge or a tower that is designed in such a way that you can prove that two triangles are congruent. Justify your image. See Mod. 14 Answer Appendix.

Higher-Order Thinking Skills

27. WHICH ONE DOESN'T BELONG? Determine which pair of triangles cannot be proved congruent using the SSS or SAS Postulates. Justify your conclusion. First pair; sample answer: The second pair can be shown congruent by SAS or SSS, and the third pair can be shown congruent by SSS.

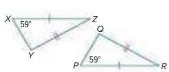
28. ANALYZE Determine whether the following statement is *true* or *false*. If true, justify your reasoning. If false, provide a counterexample.

If the congruent sides in one isosceles triangle have the same measure as the congruent sides in another isosceles triangle, then the triangles are congruent. See margin.

29. WRITE Two pairs of corresponding sides of two right triangles are congruent. Are the triangles congruent? Explain your reasoning. See margin.

30. CREATE Use a straightedge to draw obtuse triangle *ABC*. Then construct $\triangle XYZ$ so it is congruent to $\triangle ABC$ using SSS or SAS. Justify your construction mathematically and verify it using measurement. Sample answer: Using a ruler, I measured all the sides and they are congruent, so the triangles are congruent by SSS.

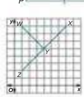
31. FIND THE ERROR Bonnie says that $\triangle PQR \cong \triangle XYZ$ by SAS. Shada disagrees. She says that there is not enough information to prove that the two triangles are congruent. Is either of them correct? Explain your reasoning. Shada; to use SAS, the angle must be the included angle.



32. PERSEVERE **Refer to the graph shown.** See margin.

a. Describe two methods you could use to prove $\triangle WYZ \cong \triangle WYX$. You may not use a ruler or protractor. Which method do you think is more efficient? Explain.

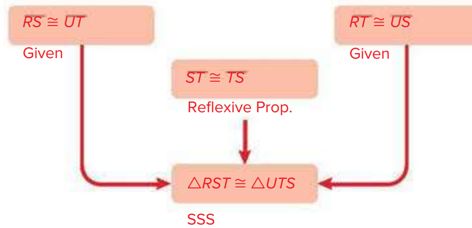
b. Are $\triangle WYZ$ and $\triangle WYX$ congruent? Explain your reasoning.



Answers

1. Statements (Reasons)
   1. $\overline{AB} \cong \overline{XY}$ (Given)
   $\overline{AC} \cong \overline{XZ}$
   $\overline{BC} \cong \overline{YZ}$
   2. $\triangle ABC \cong \triangle XYZ$ (SSS Post)

2.



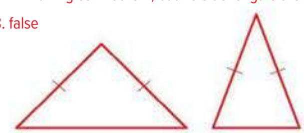
17. Yes; sample answer: $\angle GLH$ and $\angle JLK$ are vertical angles, so they are congruent. Therefore, $\triangle GLH \cong \triangle JLK$ by the SAS Congruence Postulate.

18. No; sample answer: It is not known whether $\overline{QT} \cong \overline{SR}$, so you cannot use SSS, and none of the angles are known to be congruent, so you cannot use SAS.

19. Yes; sample answer: The triangles share the side $\overline{AC}$, so they have two pairs of congruent sides. The given congruent angles are included angles, so $\triangle ABC \cong \triangle CDA$ by SAS.

20. No; both triangles must have three pairs of congruent angles from the Third Angles Theorem, but no side lengths are known.

28. false



29. Case 1: You know that the hypotenuses are congruent and that one pair of legs are congruent. Then the Pythagorean Theorem says that the other pair of legs are congruent, so the triangles are congruent by SSS. Case 2: You know that the pairs of legs are congruent and that the right angles are congruent, so the triangles are congruent by SAS.

32a. Sample answer: Method 1: You could use the Distance Formulas to find the length of each of the sides, and then use the SSS Congruence Postulate to prove the triangles congruent. Method 2: You could find the slopes of $\overline{ZX}$ and $\overline{WY}$ to prove that they are perpendicular and that $\angle WYZ$ and $\angle WYX$ are both right angles. You can use the Distance Formula to prove that $\overline{XY} \cong \overline{ZY}$. Because the triangles share the leg $\overline{WY}$, you can use the SAS Congruence Postulate; Sample answer: I think that method 2 is more efficient, because you only have two steps instead of three.

32b. Yes; sample answer: The slope of $\overline{WY}$ is –1, the slope of $\overline{ZX}$ is 1, and –1 and 1 are opposite reciprocals, so $\overline{WY}$ is perpendicular to $\overline{ZX}$. Because they are perpendicular, $\angle WYZ$ and $\angle WYX$ are both 90°. Using the Distance Formula, the length of $\overline{ZY}$ is $\sqrt{(4-1)^2+(5-2)^2}$ or $3\sqrt{2}$, and the length of $\overline{XY}$ is $\sqrt{(7-4)^2+(8-5)^2}$ or $3\sqrt{2}$. Because $\overline{WY} \cong \overline{WY}$, $\triangle WYZ \cong \triangle WYX$ by the SAS Congruence Postulate.

Lesson 14-4

G.CO.8, G.SRT.5

Proving Triangles Congruent: ASA, AAS

LESSON GOAL

Students solve problems using the ASA Congruence Postulate and the AAS Congruence Theorem.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Proving Triangles Congruent: ASA
- Use ASA to Prove Triangles Congruent
- Apply ASA Congruence

Proving Triangles Congruent: AAS
- Use AAS to Prove Triangles Congruent

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Congruence and Transformations	●	●		●
Extension: The Ambiguous Case		●	●	●

Language Development Handbook

Assign page 90 of the *Language Development Handbook* to help your students build mathematical language related to solving problems using the ASA Congruence Postulate and the AAS Congruence Theorem.

ELL You can use the tips and suggestions on page T90 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.8 Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.
4 Model with mathematics.
6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students proved congruent triangles using the SSS and SAS Postulates.
G.SRT.5

Now
Students prove that triangles are congruent using the ASA Postulate or AAS Theorem.
G.SRT.5

Next
Students will use triangle congruence criteria to prove right triangles congruent.
G.CO.10, G.SRT.5

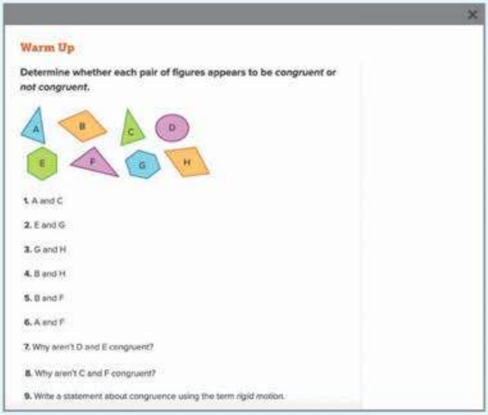
Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

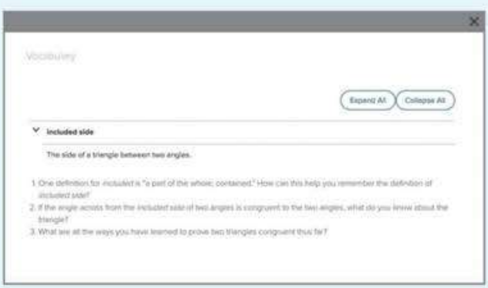
Conceptual Bridge In this lesson, students show that they understand how the criteria for triangle congruence follow from the definition of congruence in terms of rigid motion. They build fluency by using triangle congruence theorems, and they apply their understanding by solving real-world problems.

Interactive Presentation



Warm Up


Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- proving triangles congruent by using transformations

Answers:

1. congruent
2. not congruent
3. not congruent
4. congruent
5. not congruent
6. not congruent
7. different shapes
8. different sizes
9. Sample answer: Two figures are congruent if and only if there is a rigid motion or series of rigid motions that maps one figure exactly onto the other.

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of congruent triangles.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then, discuss the questions below with the class.

Mathematical Background

The Angle-Side-Angle Postulate, written as ASA, and the Angle-Angle-Side, or AAS, Theorem can also be used to prove triangle congruence.

Learn Proving T riangles Congruent: ASA

Objective

Students use the ASA congruence criterion for triangles to solve problems and to prove relationships in geometric figures.

MP Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of the Angle-Side-Angle (ASA) Congruence Postulate in this Learn.

Example 1 Use ASA to Prove T riangles Congruent

MP Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to complete the given proof.

Questions for Mathematical Discourse

AL What do you need to know to use ASA when you are proving triangles congruent? Two angles and the included side of one triangle are congruent to the corresponding two angles and included side of another triangle.

OL What does the given information $\overline{BD}$ bisects $\overline{AE}$ tell you? $\overline{AC} \cong \overline{CE}$

BL Suppose you weren't given $\angle BAC \cong \angle DEC$; instead, you were given $\angle ABC \cong \angle CDE$. Could you still prove the triangles congruent? Explain. Yes; sample answer: You could show that $\angle BAC \cong \angle DEC$ by using the Third Angles Theorem. Then proceed with ASA.

Go Online

- F ind additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 14-4

Proving Triangles Congruent: ASA, AAS

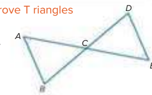
Learn Proving T riangles Congruent: ASA

An **included side** is the side of a triangle between two angles.

If two triangles are formed using the same two angle measures and included side length, then there is a series of rigid motions that will show the two triangles congruent. This leads to the postulate below.

Postulate 14.3: Angle-Side-Angle (ASA) Congruence
If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

Example 1 Use ASA to Prove T riangles Congruent



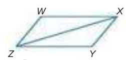
Complete the two-column proof.

Given: $\angle BAC \cong \angle DEC$; $\overline{BD}$ bisects $\overline{AE}$.

Prove: $\triangle ACB \cong \triangle ECD$

Statements	Reasons
1. $\angle BAC \cong \angle DEC$	1. Given
2. $\overline{BD}$ bisects $\overline{AE}$	2. Given
3. $\overline{AC} \cong \overline{EC}$	3. Definition of segment bisector
4. $\angle ACB \cong \angle ECD$	4. Vertical Angles Theorem
5. $\triangle ACB \cong \triangle ECD$	5. ASA

Check



Complete the two-column proof.

Given: $\overline{WX} \parallel \overline{YZ}$ and $\overline{WZ} \parallel \overline{YX}$

Prove: $\triangle WXZ \cong \triangle YZX$

Statements	Reasons
1. ? $\overline{WX} \parallel \overline{YZ}$	1. Given
2. $\overline{WZ} \parallel \overline{YX}$	2. ? Given
3. $\angle WXZ \cong \angle YZX$	3. ? Alternate Interior Angles Theorem
4. ? $\angle WZX \cong \angle YXZ$	4. Alternate Interior Angles Theorem
5. ? $\overline{XZ} \cong \overline{XZ}$	5. Reflexive Property of Congruence
6. $\triangle WXZ \cong \triangle YZX$	6. ? ASA

Go Online Y ou can complete an Extra Example online.

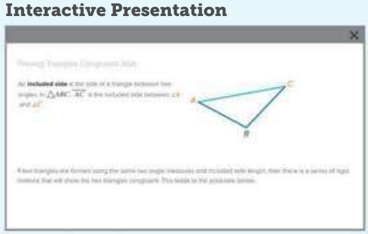
T oday's Goals
- Use the ASA congruence criterion for triangles to solve problems and prove relationships in geometric figures.
- Use the AAS congruence criterion for triangles to prove relationships in geometric figures.

T oday's Vocabulary
included side

Think About It!
Can the Vertical Angles Theorem always be used to prove angles congruent in any two triangles? Justify your argument.

No; sample answer: The Vertical Angles Theorem can only prove angles congruent when the sides of two triangles are created by intersecting segments.

Interactive Presentation



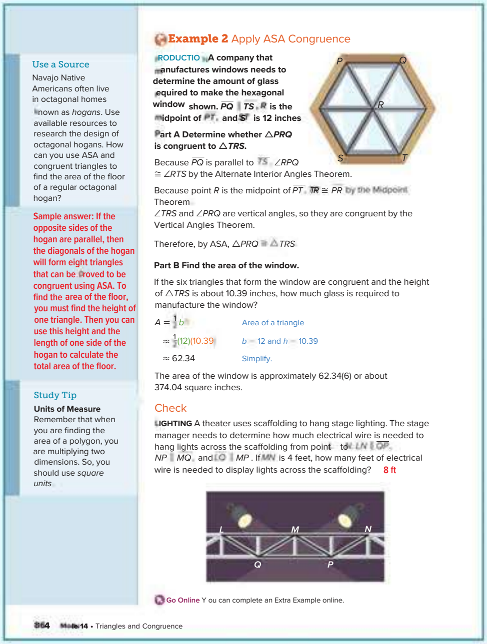
Learn

WATCH

Students watch an animation of ASA congruence.

SELECT

Students select statements and reasons to complete a proof.


Use a Source
Navajo Native Americans often live in octagonal homes known as *hogans*. Use available resources to research the design of octagonal hogans. How can you use ASA and congruent triangles to find the area of the floor of a regular octagonal hogan?

Sample answer: If the opposite sides of the hogan are parallel, then the diagonals of the hogan will form eight triangles that can be proved to be congruent using ASA. To find the area of the floor, you must find the height of one triangle. Then you can use this height and the length of one side of the hogan to calculate the total area of the floor.

Study Tip
Units of Measure Remember that when you are finding the area of a polygon, you are multiplying two dimensions. So, you should use *square units*.

Example 2 Apply ASA Congruence

PRODUCTION A company that manufactures windows needs to determine the amount of glass required to make the hexagonal window shown. $\overline{PQ} \parallel \overline{TS}$, R is the midpoint of $\overline{PT}$, and $\overline{ST}$ is 12 inches.

Part A Determine whether $\triangle PRQ$ is congruent to $\triangle TRS$.

Because $\overline{PQ}$ is parallel to $\overline{TS}$, $\angle RPQ \cong \angle RTS$ by the Alternate Interior Angles Theorem.

Because point R is the midpoint of $\overline{PT}$, $\overline{TR} \cong \overline{PR}$ by the Midpoint Theorem.

$\angle TRS$ and $\angle PRQ$ are vertical angles, so they are congruent by the Vertical Angles Theorem.

Therefore, by ASA, $\triangle PRQ \cong \triangle TRS$.

Part B Find the area of the window.

If the six triangles that form the window are congruent and the height of $\triangle TRS$ is about 10.39 inches, how much glass is required to manufacture the window?

$A = \frac{1}{2}bh$ — Area of a triangle

$\approx \frac{1}{2}(12)(10.39)$ — $b = 12$ and $h = 10.39$

≈ 62.34 — Simplify.

The area of the window is approximately 62.34(6) or about 374.04 square inches.

Check
LIGHTING A theater uses scaffolding to hang stage lighting. The stage manager needs to determine how much electrical wire is needed to hang lights across the scaffolding from point L to N. $\overline{LN} \parallel \overline{QP}$, $\overline{NP} \parallel \overline{MQ}$, and $\overline{LQ} \parallel \overline{MP}$. If MN is 4 feet, how many feet of electrical wire is needed to display lights across the scaffolding? 8 ft

Go Online You can complete an Extra Example online.

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Example 2 Apply ASA Congruence

Teaching the Mathematical Practices

5 Use a Source Guide students to find external information to answer the questions posed in the Use a Source feature.

Common Error

Students may forget to multiply the area of one pane of glass by 6 to get the area of the window. Remind students that when solving any problem, to read the statement of the problem again when they think they have found a solution to see whether it makes sense.

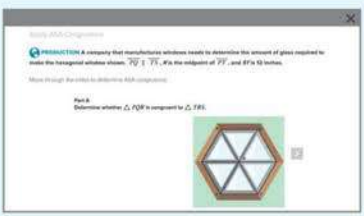
Questions for Mathematical Discourse

AL What is the relationship between $\angle SRT$ and $\angle QRP$? Explain. The two angles are congruent because they are vertical angles.

OL What does the given information R is the midpoint of $\overline{PT}$ tell you? $\overline{PR} \cong \overline{RT}$

BL Suppose you weren't given $\overline{PQ} \parallel \overline{TS}$ but instead were told that $\angle Q \cong \angle S$, could you still prove the triangles congruent? Explain. Yes; sample answer: You could use the Third Angles Theorem to show that $\angle P \cong \angle T$.

Interactive Presentation



Example 2

TAP Students tap to see a Study Tip.

CHECK Students complete the Check online to determine whether they are ready to move on.

DIFFERENTIATE

Enrichment Activity BL
Ask students to study the proofs for the examples in this lesson and to note the properties that recur, such as the reflexive properties of angles, segments, bisectors, midpoints, and so on. Students can start a list of things to watch for when they are working on proofs and include recurring properties, theorems, formulas, and methods that they can refer to in later lessons. They can also look at the order of the steps in paragraph proofs, flow proofs, and two-column proofs for similarities and differences.

DIFFERENTIATE

Reteaching Activity AL ELL
Have students create note cards with the theorems and definitions from this module to help them learn the concepts better. Also have the students create examples of these theorems.

Learn Proving Triangles Congruent: AAS

Objective

Students use the AAS congruence criterion for triangles to solve problems and to prove relationships in geometric figures.

MP Teaching the Mathematical Practices

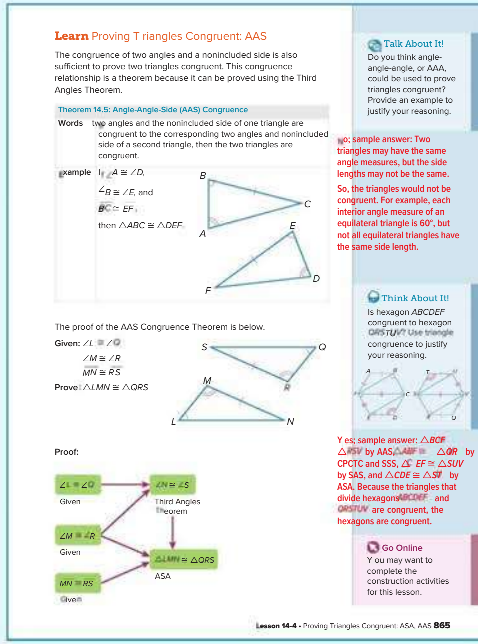
3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This Learn asks students to justify their conclusions.

What Students Are Learning

Notice that SSS, SAS, and ASA were presented as postulates. This is because the proofs of these criteria are beyond the level of most high school geometry students. However, AAS is very easy to prove using ASA, so it is listed as a theorem.

Common Misconception

Sometimes students try to list the congruent sides and angles in a circle as they move around the triangle. This could result in AAS or SAA when there are two pairs of congruent angles and one pair of congruent sides that is not between the angles. They know AAS proves congruence and want to know whether SAA does as well. When this occurs, it is best to redirect their thinking process. With two sets of angles and one set of sides, there are only two possibilities, the side is between the angles or it is another side. When it is between the angles, use ASA. If it is either of the other two sides, then use SAA. This same situation occurs with SSA, but is even more important because SSA is not a test for congruence.



Learn Proving Triangles Congruent: AAS

The congruence of two angles and a nonincluded side is also sufficient to prove two triangles congruent. This congruence relationship is a theorem because it can be proved using the Third Angles Theorem.

Theorem 14.5: Angle-Angle-Side (AAS) Congruence

Words two angles and the nonincluded side of one triangle are congruent to the corresponding two angles and nonincluded side of a second triangle, then the two triangles are congruent.

Example If $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

The proof of the AAS Congruence Theorem is below.

Given: $\angle L \cong \angle Q$, $\angle M \cong \angle R$, $\overline{MN} \cong \overline{RS}$

Prove: $\triangle LMN \cong \triangle QRS$

Proof:

$\angle L \cong \angle Q$ (Given); $\angle M \cong \angle R$ (Given) → $\angle N \cong \angle S$ (Third Angles Theorem); $\overline{MN} \cong \overline{RS}$ (Given) → $\triangle LMN \cong \triangle QRS$ (ASA)

Talk About It! Do you think angle-angle-angle, or AAA, could be used to prove triangles congruent? Provide an example to justify your reasoning.

No; sample answer: Two triangles may have the same angle measures, but the side lengths may not be the same. So, the triangles would not be congruent. For example, each interior angle measure of an equilateral triangle is 60°, but not all equilateral triangles have the same side length.

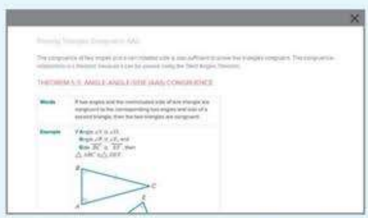
Think About It! Is hexagon *ABCDEF* congruent to hexagon *QRSTUV*? Use triangle congruence to justify your reasoning.

Yes; sample answer: $\triangle BCF \cong \triangle RSV$ by AAS, $\triangle ABF \cong \triangle QRV$ by CPCTC and SSS, $\triangle CEF \cong \triangle SUV$ by SAS, and $\triangle CDE \cong \triangle STU$ by ASA. Because the triangles that divide hexagons *ABCDEF* and *QRSTUV* are congruent, the hexagons are congruent.

Go Online You may want to complete the construction activities for this lesson.

Lesson 14-4 • Proving Triangles Congruent: ASA, AAS 865

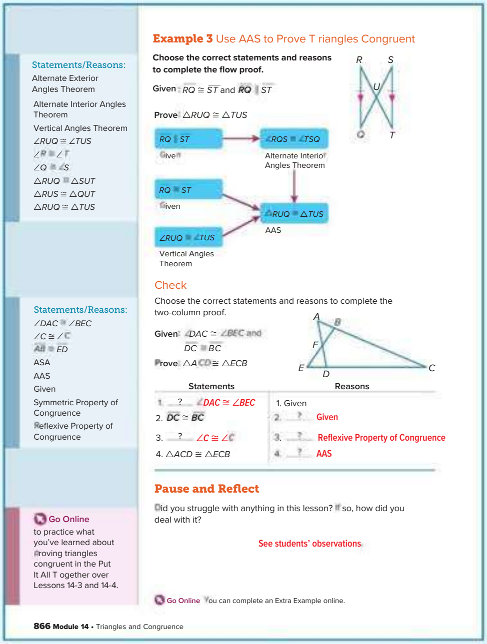
Interactive Presentation



Learn

TYPE

Students type to answer the Think About It!.



Statements/Reasons:
Alternate Exterior Angles Theorem
Alternate Interior Angles Theorem
Vertical Angles Theorem
∠RUQ ≅ ∠TUS
∠R ≅ ∠T
∠Q ≅ ∠S
△RUQ ≅ △SUT
△RUS ≅ △QUT
△RUQ ≅ △TUS

Example 3 Use AAS to Prove Triangles Congruent

Choose the correct statements and reasons to complete the flow proof.

Given: $\overline{RQ} \cong \overline{ST}$ and $\overline{RQ} \parallel \overline{ST}$

Prove: △RUQ ≅ △TUS

Check

Choose the correct statements and reasons to complete the two-column proof.

Given: ∠DAC ≅ ∠BEC and $\overline{DC} \cong \overline{BC}$

Prove: △ACD ≅ △ECB

Statements	Reasons
1. ? ∠DAC ≅ ∠BEC	1. Given
2. $\overline{DC} \cong \overline{BC}$	2. ? Given
3. ? ∠C ≅ ∠C	3. ? Reflexive Property of Congruence
4. △ACD ≅ △ECB	4. ? AAS

Statements/Reasons:
∠DAC ≅ ∠BEC
∠C ≅ ∠C
$\overline{AB} \cong \overline{ED}$
ASA
AAS
Given
Symmetric Property of Congruence
Reflexive Property of Congruence

Pause and Reflect

Did you struggle with anything in this lesson? If so, how did you deal with it?

See students' observations.

Go Online to practice what you've learned about proving triangles congruent in the Put It All Together over Lessons 14-3 and 14-4.

Go Online You can complete an Extra Example online.

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Example 3 Use AAS to Prove Triangles Congruent

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Questions for Mathematical Discourse

AL How is AAS different from ASA? Sample answer: In ASA, the side is included between the two angles; in AAS, it is not.

OL What angle is common to △ACD and △ECB? ∠C

BL Provide a counterexample to show that SSA cannot be used to show that triangles are congruent. See students' work.

Common Error

Students may list the vertices of the triangles in the wrong order in this proof based on the illustration. Remind them to continue to take care to keep triangle vertices in corresponding order.

Exit Ticket

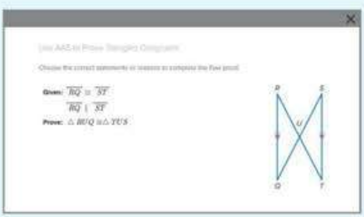
Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Interactive Presentation



Example 3

SELECT

Students select answers to complete the proof.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–15
2	exercises that use a variety of skills from this lesson	16–19
3	exercises that emphasize higher-order and critical-thinking skills	20–24

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–19 odd, 20–24
- Extension: The Ambiguous Case
- ALEKS Proving Triangle Congruence

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–23 odd
- Remediation, Review Resources: Congruence and Transformations
- Personal Tutors
- Extra Examples 1–3
- ALEKS Congruence and Similarity

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–15
- Remediation, Review Resources: Congruence and Transformations
- *Quick Review Math Handbook:* Proving Triangles Congruent (ASA, AAS)
- ALEKS Congruence and Similarity

Important to Know

Digital Exercise Alert Exercise 16 requires a construction and is not available online.

Answers

8a. Proof:

Statements (Reasons)

1. $\angle ABC \cong \angle DCB$ and $\angle ACB \cong \angle DBC$ (Given)
2. $\overline{BC} \cong \overline{BC}$ (Reflexive Property)
3. $\triangle BCA \cong \triangle CBD$ (ASA)

Practice

Go Online Y ou can complete your homework online.

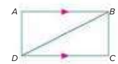
Example 1

PROOF **Write the specified type of proof.** 1–6. See Mod. 14 Answer Appendix.

1. two-column proof

Given: $\overline{AB} \parallel \overline{CD}$, $\angle CBD \cong \angle ADB$

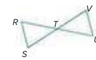
Prove: $\triangle ABD \cong \triangle CDB$



2. two-column proof

Given: $\angle S \cong \angle V$, and T is the midpoint of $\overline{SV}$.

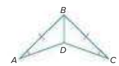
Prove: $\triangle RTS \cong \triangle UTV$



3. flow proof

Given: $\overline{AB} \cong \overline{CB}$, $\angle A \cong \angle C$, and $\overline{DB}$ bisects $\angle ABC$.

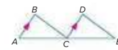
Prove: $\overline{AD} \cong \overline{CD}$



4. paragraph proof

Given: $\overline{CD}$ bisects $\overline{AE}$, $\overline{AB} \parallel \overline{CD}$, and $\angle E \cong \angle BCA$.

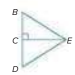
Prove: $\triangle ABC \cong \triangle CDE$



5. paragraph proof

Given: $\overline{CE}$ bisects $\angle BED$; $\angle BCE$ and $\angle ECD$ are right angles.

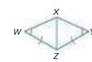
Prove: $\triangle ECB \cong \triangle ECD$



6. paragraph proof

Given: $\angle W \cong \angle Y$, $\overline{WZ} \cong \overline{YZ}$, and $\overline{XZ}$ bisects $\angle WZY$.

Prove: $\triangle XWZ \cong \triangle XYZ$



Lesson 14-4 • Proving Triangles Congruent: ASA, AAS **867**

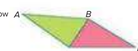
Example 2

7. REASONING Two doorstops have cross sections that are right triangles. Both have a 20° angle, and the length of the side between the 90° and 20° angles are equal.

a. Are the cross sections congruent? Explain. yes; by ASA Congruence Postulate

b. If each cross section has a height of 2 inches and $x = 5$, what is the combined area of the two cross sections? 10 in^2

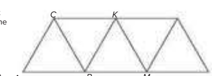
8. ARCHITECTURE An architect used the stained-glass window design in the diagram when remodeling an art studio.



a. If $\angle ABC \cong \angle DCB$ and $\angle ACB \cong \angle DBC$, prove that $\triangle BCA \cong \triangle CBD$. See margin.

b. If the height of $\triangle CBD$ is 1.4 meters and CD is 3.5 meters, how much glass is needed to make the entire window? 4.9 m^2

9. BRIDGES An engineering company that restores bridges needs to determine the amount of steel required to replace some trusses. $\overline{AC} \parallel \overline{BK}$, $\overline{CB} \parallel \overline{KM}$, and B is the midpoint of $\overline{AM}$.



a. Use the given information to confirm that $\triangle ABC \cong \triangle BMK$. See margin.

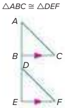
b. $\triangle ABC$ is equilateral, and AB is 18.5 feet. What is the perimeter of quadrilateral $ACKB$? 74 ft

PROOF **Write the specified type of proof.**

10. two-column proof

Given: $\overline{BC} \parallel \overline{EF}$, $\overline{AB} \cong \overline{DE}$, $\angle C \cong \angle F$

Prove: $\triangle ABC \cong \triangle DEF$



Proof:

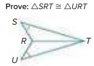
Statements (Reasons)

1. $\overline{BC} \parallel \overline{EF}$, $\overline{AB} \cong \overline{DE}$, $\angle C \cong \angle F$ (Given)
2. $\angle ABC \cong \angle DEF$ (Corresponding Angles Thm.)
3. $\triangle ABC \cong \triangle DEF$ (AAS)

11. flow proof

Given: $\angle S \cong \angle U$, and $\overline{TR}$ bisects $\angle STU$.

Prove: $\triangle SRT \cong \triangle URT$



Proof:

$\overline{TR}$ bisects $\angle STU$. (Given) → $\angle STR \cong \angle UTR$ (Def. of ∠ bisector)

$\angle S \cong \angle U$ (Given)

$\overline{RT} \cong \overline{RT}$ (Reflexive Prop.)

→ $\triangle SRT \cong \triangle URT$ (AAS)

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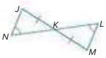
Example 3

Write the specified type of proof.

12. flow proof

Given: $\overline{JK} \cong \overline{MK}$, $\angle N \cong \angle L$

Prove: $\triangle JKN \cong \triangle MKL$



Proof:

$\angle N \cong \angle L$ — Given

$\overline{JK} \cong \overline{MK}$ — Given

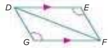
$\angle JKN \cong \angle MKL$ — Vertical ∠s are ≅.

$\triangle JKN \cong \triangle MKL$ — AAS

13. paragraph proof

Given: $\overline{DE} \parallel \overline{FG}$, $\angle E \cong \angle G$

Prove: $\triangle DFG \cong \triangle FDE$



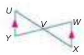
Proof:

It is given that $\angle E \cong \angle G$ and $\overline{DE} \parallel \overline{FG}$. By the Alternate Interior Angles Theorem, $\angle DFG \cong \angle FDE$. $\overline{DF} \cong \overline{DF}$ by the Reflexive Property of Congruence. Therefore, $\triangle DFG \cong \triangle FDE$ by AAS.

14. two-column proof

Given: V is the midpoint of $\overline{YW}$; $\overline{UY} \parallel \overline{XW}$,

Prove: $\triangle UVY \cong \triangle XVW$



Proof:

Statements (Reasons)

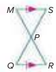
1. V is the midpoint of $\overline{YW}$, and $\overline{UY} \parallel \overline{XW}$. (Given)
2. $\overline{YV} \cong \overline{VW}$ (Midpoint Theorem)
3. $\angle VWX \cong \angle VYU$ (Alternate Interior Angles Thm.)
4. $\angle VUY \cong \angle VXW$ (Alternate Interior Angles Thm.)
5. $\triangle UVY \cong \triangle XVW$ (AAS)

15. two-column proof

Given: $\overline{MS} \cong \overline{RQ}$,

$\overline{MS} \parallel \overline{RQ}$,

Prove: $\triangle MSP \cong \triangle RQP$



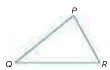
Proof:

Statements (Reasons)

1. $\overline{MS} \cong \overline{RQ}$, $\overline{MS} \parallel \overline{RQ}$ (Given)
2. $\angle SPM \cong \angle QPR$ (Vertical Angles Thm.)
3. $\angle SMP \cong \angle QRP$ (Alternate Interior Angles Thm.)
4. $\triangle MSP \cong \triangle RQP$ (AAS)

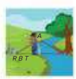
Mixed Exercises

16. USE TOOLS Use a compass and straightedge and the ASA Congruence Postulate to construct a triangle congruent to $\triangle PQR$. See margin.



17. PRECISION Two people decide to take a walk. One person is in Bombay, and the other is in Milwaukee. They start by walking straight for 1 kilometer. Then both turn right at an angle of 110° and continue to walk straight again. After a while, both turn right again, but this time at an angle of 120°. Both walk straight for a while in this new direction until they end up where they started. Each person walked in a triangular path at their location. Are the two triangles they formed congruent? Explain. Yes; sample answer: They are congruent by AAS.

18. USE ESTIMATION Delma came to a river during a hike, and she wanted to estimate the distance across it. She held her walking stick $\overline{AB}$ vertically on the ground at the edge of the river and sighted along the top of the stick across the river to the base of a tree T. Then she turned without changing the angle of her head and sighted along the top of the stick to a rock R, located on her side of the river.



a. Explain why $\triangle ABT \cong \triangle ABR$. Because Delma did not change the angle of her head, $\angle BAT \cong \angle BAR$. $\overline{AB} \cong \overline{AB}$ by the Reflexive Property of ≅. Because the walking stick is vertical, $\angle ABT$ and $\angle ABR$ are right angles, so $\angle ABT \cong \angle ABR$. Therefore, $\triangle ABT \cong \triangle ABR$ by ASA.

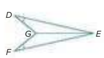
b. Delma finds that it takes 27 paces to walk from her current location to the rock. She also knows that each of her paces is 14 inches long. Explain how she can use this information to estimate the distance across the river. $BR = 27 \cdot 14 = 378$ in. or 31.5 ft. $\triangle ABT \cong \triangle ABR$, so $\overline{BT} \cong \overline{BR}$, because they are corresponding parts of congruent triangles. Therefore, the approximate distance across the river is 31.5 ft.

19. PROOF Write a paragraph proof. See margin.

Given: $\angle D \cong \angle F$

$\overline{GE}$ bisects $\angle DEF$.

Prove: $\overline{DG} \cong \overline{FG}$

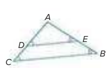


Higher-Order Thinking Skills

20. ANALYZE Find a counterexample to show why SSA (Side-Side-Angle) cannot be used to prove the congruence of two triangles. See margin.

21. FIND THE ERROR Tyrone says that it is not possible to show that $\triangle ADE \cong \triangle ACB$. Lorenzo disagrees, explaining that because $\angle ADE \cong \angle ACB$, $\angle AED \cong \angle ABC$, and $\angle A \cong \angle A$ by the Reflexive Property, $\triangle ADE \cong \triangle ACB$. Who is correct? Explain your reasoning.

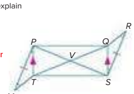
Tyrone; Lorenzo showed that all three corresponding angles were congruent, but AAA is not a proof of triangle congruence.



22. CREATE Draw and label two triangles that could be proved congruent by ASA. See margin.

23. WRITE How do you know which method (SSS, SAS, and so on) to use when you are proving triangle congruence? Use a table to explain your reasoning. See margin.

24. PROOF Using the information given in the diagram, write a flow proof to show that $\triangle PVQ \cong \triangle SVT$. See Mod. 14 Answer Appendix.



Answers

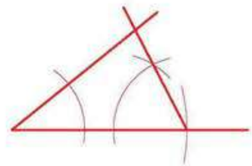
9a.

Sample answer:

- Because $\overline{AC} \parallel \overline{BK}$, $\angle CAB \cong \angle KBM$ by the Corresponding Angles Theorem.
- Because $\overline{CB} \parallel \overline{KM}$ $\angle ABC \cong \angle BMK$ by the Corresponding Angles Theorem.
- Because B is the midpoint of $\overline{AM}$, $\overline{AB} \cong \overline{BM}$ by the Midpoint Theorem.

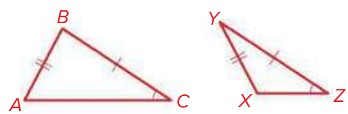
Therefore, by the ASA Congruence Postulate, $\triangle ABC \cong \triangle BMK$.

16.

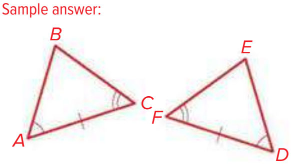


19. Proof: Because it is given that $\overline{GE}$ bisects $\angle DEF$, $\angle DEG \cong \angle FEG$ by the definition of an angle bisector. It is given that $\angle D \cong \angle F$. By the Reflexive Property, $\overline{GE} \cong \overline{GE}$. So $\triangle DEG \cong \triangle FEG$ by AAS. Therefore $\overline{DG} \cong \overline{FG}$ by CPCTC.

20. Sample answer: $\overline{AB} \cong \overline{XY}$, $\overline{BC} \cong \overline{YZ}$, $\angle C \cong \angle Z$. $\triangle ABC \ncong \triangle XYZ$.



22. Sample answer:



23.

Method	Use when...
Definition of Congruent Triangles	All corresponding parts of one triangle are congruent to the corresponding parts of the other triangle.
SSS	The three sides on one triangle must be congruent to the three sides of the other triangle.
SAS	Two sides and the included angle of one triangle must be congruent to two sides and the included angle of the other triangle.
ASA	Two angles and the included side of one triangle must be congruent to two angles and the included side of the other triangle.
AAS	Two angles and a nonincluded side of one triangle must be congruent to two angles and the corresponding nonincluded side of the other triangle.

Lesson 14-5

Proving Right Triangles Congruent

G.CO.10, G.SRT.5

LESSON GOAL

Students solve problems using the LL, HA, LA, and HL Theorems of Right Triangle Congruence.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Congruence Theorems and Right Triangles

Develop:

Right Triangles Congruence

• Problem Solving with Right Triangles

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Congruence and Transformations	●	●		●
Extension: Triangles and Area Formulas		●	●	●

Language Development Handbook

Assign page 91 of the *Language Development Handbook* to help your students build mathematical language related to solving problems using the LL, HA, LA and HL Theorems of Right Triangle Congruence.

ELL You can use the tips and suggestions on page T91 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.10 Prove theorems about triangles.

G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

5 Use appropriate tools strategically.

6 Attend to precision.

Coherence

Vertical Alignment

Previous

Students proved that triangles are congruent using the ASA Postulate or AAS Theorem.

G.SRT.5

Now

Students use triangle congruence criteria to prove right triangles congruent.

G.CO.10, G.SRT.5

Next

Students will solve problems involving isosceles and equilateral triangles using triangle congruence.

G.CO.10, G.SRT.5

Rigor

The Three Pillars of Rigor

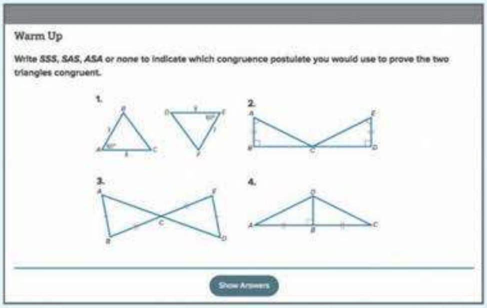
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of congruent triangles to right triangles. They build fluency and apply their understanding by solving real-world problems related to congruent right triangles.

Mathematical Background

Right triangles have their own theorems to prove congruence. The LL Congruence Theorem and the HL Postulate are used to prove right triangles congruent.

Interactive Presentation



Warm Up

Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- proving triangles congruent by using congruence criteria

Answers:

1. SAS
2. ASA
3. none
4. SAS

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of congruent right triangles.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Explore Congruence Theorems and Right Triangles

Objective

Students use dynamic geometry software to make a conjecture about the criteria needed to prove right triangles congruent.

MP Teaching the Mathematical Practices

5 Decide When to Use Tools Mathematically proficient students can make sound decisions about when to use mathematical tools such as dynamic geometry software. Help them see why using these tools will help to solve problems and what the limitations are of using these tools.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

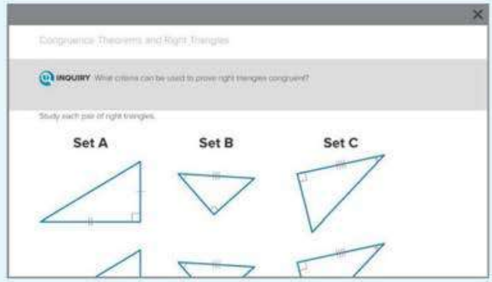
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use sets of right triangles to investigate how triangle congruence criteria work in right triangles. Next students answer some guiding exercises to investigate how triangle congruence criteria can be shortened when used for right triangles. Students then use a sketch to investigate how SSA can work if the known angle is a right angle. Next students will complete guiding exercises guiding them to conjecture the HL criterion and to write a proof. Then, students will answer the Inquiry Question.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

(*continued on the next page*)

Interactive Presentation



Explore

TYPE

Students type to complete Exercises.

WEB SKETCHPAD

Students use a sketch to explore the HL triangle congruence criterion.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and view a sample answer.

Explore Congruence Theorems and Right Triangles (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why are you not investigating the SSS criterion? Sample answer: Right triangles are special because there is already one pair of corresponding congruent angles, but SSS does not require any pairs of congruent angles.

Inquiry

What criteria can be used to prove right triangles congruent? Sample answer: You can prove that two right triangles are congruent if corresponding legs are congruent by SAS. If one pair of corresponding acute angles and the hypotenuses are congruent, you can prove the right triangles are congruent by AAS. If one pair of corresponding acute angles and one pair of corresponding legs are congruent, then you can prove that the triangles are congruent by ASA.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Right T riangle Congruence

Objective

Students use the right triangle congruence theorems to prove relationships in geometric figures.

Teaching the Mathematical Practices

3 Construct Arguments In this Learn, students will use stated assumptions, definitions, and previously established results to construct an argument.

What Students Are Learning

Three of the right triangle criteria, LL, HA, and LA, come directly from general triangle congruence criteria. HL comes from the Pythagorean Theorem and the SSS criterion.

Common Misconception

Some students may think that the LL shortcut for the congruence of right triangles comes from SSS when the Pythagorean theorem is applied. Have the students explore the situation with a drawing. They can draw two congruent right triangles and mark sides so the triangles have LL. There is already a congruence guarantee for this, SAS. What would the non right triangle congruence be for HL? Is this a guarantee? (It would be SSA, and no, this does not work in triangles that are not right.)

Essential Question Follow-Up

Students learn to apply triangle angle criteria to right angles.

Ask:

Why is it useful to be able to prove right triangles congruent? Sample answer: Right triangles can model many real-world objects, and knowing that objects are the same shape and size can help you produce them faster.

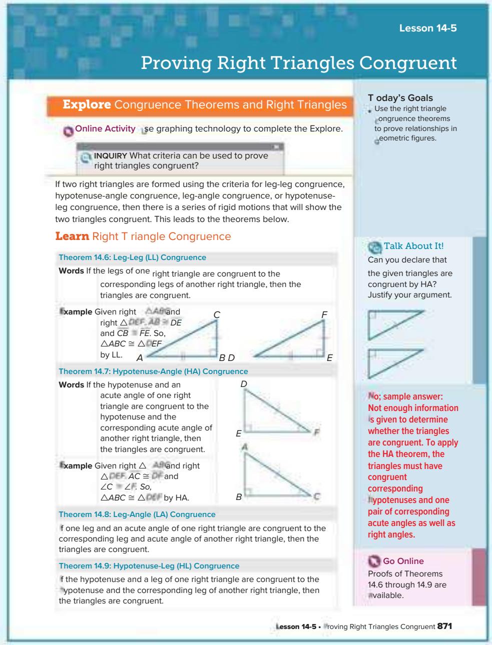
Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

DIFFERENTIATE

Enrichment Activity BL

Have students prove the right triangle congruence criteria theorems.



Lesson 14-5

Proving Right Triangles Congruent

Explore Congruence Theorems and Right Triangles

Online Activity Use graphing technology to complete the Explore.

INQUIRY What criteria can be used to prove right triangles congruent?

If two right triangles are formed using the criteria for leg-leg congruence, hypotenuse-angle congruence, leg-angle congruence, or hypotenuse-leg congruence, then there is a series of rigid motions that will show the two triangles congruent. This leads to the theorems below.

Learn Right T riangle Congruence

Theorem 14.6: Leg-Leg (LL) Congruence

Words If the legs of one right triangle are congruent to the corresponding legs of another right triangle, then the triangles are congruent.

Example Given right $\triangle ABC$ and right $\triangle DEF$, $\overline{AB} \cong \overline{DE}$ and $\overline{CB} \cong \overline{FE}$. So, $\triangle ABC \cong \triangle DEF$ by LL.

Theorem 14.7: Hypotenuse-Angle (HA) Congruence

Words If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and the corresponding acute angle of another right triangle, then the triangles are congruent.

Example Given right $\triangle ABC$ and right $\triangle DEF$, $\overline{AC} \cong \overline{DF}$ and $\angle C \cong \angle F$. So, $\triangle ABC \cong \triangle DEF$ by HA.

Theorem 14.8: Leg-Angle (LA) Congruence

If one leg and an acute angle of one right triangle are congruent to the corresponding leg and acute angle of another right triangle, then the triangles are congruent.

Theorem 14.9: Hypotenuse-Leg (HL) Congruence

If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and the corresponding leg of another right triangle, then the triangles are congruent.

T oday's Goals

- Use the right triangle congruence theorems to prove relationships in geometric figures.

Talk About It!

Can you declare that the given triangles are congruent by HA? Justify your argument.

No; sample answer: Not enough information is given to determine whether the triangles are congruent. To apply the HA theorem, the triangles must have congruent corresponding hypotenuses and one pair of corresponding acute angles as well as right angles.

Go Online

Proofs of Theorems 14.6 through 14.9 are available.

Lesson 14-5 • Proving Right Triangles Congruent 871

Interactive Presentation

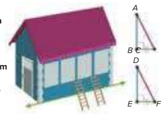
Learn

TYPE

Students type their answers to the Talk About It!.

Example 1 Problem Solving with Right T riangles

HOME IMPROVEMENT **Craig and his brother are painting a house. The brothers use ladders that are the same length. If they place their ladders an equal distance from the house, will each ladder reach the same height on the house? Construct a logical argument.**



Draw a diagram to model this situation. It is given that the length of the ladders is the same and that they are placed the same distance from the house.

Because the wall of the house is perpendicular to the ground, the triangles formed by the house, the ground, and the ladders are right triangles. The hypotenuses are congruent because the ladders are the same length. The corresponding legs along the ground are congruent because the ladders are placed the same distance from the house. So the triangles are congruent by the Hypotenuse-Leg Congruence Theorem or HL. Thus, $\overline{AB}$ and $\overline{DE}$ are congruent by CPCTC. You can conclude that the ladders reach to the same height on the house.

Check

FENCES The fence has parallel supports and a crossbar that forms two triangles. Complete the proof to show that the triangles are congruent.

Given: $\angle B$ and $\angle D$ are right angles. $\overline{AD} \parallel \overline{BC}$

Prove: $\triangle ABC \cong \triangle CDA$

Proof:

Statements	Reasons
1. $\angle B$ and $\angle D$ are right angles.	1. ? Given
2. $\triangle ABC$ and $\triangle CDA$ are right triangles.	2. ? Definition of right triangle
3. $\overline{AD} \parallel \overline{BC}$	3. ? Given
4. $\angle DAC \cong \angle BCA$	4. ? Alternate interior angles are congruent.
5. $\overline{AC} \cong \overline{AC}$	5. ? Reflexive Property of Congruence
6. $\triangle ABC \cong \triangle CDA$	6. ? HA

Reasons:
- Alternate exterior angles are congruent.
- Alternate interior angles are congruent.
- Consecutive interior angles are congruent.
- Definition of right triangle
- Given
- HA
- HL
- LA
- Reflexive Property
- Symmetric Property

Go Online You can complete an Extra Example online.

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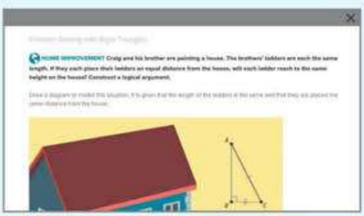
Interactive Presentation



Example 1

SELECT

Students select the correct terms to complete the solution.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Problem Solving with Right Triangles

Teaching the Mathematical Practices

4 Use Tools Point out that to solve the problem in this example, students will need to use diagrams.

Questions for Mathematical Discourse

AL How are right triangles labeled differently than other triangles? They have a right angle symbol.

OL What other unique features do right triangles have? The sides adjacent to the right angle are called legs, and the side opposite the right angle is called the hypotenuse.

BL Is there another type of triangle that has special names for its parts? isosceles

Common Error

Students may forget that they already know that a pair of corresponding angles are congruent when given a pair of right angles. Encourage students to sketch the information given in the problem, and draw in the right angle symbols on their sketch. This will give them a visual reminder so they can see that those right angles are congruent.

DIFFERENTIATE

Language Development Activity AL ELL

IF students are having difficulty determining which right triangle congruence criterion to use,

THEN have the students label the names of the corresponding parts such as “leg” on their diagrams.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–3
2	exercises that use a variety of skills from this lesson	4–11
3	exercises that emphasize higher-order and critical-thinking skills	12–14

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

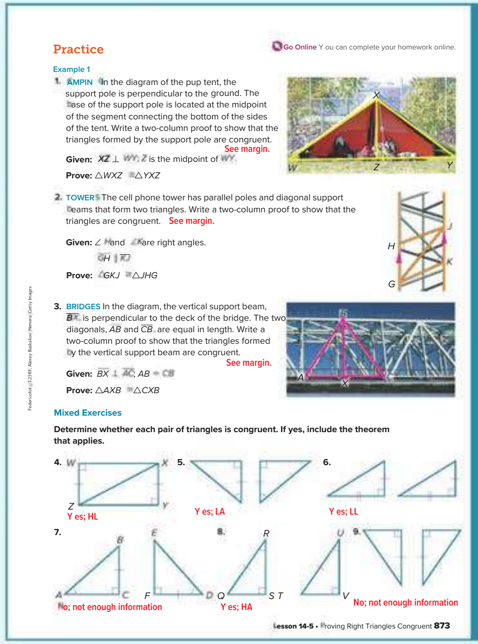
- Practice, Exercise 1, 12–14
- Extension: Triangles and Area Formulas
- ALEKS Proving Triangle Congruence

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–13 odd
- Remediation, Review Resources: Congruence and Transformations
- Personal Tutors
- Extra Example 1
- ALEKS Congruence and Similarity

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1, 3
- Remediation, Review Resources: Congruence and Transformations
- ALEKS Congruence and Similarity



Practice

Go Online Y ou can complete your homework online.

Example 1

1. AMPIN In the diagram of the pup tent, the support pole is perpendicular to the ground. The ase of the support pole is located at the midpoint of the segment connecting the bottom of the sides of the tent. Write a two-column proof to show that the triangles formed by the support pole are congruent. See margin.

Given: $XZ \perp WY$; Z is the midpoint of WY.

Prove: $\triangle WXZ \cong \triangle YXZ$

2. TOWERS The cell phone tower has parallel poles and diagonal support beams that form two triangles. Write a two-column proof to show that the triangles are congruent. See margin.

Given: $\angle H$ and $\angle K$ are right angles. $\overline{GH} \parallel \overline{KJ}$

Prove: $\triangle GKJ \cong \triangle JHG$

3. BRIDGES In the diagram, the vertical support beam, $\overline{BX}$, is perpendicular to the deck of the bridge. The two diagonals, $\overline{AB}$ and $\overline{CB}$, are equal in length. Write a two-column proof to show that the triangles formed by the vertical support beam are congruent. See margin.

Given: $\overline{BX} \perp \overline{AC}$; $AB = CB$

Prove: $\triangle AXB \cong \triangle CXB$

Federicofot 123RF, Alexey Baskakov Hemera Getty Images

Mixed Exercises

Determine whether each pair of triangles is congruent. If yes, include the theorem that applies.

4. Y es; HL

5. Y es; LA

6. Y es; LL

7. No; not enough information

8. Y es; HA

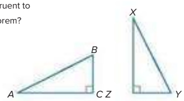
9. No; not enough information

Lesson 14-5 • Proving Right Triangles Congruent 873

10. Which pairs of corresponding parts need to be congruent to prove that $\triangle ABC \cong \triangle XYZ$ using the indicated theorem?

a. HA Sample answer: $\overline{AB} \cong \overline{XY}$ and $\angle A \cong \angle X$

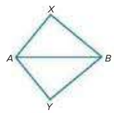
b. LL $\overline{BC} \cong \overline{YZ}$ and $\overline{AC} \cong \overline{XZ}$



11. PROOF Write a two-column proof. See margin.

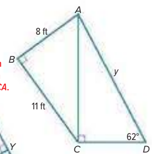
Given: $\overline{BX} \perp \overline{XA}$, $\overline{BY} \perp \overline{YA}$, and $\overline{XA} \cong \overline{YA}$

Prove: $\triangle BXA \cong \triangle BYA$



Higher-Order Thinking Skills

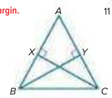
12. WRITE The sketch shows the side view of a sculpture that is being designed by an artist. Determine whether $\triangle ABC \cong \triangle DCA$. If yes, then provide a paragraph proof. If no, then explain your reasoning. No; there is not enough information provided. You would need to know another angle measure in $\triangle ABC$, the value of y, or the length of another side of $\triangle DCA$.



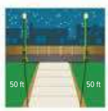
13. PROOF Write a paragraph proof. See margin.

Given: $\overline{BY} \perp \overline{AC}$; $\overline{CX} \perp \overline{AB}$; $AX = AY$

Prove: $\triangle ABY \cong \triangle ACX$



14. FIND THE ERROR The Harding Family recently hired an electrical contractor to install two light posts on opposite sides of the end of the walkway that leads from the rear of their house to the alley. They wanted the contractor to install the posts so their distances from the end of the walkway were equal in length. Suppose two triangles are drawn from the light posts to both ends of the walkway as shown. Josephine says that it can be proved with a right triangle congruency theorem that the posts are equidistant from the end of the driveway. Is Josephine's conclusion correct? Explain your reasoning.



No; sample answer: You must also know that the segments joining the light posts to the end of the walkway are perpendicular to the end of the walkway.

Answers

1. Proof:

Statements (Reasons)

1. $\overline{XZ} \perp \overline{WY}$ (Given)
2. $\angle XZW$ and $\angle XZY$ are right angles. ($\perp$ lines form right angles.)
3. $\triangle WXZ$ and $\triangle YXZ$ are right triangles. (Definition of right triangle)
4. Z is the midpoint of $\overline{WY}$. (Given)
5. $\overline{WZ} \cong \overline{ZY}$ (Definition of midpoint)
6. $\overline{XZ} \cong \overline{XZ}$ (Reflexive Property of Congruence)
7. $\triangle WXZ \cong \triangle YXZ$ (LL Congruence Theorem)

2. Proof:

Statements (Reasons)

1. $\angle H$ and $\angle K$ are right angles. (Given)
2. $\triangle GKJ$ and $\triangle JHG$ are right triangles. (Definition of right triangle)
3. $\overline{GH} \parallel \overline{KJ}$ (Given)
4. $\angle HGJ \cong \angle KJG$ (Alternate interior angles are congruent.)
5. $\overline{GJ} \cong \overline{GJ}$ (Reflexive Property of Congruence)
6. $\triangle GKJ \cong \triangle JHG$ (HA Congruence Theorem)

3. Proof:

Statements (Reasons)

1. $\overline{BX} \perp \overline{AC}$ (Given)
2. $\angle AXB$ and $\angle CXB$ are rt. $\angle s$. (Definition of $\perp$ lines)
3. $\triangle AXB$ and $\triangle CXB$ are rt. $\triangle s$. (Definition of right $\triangle s$)
4. $\overline{XB} \cong \overline{XB}$ (Reflexive Property of Congruence)
5. $AB = CB$ (Given)
6. $\overline{AB} \cong \overline{CB}$ (Definition of congruent)
7. $\triangle AXB \cong \triangle CXB$ (HL Congruence Theorem)

11. Proof:

Statements (Reasons)

1. $\overline{BX} \perp \overline{XA}$, $\overline{BY} \perp \overline{YA}$ (Given)
2. $\angle BXA$ and $\angle BYA$ are rt. $\angle s$. (Definition of $\perp$ lines)
3. $\triangle BXA$ and $\triangle BYA$ are rt. $\triangle s$. (Definition of right $\triangle s$)
4. $\overline{XA} \cong \overline{YA}$ (Given)
5. $\overline{BA} \cong \overline{BA}$ (Reflexive Property of Congruence)
6. $\triangle BXA \cong \triangle BYA$ (HL Congruence Theorem)

13. Proof: By the definition of $\perp$ segments, $\angle AYB$ and $\angle AXC$ are right angles. By the definition of right triangles, both $\triangle AYB$ and $\triangle AXC$ are right triangles. By the definition of congruent segments, $\overline{AX}$ is congruent to $\overline{AY}$. By the Reflexive Property of Congruence, $\angle BAY$ is congruent to $\angle CAX$. Therefore by LA, $\triangle ABY$ is congruent to $\triangle ACX$.

G.CO.10, G.SRT.5

Isosceles and Equilateral Triangles

LESSON GOAL

Students solve problems involving isosceles and equilateral triangles using theorems of triangle congruence.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore:

- Properties of Equilateral, Isosceles, and Scalene Triangles
- Isosceles and Equilateral Triangles

Develop:

Isosceles Triangles

- Prove Theorems About Isosceles Triangles
- Find Missing Measures in Isosceles Triangles

Equilateral Triangles

- Find Missing Measures in Equilateral Triangles
- Find Missing Values

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Triangles	●	●		●
Extension: Exterior and Interior Angles of Isosceles Triangles		●	●	●

Language Development Handbook

Assign page 92 of the *Language Development Handbook* to help your students build mathematical language related to solving problems involving isosceles and equilateral triangles.

ELL You can use the tips and suggestions on page T92 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.10 Prove theorems about triangles.

G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

4 Model with mathematics.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students used triangle congruence criteria to prove right triangles congruent.
G.CO.10, G.SRT.5

Now
Students solve problems involving isosceles and equilateral triangles using triangle congruence.
G.CO.10, G.SRT.5

Next
Students will use coordinate geometry to prove triangles congruent.
G.CO.10, G.GPE.4

Rigor

The Three Pillars of Rigor

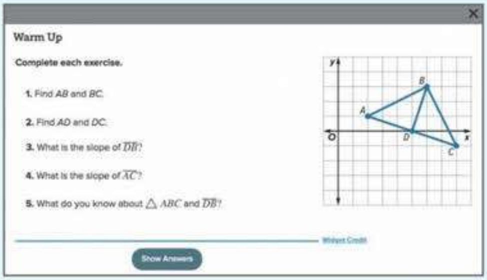
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of relationships in triangles to isosceles and equilateral triangles. They build fluency and apply their understanding by solving real-world problems related to isosceles and equilateral triangles.

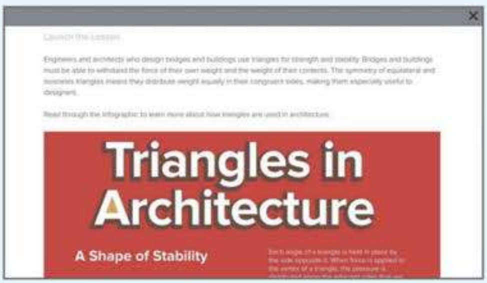
Mathematical Background

Isosceles triangles have special properties recognized in the Isosceles Triangle Theorem and its converse. If two sides of a triangle are congruent, then the angles opposite those sides are congruent. This theorem is used to prove corollaries about the angles of an equilateral triangle.

Interactive Presentation



Warm Up



Launch the Lesson

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- identifying isosceles and equilateral triangles

Answers:

1. $2\sqrt{5}$ units
2. $\sqrt{10}$ units
3. 3
4. $-\frac{1}{3}$
5. $\triangle ABC$ is isosceles; $\overline{DB}$ is a perpendicular bisector of $\overline{AC}$.

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see real-world applications of equilateral and isosceles triangles.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then, discuss the questions below with the class.

Explore Properties of Equilateral, Isosceles, and Scalene Triangles

Objective

Students use dynamic geometry software to investigate the properties of equilateral, isosceles, and scalene triangles.

Teaching the Mathematical Practices

3 Make Conjectures In this Explore, students will make conjectures and then build a logical progression of statements to validate the conjectures. Once students have made their conjectures, guide the students to validate them.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

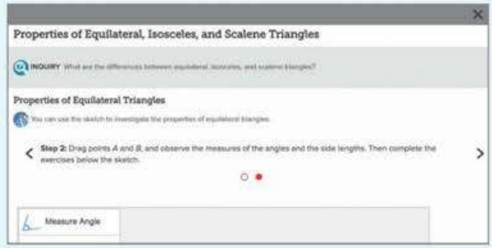
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to records their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use sketches to investigate the properties of equilateral, isosceles, and scalene triangles. With each sketch, students complete the exercises guiding them to make conjectures about equilateral, isosceles, and scalene triangles. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

WEB SKETCHPAD

Students use a sketch to investigate the properties of equilateral, isosceles, and scalene triangles.

TYPE

Students type to complete the exercises.

Interactive Presentation

Explore

TYPE

Students will respond to the Inquiry Question and view a sample answer.

Explore Properties of Equilateral, Isosceles, and Scalene Triangles (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What observation can you make about the sides and angles in the equilateral triangle? Sample answer: All three sides are congruent, and all three angles are congruent.
- What observation can you make about the sides and angles in the scalene triangle? Sample answer: The lengths of the sides are all different, and the measures of the angles are all different.

Inquiry

What are the differences between equilateral, isosceles, and scalene triangles? Sample answer: An equilateral triangle has three congruent sides and angles. An isosceles triangle has two congruent sides and angles. A scalene triangle has no congruent sides or angles.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Explore Isosceles and Equilateral T riangles

Objective

Students use dynamic geometry software to make conjectures about the relationships between the parts of isosceles and equilateral triangles.

MP Teaching the Mathematical Practices

3 Make Conjectures In this Explore, students will make conjectures and then build a logical progression of statements to validate the conjectures. Once students have made their conjectures, guide the students to validate them.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

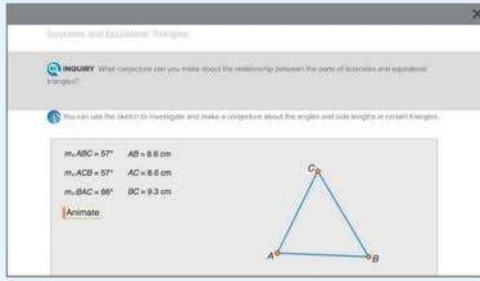
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will watch an animation of an isosceles triangle to observe relationships between sides and angles of an isosceles triangle. Then students complete the exercises guiding them to conjecture about isosceles and equilateral triangles. Then, students will answer the Inquiry Question.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

(*continued on the next page*)

Interactive Presentation



Explore

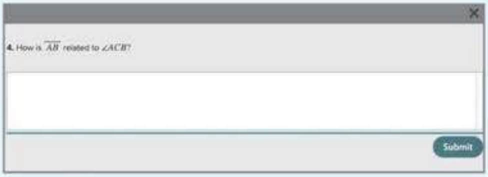
WEB SKETCHPAD

Students use a sketch to investigate isosceles and equilateral triangles.

TYPE

Students type to complete the guiding exercises.

Interactive Presentation



Explore

TYPE

Students respond to the Inquiry Question and view a sample answer.

Explore Isosceles and Equilateral Triangles (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- When you animate the triangle, what do you notice about the lengths of sides *AB* and *AC*? The lengths of sides *AB* and *AC* always remain the same.
- When you animate the triangle, what do you notice about the measures of $\angle ABC$ and $\angle ACB$? The measures of $\angle ABC$ and $\angle ACB$ always remain equal.

Inquiry

What conjecture can you make about the relationship between the parts of isosceles and equilateral triangles? Sample answer: An isosceles triangle must have two congruent angles opposite its two congruent sides. An equilateral triangle must have three congruent angles and three congruent sides.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Isosceles Triangles

Objective

Students solve problems involving isosceles triangles by using theorems of triangle congruence.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of isosceles triangles in this Learn.

What Students Are Learning

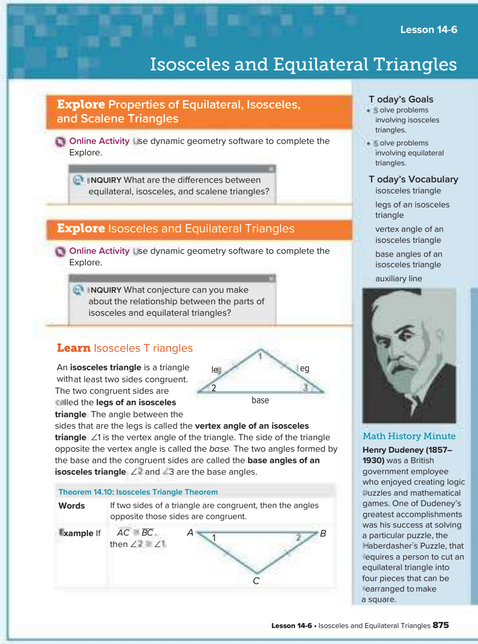
Special triangles such as right triangles and isosceles triangles have special definitions for some of their parts. The sides of right triangles adjacent to the right angle are legs, and the side opposite the right angle is the hypotenuse. Two congruent sides of an isosceles triangle are also called legs, and the third side is called the base.

Common Misconception

When they are looking at a figure, students have a hard time adjusting to the idea that even if two segments or angles look congruent, they cannot be assumed to be congruent unless they are marked. A triangle is not isosceles unless at least two of the sides are marked congruent, no matter how much it looks like an isosceles triangle. Maybe one side is a millimeter longer, but the figure is too small to show the difference. Congruent means "exactly the same." It is helpful to remind the students that they are learning a new, extremely precise language. In geometry, congruence must be communicated with the proper marks if it is known to exist.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 14-6

Isosceles and Equilateral Triangles

Explore Properties of Equilateral, Isosceles, and Scalene Triangles

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY What are the differences between equilateral, isosceles, and scalene triangles?

Explore Isosceles and Equilateral Triangles

Online Activity Use dynamic geometry software to complete the Explore.

INQUIRY What conjecture can you make about the relationship between the parts of isosceles and equilateral triangles?

Learn Isosceles Triangles

An **isosceles triangle** is a triangle with at least two sides congruent. The two congruent sides are called the **legs of an isosceles triangle**. The angle between the sides that are the legs is called the **vertex angle of an isosceles triangle**. ∠1 is the vertex angle of the triangle. The side of the triangle opposite the vertex angle is called the *base*. The two angles formed by the base and the congruent sides are called the **base angles of an isosceles triangle**. ∠2 and ∠3 are the base angles.

Theorem 14.10: Isosceles Triangle Theorem

Words	If two sides of a triangle are congruent, then the angles opposite those sides are congruent.
Example	If $\overline{AC} \cong \overline{BC}$, then $\angle 2 \cong \angle 1$.

Today's Goals
- Solve problems involving isosceles triangles.
- Solve problems involving equilateral triangles.

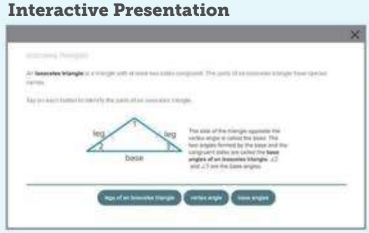
Today's Vocabulary
isosceles triangle
legs of an isosceles triangle
vertex angle of an isosceles triangle
base angles of an isosceles triangle
auxiliary line

Math History Minute

Henry Dudeney (1857–1930) was a British government employee who enjoyed creating logic puzzles and mathematical games. One of Dudeney's greatest accomplishments was his success at solving a particular puzzle, the Haberdasher's Puzzle, that requires a person to cut an equilateral triangle into four pieces that can be rearranged to make a square.

Lesson 14-6 • Isosceles and Equilateral Triangles 875

Interactive Presentation



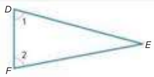
Learn

TAP

Students tap to see illustrations of definitions related to isosceles triangles.

Theorem 14.11: Converse of the Isosceles Triangle Theorem

Words	If two angles of a triangle are congruent, then the sides opposite those angles are congruent.
Example	If $\angle 1 \cong \angle 2$, then $\overline{FE} \cong \overline{DE}$



Go Online A proof of Theorem 5.11 is available.

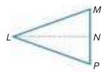
Example 1 Prove the Isosceles T riangle Theorem

Prove the Isosceles Triangle Theorem.

T o prove the Isosceles Triangle Theorem, draw an *auxiliary line* and use the two triangles that are formed. An **auxiliary line** is an extra line or segment drawn in a figure to help analyze geometric relationships.

Given: $\triangle LMP$, $\overline{LM} \cong \overline{LP}$

Prove: $\angle M \cong \angle P$



Proof

Statements	Reasons
1. Let N be the midpoint of $\overline{MP}$.	1. Every segment has exactly one midpoint.
2. Draw an auxiliary segment $\overline{LN}$.	2. Two points determine a line.
3. $\overline{MN} \cong PN$	3. Midpoint Theorem
4. $\overline{LN} \cong LN$	4. Reflexive Property of Congruence
5. $\overline{LM} \cong \overline{LP}$	5. Given
6. $\triangle LMN \cong \triangle LPN$	6. SSS
7 . $\angle M \cong \angle P$	7 . CPCTC

Statements/Reasons:
Definition of congruence
Given
Midpoint Theorem
$\overline{LN} \cong \overline{LN}$
$\overline{MP} \cong LP$

Go Online Y ou can complete an Extra Example online.

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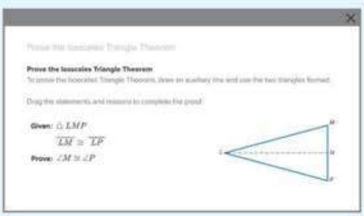
Interactive Presentation



Example 1

DRAG & DROP

Students drag statements and reasons to complete the proof.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Prove the Isosceles T riangle Theorem

MP Teaching the Mathematical Practices

7 Draw an Auxiliary Line Help students see the need to draw an auxiliary line to prove the Isosceles Triangle Theorem.

Questions for Mathematical Discourse

AL What do you know about isosceles triangles? At least two of the sides are congruent.

OL Why is it useful to draw an auxiliary line in this proof? The auxiliary line creates two triangles so we can find three pairs of corresponding congruent sides.

BL Draw the angle bisector of $\angle MLP$. Can you still prove the theorem? Explain. Y es; using the angle bisector, you can prove the theorem using SAS.

Common Misconception

Students may notice that the base angles appear congruent and think that the theorem is obvious and, therefore, does not need to be proved. Remind them that we can make conjectures based on appearances, but only a proof will let us know whether a particular mathematical statement is true.

Example 2 Find Missing Measures in Isosceles Triangles

Teaching the Mathematical Practices

1 Understand the Approaches of Others Work with students to look at the Alternate Method. Ask students to compare and contrast the original method and the alternate method.

Questions for Mathematical Discourse

AL What do you know about $m\angle B$ and $m\angle C$? They are equal.

OL Write an equation to find $m\angle B$ using the Triangle Angle-Sum Theorem. $m\angle B + m\angle B + 70° = 180°$

BL If you were given that $m\angle A = (5x + 4)°$ and $m\angle B = (7x - 7)°$, what would the value of x be? $x = 10$

Learn Equilateral Triangles

Objective

Students solve problems involving equilateral triangles by using theorems of triangle congruence.

Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of equilateral triangles in this Learn.

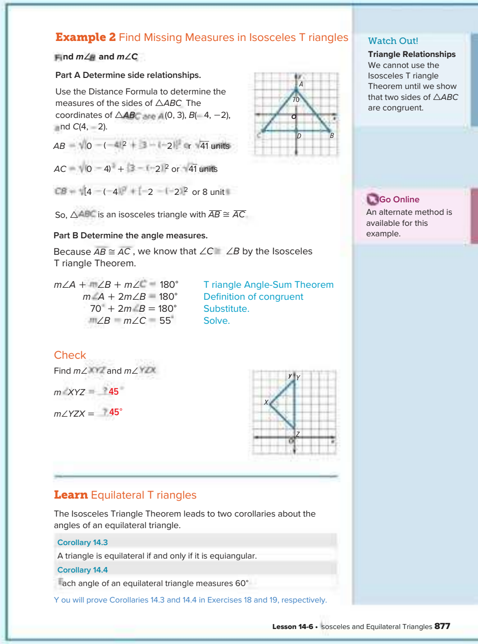
Common Misconception

Students may assume that the definition of an isosceles triangle does not include equilateral triangles. Remind them that being isosceles means that the triangle has at least two congruent sides, not exactly two.

DIFFERENTIATE

Enrichment Activity BL

Create an equilateral triangle with three unknown sides and use an algebraic equation to solve the problem.



Example 2 Find Missing Measures in Isosceles Triangles

Find $m\angle B$ and $m\angle C$.

Part A Determine side relationships.

Use the Distance Formula to determine the measures of the sides of $\triangle ABC$. The coordinates of $\triangle ABC$ are $A(0, 3)$, $B(-4, -2)$, and $C(4, -2)$.

$AB = \sqrt{(0 - (-4))^2 + (3 - (-2))^2}$ or $\sqrt{41}$ units

$AC = \sqrt{(0 - 4)^2 + (3 - (-2))^2}$ or $\sqrt{41}$ units

$CB = \sqrt{(4 - (-4))^2 + (-2 - (-2))^2}$ or 8 units

So, $\triangle ABC$ is an isosceles triangle with $\overline{AB} \cong \overline{AC}$.

Part B Determine the angle measures.

Because $\overline{AB} \cong \overline{AC}$, we know that $\angle C \cong \angle B$ by the Isosceles Triangle Theorem.

$m\angle A + m\angle B + m\angle C = 180°$	Triangle Angle-Sum Theorem
$m\angle A + 2m\angle B = 180°$	Definition of congruent
$70° + 2m\angle B = 180°$	Substitute.
$m\angle B = m\angle C = 55°$	Solve.

Check

Find $m\angle XYZ$ and $m\angle YZX$.

$m\angle XYZ =$? 45°

$m\angle YZX =$? 45°

Learn Equilateral Triangles

The Isosceles Triangle Theorem leads to two corollaries about the angles of an equilateral triangle.

Corollary 14.3
A triangle is equilateral if and only if it is equiangular.

Corollary 14.4
Each angle of an equilateral triangle measures 60°.

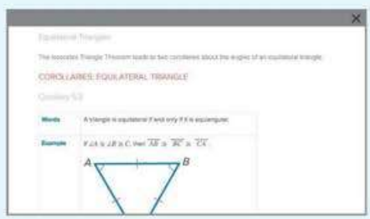
You will prove Corollaries 14.3 and 14.4 in Exercises 18 and 19, respectively.

Watch Out!
Triangle Relationships We cannot use the Isosceles Triangle Theorem until we show that two sides of $\triangle ABC$ are congruent.

Go Online An alternate method is available for this example.

Lesson 14-6 • Isosceles and Equilateral Triangles 877

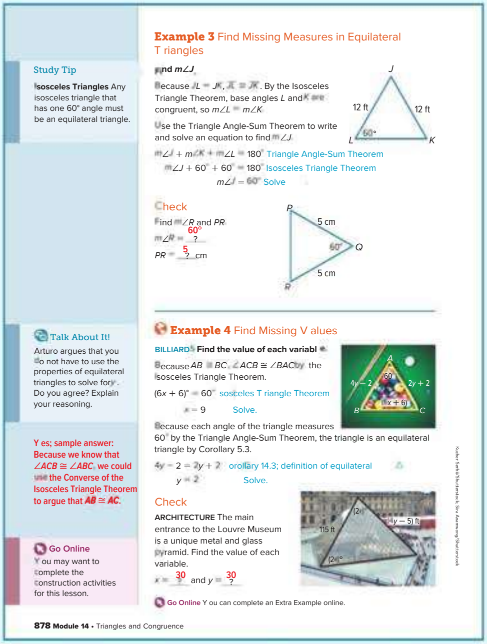
Interactive Presentation



Learn

TAP

Students tap to see a Math History Moment or to answer questions.



Study Tip

Isosceles Triangles Any isosceles triangle that has one 60° angle must be an equilateral triangle.

Example 3 Find Missing Measures in Equilateral Triangles

Find $m\angle J$.

Because $JL = JK$, $\overline{JL} \cong \overline{JK}$. By the Isosceles Triangle Theorem, base angles L and K are congruent, so $m\angle L = m\angle K$.

Use the Triangle Angle-Sum Theorem to write and solve an equation to find $m\angle J$.

$m\angle J + m\angle K + m\angle L = 180°$ Triangle Angle-Sum Theorem

$m\angle J + 60° + 60° = 180°$ Isosceles Triangle Theorem

$m\angle J = 60°$ Solve

Check

Find $m\angle R$ and PR.

$m\angle R =$ 60°

$PR =$ 5 cm

Talk About It!

Arturo argues that you do not have to use the properties of equilateral triangles to solve for y. Do you agree? Explain your reasoning.

Yes; sample answer: Because we know that $\angle ACB \cong \angle ABC$, we could use the Converse of the Isosceles Triangle Theorem to argue that $\overline{AB} \cong \overline{AC}$.

Example 4 Find Missing Values

BILLIARDS **Find the value of each variable.**

Because $AB = BC$, $\angle ACB \cong \angle BAC$ by the Isosceles Triangle Theorem.

$(6x + 6)° = 60°$ Isosceles Triangle Theorem

$x = 9$ Solve.

Because each angle of the triangle measures 60° by the Triangle Angle-Sum Theorem, the triangle is an equilateral triangle by Corollary 5.3.

$4y - 2 = 2y + 2$ Corollary 14.3; definition of equilateral △

$y = 2$ Solve.

Check

ARCHITECTURE The main entrance to the Louvre Museum is a unique metal and glass pyramid. Find the value of each variable.

$x =$ 30 and $y =$ 30

Go Online You may want to complete the construction activities for this lesson.

Go Online You can complete an Extra Example online.

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Interactive Presentation

Example 4

TAP

Students tap to reveal parts of the solution.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Find Missing Measures in Equilateral Triangles

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to solve the problem in this example, students will need to analyze the mathematical relationships in the problem to draw a conclusion.

Questions for Mathematical Discourse

AL What is the relationship between isosceles and equilateral triangles? Sample answer: All equilateral triangles are also considered to be isosceles triangles.

OL What are the legs and the vertex of the triangle? $\overline{JL}$ and $\overline{JK}$ are the legs; $\angle J$ is the vertex.

BL If you were given that $m\angle J = 6x + 6$ and $m\angle K = 7x - 3$, what would be the value of x be? $x = 9$

DIFFERENTIATE

Reteaching Activity AL ELL

Have groups of students work on Example 3. Encourage groups to discuss the properties of isosceles and equilateral triangles while they are solving the problem.

Example 4 Find Missing Values

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to solve the problem in this example, students will need to analyze the mathematical relationships in the problem to draw a conclusion.

Questions for Mathematical Discourse

AL Based on the diagram, what do you know about $m\angle C$? $m\angle C = m\angle A$

OL Write an equation to find $m\angle B$ using the Triangle Angle-Sum Theorem. $m\angle B + 60° + 60° = 180°$

BL In **Part B**, is it possible for $BC = \frac{1}{2}y + 3$? Explain. No; if $y = 2$, then $\frac{1}{2}y + 3 = 4$. Because the triangle is equilateral, the three sides must be congruent.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–16
2	exercises that use a variety of skills from this lesson	17–26
3	exercises that emphasize higher-order and critical-thinking skills	27–32

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–25 odd, 27–32
- Extension: Exterior and Interior Angles of Isosceles Triangles
- ALEKS Isosceles and Equilateral Triangles

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–32 odd
- Remediation, Review Resources: Triangles
- Personal Tutors
- Extra Examples 1–4
- ALEKS Triangle Constructions and Triangle Inequalities

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–16 odd
- Remediation, Review Resources: Triangles
- *Quick Review Math Handbook*: Proving Triangles Congruent (ASA, AAS)
- ALEKS Triangle Constructions and Triangle Inequalities

Important to Know

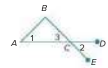
Digital Exercise Alert Exercise 25 requires constructions. Students will need to complete the constructions by using a compass and straightedge.

Practice

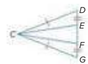
Go Online You can complete your homework online.

Example 1

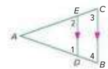
1. PROOF Write a two-column proof. See Mod. 14 Answer Appendix.
Given: $\angle 1 \cong \angle 2$
Prove: $\overline{AB} \cong \overline{CB}$



2. PROOF Write a two-column proof. See Mod. 14 Answer Appendix.
Given: $\overline{CD} \cong \overline{CG}$, $\overline{DE} \cong \overline{GF}$
Prove: $\overline{CE} \cong \overline{CF}$



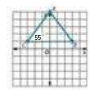
3. PROOF Write a two-column proof. See Mod. 14 Answer Appendix.
Given: $\overline{DE} \parallel \overline{BC}$, $\angle 1 \cong \angle 2$
Prove: $\overline{AB} \cong \overline{AC}$



4. ROOFS In the picture, $\overline{BD} \perp \overline{AC}$ and $\triangle ABC$ is an isosceles triangle with base $\overline{AC}$. Write a two-column proof to prove that $\overline{BD}$ bisects the angle formed by the sloped sides of the roof, $\angle ABC$. See Mod. 14 Answer Appendix.

Example 2

5. Refer to the figure. See Mod. 14 Answer Appendix.
a. Find the measures of the sides of $\triangle ABC$. Show your work.
b. Find $m\angle A$. Show your work.



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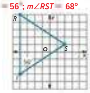
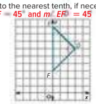
6. Find SR, ST, RT, $m\angle TRS$ and $m\angle RST$. Round to the nearest tenth, if necessary. SR = 7.2 units; ST = 7.2 units; RT = 8 units; $m\angle TRS$ = 56°; $m\angle RST$ = 68°

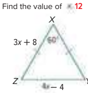


7. Find the measures of $\angle DEF$ and $\angle EFD$. Round to the nearest tenth, if necessary. $m\angle DEF$ = 45° and $m\angle EFD$ = 45

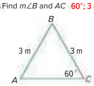


Examples 3 and 4

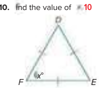
8. Find the value of x. 12



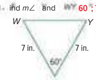
9. Find $m\angle B$ and AC. 60°; 3 m



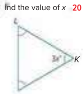
10. Find the value of x. 10



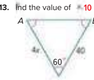
11. Find $m\angle W$ and WY. 60°; 7 in.



12. Find the value of x. 20

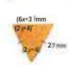


13. Find the value of x. 10

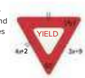


Find the value of each variable.

14. CHIPS Some tortilla chips can be modeled by a triangle.
a. Solve for x. 4
b. Solve for y. 28



15. SIGNS Yield signs notify drivers to slow down and allow oncoming vehicles to proceed first.
a. Solve for x. 7
b. Solve for y. 15



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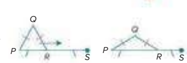
Mixed Exercises

16. ROOF Julita works for a company that makes lounge chairs. As shown in the figure, the back of each chair is an isosceles triangle that can be adjusted so the person sitting on the chair can recline.



Suppose the chair is adjusted so $m\angle Q$ = 50°. What is $m\angle QRS$? Write a paragraph proof to justify your argument. See Mod. 14 Answer Appendix.

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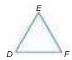
Mixed Exercises

17. STRUCTURE Each of the triangles shown is isosceles.

a. Use a ruler to find the midpoint of each side of each triangle. Copy and draw a triangle formed by connecting the midpoints of each side.

b. Look for patterns in your drawings. Make a conjecture about what you notice. Sample answer: The triangle formed by connecting the midpoints of the sides of an isosceles triangle is an isosceles triangle.

18. PROOF Write a two-column proof to prove each case of Corollary 14.3.



a. Case 1

Given: $\triangle DEF$ is an equilateral triangle.

Prove: $\triangle DEF$ is an equiangular triangle.

Proof:

Statements (Reasons)

1. $\triangle DEF$ is an equilateral triangle. (Given)
2. $\overline{DE} \cong \overline{EF} \cong \overline{DF}$ (Def. of equilateral △)
3. $\angle D \cong \angle E \cong \angle F$ (Isosceles △ Thm.)
4. $\triangle DEF$ is an equiangular triangle. (Def. of equiangular triangle)

b. Case 2

Given: $\triangle DEF$ is an equiangular triangle.

Prove: $\triangle DEF$ is an equilateral triangle.

Proof:

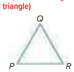
Statements (Reasons)

1. $\triangle DEF$ is an equiangular triangle. (Given)
2. $\angle D \cong \angle E \cong \angle F$ (Def. of equiangular △)
3. $\overline{DE} \cong \overline{EF} \cong \overline{DF}$ (Conv. of Isosceles △ Thm.)
4. $\triangle DEF$ is an equilateral triangle. (Def. of equilateral triangle)

19. PROOF Write a two-column proof to prove Corollary 14.4.

Given: $\triangle PQR$ is an equilateral triangle.

Prove: $m\angle P = m\angle Q = m\angle R = 60°$



Proof:

Statements (Reasons)

1. $\triangle PQR$ is an equilateral triangle. (Given)
2. $\overline{PQ} \cong \overline{QR} \cong \overline{PR}$ (Def. of equilateral triangle)
3. $\angle P \cong \angle Q \cong \angle R$ (Isosceles Triangle Theorem)
4. $m\angle P = m\angle Q = m\angle R$ (Def. of congruence)
5. $m\angle P + m\angle Q + m\angle R = 180°$ (Triangle Angle-Sum Thm.)
6. $3m\angle P = 180°$ (Substitution)
7. $m\angle P = 60°$ (Division Property)
8. $m\angle P = m\angle Q = m\angle R = 60°$ (Substitution)

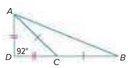
REGULARITY **Find each measure.**

20. $m\angle CAD$ 44°

21. $m\angle ACD$ 44°

22. $m\angle ACB$ 136°

23. $m\angle ABC$ 22°



24. PATHS A marble path, as shown at the right, is constructed out of several congruent isosceles triangles. All the vertex angles measure 20°. What is the measure of angle 1 in the figure? 80°

25. PRECISION Construct three different isosceles right triangles. Explain your method. Then verify your constructions using measurement and mathematics. See margin.

26. STATE YOUR ASSUMPTIONS Every day, cars drive through approximate isosceles triangles when they go over the Leonard Zakim Bridge in Boston. The ten-lane roadway forms the bases of the triangles.

a. If $m\angle A = 67°$, find $m\angle B$. 46°

b. Find $m\angle C$. 67°

c. What assumption is made when approximating that the bridge forms isosceles triangles? Sample answer: I assumed that the cables representing the legs of the triangles were the exact same length.

Higher-Order Thinking Skills

ANAL YZE **Determine whether the following statements are *sometimes*, *always*, or *never* true. Justify your argument.**

27. If the measure of the vertex angle of an isosceles triangle is an integer, then the measure of each base angle is an integer. Sometimes; sample answer: Only if the measure of the vertex angle is even.

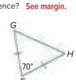
28. If the measures of the base angles of an isosceles triangle are integers, then the measure of its vertex angle is odd. Never; sample answer: The measure of the vertex angle will be 180 − 2 (measure of the base angle). So, if the base angles are integers, then 2(measure of base angle) will be even and 180 − 2(measure of the base angle) will be even.

29. CREATE If possible, draw an isosceles triangle with base angles that are obtuse. If it is not possible, explain why not. Sample answer: It is not possible because a triangle cannot have more than one obtuse angle.

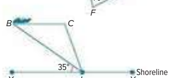
30. WRITE How can triangle classifications help you prove triangle congruence? See margin.

31. FIND THE ERROR Darshan and Miguela are finding $m\angle G$ in the figure shown. Darshan says that $m\angle G = 35°$, and Miguela says that $m\angle G = 60°$. Is either of them correct? Explain your reasoning.

No; $m\angle G = \frac{180 - 70}{2}$ or 55°



32. PERSEVERE A boat is traveling at 25 mi/h parallel to a straight section of the shoreline, $\overline{XY}$, as shown. An observer in a lighthouse L spots the boat when the angle formed by the boat, the lighthouse, and the shoreline is 35°. The observer spots the boat again when $m\angle CLX = 70°$.

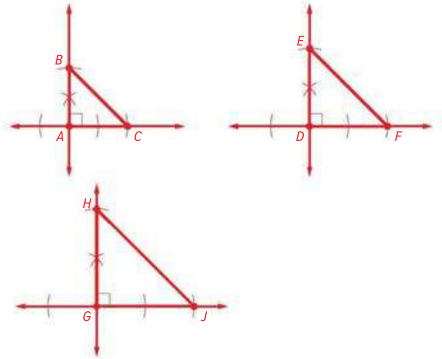


a. Explain how you can prove that $\triangle BCL$ is isosceles. See margin.

b. It takes the boat about 15 minutes to travel from point B to point C. When the boat is at point C, what is the distance to the lighthouse? 6.25 mi

Answers

25.



Sample answer: I constructed a pair of perpendicular segments and then used the same compass setting to mark points that are equidistant from their intersection. I measured both legs for each triangle. Because $AB = AC = 1.3$ cm, $DE = DF = 1.9$ cm, and $GH = GJ = 2.3$ cm, the triangles are isosceles. I used a protractor to confirm that $\angle A$, $\angle D$, and $\angle G$ are all right angles.

30. Sample answer: If a triangle is already classified, you can use the previously proven properties of that type of triangle in the proof. Doing this can save you steps when writing the proof.

32a. Because $\overline{BC} \parallel \overline{XY}$, $m\angle CBL = 35°$ by the Alternate Interior Angles Theorem. Because $\angle CBL \cong \angle CLB$, $\triangle BCL$ is isosceles by the Converse of the Isosceles Triangle Theorem.

Lesson 14-7

Triangles and Coordinate Proof

G.CO.10, G.GPE.4

LESSON GOAL

Students write coordinate proofs using theorems of triangle congruence.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Position and Label Triangles
- Position and Label a Triangle
- Identify Missing Coordinates

Explore: Triangles and Coordinate Proofs

Develop:

Triangles and Coordinate Proof
- Write a Coordinate Proof
- Prove a Theorem by Using Coordinate Geometry
- Classify a Triangle

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Proving Triangles Congruent: ASA, AAS	●	●		●
Extension: Rectangle Paradox		●	●	●

Language Development Handbook

Assign page 93 of the *Language Development Handbook* to help your students build mathematical language related to writing coordinate proofs.

ELL You can use the tips and suggestions on page T93 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Geometry

Standards for Mathematical Content:

G.CO.10 Prove theorems about triangles.

G.GPE.4 Use coordinates to prove simple geometric theorems algebraically.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

4 Model with mathematics.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students solved problems involving isosceles and equilateral triangles using triangle congruence.
G.CO.10, G.SRT.5

Now
Students use coordinate geometry to prove triangles congruent.
G.CO.10, G.GPE.4

Rigor

The Three Pillars of Rigor

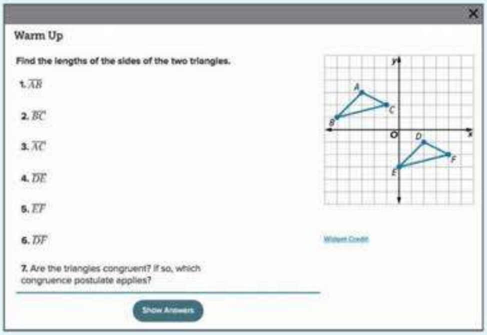
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students draw on their understanding of relationships in triangles and build fluency by using coordinates to prove theorems about triangles.

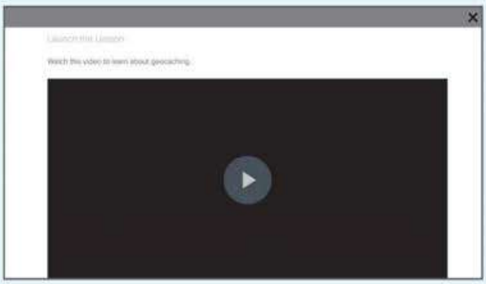
Mathematical Background

A coordinate proof uses the coordinate plane in combination with algebra to prove theorems. The Distance Formula, Slope Formula, and Midpoint Formula are often used in coordinate proofs.

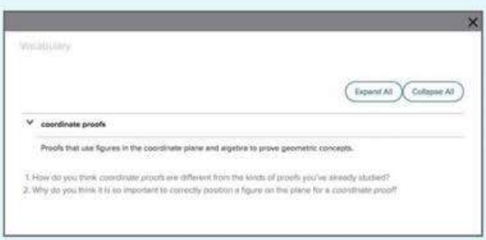
Interactive Presentation



Warm Up



Launch the Lesson



Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

• identifying SSS

Answers:
1. $2\sqrt{2}$
2. $\sqrt{17}$
3. $\sqrt{5}$
4. $2\sqrt{2}$
5. $\sqrt{17}$
6. $\sqrt{5}$
7. yes; SSS

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In this Launch the Lesson, students can see a real-world application of coordinates.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then, discuss the questions below with the class.

Explore Triangles and Coordinate Proofs

Objective

Students apply properties of triangles on the coordinate plane to label the vertices using algebra.

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. This Explore asks students to justify their conclusions.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

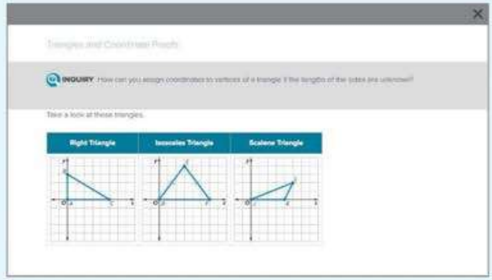
Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will examine three triangles on the coordinate plane and find the coordinates of their vertices. Then students complete the Exercises guiding them to use variables for coordinates given the base length and height of an isosceles triangle. Then, students will answer the Inquiry Question.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

(*continued on the next page*)

Interactive Presentation



Explore

TYPE

Students type to complete the guiding exercises.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and view a sample answer.

Explore Triangles and Coordinate Proofs (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- If a right triangle has base length c and height d, what would the coordinates of point A, B, and C be? $A(0, 0)$, $B(0, d)$, $C(c, 0)$

Inquiry

How can you assign coordinates to vertices of a triangle if the lengths of the sides are unknown? Sample answer: You can use variables to represent the *x*- and *y*-coordinates of each vertex. You can use the properties of the triangle to determine the relationship between the coordinates to reduce the number of variables needed.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Position and Label Triangles

Objective

Students position a triangle on the coordinate plane and label the vertices.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions when they discuss their solutions with others.

Example 1 Position and Label a Triangle

MP Teaching the Mathematical Practices

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. In this example, notice the relationship between the problem variables and the triangle in question.

Questions for Mathematical Discourse

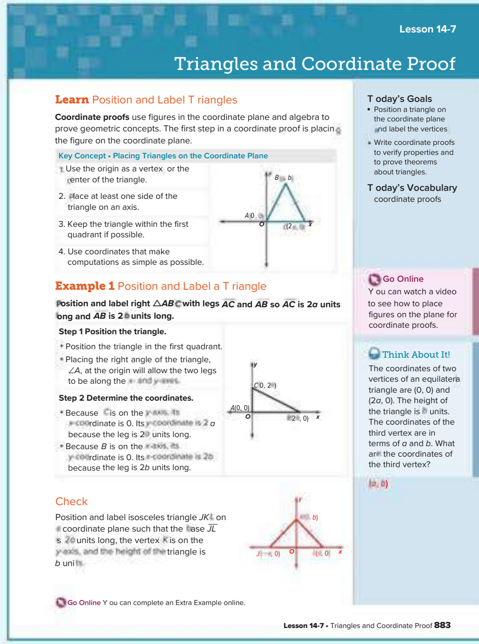
AL What lines on a coordinate plane make it easier to find distance? Sample answer: the x-axis and the y-axis.

OL If two sides of the triangle are placed along each axis, what do you know about the measure of the included angle? It is a right angle.

BL What is BC? $\sqrt{4a^2 + 4b^2}$

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.


Lesson 14-7

Triangles and Coordinate Proof

Learn Position and Label Triangles

Coordinate proofs use figures in the coordinate plane and algebra to prove geometric concepts. The first step in a coordinate proof is placing the figure on the coordinate plane.

Key Concept • Placing Triangles on the Coordinate Plane

1. Use the origin as a vertex or the center of the triangle.
2. Place at least one side of the triangle on an axis.
3. Keep the triangle within the first quadrant if possible.
4. Use coordinates that make computations as simple as possible.

Example 1 Position and Label a Triangle

Position and label right $\triangle ABC$ with legs $\overline{AC}$ and $\overline{AB}$ so $\overline{AC}$ is $2a$ units long and $\overline{AB}$ is $2b$ units long.

Step 1 Position the triangle.

- Position the triangle in the first quadrant.
- Placing the right angle of the triangle, $\angle A$, at the origin will allow the two legs to be along the x- and y-axes.

Step 2 Determine the coordinates.

- Because C is on the y-axis, its x-coordinate is 0. Its y-coordinate is $2a$ because the leg is $2a$ units long.
- Because B is on the x-axis, its y-coordinate is 0. Its x-coordinate is $2b$ because the leg is $2b$ units long.

Check

Position and label isosceles triangle JKL on a coordinate plane such that the base $\overline{JL}$ is $2a$ units long, the vertex K is on the y-axis, and the height of the triangle is b units.

Go Online You can complete an Extra Example online.

Today's Goals
- Position a triangle on the coordinate plane and label the vertices.
- Write coordinate proofs to verify properties and to prove theorems about triangles.

Today's Vocabulary
coordinate proofs

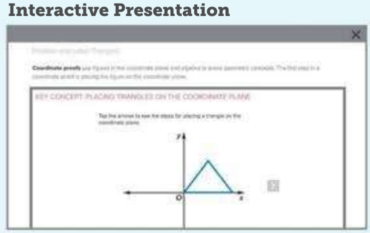
Go Online You can watch a video to see how to place figures on the plane for coordinate proofs.

Think About It! The coordinates of two vertices of an equilateral triangle are (0, 0) and $(2a, 0)$. The height of the triangle is b units. The coordinates of the third vertex are in terms of a and b. What are the coordinates of the third vertex?

(a, b)

Lesson 14-7 • Triangles and Coordinate Proof 883

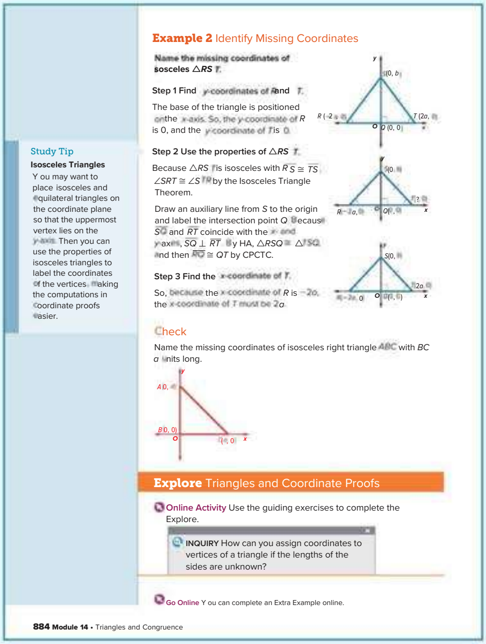
Interactive Presentation



Learn

TAP

Students tap to see steps in positioning and labeling triangles on the coordinate plane.



Example 2 Identify Missing Coordinates

Name the missing coordinates of isosceles △RST.

Step 1 Find y-coordinates of R and T.

The base of the triangle is positioned on the x-axis. So, the y-coordinate of R is 0, and the y-coordinate of T is 0.

Step 2 Use the properties of △RST.

Because △RST is isosceles with $\overline{RS} \cong \overline{TS}$, ∠SRT ≅ ∠STR by the Isosceles Triangle Theorem.

Draw an auxiliary line from S to the origin and label the intersection point Q. Because $\overline{SQ}$ and $\overline{RT}$ coincide with the x- and y-axes, $\overline{SQ} \perp \overline{RT}$. By HL, △RSQ ≅ △TSQ, and then $\overline{RQ} \cong \overline{QT}$ by CPCTC.

Step 3 Find the x-coordinate of T.

So, because the x-coordinate of R is −2a, the x-coordinate of T must be 2a.

Study Tip

Isosceles Triangles You may want to place isosceles and equilateral triangles on the coordinate plane so that the uppermost vertex lies on the y-axis. Then you can use the properties of isosceles triangles to label the coordinates of the vertices, making the computations in coordinate proofs easier.

Check

Name the missing coordinates of isosceles right triangle ABC with BC a units long.

Explore Triangles and Coordinate Proofs

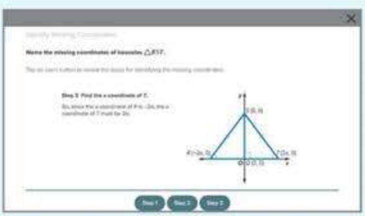
Online Activity Use the guiding exercises to complete the Explore.

INQUIRY How can you assign coordinates to vertices of a triangle if the lengths of the sides are unknown?

Go Online You can complete an Extra Example online.

884 Module 14 • Triangles and Congruence

Interactive Presentation



Example 2

TAP

Students tap to choose answers and to see the next steps in a proof.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Identify Missing Coordinates

Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to construct an argument.

Common Error

Students may incorrectly identify the quadrant in which the triangle is located. Remind them that whenever possible they should use coordinates in the first quadrant because all coordinates there are positive.

Questions for Mathematical Discourse

AL Why is the x-coordinate of S 0? Sample answer: S is on the y-axis.

OL What is the y-coordinate of any point on the x-axis? 0

BL How does the Hypotenuse-Leg Theorem help you find the coordinates of T? If you draw a perpendicular segment from S to the origin O and label the intersection point Q, you can use the theorem to show that $\triangle RSQ \cong \triangle TSQ$; so, that Q is the midpoint of $\overleftrightarrow{RT}$.

DIFFERENTIATE

Enrichment Activity BL

Supply students with an overhead or translucent copy of a map. Have students choose three destinations and use these vertices to draw a triangle. Next, students place the translucent map on a coordinate plane. Encourage students to experiment with this placement. Finally, have students use coordinate proof to classify the triangle.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Triangles and Coordinate Proof

Objective

Students write coordinate proofs to verify properties and to prove theorems about triangles.

Teaching the Mathematical Practices

3 Construct Arguments In this Learn, students will see how to use stated assumptions, definitions, and previously established results to write a coordinate proof.

Common Misconception

Students may think that for proof purposes a triangle on the coordinate plane must be completely random in terms of where it is located and how it is oriented. Explain to them that any triangle is congruent to one that is in the position and orientation that makes the coordinates simple. This is why we can position a triangle and label its vertices this way.

Essential Question Follow-Up

Students prove theorems about triangles on the coordinate plane.

Ask:

Why is it important to know how to prove theorems using the coordinate plane? Sample answer: Sometimes the theorem is easier to prove using coordinates than without coordinates.

Example 3 Write a Coordinate Proof

Teaching the Mathematical Practices

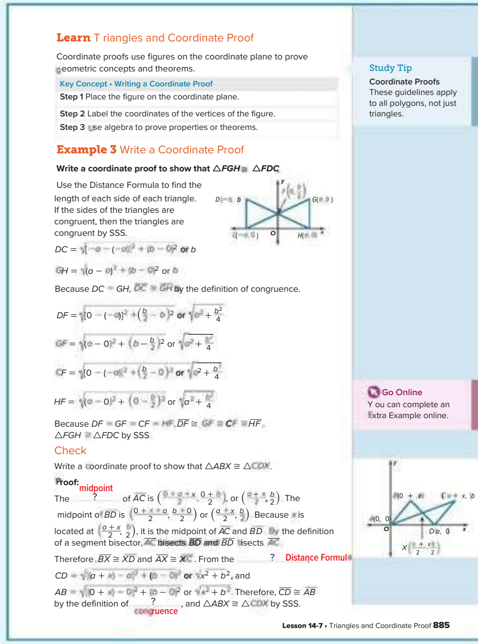
3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to construct an argument.

Questions for Mathematical Discourse

AL How else can you write the congruence statement? Sample answer: $\triangle GHF \cong \triangle DCF$

OL How did using a coordinate proof make it easier to complete this proof? Sample answer: On the coordinate plane, you can use the Distance Formula to determine whether two line segments are congruent.

BL Could you prove the statement using SAS? Explain. Yes; $\angle DFC \cong \angle GFH$ because they are vertical angles.



Learn Triangles and Coordinate Proof

Coordinate proofs use figures on the coordinate plane to prove geometric concepts and theorems.

Key Concept • Writing a Coordinate Proof

Step 1 Place the figure on the coordinate plane.

Step 2 Label the coordinates of the vertices of the figure.

Step 3 Use algebra to prove properties or theorems.

Study Tip: Coordinate Proofs These guidelines apply to all polygons, not just triangles.

Example 3 Write a Coordinate Proof

Write a coordinate proof to show that $\triangle FGH \cong \triangle FDC$.

Use the Distance Formula to find the length of each side of each triangle. If the sides of the triangles are congruent, then the triangles are congruent by SSS.

$DC = \sqrt{[-a-(-a)]^2 + (b-0)^2}$ or b

$GH = \sqrt{(a-a)^2 + (b-0)^2}$ or b

Because $DC = GH$, $\overline{DC} \cong \overline{GH}$ by the definition of congruence.

$DF = \sqrt{[0-(-a)]^2 + \left(\frac{b}{2}-b\right)^2}$ or $\sqrt{a^2 + \frac{b^2}{4}}$

$GF = \sqrt{(a-0)^2 + \left(b-\frac{b}{2}\right)^2}$ or $\sqrt{a^2 + \frac{b^2}{4}}$

$CF = \sqrt{[0-(-a)]^2 + \left(\frac{b}{2}-0\right)^2}$ or $\sqrt{a^2 + \frac{b^2}{4}}$

$HF = \sqrt{(a-0)^2 + \left(0-\frac{b}{2}\right)^2}$ or $\sqrt{a^2 + \frac{b^2}{4}}$

Because $DF = GF = CF = HF$, $\overline{DF} \cong \overline{GF} \cong \overline{CF} \cong \overline{HF}$. $\triangle FGH \cong \triangle FDC$ by SSS.

Go Online You can complete an Extra Example online.

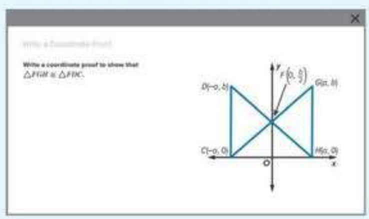
Check

Write a coordinate proof to show that $\triangle ABX \cong \triangle CDX$.

Proof: The __?__ (midpoint) of $\overline{AC}$ is $\left(\frac{0+a+x}{2}, \frac{0+b}{2}\right)$, or $\left(\frac{a+x}{2}, \frac{b}{2}\right)$. The midpoint of $\overline{BD}$ is $\left(\frac{0+x+a}{2}, \frac{b+0}{2}\right)$ or $\left(\frac{a+x}{2}, \frac{b}{2}\right)$. Because X is located at $\left(\frac{a+x}{2}, \frac{b}{2}\right)$, it is the midpoint of $\overline{AC}$ and $\overline{BD}$. By the definition of a segment bisector, $\overline{AC}$ bisects $\overline{BD}$ and $\overline{BD}$ bisects $\overline{AC}$. Therefore, $\overline{BX} \cong \overline{XD}$ and $\overline{AX} \cong \overline{XC}$. From the __?__ (Distance Formula), $CD = \sqrt{(a+x)-a]^2 + (b-0)^2}$ or $\sqrt{x^2+b^2}$, and $AB = \sqrt{(0+x)-0]^2 + (b-0)^2}$ or $\sqrt{x^2+b^2}$. Therefore, $\overline{CD} \cong \overline{AB}$ by the definition of __?__ (congruence), and $\triangle ABX \cong \triangle CDX$ by SSS.

Lesson 14-7 • Triangles and Coordinate Proof 885

Interactive Presentation



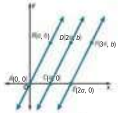
Example 3

TAP

Students tap to reveal steps in the proof and to select answer choices.

Example 4 Prove a Theorem by Using Coordinate Geometry

Write a coordinate proof to show that if two lines are each equidistant from a third line, then the two lines are parallel to each other.



Given: $\overleftrightarrow{AB}$ **and** $\overleftrightarrow{EF}$ **are equidistant from** $\overleftrightarrow{CD}$**.**

Prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{EF}$

Proof:

The slope of $\overleftrightarrow{AB} = \frac{b-0}{a-0} = \frac{b}{a}$. The slope of $\overleftrightarrow{EF} = \frac{b-0}{3a-2a} = \frac{b}{a}$.

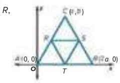
Because the slopes of $\overleftrightarrow{AB}$ and $\overleftrightarrow{EF}$ are the same, $\overleftrightarrow{AB} \parallel \overleftrightarrow{EF}$.

Check

Write a coordinate proof to show that the three segments joining the midpoints of the sides of an isosceles triangle form another isosceles triangle.

Given: Isosceles triangle ABC, $\overline{BC} \cong \overline{AC}$; R, S, and T are midpoints of their respective sides.

Prove: $\triangle RST$ is isosceles.



Proof:

Midpoint R is $\left(\frac{a+0}{2}, \frac{b+0}{2}\right)$ or $\left(\frac{a}{2}, \frac{b}{2}\right)$.

Midpoint S is $\left(\frac{a+2a}{2}, \frac{b+0}{2}\right)$ or $\left(\frac{3a}{2}, \frac{b}{2}\right)$.

Midpoint T is $\left(\frac{2a+0}{2}, \frac{0+0}{2}\right)$ or $(a, 0)$.

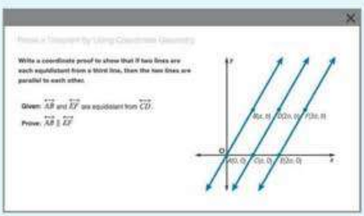
$RT = \sqrt{\left(\frac{a}{2} - a\right)^2 + \left(\frac{b}{2} - 0\right)^2}$ or $\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$

$ST = \sqrt{\left(\frac{3a}{2} - a\right)^2 + \left(\frac{b}{2} - 0\right)^2}$ or $\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$

$RT = ST$ and $\overline{RT} \cong \overline{ST}$ and $\triangle RST$ is isosceles.

Go Online You can complete an Extra Example online.

Interactive Presentation



Example 4

TAP

Students tap to reveal steps in the solution.

Example 4 Prove a Theorem by Using Coordinate Geometry

Teaching the Mathematical Practices

3 Construct Arguments In this example, students will use stated assumptions, definitions, and previously established results to complete the given proof.

Questions for Mathematical Discourse

AL What is the formula for slope on the coordinate plane? $m = \frac{y_2 - y_1}{x_2 - x_1}$

OL How do you know that $\overleftrightarrow{AB}$ is equidistant from $\overleftrightarrow{CD}$ in the diagram? Sample answer: Along two different horizontal lines, the lines are the same number of units apart horizontally.

BL How did using a coordinate proof make it easier to complete this proof? Sample answer: On the coordinate plane, you can use the concept of slope to show that the two lines are parallel.

DIFFERENTIATE

Reteaching Activity AL ELL

Use coordinates geometry to prove the Midpoint Formula.

Example 5 Classify a T riangle

Teaching the Mathematical Practices

6 Use Quantities Use the Study Tip to guide students to clarify their use of quantities in this example. Ensure that they specify the units of measure used in the problem and label axes appropriately.

Questions for Mathematical Discourse

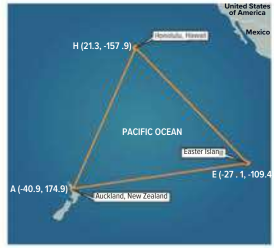
AL What do you know about scalene triangles? All the sides of a scalene triangle are different lengths.

OL Easter Island has a negative *y*-coordinate while Hawaii has a positive *y*-coordinate. What does that tell you? Easter Island is south of the equator and Hawaii is north of the equator.

BL New Zealand is west of the International Date Line, and Easter Island and Hawaii are east of the line. What part of their coordinates tells you this? Explain. Sample answer: The *x*-coordinate of New Zealand is less than 180, and the *x*-coordinates of Easter Island and Hawaii are more than 180. This is because the International Date Line is 180° around the Earth from the origin.

Example 5 Classify a T riangle

NAVIGATION **The Polynesian Triangle is a triangle formed between the three Pacific island groups that form the South Pacific region known as Polynesia. The approximate coordinates in latitude and longitude of each vertex are Auckland, New Zealand (−40.9, 174.9), Honolulu, Hawaii (21.3, −157.9), and Easter Island (−27.1, −109.4).**



Part A Estimate the type of triangle formed by the Polynesian islands.

The triangle appears to be a(n) acute scalene triangle.

Part B Use coordinate geometry to determine the type of triangle formed.

Use the Distance Formula to determine the length of each side of the triangle.

Round to the nearest tenth.

$$AE = \sqrt{(-40.9 - (-27.1))^2 + (174.9 - (-109.4))^2} \approx 284.6$$

$$EH = \sqrt{(-27.1 - 21.3)^2 + (-109.4 - (-157.9))^2} \approx 68.5$$

$$AH = \sqrt{(-40.9 - 21.3)^2 + (174.9 - (-157.9))^2} \approx 338.6$$

Because the length of each side is different, the triangle is scalene.

Go Online Y ou can complete an Extra Example online.

Study Tip

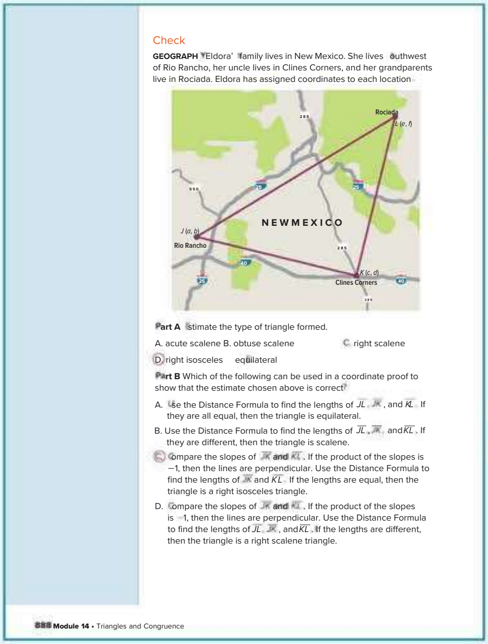
Units of Measure While the distance between cities is usually measured in miles or kilometers, latitude and longitude are measured in degrees relative to the Prime Meridian and the Equator.

Interactive Presentation

Example 5

TAP

Students tap to reveal steps in the proof and to select answer choices.



Check

GEOGRAPHY Eldora's family lives in New Mexico. She lives southwest of Rio Rancho, her uncle lives in Clines Corners, and her grandparents live in Rociada. Eldora has assigned coordinates to each location.

Part A Estimate the type of triangle formed.

A. acute scalene B. obtuse scalene C. right scalene

D. right isosceles equilateral

Part B Which of the following can be used in a coordinate proof to show that the estimate chosen above is correct?

A. Use the Distance Formula to find the lengths of $\overline{JL}$, $\overline{JK}$, and $\overline{KL}$. If they are all equal, then the triangle is equilateral.

B. Use the Distance Formula to find the lengths of $\overline{JL}$, $\overline{JK}$, and $\overline{KL}$. If they are different, then the triangle is scalene.

C. Compare the slopes of $\overline{JK}$ and $\overline{KL}$. If the product of the slopes is −1, then the lines are perpendicular. Use the Distance Formula to find the lengths of $\overline{JK}$ and $\overline{KL}$. If the lengths are equal, then the triangle is a right isosceles triangle.

D. Compare the slopes of $\overline{JK}$ and $\overline{KL}$. If the product of the slopes is −1, then the lines are perpendicular. Use the Distance Formula to find the lengths of $\overline{JL}$, $\overline{JK}$, and $\overline{KL}$. If the lengths are different, then the triangle is a right scalene triangle.

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Interactive Presentation

Example 6

CHECK

Students complete the Check online to determine whether they are ready to move on.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–16
2	exercises that use a variety of skills from this lesson	17–24
3	exercises that emphasize higher-order and critical-thinking skills	25–30

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

- Practice, Exercises 1–23 odd, 25–30
- Extension: Rectangle Paradox

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–29 odd
- Remediation, Review Resources: Proving Triangles Congruent: ASA, AAS
- Personal Tutors
- Extra Examples 1–5
- ALEKS Proving Triangle Congruence

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–15 odd
- Remediation, Review Resources: Proving Triangles Congruent: ASA, AAS
- *Quick Review Math Handbook*: Isosceles and Equilateral Triangles
- ALEKS Proving Triangle Congruence

Practice

Go Online You can complete your homework online.

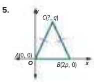
Example 1

REGULARITY **Position and label each triangle on the coordinate plane.** 1–4. See margin.

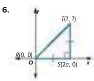
1. isosceles $\triangle ABC$ with base $\overline{AB}$ that is a units long and height that is b units
2. right $\triangle XYZ$ with hypotenuse $\overline{YZ}$, leg $\overline{XY}$ that is b units long, and leg $\overline{XZ}$ that is three times the length of $\overline{XY}$
3. isosceles right $\triangle RST$ with hypotenuse $\overline{RS}$ and legs $3a$ units long
4. right $\triangle JKL$ with legs $\overline{JK}$ and $\overline{KL}$ such that $\overline{JK}$ is a units long and leg $\overline{KL}$ is $4b$ units long

Example 2

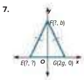
Name the missing coordinate(s) of each triangle.

5. 

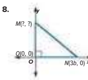
$C(p, q)$

6. 

$T(2a, 2a)$

7. 

$E(-2g, 0)$; $F(0, b)$

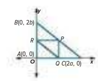
8. 

Sample answer: $M(0, c)$

Examples 3 and 4

PROOF **For Exercises 9–13, write a coordinate proof for each statement.**

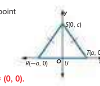
9. The segments joining the midpoints of the sides of a right triangle form a right triangle. See Mod. 14 Answer Appendix.



Given: Point R is the midpoint of $\overline{AB}$.
Point P is the midpoint of $\overline{BC}$.
Point Q is the midpoint of $\overline{AC}$.

Prove: $\triangle RPQ$ is a right triangle.

10. A segment from the vertex angle of an isosceles triangle to the midpoint of the base is perpendicular to the base.

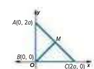


Given: Isosceles $\triangle RST$; U is the midpoint of base $\overline{RT}$.

Prove: $\overline{SU} \perp \overline{RT}$

Proof:
U is the midpoint of $\overline{RT}$, so the coordinates of U are $\left(\frac{-a+a}{2}, \frac{0+0}{2}\right) = (0, 0)$. Thus, $\overline{SU}$ lies on the y-axis, and $\triangle RST$ was placed so $\overline{RT}$ lies on the x-axis. The axes are perpendicular, so $\overline{SU} \perp \overline{RT}$.

11. In an isosceles right triangle, the segment from the vertex of the right angle to the midpoint of the hypotenuse is perpendicular to the hypotenuse.

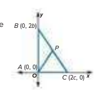


Given: isosceles right $\triangle ABC$ with right angle $\angle ABC$; M the midpoint of $\overline{AC}$.

Prove: $\overline{BM} \perp \overline{AC}$

Proof:
The Midpoint Formula shows that the coordinates of M are $\left(\frac{0+2a}{2}, \frac{2a+0}{2}\right)$ or (a, a). The slope of $\overline{AC}$ is $\frac{2a-0}{0-2a} = -1$. The slope of $\overline{BM}$ is $\frac{a-0}{a-0} = 1$. The product of the slopes is -1, so $\overline{BM} \perp \overline{AC}$.

12. The measure of the segment that joins the vertex of the right angle in a right triangle to the midpoint of the hypotenuse is one-half the measure of the hypotenuse.



Given: right $\triangle ABC$; P is the midpoint of $\overline{BC}$.

Prove: $AP = \frac{1}{2}BC$

Proof:
Midpoint P is $\left(\frac{0+2c}{2}, \frac{2b+0}{2}\right)$ or (c, b).

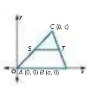
$AP = \sqrt{(c-0)^2 + (b-0)^2}$ or $\sqrt{c^2 + b^2}$

$BC = \sqrt{(2c-0)^2 + (0-2b)^2} = \sqrt{4c^2 + 4b^2}$ or $2\sqrt{c^2 + b^2}$

$\frac{1}{2}BC = \sqrt{c^2 + b^2}$

So, $AP = \frac{1}{2}BC$.

13. If a line segment joins the midpoints of two sides of a triangle, then its length is equal to one-half the length of the third side.



Given: S is the midpoint of $\overline{AC}$.
T is the midpoint of $\overline{BC}$.

Prove: $ST = \frac{1}{2}AB$

Proof:
The coordinates of S are $\left(\frac{b}{2}, \frac{c}{2}\right)$, and the coordinates of T are $\left(\frac{a+b}{2}, \frac{c}{2}\right)$.

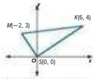
$ST = \sqrt{\left(\frac{a+b}{2} - \frac{b}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2}$ or $\frac{a}{2}$

$AB = \sqrt{(a-0)^2 + (0-0)^2}$ or a

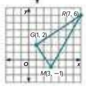
$ST = \frac{1}{2}AB$

Example 5

14. NEIGHBORHOODS Kalini lives 6 miles east and 4 miles north of her high school. After school, she works part time at the mall in a music store. The mall is 2 miles west and 3 miles north of the school. Use coordinate geometry to determine the type of triangle formed by Kalini's high school, her home, and the mall.
See Mod. 14 Answer Appendix.



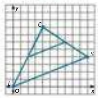
15. COUNTY FAIR The fair committee wants to print a map to distribute to vendors as they arrive to set up their booths at the fairgrounds. On a coordinate grid, the main gate is located at (3, −1), the grandstand is located at (1, 2), and the rides and games are located at (7, 6). Use coordinate geometry to determine the type of triangle formed by these locations. See Mod. 14 Answer Appendix.



16. USE ESTIMATION A town is preparing for a 5K run. The race will start at city hall C. The course will take runners along straight streets to the library L, to the science museum S, and back to city hall for the finish.

a. Estimate the type of triangle formed by the course. acute scalene

b. Use coordinate geometry to determine the type of triangle formed. See margin.



Mixed Exercises

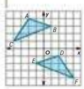
REASONING **For Exercises 17 and 18, determine whether the triangle can be a right triangle. Explain.** 17–18. See margin.

17. $X(0, 0)$, $Y(2h, 2h)$, $Z(4h, 0)$ **18.** $X(0, 0)$, $Y(1, h)$, $Z(2h, 0)$

19. SHELVES Martha has a shelf bracket shaped like a right isosceles triangle. She wants to know the length of the hypotenuse relative to the sides. She does not have a ruler but remembers the Distance Formula. She places the bracket in Quadrant I of a coordinate grid with the right angle at the origin. The length of each leg is a. What are the coordinates of the vertices that form the two acute angles? $(a, 0)$ and $(0, a)$

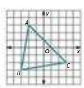
20. FLAGS A flag is shaped like an isosceles triangle. A designer would like to make a drawing of the flag on a coordinate plane. She positions it so the base of the triangle is on the y-axis with one endpoint located at (0, 0). She locates the tip of the flag at $\left(a, \frac{b}{2}\right)$. What are the coordinates of the third vertex? $(0, b)$

21. DESIGN Andrew is using a coordinate plane to design a quilt. Two of the triangular patches for the quilt are shown in the figure. Andrew wants to be sure that $\angle A$ and $\angle D$ have the same measure. Describe the main steps you can use to prove that $\angle A \cong \angle D$.
Sample answer: Use the Distance Formula to find the length of each side of each triangle. Show that the triangles are congruent by SSS. Conclude that $\angle A \cong \angle D$ using CPCTC.

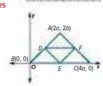


Lesson 14-7 • Triangles and Coordinate Proof **891**

22. COMMUNITY A landscape architect is using a coordinate plane to design a triangular community garden. The fence that will surround the garden is modeled by $\triangle ABC$. The architect wants to know whether any of the three angles in the fence will be congruent. Determine the answer for the architect and give a coordinate proof to justify your response.
Y es, $\angle A \cong \angle C$; sample answer: Use the Distance Formula to show that $AB = \sqrt{37}$, $BC = \sqrt{37}$, and $AC = \sqrt{50}$. Because $AB = BC$, two sides of the triangle are congruent. By the Isosceles Triangle Theorem, the angles opposite these sides are congruent, so $\angle A \cong \angle C$.



23. $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{AC}$. D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{BC}$, and F is the midpoint of $\overline{AC}$. What are the coordinates of D, E, and F? $D(a, b)$, $E(2a, 0)$, $F(3a, b)$



24. DRAFTING An engineer is designing a roadway. Three roads intersect to form a triangle. The engineer marks two vertices of the triangle at (−5, 0) and (5, 0) on a coordinate plane.

a. Describe the set of points in the coordinate plane that could not be used as the third vertex of the triangle. the x-axis

b. Describe the set of points in the coordinate plane that could be the vertex of an isosceles triangle. the y-axis except for the origin

c. Describe the set of points in the coordinate plane that would make a right triangle with the other two points if the right angle is located at (−5, 0).
the points with x-coordinate −5, except for (−5, 0)

Higher-Order Thinking Skills

25. CREATE Draw an isosceles right triangle on the coordinate plane so the midpoint of its hypotenuse is the origin. Label the coordinates of the vertex. See margin.

26. WRITE Explain why following each guideline for placing a triangle on the coordinate plane is helpful in proving coordinate proofs. a–c. See margin.

a. Use the origin as a vertex of the triangle.

b. Place at least one side of the triangle on the x- or y-axis.

c. Keep the triangle within the first quadrant if possible.

PERSEVERE **Find the coordinates of point L so $\triangle JKL$ is the indicated type of triangle. Point J has coordinates (0, 0), and point K has coordinates $(2a, 2b)$.**
27–29. Sample answers given.

27. scalene triangle $(a, 0)$ **28.** right triangle $(2a, 0)$ or $(0, 2b)$ **29.** isosceles triangle $(4a, 0)$ or $(0, 4b)$

30. ANAL YZE The midpoints of the sides of a triangle are located at $(a, 0)$, $(2a, b)$ and (a, b). If one vertex is located at the origin, what are the coordinates of the other vertices? Explain your reasoning. See margin.

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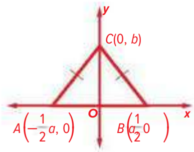
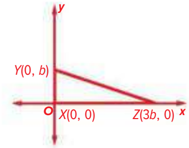
Answers

1.



2.



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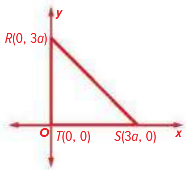

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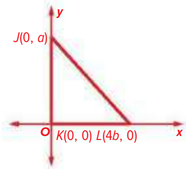
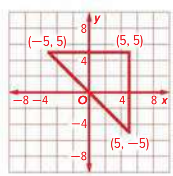


16b. $LS = 2\sqrt{29}$, $CS = \sqrt{52}$, $LC = 4\sqrt{5}$; Because none of these lengths are the same, the triangle is scalene. Slope $\overline{LC} = 2$, slope $\overline{CS} = -\frac{2}{3}$ because these slopes are not opposite reciprocals, the angle is not right. Therefore, the triangle is not a right triangle. Using a protractor, I can confirm that it is acute.

17. Slope of $\overline{XY} = 1$, slope of $\overline{YZ} = -1$, slope of $\overline{ZX} = 0$; because $1(-1) = -1$, $\overline{XY} \perp \overline{YZ}$. Therefore, $\triangle XYZ$ is a right triangle.

18. Slope of $\overline{XY} = h$, slope of $\overline{YZ} = \frac{h}{1-2h}$, slope of $\overline{ZX} = 0$; when $h = \frac{1}{2}$, $\overline{YZ}$ is a vertical segment and $\overline{ZX} \perp \overline{YZ}$. When $h = 1$, $h \cdot \frac{h}{1-2h} = -1$ and $\overline{XY} \perp \overline{YZ}$. So, $\triangle XYZ$ is only a right triangle if $h = \frac{1}{2}$ or 1.

25. Sample answer:



26a. Using the origin as a vertex of the triangle makes calculations easier because the coordinates are (0, 0).

26b. Placing at least one side of the triangle on the x- or y-axis makes it easier to calculate the length of the side because one of the coordinates will be 0.

26c. Keeping a triangle within the first quadrant makes all of the coordinates positive, and it makes the calculations easier.

30. $(2a, 0)$, $(2a, 2b)$; Using the Midpoint Formula,

$$(a, 0) = \left(\frac{0 + x_1}{2}, \frac{0 + y_1}{2}\right), \text{ so } x_1 = 2a \text{ and } y_1 = 0.$$

$$(a, b) = \left(\frac{0 + x_2}{2}, \frac{0 + y_2}{2}\right), \text{ so } x_2 = 2a \text{ and } y_2 = 2b.$$

Review

Rate Yourself

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why is it useful to know when two triangles are congruent?
- Why is it useful to be able to prove right triangles congruent?
- Why is it important to know how to prove theorems using the coordinate plane?

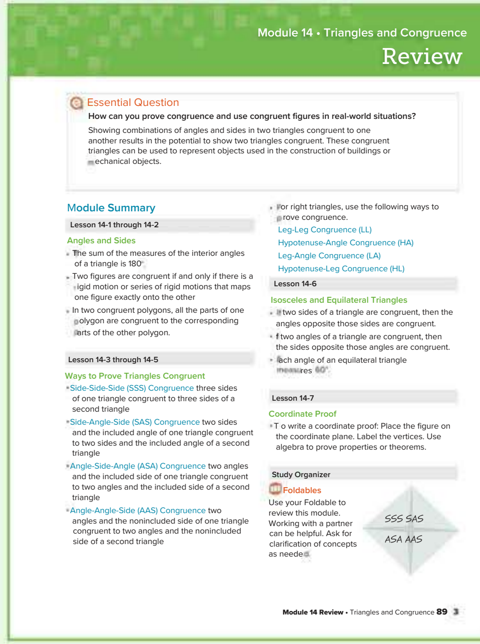
Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to triangle congruence.

LS **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Congruence, Proof, and Constructions** and **Connecting Algebra and Geometry Through Coordinates.**

- Understand congruence is terms of rigid motions
- Prove Geometric Theorems
- Use coordinates to prove simple geometric theorems algebraically


Module 14 • Triangles and Congruence

Review

Essential Question

How can you prove congruence and use congruent figures in real-world situations?

Showing combinations of angles and sides in two triangles congruent to one another results in the potential to show two triangles congruent. These congruent triangles can be used to represent objects used in the construction of buildings or mechanical objects.

Module Summary

Lesson 14-1 through 14-2

Angles and Sides

- The sum of the measures of the interior angles of a triangle is 180°.
- Two figures are congruent if and only if there is a rigid motion or series of rigid motions that maps one figure exactly onto the other
- In two congruent polygons, all the parts of one polygon are congruent to the corresponding parts of the other polygon.

Lesson 14-3 through 14-5

Ways to Prove Triangles Congruent

- Side-Side-Side (SSS) Congruence three sides of one triangle congruent to three sides of a second triangle
- Side-Angle-Side (SAS) Congruence two sides and the included angle of one triangle congruent to two sides and the included angle of a second triangle
- Angle-Side-Angle (ASA) Congruence two angles and the included side of one triangle congruent to two angles and the included side of a second triangle
- Angle-Angle-Side (AAS) Congruence two angles and the nonincluded side of one triangle congruent to two angles and the nonincluded side of a second triangle
- For right triangles, use the following ways to prove congruence.
  - Leg-Leg Congruence (LL)
  - Hypotenuse-Angle Congruence (HA)
  - Leg-Angle Congruence (LA)
  - Hypotenuse-Leg Congruence (HL)

Lesson 14-6

Isosceles and Equilateral Triangles

- If two sides of a triangle are congruent, then the angles opposite those sides are congruent.
- If two angles of a triangle are congruent, then the sides opposite those angles are congruent.
- Each angle of an equilateral triangle measures 60°.

Lesson 14-7

Coordinate Proof

- To write a coordinate proof: Place the figure on the coordinate plane. Label the vertices. Use algebra to prove properties or theorems.

Study Organizer

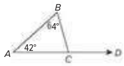
Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

Module 14 Review • Triangles and Congruence 893

Test Practice

1. OPEN RESPONSE Find the measure of $\angle BCD$ in degrees. (Lesson 14-1)



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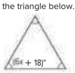
2. MULTIPLE CHOICE Find the value of x given the triangle below. (Lesson 14-1)



A. 7
B. 12
C. 60
D. 126

3. MULTI-SELECT In $\triangle PQR$, $\angle Q$ is a right angle. Select all the statements about $\angle P$ and $\angle R$ that must be true. (Lesson 14-1)

A. $\angle P$ and $\angle R$ are complementary.
B. $\angle P$ and $\angle R$ are supplementary.
C. $\angle P$ and $\angle R$ are congruent.
D. $\angle P$ and $\angle R$ are acute.
E. $\angle P$ or $\angle R$ is obtuse.

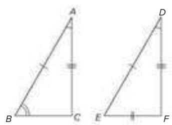
4. OPEN RESPONSE $\triangle PRQ$ has side lengths $PR = 6$, $QR = 8$, and $PQ = 5$. If $\triangle PRQ \cong \triangle CBA$, then put the side lengths of $\triangle CBA$ in order from shortest to longest. (Lesson 14-2)

AC, BC, AB

5. MULTI-SELECT Given $\triangle DEF$ and $\triangle JLK$ where $\overline{DE} \cong \overline{JL}$, $\overline{FD} \cong \overline{KJ}$, $\overline{LK} \cong \overline{EF}$, $\angle D \cong \angle J$, $\angle E \cong \angle L$, and $\angle F \cong \angle K$, which of the following conclusions can be made? Select all that apply. (Lesson 14-2)

A. $\triangle DEF$ and $\triangle JLK$ are congruent.
B. $\triangle DEF$ and $\triangle JLK$ are not congruent.
C. A series of rigid motions will map $\triangle DEF$ onto $\triangle JLK$.
D. A series of rigid motions will not map $\triangle DEF$ onto $\triangle JLK$.
E. $\triangle FDE$ and $\triangle KJL$ are congruent.
F. $\triangle FDE$ and $\triangle KJL$ are not congruent.

6. MULTIPLE CHOICE Which postulate shows $\triangle ABC \cong \triangle DEF$? (Lesson 14-3)



A. Side-Side-Side
B. Angle-Side-Angle
C. Side-Angle-Side
D. Angle-Angle-Side

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Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Put It All Together: Lessons 14-3 through 14-4
Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

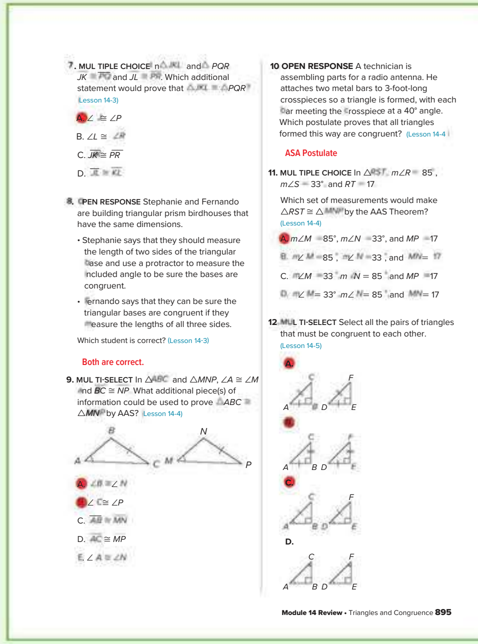
Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–16 mirror the types of questions that your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	2, 6, 7, 11, 14, 16–19
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	3, 5, 9
Table Item	Students complete a table by entering in the correct values.	12
Open Response	Students construct their own response.	1, 4, 8, 10, 13, 15

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
G.CO.7	14-2	5
G.CO.10	14-1, 14-6	1–3, 13–15
G.SRT.5	14-2 through 14-5	4, 6–12
G.GPE.4	14-7	16–19



7. MULTIPLE CHOICE In $\triangle JKL$ and $\triangle PQR$, $JK \cong PQ$ and $JL \cong PR$. Which additional statement would prove that $\triangle JKL \cong \triangle PQR$? (Lesson 14-3)

A. $\angle J \cong \angle P$
B. $\angle L \cong \angle R$
C. $\overline{JK} \cong \overline{PR}$
D. $\overline{JL} \cong \overline{KL}$

8. OPEN RESPONSE Stephanie and Fernando are building triangular prism birdhouses that have the same dimensions.

- Stephanie says that they should measure the length of two sides of the triangular base and use a protractor to measure the included angle to be sure the bases are congruent.
- Fernando says that they can be sure the triangular bases are congruent if they measure the lengths of all three sides.

Which student is correct? (Lesson 14-3)

Both are correct.

9. MULTI-SELECT In $\triangle ABC$ and $\triangle MNP$, $\angle A \cong \angle M$ and $\overline{BC} \cong \overline{NP}$. What additional piece(s) of information could be used to prove $\triangle ABC \cong \triangle MNP$ by AAS? (Lesson 14-4)

A. $\angle B \cong \angle N$
B. $\angle C \cong \angle P$
C. $\overline{AB} \cong \overline{MN}$
D. $\overline{AC} \cong \overline{MP}$
E. $\angle A \cong \angle N$

10. OPEN RESPONSE A technician is assembling parts for a radio antenna. He attaches two metal bars to 3-foot-long crosspieces so a triangle is formed, with each bar meeting the crosspiece at a 40° angle. Which postulate proves that all triangles formed this way are congruent? (Lesson 14-4)

ASA Postulate

11. MULTIPLE CHOICE In $\triangle RST$, $m\angle R = 85°$, $m\angle S = 33°$, and $RT = 17$.

Which set of measurements would make $\triangle RST \cong \triangle MNP$ by the AAS Theorem? (Lesson 14-4)

A. $m\angle M = 85°$, $m\angle N = 33°$, and $MP = 17$
B. $m\angle M = 85°$, $m\angle N = 33°$, and $MN = 17$
C. $m\angle M = 33°$, $m\angle N = 85°$, and $MP = 17$
D. $m\angle M = 33°$, $m\angle N = 85°$, and $MN = 17$

12. MULTI-SELECT Select all the pairs of triangles that must be congruent to each other. (Lesson 14-5)

A.
B.
C.
D.

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13. **OPEN RESPONSE** If the vertex angle of an isosceles triangle measures 86°, what is the angle measure in degrees of one of the base angles? (Lesson 14-6)

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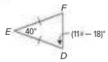
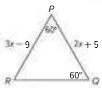
14. **MULTIPLE CHOICE** Find the value of x. (Lesson 14-6)



A. 5.3
B. 8
C. 14.4
D. 70

15. **OPEN RESPONSE** What is the length of segment QR? (Lesson 14-6)



33 units

16. **MULTIPLE CHOICE** An air traffic control tower is located at $O(0, 0)$ on a coordinate plane. Aircraft A is located at $A(39, 52)$ and aircraft B is located at $B(25, 60)$.

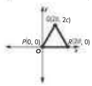
What statement is true about this situation? (Lesson 14-7)

A. Aircraft A is closer to the control tower.
B. Aircraft B is closer to the control tower.
C. Both aircraft are the same distance from the control tower.
D. $\triangle OAB$ is an equilateral triangle.

17. **MULTIPLE CHOICE** A triangle drawn on a coordinate plane has vertices $A(0, 0)$, $B(0, 2b)$, and $C(2c, 0)$. Which expression represents the slope of $\overline{BC}$? (Lesson 14-7)

A. $-\frac{b}{c}$
B. $\frac{b}{c}$
C. $\frac{c}{b}$
D. $-\frac{c}{b}$

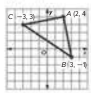
18. **MULTIPLE CHOICE** The given triangle will be used in a coordinate proof.



What are the coordinates of the midpoint of $\overline{QR}$? (Lesson 14-7)

A. $(b - d, 0)$
B. $(b + d, 0)$
C. $(b - d, c)$
D. $(b + d, c)$

19. **MULTIPLE CHOICE** Use coordinate geometry to determine the type of triangle formed below. (Lesson 14-7)



A. equilateral
B. isosceles
C. right
D. scalene

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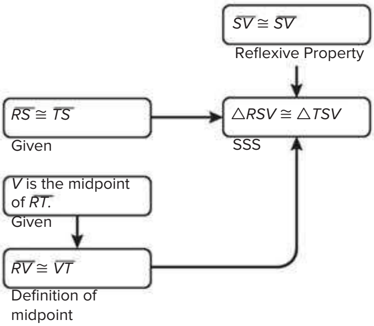
Lesson 14-3

3. Proof:

Statements (Reasons)

1. $\overline{AB} \cong \overline{CB}$, D is the midpoint of $\overline{AC}$. (Given)
2. $\overline{AD} \cong \overline{DC}$ (Definition of midpoint)
3. $\overline{BD} \cong \overline{BD}$ (Reflexive Property of Congruence)
4. $\triangle ABD \cong \triangle CBD$ (SSS)

4.



5. Proof: We know that $\overline{QR} \cong \overline{SR}$ and $\overline{ST} \cong \overline{QT}$. $\overline{RT} \cong \overline{RT}$ by the Reflexive Property. Because $\overline{QR} \cong \overline{SR}$, $\overline{ST} \cong \overline{QT}$, and $\overline{RT} \cong \overline{RT}$, $\triangle QRT \cong \triangle SRT$ by SSS.

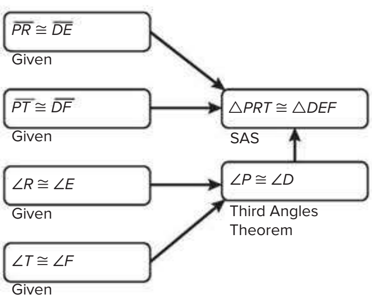
6. Proof:

Statements (Reasons)

1. $\overline{AB} \cong \overline{ED}$, $\overline{CA} \cong \overline{CE}$, $\overline{AC}$ bisects $\overline{BD}$ (Given)
2. C is the midpoint of $\overline{BD}$. (Definition of segment bisector)
3. $\overline{BC} \cong \overline{CD}$ (Midpoint Thm.)
4. $\triangle ABC \cong \triangle EDC$ (SSS)

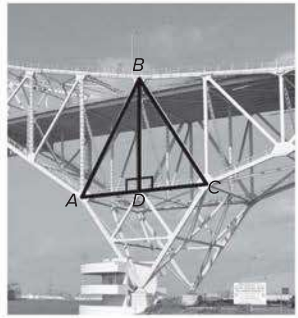
13. Proof: Because V is the midpoint of $\overline{YZ}$ and the midpoint of $\overline{WX}$, by the Midpoint Theorem, $\overline{YV} = \overline{VZ}$ and $\overline{WV} = \overline{XV}$. Because $\angle YVW$ and $\angle ZVX$ are vertical angles, by the Vertical Angle Theorem, the angles are congruent. Therefore, by SAS, $\triangle XVZ \cong \triangle WVY$.

14.



16. Because R is the midpoint of $\overline{QS}$ and $\overline{PT}$, $\overline{PR} \cong \overline{RT}$ and $\overline{RQ} \cong \overline{RS}$ by definition of a midpoint. $\angle PRQ \cong \angle TRS$ by the Vertical Angles Theorem. So, $\triangle PRQ \cong \triangle TRS$ by SAS.

26b. Sample answer: $\overline{BD} \cong \overline{BD}$ by the Reflexive Property, and $\angle BDA \cong \angle BDC$ because they are both right angles. If I can prove that $\overline{AD} \cong \overline{CD}$, then I can prove that these two triangles are congruent.



Lesson 14-4

1. Proof:

Statements (Reasons)

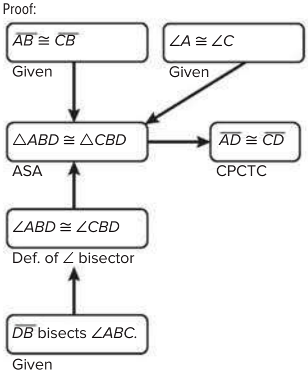
1. $\overline{AB} \parallel \overline{CD}$ (Given)
2. $\angle CBD \cong \angle ADB$ (Given)
3. $\angle ABD \cong \angle BDC$ (Alternate Interior Angles Theorem)
4. $\overline{BD} \cong \overline{BD}$ (Reflexive Property of Congruence)
5. $\triangle ABD \cong \triangle CDB$ (ASA)

2. Proof:

Statements (Reasons)

1. $\angle S \cong \angle V$
   T is the midpoint of $\overline{SV}$. (Given)
2. $\overline{ST} \cong \overline{TV}$ (Definition of Midpoint)
3. $\angle RTS \cong \angle VTU$ (Vertical Angle Theorem)
4. $\triangle RTS \cong \triangle UTV$ (ASA)

3. Proof:

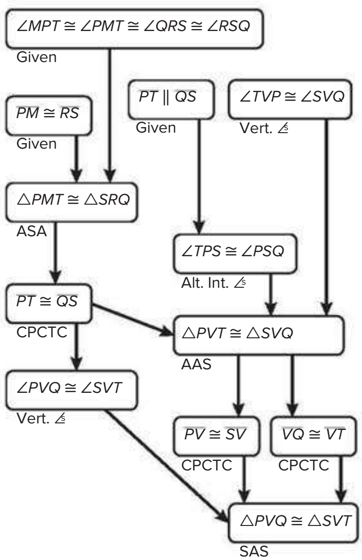


4. Proof: We know that $\angle E \cong \angle BCA$ and $\overline{CD}$ bisects $\overline{AE}$. Because $\overline{CD}$ bisects $\overline{AE}$, by the definition of bisector, $\overline{AC} = \overline{CE}$. We are also given that $\overline{AB} \parallel \overline{CD}$. From this we can determine that $\angle A$ is congruent to $\angle DCE$ by the Corresponding Angles Theorem. From this we know that $\triangle ABC \cong \triangle CDE$ by the ASA Congruence Postulate.

5. Proof: We are given that $\overline{CE}$ bisects $\angle BED$ and that $\angle BCE$ and $\angle ECD$ are right angles. Because all right angles are congruent, $\angle BCE \cong \angle ECD$. By the definition of angle bisector, $\angle BEC \cong \angle DEC$. The Reflexive Property tells us that $\overline{EC} \cong \overline{EC}$. By Angle-Side-Angle Congruence Postulate, $\triangle ECB \cong \triangle ECD$.

6. Proof: It is given that $\angle W \cong \angle Y$, $\overline{WZ} \cong \overline{YZ}$, and $\overline{XZ}$ bisects $\angle WZY$. By the definition of angle bisector, $\angle WZX \cong \angle YZX$. The Angle-Side-Angle Congruence Postulate tells us that $\triangle XWZ \cong \triangle XYZ$.

24. Proof:



Lesson 14-6

1. Proof:

Statements (Reasons)

1. $\angle 1 \cong \angle 2$ (Given)
2. $\angle 2 \cong \angle 3$ (Vertical Angles Thm.)
3. $\angle 1 \cong \angle 3$ (Transitive Prop. of $\cong$)
4. $\overline{AB} \cong \overline{CB}$ (Conv. of Isos. Triangle Thm.)

2. Proof:

Statements (Reasons)

1. $\overline{CD} \cong \overline{CG}$ (Given)
2. $\angle D \cong \angle G$ (Isosceles Triangle Theorem)
3. $\overline{DE} \cong \overline{GF}$ (Given)
4. $\triangle CDE \cong \triangle CGF$ (SAS)
5. $\overline{CE} \cong \overline{CF}$ (CPCTC)

3. Proof:

Statements (Reasons)

1. $\overline{DE} \parallel \overline{BC}$ (Given)
2. $\angle 1 \cong \angle 4$, $\angle 2 \cong \angle 3$ (Corresponding angles are $\cong$.)
3. $\angle 1 \cong \angle 2$ (Given)
4. $\angle 1 \cong \angle 3$ (Transitive Property of $\cong$)
5. $\angle 3 \cong \angle 4$ (Transitive Property of $\cong$)
6. $\overline{AB} \cong \overline{AC}$ (Converse of Isosceles Triangle Thm.)

4. Proof:

Statements (Reasons)

1. $\overline{BD} \perp \overline{AC}$, $\triangle ABC$ is an isosceles triangle with base AC. (Given)
2. $\angle BDA$ and $\angle BDC$ are right angles. (Definition of perpendicular lines)
3. $\angle BDA \cong \angle BDC$ (All right angles are congruent.)
4. $\overline{AB} \cong \overline{BC}$ (Definition of isosceles triangle)
5. $\angle BAD \cong \angle BCD$ (Isosceles Triangle Theorem)
6. $\triangle BAD \cong \triangle BCD$ (AAS)
7. $\angle ABD \cong \angle CBD$ (CPCTC)
8. $\overline{BD}$ bisects the angle formed by the sloped sides of the roof, $\angle ABC$. (Definition of angle bisector)

5a. The coordinates of $\triangle ABC$ are $A(0, 5)$, $B(3, 1)$, and $C(-3, 1)$.

$AC = \sqrt{[0-(-3)]^2 + (5-1)^2}$ or 5 units

$AB = \sqrt{(0-3)^2 + (5-1)^2}$ or 5 units

$BC = 6$ units

So, $\triangle ABC$ is an isosceles triangle with $\overline{AB} \cong \overline{AC}$.

5b. Because $\overline{AB} \cong \overline{AC}$, we know that $\angle C \cong \angle B$ by the Isosceles Triangle Theorem.

$m\angle A + m\angle B + m\angle C = 180°$	Triangle Angle-Sum Theorem
$m\angle A + 2m\angle C = 180°$	Definition of congruent
$m\angle A + 2(55) = 180°$	Substitute.
$m\angle A + 110 = 180°$	Multiply.
$m\angle A = 70°$	Solve.

16. 115°; Sample answer: Because $\triangle PQR$ is isosceles, base angles are congruent, so $m\angle P = m\angle QRP$. It is given that $m\angle Q = 50°$, so by the Triangle Angle-Sum Theorem, $m\angle P + m\angle QRP + 50° = 180°$. By substitution $2m\angle QRP + 50° = 180°$. $2m\angle QRP = 130°$, so $m\angle QRP = 65°$. Because $\angle QRP$ and $\angle QRS$ are supplementary, $m\angle QRS = 180° - 65° = 115°$.

Lesson 14-7

9. Sample answer: The midpoint P of $\overline{BC}$ is $\left(\frac{0+2a}{2}, \frac{2b+0}{2}\right) = (a, b)$. The midpoint Q of $\overline{AC}$ is $\left(\frac{0+2a}{2}, \frac{0+0}{2}\right) = (a, 0)$. The midpoint R of $\overline{AB}$ is $\left(\frac{0+0}{2}, \frac{0+2b}{2}\right) = (0, b)$. The slope of $\overline{RP}$ is $\frac{b-b}{a-0} = \frac{0}{a} = 0$, so the segment is horizontal. The slope of $\overline{PQ}$ is $\frac{b-0}{a-a} = \frac{b}{0}$, which is undefined, so the segment is vertical. $\angle RPQ$ is a right angle because any horizontal line is perpendicular to any vertical line. $\triangle PRQ$ has a right angle, so $\triangle PRQ$ is a right triangle.

14. Slope of $\overline{SK} = \frac{4-0}{6-0}$ or $\frac{2}{3}$

Slope of $\overline{SM} = \frac{3-0}{-2-0}$ or $-\frac{3}{2}$

Because the slope of $\overline{SM}$ is the negative reciprocal of the slope of $\overline{SK}$, $\overline{SM} \perp \overline{SK}$. Therefore, $\triangle SKM$ is right triangle.
Therefore, the triangle formed by Kalini's high school, her home, and the mall is a right triangle.

15. The slope between the grandstand and the rides and games is $\frac{2}{3}$. The slope between the grandstand and the main gate is $-\frac{3}{2}$. Because $\frac{2}{3} \cdot -\frac{3}{2} = -1$, the triangle formed by these three locations is a right triangle.

MODULE 14 ANSWER APPENDIX

Module 1

Quick Check

1. $\frac{2}{3}$ **3.** $\frac{1}{13}$ **5.** 4 **7.** $\frac{3}{2}$ or $1\frac{1}{2}$

Lesson 1-1

1. $2 + 12 \div 4$ **3.** $3 \times 11 + 7$ **5.** $6 - 3 - 1$ **7.** $24 \div 6 + 7$ **9.** $116 \div 4 + 28 - 33$ **11.** $\frac{3+7}{2}$ **13.** $(4 + 9) \times 3$ **15.** $\frac{10}{4 \times 5}$ **17.** $\frac{1+2}{20}$ **19.** $\frac{12+16}{3+4}$ **21.** $\frac{36+14}{2 \times 5}$ **23.** $\frac{6+15}{13-9}$ **25.** 365×85 **27.** $2 \times 35 - 5$ **29.** 12×7 **31.** 2744 **33.** 11 **35.** 42 **37.** 7 **39.** 56 **41.** 22 **43.** 8 **45.** 324 **47.** 29 **49a.** $20 \times 5 + 9$ **49b.** 109 flies **51a.** $2 \times \frac{1}{2}(30 + 50)24$ **51b.** 1920 in^2 **53.** $8^4 + 6$ **55.** Tamara; when evaluating, first perform the multiplication and division from left to right, and then the addition and subtraction from left to right. **57.** The cashier; Kelly should have entered the expression into her calculator as $3(18.95 - 2) + 2(11.50)$. **59.** $10 \times 18 + 8 \times 18$; You can also find the length of each side of the apartment, 18 and $10 + 8$, and then multiply: $18(10 + 8)$. **61.** The student should have added/subtracted from left to right. The correct value is 30.

Lesson 1-2

1. four times a number *q* **3.** 15 plus *r* **5.** 3 times *x* squared **7.** two times *a* plus six **9.** twenty-five plus six times a number squared **11.** three times a number raised to the fifth power divided by two **13.** 5 times *g* to the sixth power **15.** four minus five times *h* **17.** 1 less than 7 times *x* cubed **19.** 3 times *n* squared minus *x* **21.** 18 times the quantity of *p* plus 5 **23.** $n - 35$ **25.** $\frac{1}{3}n$ **27.** $\frac{45}{r}$ **29.** $18 - 3d$ **31.** $\frac{20}{t^6}$ **33.** $(k + 2) - 15$

35. $2m + 6$ **37.** $\frac{6136}{y}$ **39.** $19.95 \times t - 10$ or $19.95t - 10$ **41.** $5f + 45h$ **43.** 1 **45.** 7 **47.** 149 **49.** $\frac{65}{4}$ **51.** 44 **53.** $\frac{1}{2}$ **55.** 18 **57.** 10 **59.** 16 **61.** 13 **63a.** $5t - 100$ **63b.** 1400 students **65a.** $1.75 + 3.45m$ **65b.** \$29.35 **67a.** Sample answer: the quotient of 5*x* and 2 plus *y* cubed; 5*x* divided by 2 plus *y* to the third power **67b.** 18 **69.** 89 **71.** 52 **73.** Sample answer: the quotient of *x* minus 1 and 2; $\frac{x-1}{2}$ **75a.** $x - (36 \times 4)$ **75b.** $\frac{x - (36 \times 4)}{0.20}$ **75c.** 350 mi **77.** Sample answer: An algebraic expression is a math phrase that contains one or more numbers or variables. To write an algebraic expression from a real-world situation, first assign variables. Then determine arithmetic operations done on the variables. Finally, put the terms in order. **79.** Sample answer: Movie tickets cost \$10 and a box of popcorn cost \$5.25. You buy *t* movie tickets and a box of popcorn. What is the greatest number of movies tickets you can purchase with \$50?

Lesson 1-3

1. Symmetric Property of Equality **3.** Symmetric Property of Equality **5.** 14 **7.** 34 **9a.** Exit 15 to Exit 8 **9b.** Symmetric Property of Equality

11.

$= (3 \div 2)\frac{2}{3}$	Multiplicative Identify
$= \frac{3}{2} \cdot \frac{2}{3}$	Substitution
$= 1$	Multiplicative Identify

13.

$= 2(5 - 5)$	Substitution
$= 2(0)$	Substitution
$= 0$	Multiplicative Property of Zero

15.

$= 2(2 - 1) \cdot \frac{1}{2}$	Substitution
$= 2(1) \cdot \frac{1}{2}$	Substitution
$= 2 \cdot \frac{1}{2}$	Multiplicative Identity
$= 1$	Multiplicative Inverse

17. $= 4 + \frac{4}{9} + 7 + \frac{2}{9}$	Substitution
$= 4 + 7 + \frac{4}{9} + \frac{2}{9}$	Commutative (+)
$= 4 + 7 + \left(\frac{4}{9} + \frac{2}{9}\right)$	Associative (+)
$= 11 + \frac{6}{9}$	Substitution
$= 11\frac{6}{9}$	Substitution
$= 11\frac{2}{3}$	Substitution
19. $= (2 \cdot 8) \cdot (10 \cdot 2)$	Associative (×)
$= 16 \cdot 20$	Substitution
$= 320$	Substitution
21. $= \left(2\frac{3}{4} \cdot 1\frac{1}{8}\right) \cdot 32$	Associative (×)
$= \left(\frac{11}{4} \cdot \frac{9}{8}\right) \cdot 32$	Substitution
$= \frac{99}{32} \cdot 32$	Substitution
$= 99$	Substitution
23. $= 2 \cdot 5 \cdot 4 \cdot 3$	Commutative (×)
$= (2 \cdot 5) \cdot (4 \cdot 3)$	Associative (×)
$= 10 \cdot 12$ or 120	Substitution
25. $= \frac{4}{3} \cdot 3 \cdot 7 \cdot 10$	Commutative (×)
$= \left(\frac{4}{3} \cdot 3\right) \cdot (7 \cdot 10)$	Associative (×)
$= 4 \cdot 70$ or 280	Substitution

27. −64 **29.** −5 **31.** −9 **33.** Sample answer: Multiplicative Identity and Multiplicative Inverse **35.** 0; Additive Identity **37.** 1; Multiplicative Identity **39.** 5; Additive Identity **41.** 1; Multiplicative Inverse **43.** 3; Reflexive Property **45.** Yes; the Commutative and Associative Properties of Multiplication allow it to be rewritten. **47.** Sample answer: $126 + 28 + 52 = 126 + (28 + 52) = 206$ **49.** Sample answer: $5 = 3 + 2$ and $3 + 2 = 4 + 1$, so $5 = 4 + 1$; $5 + 7 = 8 + 4$ and $8 + 4 = 12$, so $5 + 7 = 12$. **51.** $4 \div 8 \neq 8 \div 4$ because $4 \div 8 = \frac{1}{2}$ and $8 \div 4 = 2$, so there is no Commutative Property for division. $16 \div (8 \div 4) \neq (16 \div 8) \div 4$ because $16 \div (8 \div 4) = 16 \div 2 = 8$ and $(16 \div 8) \div 4 = 2 \div 4 = \frac{1}{2}$, so there is no Associative Property for division. As long as neither number is 0, when the order of division of two numbers is switched, the results are multiplicative inverses of each other. **53a.** False; sample answer: $3 - 4 = -1$, which is not a whole number. **53b.** True **53c.** False; sample answer: $2 \div 3 = \frac{2}{3}$, which is not a whole number. **55.** $(2j)k = 2(jk)$; The other three equations illustrate the Commutative Property of Addition or Multiplication. This equation represents the Associative Property of Multiplication.

Lesson 1-4

1. $4(6) + 5(6)$; 54 **3.** $6(6) - 6(1)$; 30 **5.** $14(8) - 14(5)$; 42 **7a.** $39(23 + 2)$ **7b.** \$975 **9a.** $10\left(3\frac{3}{5}\right)$ **9b.** $10\left(3\frac{3}{5}\right) = 10\left(3 + \frac{3}{5}\right) = 10(3) + 10\left(\frac{3}{5}\right) = 30 + 6 = 36$ yards of fabric **11.** $7(500 - 3)$; 3479 **13.** $36\left(3 + \frac{1}{4}\right)$; 117 **15.** $5(90 - 1)$; 445 **17.** $15(100 + 4)$; 1560 **19.** $12(100 - 2)$; 1176 **21.** $3(10 + 0.2)$; 30.6 **23.** $2(x) + 2(4)$; $2x + 8$ **25.** $4(8) + (-3m)(8)$; $32 - 24m$ **27.** $2(17) + (-4n)(17)$; $34 - 68n$ **29.** $\frac{1}{3}(27) + (-2b)(27)$; $9 - 54b$ **31.** $6(2c) + 6(-cd^2) + 6(d)$; $12c - 6cd^2 + 6d$ **33.** $3(m) + 3(n)$; $3m + 3n$ **35.** $\frac{1}{2}(14) + (6a)(14)$; $7 + 84a$ **37.** $0.3(9) + (-6x)(9)$; $2.7 - 54x$ **39.** $18r$ **41.** $2m + 7$ **43.** $13m + 5p$ **45.** $14m + 11g$ **47.** $12k^3 + 12k$ **49.** $18g$ **51.** $5a^2$ **53.** $2q^2 + q$ **55a.** $3a + 5(a - b)$

55b. $3a + 5(a - b) = 3a + 5a - 5b$	Distributive Property
$= (3a + 5a) - 5b$	Associative (+)
$= 8a - 5b$	Substitution

57. $24x + 28$ **59.** $18d + 20$ **61.** $7y^3 + y^4$ **63.** $4b$ **65.** $20x + 37y$ **67.** $2n + 2m$ and $2(n + m)$ **69.** No; sample answer: 10 pounds 5 ounces is $10(16) + 5 = 165$ ounces, but Ariana used the Distributive Property incorrectly. She should have written $8(20 + 2) = 8(20) + 8(2) = 160 + 16 = 176$ ounces. **71.** Sample answer: Algebraic expressions are helpful because they are easier to interpret and apply than verbal expressions. They can also be written in a more simplified form.

Lesson 1-5

1. $|p - t|$ and $|t - p|$ **3.** $|r - w|$ and $|w - r|$ **5.** 15 **7.** 22 **9.** 37 **11.** 32 **13.** −62 **15.** 11 **17.** 10 **19.** 5 **21.** 5 **23.** 6 **25.** −7.4 **27.** 8.4 **29.** −15 **31.** 22 **33.** 14.5 **35a.** $|g - d|$ and $|d - g|$ **35b.** 5 meters **37.** Sample answer: A meteorologist says that the high temperature is going to be 89 degrees. If the actual high temperature that day is x, then $|x - 89|$ represents the number of degrees the meteorologist is away from the actual high temperature. **39.** False; sample answer: Suppose $a = 5$ and $b = -3$, then $|a + b| = |5 + (-3)| = |5 - 3| = |2| = 2$ and $|a| + |b| = |5| + |-3| = 5 + 3 = 8$. $2 \neq 8$, so Diaz's claim is not correct.

Lesson 1-6

1. 32 **3.** 11.4 seconds **5.** 0.512 **7.** Automatic Method: \$6000; Exact Method: \$5625 **9.** $2\frac{1}{3}$ snack bars **11.** 6 **13.** \$333.33 **15.** \$10,500 **17.** Because the number of students enrolled at Hartgrove High School can be counted, giving an exact enrollment is accurate. **19.** The map maker is probably accurate because the number of traffic lights in New York City is not very specific. **21.** Sample answer: Steve Nash would be selected as a free throw shooter. Michael Jordan would be selected as a free throw shooter. Shaquille O'Neal would not be selected as a free throw shooter. **23.** Light-years; sample answer: The distance from Earth to the star is very great so using the largest distance unit is appropriate in this situation. most accurate number of visitors at the zoo for a given temperature. **27.** Sample answer: The number of visitors does not increase at the same rate for each average daily temperature. **29.** Sample answer: An employer might consider the number of sick days an employee takes or the amount of sales an employee generates. **31.** Sample answer: \$28.43 because $3.299 \times 8.618 = 28.430782$. The answer could be accurate to the thousands place, but it is only necessary to round to the nearest hundredths place because the penny is the smallest unit of money.

Module 1 Review

1. D **3.** $5(x + 7) - 4^3$ **5.** A **7.** B **9.** A. True B. False C. True **11.** B, D **13.** C, E **15.** 67 **17.** No; sample answer: The manager rounded down, but actually spent much more than \$4000. It would have been better to report a greater amount so that it was clear her budget was not overspent.

Module 2

Quick Check

1. $6n + 2$ **3.** $4b + 9$ **5.** 8 **7.** 32 **9.** 36

Lesson 2-1

1. $3m + 2 = 18$ **3.** $\frac{24}{x} = 14 - 2x$ **5.** $2 + 3h = 6$ **7.** $(48 + 33) + n = 107$ **9.** $2a + a^3 = b$ **11.** $x + x^2 = yz$ **13.** $A = \ell^2$ **15.** $P = 2\ell + 2w$ **17.** $I = prt$ **19.** The sum of j and sixteen is thirty-five. **21.** Seven times the sum of p and twenty-three is the same as one hundred two. **23.** Two-fifths of v plus three-fourths is identical to two-thirds of x squared. **25.** g plus 10 is the same as 3 times g. **27.** 4 times the sum of a and b is 9 times a. **29.** Half of the sum of f and y is f minus 5. **31.** Sample answer: The volume equals π times the radius squared times the height. The base is a circle so the expression πr^2 represents the area of the base. **33.** Sample answer: The interest equals the product of the principal, the rate, and the time. **35.** Sample answer: Force equals mass times acceleration. The expression ma represents the force on an object with mass m that is accelerating. **37.** B **39.** A **41.** $y^2 - 12 = 5x$ **43.** $100 - 3b = 6b$ **45.** Four times n equals x times the difference of five and n. **47.** The sum of y and the product of 3 and the square of x is 5 times x. **49.** $V = \ell wh$ **51.** $m + 2m = 24$ or $3m = 24$ **53.** $c = 10w + 0.1(10w)$ or $c = 11w$ **55a.** It is correct. The product is squared, so parentheses are needed. **55b.** It is not correct. One-half of a number means to multiply, not divide, by one-half. It should be $\frac{1}{2}n + 3 = n - 2$. **57.** Sample answer: A teacher ordered 188 math books. The algebra books were packed in boxes of 12. The geometry books were packed in boxes of 10. He ordered one more box of algebra books than geometry books. How many books of each type book did he order? Let a = number of algebra books. **59.** $S = 6\ell^2$ **61.** Sample answer: First, you should identify the unknown quantity or quantities for which you are trying to solve, and assign variables. Then, you should look for key words or phrases that can help you to determine operations that are being used. You can then write the equation using the numbers that you are given and the variables and operations that you assigned.

Lesson 2-2

1. 23 **3.** −43 **5.** −12 **7.** 73 **9.** −15 **11.** −54 **13.** $\frac{7}{20}$ **15.** $-\frac{7}{15}$ **17.** −937 **19.** −147 **21.** −25 **23.** $-\frac{9}{2}$ **25.** 15 **27.** 10 **29.** 64 **31.** 28 **33.** 18 **35.** 24 **37.** 27 **39.** 39 **41.** 64 **43.** 9 **45.** −12 **47.** 7 **49.** 64 **51.** −252 **53.** −52 **55.** $x + 33 = 2005$; $x = 1972$ **57.** $x - 21 = -9$; $x = 12$°C **59a.** Let p = the number of players who signed up for the soccer league. If 13% of the players who signed up for the soccer league dropped out, then 100% −13%, or 87% of the players finished the season. So, $0.87p$ represents the number of players who finished the season. **59b.** $0.87p = 174$ **59c.** $p = 200$; 200 players signed up for the soccer league **61.** $\frac{2}{3} = -8n$; $-\frac{1}{12}$ **63.** $\frac{4}{5} = \frac{10}{16}n$; $\frac{32}{25}$ **65.** $4\frac{4}{5}n = 1\frac{1}{5}$; $\frac{1}{4}$ **67.** −77 **69.** $\frac{16}{3}$ **71.** −10 **73.** $-\frac{10}{7}$ or $-1\frac{3}{7}$ **75.** 18 **77.** 225 **79.** −14 **81.** 4 **83.** −49 **85.** 40 **87.** −15 **89.** $-\frac{8}{15}$ **91a.** $12x = 780$; $x = 65$ **91b.** $20 **93.** $x = 216$; Multiplication Property of Equality **95.** $y = -224$; Subtraction Property of Equality **97.** $15 = b$; Division Property of Equality **99.** $n - 16 = 29$ does not belong because for the other three, $n = 13$, and for this one $n = 45$. **101.** Sample answer: $x - 4 = 10$ **103.** Sample answer: To solve $5x = 35$, I would divide each side by 5 to get $x = 7$. To solve $5 + x = 35$, I would subtract 5 from each side to get $x = 30$. In both equations I used properties of equality to isolate the variable. In the first equation I used the Division Property of Equality and I used the Subtraction Property of Equality in the second equation.

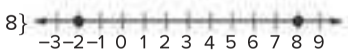
Lesson 2-3

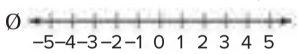
1. −5 **3.** −5 **5.** 70 **7.** 27 **9.** 16 **11.** −61 **13.** $\frac{1}{2}a - 5.25 = 22.50$; $55.50 **15.** $\frac{t - 10}{15} = 4$; 70 treats **17.** $71 = 2h - 1$; 36 inches **19.** $\frac{18}{a}$ **21.** $\frac{-35}{a}$ **23.** $\frac{-24}{a}$ **25.** $\frac{-14}{a}$ **27.** 7 **29.** 10 **31.** −16 **33.** −2 **35.** 18 **37.** $(n - 2) \div 3 = 30$; 92 **39.** Sample answer: Both are correct. Dividing by a number and multiplying by that number's reciprocal are equivalent operations. **41a.** $x = \frac{-2}{a}$ **41b.** $x = 13a$ **41c.** $x = \frac{10}{a}$ **43.** Never; whenever three odd integers are added together, the sum is always odd.

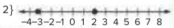
Lesson 2-4

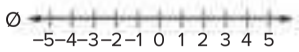
1. 6 **3.** 1 **5.** −2 **7.** −2 **9.** 14 **11.** 4 **13.** −5 **15.** 0 **17.** $7 + F = 4F + 1$; France won 2 gold medals and the U.S won 9 gold medals. **19.** $38 + 4x = 45.5 + 2.5x$; 5 years **21.** $180 - x = 10 + 2(90 - x)$; 10° **23.** $9(5 + x) = 15\frac{3}{7}x$; 7 **25.** no solution **27.** identity **29.** no solution **31.** one solution **33.** identity **35.** no solution **37.** no solution **39.** all numbers **41.** −25 **43.** 3 **45.** −2 **47.** 15 **49a.** Let n = the first odd integer; $2(n + 2) = 3n - 13$ **49b.** 17 and 19 **51a.** Let k = the number; $4k - 3 = 2k + 5$ **51b.** $k = 4$ **51c.** Substitute 4 for k in the expression for the perimeter of Figure 2, $2k + 5$. So the perimeter of Figure 2 is $2(4) + 5 = 8 + 5 = 13$. **51d.** Substitute 4 for k in the expression for the perimeter of Figure 1, $4k - 3$. So the perimeter of Figure 1 is $4(4) - 3 = 16 - 3 = 13$. The perimeter for Figure 1 and Figure 2 is the same, so the value of k is correct. **53.** Anthony is correct. When Patty added m to each side, she subtracted the terms instead of adding them. **55.** Sample answer: $2(3x + 6) = 3(2x + 5)$ **57a.** Incorrect; the 2 must be distributed over both g and 5, then 10 must be subtracted from each side; 6. **57b.** Correct; the Subtraction Property was used to combine the variable terms on the left side of the equation. The Division Property was used to isolate the variable on one side. **57c.** Incorrect; to eliminate $-6z$ on the left side of the equal sign, $6z$ must be added to each side of the equation; 1. **59.** Sample answer: $2x + 1 = x + 9$

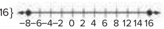
Lesson 2-5

1. {−2, 8}


3. ∅


5. {−3.25, 2}


7. ∅


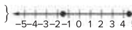
9. {−8, 16}


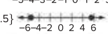
11. {2, 5} **13.** {−6, 4} **15.** {0, 4} **17.** {−5, −1} **19.** $|t - 400| = 15$; min = 385°F; max = 415°F **21a.** $|t - 20.9| = 5.3$ **21b.** 15.6 to 26.2; 10.3 to 31.5

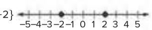
23.

$\|x - 35\| = 0.5$	Absolute value equation
Case 1: $x - 35 = 0.5$	Definition of absolute value.
$x = 35.5$	Simplify
Case 2: $x - 35 = -0.5$	Definition of absolute value.
$x = 34.5$	Simplify

The bags of rock salt weigh no less than 34.5 pounds and no more than 35.5 pounds. **25.** $|x| = 6$ **27.** $|x + 2| = 4$ **29.** $|x + 3| = 2$ **31.** $|x| = 4$

33. $\left\{-\frac{3}{2}, \frac{9}{2}\right\}$


35. {5.5, −5.5}


37. {2, −2}


39. $|x| = 1\frac{1}{2}$ **41.** $|x - \frac{1}{4}| = \frac{1}{4}$ **43.** $|x + \frac{1}{3}| = 1$ **45a.** $|x - 38| = 2$ **45b.** 40°F, 36°F **47.** $|x - 3| = 1$ **49.** $|a - b| + |b - c| = |a - c|$ **51.** Cami; The absolute value of a number cannot be negative. **53.** Sample answer: Let x = the temperature at night. Then the temperature is 4 ± 10 degrees.

Lesson 2-6

1. 40 **3.** 29.25 **5.** 9.8 **7.** 1.32 **9.** 0.84 **11.** 0.57 **13.** 6 **15.** 11 **17.** 18 **19.** 0.8 **21.** 11 **23.** −2 **25.** 1.44 **27.** −2.29 **29.** −2.2 **31.** 10 **33.** 3 **35.** −8.4 **37.** 12.5 gal **39.** $46.27 **41a.** 60 free throws **41b.** Sample answer: I assumed that Brent continues to make free throws at the same rate. **43.** 22.5 in. **45.** $3333.33 **47.** 204.55 mL **49.** 6 **51.** 10 **53.** 21 **55.** 8 **57.** 42 **59.** 27 **61.** 3 **63.** 15 **65.** −3 **67.** −0.4 **69.** −6 **71a.** Sample answer: 2.2 cm **71b.** Sample answer: about 6.6 miles **71c.** Sample answer: about 453.7 mi^2 **73a.** $4.50; because 8 potatoes cost $1.50, multiply by 3 to get a cost of $4.50 for 24 potatoes. **73b.** $4.13; Sample answer: $4.13 is slightly less than $4.50, which aligns with my estimate. **73c.** 37 **73d.** $0.19 **75.** $\frac{2}{4}$ or $\frac{1}{2}$ **77.** Ratios and rates each compare two numbers by using division. However, rates compare two measurements that involve different units of measure. **79.** $\frac{x}{100} = \frac{z}{y}$

Lesson 2-7

1. $y = \frac{x-1}{2}$ **3.** $f = \frac{5-g}{7}$ **5.** $t = \frac{x}{7}$ **7.** $r = \frac{q}{2}$ **9.** $a = -\frac{b}{8}$ **11.** $v = \frac{u-z}{w}$ **13.** $g = \frac{10j+9h}{f}$ **15.** $t = \frac{3}{2}(r - v)$ **17.** $a = \frac{-33+x}{10c}$ **19a.** $\ell = \frac{P-2w}{2}$ **19b.** 14 m **21a.** $g = \frac{c-p}{13.50}$ **21b.** 6 games **25.** ≈ 12.96 trillion pounds **27.** 0.44 ft **29.** 24 miles **31.** 90.2 gallons **33.** 82 students **35.** $c = \frac{2k-3g}{b}$ **37.** $c = \frac{5p-6j}{8}$ **39.** $c = x - 2d$ **41.** $t = \frac{w-11v}{31}$ **43.** $c = \frac{-13+f}{10-d}$ **45.** $r = \frac{A}{P} - 1 = \frac{2182.25}{2150} - 1 = 1.015 - 1 = 0.015$; The interest rate is 1.5%. **47a.** $y = \frac{mx+mt-z}{r}$ **47b.** Division by 0 is undefined, so in the original equation $m \neq 0$, and in the final equation $r \neq 0$. **49.** No; Sasha does not have a correct solution. When she multiplied F by $\frac{5}{9}$, she should have multiplied 32 by $\frac{5}{9}$.

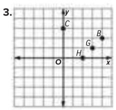
Module 2 Review

1. A **3.** D **5.** D **7.** C **9.** B, D **11.** C **13.** Let d = age of dogwood tree (in years); $3d - 2 = \frac{1}{2}(d + 8 + 8)$; The gingko tree is 10 years old, and the dogwood tree is 4 years old. **15.** A **17.** $b = 8$ **19.** B **21.** D **23A.** $h = 3\frac{V}{B}$ **23B.** 12

Module 3

Quick Check

1, 3.

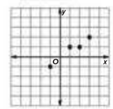


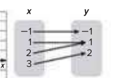
5. 20 **7.** −3

Lesson 3-1

1.

x	y
−1	−1
1	1
2	1
3	2

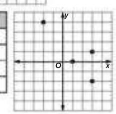


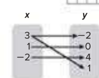


D = {−1, 1, 2, 3}; R = {−1, 1, 2}

3.

x	y
3	−2
1	0
−2	4
3	1

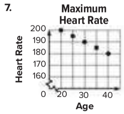




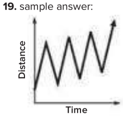
D = {−2, 1, 3}; R = {−2, 0, 1, 4}

5a. independent: price of item, dependent: number of items purchased **5b.** As the price of an item increases, the number of items purchased decreases.

7.



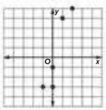
9. The x-axis represents the time in seconds. The y-axis represents the height of the elevator in feet. The x-axis has a scale of 1 mark = 1 second. The y-axis has a scale of 1 mark = 10 feet. The origin (0, 0) represents a height of 0 feet in 0 seconds. **11.** {(1, 7), (3, 45), (5, 11), (13, 15)} **13.** {(2, 5), (5, 0), (7, 8), (7, 10), (10, 2)} **15.** {(2, 80), (3, 120), (6, 240), (8, 320)}; The x-axis represents the number of gallons of syrup. The y-axis represents the number of gallons of sap. The x-axis has a scale of 1 mark = 1 gallon of syrup. The y-axis has a scale of 1 mark = 40 gallons of sap. The origin (0, 0) represents 0 gallons of sap makes 0 gallons of syrup. **17a.** {(0, 12), (1, 8), (2, 23), (3, 28), (4, 11), (5, 11)} **17b.** {0, 1, 2, 3, 4, 5} **17c.** {8, 11, 12, 23, 28} **19.** sample answer:

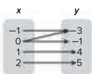


21. Tim drives away from the pizzeria, stops to make a delivery, continues to drive away from the pizzeria, stops to make another delivery, and then returns to the pizzeria. **23.** Disagree; The intersection point represents a time when Tim and Lauren were both at the same distance from the pizzeria. **25.** Disagree; sample counterexample: In the relation {(1, 2), (1, 3)}, the domain is {1}, so it has one element, while the range is {2, 3}, which has two elements. **27a.** Sample answer: {(−1, −3), (0, −3), (0, −1), (1, 4), (2, 5)}

27b. sample answer:

x	y
−1	−3
0	−1
0	−3
1	4
2	5





29. Sample answer: A dependent variable is determined by the independent variable for a given relation.

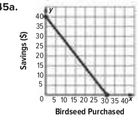
Lesson 3-2

1. Yes; for each element of the domain, there is only one element of the range. **3.** No; the element 4 in the domain is paired with both 2 and 5 in the range. **5.** No; the element 5 in the domain is paired with both −3 and 2 in the range. **7.** Yes; for each element of the domain, there is only one element of the range.
9. Yes; for any value *x*, the vertical line passes through no more than one point on the graph.
11. Yes; for any value *x*, the vertical line passes through no more than one point on the graph.
13. No; for $x > 0$, the vertical line passes through more than one point on the graph.
15a.

Year	2014	2015	2016	2017
Value ($)	254,000	293,000	338,000	372,000

15b. Domain: {2014, 2105, 2016, 2017}; Range: {254,000; 293,000; 338,000; 372,000}
15c. For each element of the domain, there is only one element of the range. So, this relation is a function. **17.** 26 **19.** 2 **21.** 42 **23.** 4
25. 6 **27.** $9b^2 - 3b$ **29.** $f(3.5) = 12.25$, which means the area of a square with a side of length 3.5 units is 12.25 square units.
31. $f(12) = \$435$, which is the cost of a gym membership for 12 months, or 1 year. **33.** −1
35. 14 **37.** −4 **39.** $-8y - 3$ **41.** $-2c - 8$
43. $-10d - 15$

45a.

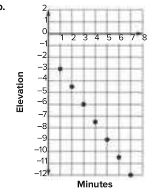


45b. Yes; for any value *x*, the vertical line passes through no more than one point on the graph. **45c.** $f(3) = 36.25$, which means if Aisha buys 3 pounds of birdseed, she saves \$36.25; $f(18) = 17.50$, which means if Aisha buys 18 pounds of birdseed, she saves \$17.50; $f(36) = -5$, which means if Aisha wants to buy 36 pounds of birdseed, she needs \$5 extra. **45d.** 8 pounds **47a.** $h(20) = 46$; The height of the balloon 20 seconds after it is released is 46 feet. **47b.** 2 minutes is 2(60) = 120 seconds, so calculate $h(120)$ by substituting $t = 120$ in the equation; $h(120) = 2(120) + 6 = 246$; The height of the balloon is 246 feet. **47c.** 6 feet; $t = 0$ before the balloon is released, and $h(0) = 6$. **47d.** Sample answer: The values of *t* must be greater than or equal to zero because a negative value for the time does not make sense for the given situation. The graph would start at the vertical axis and go only to the right. **49.** Sample answer: You can determine whether each element of the domain is paired with exactly one element of the range. For example, if given a graph, you could use the vertical line test; if a vertical line intersects the graph more than once, then the relation that the graph represents is not a function. **51.** $f(g + 3.5) = -4.3g - 17.05$
53. Sample answer: $f(x) = 3x + 2$

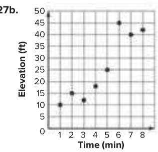
Lesson 3-3

1. Neither; because the function has continuous sections but is not a single line or curve, it is neither continuous or discrete **3.** Discrete; because the function is made up entirely of individual points, it is discrete. **5.** Continuous; because the function is graphed with a single line, it is continuous. **7.** Discrete; because the function is made up entirely of individual points, it is discrete. **9.** discrete **11.** discrete

13. linear **15.** nonlinear **17.** nonlinear
19. nonlinear **21.** nonlinear **23.** linear
25a. linear
25b.



27a. nonlinear
27b.

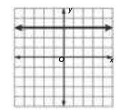


29. continuous; nonlinear **31.** neither; nonlinear **33.** discrete; nonlinear
35. Sample answer: A studio charges musicians to use the space and recording equipment by the hour, rounding a fraction of an hour up. So, for up to 1 hour, the studio charges \$100, but for up to 2 hours, the studio charges \$200, and so on. The function that models this situation is neither discrete nor continuous.

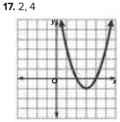
Lesson 3-4

1. *x*-intercept: (−0.75, 0) *y*-intercept: (0, 2) positive: when $x > -0.75$ negative: when $x < -0.75$ **3.** *x*−intercepts: (0, 0) and (2, 0) *y*-intercept: (0, 0) positive: when $x < 0$ and when $x > 2$ negative: $0 < x < 2$
5. *x*-intercepts: (−2, 0) *y*-intercept: (0, 4) positive: when $x > -2$ negative: when $x < -2$

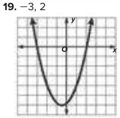
7. *x*-intercepts: (−5, 0) and (3, 0) *y*-intercept: (0, 3) positive: $-5 < x < 3$ negative: $x < -5$ and when $x > 3$
9. *x*-intercepts: none *y*-intercept: (0, −3) positive: never negative: always
11. The *x*-intercept is 0. The *y*-intercept is 0. This means that Ryan earns \$0 for working 0 hours. The function is positive when *x* is greater than 0, which means that Ryan earns money for working. No portion of the graph shows that the function is negative.
13a. The *x*-intercept is 6. The *y*-intercept is 1950. **13b.** The *x*-intercept means that after 6 months, Javier's remaining balance will be \$0, or it will take Javier 6 months to repay his parents. The *y*-intercept means that Javier owes his parents \$1950 after 0 months, or Javier initially borrowed \$1950 from his parents.
15. Sample graph; no solution



17. 2, 4



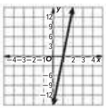
19. −3, 2



21. The zero of the function is at 32. This represents that after tying ribbon on 32 gift bags, Juanita will have no ribbon left.

23. x-intercepts: $(-2, 0)$ and $(2, 0)$ y-intercept: $(0, 4)$ positive: $-2 < x < 2$ negative: when $x < -2$ and when $x > 2$ **25.** The x-intercepts are 3 and 7. That means that the bird will be at sea level at 3 seconds and at 7 seconds. The y-intercept is 4.5. This means that at time 0, the bird was at a height of 4.5 feet. The function is positive when x is less than 3 and when x is greater than 7, which means that the bird is above sea level from 0 to 3 seconds and after 7 seconds. The function is negative when x is between 3 and 7, which means that the bird is below sea, or under water, for 4 seconds.

27. 1; x-int: 1; y-int: -15



29. To find the x-intercept in a graph, find the place where the function crosses the x-axis. To find the y-intercept in a graph, find the place where the function crosses the y-axis. To find the x-intercept in a table, find the x-value when the y-value is 0. To find the y-intercept in a table, find the y-value when the x-value is 0. **31.** Find the related function. Subtract 16 from each side: $0 = x + 4 + (2^4 - 6) - 16$. Evaluate the exponent: $0 = x + 4 + (16 - 6) - 16$. Evaluate the expression in parentheses: $0 = x + 4 + 10 - 16$. Add and subtract: $0 = x - 2$. Replace 0 for $f(x)$. The related function is $f(x) = x - 2$. The graph of the related function intersects the x-axis at 2. This is the x-intercept, or zero. So the solution of the equation is 2. Check the solution by solving the equation algebraically. Evaluate the exponent: $16 = x + 4 + (16 - 6)$. Evaluate the expression in parentheses: $16 = x + 4 + 10$. Add: $16 = x + 14$. Subtract 14 from each side: $2 = x$.

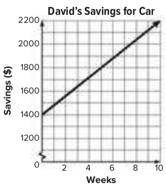
Lesson 3-5

1. This function is symmetric in the line $x = -1$.
3. This function is symmetric in the line $x = 2.5$.

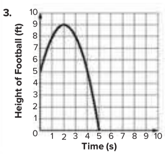
5. The graph is symmetric in the line $x = 5$. In the context of the situation, the symmetry of the graph tells you that the area is the same when the width is a number less than or greater than 5. **7.** always decreasing
9. decreasing: $x < 1.5$; increasing: $x > 1.5$
11. extrema: B and D; rel min: D; rel max: B
13. extrema: B and D; rel min: B; rel max: D
15. Point A is a relative maximum. Point A represents the greatest height of the golf ball given the distance from the tee. **17.** As x decreases, y increases. As x increases, y increases. **19.** As x decreases, y decreases. As x increases, y increases. **21.** no line symmetry; always decreasing; extrema: none; As x decreases, y increases. As x increases, y decreases. **23.** The approximate point (2.5, 114) is a relative maximum. This represents the greatest height of the rock given the time.
25. The graph has one relative minimum at about $(-2.25, -16)$; This statement is not true because there are two relative minimums: one at about $(-2.25, -16)$ and one at about $(2.25, -16)$.

Lesson 3-6

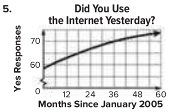
1.



3.

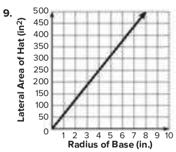


5.



7. Sample answer: Internet use at home initially has a higher number of users than Internet use away from home. Both Internet use at home and Internet use away from home increase after 36 months since March 2004. Neither Internet use at home nor Internet use away from home reaches 0 users.

9.



11. Sample answer: The graph on the calculator and the graph I sketched are both linear, increasing, and have an x- and y-intercept at 0. **13.** $P(x) = 28x - 840$; $P(x)$ is Aidan's profit from fixing and selling x bicycles. **15.** The x-intercept; To find the x-intercept, locate the point on the graph when $P(x) = 0$, which is 30. So, when 30 bicycles are bought and sold, Aidan makes a profit of $0.

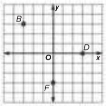
Module 3 Review

1. B, C, F
3. C
5. Sample answer: The element -4 in the domain is paired with both 8 and 13 in the range. This relation is not a function. **7.** B
9. $(-1, 0)$ and $(0, -1)$ **11.** 40; 60 **13.** Sample answer: In 1900 the population of Ohio was nearly 4 million more than the population of Florida. Both populations grew between 1900 and 1950. At this point, the population of Ohio exceeded that of Florida by approximately 5 million, indicating a greater growth rate for Ohio than Florida during those decades. Then from 1950 to 2000, the population of Ohio grew by about 3.4 million, whereas the population of Florida grew by about 13 million, indicating a significantly greater growth rate for Florida during those decades. In fact, by 2000, the population of Florida surpassed Ohio by more than 4 million people.

Module 4

Quick Check

1, 3, 5.

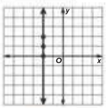


7. $y = -3x + 1$ **9.** $y = \frac{5}{2}x - 6$ **11.** $y = -10x + 6$

Lesson 4-1

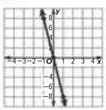
1.

x	y
−2	0
−2	1
−2	2



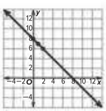
3.

x	y
−1	8
0	0
1	−8



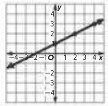
5.

x	y
0	8
1	7
2	6

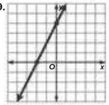


7.

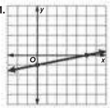
x	y
0	1
2	2
4	3



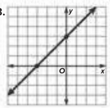
9.



11.

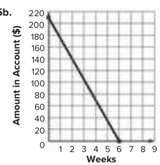


13.

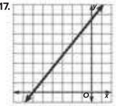


15a. The x-intercept is 6. This means that after 6 weeks, Amanda will have $0 in her school lunch account. The y-intercept is 210. This means that there was initially $210 in Amanda's school lunch account.

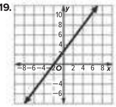
15b.



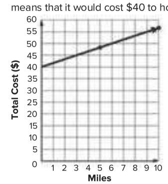
17.



19.



21. x-int: 7; y-int: −2
23. x-int: $1\frac{1}{3}$; y-int: 4
25. x-int: $-1\frac{1}{2}$; y-int: 1
27. $y = 1.7x + 40$; The y-intercept is 40. This means that it would cost $40 to hook up the car.



29. Sample answer: The x-intercept is −4. The x-intercept is not reasonable because the football team cannot lose −4 games. The y-intercept is 4. The y-intercept is reasonable because the y-intercept means that if the football team won 4 games, they lost 0 games. **31.** No; sample answer: A horizontal line only has a y-intercept and a vertical line only has an x-intercept. **33.** In the equation, let $y = 0$ to find the x-intercept: $2x + (0) = 4$. So the x-intercept is 2. In the equation, let $x = 0$ to find the y-intercept: $2(0) + y = 4$. So the y-intercept is 4. Robert graphed points at (2, 0) and (0, 4) and connected the points with a line. **35.** Sample answer: $y = 8$; horizontal line **37.** Sample answer: $x - y = 0$; line through (0, 0)

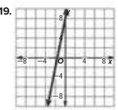
Lesson 4-2

1. $\frac{1}{5}$ **3.** increased about 1.9 people per square mile **5a.** −5; This means the temperature decreased 5°F per hour from 6 A.M. to 7 A.M. **5b.** −5; This means the temperature decreased 5°F per hour from 1 P.M. to 2 P.M. **7.** linear; $-\frac{1}{1}$ or −1 **9.** not linear **11.** $-\frac{3}{5}$ **13.** 1 **15.** 0 **17.** $\frac{1}{6}$ **19.** $\frac{4}{3}$ **21.** undefined **23.** 1 **25.** undefined **27.** 2 **29.** undefined **31.** −1 **33.** undefined **35.** $-\frac{7}{2}$ **37.** $\frac{5}{2}$ **39.** $\frac{3}{4}$ **41.** 6 **43.** 8 **45.** 11 **47.** $\frac{1}{20}$ **49.** $-\frac{1}{2}$ **51.** $\frac{1}{3}$ **53.** $\frac{1}{2}$ **55.** −1 **57.** $\frac{7}{4}$ **59.** 3 **61.** After drawing a graph, use the two points on the graph to determine the slope. This can be done by counting squares for the rise and run of the line or by using the coordinates of the points in the slope formula. **63.** The rate of change is $2\frac{1}{4}$ inches of growth per week. **65.** Step 1; she reversed the order of the x-coordinates in the formula. **67.** The difference in the x-values is always 0, and division by 0 is undefined.

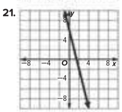
Lesson 4-3

1. $y = 5x - 3$ **3.** $y = -6x - 2$ **5.** $y = 3x + 2$ **7.** $y = x - 12$ **9.** $y = 5x + 6$ **11.** $y = \frac{1}{3}x - 2$ **13.** $y = -0.25x - 3$ **15.** $y = 25x + 100$ **17.** $y = 0.12x + 9$

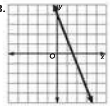
19.



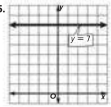
21.



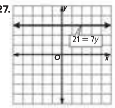
23.



25.

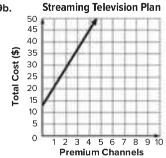


27.



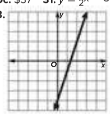
29a. $c = 13 + 8p$

29b.

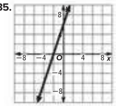


29c. \$37 **31.** $y = \frac{1}{2}x - 3$

33.



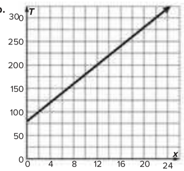
35.



37. $y = 2x - 3$ **39.** $y = -x - 1$

41a. $T = 10x + 80$

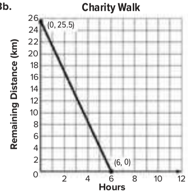
41b.



41c. 300°F **41d.** $T = -5x + 300$

43a. $y = -4.25x + 25.5$

43b.



43c. The x-intercept (6) represents the number of hours it will take Jazmin to complete the walk. The y-intercept (25.5) represents the length of the walk. **43d.** Using the graph, I can determine the value of x when y equals $-17 + 25.5$ or 8.5 km, and use the value of the x-intercept. The value of x is 4 when $y = 8.5$ and the x-intercept is 6. Therefore, Jazmin has $6 - 4$ or 2 hours more to walk. **45.** Yes; you can find the value of x on the graph when $y = 0$; $x = \frac{1}{2}$.

47. Sample answer: $y = 25x + 200$; I have \$200 in savings and will save \$25 per week until I have enough money to buy a new phone. I can predict how much money I'll have after x number of weeks.

Lesson 4-4

1. $g(x)$ is a translation of the parent function 11 units up **3.** $g(x)$ is a translation of the parent function 7 units right **5.** $g(x)$ is a translation of the parent function 10 units left and 1 unit down **7.** $g(x) = 4x$; $g(x)$ is the translation of $f(x)$ 3.5 units down. **9.** $g(h) = 8h + 15$; $g(h)$ is the translation $f(h)$ of 5 units up. **11.** $g(x)$ is a vertical compression of the parent function by a factor of $\frac{1}{3}$ **13.** $g(x)$ is a horizontal compression of the parent function by a factor of $\frac{1}{3}$ **15.** $g(x)$ is a horizontal stretch of the parent function by a factor of 2.5 **17.** $g(x)$ is a vertical stretch of the parent function by a factor of 8 and a reflection across the x-axis **19.** $g(x)$ is a horizontal stretch of the parent function by a factor of $\frac{5}{4}$ and a reflection across the y-axis **21.** $g(x)$ is a horizontal compression of the parent function by a factor of $\frac{2}{3}$ and a reflection across the y-axis **23.** $g(x)$ is a translation of the parent function 2 units right and 8 units down **25.** $g(x)$ is a vertical compression of the parent function by a factor of $\frac{1}{5}$ **27.** $g(x)$ is a horizontal compression of the parent function by a factor of 0.4 **29.** $g(x) = x - 7$ **31.** $g(x) = 1.5x$; The graph of $g(x) = 1.5x$ is the graph of $f(x) = 0.50x$ stretched vertically by a factor of 3. **33a.** $g(x) = 1.29x$ **33b.** The graph of $g(x) = 1.29x$ is the graph of $f(x) = x$ stretched vertically by a factor of 1.29. **35.** $y = \frac{1}{a}x$; The function is horizontally stretched by a factor of a.

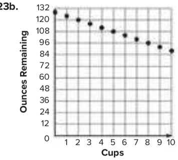
Lesson 4-5

1. This sequence has a common difference of 4 between its terms. This is an arithmetic sequence. **3.** This sequence does not have a common difference between its terms. This is not an arithmetic sequence. **5.** This sequence does not have a common difference between its terms. This is not an arithmetic sequence.

7. This sequence has a common difference of 3 between its terms. This is an arithmetic sequence. **9.** 1.06; 4.26, 5.32, 6.38 **11.** −2; 13, 11, 9 **13.** $\frac{1}{3}$; $3\frac{2}{3}$, 4, $4\frac{1}{3}$ **15.** 4; 19, 23, 27 **17.** 2; −5, −3, −1 **19.** $a_n = -5n + 2$; −33 **21.** $a_n = -4n - 7$; −35

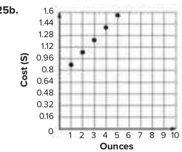
23a. $f(n) = -4n + 128$

23b.



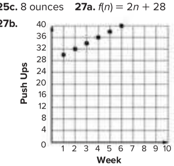
23c. 72 ounces **25a.** $f(n) = 0.17n + 0.71$

25b.



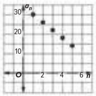
25c. 8 ounces **27a.** $f(n) = 2n + 28$

27b.



27c. 11th week **29.** This sequence does not have a common difference between its terms. This is not an arithmetic sequence.
31. This sequence has a common difference of 2 between its terms. This is an arithmetic sequence.

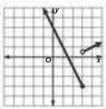
33. $a_n = -4n + 34$



35a. $a_n = 3000 + 500n$ **35b.** $15,000 **37a.** $a_n = 3n - 1$ **37b.** 59 **39.** Sample answer: 5, 3, 8, 6, 11, 9, 14, ...; The pattern is to subtract 2 from the first term to find the second term, then add 5 to the second term to find the third term. **41.** Sample answer: 2, −8, −18, −28, ... **43a.** Sample answer: $a_n = -2 - 3n$ **43b.** $a_n = -19 + 7n$ **43c.** $a_n = 12 - 2n$ **45.** On day 9, Andre has read 270 pages while Sam has 270 pages left to read. The table shows that both functions have a value of 270 when $x = 9$.

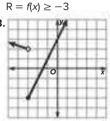
Lesson 4-6

1.



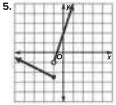
D = all real numbers,
R = $f(x) \geq -3$

3.



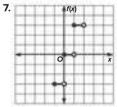
D = all real numbers,
R = $f(x) \geq -3$

5.



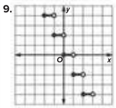
D = all real numbers,
R = $f(x) \geq -2.5$

7.



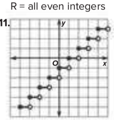
D = all real numbers,
R = all integer multiples of 3

9.



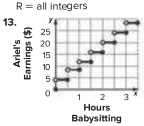
D = all real numbers,
R = all even integers

11.



D = all real number,
R = all integers

13.



15.

$$f(x) = \begin{cases} 16.20 \text{ if } 0 < x \leq 1 \\ 19.30 \text{ if } 1 < x \leq 2 \\ 22.40 \text{ if } 2 < x \leq 3 \\ 25.50 \text{ if } 3 < x \leq 4 \\ 28.60 \text{ if } 4 < x \leq 5 \end{cases}$$

D = {x | 0 < x < 5};
R = {16.20, 19.30, 22.40, 25.50, 28.60}

17. $g(x) = \begin{cases} 2x + 1 & \text{if } x \leq 2 \\ x - 2 & \text{if } x > 2 \end{cases}$

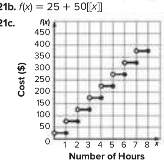
19. $133.00

21a.

x	0	2	4	6	8
f(x)	0	75	175	275	375

21b. $f(x) = 25 + 50[[x]]$

21c.



21d. $2 < x \leq 3$

23. Sample answer:

$$y = \begin{cases} -x & x < -4 \\ 2x & -4 \leq x \leq 2 \\ x & x > 2 \end{cases}$$

25. A step function has different constants over different intervals of its domain. A piecewise-defined function can have different algebraic rules over different intervals of its domain.

27. $f(x) = \begin{cases} \frac{1}{2}x - 3 & x > 6 \\ -\frac{1}{2}x + 3 & x \leq 6 \end{cases}$

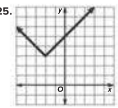
29. R = $f(x) \geq 0$

31. 2.4

Lesson 4-7

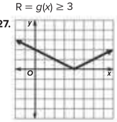
1. The graph of $g(x)$ is the parent function translated 5 units down. **3.** The graph of $g(x)$ is the parent function translated 2 units right and 7 units up. **5.** The graph of $g(x)$ is the parent function translated 1 unit up. **7.** $f(x) = |x + 2|$ **9.** $f(x) = |x| - 3$ **11.** $f(x) = |x| + 1$ **13.** The graph of $g(x)$ is a horizontal compression of the parent function. **15.** The graph of $g(x)$ is a vertical stretch of the parent function. **17.** The graph of $g(x)$ is a horizontal stretch of a parent function. **19.** The graph of $g(x)$ is a reflection of the parent function across the x-axis and a vertical stretch. **21.** The graph of $g(x)$ is a reflection of the parent function across the y-axis and a horizontal stretch. **23.** The graph of $g(x)$ is a reflection of the parent function across the y-axis and a horizontal compression.

25.



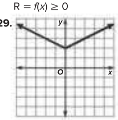
D = all real numbers,
R = $g(x) \geq 3$

27.



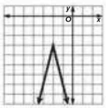
D = all real numbers,
R = $f(x) \geq 0$

29.

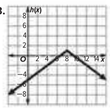


D = all real numbers,
R = $f(x) \geq 2$

Selected Answers

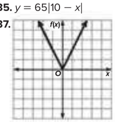
31. 

D = all real numbers,
R = $f(x) \le -3$

33. 

D = all real numbers,
R = $h(x) \le 1$

35. $y = 65|10 - x|$

37. 

D = all real numbers,
R = $f(x) \ge 0$
The graph of $f(x)$ is the parent function horizontally compressed by a factor of $\frac{1}{2}$.

39. $f(x) = |-3x - 5|$ 41. $f(x) = \left|\frac{1}{3}x + 2\right|$

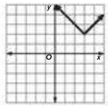
43. $x = |s - 16|$

45. $x = |t - 21.7|$; The range of times is twice the value of x, 3.2(2) = 6.4 s; The solution to the equation is 24.9 and 18.5, which has a range of 24.9 − 18.5 = 6.4 s.

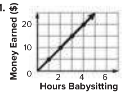
47. $x = |b - 12|$

49. To get the graph of $h(x)$, the parent absolute value function is reflected in the x-axis, then translated 2 units left and 3 units down.

51. $f(x) = \begin{cases} -x + 5 & \text{if } x < 3 \\ x - 1 & \text{if } x \ge 3 \end{cases}$



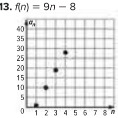
Module 4 Review

1. 

3. −500 gallons/hr 5. B 7. A

9. dilation 11. C

13. $f(n) = 9n - 8$



15.

Hours Worked, x	Money Earned, $f(x)$
30	270
35	315
40	360
45	427.5
50	495

17. B

19. Sample answer: It is translated 5 units up.

21. $f(x) = -|x - 4| + 3$

Module 5

Quick Check

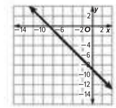
1. $y = 5 - x$ 3. $y = x + 5$ 5. (4, 2) 7. (2, −4)
9. (−3, −3)

Lesson 5-1

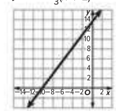
1. $y = \frac{1}{2}x$ 3. $-\frac{3}{4}x + \frac{17}{2}$ 5. $y = \frac{1}{2}x + 1$
7. $d = 3t + 12$ 9. $C = 2.54y + 62.38$
11. $y = -4$ 13. $y = \frac{4}{3}x - \frac{1}{3}$ 15. $y = -\frac{3}{2}x - \frac{9}{2}$
17. $y = -\frac{4}{11}x + \frac{58}{11}$ 19. $y = -\frac{1}{2}x - \frac{9}{2}$
21. $y = \frac{1}{6}x + \frac{19}{24}$ 23. $C = 10d + 12$
25. $T = -4.5x + 103$ 27. $y = 3x - 1$
29. $y = -x - 4$ 31. $y = -x + 3$ 33. No; substituting 3 and −1 for x and y results in an equation that is not true. 35. Yes; substituting 15 and −13 for x and y results in an equation that is true. 37. Sample answer: (3, −3)
39. Sample answer: (0, −5) 41. Sample answer: (0, 4) 43. C; x represents the number of plane tickets per order and y represents the total cost of an order. 45. A; x represents the number of hours and y represents the oil level in the tank, in inches. 47a. $y = x + 2.5$
47b. 10 47c. $y = x + 1.5$ 49a. $y = 7.5x + 1$
49b. 1; Koby's puppy weighed 1 pound at birth (0 months) 49c. 7.5; Koby's puppy gained 7.5 pounds a month for the first 6 months.
51. Jacinta; Tess switched the x- and y-coordinates on the point that she entered in Step 3. 53. Sample answer: Let y represent the number of quarts of water in a pitcher, and let x represent the time in seconds that water is pouring from the pitcher. As time increase by 1 second, the amount of water in the pitcher decrease by $\frac{1}{2}$ qt. An equation is $y = -\frac{1}{2}x + 4$. The slope is the rate at which the water is leaving the pitcher, $\frac{1}{2}$ quart per second. The y-intercept represents the amount of water in the pitcher when it is full, 4 qt.

Lesson 5-2

1. $y + 3 = -1(x + 6)$



3. $y - 11 = \frac{4}{3}(x + 2)$



5. Sample answer: $y + 3 = -4(x - 1)$
7. Sample answer: $y - 3 = \frac{4}{3}(x - 3)$
9. $y = -6x - 47$ 11. $y = \frac{1}{6}x - \frac{8}{3}$
13. $y - 18 = 3.5(x - 5)$ 15. $2x - y = 6$
17. $x + 6y = -7$ 19. $x - y = -1$
21. Sample answer: $2x + 3y = -13$
23. Sample answer: $3x + y = -3$
25. Sample answer: $y = x - 5$; $y = -x + 1$
27. Sample answer: $y = -5x + 2$; $y = \frac{1}{5}x + 2$
29. Sample answer:
$y = -\frac{3}{4}x + \frac{3}{2}$; $y = \frac{4}{3}x + \frac{17}{3}$
31. neither
33. perpendicular 35. neither
37. Sample answer: $5x + 4y = 20$
39. $y = 9x + 5$; $9x - y = -5$
41. $y = -6x - 45$; $6x + y = -45$
43. $y = \frac{9}{10}x - 4\frac{3}{10}$; $9x - 10y = 43$
45. Yes; sample answer: The line that represents one of the ceiling walls has a slope of $-\frac{1}{4}$ and the line that represents the other ceiling wall has a slope of 4.
47a. Sample answer: $y - 0 = 0.5(x - 0)$
47b. $y = 0.5x$ 47c. $x - 2y = 0$
49. Sample answer: You need to know the slope of the line and the y-intercept of the line, the slope and the coordinates of another point on the line, or the coordinates of two points on the line.

51. No; the line through (7, −10) and (3, −2) has a slope of −2 and 2x − y = −5 has a slope of 2.

53. Sample answer: $y - g = \frac{j-g}{h-f}(x - f)$

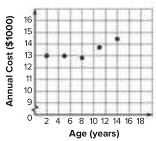
55. Sample answer: Jocari spent $18 to go to a carnival and play games. The price she paid included admission. The games cost $2 each; $y - 18 = 2(x - 5)$, $y = 2x + 8$.

Lesson 5-3

1. Positive; as time spent exercising increases, the more Calories are burned. 3. Negative; as weight increases, the number of repetitions decreases. 5a. $y = -328.275x + 3142.15$

5b. about 187.675 million

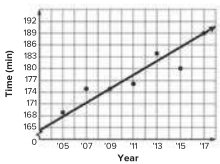
7a. There is a positive correlation between the child's age and annual cost.



7b. $y = 270x + 10{,}640$ 7c. about $15,230

9. no correlation

11a.

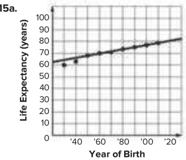


11b. Sample answer: x represents the number of years since 2005, so year 2005 is represented by x = 0 and year 2020 is represented by x = 15. Two points on the line of best fit are (4, 175) and (17, 189). Use these two points to find the slope to be 1.75 and the equation of the line of best fit to be $y = 1.75x + 168$. 11c. Sample answer: about 196 minutes 11d. Sample answer: Not all of the data points are close to the line of best fit, so there is not a consistent trend regarding the length of games. Therefore, the predicted game length may or may not be accurate.

13a. positive correlation; As the number of years since 2007 increases, the price of a ticket increases.

13b. $y = 2.87x + 64.24$ 13c. about $133.12

15a.

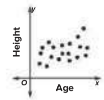


15b. Sample answer: About 81.8; The data show a positive correlation, so as the years increase, the life expectancy also increases. Therefore, the life expectancy should be higher than that of a baby born in 2010. 15c. Sample answer: I assumed that the trend continues, so as the year increases, the life expectancy also increases.

17. Sample answer: The salary of an individual and the years of experience that he or she has could be modeled using a scatter plot. This would be a positive correlation because the more experience an individual has, the higher the salary would likely be.

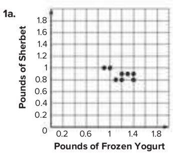
19. Neither; line g has the same number of points above the line and below the line. Line f is close to 2 of the points; but for the rest of the data, there are 3 points above and 3 points below the line.

21. Sample answer: You can visualize a line to determine whether the data has a positive or negative correlation. The graph shows the ages and heights of people. To predict a person's age given his or her height, write a linear equation for the line of fit. Then substitute the person's height and solve for the corresponding age. You can use the pattern in the scatter plot to make decisions.



Lesson 5-4

1a.

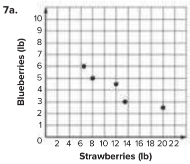


1b. Negative; as the number of pounds of frozen yogurt consumed increases, the number of pounds of sherbet consumed decreases.

1c. The relationship may be a causation. Since both are frozen desserts, eating more frozen yogurt may cause people to decrease the amount of sherbet they eat. Other things that might influence the data are an increase in frozen yogurt stores and a decrease in popularity or availability of sherbet.

3. Correlation, sample answer: Having a wider palm does not cause someone to watch less television. 5. Causation; sample answer: An increase in the price of cereal likely causes customers to buy less cereal.

7a.



7b. Negative; as the number of pounds of strawberries produced increases, the number of pounds of blueberries produced decreases.

7c. The relationship is a correlation, but not a causation. A better yield of strawberries does not cause the blueberries to grow poorly. Other factors, such as temperature and rain, could be affecting the plants that week.

9. positive correlation and causation; Sample answer: Because pizzas are topped with cheese, an increase in the number of pizzas made cause more cheese to be used.

11. Sample answer: Two elements can have a strong correlation, but it does not mean that one causes the other. There could be an unknown factor affecting the elements.

13. Sample answer: Correlation does not mean causation. Even though there is a strong correlation that does not mean buying swimsuits causes the use of air conditioners. Another factor, like the temperature, could be affecting both swimsuit sales and use of air conditioners.

Lesson 5-5

1a. $y = -1.31x + 50.95$ 1b. $r \approx -0.714$; The equation models the data fairly well. Its negative value means that as the years since 2010 increase, the total number of goals the soccer team scores each season decreases.

3a. $y = 8.52x + 3.18$

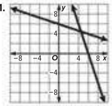
3b. $r \approx 0.999$; The equation models the data very well. Its positive value means that as the years since 2010 increase, sales, in millions of dollars, increase.

5a. $y = 103.77x + 108.06$ **5b.** about \$3221.16 **7a.** $y = 0.59x + 1.51$ **7b.** The residuals are randomly scattered and are centered about the line $y = 0$. So, the best-fit line models that data well. **9a.** $y = 0.26x + 21.21$
9b. $r \approx 0.359$; The equation does not model the data well. Its value means that as the years since the 2011–2012 school year increase, the percentage of students in public school who met all six of California's physical fitness standards each year varies. **9c.** Because the data on the students who meet all six standards is reported as a percentage, it cannot exceed 100. **11a.** $y = 140.4x + 13.8$
11b. $r \approx 0.999$; The equation models the data very well. Its positive value means that as the number of games increases, the cumulative number of yards increases.
11c. Sample answer: Because the data have a positive correlation, the total number of yards will increase and then number of games increases. So, the running back will have run for 950 yards between games 6 and 9.
11d. during game 7
13a. $y = 9619x + 443{,}918.8$ **13b.** $r \approx 0.999$; The equation models the data very well. Its positive value means that as the number of years since the 2010-2011 school year increases, the number of student athletes participating in college athletics each year increases.
13c. The residuals are randomly scattered and are centered about the line $y = 0$. So, the best-fit line models that data well.
13d. about 684,394
15. Apply a linear regression model to the data. Use the number of each test as the independent variable. If there is no correlation, the r-value will not be close enough to 1 or −1. If this is the case, the line of fit could not be used to predict the scores of the other students.
17a. $y = 84{,}345.0x + 5{,}003{,}868.3$
17b. about 7,365,528

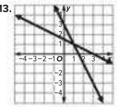
Lesson 5-6

1. {(−1, −9), (−4, −7), (−7, −5), (−10, −3), (−13, −1)}
3. {(−2, −4), (−1, −2), (1, 0), (0, 2), (2, 4)}
5. {(−3, 5), (−9, 2), (−15, −1), (−21, −4)}
7. {(16, −1), (12, −2), (8, −3), (4, −4)}
9. {(−49, −4), (35, 8), (−28, −1), (7, 4)}

11.



13.



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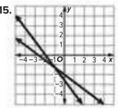
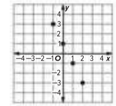


17. $f^{-1}(x) = \frac{x}{6} - 7$
19. $f^{-1}(x) = \frac{5}{2}(x + 16)$ **21.** $f^{-1}(x) = \frac{1 - 5x}{4}$
23a. $P^{-1}(x) = \frac{x + 36}{7.6}$ **23b.** x represents Alisha's profit and represents the number of dozens of brownies sold. **23c.** 5
25a. $C^{-1}(x) = \frac{x - 125}{16}$ **25b.** 108 feet
27. $f^{-1}(x) = \frac{1}{4}x + 6$ **29.** $f^{-1}(x) = 6x - 42$
31. $f^{-1}(x) = \frac{7}{2}x - 14$ **33.** $f^{-1}(x) = \frac{1}{7}x - \frac{6}{7}$
35. $f^{-1}(x) = 2x - 22$ **37.** B **39.** A
41. {(−k, b), (p, −g), (−m, −w), (q, r)} **43.** The slopes are reciprocals. For example, if the slope of one line is $\frac{2}{3}$, then the slope of the inverse function is $\frac{3}{2}$. **45.** Sample answer: This claim is incorrect. The −1 in the inverse function notation is not an exponent. As an example, the inverse function for $y = x + 1$ is found by switching x and y and solving for y, which gives $y = x - 1$. $y = x - 1$ is not the same as $y = \frac{1}{(x + 1)}$, which is not a line. This method does not work.

47. $a = 2$; $b = 14$
49. sometimes; Sample answer: $f(x)$ and $g(x)$ do not need to be inverse functions for $f(a) = b$ and $g(b) = a$. For example, if $f(x) = 2x + 10$, then $f(2) = 14$ and if $g(x) = x - 12$, then $g(14) = 2$, but $f(x)$ and $g(x)$ are not inverse functions. However, if $f(x)$ and $g(x)$ are inverse functions, then $f(a) = b$ and $g(b) = a$.
51. Sample answer: A situation may require substituting values for the dependent variable into a function. By finding the inverse of the function, the dependent variable becomes the independent variable. This makes the substitution an easier process.

Module 5 Review

1. A **3.** $y = 1.5x + 11$ **5.** A
7. $y - 4 = 2.5(x - 2)$ **9.** A
11. a positive correlation **13.** C
15. $f^{-1}(x) = -2x + 1$;



17. A

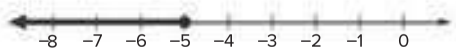
Module 6

Quick Check

1. 4 **3.** −2 **5.** {−29, 7} **7.** {−1, 15}

Lesson 6-1

1.



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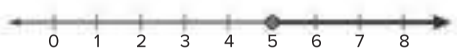
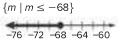


5.

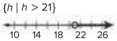
7. $t < -1$ **9.** $w < 5$ **11.** $b \geq -5$
13. $\{m \mid m < 7\}$ **15.** $\{r \mid r \leq 15\}$ **17.** $\{b \mid b \geq 2\}$
19. $\{c \mid c \leq -4\}$ **21.** $\{m \mid m \geq 4\}$
23. $\{r \mid r \geq 22\}$ **25.** $\{a \mid a \leq -4\}$
27. $\{w \mid w \geq -5\}$ **29.** $\{x \mid x \leq 5\}$
31. $\frac{3}{10}x \leq 4.50$, $x \leq \$15$ **33.** no more than 2.1 pounds per day **35.** at least 500 pieces
37. $\{m \mid m \leq -68\}$



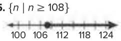
39. $\{c \mid c > 121\}$

41. $\{x \mid x \leq 20\}$

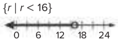
43. $\{h \mid h > 21\}$



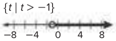
45. $\{n \mid n \geq 108\}$



47. $\{r \mid r < 16\}$

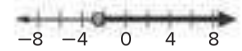


49. $\{t \mid t > -1\}$

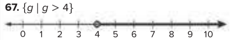


51. $\{z \mid z \geq 11\}$

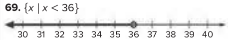
53. $\{d \mid d > -2\frac{1}{2}\}$



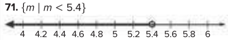
55. d **57.** a **59.** b
61. Sample answer: Let n = the number. $n + 7 \leq -18$; $\{n \mid n \leq -25\}$
63. Sample answer: Let n = the number. $n + 2 \leq 1$; $\{n \mid n \leq -1\}$
65. Sample answer: Let n = the number. $-12n \leq 84$; $\{n \mid n \geq -7\}$
67. $\{g \mid g > 4\}$



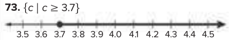
69. $\{x \mid x < 36\}$



71. $\{m \mid m < 5.4\}$



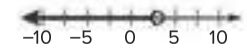
73. $\{c \mid c \geq 3.7\}$



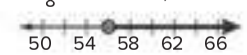
75. \$22.23.
77. Sample answer: Let x represent the decibel level of the calls of a blue whale; $x - 83 \leq 105$; $x \leq 188$. The calls of a blue whale are less than or equal to 188 decibels.
79. $-\frac{x}{2} < 1$ **81a.** $x < \frac{7}{a}$ **81b.** $x \geq \frac{12}{a}$
81c. $x > 3$ **81d.** $x \geq \frac{1}{4}$

Lesson 6-2

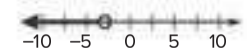
1a. $15 + 2h \leq 35$ **1b.** $h \leq 10$; 10 hours
3a. $1.50 + 0.25(5x - 1) \leq 3.75$ **3b.** $x \leq 2$; 2 mi
3c. Because the service charges per $\frac{1}{a}$ mile, multiply a by the number of miles, x, to find the number of $\frac{1}{a}$ miles. Subtract 1 from the total number of $\frac{1}{a}$ miles, ax, to find the number of additional $\frac{1}{a}$ miles. Multiply the difference by the cost per additional $\frac{1}{a}$ mile, \$0.25, and add the cost for the first $\frac{1}{a}$ mile, \$1.50. This sum is less than or equal to the total amount Demetri has in his pocket, \$3.75, so $1.50 + 0.25(ax - 1) \leq 3.75$. **5a.** $100 + 40x \leq 250$
5b. $x \leq 3.75$; 3 people
7. $21 > 15 + 2x$; $x < 3$



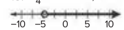
9. $\frac{x}{8} - 13 > -6$; $x > 56$



11. $37 < 7 - 10x$; $x < -3$

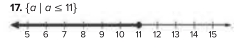


13. $-\frac{5}{4}x + 6 < 12$; $x > -\frac{24}{5}$

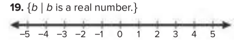


15. $15x + 30 < 10x - 45$; $x < -15$

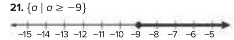
17. $\{a \mid a \leq 11\}$



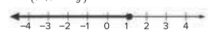
19. $\{b \mid b$ is a real number.$\}$



21. $\{a \mid a \geq -9\}$



23. $\{x \mid x \geq \frac{1}{2}\}$ **25.** $\{m \mid m \geq 18\}$
27. $\{w \mid w > -2\}$ **29.** $\{x \mid x \leq 8\}$
31. $\{x \mid x > -6\}$ **33.** $\{x \mid x \geq 1.5\}$
35. $\left\{p \mid p \leq 1\frac{1}{9}\right\}$

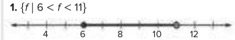


37a. $2x + 4 \leq 13$; $x \leq 4.5$
37b. 4.5 ft.
37c. 5ft.
39. Eric does not have any pencils. Based on his statement, the inequality is $6p + 15 < 20$, where p is the number of pencils. The solution of the inequality is $p < \frac{5}{6}$. However, the number of pencils must be a whole number, so $p = 0$.

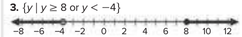
41. $10n - 7(n + 2) > 5n - 12$
(Original inequality)
$10n - 7n - 14 > 5n - 12$
(Distributive Property)
$3n - 14 > 5n - 12$
(Combine like terms.)
$3n - 14 - 5n > 5n - 12 - 5n$
(Subtract $5n$ from each side.)
$-2n - 14 > -12$
(Simplify.)
$-2n - 14 + 14 > -12 + 14$
(Add 14 to each side.)
$-2n > 2$
(Simplify.)
$\frac{-2n}{-2} < \frac{2}{-2}$
Divide each side by −2. Change > to <.
$n < -1$
(Simplify.)
The solution set is $\{n \mid n < -1\}$.
43. $\frac{78 + 80 + 78 + x}{4} \geq 82$; $x \geq 94$; Mei needs a score of at least 94 on the next exam.
45. Sample answer: $2(2x - 1) < 10$
47. Let c = the number of baseball cards Ted has; $4c > 5c - 15$; $15 > c$; Ted has fewer than 15 cards. **49.** Ø; If the inequality is always true, the opposite inequality will always be false.
51. Sample answer: The solution set for the inequality that results in a false statement is the empty set, as in $12 \leq -15$. The solution set for an inequality in which any value of x results in a true statement is all real numbers, as in $12 \leq 12$.

Lesson 6-3

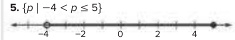
1. $\{f \mid 6 < f < 11\}$



3. $\{y \mid y \geq 8$ or $y < -4\}$



5. $\{p \mid -4 < p \leq 5\}$

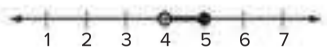


Selected Answers

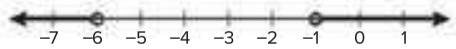
7. $\{h \mid 2 \le h < 3\}$

9. $\{y \mid y < -3\}$

11. $\{b \mid 4 < b \le 5\}$



13. $\{m \mid m < -6 \text{ or } m > -1\}$



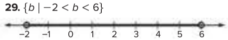
15. $\{m \mid 2 \le m < 4\}$

17a. $x + 8 < 20$ or $x + 8 > 35$ **17b.** $0 < x < 12$ or $x > 27$; Because the combined height of the sign and pole cannot be negative, the value of x must be greater than 0.

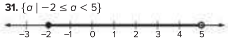
17c.

19. $-3 < x \le 3$ **21.** $x < -2$ or $x \ge 1$
23. $b > 3$ or $b \le 0$ **25.** $y < -1$ or $y \ge 1$
27. $f \mid -2 < f < -1\}$

29. $\{b \mid -2 < b < 6\}$



31. $\{a \mid -2 \le a < 5\}$



33. Sample answer: Let n = the number. $n - 2 \le 4$ or $n - 2 \ge 9$; $\{n \mid n \le 6 \text{ or } n \ge 11\}$
35. $54° \le x \le 68°$ **37.** The minimum is 67, since the solution of the inequality $2000 \le 1000 + 15x$ is $66\frac{2}{3} \le x$, and the number of students must be a whole number. The maximum is 100, since the solution of the inequality $1000 + 15x \le 3000$ is $x \le 133\frac{1}{3}$, but the bus can only hold 100 students. **39a.** The side lengths must be 5, x, and $9 - x$. Using the Triangle Inequality results in the compound inequality $x + 5 > 9 - x$ and $14 - x > x$.

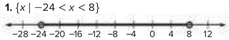
39b. The solution of the compound inequality is $2 < x < 7$, so each of the lengths must be greater than 2 m but less than 7 m. The sum of the two lengths must be 9 m.

41. $\{x \mid -2 < x < 5\}$

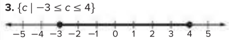
43a. $\$400 \le x \le \$800$ **43b.** $\$428 \le x \le \$856$
45. B
47. $x > -1$ or $x \ge 4$; This can be written as $x > -1$ because this is the union of two graphs.
49. The union of the two graphs is the graph on the left, so the graph on the left is the graph of the solution set for **Exercise 47.** The intersection of the two graphs is the graph on the right, so the graph on the right is the graph of the solution set for **Exercise 48.**
51a. $x > -\frac{4}{a}$ and $x \le \frac{4}{a}$
51b. $x < -\frac{6}{a}$ or $x > 5a$
53. Sometimes; The graph of $x > 2$ or $x < 5$ includes the entire number line.

Lesson 6-4

1. $\{x \mid -24 < x < 8\}$

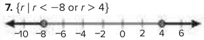


3. $\{c \mid -3 \le c \le 4\}$

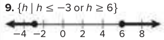


5. $\{\varnothing\}$

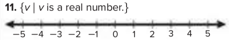
7. $\{r \mid r < -8 \text{ or } r > 4\}$



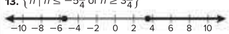
9. $\{h \mid h \le -3 \text{ or } h \ge 6\}$



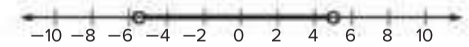
11. $\{v \mid v$ is a real number.$\}$



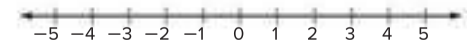
13. $\left\{n \mid n \le -5\frac{1}{4} \text{ or } n \ge 3\frac{3}{4}\right\}$



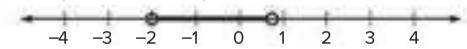
15. $\left\{h \mid -5\frac{2}{3} < h < 5\right\}$



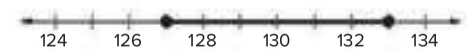
17. $\{\varnothing\}$



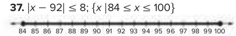
19. $\left\{r \mid -2 < r < \frac{2}{3}\right\}$



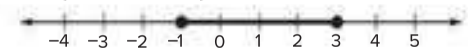
21. $\{x \mid 58.5 \le x \le 61.5\}$ **23a.** $|p - 130| \le 3.05$
23b.



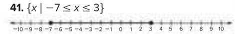
25a. $|x - 515| \le 114$ **25b.** 287 to 743
27. $|n + 2| \ge 1$ **29.** $|w - 2| < 2$ **31.** $|x| > 1$
33. d **35.** c
37. $|x - 92| \le 8$; $\{x \mid 84 \le x \le 100\}$



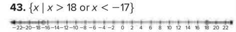
39. $\{x \mid -1 \le x \le 3\}$



41. $\{x \mid -7 \le x \le 3\}$

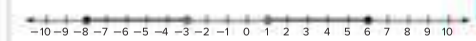


43. $\{x \mid x > 18 \text{ or } x < -17\}$



45. By definition, the absolute value is always greater than a negative number. Therefore, no matter what number is chosen, it will always be greater than −1 when evaluated in the absolute value inequality given. **47a.** Set the absolute value of an unknown variable, x, minus the recommended weight, 516, to be less than or equal to the variance of 4. So, the inequality $|x - 516| \le 4$ represents the situation.
47b. Write two inequalities, one for each case: $x - 516 \le 4$ and $-(x - 516) \le 4$. For the first case, add 516 to both sides: $x \le 520$. For the second case, distribute the negative on the left side, subtract 516 from both sides and divide by a negative 1 remembering to switch the inequality sign: $x \ge 512$. This means a box of cereal should have a minimum weight of 512 g and a maximum weight of 520 g.

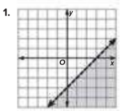
49. The solution set for $|x - 2| > 4$ is $\{x \mid x < -2 \text{ or } x > 6\}$. The solution set for $-2x < 4$ or $x > 6$ is $\{x \mid x > -2\}$. One includes numbers greater than −2, and the other includes numbers less than −2 or greater than 6. These solution sets are not the same.
51. Jordan is correct. Chloe did not distribute the negative to both x and 3.
53. $(-8 \le n < -3)$ or $(1 < n \le 6)$. To solve this compound inequality, split it into two inequalities. The first one to solve is $|n + 1| > 2$ and the second one is $|n + 1| \le 7$. The solution set of the entire problem is the overlap of the individual solutions.



55. No; Sample answer: Lucita forgot to change the direction of the inequality sign for the negative case of the absolute value.
57. Sample answer: If $t = 0$, then the absolute value is equal to 0, not greater than 0.
59. Sample answer: When an absolute value is on the left and the inequality symbol < or ≤, the compound sentence uses *and*, and if the inequality symbol is > or ≥, the compound sentence uses *or*. To solve, if $|x| < n$, then set up and solve the inequalities $x < n$ and $x > -n$, and if $|x| > n$, then set up and solve the inequalities $x > n$ or $x < -n$.

Lesson 6-5

1.



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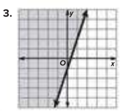

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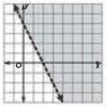

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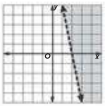

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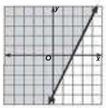

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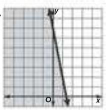
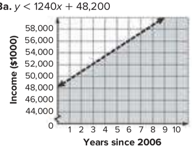
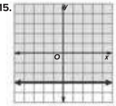


13a. $y < 1240x + 48{,}200$

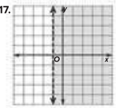


13b. no, no, yes, no

15.



17.



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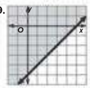

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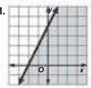
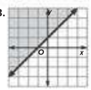
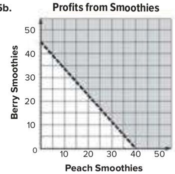


23.



25a. $2.25p + 2b > 90$; $p \geq 0$ and $b \geq 0$

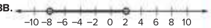
25b.



25c. The café sold more than 22 berry smoothies. **25d.** 41; Sample explanation: 40 peach smoothies results in a profit of exactly \$90, so to make a profit of more than \$90, the café must have sold 41 smoothies. **27.** The value of c must be positive. Since (0, 0) is a solution of the inequality, $a(0) + b(0) < c$ must be a true statement, so $0 < c$. **29.** Sample answer: $y < -x + 1$ **31.** Sample answer: The inequality $y > 10x + 45$ represents the cost of a monthly smartphone data plan with a one-time fee of \$45, plus \$10 per GB of data used. Both the domain and range are nonnegative real numbers because the GB used, and the total cost cannot be negative.

Module 6 Review

1. A
3. 8 rows
5. C
7. $\{t \mid t < -3\}$
9. A, B
11. $\{g \mid -5 \leq g\}$
13A. $\{h \mid -8 < h < 2\}$
13B.



15. B
17. C, D, B, A

Selected Answers

Selected Answers

Module 7

Quick Check

1. (4, 0) **3.** (0, 0) **5.** $x = 6 - 2y$ **7.** $m = 2n + 6$

Lesson 7-1

1. 1; consistent; independent

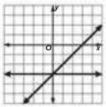
3. 0; inconsistent

5. 1; consistent; independent

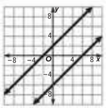
7. 1; consistent; independent

9. 1; consistent; independent

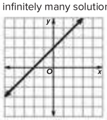
11. 1 solution; (0, −3)



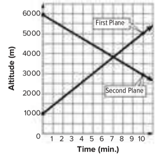
13. no solution



15. infinitely many solutions

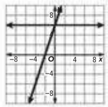


17a. $y = 400x + 1000$; $y = 5900 - 300x$

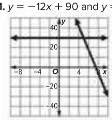


17b. After 7 minutes the planes will be at the same altitude.

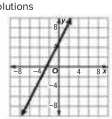
19. $y = 3x + 6$ and $y = 6$; (0, 6)



21. $y = -12x + 90$ and $y = 30$; (5, 30)



23. $y = 2x + 5$ and $y = 2x + 5$; infinitely many solutions

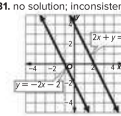


25. approximately (2.68, 1.01)

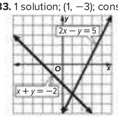
27. approximately (2.67, −0.88)

29. Sample answer: $x + y = 260$; $2.5x + 0.75y = 450$; approximately (145.71, 114.29); The bookstore will make a weekly profit of $450 with total weekly sales of 260 items when about 146 books and about 114 magazines are sold.

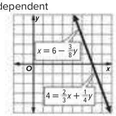
31. no solution; inconsistent



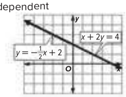
33. 1 solution; (1, −3); consistent; independent



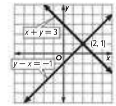
35. infinitely many solutions; consistent; dependent



37. infinitely many solutions; consistent; dependent

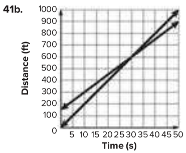


39. 1 solution; (2, 1); consistent; independent



41a. x = time, in seconds, y = distance from where Olivia started to the finish line, in feet; $y = 20x$; $y = 15x + 150$

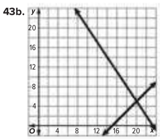
41b.



41c. 600 ft

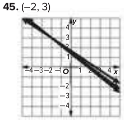
43a. x = time walking in minutes, y = time on bike in minutes; $3x + 2y = 70$, $x = y + 15$

43b.



43c. 20 minutes

45. (−2, 3)



47. Sample answer: $4x + 2y = 14$, $12x + 6y = 18$; This system is inconsistent, while the others are consistent and independent.

49. Graphing clearly shows whether a system of equations has one solution, no solution, or infinitely many solutions. However, finding the exact value of x and y from a graph can be difficult.

51. Francisca; If the item is less than \$100, then \$10 off is better. Of the item is more than \$100, then the 10% is better.

Lesson 7-2

1. (1, 6) **3.** (29, 53) **5.** (1, 1) **7.** infinitely many

9. no solution **11.** (0, 1) **13.** (2, 5)

15. infinitely many

17a. Sample answer: $a + b = 5$; $0.7a + 0.2b = 0.65(5)$

17b. 4.5 mL from Beaker A and 0.5 mL from Beaker B

19. $\left(\frac{1}{2}, -\frac{3}{8}\right)$

21. Sample answer: In 2011, the population of Ecuador was about 15,180,000 and the population of Chile was about 17,150,000. The population of Ecuador increased by 1,210,000 and the population of Chile increased by 760,000 from 2011 to 2016. Let x = the number of 5-year periods and y = population. The system is $y = 15{,}180{,}000 + 1{,}210{,}000x$ and $y = 17{,}150{,}000 + 760{,}000x$. Solve by substitution to find that $x \approx 4.4$, or $4.4 \times 5 = 22$ years. So, the population of Ecuador and Chile will be equal in about $2011 + 22 = 2033$. (Source: World Bank)

23. Let x = tens digit and y = units digit of the original number; $10y + x = 10x + y - 45$; $x = 3y + 1$; (7, 2); The original number is 72.

25. Neither; Guillermo substituted incorrectly for b. Cara solved correctly for b, but misinterpreted the pounds of apples bought.

27. Sample answer: The solutions found by each of these methods should be the same. However, it may be necessary to estimate when using a graph. So, when a precise solution is needed, you should use substitution.

29. An equation containing a variable with a coefficient of 1 can easily be solved for the variable. That expression can then be substituted into the second equation for the variable.

Lesson 7-3

1. (−3, 4) **3.** (−3, 1) **5.** (4, −2) **7.** (8, −7)

9. (4, 7) **11.** (4, 1.5) **13.** (2, 1) **15.** (11, 0)

17. (−3, 7) **19.** (2, −1) **21.** (−3, −5)

23. (10, 4) **25.** (7, 5) **27.** (2, −3)

29. −2 and −4

31a. $r + s = 181$ and $r - s = 119$

31b. 31 state senators and 150 state representatives

33. (4, −1) **35.** $\left(-1, 3\frac{1}{3}\right)$

37. (−36, −4) **39.** 34 games

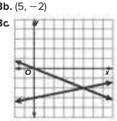
41a. Sample answer: $4p + 2n = 18.50$, $7p + 2n = 26.75$, where p is the price of a bag of popcorn and n is the price of a plate of nachos

41b. (2.75, 3.75); A bag of popcorn costs \$2.75 and a plate of nachos costs \$3.75.

43a. Add the equations because this will eliminate the variable y, and then you can solve for x.

43b. (5, −2)

43c.



Sample answer: The point of intersection on the graph will match the solution (5, −2).

43d. The equations would be equivalent. There would be infinitely many solutions, all real numbers x and y satisfying the equation $x - 5y = 15$.

43e. There would be no solution because the lines would be parallel and would never intersect.

45. Sample answer: $x + y = 1$ and $-x - y = 1$; This system of equations has no solutions.

47. Sample answer: $-x + y = 5$, I used the solution to create another equation with the coefficient of the x-term being opposite of its corresponding coefficient.

49. Sample answer: It would be most beneficial to use elimination to solve a system of equations when one variable has either the same coefficient in both equations or one variable has coefficients that are additive inverses in the equations.

Lesson 7-4

1. (−1, 3) **3.** (−3, 4) **5.** (−2, 3) **7.** (3, 5)

9. (1, −5) **11.** (0, 1)

13a. $2x + y = 592.30$ and $x + 2y = 691.31$, where x is the number of MLB games and y is the number of NBA games

13b. MLB: \$164.43, NBA: \$263.44

15. 8 and −1

17. wash: \$6, vacuum: \$2

19a.

	Tropical Breeze	Kona Cooler	Total
Amount of Juice (qt)	t	k	10
Amount of Pineapple Juice (qt)	$0.2t$	$0.5k$	4

19b. $\left(3\frac{1}{3}, 6\frac{2}{3}\right)$; The owner should mix $3\frac{1}{3}$ qt of Tropical Breeze and $6\frac{2}{3}$ qt of Kona Cooler.

19c. $3\frac{1}{3}$ qt + $6\frac{2}{3}$ qt = 10 qt, so the total amount is correct, and $0.2\left(3\frac{1}{3}\text{ qt}\right) + 0.5\left(6\frac{2}{3}\text{ qt}\right) = 4$ qt, so the amount of pineapple juice in the new drink is correct.

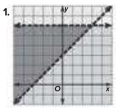
21. Jason; In order to eliminate the t-terms, you can multiply the second equation by 2 and then subtract, or multiply the equation by −2 and then add. Daniela did not subtract the equations correctly.

23. Sample answer: $2x + 3y = 6$ and $4x + 9y = 5$

25. Sample answer: It is more helpful to use substitution when one of the variables has a coefficient of 1 or if a coefficient can be reduced to 1 without turning other coefficients into fractions. Otherwise, elimination is more helpful because it will avoid the use of fractions when solving the system.

Lesson 7-5

1.



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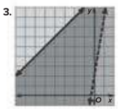
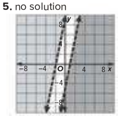


5. no solution



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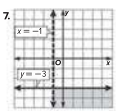
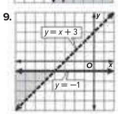
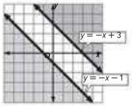


9.



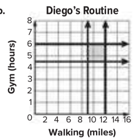
Selected Answers

11. no solution



13a. Sample answer: Let x = hours at gym and y = miles of walking; $x \geq 9$, $x \leq 12$, $y \geq 4.5$, $y \leq 6$

13b.



13c. Sample answers: gym 5 h, walk 9 mi; gym 6 h, walk 10 mi, gym 5.5 h, walk 11 mi

15. $y \leq x + 2$, $y \geq x - 3$

17. $y \geq x + 1$, $y < 1$

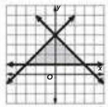
19. The solution set is the region where the graphs of the inequalities overlap. The point (2.5, 1) is not in the overlapping region, so it is not a solution. A solution must make all of the inequalities in the system true statements: $4x - 5y \geq 2 \rightarrow 4(2.5) - 5(1) \geq 2 \rightarrow 10 - 5 \geq 2 \rightarrow 5 \geq 2$; $2x + 3y > 8 \rightarrow 2(2.5) + 3(1) > 8 \rightarrow 5 + 3 > 8 \rightarrow 8 > 8$; The first inequality is true, but the second inequality is false. So, (2.5, 1) is not a solution.

21. Let x = tins of popcorn and y = tins of peanuts; $x + y \leq 200$; $x \geq y$; $3x + 4y \leq 900$; $x \geq 0$ and $y \geq 0$.

23. Sample answer: (3, 3)

25. Sometimes; sample answer: $y > 3$, $y < -3$ will have no solution, but $y < -3$, $y < 3$ will have solutions.

27. 9 units^2



Module 7 Review

1. B, C, D **3.** B

5. 4 wooden frames and 3 plastic frames

7. one solution; (−2, 7)

9. (10, 4)

11. A, B **13.** $r = 6$, $t = 5$ **15.** D **17.** C

Module 8

Quick Check

1. −196

3. 0.25

5. 32

7. −3

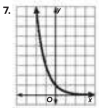
Lesson 8-1

1. No; the domain values are at regular intervals and the range values have a common difference 3.

3. Yes; the domain values are at regular intervals and the range values have a common factor 2.

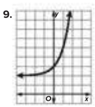
5. No; there is no common factor between the picture areas.

7.



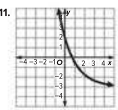
1; D = {all real numbers}, R = $\{y \mid y > 0\}$

9.



3; D = {all real numbers}, R = $\{y \mid y > 2\}$

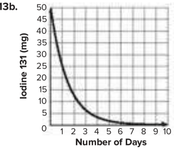
11.



2; D = {all real numbers}, R = $\{y \mid y > -3\}$

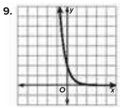
13a. y-intercept = 50; D = {all real numbers}, R = $\{y \mid y > 0\}$

13b.



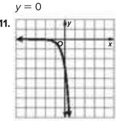
13c. Because time cannot be negative, the relevant domain is $\{x \mid x \geq 0\}$. Because the amount of Iodine 131 cannot be negative, and the amount when $x = 0$ is 50 mg, the relevant range is $\{y \mid 0 < y \leq 50\}$.

9.



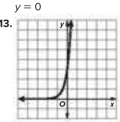
2; D = {all real numbers}, R = $\{y \mid y > 0\}$; $y = 0$

11.



−3; D = {all real numbers}, R = $\{y \mid y < 0\}$; $y = 0$

13.



3; D = {all real numbers}, R = $\{y \mid y > 0\}$; $y = 0$

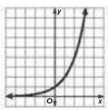
15a. 1038 millibars

15b. about 794 millibars

15c. It decreases.

17. $f(x) = 3(2^x)$

19. Sample answer: The number of teams competing in a basketball tournament can be represented by $y = 2^x$, where the number of teams competing is y and the number of rounds is x. The y-intercept of the graph is 1. The graph increases rapidly for $x > 0$. With an exponential model, each team that joins the tournament will play all of the other teams. If the scenario were modeled with a linear function, each team that joined would play a fixed number of teams.



Lesson 8-2

1. translated up 8 units **3.** compressed horizontally **5.** reflected across the x-axis; translated 1 unit right **7.** reflected across the y-axis; translated 4 units up **9.** stretched vertically **11.** translated right 3 units **13.** $y = -2^x$ **15.** $y = 2^{-x} + 5$ **17.** stretched vertically by a factor of 2000 **19.** stretched vertically by a factor of 20 **21.** translated up 6 units **23.** reflected across the x-axis; compressed vertically **25.** reflected across the y-axis **27.** $g(x) = 2^x + 3$

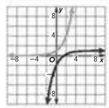
29. $g(x) = 5^{x-2}$ **31.** $g(x) = 6^x + 5$

33. $g(x) = \frac{1}{2}(4^x)$ **35.** $g(x) = 2^{3x}$

37. $g(x) = 5^x - 2$ **39.** $g(x) = 5^{x-4}$

41a. translated up 500 units **41b.** $500

43. The graph has been reflected over the x-axis and reflected over the y-axis. It has been stretched vertically by a factor of 3 and shifted up 1 unit.



45. The graphs of these two exponential functions are the same. $f(x) = 4^{x+2} = 4^x \cdot 4^2 = 16 \cdot 4^x = g(x)$.

47. Jennifer is correct. Sample answer: As it is written, the function is multiplied by 2, which causes the graph to rise more rapidly than the parent graph, so Jennifer is correct. However, $g(x) = 2(2^x)$ is equivalent to $g(x) = 2^{x+1}$. This graph is the parent graph of $f(x) = 2^x$ shifted to the left one unit, but it still rises at the same rate.

49. The first pair; $g(x)$ is shifted right 3 units instead of left 3 units.

Lesson 8-3

1. $y = 4 \cdot 2^x$ **3.** $y = 10 \cdot 3^x$ **5.** $y = 3 \cdot 4^x$
7. $y = 3 \cdot 2^x$ **9.** $y = \left(\frac{1}{4}\right)^x$ **11.** $f(x) = 50 \cdot 2^x$, where x is the number of 30-minute time periods **13.** $f(x) = 43 \cdot (1.23)^x$, where x is the number of years since 2010.
15a. $P = 8{,}192{,}426(1.009)^t$ **15b.** about 9,370,872
17a. $Z = 60{,}000(0.90)^t$ **17b.** about $31,886

19. $2200 **21.** 360 million
23. $y = 2.6 \cdot 4^x$ **25.** about 77,529

27. Sample answer: The equation can be rewritten in the form $y = a(1 + r)^x$ to find the amount of original investment, a, and the rate of increase or decrease. Because $a = 2400$, he invested $2400. Because $1 + r = 0.95$ and is less than 1, his investment is decreasing in value. A graphing calculator can be used to find that the investment will be worth $1200 in about 13.5 years.

29a. $P(t) = 128(1.25)^t$

29b. an increase of approximately 41 deer per year **29c.** No; the amount of increase is exponential, not linear. **29d.** There is no common difference over equal intervals (differences are 32, 40 and 50). There is a common factor (factor is 1.25 in each case.)

31. about 9.2 years **33.** Sample answer: Exponential models can grow without bound, which is usually not the case for the situation that is being modeled. For instance, a population cannot grow without bound due to space and food constraints. Therefore, the situation that is being modeled should be carefully considered when used to make decisions.

35a. Sample answer: 5%; about 14.2 years

35b. Sample answer: 10%; about 6.6 years

35c. Sample answer: about 10.4 years; about $8320

Lesson 8-4

1a. $A(t) = (1.021)^t$; $A(t) = (1.0052)^{4t}$

1b. Bank B has the better plan because the effective quarterly interest rate is 0.8%, which is greater than the quarterly interest rate of about 0.52% for Bank A.

1c. About 3.2%; sample answer: This confirms the result of part **b** because 3.2% is greater than the annual interest rate at Bank A, so Bank B has the better plan.

3. Bank A; Bank A has a quarterly interest rate of 0.95%. Bank B has a quarterly interest rate of about 0.92%. Bank A's quarterly interest rate is higher.

5. Species B; the population of Species A is decreasing at a rate of about 0.25% per quarter. The population of Species B is decreasing at a rate of about 0.4% per quarter. The population of Species B is decreasing at a faster rate.

7. Plan A

9. Account A; Account A has a semi-annual interest rate of 2.3%. Account B has a semi-annual interest rate of about 2.1%. Account A's semi-annual interest rate is greater.

11. Account A; Account A has a monthly interest rate of 0.5%. Account B has a monthly interest rate of about 0.21%. Account A's monthly interest rate is greater.

13. $T(t) = 72 + 140(0.67)^t$

15. Sample answer: Bank A offers a savings account with a 0.6% interest rate compounded quarterly. Bank B offers a savings account with a 2% interest rate compounded annually. Bank A offers the better interest rate because it has a higher effective annual interest rate of about 2.4%.

Lesson 8-5

1. The ratios are not the same, so the sequence is not geometric.

3. Since the ratio is the same for all of the terms, 5, the sequence is geometric.

5. The ratios are not the same, so the sequence is not geometric.

7. Because the ratio is the same for all of the terms, $\frac{1}{2}$, the sequence is geometric.

9. The ratios are not the same, so the sequence is not geometric.

11. The ratios are not the same, so the sequence is not geometric.

13. –250, 1250, –6250

15. 108, 324, 972

17. –2058; –14,406; –100,842

19. 54, 162, 486

21. $\frac{1}{10}, \frac{1}{20}, \frac{1}{40}$

23. $\frac{1}{3}, \frac{1}{18}, \frac{1}{108}$

25. 387,420,489

27. 177,147

29. $a_n = 4 \cdot \left(\frac{3}{2}\right)^{n-1}$

31. $1310.72

33a. $a_n = P \cdot 1.005^n$

33b. $538.84

35. $a_n = \frac{9}{16}\left(\frac{2}{3}\right)^{n-1}$; $\frac{4}{81}$

37. $a_n = -8\left(\frac{1}{4}\right)^{n-1}$; $-\frac{1}{2048}$

39. Sample answer: $a_n = 39{,}416(1.03)^{n-1}$; ≈ 69,116.19; This means that after 20 years of employment the average annual salary will be about $69,116.19.

41. −3, −12, −48

43a. The first method provides a starting salary of $100 and an $8 per month raise. The second method provides a starting salary of $0.01 and doubles it each month.

43b. The first situation is linear because there is a common difference of $8. The equation is $y = 8x + 92$. The second situation is exponential because it is a geometric sequence with a common ratio of 2. The equation is $y = 0.01(2)^{x-1}$.

43c. Sample answer: As long as I do not need money immediately, I would use the second method. In the last month, I would make $y = 0.01(2)^{23} = \$83{,}886.08$ due to the fact that the payment is growing exponentially. In the last month, in the first method I would make $y = 8(24) + 92 = \$284$.

45. If the values fit a geometric sequence, then $r = \sqrt{\frac{540}{180}} = \sqrt{3}$. This would mean that the interior angles of a square would have a sum of $180\sqrt{3} \approx 312°$. Since the sum of the angles in a square is 360°, this is not a geometric sequence.

47. Neither; Haro calculated the exponent incorrectly. Matthew did not calculate $(-2)^8$ correctly.

49. Sample answer: When graphed, the terms of a geometric sequence lie on a curve that can be represented by an exponential function. They are different in that the domain of a geometric sequence is the set of natural numbers, while the domain of an exponential function is all real numbers. Thus, geometric sequences are discrete, while exponential functions are continuous.

51. Sample answer: In the geometric sequence 6, 3, 1.5, ..., the value of r is 0.5 and the absolute value of a_{n+1} will be closer to zero than the value of a_n.

Lesson 8-6

1. 23, 30, 37, 44, 51

3. 8, 20, 50, 125, 312.5

5. 13, −29, 55, −113, 223

7. $a_1 = 12$, $a_n = a_{n-1} - 13$, $n \geq 2$

9. $a_1 = 2$, $a_n = a_{n-1} + 9$, $n \geq 2$

11. $a_1 = 40$, $a_n = -1.5a_{n-1}$, $n \geq 2$

13. $a_1 = 3$, $a_n = a_{n-1} - 1$, $n \geq 2$

15. $a_1 = 2$, $a_n = a_{n-1} + 1$, $n \geq 2$

17. $a_1 = \frac{5}{2}$, $a_n = a_{n-1} - \frac{1}{2}$, $n \geq 2$

19a. 875, 1050, 1225, 1400, 1575

19b. $a_1 = 175$, $a_n = a_{n-1} + 175$, $n \geq 2$

19c. $a_n = 175n$

21a. $a_1 = 6$, $a_n = 0.9a_{n-1}$, $n \geq 2$

21b. $a_n = 6(0.9)^{n-1}$

23. $a_n = -12n + 10$

25. $a_1 = 45$, $a_n = a_{n-1} - 7$, $n \geq 2$

27. $a_1 = -11$, $a_n = a_{n-1} + 5$, $n \geq 2$

29. $a_n = 16(4)^{n-1}$

31. $a_n = 500(1.05)^{n-1}$

33. Ramon has 2 parents, 4 grandparents, 8 great-grandparents, and so on. We can write a geometric sequence to count the number of ancestors in a given generation. The recursive formula is $a_1 = 2$, $a_n = 2_{n-1}$, $n >= 2$. The explicit formula is $a_n = 2^n$. Ramon's claim is about the 8th generation back: $a_8 = 2^8 = 256$. Ramon is correct.

35a. Sample answer: B3 = B2 + B1 and C2 = B2 ÷ B1

35b. The ratio approaches a constant value of 1.618034.... For larger values of n, the Fibonacci numbers behave like a geometric sequence with a common ratio of 1.618034....

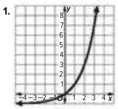
37. Both; Sample answer: The sequence can be written as the recursive formula $a_1 = 2$, $a_n = (-1)a_{n-1}$, $n \geq 2$. The sequence can also be written as the explicit formula $a_n = 2(-1)^{n-1}$.

39. False; sample answer: A recursive formula for the sequence 1, 2, 3, ... can be written as $a_1 = 1$, $a_n = a_{n-1} + 1$, $n \geq 2$ or as $a_1 = 1$, $a_2 = 2$, $a_n = a_{n-2} + 2$, $n \geq 3$.

41. Sample answer: In an explicit formula, the nth term a_n is given as a function of n. In a recursive formula, the nth term a_n is found by performing operations to one or more of the terms that precede it.

Module 8 Review

1.



3. As x increases, y increases; and, as x decreases, y approaches 0.

5. A

7. A

9. D

11. The Local Credit Union offers Joey the better savings plan; sample answer: the monthly interest rate is 0.12% higher than at First & Loan, and the annual interest rate is 1.6% higher than at First & Loan.

13. 146 people

15. $a_1 = 20$, $a_n = a_{n-1} + 15$, $n \geq 2$

17. row 2: 12; row 3: 12, 48; row 4: 48, 192

Module 9

Quick Check

1. 45.88 **3.** $3\frac{3}{20}$ **5.** 82.4% **7.** 85.6%

Lesson 9-1

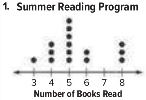
1. mean: 15.625, median: 15.5, mode: none
3. mean: 3.3, median: 2.5, mode: 2
5. mean: 5 students; median: 4 students; mode: 3 students **7.** mean: 54.75 mph; median: 54 mph; mode: 53 mph **9.** mean: about 2.8; median: 2.75; mode: 2 **11.** 25th percentile
13. Sample answer: The mean could be slightly higher because on a few of Saturday nights throughout the year, there were a very large number of people at the movies, which caused the mean to increase but did not affect the median. **15.** mean: 252; median: 245; mode: none **17.** 23 **19.** mean: 51.5, median: 51, mode: none **21.** 20 points **23.** 90th percentile **25a.** mean = 109,633; median = 66,556, no mode **25b.** Sample answer: The novels lower than the 50th percentile would be those consisting of words in the thirty-thousands and in the upper fifty-thousands. My prediction is correct because those three books are in the 47th percentile, which is just under the 50th percentile. **25c.** The median will change from 66,556 to 69,920, a difference of 3364 words. The mean will change from 109,633 to 111,065, a difference of 1432 words. **27.** Canada: 20th percentile; France: 50th percentile; Japan: 40th percentile; Russia: 60th percentile; Brazil: 10th percentile; Great Britain: 70th percentile

Olympic Medal Counts	
Country	**Total Medals**
Australia	29
Brazil	19
Canada	22
China	70
France	42
Great Britain	67
Japan	41
New Zealand	18
Russia	56
United States	121

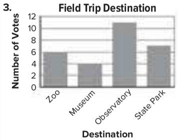
29. Sample answer: I can assume that the data is tightly clustered around 37 because all three measures of center are close. **31.** 75th percentile
33. Because the mean is an average of all the numbers in the data set, it is most affected by outliers. An outlier on the high end will cause the mean to increase. The median is the middle value in the dataset, adding one high number should not have much effect on the median unless the dataset has values, which are widely spread. The mode is the most frequent number so the outlier will have no effect on the mode unless the outlier is the same as the mode.
35. The mean, median, and mode will all be multiplied by the number. **37.** Julio should have chosen the mean because all the growth values are close together. **39.** Sample answer: To find a percentile rank, order the data set in decreasing order. Count the number of items below the item you are ranking, and divide that by the total number of items. Multiply this answer by 100 to arrive at the percentile rank.

Lesson 9-2

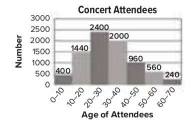
1.



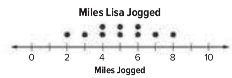
3.



5. histogram

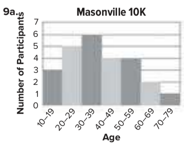


7a.



7b. 2 **7c.** 8

9a.



9b. 8 **9c.** 30-39
11. Sample answer: The scientist should break down the data into increments of two-tenths starting at 1 and going through 2.6.
13. Sample answer: 1) Because the data is clustered around ratings 7-10, it can be concluded that the product is well-liked by most customers and may have minor inconsistencies that certain people did not like. 2) Because there are only two low ratings, it can be concluded that dissatisfaction with the product could be a result of personal preference or manufacturer defect in a specific item.
15. Sample answer: If the range of the data is broad with specific, unrepeating values, then it makes the dot plot more meaningful if the range is divided up into equal intervals.
17. Sample answer: Bar graphs and histograms are similar because each displays data with bars. They are different because a bar graph is best used with data that are discrete and a histogram represents data that are continuous. For this reason, the bars in a bar graph do not touch and represent single values while the bars in a histogram touch and represent a range of values.

Lesson 9-3

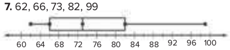
1. Sample answer: The intended population is all students. By asking only students leaving basketball practice, Awan is not getting a representative example of the entire student body. **3.** Sample answer: The first sentence states a positive outcome of music education, which may bias the respondent toward support. This bias may serve people trying to keep music education in schools. **5.** Mean: 4, median: 4, mode: 2; The mean and median are appropriate measures to use to accurately summarize the data. **7.** Sample answer: The scale for vendor 1 starts at 70, and because of the size of the bars, it looks like their sales doubled in one year, when they increased about 50%. Vendor 2 had a larger increase in sales of approximately 67%.
9. Sample answer: The required class would be better because it is more likely to contain a representative sample of students. The elective class might not be representative of the whole student body because these courses are chosen for reasons such as personal preference or future career aspirations.
11. Median; sample answer: The two lowest weights are much lower than the others, so the mean will be affected by those outliers.
13. Sample answer: The original data are very close together, so it is likely that the measures of center will all be the same or very close. Adding an outlier of 24 to the data set will cause the mean to go up, but the median and mode would likely stay unchanged or very close to the original number. So, in this case the median or mode would best represent the center of data.

15. Sample answer: To assess a sample for bias, identify the intended population and sample method; then, based on this information assess whether there is potential sample bias. **21.** There are more extreme values in the lower end, which cause the mean to be lower than the median.

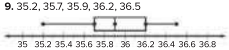
Lesson 9-4

1. 41 **3.** 62 **5.** 20

7. 62, 66, 73, 82, 99

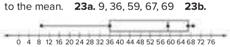


9. 35.2, 35.7, 35.9, 36.2, 36.5

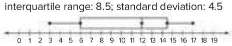


11. 25 **13.** 13 **15.** 20 **17.** 2.83 **19.** 2.97

21. 2.16; Since the standard deviation is large compared to the mean of 3, the number of goals scored each game is not relatively close to the mean. **23a.** 9, 36, 59, 67, 69 **23b.**



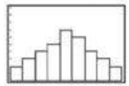
25. range: 14; minimum: 3; lower quartile: 6; median: 12; upper quartile: 14.5; maximum: 17; interquartile range: 8.5; standard deviation: 4.5



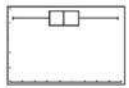
27. Both; sample answer; When an outlier is removed from a set of data, the spread and standard deviation of the data will decrease. When more values that are equal to the mean of a data set are added to the data set, the mean will be stronger and outliers will have less influence.

Lesson 9-5

1. symmetric

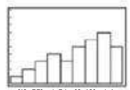


[24, 78] scl: 6 by [0, 10] scl: 1



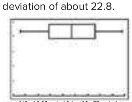
[24, 78] scl: 6 by [0, 5] scl: 1

3. Sample answer: The distribution is skewed, so use the five-number summary. The range is 53 – 12, or 41. The median is 39.5, and half of the data are between 28 and 48.



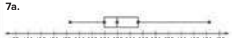
[10, 55] scl: 5 by [0, 10] scl: 1

5. Sample answer: The distribution is symmetric, so use the mean and standard deviation. The mean is about 58.7 with standard deviation of about 22.8.



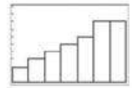
[10, 100] scl: 10 by [0, 5] scl: 1

7a.

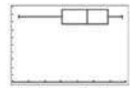


7b. min: 182, Q_1: 249, median: 274, Q_3: 315.5, max: 455 **7c.** The outlier mainly affects the mean. When the outlier is removed, the median decreases, but only $3 to $271. However, the mean changes from $289 to $266, which is more representative of the data as a whole.

9. negatively skewed

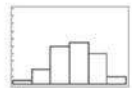


[10, 80] scl: 10 by [0, 10] scl: 1



[10, 80] scl: 10 by [0, 10] scl: 1

11. symmetric



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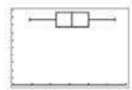
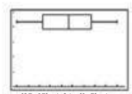


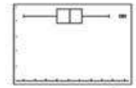
[10, 70] scl: 10 by [0, 20] scl: 2

13. symmetric



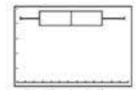
[35, 65] scl: 3 by [0, 5] scl: 1

15. Sample answer: The distribution is approximately symmetric, so use the mean and standard deviation. The mean is about 54.7 years with standard deviation of about 6.2 years.



[40, 70] scl: 3 by [0, 5] scl: 1

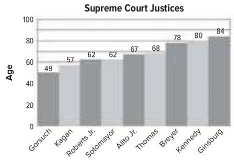
17a. Sample answer: The distribution is skewed, so use the five-number summary. min: 62, max: 525, med: 103, Q1: 84, Q3: 290 **17b.** Sample answer: The distribution is symmetric, so use the mean and standard deviation. The mean is about 92.4 with standard deviation of about 18.4.



[60, 125] scl: 5 by [0, 5] scl: 1

17c. Original: mean 171.5, median 103; altered: mean about 92.4, median 92. The means differ by about 79.1, while the medians differ by 11. **19a.** Sample answer: 225–230 g would be a reasonable advertised weight for either brand, so it is quite likely that they have the same advertised weight. Rafaello appears to have better control over the exact quantity in each package because its distribution is grouped more closely about the mean. **19b.** Sample answer: Both distributions have an inverted, symmetric U-shape with “tails” on either side. Leonardo’s distribution is lower and wider. **21.** Currently, Gerardo’s distribution would be positively skewed. If he lost his longest streaks, the data would represent a symmetric distribution.

23a. negatively skewed

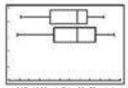


23b. The data is skewed, so use the five-number summary; min: 49, Q_1: 59.5, median: 67, Q_3: 79, max: 84. **23c.** There are no outliers in the data. **25.** Sample answer: A bimodal distribution is a distribution of data that is characterized by having data divided into two clusters, thus producing two modes and having two peaks. The distribution can be described by summarizing the center and spread of each cluster of data. **27.** Sample answer: In a symmetrical distribution, the majority of the data are located near the center of the distribution. The mean of the distribution is also located near the center of the distribution. Therefore, the mean and standard deviation should be used to describe the data. In a skewed distribution, the majority of the data lie either on the right or left side of the distribution. Because the distribution has a tail or may have outliers, the mean is pulled away from the majority of the data. The median is less affected. Therefore, the five-number summary should be used to describe the data.

Lesson 9-6

1. 60.9; 60; 60; 14; 4.7 **3.** 22.5; 21.5; no mode; 24; 7.4 **5.** 36.8; 38; 12; 56; 20.0 **7.** 26.8; 27.2; 29.6; 10.4; 3.5

9a. both negatively skewed



9b. Sample answer: The distributions are skewed, so use the five-number summaries. The medians for both teams are 79. The upper quartile and maximum for the Marlins are 83.5 and 92. The upper quartile and maximum for the Cubs are 88 and 97. This means that the upper 50% of data for the Cubs is slightly higher than the upper 50% of data for the Marlins. Overall, we can conclude that the Cubs were slightly more successful than the Marlins during this time period. **11.** 93.5; 94.5; 97; 17; 5.1 **13.** 60; 62.5; 45; 50; 16.9 **15.** 25.9; 21.5; 17; 30; 10.3 **17.** 13.9; 14; 14.4; 1.4; 0.5 **19.** 75.8; 72; no mode; 48, 16.1 **21a.** 160.5; 166; no mode; 115; 33.9 **21b.** 216.5; 222; no mode; 115; 33.9 **23a.** 64.7, 66, 55, 46, 15.9 **23b.** 18.1, 18.9, 12.8, 25.6, 8.9

25a. Sample answer: The mean of Saeed's prices is $11.79, which is $0.80 more than his rival's mean price. The new prices come from subtracting $0.80 from each price, which will reduce the mean price to be the same as his rival's.

New Prices				
14.19	3.69	9.19	17.69	12.19
6.19	7.69	21.19	12.69	13.19
9.19	10.19	11.69	3.69	12.19

25b. Current prices: $\mu = 11.79$, $\sigma = 4.60$
New prices: $\mu = 10.99$, $\sigma = 4.60$ The mean has dropped by 0.8, but the standard deviation has remained constant **27.** Sample answer: Male students, $\bar{x} = 70.0$ in., $\sigma = 2.0$ in. Female students, $\bar{x} = 66.3$ in., $\sigma = 2.7$ in. Sample answer: For male students, the mean is 70.0 in., and the standard deviation is 2.0 in. For female students, the mean is 66.3 in., and the standard deviation is 2.7 in. On average, males are taller. However, because the standard deviation of males is smaller than that of females, the heights of females are more spread out.

29. Sample answer: Histograms show the frequency of values occurring within set intervals. This makes the shape of the distribution easy to recognize. However, no specific values of the data set can be identified from looking at the histogram, and the overall spread of the data can be difficult to determine. The box plot show the data divided into four sections. This aids when comparing the spread of one set of data to another. However, the box plots are limited because they cannot display the data any more specifically than showing it divided into four sections. **31.** $37,750 **33.** Sample answer: When two distributions are symmetric, determine how close the averages are and how spread out each set of data is. The mean and standard deviation are the best values to use for this comparison. When distributions are skewed, determine which direction the data is skewed and the degree to which the data is skewed. The mean and standard deviation cannot provide information in this regard, but get this information by comparing the range, quartiles, and medians found in the five-number summaries. So if one or both sets of data are skewed, it is best to compare their five-number summaries.

Lesson 9-7

1.

	Small	Large	Total
Cherry	35	20	55
Grape	25	15	40
Watermelon	15	15	30
Total	75	50	125

3. 30

5.

	Male	Female	Total
Spanish	22.5%	25%	47.5%
French	20%	15%	35%
German	7.5%	10%	17.5%
Total	50%	50%	100%

7. Sample answer: Most of the students are studying Spanish. **9.** Sample answer: Each conditional relative frequency represents the proportion of each candidate's support from each gender. **11.** 12

13.

	Male	Female	Total
Tree Swallow	5	7	12
Cardinal	5	10	15
Goldfinch	8	5	13
Total	18	22	40

15. 18

17.

	Sports or Clubs	No Sports or Clubs	Total
Freshmen	10%	12.5%	22.5%
Sophomores	12.5%	15%	27.5%
Juniors	10.6%	14.4%	25%
Seniors	11.9%	13.1%	25%
Total	45%	55%	100%

19. 55.6% **21.** 66 **23.** 100 **25.** 31 **27.** 38%

29.

Region	Apple	Sweet Potato	Pumpkin	Totals
West	77 ≈ 19.0%	4 ≈ 1.0%	13 ≈ 3.2%	94 ≈ 23.2%
Midwest	32 ≈ 7.9%	6 ≈ 1.5%	54 ≈ 13.3%	92 ≈ 22.7%
South	12 ≈ 3.0%	63 ≈ 15.6%	24 ≈ 5.9%	99 ≈ 24.4%
Northeast	92 ≈ 22.7%	2 ≈ 0.5%	26 ≈ 6.4%	120 ≈ 29.6%
Total	213 ≈ 52.6%	75 ≈ 18.5%	117 ≈ 28.9%	405 = 100%

31. Sample answer: The conditional relative frequencies based on pie preference give the probability of a person preferring a particular pie choice being from one of the U.S. regions. For example, there is an 84% probability that a person who prefers sweet potato pie is from the south.

Region	Apple	Sweet Potato	Pumpkin
West	36.2%	5.3%	11.1%
Midwest	15.0%	8.0%	46.2%
South	5.6%	84%	20.5%
Northeast	43.2%	2.7%	22.2%
Total	100%	100%	100%

33.

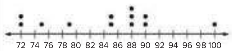
Vehicle Type	2WD	AWD	Totals
Hatchbacks	90	9	99
Sedans	60	13	73
SUVs	2	41	43
Total	152	63	215

35. Sample answer: Yes, there does appear to be an association. When the gasoline prices are higher, the distances traveled appear to be lower; when the gasoline prices are lower, the distances traveled appear to be higher.

37. Sample answer: A relative frequency is the ratio of the number in a category to the overall total of both categories. A conditional relative frequency is the ratio of the joint frequency to the marginal frequency. Therefore, it is important to understand what relationship is being analyzed because each two-way relative frequency table can provide two different conditional relative frequency tables.

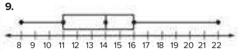
Module 9 Review

1.



3. C, D **5.** The data could be separated into intervals of 10, from 0–9, 10–19, 20–29, and so on through 70–79. **7.** scaled dot plot; histogram

9.

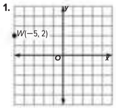


11. D **13.** C

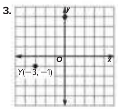
Module 10

Quick Check

1.



3.



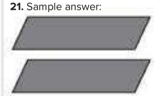
5. 20 **7.** 68

Lesson 10-1

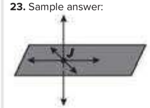
1. Sample answer: Kelsey's jersey number is greater than 5 and less than 11. Marie and Kelsey are on the same team. Kylie's team scored 26 points. **3.** Sample answer: Tom mulched the yard of all three clients this week. Ms. Martinez paid Tom $115 this week. Mr. Hansen paid Tom to mow his lawn and mulch his yard. Mrs. Johnson used all of Tom's services this week. **5.** analytic geometry **7.** synthetic geometry **9.** synthetic geometry **11.** Sample answer: Pedro and Rafael ate the same type of salad. **13.** Sample answer: Theo is likely doing analytic geometry, because he is using a graph with points. **15.** Sample answer: Because Sydney's plan is on a grid, she is likely using analytic geometry; that is, assuming the grid is used as a coordinate system. **17.** Sample answer: The three routes are not a model for the axiomatic system. Axioms 1, 2, and 4 are satisfied. Axiom 3 is not satisfied because Route 3 visits Stadium District twice and it is not the first/last stop. Axiom 5 is not satisfied because all three routes visit Stadium District. **19.** Sample answer: The rules of a game are like the axioms of an axiomatic system. They establish what can happen within the game. Plays are like theorems. They are tested against the rules or axioms to see whether they are legal in the game. In basketball, it is a rule that during playing time 5 players from each team shall be on the playing court. A play in which 6 players are on the court is a violation because the rules allow exactly 5 players. **21.** Sample answer: The second figure does not satisfy all the axioms. The axioms do not specify that the line segments connecting the points need to be straight, so the first and third figures would work. **23.** Sample answer: The triangle on the coordinate grid does not belong because it illustrates analytic geometry, while the other two figures illustrate synthetic geometry.

Lesson 10-2

1. Sample answer: n and q **3.** plane R **5.** Sample answer: point P **7.** Yes; sample answer: Line n intersects line q when the lines are extended. **9.** plane **11.** plane **13.** point on a line **15.** line **17.** plane **19.** plane **21.** Sample answer:

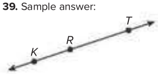


23. Sample answer:

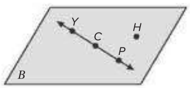


25. 5 **27.** A, B, E, F or B, C, D, E or A, C, D, F
29. $\overleftrightarrow{AB}, \overleftrightarrow{AG}, \overleftrightarrow{AH}, \overleftrightarrow{BC}, \overleftrightarrow{BH}, \overleftrightarrow{CD}, \overleftrightarrow{DE}, \overleftrightarrow{DH}, \overleftrightarrow{EF}, \overleftrightarrow{FG}, \overleftrightarrow{FH}$ **31a.** The lines intersect at the vanishing points. **31b.** Sample answer: The walls of the building and the ground form planes.
33. Sample answer:

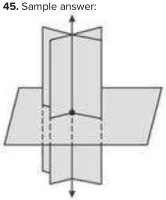
35. Sample answer: line q **37.** $\overleftrightarrow{CD}$ or $\overleftrightarrow{DC}$
39. Sample answer:



41. Sample answer:



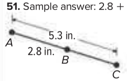
43. planes intersecting in lines
45. Sample answer:



47. Sample answer: Hiroshi is correct. After you draw the line from the first point to the other three, one of the lines from the second point is already drawn. **49.** Sample answer: A table is a finite plane. It is not possible to have a real-life object that is an infinite plane because all real-life objects have boundaries.

Lesson 10-3

1. 2.1 mm **3.** 1.1 cm **5.** 2.0 m **7.** $2\frac{1}{4}$ in. **9.** 5.3 mm **11.** $b = 12.5$; $YZ = 100$ **13.** $c = 1.7$; $YZ = 3.4$ **15.** $w = 4$; $YZ = 24$ **17.** $n = 4$; $YZ = 1$ **19.** $k = 6$; $YZ = 46$ **21.** $x = 6$; $YZ = 18$ **23.** $x = 10$; $YZ = 60$ **25.** 13 in. and 65 in. **27a.** 6 mi **27b.** Sample answer: I assumed the three locations were in a straight line. **29.** 4.4 mm **31.** 10.8 in. **33.** 66 units **35.** 3 **37.** 4
39. Sample answer: $AP + PM = AM$
41. 5184 ft **43.** $x = 3$; 13 mi **45.** 40 ft
47. $JK = 12$, $KL = 16$ **49.** $x = 3$; $y = 4$
51. Sample answer: $2.8 + BC = 5.3$; $BC = 2.5$ in.



Lesson 10-4

1. 5 **3.** 9 **5.** 12 **7.** 3 **9.** 3 **11.** 9 **13.** 6 **15.** yes **17.** no **19.** no **21.** 10 units
23. $\sqrt{89}$ or about 9.4 units
25. $\sqrt{20}$ or about 4.5 units **27.** $\sqrt{20}$ or approximately 4.5 units **29.** Yes; sample answer: The distance between Mariah's house and the library is $\sqrt{74}$ or about 8.6 miles. Because $\frac{2}{3}$ of 12 miles is 8 miles, Mariah's bike ride is more than $\frac{2}{3}$ of the cycling portion of the triathlon. **31.** $\sqrt{37}$ or about 6.1 units **33.** $\sqrt{29}$ or about 5.4 units **35.** $\sqrt{13}$ or about 3.6 units **37.** $\sqrt{89}$ or about 9.4 units **39.** $3\sqrt{37}$ or about 18.2 units **41.** $\sqrt{52}$ or about 7.2 units **43.** no **45.** (0, −4), (0, 10) **47.** 10 in. **49.** No; sample answer: We know that $QU + UR = QR = 4$ and $QU = UR$, so $QU = 2$. Further, we know that $RV + VS = RS = 2$, and $RV = VS$, so $RV = 1$. Because QU is not equal to RV, we know that $\overline{QU}$ is not congruent to $\overline{RV}$. **51.** (5, 10)

53. Sample answer: Substitute 10 for d, (1, 3) for (x_1, y_1), and (9, y) for (x_2, y_2) in the Distance Formula: $10 = \sqrt{(9-1)^2 + (y-3)^2}$. Solve for y:

$$100 = (9-1)^2 + (y-3)^2$$
$$= 8^2 + (y-3)^2$$
$$= 64 + (y-3)^2$$
$$36 = (y-3)^2$$
$$6 = y - 3 \text{ or } -6 = y - 3$$
$$9 = y \text{ or } -3 = y$$

So, the y-coordinate of point B is 9 or −3.

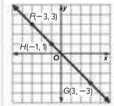
Lesson 10-5

1. 6 **3.** 9 **5.** 8.4 **7.** −1 **9.** −5.5 **11.** −4 **13.** −1 **15.** −2 **17.** Y **19.** −4 **21.** −3 **23a.** 4.36 mi **23b.** 10 mi **25.** 720 mi **27.** $2\frac{2}{5}$ in. **29.** Sample answer: Draw $\overline{AB}$. Next, draw a construction line and place point C on it. From C, strike 6 arcs in succession of length AB. On the sixth segment of length AB, perform a segment bisector two times to create a $\frac{1}{4}AB$ length. Label the endpoint D. **31.** Sometimes; sample answer: If the coordinate of X is 0 and the coordinate of Y is negative, then the coordinate of W will be negative and less than the coordinate of X. If the coordinate of X is positive and the coordinate of Y is greater than the coordinate of X, then the coordinate of W will be greater than the coordinate of X.

Lesson 10-6

1. (−3.6, −2.2) **3.** $\left(1, 1\frac{2}{3}\right)$ **5.** $\left(\frac{14}{3}, 1\right)$
7. $\left(-\frac{7}{5}, 4\right)$ **9.** $\left(\frac{16}{7}, -3\right)$ **11.** $\left(1, \frac{5}{4}\right)$
13. $\left(\frac{20}{7}, -1\right)$ **15.** (−7, 11) and (−1, 1)
17. (−3, −2) **19.** $\frac{3}{4}$ **21a.** Julianne substituted the wrong values for (x_1, y_1) and (x_2, y_2).
21b. (1.6, 4.2) **23.** Sample answer: Because the distance from A to P is twice the distance from P to D, the distance from A to P could be 2 and the distance from P to D could be 1. Therefore, the fractional distance that P is from A to D is $\frac{2}{2+1}$ or $\frac{2}{3}$. The coordinates of point P are (5, 10).

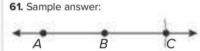
25. Sample answer:



H is $\frac{1}{3}$ of the distance from F to G.

Lesson 10-7

1. −2 **3.** 0.5 **5.** 1.5 **7.** 3 **9.** −4.5 **11.** 8.5 **13.** 9 **15.** 3 **17.** 9 yd **19.** (4, 6) **21.** (−3, −3) **23.** (−1.5, 3.5) **25.** (5, 4) **27.** (18.5, 5.5) **29.** (−6.5, −3) **31.** (4.2, 10.4) **33.** A(1, 6) **35.** C(16, −4) **37.** C(−12, 13.25) **39.** 16 **41.** 2.5 **43.** 8 **45.** 7 **47.** 0.5 **49.** Q(8, 4) **51.** N(−5, 1) **53.** (1, 1) **55.** (6, 7.5) **57.** 48 m **59.** Sample answer: The midpoint of a segment is the average of the coordinates of the endpoints. Divide each coordinate of the endpoint that is not located at the origin by 2. For example, if the segment has coordinates (0, 0) and (−10, 6), the midpoint is located at $\left(-\frac{10}{2}, \frac{6}{2}\right)$ or (−5, 3). Using the Midpoint Formula, if the endpoints of the segment are (0, 0) and (a, b), the midpoint is $\left(\frac{a+0}{2}, \frac{b+0}{2}\right)$ or $\left(\frac{a}{2}, \frac{b}{2}\right)$.
61. Sample answer:



Module 10 Review

1. B, D
3. intersecting planes
5. 11
7. A
9. B
11. D
13. (3, 6)
15. B
17. C

Module 11

Quick Check

1. 63 **3.** 3 **5.** 1980 **7.** 344

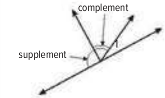
Lesson 11-1

1. *A* **3.** ∠*ADC*, ∠*CDA* **5.** 66° **7.** 78° **9.** 61° **11.** 56° **13.** Sample answer: ∠*SRQ* and ∠*TRP* **15.** 65 **17.** $x = 35, y = 85$ **19.** *R* **21.** *P* **23.** $\overrightarrow{NM}$ and $\overrightarrow{NV}$ **25.** $\overrightarrow{RP}$ and $\overrightarrow{RQ}$ or $\overrightarrow{RT}$ and $\overrightarrow{RQ}$ **27.** ∠*TPQ* **29.** ∠*TPN*, ∠*NPT*, ∠*TPM*, ∠*MPT* **31.** *S*, *Q* **33.** Sample answer: ∠*MPR*, ∠*PRQ* **35.** $x = 48, y = 21$ **37.** Sample answer: ∠*HGE*, ∠*DGE* **39.** Sample answer: ∠*BFC*, ∠*BFD* **41.** Sample answer: ∠*STV*, ∠*VTW*, ∠*UTW* **43a.** yes **43b.** no **43c.** no **43d.** no **45.** 45° **47a.** 45° **47b.** $\overrightarrow{EF}$ **49.** 168; sample answer: If $m\angle RMP = 21°$ and $\overrightarrow{MR}$ bisects ∠*QMP*, then $m\angle QMP = 2(21)$ or 42°. If $m\angle QMP = 42°$ and $\overrightarrow{MQ}$ bisects ∠*LMP*, then $m\angle LMP = 2(42)$ or 84°. If $m\angle LMP = 84°$ and $\overrightarrow{MP}$ bisects ∠*LMN*, then $m\angle LMN = 2(84) = 168°$.

Lesson 11-2

1. 72.5°, 107.5° **3.** 128°; 52° **5.** 45°; 135° **7.** $m\angle ABD = 47°$; $m\angle DBC = 43°$ **9.** $a = 20$ **11.** Yes; because ∠7 is a right angle, ∠6 and ∠8 must form a right angle. **13.** Yes; the angles are nonadjacent and are formed by two intersecting lines. **15.** 40° **17.** 7 **19.** 92° **21.** No; the measures of the angles are unknown. **23.** No; the angles are not adjacent. **25.** $x = 25$ **27.** $x = 94, y = 79, z = 26$ **29.** Yes; sample answer: Angles that are right or obtuse do not have complements because their measures are greater than or equal to 90°. **31.** Sample answer: You can determine whether an angle is right if it is marked with a right angle symbol, if the angle is a vertical pair with a right angle, or if the angle forms a linear pair with a right angle.

33. Sample answer:



35. No; sample answer: Straight angles or angles that are greater than 180° do not have supplementary angles because their measures are greater than or equal to 180°.

Lesson 11-3

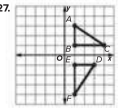
1. 20.9 cm; 16 cm² **3.** 17.8 cm; 25.1 cm² **5.** 11.3 cm; 8 cm² **7.** 15.7 in.; 19.6 in² **9.** 42.1 ft; 92.5 ft² **11.** 35.1 m² **13a.** 398 m **13b.** In part a, I assumed that there was no space between the field and the first lane of the track. I also assumed the athlete's body was centered on the border of the track. **13c.** 402 m **13d.** 30 cm **15.** quadrilateral; 20 units; 24 units 2 **17.** quadrilateral; 20 units 2; 36 units 2 **19.** $650; sample answer: The side of the play area that is adjacent to the house does not need fencing. The remaining three sides of the play area on the grid have lengths 4, 5, and 4 units. The perimeter of the play area on the grid is $P = 4 + 5 + 4 = 13$ units. Each unit on the grid represents 5 feet, so Derek will need 13(5 ft) or 65 ft of fencing. The cost of the fencing is $10 per foot, so the total cost will be 65($10) = $650. **21.** 78.5 square miles **23a.** $s = 4$, so $A = 4^2 = 16$ units² **23b.** $b = 2$, $h = \sqrt{3}$, so $A = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$ units² **23c.** 14.3 units²; sample answer: The area not covered by the triangle is equal to the area of the square minus the area of the triangle. So, $A = (16 - \sqrt{3})$ units², 14.3 units². **25.** 290.93 units² **27.** Sample answer: An equilateral triangle has congruent side lengths. Use the Distance Formula to find $KM = \sqrt{(10-(-2))^2 + (6-1)^2} = 13$. So, $P = 3(KM) = 3(13)$ or 39 units.

Lesson 11-4

1. reflection **3.** rotation **5.** translation **7.** rotation **9.** $A'(2, 0)$, $B'(-1, -5)$, and $C'(4, -3)$

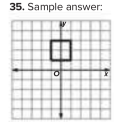
11. $A'(2, 2)$, $B'(-1, 7)$, and $C'(4, 5)$ **13.** $A'(-2, 0)$, $B'(1, -5)$, and $C'(-4, -3)$ **15.** $D'(4, 1)$, $E'(5, -2)$, and $F'(1, -2)$ **17.** $D'(5, -1)$, $E'(6, 2)$, and $F'(2, 2)$ **19.** $D'(-4, 1)$, $E'(-5, -2)$, and $F'(-1, -2)$ **21.** translation along vector ⟨−10, 35⟩ **23a.** rotation **23b.** translation **25.** reflection; translation

27.



$D(3, -1)$, $E(1, -1)$, $F(1, -4)$

29. reflection in the *x*-axis **31.** translation 3 units right and 2 units down **33.** Antwan; sample answer: When you reflect a point across the *x*-axis, the reflected point is in the same place horizontally, but not vertically. When (2, 3) is reflected across the *x*-axis, the coordinates of the reflected point are (2, −3) because it is in the same location horizontally, but the other side of the *x*-axis vertically. **35.** Sample answer:

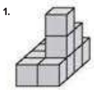


Lesson 11-5

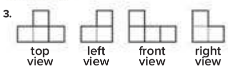
1. not a polyhedron; cone **3.** polyhedron; rectangular pyramid; base *WXYZ*; faces ▭*WXYZ*, △*VWX*, △*VXY*, △*VYZ*, △*VZW*; edges $\overline{WX}$, $\overline{XY}$, $\overline{YZ}$, $\overline{ZW}$, $\overline{WV}$, $\overline{XV}$, $\overline{YV}$, $\overline{ZV}$; vertices *W*, *X*, *Y*, *Z*, *V* **5.** sphere; not a polyhedron **7.** 26.9 cm²; 7.7 cm³ **9.** 800 ft²; 1280 ft³ **11.** 90π or about 282.7 yd²; 100π or about 314.2 yd³ **13a.** 35 ft² **13b.** 13 bags **15.** 2004 in² **17.** 3.9 cm **19.** 3 in. **21a.** 5027 ft³ **21b.** 402 ft³ **21c.** 5429 ft³ **23a.** 0.012 m³; $V = \frac{1}{3}Bh$, so $V = \frac{1}{3}(900)(40)$ or 12,000 cm³. One cubic meter equals 1 million cubic centimeters, so the volume of one pyramid shaped lawn ornament is 0.012 m³. **23b.** concrete: 28.5 kg; granite: 32.2 kg; marble 32.5 kg **23c.** If the volume of the lawn ornament stays the same, then the weight of the ornament increases as the density of the material used to make it increases. **25.** Neither; sample answer: The surface area is twice the sum of the areas of the top, front, and left side of the prism or 2(5 • 3 + 5 • 4 + 3 • 4), which is 94 square inches. **27.** Sample answer: The cone and pyramid have nearly the same volumes. Cone: The area of the base is approximately 154 square centimeters, so $V \approx \frac{1}{3}(154)(28)$ or about 1437 cubic centimeters. The volume of the pyramid is greater by such a small amount that we can say the volumes are approximately equal. **29.** 27 mm³

Lesson 11-6

1.



3.



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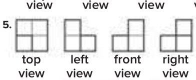
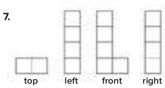
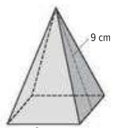


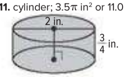
7.



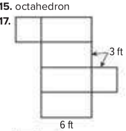
9. square pyramid, 144 cm^2



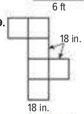
11. cylinder; 3.5π in^2 or 11.0 in^2



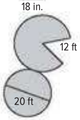
13. tetrahedron
15. octahedron
17.



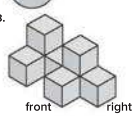
19.



21.



23.



25. Sample answer: ice cream cone

27. No; the area of this net is 48 square centimeters.

29. Julian; All of Julian's triangles are congruent. Caleb's top triangle will not match with the others when folded. **31.** Orthographic drawings and nets are both two-dimensional shapes used to describe three-dimensional figures. Orthographic drawings are views of the top, left, front, and right sides of an object, whereas nets can be folded to create a three-dimensional object.

Lesson 11-7

1. The sample is precise because there are consistently 17 or 18 rice cakes in each package. The sample indicates an innacurate claim of 20 rice cakes per package. **3.** 0.05 milliamp **5.** 1.1° **7a.** 16.5 yd **7b.** 12.5 yd **7c.** least possible area = 189.75 yd^2; greatest possible area = 218.75 yd^2 **7d.** The cost would be at least \$75.90 but less than \$87.50. **9.** 1 in. **11a.** 0.11 lb **11b.** 0.01 lb **11c.** 0.64 lb **13.** The cost would be at least \$955.94 but less than \$961.36. **15.** The cost would be at least \$107.81 but less than \$132.81. **17.** Accuracy is how well the information or data matches the true values. Precision is the repeatability of the measurement and level of measurement. Sample answer: 16.1 oz, 16.3 oz, 15.93 oz, 15.8 oz. **19a.** There are two faces that have an area of 111.4 in^2, two faces that have an area 35.82 in^2, and two faces that have an area 75.3 in^2. **19b.** 445.0 in^2 **19c.** The calculation of surface area is accurate to the nearest tenth. The true surface area falls between 444.95 in^2 and 445.05 in^2. **19d.** 440.1 in^2

Lesson 11-8

1. 5 **3.** 2 **5.** 1 **7.** Michelle **9.** 4.5 to 5.5 in. **11.** 2.5 to 3.5 cm **13.** 4.375 to 4.625 in. **15.** 2 **17.** 4 **19.** 2 **21.** 7.2 mL **23.** 15.1 cm^2 **25.** The area of the circle is between 31.2 and 33.2 cm^2. **27.** 4 **29.** 5 **31.** 5 **33.** 0.663 gram **35.** Scale 1: 4; Scale 2: 3; Scale 3: 4 **37.** 79.40 yd^2 **39.** 8.3 g **41.** 624 in^3

43. Sample answer: The 0 in each dimension, 1.40 cm and 1.60 cm, is significant. The answer should be given with 3 significant figures as 2.24 square centimeters. **45.** Yes; sample answer: The zeros before and after the decimal are not significant because a nonzero number did not come before them. Therefore, both numbers have two significant figures. **47.** Sometimes; sample answer: A zero between two nonzero significant figures is always significant, a leading zero is never significant, and a zero at the end of a number is only significant when a decimal point is given in the number.

Module 11 Review

1. A, C **3.** Sample answer: Use paddy paper or wax paper. Draw an angle on the paper. Fold the paper so that one side of the angle is directly on top of the other side. Draw the angle bisector in the crease of the fold. **5.** B **7.** Perimeter: 20.6 units; Area: 24.2 square units **9.** B **11.** C **13.** 2144.66 in^3 **15.** B

Module 12

Quick Check

1. 5 **3.** 2.3 **5.** $\angle BXD$, $\angle AXE$ **7.** 6

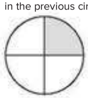
Lesson 12-1

1. Each term in the pattern is four more than the previous term; 24.

3. Each term is one half the previous term; $\frac{1}{16}$.

5. Each arrival time is 2 hours and 30 minutes prior to the previous arrival time; 7:30 A.M.

7. The shaded section in each circle has moved one section counterclockwise from its location in the previous circle.



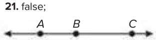
9. The product is an odd number.

11. They are equal. **13.** The lines are perpendicular.

15. $NP = PQ$ **17.** 180 cm

19. False; sample answer: Suppose $x = 2$, then $-x = -2$.

21. false;



23. False; sample answer: The length could be 4 m, and the width could be 5 m.

25. Yes; sample answer: If no team got more than 5 medals, then the total number of medals could not be more than 5×6 or 30 medals.

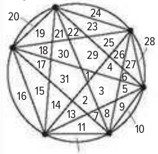
27a. 1, 3, 5 **27b.** You get all the odd numbers.

27c. Their sum equals n^2.

29. Sample answer: When $n = 4$, $2^n - 1 = 15$ and $15 = 3 \times 5$.

31a. Sample answer: The number of regions doubles when you add a point on the circle.

31b. For six points, there should be 32 regions; however only 31 regions are formed. The conjecture is false.



33. False; sample answer: If the two points create a straight angle that includes the third point, then the conjecture is true. If the two points do not create a straight angle with the third point, then the conjecture is false.

35. Sample answer: A postulate states that a line contains at least two points. These two points and all the points between them are line segments, by the definition of a line segment.

Lesson 12-2

1. $-3 - 2 = -5$, and vertical angles are congruent; p is true, and q is true, so p and q is true.

3. Vertical angles are congruent, or $2 + 8 \leq 10$; q is true, and $\sim r$ is true, so $q \vee \sim r$ is true.

5. $-3 - 2 \neq -5$, and not all vertical angles are congruent; $\sim p$ is false, and $\sim q$ is false, so $\sim p \wedge \sim q$ is false. **7.** H: there is no struggle; C: there is no progress **9.** H: you lead; C: I will follow **11.** H: two angles are vertical; C: they are congruent **13.** H: you were at the party; C: you received a gift; If you were at the party, then you received a gift. **15.** H: a figure is a circle; C: the area is πr^2; If a figure is a circle, then the area is πr^2. **17.** H: an angle is right; C: the angle measures 90°; If an angle is right, then the angle measures 90°. **19.** Converse: If you can buy five raffle tickets, then you have five dollars. The converse is true. Inverse: If you do not have five dollars, then you cannot buy five raffle tickets. The inverse is true.

Contrapositive: If you cannot buy five raffle tickets, then you do not have five dollars. The contrapositive is true.

21. Converse: If you do not take this medicine, then you can drive. The converse is true. Inverse: If you are not driving, then you can take this medicine. The inverse is true. Contrapositive: If you take this medicine, then you are not driving. The contrapositive is true.

23. Conditional: If it is raining, then the game will be cancelled. The conditional is true. Converse: If the game is cancelled, then it is raining. Counterexample: The game could be cancelled, and it is not raining. The converse is false. Because the converse is false, a biconditional statement cannot be written.

25. Conditional: If a polygon has four sides, then it is a quadrilateral. Converse: If a polygon is a quadrilateral, then it has four sides. The conditional and the converse are true, so the biconditional is true. **27.** true **29.** false

31. yes **33.** nothing **35.** If a ray does not bisect an angle, then it does not divide the angle into two congruent angles.

37. If two angles are right angles, then they are congruent.

39a. Sample answer: If you are in Houston, then you are in Texas. **39b.** Sample answer: Converse: If you are in Texas, then you are in Houston. Inverse: If you are not in Houston, then you are not in Texas. Contrapositive: If you are not in Texas, then you are not in Houston. **39c.** Converse: false; Inverse: false; Contrapositive: true

41. There exists at least one student at Hammond High school that does not have a locker.

43. For every real number x, $x^2 \neq x$. **45.** There exists a real number that does not have a real square root.

47. Truth table with the following columns:

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Because column 5 is the same as column 8, the conditional is equivalent to its contrapositive. Because column 6 is the same as column 7, the converse and the inverse are equivalent.

49. Kiri; sample answer: When the hypothesis of a conditional is false, the conditional is always true.

Lesson 12-3

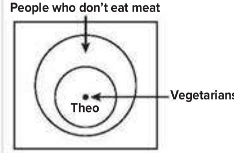
1. deductive **3.** deductive **5.** inductive

7. valid; Law of Detachment **9.** Invalid; your battery could be dead because it was old.

11. valid; Law of Detachment **13.** If Tina has a grade point average of 3.0 or greater, then she will have her name in the school paper.

15. If the measure of an angle is between 90° and 180°, then it is not acute. **17.** No valid conclusion; the conclusion of statement (1) is not the hypothesis of statement (2). **19.** The sum of the measures of $\angle 1$ and $\angle 2$ is 90°; Law of Detachment **21.** No valid conclusion; the conclusion of statement (1) is not the hypothesis of statement (2).

23. If Terryl completes a course with a grade of C, then he will have to take the course again; Law of Syllogism. **25.** Valid; Theo is inside the small and large circles, so the conclusion is valid.



27. then Mozart did not live in Vienna in the early 1800s. **29.** The child is at least 5 years old. **31.** Energy costs will be higher in Florida.

33. Law of Detachment: $[(p \rightarrow q) \wedge p] \rightarrow q$. Law of Syllogism: $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$.

35. Sample answer: Jonah's statement can be restated as, "Jonah is in Group B, and Janeka is in Group B." For this compound statement to be true, both parts of the statement must be true. If Jonah was in Group A, he would not be able to say that he is in Group B, because students in Group A must always tell the truth. Therefore, the statement that Jonah is in Group B is true. For the compound statement to be false, the statement that Janeka is in Group B must be false. Therefore, Jonah is in Group B, and Janeka is in Group A.

37. Sample answer: Given: If you are at the Willis Tower, then you are in Chicago. If you are in Chicago, then you are in Illinois. Conclusion: Therefore, if you are at the Willis Tower, then you are in Illinois.

39. D; Statement D follows logically from statements (1) and (2). Statements A, B, and C do not follow logically from statements (1) and (2).

Lesson 12-4

1. The two planes meet at the edge, which lies on line t. Postulate: If two planes intersect, then their intersection is a line.

3. Postulate 3.1; Through any two points, there is exactly one line.

5. Always; Postulate 3.2 states that through any three noncollinear points, there is exactly one plane.

7. Sometimes; the points do not have to be collinear to lie in a plane.

9. Never; Postulate 3.7 states that if two planes intersect, then their intersection is a line.

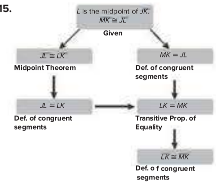
11. Statements (Reasons)

1. Y is the midpoint of $\overline{XZ}$. W is collinear with X, Y, and Z. Z is the midpoint of $\overline{YW}$. (Given)
2. $\overline{XY} \cong \overline{YZ}$ and $\overline{YZ} \cong \overline{ZW}$ (Midpoint Theorem)
3. $XY = YZ$ and $YZ = ZW$ (Definition of congruent segments)
4. $XY = ZW$ (Transitive Property of Equality)
5. $\overline{XY} \cong \overline{ZW}$ (Definition of congruent segments)

13. Statements (Reasons)

1. $SR = RT$ (Given)
2. R is the midpoint of $\overline{ST}$. (Definition of midpoint)
3. $SR = UR$ and $RT = RV$ (Given)
4. $SR = RT$, so $UR = RV$. (Substitution)
5. R is the midpoint of $\overline{UV}$. (Definition of midpoint)

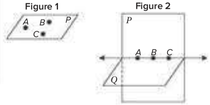
15.



17. Given: B is the midpoint of $\overline{AC}$. C is the midpoint of $\overline{BD}$.
Prove: $AB = CD$
Proof: Because B is the midpoint of $\overline{AC}$ and C is the midpoint of $\overline{BD}$, we know by the Midpoint Theorem that $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{CD}$. Because congruent segments have equal measures, $AB = BC$ and $BC = CD$. Thus, by the Transitive Property of Equality, $AB = CD$.

19. Postulate 3.3: A line contains at least two points. **21.** Iza is because a line can lie in a plane and intersect it in infinite points. **23.** 15 **25.** Ana is correct. Sample answer: The proof should begin with the given, which is that $\overline{AB}$ is congruent to $\overline{BD}$ and A, B, and D are collinear. Therefore, Ana began the proof correctly. **27a.** Plane Q is perpendicular to plane P. **27b.** Line a is perpendicular to plane P. **29.** Sometimes; if the points were noncollinear, then there would be exactly one plane by Postulate 3.2 shown by Figure 1. If the points were collinear, then there would be infinitely many planes. Figure 2 shows what two planes through collinear points would look like. More planes would rotate around the three points.

Figure 1 Figure 2



Lesson 12-5

1a. C is the midpoint of $\overline{AE}$.
C is the midpoint of $\overline{BD}$.
$\overline{AE} \cong \overline{BD}$

1b. Definition of midpoint **1c.** Definition of ≅ segments

1d. $AE = AC + CE$
$BD = BC + CD$

1e. Substitution Property **1f.** Substitution Property **1g.** $2AC = 2CD$ **1h.** $AC = CD$

1i. Definition of ≅ segments

3. Given: $\overline{VZ} \cong \overline{VY}$ and $\overline{WY} \cong \overline{XZ}$
Prove: $\overline{VW} \cong \overline{VX}$
Proof
Statements (Reasons)

1. $\overline{VZ} \cong \overline{VY}$ and $\overline{WY} \cong \overline{XZ}$ (Given)
2. $VZ = VY$ and $WY = XZ$ (Def. of ≅ segments)
3. $VZ = VX + XZ$ and $VY = VW + WY$ (Segment Addition Postulate)
4. $VX + XZ = VW + WY$ (Substitution Prop.)
5. $VX + WY = VW + WY$ (Substitution Prop.)
6. $VX = VW$ (Sub. Prop. of =)
7. $VW = VX$ (Symmetric Property)
8. $\overline{VW} \cong \overline{VX}$ (Def. of ≅ segments)

5. Clara is shorter than Chad when Maria is shorter than Luna; Clara and Chad are the same height when Maria is the same height as Luna.

7. No, it's not possible. Lola's house must be less than a mile from each house because she lives between them.

9a. Given: $\overline{AC} \cong \overline{GI}$, $\overline{FE} \cong \overline{LK}$, $AC + CF + FE = GI + IL + LK$
Prove: $\overline{CF} \cong \overline{IL}$
Proof
Statements (Reasons)

1. $\overline{AC} \cong \overline{GI}$, $\overline{FE} \cong \overline{LK}$, $AC + CF + FE = GI + IL + LK$ (Given)
2. $AC = GI$ and $FE = LK$ (Def. of ≅ segments)
3. $AC + CF + FE = AC + IL + LK$ (Substitution Property)
4. $AC - AC + CF + FE = AC - AC + IL + LK$ (Subtraction Property of Equality)
5. $CF + FE = IL + LK$ (Substitution Property)
6. $CF + FE = IL + FE$ (Substitution Property)
7. $CF + FE - FE = IL + FE - FE$ (Subtraction Property of Equality)
8. $CF = IL$ (Substitution Property)
9. $\overline{CF} \cong \overline{IL}$ (Def. of ≅ segments)

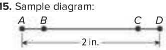
11a. Both segments are half the length of two congruent segments, so the lengths of the shorter segments must be the same.

11b.

1. $\overline{AB} \cong \overline{CD}$; M is the midpoint of $\overline{AB}$ and $\overline{CD}$.
2. Congruent segments have equal lengths.
3. Definition of midpoint
5. Segment Addition Postulate
6. Substitution Property of Equality
7. Substitution Property of Equality
8. Distributive Property
9. $AM = CM$
10. $\overline{AM} \cong \overline{CM}$

13. Neither; because $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{BF}$, then $\overline{AB} \cong \overline{BF}$ by the Transitive Property of Congruence.

15. Sample diagram:



17. No; The Segment Addition Postulate only applies to points that are collinear, but points P, Q, and R are not collinear.

19. Because $\overline{PQ} \cong \overline{RS}$ and congruent segments have equal lengths, $PQ = RS$. Because Q is the midpoint of $\overline{PR}$, $PQ = QR$. By the Substitution Property of Equality, $QR = RS$ so R is the midpoint of $\overline{QS}$.

Lesson 12-6

1. 113° **3.** 74° **5.** 46°

7. Statements (Reasons)

1. ∠1 and ∠2 form a linear pair.
∠3 and ∠4 form a linear pair. (Def. of linear pair)
2. ∠1 and ∠2 are supplementary.
∠3 and ∠4 are supplementary. (Supp. Thm)
3. ∠1 ≅ ∠3 (Given)
4. ∠2 ≅ ∠4 (≅ Supp. Thm)

9. Statements (Reasons)
1. $m\angle ABC = m\angle DEF$ (Given)
2. $\angle ABC \cong \angle DEF$ (Def. of ≅ angles)
3. $\angle ABC$ and $\angle DEF$ are supplementary. (Given)
4. $\angle ABC$ and $\angle DEF$ are rt. angles. (If two ∠s are ≅ and supp., then each ∠ is a rt. ∠.)

11. 36°; 84°

13. $m\angle 6 = m\angle 8 = 73°$, $m\angle 7 = 107°$ (≅ Supp. Thm. and Vert. ∠s Thm.)

15. $m\angle 3 = 62°$, $m\angle 1 = m\angle 4 = 45°$ (≅ Comp. and Supp. Thm.)

17. $m\angle 9 = 156°$, $m\angle 10 = 24°$ (≅ Supp. Thm.)

19. Statements (Reasons)
1. $\angle PQT$ and $\angle TQR$ form a linear pair. (Given)
2. $\angle PQR$ is a straight angle. (Given from figure)
3. $m\angle PQR = 180°$ (Def. of straight angle)
4. $m\angle PQT + m\angle TQR = m\angle PQR$ (Angle Add. Post.)
5. $m\angle PQT + m\angle TQR = 180°$ (Subs.)
6. $\angle PQT$ and $\angle TQR$ are supplementary. (Def. of supp. angles)

21. Statements (Reasons)
1. ∠1 and ∠2 are supplementary.
∠2 and ∠3 are supplementary. (Given)
2. $m\angle 1 + m\angle 2 = 180°$
$m\angle 2 + m\angle 3 = 180°$ (Def. of supp. angles)
3. $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$ (Subs.)
4. $m\angle 1 = m\angle 3$ (Subtraction Prop. of =)
5. $\angle 1 \cong \angle 3$ (Def. of ≅ angles)

23. Statements (Reasons)
1. $\ell \perp m$ (Given)
2. ∠1 is a right angle. (Def. of ⊥)
3. $m\angle 1 = 90°$ (Def. of rt. angles)
4. $\angle 1 \cong \angle 4$ (Vert. Angles Thm)
5. $m\angle 1 = m\angle 4$ (Def. of ≅ angles)
6. $m\angle 4 = 90°$ (Subs.)
7. ∠1 and ∠2 form a linear pair.
∠3 and ∠4 form a linear pair. (Def. of linear pairs)
8. $m\angle 1 + m\angle 2 = 180°$, $m\angle 4 + m\angle 3 = 180°$ (Linear pairs are supp.)
9. $90° + m\angle 2 = 180°$, $90° + m\angle 3 = 180°$ (Subs.)
10. $m\angle 2 = 90°$, $m\angle 3 = 90°$ (Subtraction)
11. ∠2, ∠3, and ∠4 are rt. angles. (Def. of rt. angles (Steps 6, 10))

25. Statements (Reasons)
1. $\angle 1 \cong \angle 2$, ∠1 and ∠2 are supplementary. (Given)
2. $m\angle 1 + m\angle 2 = 180°$ (Def. of supp. angles)
3. $m\angle 1 = m\angle 2$ (Def. of ≅ angles)
4. $m\angle 1 + m\angle 1 = 180°$ (Subs.)
5. $2(m\angle 1) = 180°$ (Subs.)
6. $m\angle 1 = 90°$ (Div. Prop.)
7. $m\angle 2 = 90°$ (Subs. (steps 3, 6))
8. ∠1 and ∠2 are rt. angles. (Def. of rt. angles)

27. By the Transitive Property, if any two angles are equal to the angle of the template, then they must be equal to each other.

29. Sample answer: $m\angle WXY = 90°$

Given: $m\angle WXZ = 45°$, $\angle WXZ \cong \angle YXZ$

Prove: $m\angle WXY = 90°$

Proof:

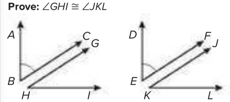
Statements (Reasons)
1. $m\angle WXZ = 45°$, $\angle WXZ \cong \angle YXZ$ (Given)
2. $m\angle WXZ = m\angle YXZ$ (Def. of ≅ ∠s)
3. $m\angle YXZ = 45°$ (Substitution)
4. $m\angle WXY = m\angle WXZ + m\angle YXZ$ (Angle Add. Post.)
5. $m\angle WXY = 45° + 45°$ (Substitution)
6. $m\angle WXY = 90°$ (Substitution)

31. Each of these theorems uses the words *or to congruent angles* indicating that this case of the theorem must also be proved true. The first proof of each theorem only addressed the *to the same angle* case of the theorem.

Proof of the Congruent Complements Theorem (Case 2: Congruent Angles)

Given: $\angle ABC \cong \angle DEF$, ∠GHI is complementary to ∠ABC, ∠JKL is complementary to ∠DEF.

Prove: $\angle GHI \cong \angle JKL$



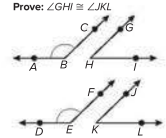
Proof:

Statements (Reasons)
1. $\angle ABC \cong \angle DEF$, ∠GHI is complementary to ∠ABC, ∠JKL is complementary to ∠DEF. (Given)
2. $m\angle ABC + m\angle GHI = 90°$,
$\angle DEF + \angle JKL = 90°$ (Def. of compl. angles)
3. $m\angle ABC = m\angle DEF$ (Def. of ≅ angles)
4. $m\angle ABC + m\angle JKL = 90°$(Subs.)
5. $90° = m\angle ABC + m\angle JKL$ (Symm. Prop.)
6. $m\angle ABC + m\angle GHI = m\angle ABC + m\angle JKL$ (Trans. Prop.)
7. $m\angle ABC - m\angle ABC + m\angle GHI =$ $m\angle ABC - m\angle ABC + m\angle JKL$ (Subt. Prop.)
8. $m\angle GHI = m\angle JKL$ (Subs.)
9. $\angle GHI \cong \angle JKL$ (Def. of ≅ angles)

Proof of the Congruent Supplements Theorem (Case 2: Congruent Angles)

Given: $\angle ABC \cong \angle DEF$, ∠GHI is supplementary to ∠ABC, ∠JKL is supplementary to ∠DEF.

Prove: $\angle GHI \cong \angle JKL$

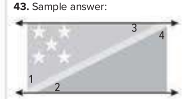


Proof:

Statements (Reasons)
1. $\angle ABC \cong \angle DEF$, ∠GHI is suppl. to ∠ABC, ∠JKL is suppl. to ∠DEF. (Given)
2. $m\angle ABC + m\angle GHI = 180°$,
$m\angle DEF + m\angle JKL = 180°$ (Def. of suppl. angles)
3. $m\angle ABC = m\angle DEF$ (Def. of ≅ angles)
4. $m\angle ABC + m\angle JKL = 180°$ (Subs.)
5. $180° = m\angle ABC + m\angle JKL$ (Symm. Property)
6. $m\angle ABC + m\angle GHI = m\angle ABC + m\angle JKL$ (Trans. Prop.)
7. $m\angle ABC - m\angle ABC + m\angle GHI =$ $m\angle ABC - m\angle ABC + m\angle JKL$ (Subt. Prop.)
8. $m\angle GHI = m\angle JKL$ (Subs.)
9. $\angle GHI \cong \angle JKL$ (Def. of ≅ angles)

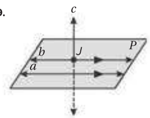
Lesson 12-7

1. $\overline{BF}$, $\overline{CG}$, and $\overline{DH}$ **3.** *ABCD* and *EFGH* or *ABFE* and *CDHG* **5.** $\overline{EF}$, $\overline{AB}$, and $\overline{DC}$
7. Sample answer: *ABCD* and *DCGH* could be characterized as perpendiculars, because *DCGH* contains segment $\overline{CG}$, which is perpendicular to *ABCD*. **9.** line *d*; alternate exterior **11.** line *d*; corresponding **13.** line *a*; consecutive interior **15.** line *c*; alternate interior **17.** line *d*; corresponding **19.** line *p*
21. 100° **23.** 80° **25.** 100° **27.** 170°
29. $x = 28$, $y = 47$; Use supplementary angles to find *x*. Then use alternate exterior angles to find *y*. **31.** $x = 12$, $y = 31$ **33.** 105°
35. 105° **37.** 75° **39.** 94°
41. 64°; sample answer: Opposite sides of a rectangle are parallel. So, the top and bottom lines on the side panel are parallel and cut by a transversal, which is the dashed line. Therefore, ∠1 and the 116°-angle are consecutive interior angles, so their sum is 180°. $m\angle 1 + 116° = 180°$, so $m\angle 1 = 64°$.
43. Sample answer:



∠1 and ∠4 are alternate interior angles. ∠2 and ∠3 are also alternate interior angles. ∠1 and ∠2 are complementary angles, and ∠3 and ∠4 are complementary angles.
45. By the Corresponding Angles Postulate, $\angle 1 \cong \angle 13$ and $\angle 13 \cong \angle 9$. By the Transitive Property, $\angle 1 \cong \angle 9$. So, $m\angle 1 = m\angle 9$. By the Corresponding Angles Postulate, $\angle 4 \cong \angle 8$ and $\angle 8 \cong \angle 12$. By the Transitive Property, $\angle 4 \cong \angle 12$. So, $m\angle 4 = m\angle 12$. It is given that $m\angle 1 - m\angle 4 = 25°$. By the Substitution Property, $m\angle 9 - m\angle 12 = 25°$.
47. By the Vertical Angles Theorem, $\angle 7 \cong \angle 5$. By the Corresponding Angles Theorem $\angle 5 \cong \angle 1$. By the Transitive Property, $\angle 1 \cong \angle 7$.

49.



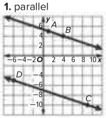
51. Sometimes; sample answer: $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{EF}$ depending on where the planes intersect.

53. $x = 171$ or $x = 155$; $y = 3$ or $y = 5$

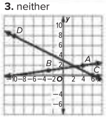
55. No; sample answer: From the definition of skew lines, the lines must not intersect and cannot be coplanar. Different planes cannot be coplanar, but they are always parallel or intersecting. Therefore, planes cannot be skew.

Lesson 12-8

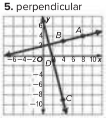
1. parallel



3. neither

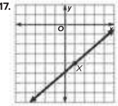


5. perpendicular

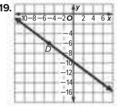


7. perpendicular **9.** parallel **11.** perpendicular **13.** neither **15.** parallel

17.



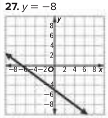
19.



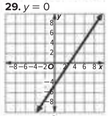
21a. $y = -\frac{2}{3}x + \frac{11}{3}$ **21b.** When the drummer crosses the x-axis, the y-coordinate will be 0, so solve $0 = -\frac{2}{3}x + \frac{11}{3}$ to find the x-coordinate; solving shows that $x = \frac{11}{2}$ or $5\frac{1}{2}$, so the x-coordinate will be greater than 5.

23. $y = -10$ **25.** $y = \frac{1}{5}x + \frac{12}{5}$

27. $y = -8$



29. $y = 0$



31. $y = -\frac{3}{2}x - 7$; $y = \frac{2}{3}x - \frac{8}{3}$

33. No; none of the slopes are equal, and no two of the slopes have a product of -1.

35a. 450 yd **35b.** -3; Ford Street and 6th Street are parallel, so they have the same slope. **35c.** Both have a slope of $\frac{1}{3}$ because both are perpendicular to Ford and 6th, and the slope of a perpendicular is given by the negative reciprocal. **35d.** 200 yd

37. $S\left(0, -5\frac{1}{2}\right)$; The slope of $\overline{QR}$ is $\frac{2-4}{3-(-2)} = -\frac{2}{5}$, so the slope of $\overline{RS}$ is $\frac{5}{2}$. Let the coordinates of S be $(0, y)$ because S must be on the y-axis. Solve $\frac{5}{2} = \frac{y-2}{0-3}$ for y. $y = -5\frac{1}{2}$, so the coordinates of S are $\left(0, -5\frac{1}{2}\right)$.

39a. $y - 1 = a(x - 5)$; the line must have slope a to be parallel to line p.

39b. $y = -\frac{1}{a}x + \frac{2}{a} + 3$; the line must have slope $-\frac{1}{a}$ to be perpendicular to line p.

41a. $B(2, 4)$ and $D(10, -4)$

41b. Sample answer: The slopes of $\overline{AB}$ and $\overline{DC}$ are undefined, so they are parallel to each other. The slopes of $\overline{AD}$ and $\overline{BC}$ are 0, so they are parallel to each other.

41c. Sample answer: Because the slope of $\overline{AB}$ is undefined and the slope of $\overline{BC}$ is zero, the lines are perpendicular to each other. Therefore, they form a right angle, which measures 90°. The same logic applies to all the sides.

43. Yes; the slope of the line through the points $(-2, 2)$ and $(2, 5)$ is $\frac{3}{4}$. The slope of the line through the points $(2, 5)$ and $(6, 8)$ is $\frac{3}{4}$. Because these lines have the same slope and have a point in common, their equations would be the same. Therefore, all the points are on the same line, and all the points are collinear.

45. Two nonvertical lines are parallel if and only if they have the same slope. Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

Lesson 12-9

1. $a \parallel b$; Alternate Interior Angles Converse

3. $\ell \parallel m$; Alternate Exterior Angles Converse

5. $g \parallel h$; Converse of Corresponding Angles Thm.

7. 22 **9.** 6 **11.** 13 **13.** 20

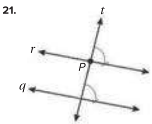
15. Parallelogram; sample answer: The top edges are perpendicular to the vertical line, so they are a single line. The bottom edge is also a single line and perpendicular to the same line as the top, so it is parallel to the top. The top edge is a transversal to the left and right slanted edges, and the angles are supplementary. So, the left and right edges are parallel.

17a. 108°; sample answer: To ensure that the horizontal part of the A is truly horizontal, it should be parallel to the dashed line. Therefore, $\angle 2$ and the 108°-angle are alternate interior angles, and $m\angle 2 = 108°$. $\angle 1$ and $\angle 2$ are congruent angles, so $m\angle 1 = 108°$.

17b. Sample answer: One side of the A is longer than the other.

19. Sample answer: It is given that $\angle 1 \cong \angle 2$. Also, $\angle 1 \cong \angle 3$, because these are vertical angles. Therefore, $\angle 2 \cong \angle 3$ by the Transitive Property of Congruence. This shows that $\ell \parallel m$ by the Converse of Corresponding Angles Theorem.

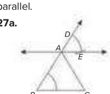
21.



23. Sample answer: Because the corners are right angles, each pair of opposite sides is perpendicular to the same line. Therefore, each pair of opposite sides is parallel.

25. Daniela is correct. $\angle 1$ and $\angle 2$ are alternate interior angles for $\overline{WX}$ and $\overline{YZ}$. So, if alternate interior angles are congruent, then the lines are parallel.

27a.



27b. Sample answer: Using a straightedge, the lines are equidistant. So, they are parallel.

27c. Sample answer: $\angle ABC$ was copied to construct $\angle DAE$. So, $\angle ABC \cong \angle DAE$. $\angle ABC$ and $\angle DAE$ are corresponding angles, so by the Converse of the Corresponding Angles Theorem, $\overleftrightarrow{AE} \parallel \overleftrightarrow{BC}$.

Selected Answers

29. Yes; sample answer: A pair of angles can be supplementary and congruent if the measure of both angles is 90°, because the sum of the angle measures would be 180°.

Lesson 12-10

1. $\sqrt{2}$ or about 1.41 units **3.** 6 units **5.** $\sqrt{10}$ or about 3.16 units

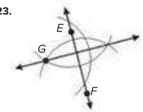
7. yes; 4.24 in.

9. 8 units **11.** $\sqrt{10}$ or about 3.16 units **13.** $3\sqrt{2}$ or about 4.24 units

15. $4\sqrt{17}$ or about 16.49 units **17.** $\sqrt{14.76}$ or about 3.84 units **19.** 0 units

21. Sample answer: Isaiah can measure the perpendicular distance between the wires in two different places. If the distances are equal, then the wires are parallel.

23.



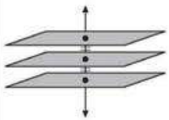
25. No; Sample answer: A path that is perpendicular to the tee box would be the shortest. The angle that the tee box makes with the path that Mark walked is less than 90°, so it is not the shortest possible path.

27. Sample answer: The point is on the line. The two lines are the same line.

29. Sample answer: First, a point on one of the parallel lines is found. Then the line that is perpendicular to the line through the point is found. Then the point of intersection is found between the perpendicular line and the other line that is not used in the first step. Last, the Distance Formula is used to determine the distance between the pair of intersection points. This value is the distance between the pair of parallel lines.

31. Sometimes; sample answer: The distance can only be found if the line is parallel to the plane.

33. If two planes are each equidistant from a third plane, then the two planes are parallel to each other.



Module 12 Review

1. 6

3. B

5. $x = 29$, $m\angle ABD = 66°$, $m\angle DBC = 24°$

7. A, D, E

9. Lines *m* and *p*

11. 102° **13.** A, B, C, D **15.** B

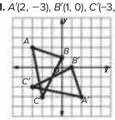
Module 13

Quick Check

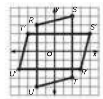
1. −14 **3.** −2 **5.** (3, 0) **7.** (1, −1)

Lesson 13-1

1. $A'(2, -3)$, $B'(1, 0)$, $C'(-3, -2)$

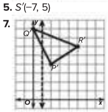


3. $R'(3, -2)$, $S'(4, 2)$, $T'(-3, 2)$, $U'(-4, -2)$



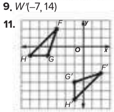
5. $S'(-7, 5)$

7.

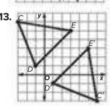


9. $W'(-7, 14)$

11.



13.



15. reflection in the line $y = -1$

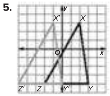
17. Sample answer: The M can be represented with the points (0, 0), (0, 3), (1, 1), (2, 0), and (2, 3). Reflecting in the line $y = x$ gives (0, 0), (3, 0), (1, 1), (0, 2), and (3, 2). **19.** $(-b, a)$

Lesson 13-2

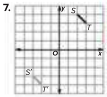
1. $\triangle J'K'L'$ is a translation of $\triangle JKL$. This translation vector can be represented as $\langle 2, 5\rangle$.

3. (4, 3), (5, 5), (6, 3)

5.



7.

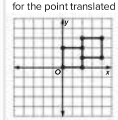


9. $Q'(2, -7)$ **11.** $B'(12, 2)$

13. No; sample answer: The size has been changed.

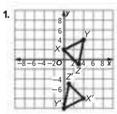
15. $(x, y) \rightarrow (x, y + 4)$

17. (0, 2), (2, 2), (0, 0), (2, 0); Sample answer: To minimize the length of the vector, I used the vertex closest to the origin, (2, 1), as the preimage for the point translated to the origin.

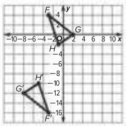


Lesson 13-3

1.



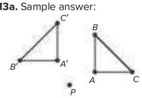
3.



5. home plate: (3, 23), first base: (3, 13), second base: (13, 13), third base: (13 , 23)

7. $K'(0, -10)$ **9.** $X'(-3, 13)$, $Y'(0, 12)$ **11.** 10 times

13a. Sample answer:



13b. Sample answer: 270° clockwise rotation about point P

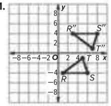
15. Sample answer: A rotation followed by another rotation is still a rotation. For example, a rotation of 30° clockwise followed by a rotation of 20° counterclockwise is the same as a rotation of 10° clockwise.

17. Sample answer: Distance is preserved because the lengths of segments remain the same measure. Angle measures are preserved because angle measures remain the same measure. Parallelism is preserved because parallel lines remain parallel. Collinearity is preserved because points remain on the same lines.

19. Yes; sample answer: A rotation is a transformation that maintains congruence of the original figure and its image. So, the preimage can be mapped onto the image, and corresponding segments will be congruent. Therefore, collinearity and betweenness of points are maintained in rotations.

Lesson 13-4

1.



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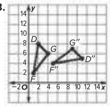

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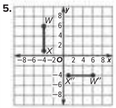

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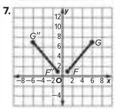
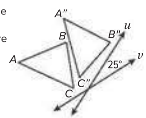
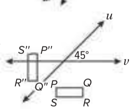


9. a 50° clockwise rotation about the point where lines u and v intersect



11. a 90° clockwise rotation about the point where lines u and v intersect

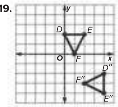


13. $\triangle JKL \cong \triangle MNP$; reflection in y-axis followed by translation along $\langle -1, 2\rangle$

15. $\triangle JKL \cong \triangle MNP$; translation along $\langle 2, 0\rangle$ followed by 180° rotation about the origin

17. Sample answer: Reflect the first two shapes vertically, and then translate to the right and repeat.

19.



21. (3, −1), (2, −5), (−5, −2)

23. (1, 1), (0, 5), (−7, 2)

25. Sometimes; sample answer: If the lines of reflection intersect, the composition is a rotation.

27. Sometimes; sample answer: This is true if the point is the origin.

29. Sample answer: Proof: It is given that a translation along $\langle a, b\rangle$ maps R to R' and S to S'. Using the definition of a translation, points R and S move the same distance in the same direction, therefore $\overline{RS} \cong \overline{R'S'}$. It is also given that a reflection in a maps R' to R'' and S' to S''. Using the definition of a reflection, points R and S are the same distance from line a, so $\overline{R'S'} \cong \overline{R''S''}$. By the Transitive Property of Congruence, $\overline{RS} \cong \overline{R''S''}$.

31. Sometimes; sample answer: The order of rotating by 180° about the origin and reflecting in the line $y = x$ does not change the location of the final image.

33. $P'''(1, -2)$, $Q'''(2, 1)$, $R'''(-1, 3)$, $S'''(-2, 0)$

Lesson 13-5

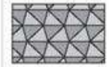
1. $x = \frac{180(5-2)}{5} = 108°$; Because 108° is not a factor of 360°, a regular pentagon will not tessellate the plane.

3. $x = \frac{180(9-2)}{9} = 140°$; Because 140° is not a factor of 360°, a regular 9-gon will not tessellate the plane.

5. yes; 2 regular hexagons, 2 equilateral triangles

7. tessellation; uniform, semi-regular

9. Yes; sample answer: reflection, rotation, translation

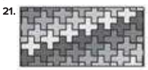


11. Never; sample answer: Each interior angle of a regular dodecagon is $\frac{180°(12-2)}{12} = 150°$. Because 150° is not a factor of 360°, a regular dodecagon will not tessellate the plane.

13. Never; sample answer: Each interior angle of a regular 15-gon is $\frac{180°(15-2)}{15} = 156°$. Because 156° is not a factor of 360°, a regular 15-gon will not tessellate the plane.

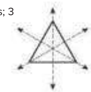
15. translation **17.** rotation **19.** Sample answer: Translations can be performed because the pieces slide. Rotations can be performed because each piece can be turned. Reflections cannot be performed because the back of a piece cannot be used to create the puzzle.

21.



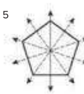
Lesson 13-6

1. yes; 3



3. no

5. yes; 5



7a. Yes; the hubcap can map onto itself with a rotation that is less than 360°.

7b. Yes; the hubcap can map onto itself with a rotation that is less than 360°.

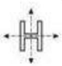
7c. Yes; the hubcap can map onto itself with a rotation that is less than 360°.

9. Yes; the symbol can map onto itself with a rotation that is less than 360°.

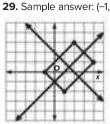
11. yes; 3; 120°

13. yes; 2; 180°

15.



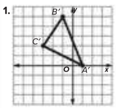
17. 2; 180° **19.** pentagon **21.** yes; order of symmetry: 3; magnitude of symmetry: 120° **23.** yes; order of symmetry: 6; magnitude of symmetry: 60°
25. Sample answer: A rectangular mirror with two lines of symmetry, one vertical and one horizontal, through the middle or a spoon with one line of symmetry down the middle.
27. 24; 360° ÷ 15° = 24, so the order of symmetry is 24. This means there are 24 sides.
29. Sample answer: (–1, 0), (2, 3), (4, 1), and (1, –2);



31. Sample answer: In both rotational and line symmetry a figure is mapped onto itself. However, in line symmetry the figure is mapped onto itself by a reflection, and in rotational symmetry a figure is mapped onto itself by a rotation. A figure can have line symmetry and rotational symmetry.

Module 13 Review

1.



3. (–9, 6)
5. Sample answer: The length of $\overline{AC}$ is not the same as the length of $\overline{EG}$.
7. C **9.** Quadrant II **11.** false **13.** A, D
15. C
17. order = 24; magnitude = 15°

Module 14

Quick Check

1. right **3.** obtuse **5.** 10.8 **7.** 18.0

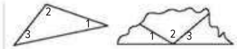
Lesson 14-1

1. $m\angle 1 = 30°$, $m\angle 2 = 60°$
3. $m\angle 1 = 109°$, $m\angle 2 = 29°$, $m\angle 3 = 71°$
5. 50° **7.** 98° **9.** 62° **11.** 26° **13.** 55°
15. $x = 20$; 40°, 60°, 80° **17.** $x = 11$; 80°, 117°
19.

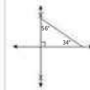
Proof:

$\angle R$ is a rt. $\angle$. — **Given**
→ $m\angle R = 90°$ — **Def. of rt. ∠**
$m\angle R + m\angle S + m\angle T = 180°$ — **Triangle Angle-Sum Thm.**
→ $90 + m\angle S + m\angle T = 180°$ — **Substitution**
→ $m\angle S + m\angle T = 90°$ — **Subtraction Prop.**
→ $\angle S$ and $\angle T$ are complementary. — **Def. of comp. angles**

21. $m\angle D = 27°$, $m\angle E = 81°$, $m\angle F = 72°$
23. $m\angle J = 18°$, $m\angle K = 72°$, $m\angle L = 90°$
25. $m\angle Z < 23°$; Sample answer: Because the sum of the measures of the angles of a triangle is 180° and $m\angle X = 157°$, $157° + m\angle Y + m\angle Z = 180°$, so $m\angle Y + m\angle Z = 23°$. If $m\angle Y$ was 0°, then $m\angle Z$ would equal 23°. But because an angle must have a measure greater than 0°, $m\angle Z$ must be less than 23°, so $m\angle Z < 23°$.
27. 90° **29.** $m\angle 1 = 55°$, $m\angle 2 = 75°$, $m\angle 3 = 55°$, $m\angle 4 = 15°$ **31.** Sample answer: Draw a triangle and then tear the corners off the triangle. Arrange the three corners so the angles are adjacent. The angles now form a straight angle. Because a straight angle measures 180°, the sum of the measures of the angles of a triangle is 180°.



33. Sample answer: Because an exterior angle is acute, the adjacent angle must be obtuse. Because another exterior angle is right, the adjacent angle must be right. A triangle cannot contain a right angle and an obtuse angle because the sum would be greater than 180°. Therefore, a triangle cannot have an obtuse, an acute, and a right exterior angle.
35. Sample answer: I found the measure of the second angle by subtracting the first angle from 90° because the acute angles of a right triangle are complementary.



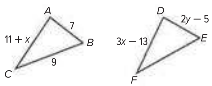
Lesson 14-2

1. $\angle A \cong \angle D$; $\angle ABC \cong \angle DCB$; $\angle ACB \cong \angle DBC$; $\overline{AC} \cong \overline{DB}$; $\overline{AB} \cong \overline{DC}$; $\triangle ABC \cong \triangle DCB$
3. $\angle X \cong \angle A$; $\angle Y \cong \angle B$; $\angle Z \cong \angle C$; $\overline{XY} \cong \overline{AB}$; $\overline{XZ} \cong \overline{AC}$; $\overline{YZ} \cong \overline{BC}$; $\triangle XYZ \cong \triangle ABC$
5. $\angle R \cong \angle J$; $\angle T \cong \angle K$; $\angle S \cong \angle L$; $\overline{RT} \cong \overline{JK}$; $\overline{TS} \cong \overline{KL}$; $\overline{RS} \cong \overline{JL}$; $\triangle RTS \cong \triangle JKL$
7. 48 **9.** 5 **11.** 35 **13.** 41°
15. Proof:

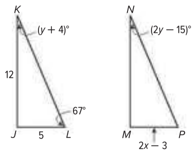
Statements (Reasons)
1. $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{CD}$ (Given)
2. $\overline{BD} \cong \overline{BD}$ (Reflexive Prop. of Congruence)
3. $\angle ABD \cong \angle CBD$, $\angle ADB \cong \angle CDB$ (Given)
4. $\angle A \cong \angle C$ (Third Angles Theorem)
5. $\triangle ABD \cong \triangle CBD$ (Def. of congruent triangles)

17. Proof: It is given that $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$. Therefore, $\angle ABD \cong \angle CBD$ and $\angle ADB \cong \angle CDB$ by the definition of angle bisector. By the Third Angles Theorem, $\angle A \cong \angle C$. It is given that $\overline{AB} \cong \overline{CB}$ and $\overline{AD} \cong \overline{CD}$. By the Reflexive Property of Congruence, $\overline{BD} \cong \overline{BD}$. Therefore, $\triangle ABD \cong \triangle CBD$ by Def. of congruent triangles.

19. $x = 12$; $y = 6$



21. $x = 4$; $y = 19$



23a. $\triangle ABI \cong \triangle EBF$, $\triangle CBD \cong \triangle HBG$
23b. Sample answer: $\angle A \cong \angle E$, $\angle ABI \cong \angle EBF$, $\angle I \cong \angle F$; $\overline{AB} \cong \overline{EB}$, $\overline{BI} \cong \overline{BF}$, $\overline{AI} \cong \overline{EF}$
25. Statements (Reasons)

1. $\angle P \cong \angle X$, $\angle Q \cong \angle Y$ (Given)
2. $m\angle P = m\angle X$, $m\angle Q = m\angle Y$ (Def. of congruent angles)
3. $m\angle P + m\angle Q + m\angle R = 180$
   $180 = m\angle X + m\angle Y + m\angle Z$ (Triangle Angle-Sum Thm.)
4. $m\angle P + m\angle Q + m\angle R = m\angle X + m\angle Y + m\angle Z$ (Transitive Property)
5. $m\angle X + m\angle Y + m\angle R = m\angle X + m\angle Y + m\angle Z$ (Substitution Property)
6. $m\angle R = m\angle Z$ (Subtraction Prop. of Eq.)
7. $\angle R \cong \angle Z$ (Def. of congruent angles)

27. Sample answer: Do you think that the sum of the angles of a quadrilateral is constant? If so, do you think that the final pair of corresponding angles will be congruent if three other pairs of corresponding angles are congruent for a pair of quadrilaterals?

29. Sample answer: When naming congruent triangles, it is important that the corresponding vertices be in the same location for both triangles because the location indicates congruence. For example, if $\triangle ABC$ is congruent to $\triangle DEF$, then $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$.

Lesson 14-3

1. Statements (Reasons)

1. $\overline{AB} \cong \overline{XY}$
   $\overline{AC} \cong \overline{XZ}$
   $\overline{BC} \cong \overline{YZ}$ (Given)
2. $\triangle ABC \cong \triangle XYZ$ (SSS Post.)

3. Proof:

Statements (Reasons)

1. $\overline{AB} \cong \overline{CB}$, D is the midpoint of $\overline{AC}$. (Given)
2. $\overline{AD} \cong \overline{DC}$ (Definition of midpoint)
3. $\overline{BD} \cong \overline{BD}$ (Reflexive Property of Congruence)
4. $\triangle ABD \cong \triangle CBD$ (SSS)

5. Proof: We know that $\overline{QR} \cong \overline{SR}$ and $\overline{ST} \cong \overline{QT}$. $\overline{RT} \cong \overline{RT}$ by the Reflexive Property. Because $\overline{QR} \cong \overline{SR}$, $\overline{ST} \cong \overline{QT}$, and $\overline{RT} \cong \overline{RT}$, $\triangle QRT \cong \triangle SRT$ by SSS.
7. $DE = 5\sqrt{2}$, $PQ = 5\sqrt{2}$, $EF = 2\sqrt{10}$, $QR = 2\sqrt{10}$, $DF = 5\sqrt{2}$, $PR = 5\sqrt{2}$; $\triangle DEF \cong \triangle PQR$ by SSS because corresponding sides have the same measure and are congruent. **9.** $AB = 2$, $KL = 2$, $BC = 2\sqrt{2}$, $LM = 2\sqrt{2}$, $AC = 2$, $KM = 2$; The corresponding sides have the same measure and are congruent, so $\triangle ABC \cong \triangle KLM$ by SSS.
11. Proof:

Statements (Reasons)

1. $NP = PM$, $\overline{NP} \perp \overline{PL}$ (Given)
2. $\angle MPL$ and $\angle NPL$ are rt. angles. ($\perp$ lines form rt. angles.)
3. $\angle MPL \cong \angle NPL$ (All right angles are congruent.)
4. $\overline{PL} \cong \overline{PL}$ (Reflexive Property of $\cong$)
5. $\triangle NPL \cong \triangle MPL$ (SAS)

13. Proof: Because V is the midpoint of $\overline{YZ}$ and the midpoint of $\overline{WX}$, by the definition of midpoint, $\overline{YV} \cong \overline{VZ}$ and $\overline{WV} \cong \overline{XV}$. Because $\angle YVW$ and $\angle ZVX$ are vertical angles, by the Vertical Angles Theorem, the angles are congruent. Therefore, by SAS, $\triangle XVZ \cong \triangle WVY$.
15. Proof:

Statements (Reasons)

1. $\overline{BD} \perp \overline{AC}$; $\overline{BD}$ bisects $\overline{AC}$. (Given)
2. $\angle BDA$ and $\angle BDC$ are rt. angles. ($\perp$ lines form rt. angles.)
3. $\angle BDA \cong \angle BDC$ (All right angles are congruent.)
4. $\overline{AD} \cong \overline{DC}$ (Def. of segment bisector)
5. $\overline{BD} \cong \overline{BD}$ (Reflexive Property of $\cong$)
6. $\triangle ABD \cong \triangle CBD$ (SAS)

17. Yes; sample answer: $\angle GLH$ and $\angle JLK$ are vertical angles, so they are congruent. Therefore, $\triangle GLH \cong \triangle JLK$ by the SAS Congruence Postulate. **19.** Yes; sample answer: The triangles share the side $\overline{AC}$, so they have two pairs of congruent sides. The given congruent angles are included angles, so $\triangle ABC \cong \triangle CDA$ by SAS. **21.** No; sample answer: The sticks do not change size, so any arrangement will yield a congruent triangle. **23.** Sample answer: She needs to measure one side of each tile because all the tiles are equilateral triangles. **25.** No; sample answer: You cannot use SAS because the angle congruence that we are given is not an included angle between two sides that are known to be congruent, and SSS cannot be used because only 2 sides of each triangle are known to be congruent. **27.** First pair; sample answer: The second pair can be shown congruent by SAS or SSS, and the third pair can be shown congruent by SSS. **29.** Case 1: You know that the hypotenuses are congruent and that one pair of legs are congruent. Then the Pythagorean Theorem says that the other pair of legs are congruent, so the triangles are congruent by SSS. Case 2: You know that the pairs of legs are congruent and that the right angles are congruent, so the triangles are congruent by SAS. **31.** Shada; to use SAS, the angle must be the included angle.

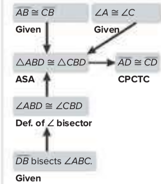
Lesson 14-4

1. Proof:

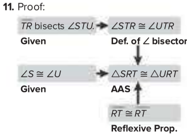
Statements (Reasons)

1. $\overline{AB} \parallel \overline{CD}$ (Given)
2. $\angle CBD \cong \angle ADB$ (Given)
3. $\angle ABD \cong \angle CDB$ (Alternate Interior Angles Theorem)
4. $\overline{BD} \cong \overline{BD}$ (Reflexive Property of Congruence)
5. $\triangle ABD \cong \triangle CDB$ (ASA)

3. Proof:



5. Proof: We are given that $\overline{CE}$ bisects $\angle BED$ and that $\angle BCE$ and $\angle ECD$ are right angles. Because all right angles are congruent, $\angle BCE \cong \angle ECD$. By the definition of angle bisector, $\angle BEC \cong \angle DEC$. The Reflexive Property tells us that $\overline{EC} \cong \overline{EC}$. By the Angle-Side-Angle Congruence Postulate, $\triangle ECB \cong \triangle ECD$.
7a. yes; by the ASA Congruence Postulate
7b. 10 in^2 **9a.** Sample answer: Because $\overline{AC} \parallel \overline{BK}$, $\angle CAB \cong \angle KBM$ by the Corresponding Angles Theorem. Because $\overline{CB} \parallel \overline{KM}$, $\angle ABC \cong \angle BMK$ by the Corresponding Angles Theorem. Because B is the midpoint of $\overline{AM}$, $\overline{AB} \cong \overline{BM}$ by the Midpoint Theorem. Therefore, by the ASA Congruence Postulate, $\triangle ABC \cong \triangle BMK$.
9b. 74 ft
11. Proof:



13. Proof: It is given that $\angle E \cong \angle G$ and $\overline{DE} \parallel \overline{FG}$. By the Alternate Interior Angles Theorem, $\angle DFG \cong \angle FDE$. $\overline{DF} \cong \overline{DF}$ by the Reflexive property of Congruence. Therefore, $\triangle DFG \cong \triangle FDE$ by AAS.

15. Proof:

Statements (Reasons)

1. $\overline{MS} \cong \overline{RQ}$, $\overline{MS} \parallel \overline{RQ}$ (Given)
2. $\angle SPM \cong \angle QPR$ (Vertical Angles Theorem)
3. $\angle SMP \cong \angle QRP$ (Alternate Interior Angles Theorem)
5. $\triangle MSP \cong \triangle RQP$ (AAS)

17. Yes; sample answer: They are congruent by AAS. **19.** Proof: Because it is given that $\overline{GE}$ bisects $\angle DEF$, $\angle DEG \cong \angle FEG$ by the definition of an angle bisector. It is given that $\angle D \cong \angle F$. By the Reflexive Property, $\overline{GE} \cong \overline{GE}$. So, $\triangle DEG \cong \triangle FEG$ by AAS. Therefore, $\overline{DG} \cong \overline{FG}$ by CPCTC. **21.** Tyrone; Lorenzo showed that all three corresponding angles were congruent, but AAA is not a proof of triangle congruence.

23.

Method	Use when...
Definition of Congruent Triangles	All corresponding parts of one triangle are congruent to the corresponding parts of the other triangle.
SSS	The three sides on one triangle must be congruent to the three sides of the other triangle.
SAS	Two sides and the included angle of one triangle must be congruent to two sides and the included angle of the other triangle.
ASA	Two angles and the included side of one triangle must be congruent to two angles and the included side of the other triangle.
AAS	Two angles and a non-included side of one triangle must be congruent to two angles and the corresponding non-included side of the other triangle.

Lesson 14-5

1. Statements (Reasons)

1. $\overline{XZ} \perp \overline{WY}$ (Given)
2. $\angle XZW$ and $\angle XZY$ are right angles. ($\perp$ lines form right angles.)
3. $\triangle WXZ$ and $\triangle YXZ$ are right triangles. (Definition of right triangle)
4. Z is the midpoint of $\overline{WY}$. (Given)
5. $\overline{WZ} \cong \overline{ZY}$ (Definition of midpoint)
6. $\overline{XZ} \cong \overline{XZ}$ (Reflexive Property of Congruence)
7. $\triangle WXZ \cong \triangle YXZ$ (LL Congruence Theorem)

3. Proof:

Statements (Reasons)

1. $\overline{BX} \perp \overline{AC}$ (Given)
2. $\angle AXB$ and $\angle CXB$ are rt. $\angle$s. (Definition of $\perp$ lines)
3. $\triangle AXB$ and $\triangle CXB$ are rt. $\triangle$s. (Definition of right $\triangle$s)
4. $\overline{XB} \cong \overline{XB}$ (Reflexive Property of Congruence)
5. $AB = CB$ (Given)
6. $\overline{AB} \cong \overline{CB}$ (Definition of congruent)
7. $\triangle AXB \cong \triangle CXB$ (HL Congruence Thm.)

5. Yes; LA **7.** No; not enough information
9. No; not enough information
11. Proof:

Statements (Reasons)

1. $\overline{BX} \perp \overline{XA}$, $\overline{BY} \perp \overline{YA}$ (Given)
2. $\angle BXA$ and $\angle BYA$ are rt. $\angle$s. (Definition of $\perp$ lines)
3. $\triangle BXA$ and $\triangle BYA$ are rt. $\triangle$s. (Definition of right $\triangle$s)
4. $\overline{XA} \cong \overline{YA}$ (Given)
5. $\overline{BA} \cong \overline{BA}$ (Reflexive Property of Congruence)
6. $\triangle BXA \cong \triangle BYA$ (HL Congruence Theorem)

13. Proof: By the definition of $\perp$ segments, $\angle AYB$ and $\angle AXC$ are right angles. By the definition of right triangles, $\triangle AYB$ and $\triangle AXC$ are right triangles. By the definition of congruent segments, $\overline{AX}$ is congruent to $\overline{AY}$.

By the Reflexive Property of Congruence, $\angle BAY$ is congruent to $\angle CAX$. Therefore by LA, $\triangle ABY$ is congruent to $\triangle ACX$.

Lesson 14-6

1. Proof:

Statements (Reasons)

1. $\angle 1 \cong \angle 2$ (Given)
2. $\angle 2 \cong \angle 3$ (Vertical Angles Thm.)
3. $\angle 1 \cong \angle 3$ (Transitive Prop. of $\cong$)
4. $\overline{AB} \cong \overline{CB}$ (Conv. of Isos. Triangle Thm.)

3. Proof:

Statements (Reasons)

1. $\overline{DE} \parallel \overline{BC}$ (Given)
2. $\angle 1 \cong \angle 4$, $\angle 2 \cong \angle 3$ (Corresponding angles are $\cong$.)
3. $\angle 1 \cong \angle 2$ (Given)
4. $\angle 1 \cong \angle 3$ (Transitive Property of $\cong$.)
5. $\angle 3 \cong \angle 4$ (Substitution.)
6. $\overline{AB} \cong \overline{AC}$ (Converse of Isosceles Triangle Theorem.)

5a. The coordinates of $\triangle ABC$ are $A(0, 5)$, $B(3, 1)$, and $C(-3, 1)$.
$AC = \sqrt{[0-(-3)]^2 + (5-1)^2}$ or 5 units
$AB = \sqrt{(0-3)^2 + (5-1)^2}$ or 5 units
$BC = 6$ units
So, $\triangle ABC$ is an isosceles triangle with $\overline{AB} \cong \overline{AC}$.

5b. Because $\overline{AB} \cong \overline{AC}$, we know that $\angle C \cong \angle B$ by the Isosceles Triangle Theorem.

$m\angle A + m\angle B + m\angle C = 180°$	Triangle Angle-Sum Theorem
$m\angle A + 2m\angle C = 180°$	Definition of congruent
$m\angle A + 2(55) = 180°$	Substitute.
$m\angle A + 110 = 180°$	Multiply.
$m\angle A = 70°$	Solve.

7. $m\angle DEF = 45°$ and $m\angle EFD = 45°$ **9.** 60°; 3 m **11.** 60°; 7 in. **13.** 10 **15a.** 7 **15b.** 15
17a.

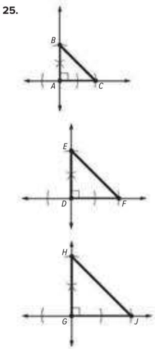
17b. Sample answer: The triangle formed by connecting the midpoints of the sides of an isosceles triangle is an isosceles triangle.

19. Proof:

Statements (Reasons)

1. $\triangle PQR$ is an equilateral triangle. (Given)
2. $\overline{PQ} \cong \overline{QR} \cong \overline{PR}$ (Def. of equilateral triangle)
3. $\angle P \cong \angle Q \cong \angle R$ (Isosceles Triangle Theorem)
4. $m\angle P = m\angle Q = m\angle R$ (Def. of congruence)
5. $m\angle P + m\angle Q + m\angle R = 180°$ (Triangle Angle-Sum Thm.)
6. $3m\angle P = 180°$ (Substitution)
7. $m\angle P = 60°$ (Division Property)
8. $m\angle P = m\angle Q = m\angle R = 60°$ (Substitution)

21. 44° **23.** 22°
25.



Sample answer: I constructed a pair of perpendicular segments and then used the same compass setting to mark points that are equidistant from their intersection. I measured both legs for each triangle. Because $AB = AC = 1.3$ cm, $DE = DF = 1.9$ cm, and

Selected Answers

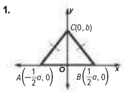
GH = GJ = 2.3 cm, the triangles are isosceles. I used a protractor to confirm that ∠A, ∠D, and ∠G are all right angles.

27. Sometimes; sample answer: Only if the measure of the vertex angle is even.

29. Sample answer: It is not possible because a triangle cannot have more than one obtuse angle. **31.** No; $m\angle G = \frac{180-70}{2}$ or 55°.

Lesson 14-7

1.



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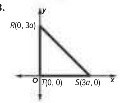


5. C(p, q) **7.** E(−2g, 0); F(0, b) **9.** Sample answer: The midpoint P of $\overline{BC}$ is $\left(\frac{0+2a}{2}, \frac{2b+0}{2}\right) = (a, b)$. The midpoint Q of $\overline{AC}$ is $\left(\frac{0+2a}{2}, \frac{0+0}{2}\right) = (a, 0)$. The midpoint R of $\overline{AB}$ is $\left(\frac{0+0}{2}, \frac{0+2b}{2}\right) = (0, b)$. The slope of $\overline{RP}$ is $\frac{b-b}{a-0} = \frac{0}{a} = 0$, so the segment is horizontal. The slope of $\overline{PQ}$ is $\frac{b-0}{a-a} = \frac{0}{a}$, which is undefined, so the segment is vertical. ∠RPQ is a right angle because any horizontal line is perpendicular to any vertical line. △PRQ has a right angle, so △PRQ is a right triangle.

11. Proof: The Midpoint Formula shows that the coordinates of M are $\left(\frac{0+2a}{2}, \frac{2a+0}{2}\right)$ or (a, a). The slope of $\overline{AC}$ is $\frac{2a-0}{0-2a} = -1$. The slope of $\overline{BM}$ is $\frac{a-0}{a-0} = 1$. The product of the slopes is −1, so $\overline{BM} \perp \overline{AC}$. **13.** Proof: The coordinates of S are $\left(\frac{b}{2}, \frac{c}{2}\right)$, and the coordinates of T are $\left(\frac{a+b}{2}, \frac{c}{2}\right)$.

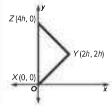
$ST = \sqrt{\left(\frac{a+b}{2} - \frac{b}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2}$ or $\frac{a}{2}$

$AB = \sqrt{(a-0)^2 + (0-0)^2}$ or a

$ST = \frac{1}{2}AB$

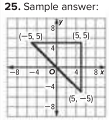
15. The slope between the grandstand and the rides and games is $\frac{2}{3}$. The slope between the grandstand and the main gate is $-\frac{3}{2}$. Because $\frac{2}{3} \cdot -\frac{3}{2} = -1$, the triangle formed by these three locations is a right triangle.

17. Slope of $\overline{XY} = 1$, slope of $\overline{YZ} = -1$, slope of $\overline{ZX} = 0$; because $1(-1) = -1$, $\overline{XY} \perp \overline{YZ}$. Therefore, △XYZ is a right triangle.



19. (a, 0) and (0, a) **21.** Sample answer: Use the Distance Formula to find the length of each side of each triangle. Show that the triangles are congruent by SSS. Conclude that ∠A ≅ ∠D using CPCTC. **23.** D(a, b), E(2a, 0), F(3a, b)

25. Sample answer:



27. Sample answer: (a, 0)

29. Sample answer: (4a, 0) or (0, 4b)

Module 14 Review

1. 106 **3.** A, D **5.** A, C, E **7.** A **9.** A, B
11. A **13.** 47 **15.** 33 units **17.** A **19.** B

Glossary

English	Español
A	
30°-60°-90° triangle A right triangle with two acute angles that measure 30° and 60°.	triángulo 30°-60°-90° Un triángulo rectángulo con dos ángulos agudos que miden 30° y 60°.
45°-45°-90° triangle A right triangle with two acute angles that measure 45°.	triángulo 45°-45°-90° Un triángulo rectángulo con dos ángulos agudos que miden 45°.
absolute value The distance a number is from zero on the number line.	valor absoluto La distancia que un número es de cero en la línea numérica.
absolute value function A function written as $f(x) = \lvert x \rvert$, in which $f(x) \geq 0$ for all values of x.	función del valor absoluto Una función que se escribe $f(x) = \lvert x \rvert$, donde $f(x) \geq 0$, para todos los valores de x.
accuracy The nearness of a measurement to the true value of the measure.	exactitud La proximidad de una medida al valor verdadero de la medida.
additive identity Because the sum of any number a and 0 is equal to a, 0 is the additive identity.	identidad aditiva Debido a que la suma de cualquier número a y 0 es igual a, 0 es la identidad aditiva.
additive inverses Two numbers with a sum of 0.	inverso aditivos Dos números con una suma de 0.
adjacent angles Two angles that lie in the same plane and have a common vertex and a common side but have no common interior points.	ángulos adyacentes Dos ángulos que se encuentran en el mismo plano y tienen un vértice común y un lado común, pero no tienen puntos comunes en el interior.
adjacent arcs Arcs in a circle that have exactly one point in common.	arcos adyacentes Arcos en un circulo que tienen un solo punto en común.
algebraic expression A mathematical expression that contains at least one variable.	expresión algebraica Una expresión matemática que contiene al menos una variable.
algebraic notation Mathematical notation that describes a set by using algebraic expressions.	notación algebraica Notación matemática que describe un conjunto usando expresiones algebraicas.
alternate exterior angles When two lines are cut by a transversal, nonadjacent exterior angles that lie on opposite sides of the transversal.	ángulos alternos externos Cuando dos líneas son cortadas por un ángulo transversal, no adyacente exterior que se encuentran en lados opuestos de la transversal.
alternate interior angles When two lines are cut by a transversal, nonadjacent interior angles that lie on opposite sides of the transversal.	ángulos alternos internos Cuando dos líneas son cortadas por un ángulo transversal, no adyacente interior que se encuentran en lados opuestos de la transversal.
altitude of a parallelogram A perpendicular segment between any two parallel bases.	altitud de un paralelogramo Un segmento perpendicular entre dos bases paralelas.

Glossary

altitude of a prism or cylinder A segment perpendicular to the bases that joins the planes of the bases.

altitud de un prisma o cilindro Un segmento perpendicular a las bases que une los planos de las bases.

altitude of a pyramid or cone A segment perpendicular to the base that has the vertex as one endpoint and a point in the plane of the base as the other endpoint.

altitud de una pirámide o cono Un segmento perpendicular a la base que tiene el vértice como un punto final y un punto en el plano de la base como el otro punto final.

altitude of a triangle A segment from a vertex of the triangle to the line containing the opposite side and perpendicular to that side.

altitud de triángulo Un segmento de un vértice del triángulo a la línea que contiene el lado opuesto y perpendicular a ese lado.

ambiguous case When two different triangles could be created or described using the given information.

caso ambiguo Cuando dos triángulos diferentes pueden ser creados o descritos usando la información dada.

amplitude For functions of the form $y = a \sin b\theta$ or $y = a \cos b\theta$, the amplitude is $|a|$.

amplitud Para funciones de la forma $y = a$ sen $b\theta$ o $y = a \cos b\theta$, la amplitud es $|a|$.

analytic geometry The study of geometry that uses the coordinate system.

geometría analítica El estudio de la geometría que utiliza el sistema de coordenadas.

angle The intersection of two noncollinear rays at a common endpoint.

ángulo La intersección de dos rayos no colineales en un extremo común.

angle bisector A ray or segment that divides an angle into two congruent angles.

bisectriz de un ángulo Un rayo o segmento que divide un ángulo en dos ángulos congruentes.

angle of depression The angle formed by a horizontal line and an observer's line of sight to an object below the horizontal line.

ángulo de depresión El ángulo formado por una línea horizontal y la línea de visión de un observador a un objeto por debajo de la línea horizontal.

angle of elevation The angle formed by a horizontal line and an observer's line of sight to an object above the horizontal line.

ángulo de elevación El ángulo formado por una línea horizontal y la línea de visión de un observador a un objeto por encima de la línea horizontal.

angle of rotation The angle through which a figure rotates.

ángulo de rotación El ángulo a través del cual gira una figura.

apothem A perpendicular segment between the center of a regular polygon and a side of the polygon or the length of that line segment.

apotema Un segmento perpendicular entre el centro de un polígono regular y un lado del polígono o la longitud de ese segmento de línea.

approximate error The positive difference between an actual measurement and an approximate or estimated measurement.

error aproximado La diferencia positiva entre una medida real y una medida aproximada o estimada.

arc Part of a circle that is defined by two endpoints.

arco Parte de un círculo que se define por dos puntos finales.

arc length The distance between the endpoints of an arc measured along the arc in linear units.

longitude de arco La distancia entre los extremos de un arco medido a lo largo del arco en unidades lineales.

area The number of square units needed to cover a surface.

área El número de unidades cuadradas para cubrir una superficie.

arithmetic sequence A pattern in which each term after the first is found by adding a constant, the common difference *d*, to the previous term.

secuencia aritmética Un patrón en el cual cada término después del primero se encuentra añadiendo una constante, la diferencia común *d*, al término anterior.

asymptote A line that a graph approaches.

asíntota Una línea que se aproxima a un gráfico.

auxiliary line An extra line or segment drawn in a figure to help analyze geometric relationships.

línea auxiliar Una línea o segmento extra dibujado en una figura para ayudar a analizar las relaciones geométricas.

average rate of change The change in the value of the dependent variable divided by the change in the value of the independent variable.

tasa media de cambio El cambio en el valor de la variable dependiente dividido por el cambio en el valor de la variable independiente.

axiom A statement that is accepted as true without proof.

axioma Una declaración que se acepta como verdadera sin prueba.

axiomatic system A set of axioms from which theorems can be derived.

sistema axiomático Un conjunto de axiomas de los cuales se pueden derivar teoremas.

axis of symmetry A line about which a graph is symmetric.

eje de simetría Una línea sobre la cual un gráfica es simétrico.

axis symmetry If a figure can be mapped onto itself by a rotation between 0° and 360° in a line.

eje simetría Si una figura puede ser asignada sobre sí misma por una rotación entre 0° y 360° en una línea.

B

bar graph A graphical display that compares categories of data using bars of different heights.

gráfico de barra Una pantalla gráfica que compara las categorías de datos usando barras de diferentes alturas.

base In a power, the number being multiplied by itself.

base En un poder, el número se multiplica por sí mismo.

base angles of a trapezoid The two angles formed by the bases and legs of a trapezoid.

ángulos de base de un trapecio Los dos ángulos formados por las bases y patas de un trapecio.

base angles of an isosceles triangle The two angles formed by the base and the congruent sides of an isosceles triangle.

ángulo de la base de un triángulo isosceles Los dos ángulos formados por la base y los lados congruentes de un triángulo isosceles.

base edge The intersection of a lateral face and a base in a solid figure.

arista de la base La intersección de una cara lateral y una base en una figura sólida.

base of a parallelogram Any side of a parallelogram.

base de un paralelogramo Cualquier lado de un paralelogramo.

base of a pyramid or cone The face of the solid opposite the vertex of the solid.

base de una pirámide o cono La cara del sólido opuesta al vértice del sólido.

English	Español
bases of a prism or cylinder The two parallel congruent faces of the solid.	**bases de un prisma o cilindro** Las dos caras congruentes paralelas de la figura sólida.
bases of a trapezoid The parallel sides in a trapezoid.	**bases de un trapecio** Los lados paralelos en un trapecio.
best-fit line The line that most closely approximates the data in a scatter plot.	**línea de ajuste óptimo** La línea que más se aproxima a los datos en un diagrama de dispersión.
betweenness of points Point C is between A and B if and only if A, B, and C are collinear and $AC + CB = AB$.	**intermediación de puntos** El punto C está entre A y B si y sólo si A, B, y C son colineales y $AC + CB = AB$.
bias An error that results in a misrepresentation of a population.	**sesgo** Un error que resulta en una tergiversación de una población.
biconditional statement The conjunction of a conditional and its converse.	**declaración bicondicional** La conjunción de un condicional y su inverso.
binomial The sum of two monomials.	**binomio** La suma de dos monomios.
bisect To separate a line segment into two congruent segments.	**bisecar** Separe un segmento de línea en dos segmentos congruentes.
bivariate data Data that consists of pairs of values.	**datos bivariate** Datos que constan de pares de valores.
boundary The edge of the graph of an inequality that separates the coordinate plane into regions.	**frontera** El borde de la gráfica de una desigualdad que separa el plano de coordenadas en regiones.
bounded When the graph of a system of constraints is a polygonal region.	**acotada** Cuando la gráfica de un sistema de restricciones es una región poligonal.
box plot A graphical representation of the five-number summary of a data set.	**diagram de caja** Una representación gráfica del resumen de cinco números de un conjunto de datos.

C

English	Español
categorical data Data that can be organized into different categories.	**datos categóricos** Datos que pueden organizarse en diferentes categorías.
causation When a change in one variable produces a change in another variable.	**causalidad** Cuando un cambio en una variable produce un cambio en otra variable.
center of a circle The point from which all points on a circle are the same distance.	**centro de un círculo** El punto desde el cual todos los puntos de un círculo están a la misma distancia.
center of a regular polygon The center of the circle circumscribed about a regular polygon.	**centro de un polígono regular** El centro del círculo circunscrito alrededor de un polígono regular.
center of dilation The center point from which dilations are performed.	**centro de dilatación** Punto fijo en torno al cual se realizan las homotecias.
center of rotation The fixed point about which a figure rotates.	**centro de rotación** El punto fijo sobre el que gira una figura.
center of symmetry A point in which a figure can be rotated onto itself.	**centro de la simetría** Un punto en el que una figura se puede girar sobre sí misma.
central angle of a circle An angle with a vertex at the center of a circle and sides that are radii.	**ángulo central de un círculo** Un ángulo con un vértice en el centro de un círculo y los lados que son radios.
central angle of a regular polygon An angle with its vertex at the center of a regular polygon and sides that pass through consecutive vertices of the polygon.	**ángulo central de un polígono regular** Un ángulo con su vértice en el centro de un polígono regular y lados que pasan a través de vértices consecutivos del polígono.
centroid The point of concurrency of the medians of a triangle.	**baricentro** El punto de intersección de las medianas de un triángulo.
chord of a circle or sphere A segment with endpoints on the circle or sphere.	**cuerda de un círculo o esfera** Un segmento con extremos en el círculo o esfera.
circle The set of all points in a plane that are the same distance from a given point called the center.	**círculo** El conjunto de todos los puntos en un plano que están a la misma distancia de un punto dado llamado centro.
circular function A function that describes a point on a circle as the function of an angle defined in radians.	**función circular** Función que describe un punto en un círculo como la función de un ángulo definido en radianes.
circumcenter The point of concurrency of the perpendicular bisectors of the sides of a triangle.	**circuncentro** El punto de concurrencia de las bisectrices perpendiculares de los lados de un triángulo.
circumference The distance around a circle.	**circunferencia** La distancia alrededor de un círculo.
circumscribed angle An angle with sides that are tangent to a circle.	**ángulo circunscrito** Un ángulo con lados que son tangentes a un círculo.
circumscribed polygon A polygon with vertices outside the circle and sides that are tangent to the circle.	**polígono circunscrito** Un polígono con vértices fuera del círculo y lados que son tangentes al círculo.
closed If for any members in a set, the result of an operation is also in the set.	**cerrado** Si para cualquier número en el conjunto, el resultado de la operación es también en el conjunto.
closed half-plane The solution of a linear inequality that includes the boundary line.	**semi-plano cerrado** La solución de una desigualdad linear que incluye la línea de límite.
codomain The set of all the y-values that could possibly result from the evaluation of the function.	**codominar** El conjunto de todos los valores y que podrían resultar de la evaluación de la función.
coefficient The numerical factor of a term.	**coeficiente** El factor numérico de un término.
coefficient of determination An indicator of how well a function fits a set of data.	**coeficiente de determinación** Un indicador de lo bien que una función se ajusta a un conjunto de datos.

cofunction identities Identities that show the relationships between sine and cosine, tangent and cotangent, and secant and cosecant.

collinear Lying on the same line.

combination A selection of objects in which order is not important.

combined variation When one quantity varies directly and/or inversely as two or more other quantities.

common difference The difference between consecutive terms in an arithmetic sequence.

common logarithms Logarithms of base 10.

common ratio The ratio of consecutive terms of a geometric sequence.

common tangent A line or segment that is tangent to two circles in the same plane.

complement of *A* All of the outcomes in the sample space that are not included as outcomes of event *A*.

complementary angles Two angles with measures that have a sum of 90°.

completing the square A process used to make a quadratic expression into a perfect square trinomial.

complex conjugates Two complex numbers of the form $a + bi$ and $a - bi$.

complex fraction A rational expression with a numerator and/or denominator that is also a rational expression.

complex number Any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit.

component form A vector written as $<x, y>$, which describes the vector in terms of its horizontal component x and vertical component y.

identidades de cofunción Identidades que muestran las relaciones entre seno y coseno, tangente y cotangente, y secante y cosecante.

colineal Acostado en la misma línea.

combinación Una selección de objetos en los que el orden no es importante.

variación combinada Cuando una cantidad varía directamente y / o inversamente como dos o más cantidades.

diferencia común La diferencia entre términos consecutivos de una secuencia aritmética.

logaritmos comunes Logaritmos de base 10.

razón común El razón de términos consecutivos de una secuencia geométrica.

tangente común Una línea o segmento que es tangente a dos círculos en el mismo plano.

complemento de *A* Todos los resultados en el espacio muestral que no se incluyen como resultados del evento *A*.

ángulo complementarios Dos ángulos con medidas que tienen una suma de 90°.

completar el cuadrado Un proceso usado para hacer una expresión cuadrática en un trinomio cuadrado perfecto.

conjugados complejos Dos números complejos de la forma $a + bi$ y $a - bi$.

fracción compleja Una expresión racional con un numerador y / o denominador que también es una expresión racional.

número complejo Cualquier número que se puede escribir en la forma $a + bi$, donde a y b son números reales e i es la unidad imaginaria.

forma de componente Un vector escrito como $<x, y>$, que describe el vector en términos de su componente horizontal x y componente vertical y.

composite figure A figure that can be separated into regions that are basic figures, such as triangles, rectangles, trapezoids, and circles.

composite solid A three-dimensional figure that is composed of simpler solids.

composition of functions An operation that uses the results of one function to evaluate a second function.

composition of transformations When a transformation is applied to a figure and then another transformation is applied to its image.

compound event Two or more simple events.

compound inequality Two or more inequalities that are connected by the words *and* or *or*.

compound interest Interest calculated on the principal and on the accumulated interest from previous periods.

compound statement Two or more statements joined by the word *and* or *or*.

concave polygon A polygon with one or more interior angles with measures greater than 180°.

concentric circles Coplanar circles that have the same center.

conclusion The statement that immediately follows the word *then* in a conditional.

concurrent lines Three or more lines that intersect at a common point.

conditional probability The probability that an event will occur given that another event has already occurred.

conditional relative frequency The ratio of the joint frequency to the marginal frequency.

conditional statement A compound statement that consists of a premise, or hypothesis, and a conclusion, which is false only when its premise is true and its conclusion is false.

figura compuesta Una figura que se puede separar en regiones que son figuras básicas, tales como triángulos, rectángulos, trapezoides, y círculos.

solido compuesta Una figura tridimensional que se compone de figuras más simples.

composición de funciones Operación que utiliza los resultados de una función para evaluar una segunda función.

composición de transformaciones Cuando una transformación se aplica a una figura y luego se aplica otra transformación a su imagen.

evento compuesto Dos o más eventos simples.

desigualdad compuesta Dos o más desigualdades que están unidas por las palabras *y* u *o*.

interés compuesto Intereses calculados sobre el principal y sobre el interés acumulado de períodos anteriores.

enunciado compuesto Dos o más declaraciones unidas por la palabra *y* o *o*.

polígono cóncavo Un polígono con uno o más ángulos interiores con medidas superiores a 180°.

círculos concéntricos Círculos coplanarios que tienen el mismo centro.

conclusión La declaración que inmediatamente sigue la palabra *entonces* en un condicional.

líneas concurrentes Tres o más líneas que se intersecan en un punto común.

probabilidad condicional La probabilidad de que un evento ocurra dado que otro evento ya ha ocurrido.

frecuencia relativa condicional La relación entre la frecuencia de la articulación y la frecuencia marginal.

enunciado condicional Una declaración compuesta que consiste en una premisa, o hipótesis, y una conclusión, que es falsa solo cuando su premisa es verdadera y su conclusión es falsa.

cone A solid figure with a circular base connected by a curved surface to a single vertex.

cono Una figura sólida con una base circular conectada por una superficie curvada a un solo vértice.

confidence interval An estimate of the population parameter stated as a range with a specific degree of certainty.

intervalo de confianza Una estimación del parámetro de población se indica como un rango con un grado específico de certeza.

congruent Having the same size and shape.

congruente Tener el mismo tamaño y forma.

congruent angles Two angles that have the same measure.

ángulo congruentes Dos ángulos que tienen la misma medida.

congruent arcs Arcs in the same or congruent circles that have the same measure.

arcos congruentes Arcos en los mismos círculos o congruentes que tienen la misma medida.

congruent polygons All of the parts of one polygon are congruent to the corresponding parts or matching parts of another polygon.

polígonos congruentes Todas las partes de un polígono son congruentes con las partes correspondientes o partes coincidentes de otro polígono.

congruent segments Line segments that are the same length.

segmentos congruentes Línea segmentos que son la misma longitud.

congruent solids Solid figures that have exactly the same shape, size, and a scale factor of 1:1.

sólidos congruentes Figuras sólidas que tienen exactamente la misma forma, tamaño y un factor de escala de 1:1.

conic sections Cross sections of a right circular cone.

secciones cónicas Secciones transversales de un cono circular derecho.

conjecture An educated guess based on known information and specific examples.

conjetura Una suposición educada basada en información conocida y ejemplos específicos.

conjugates Two expressions, each with two terms, in which the second terms are opposites.

conjugados Dos expresiones, cada una con dos términos, en la que los segundos términos son opuestos.

conjunction A compound statement using the word *and*.

conjunción Una declaración compuesta usando la palabra *y*.

consecutive interior angles When two lines are cut by a transversal, interior angles that lie on the same side of the transversal.

ángulos internos consecutivos Cuando dos líneas se cortan por un ángulo transversal, interior que se encuentran en el mismo lado de la transversal.

consistent A system of equations with at least one ordered pair that satisfies both equations.

consistente Una sistema de ecuaciones para el cual existe al menos un par ordenado que satisfice ambas ecuaciones.

constant function A linear function of the form $y = b$; The function $f(x) = a$, where a is any number.

función constante Una función lineal de la forma $y = b$; La función $f(x) = a$, donde a es cualquier número.

constant of variation The constant in a variation function.

constante de variación La constante en una función de variación.

constant term A term that does not contain a variable.

término constante Un término que no contiene una variable.

constraint A condition that a solution must satisfy.

restricción Una condición que una solución debe satisfacer.

constructions Methods of creating figures without the use of measuring tools.

construcciones Métodos de creación de figuras sin el uso de herramientas de medición.

continuous function A function that can be graphed with a line or an unbroken curve.

función continua Una función que se puede representar gráficamente con una línea o una curva ininterrumpida.

continuous random variable The numerical outcome of a random event that can take on any value.

variable aleatoria continua El resultado numérico de un evento aleatorio que puede tomar cualquier valor.

contrapositive A statement formed by negating both the hypothesis and the conclusion of the converse of a conditional.

antítesis Una afirmación formada negando tanto la hipótesis como la conclusión del inverso del condicional.

convenience sample Members that are readily available or easy to reach are selected.

muestra conveniente Se seleccionan los miembros que están fácilmente disponibles o de fácil acceso.

converse A statement formed by exchanging the hypothesis and conclusion of a conditional statement.

recíproco Una declaración formada por el intercambio de la hipótesis y la conclusión de la declaración condicional.

convex polygon A polygon with all interior angles measuring less than 180°.

polígono convexo Un polígono con todos los ángulos interiores que miden menos de 180°.

coordinate proofs Proofs that use figures in the coordinate plane and algebra to prove geometric concepts.

pruebas de coordenadas Pruebas que utilizan figuras en el plano de coordenadas y álgebra para probar conceptos geométricos.

coplanar Lying in the same plane.

coplanar Acostado en el mismo plano.

corollary A theorem with a proof that follows as a direct result of another theorem.

corolario Un teorema con una prueba que sigue como un resultado directo de otro teorema.

correlation coefficient A measure that shows how well data are modeled by a regression function.

coeficiente de correlación Una medida que muestra cómo los datos son modelados por una función de regresión.

corresponding angles When two lines are cut by a transversal, angles that lie on the same side of a transversal and on the same side of the two lines.

ángulos correspondientes Cuando dos líneas se cortan transversalmente, los ángulos que se encuentran en el mismo lado de una transversal y en el mismo lado de las dos líneas.

corresponding parts Corresponding angles and corresponding sides of two polygons.

partes correspondientes Ángulos correspondientes y lados correspondientes.

cosecant The ratio of the length of a hypotenuse to the length of the leg opposite the angle.

cosecante Relación entre la longitud de la hipotenusa y la longitud de la pierna opuesta al ángulo.

cosine The ratio of the length of the leg adjacent to an angle to the length of the hypotenuse.

coseno Relación entre la longitud de la pierna adyacente a un ángulo y la longitud de la hipotenusa.

cotangent The ratio of the length of the leg adjacent to an angle to the length of the leg opposite the angle.

cotangente La relación entre la longitud de la pata adyacente a un ángulo y la longitud de la pata opuesta al ángulo.

coterminal angles Angles in standard position that have the same terminal side.

ángulos coterminales Ángulos en posición estándar que tienen el mismo lado terminal.

counterexample An example that contradicts the conjecture showing that the conjecture is not always true.

contraejemplo Un ejemplo que contradice la conjetura que muestra que la conjetura no siempre es cierta.

critical values The *z*-values corresponding to the most common degrees of certainty.

valores críticos Los valores z correspondientes a los grados de certeza más comunes.

cross section The intersection of a solid and a plane.

sección transversal Intersección de un sólido con un plano.

cube root One of three equal factors of a number.

raíz cúbica Uno de los tres factores iguales de un número.

cube root function A radical function that contains the cube root of a variable expression.

función de la raíz del cubo Función radical que contiene la raíz cúbica de una expresión variable.

curve fitting Finding a regression equation for a set of data that is approximated by a function.

ajuste de curvas Encontrar una ecuación de regresión para un conjunto de datos que es aproximado por una función.

cycle One complete pattern of a periodic function.

ciclo Un patron completo de una función periódica.

cylinder A solid figure with two congruent and parallel circular bases connected by a curved surface.

cilindro Una figura sólida con dos bases circulares congruentes y paralelas conectadas por una superficie curvada.

D

decay factor The base of an exponential expression, or $1 - r$.

factor de decaimiento La base de una expresión exponencial, o $1 - r$.

decomposition Separating a figure into two or more nonoverlapping parts.

descomposición Separar una figura en dos o más partes que no se solapan.

decreasing Where the graph of a function goes down when viewed from left to right.

decreciente Donde la gráfica de una función disminuye cuando se ve de izquierda a derecha.

deductive argument An argument that guarantees the truth of the conclusion provided that its premises are true.

argumento deductivo Un argumento que garantiza la verdad de la conclusión siempre que sus premisas sean verdaderas.

deductive reasoning The process of reaching a specific valid conclusion based on general facts, rules, definitions, or properties.

razonamiento deductivo El proceso de alcanzar una conclusión válida específica basada en hechos generales, reglas, definiciones, o propiedades.

define a variable To choose a variable to represent an unknown value.

definir una variable Para elegir una variable que represente un valor desconocido.

defined term A term that has a definition and can be explained.

término definido Un término que tiene una definición y se puede explicar.

definitions An explanation that assigns properties to a mathematical object.

definiciones Una explicación que asigna propiedades a un objeto matemático.

degree The value of the exponent in a power function; $\frac{1}{360}$ of the circular rotation about a point.

grado Valor del exponente en una función de potencia. $\frac{1}{360}$ de la rotación circular alrededor de un punto.

degree of a monomial The sum of the exponents of all its variables.

grado de un monomio La suma de los exponents de todas sus variables.

degree of a polynomial The greatest degree of any term in the polynomial.

grado de un polinomio El grado mayor de cualquier término del polinomio.

density A measure of the quantity of some physical property per unit of length, area, or volume.

densidad Una medida de la cantidad de alguna propiedad física por unidad de longitud, área o volumen.

dependent A consistent system of equations with an infinite number of solutions.

dependiente Una sistema consistente de ecuaciones con un número infinito de soluciones.

dependent events Two or more events in which the outcome of one event affects the outcome of the other events.

eventos dependientes Dos o más eventos en que el resultado de un evento afecta el resultado de los otros eventos.

dependent variable The variable in a relation, usually *y*, with values that depend on *x*.

variable dependiente La variable de una relación, generalmente *y*, con los valores que depende de *x*.

depressed polynomial A polynomial resulting from division with a degree one less than the original polynomial.

polinomio reducido Un polinomio resultante de la división con un grado uno menos que el polinomio original.

descriptive modeling A way to mathematically describe real-world situations and the factors that cause them.

modelado descriptivo Una forma de describir matemáticamente las situaciones del mundo real y los factores que las causan.

descriptive statistics The branch of statistics that focuses on collecting, summarizing, and displaying data.

estadística descriptiva Rama de la estadística cuyo enfoque es la recopilación, resumen y demostración de los datos.

diagonal A segment that connects any two nonconsecutive vertices within a polygon.

diagonal Un segmento que conecta cualquier dos vértices no consecutivos dentro de un polígono.

diameter of a circle or sphere A chord that passes through the center of a circle or sphere.

difference of squares A binomial in which the first and last terms are perfect squares.

difference of two squares The square of one quantity minus the square of another quantity.

dilation A nonrigid motion that enlarges or reduces a geometric figure; A transformation that stretches or compresses the graph of a function.

dimensional analysis The process of performing operations with units.

direct variation When one quantity is equal to a constant times another quantity.

directed line segment A line segment with an initial endpoint and a terminal endpoint.

directrix An exterior line perpendicular to the line containing the foci of a curve.

discontinuous function A function that is not continuous.

discrete function A function in which the points on the graph are not connected.

discrete random variable The numerical outcome of a random event that is finite and can be counted.

discriminant In the Quadratic Formula, the expression under the radical sign that provides information about the roots of the quadratic equation.

disjunction A compound statement using the word *or*.

distance The length of the line segment between two points.

distribution A graph or table that shows the theoretical frequency of each possible data value.

domain The set of the first numbers of the ordered pairs in a relation; The set of *x*-values to be evaluated by a function.

diámetro de un círculo o esfera Un acorde que pasa por el centro de un círculo o esfera.

diferencia de cuadrados Un binomio en el que los términos primero y último son cuadrados perfectos.

diferencia de dos cuadrados El cuadrado de una cantidad menos el cuadrado de otra cantidad.

dilatación Un movimiento no rígido que agranda o reduce una figura geométrica; Una transformación que estira o comprime el gráfico de una función.

análisis dimensional El proceso de realizar operaciones con unidades.

variación directa Cuando una cantidad es igual a una constante multiplicada por otra cantidad.

segmento de línea dirigido Un segmento de línea con un punto final inicial y un punto final terminal.

directriz Una línea exterior perpendicular a la línea que contiene los focos de una curva.

función discontinua Una función que no es continua.

función discreta Una función en la que los puntos del gráfico no están conectados.

variable aleatoria discreta El resultado numérico de un evento aleatorio que es finito y puede ser contado.

discriminante En la Fórmula cuadrática, la expresión bajo el signo radical que proporciona información sobre las raíces de la ecuación cuadrática.

disyunción Una declaración compuesta usando la palabra *o*.

distancia La longitud del segmento de línea entre dos puntos.

distribución Un gráfico o una table que muestra la frecuencia teórica de cada valor de datos posible.

dominio El conjunto de los primeros números de los pares ordenados en una relación; El conjunto de valores *x* para ser evaluados por una función.

dot plot A diagram that shows the frequency of data on a number line.

double root Two roots of a quadratic equation that are the same number.

gráfica de puntos Una diagrama que muestra la frecuencia de los datos en una línea numérica.

raíces dobles Dos raíces de una función cuadrática que son el mismo número.

E

e An irrational number that approximately equals 2.7182818....

edge of a polyhedron A line segment where the faces of the polyhedron intersect.

elimination A method that involves eliminating a variable by combining the individual equations within a system of equations.

empty set The set that contains no elements, symbolized by { } or ∅.

end behavior The behavior of a graph at the positive and negative extremes in its domain.

enlargement A dilation with a scale factor greater than 1.

equation A mathematical statement that contains two expressions and an equal sign, =.

equiangular polygon A polygon with all angles congruent.

equidistant A point is equidistant from other points if it is the same distance from them.

equidistant lines Two lines for which the distance between the two lines, measured along a perpendicular line or segment to the two lines, is always the same.

equilateral polygon A polygon with all sides congruent.

equivalent equations Two equations with the same solution.

equivalent expressions Expressions that represent the same value.

evaluate To find the value of an expression.

e Un número irracional que es aproximadamente igual a 2.7182818 ....

arista de un poliedro Un segmento de línea donde las caras del poliedro se cruzan.

eliminación Un método que consiste en eliminar una variable combinando las ecuaciones individuales dentro de un sistema de ecuaciones.

conjunto vacio El conjunto que no contiene elementos, simbolizado por { } o ∅.

comportamiento extremo El comportamiento de un gráfico en los extremos positivo y negativo en su dominio.

ampliación Una dilatación con un factor de escala mayor que 1.

ecuación Un enunciado matemático que contiene dos expresiones y un signo igual, =.

polígono equiangular Un polígono con todos los ángulos congruentes.

equidistante Un punto es equidistante de otros puntos si está a la misma distancia de ellos.

líneas equidistantes Dos líneas para las cuales la distancia entre las dos líneas, medida a lo largo de una línea o segmento perpendicular a las dos líneas, es siempre la misma.

polígono equilátero Un polígono con todos los lados congruentes.

ecuaciones equivalentes Dos ecuaciones con la misma solución.

expresiones equivalentes Expresiones que representan el mismo valor.

evaluar Calcular el valor de una expresión.

even functions Functions that are symmetric in the y-axis.

incluso funciones Funciones que son simétricas en el eje y.

event A subset of the sample space.

evento Un subconjunto del espacio de muestra.

excluded values Values for which a function is not defined.

valores excluidos Valores para los que no se ha definido una función.

experiment A sample is divided into two groups. The experimental group undergoes a change, while there is no change to the control group. The effects on the groups are then compared; A situation involving chance.

experimento Una muestra se divide en dos grupos. El grupo experimental experimenta un cambio, mientras que no hay cambio en el grupo de control. A continuación se comparan los efectos sobre los grupos; Una situación de riesgo.

experimental probability Probability calculated by using data from an actual experiment.

probabilidad experimental Probabilidad calculada utilizando datos de un experimento real.

exponent When n is a positive integer in the expression x^n, n indicates the number of times x is multiplied by itself.

exponente Cuando n es un entero positivo en la expresión x^n, n indica el número de veces que x se multiplica por sí mismo.

exponential decay Change that occurs when an initial amount decreases by the same percent over a given period of time.

desintegración exponencial Cambio que ocurre cuando una cantidad inicial disminuye en el mismo porcentaje durante un período de tiempo dado.

exponential decay function A function in which the independent variable is an exponent, where $a > 0$ and $0 < b < 1$.

función exponenciales de decaimiento Una ecuación en la que la variable independiente es un exponente, donde $a > 0$ y $0 < b < 1$.

explicit formula A formula that allows you to find any term a_n of a sequence by using a formula written in terms of n.

fórmula explícita Una fórmula que le permite encontrar cualquier término a_n de una secuencia usando una fórmula escrita en términos de n.

exponential equation An equation in which the independent variable is an exponent.

ecuación exponencial Una ecuación en la que la variable independiente es un exponente.

exponential form When an expression is in the form x^n.

forma exponencial Cuando una expresión está en la forma x^n.

exponential function A function in which the independent variable is an exponent.

función exponencial Una función en la que la variable independiente es el exponente.

exponential growth Change that occurs when an initial amount increases by the same percent over a given period of time.

crecimiento exponencial Cambio que ocurre cuando una cantidad inicial aumenta por el mismo porcentaje durante un período de tiempo dado.

exponential growth function A function in which the independent variable is an exponent, where $a > 0$ and $b > 1$.

función de crecimiento exponencial Una función en la que la variable independiente es el exponente, donde $a > 0$ y $b > 1$.

exponential inequality An inequality in which the independent variable is an exponent.

desigualdad exponencial Una desigualdad en la que la variable independiente es un exponente.

exterior angle of a triangle An angle formed by one side of the triangle and the extension of an adjacent side.

ángulo exterior de un triángulo Un ángulo formado por un lado del triángulo y la extensión de un lado adyacente.

exterior angles When two lines are cut by a transversal, any of the four angles that lie outside the region between the two intersected lines.

ángulos externos Cuando dos líneas son cortadas por una transversal, cualquiera de los cuatro ángulos que se encuentran fuera de la región entre las dos líneas intersectadas.

exterior of an angle The area outside of the two rays of an angle.

exterior de un ángulo El área fuera de los dos rayos de un ángulo.

extraneous solution A solution of a simplified form of an equation that does not satisfy the original equation.

solución extraña Una solución de una forma simplificada de una ecuación que no satisface la ecuación original.

extrema Points that are the locations of relatively high or low function values.

extrema Puntos que son las ubicaciones de valores de función relativamente alta o baja.

extreme values The least and greatest values in a set of data.

valores extremos Los valores mínimo y máximo en un conjunto de datos.

F

face of a polyhedron A flat surface of a polyhedron.

cara de un poliedro Superficie plana de un poliedro.

factored form A form of quadratic equation, $0 = a(x - p)(x - q)$, where $a \neq 0$, in which p and q are the x-intercepts of the graph of the related function.

forma factorizada Una forma de ecuación cuadrática, $0 = a(x - p)(x - q)$, donde $a \neq 0$, en la que p y q son las intercepciones x de la gráfica de la función relacionada.

factorial of n The product of the positive integers less than or equal to n.

factorial de n El producto de los enteros positivos inferiores o iguales a n.

factoring The process of expressing a polynomial as the product of monomials and polynomials.

factorización por agrupamiento Utilizando la Propiedad distributiva para factorizar polinomios que possen cuatro o más términos.

factoring by grouping Using the Distributive Property to factor some polynomials having four or more terms.

factorización El proceso de expresar un polinomio como el producto de monomios y polinomios.

family of graphs Graphs and equations of graphs that have at least one characteristic in common.

familia de gráficas Gráficas y ecuaciones de gráficas que tienen al menos una característica común.

feasible region The intersection of the graphs in a system of constraints.

región factible La intersección de los gráficos en un sistema de restricciones.

finite sample space A sample space that contains a countable number of outcomes.

espacio de muestra finito Un espacio de muestra que contiene un número contable de resultados.

finite sequence A sequence that contains a limited number of terms.

secuencia finita Una secuencia que contiene un número limitado de términos.

five-number summary The minimum, quartiles, and maximum of a data set.

resumen de cinco números El mínimo, cuartiles y máximo de un conjunto de datos.

flow proof A proof that uses boxes and arrows to show the logical progression of an argument.

demostración de flujo Una prueba que usa cajas y flechas para mostrar la progresión lógica de un argumento.

focus A point inside a parabola having the property that the distances from any point on the parabola to them and to a fixed line have a constant ratio for any points on the parabola.

foco Un punto dentro de una parábola que tiene la propiedad de que las distancias desde cualquier punto de la parábola a ellos ya una línea fija tienen una relación constante para cualquier punto de la parábola.

formula An equation that expresses a relationship between certain quantities.

fórmula Una ecuación que expresa una relación entre ciertas cantidades.

fractional distance An intermediary point some fraction of the length of a line segment.

distancia fraccionaria Un punto intermediario de alguna fracción de la longitud de un segmento de línea.

frequency The number of cycles in a given unit of time.

frecuencia El número de ciclos en una unidad del tiempo dada.

function A relation in which each element of the domain is paired with exactly one element of the range.

función Una relación en que a cada elemento del dominio de corresponde un único elemento del rango.

function notation A way of writing an equation so that $y = f(x)$.

notación functional Una forma de escribir una ecuación para que $y = f(x)$.

G

geometric means The terms between two nonconsecutive terms of a geometric sequence; The nth root, where n is the number of elements in a set of numbers, of the product of the numbers.

medios geométricos Los términos entre dos términos no consecutivos de una secuencia geométrica; La enésima raíz, donde n es el número de elementos de un conjunto de números, del producto de los números.

geometric model A geometric figure that represents a real-life object.

modelo geométrico Una figura geométrica que representa un objeto de la vida real.

geometric probability Probability that involves a geometric measure such as length or area.

probabilidad geométrica Probabilidad que implica una medida geométrica como longitud o área.

geometric sequence A pattern of numbers that begins with a nonzero term and each term after is found by multiplying the previous term by a nonzero constant r.

secuencia geométrica Un patrón de números que comienza con un término distinto de cero y cada término después se encuentra multiplicando el término anterior por una constante no nula r.

geometric series The indicated sum of the terms in a geometric sequence.

series geométricas La suma indicada de los términos en una secuencia geométrica.

glide reflection The composition of a translation followed by a reflection in a line parallel to the translation vector.

reflexión del deslizamiento La composición de una traducción seguida de una reflexión en una línea paralela al vector de traslación.

greatest integer function A step function in which $f(x)$ is the greatest integer less than or equal to x.

función entera más grande Una función del paso en que $f(x)$ es el número más grande menos que o igual a x.

growth factor The base of an exponential expression, or $1 + r$.

factor de crecimiento La base de una expresión exponencial, o $1 + r$.

H

half-plane A region of the graph of an inequality on one side of a boundary.

semi-plano Una región de la gráfica de una desigualdad en un lado de un límite.

height of a parallelogram The length of an altitude of the parallelogram.

altura de un paralelogramo La longitud de la altitud del paralelogramo.

height of a solid The length of the altitude of a solid figure.

altura de un sólido La longitud de la altitud de una figura sólida.

height of a trapezoid The perpendicular distance between the bases of a trapezoid.

altura de un trapecio La distancia perpendicular entre las bases de un trapecio.

histogram A graphical display that uses bars to display numerical data that have been organized in equal intervals.

histograma Una exhibición gráfica que utiliza barras para exhibir los datos numéricos que se han organizado en intervalos iguales.

horizontal asymptote A horizontal line that a graph approaches.

asíntota horizontal Una línea horizontal que se aproxima a un gráfico.

hyperbola The graph of a reciprocal function.

hipérbola La gráfica de una función recíproca.

hypothesis The statement that immediately follows the word *if* in a conditional.

hipótesis La declaración que sigue inmediatamente a la palabra *si* en un condicional.

I

identity An equation that is true for every value of the variable.

identidad Una ecuación que es verdad para cada valor de la variable.

identity function The function $f(x) = x$.

función identidad La función $f(x) = x$.

if-then statement A compound statement of the form *if p, then q*, where p and q are statements.

enunciado si-entonces Enunciado compuesto de la forma *si p, entonces q*, donde p y q son enunciados.

image The new figure in a transformation.

imagen La nueva figura en una transformación.

imaginary unit i The principal square root of -1.

unidad imaginaria i La raíz cuadrada principal de -1.

incenter The point of concurrency of the angle bisectors of a triangle.

incentro El punto de intersección de las bisectrices interiors de un triángulo.

included angle The interior angle formed by two adjacent sides of a triangle.

ángulo incluido El ángulo interior formado por dos lados adyacentes de un triángulo.

included side The side of a triangle between two angles.

lado incluido El lado de un triángulo entre dos ángulos.

inconsistent A system of equations with no ordered pair that satisfies both equations.

inconsistente Una sistema de ecuaciones para el cual no existe par ordenado alguno que satisfaga ambas ecuaciones.

increasing Where the graph of a function goes up when viewed from left to right.

creciente Donde la gráfica de una función sube cuando se ve de izquierda a derecha.

independent A consistent system of equations with exactly one solution.

independiente Un sistema consistente de ecuaciones con exactamente una solución.

independent events Two or more events in which the outcome of one event does not affect the outcome of the other events.

eventos independientes Dos o más eventos en los que el resultado de un evento no afecta el resultado de los otros eventos.

independent variable The variable in a relation, usually x, with a value that is subject to choice.

variable independiente La variable de una relación, generalmente x, con el valor que sujeta a elección.

index In nth roots, the value that indicates to what root the value under the radicand is being taken.

índice En enésimas raíces, el valor que indica a qué raíz está el valor bajo la radicand.

indirect measurement Using similar figures and proportions to measure an object.

medición indirecta Usando figuras y proporciones similares para medir un objeto.

indirect proof One assumes that the statement to be proven is false and then uses logical reasoning to deduce that a statement contradicts a postulate, theorem, or one of the assumptions.

demostración indirecta Se supone que la afirmación a ser probada es falsa y luego utiliza el razonamiento lógico para deducir que una afirmación contradice un postulado, teorema o uno de los supuestos.

indirect reasoning Reasoning that eliminates all possible conclusions but one so that the one remaining conclusion must be true.

razonamiento indirecto Razonamiento que elimina todas las posibles conclusiones, pero una de manera que la conclusión que queda una debe ser verdad.

inductive reasoning The process of reaching a conclusion based on a pattern of examples.

razonamiento inductive El proceso de llegar a una conclusión basada en un patrón de ejemplos.

inequality A mathematical sentence that contains $<, >, \leq, \geq$, or $\neq$.

desigualdad Una oración matemática que contiene uno o más de $<, >, \leq, \geq$, o $\neq$.

inferential statistics When the data from a sample is used to make inferences about the corresponding population.

estadísticas inferencial Cuando los datos de una muestra se utilizan para hacer inferencias sobre la población correspondiente.

infinite sample space A sample space with outcomes that cannot be counted.

espacio de muestra infinito Un espacio de muestra con resultados que no pueden ser contados.

infinite sequence A sequence that continues without end.

secuencia infinita Una secuencia que continúa sin fin.

informal proof A paragraph that explains why the conjecture for a given situation is true.

prueba informal Un párrafo que explica por qué la conjetura para una situación dada es verdadera.

initial side The part of an angle that is fixed on the x-axis.

lado inicial La parte de un ángulo que se fija en el eje x.

inscribed angle An angle with its vertex on a circle and sides that contain chords of the circle.

ángulo inscrito Un ángulo con su vértice en un círculo y lados que contienen acordes del círculo.

inscribed polygon A polygon inside a circle in which all of the vertices of the polygon lie on the circle.

polígono inscrito Un polígono dentro de un círculo en el que todos los vértices del polígono se encuentran en el círculo.

intercept A point at which the graph of a function intersects an axis.

interceptar Un punto en el que la gráfica de una función corta un eje.

intercepted arc The part of a circle that lies between the two lines intersecting it.

arco intersecado La parte de un círculo que se encuentra entre las dos líneas que se cruzan.

interior angle of a triangle An angle at the vertex of a triangle.

ángulo interior de un triángulo Un ángulo en el vértice de un triángulo.

interior angles When two lines are cut by a transversal, any of the four angles that lie inside the region between the two intersected lines.

ángulos interiores Cuando dos líneas son cortadas por una transversal, cualquiera de los cuatro ángulos que se encuentran dentro de la región entre las dos líneas intersectadas.

interior of an angle The area between the two rays of an angle.

interior de un ángulo El área entre los dos rayos de un ángulo.

interquartile range The difference between the upper and lower quartiles of a data set.

rango intercuartil La diferencia entre el cuartil superior *y* el cuartil inferior de un conjunto de datos.

intersection A set of points common to two or more geometric figures; intersection The graph of a compound inequality containing *and*.

intersección Un conjunto de puntos comunes a dos o más figuras geométricas; intersección La gráfica de una desigualdad compuesta que contiene la palabra *y*.

intersection of *A* and *B* The set of all outcomes in the sample space of event *A* that are also in the sample space of event *B*.

intersección de *A* y *B* El conjunto de todos los resultados en el espacio muestral del evento *A* que también se encuentran en el espacio muestral del evento *B*.

interval The distance between two numbers on the scale of a graph.

intervalo La distancia entre dos números en la escala de un gráfico.

interval notation Mathematical notation that describes a set by using endpoints with parentheses or brackets.

notación de intervalo Notación matemática que describe un conjunto utilizando puntos finales con paréntesis o soportes.

inverse A statement formed by negating both the hypothesis and conclusion of a conditional statement.

inverso Una declaración formada negando tanto la hipótesis como la conclusión de la declaración condicional.

inverse cosine The ratio of the length of the hypotenuse to the length of the leg adjacent to an angle.

inverso del coseno Relación de la longitud de la hipotenusa con la longitud de la pierna adyacente a un ángulo.

inverse functions Two functions, one of which contains points of the form (a, b) while the other contains points of the form (b, a).

funciones inversas Dos funciones, una de las cuales contiene puntos de la forma (a, b) mientras que la otra contiene puntos de la forma (b, a).

inverse relations Two relations, one of which contains points of the form (a, b) while the other contains points of the form (b, a).

relaciones inversas Dos relaciones, una de las cuales contiene puntos de la forma (a, b) mientras que la otra contiene puntos de la forma (b, a).

inverse sine The ratio of the length of the hypotenuse to the length of the leg opposite an angle.

inverso del seno Relación de la longitud de la hipotenusa con la longitud de la pierna opuesta a un ángulo.

inverse tangent The ratio of the length of the leg adjacent to an angle to the length of the leg opposite the angle.

inverso del tangente Relación de la longitud de la pierna adyacente a un ángulo con la longitud de la pierna opuesta a un ángulo.

inverse trigonometric functions Arcsine, Arccosine, and Arctangent.

funciones trigonométricas inversas Arcsine, Arccosine y Arctangent.

inverse variation When the product of two quantities is equal to a constant k.

variación inversa Cuando el producto de dos cantidades es igual a una constante k.

isosceles trapezoid A quadrilateral in which two sides are parallel and the legs are congruent.

trapecio isósceles Un cuadrilátero en el que dos lados son paralelos y las patas son congruentes.

isosceles triangle A triangle with at least two sides congruent.

triángulo isósceles Un triángulo con al menos dos lados congruentes.

J

joint frequencies Entries in the body of a two-way frequency table. In a two-way frequency table, the frequencies in the interior of the table.

frecuencias articulares Entradas en el cuerpo de una tabla de frecuencias de dos vías. En una tabla de frecuencia bidireccional, las frecuencias en el interior de la tabla.

joint variation When one quantity varies directly as the product of two or more other quantities.

variación conjunta Cuando una cantidad varía directamente como el producto de dos o más cantidades.

K

kite A convex quadrilateral with exactly two distinct pairs of adjacent congruent sides.

cometa Un cuadrilátero convexo con exactamente dos pares distintos de lados congruentes adyacentes.

L

lateral area The sum of the areas of the lateral faces of the figure.

área lateral La suma de las áreas de las caras laterales de la figura.

lateral edges The intersection of two lateral faces.

aristas laterales La intersección de dos caras laterales.

lateral faces The faces that join the bases of a solid.

caras laterales Las caras que unen las bases de un sólido.

lateral surface of a cone The curved surface that joins the base of a cone to the vertex.

superficie lateral de un cono La superficie curvada que une la base de un cono con el vértice.

lateral surface of a cylinder The curved surface that joins the bases of a cylinder.

superficie lateral de un cilindro La superficie curvada que une las bases de un cilindro.

leading coefficient The coefficient of the first term when a polynomial is in standard form.

coeficiente líder El coeficiente del primer término cuando un polinomio está en forma estándar.

legs of a trapezoid The nonparallel sides in a trapezoid.

patas de un trapecio Los lados no paralelos en un trapezoide.

legs of an isosceles triangle The two congruent sides of an isosceles triangle.

patas de un triángulo isósceles Los dos lados congruentes de un triángulo isósceles.

like radical expressions Radicals in which both the index and the radicand are the same.

expresiones radicales semejantes Radicales en los que tanto el índice como el radicand son iguales.

like terms Terms with the same variables, with corresponding variables having the same exponent.

términos semejantes Términos con las mismas variables, con las variables correspondientes que tienen el mismo exponente.

line A line is made up of points, has no thickness or width, and extends indefinitely in both directions.

línea Una línea está formada por puntos, no tiene espesor ni anchura, y se extiende indefinidamente en ambas direcciones.

line of fit A line used to describe the trend of the data in a scatter plot.

línea de ajuste Una línea usada para describir la tendencia de los datos en un diagrama de dispersión.

line of reflection A line midway between a preimage and an image; The line in which a reflection flips the graph of a function.

línea de reflexión Una línea a medio camino entre una preimagen y una imagen; La línea en la que una reflexión voltea la gráfica de una función.

line of symmetry An imaginary line that separates a figure into two congruent parts.

línea de simetría Una línea imaginaria que separa una figura en dos partes congruentes.

line segment A measurable part of a line that consists of two points, called endpoints, and all of the points between them.

segmento de línea Una parte medible de una línea que consta de dos puntos, llamados extremos, y todos los puntos entre ellos.

line symmetry A graph has line symmetry if it can be reflected in a vertical line so that each half of the graph maps exactly to the other half.

simetría de línea Un gráfico tiene simetría de línea si puede reflejarse en una línea vertical, de modo que cada mitad del gráfico se asigna exactamente a la otra mitad.

linear equation An equation that can be written in the form $Ax + By = C$ with a graph that is a straight line.

ecuación lineal Una ecuación que puede escribirse de la forma $Ax + By = C$ con un gráfico que es una línea recta.

linear extrapolation The use of a linear equation to predict values that are outside the range of data.

extrapolación lineal El uso de una ecuación lineal para predecir valores que están fuera del rango de datos.

linear function A function in which no independent variable is raised to a power greater than 1; A function with a graph that is a line.

función lineal Una función en la que ninguna variable independiente se eleva a una potencia mayor que 1; Una función con un gráfico que es una línea.

linear inequality A half-plane with a boundary that is a straight line.

desigualdad lineal Un medio plano con un límite que es una línea recta.

linear interpolation The use of a linear equation to predict values that are inside the range of data.

interpolación lineal El uso de una ecuación lineal para predecir valores que están dentro del rango de datos.

linear pair A pair of adjacent angles with noncommon sides that are opposite rays.

par lineal Un par de ángulos adyacentes con lados no comunes que son rayos opuestos.

linear programming The process of finding the maximum or minimum values of a function for a region defined by a system of inequalities.

programación lineal El proceso de encontrar los valores máximos o mínimos de una función para una región definida por un sistema de desigualdades.

linear regression An algorithm used to find a precise line of fit for a set of data.

regresión lineal Un algoritmo utilizado para encontrar una línea precisa de ajuste para un conjunto de datos.

linear transformation One or more operations performed on a set of data that can be written as a linear function.

transformación lineal Una o más operaciones realizadas en un conjunto de datos que se pueden escribir como una función lineal.

literal equation A formula or equation with several variables.

ecuación literal Un formula o ecuación con varias variables.

logarithm In $x = b^y$, y is called the logarithm, base b, of x.

logaritmo En $x = b^y$, y se denomina logaritmo, base b, de x.

logarithmic equation An equation that contains one or more logarithms.

ecuación logarítmica Una ecuación que contiene uno o más logaritmos.

logarithmic function A function of the form $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.

función logarítmica Una función de la forma $f(x) = \log_b x$, donde $b > 0$ y $b \neq 1$.

logically equivalent Statements with the same truth value.

lógicamente equivalentes Declaraciones con el mismo valor de verdad.

lower quartile The median of the lower half of a set of data.

cuartil inferior La mediana de la mitad inferior de un conjunto de datos.

M

magnitude The length of a vector from the initial point to the terminal point.

magnitud La longitud de un vector desde el punto inicial hasta el punto terminal.

magnitude of symmetry The smallest angle through which a figure can be rotated so that it maps onto itself.

magnitud de la simetria El ángulo más pequeño a través del cual una figura se puede girar para que se cargue sobre sí mismo.

major arc An arc with measure greater than 180°.

arco mayor Un arco con una medida superior a 180°.

mapping An illustration that shows how each element of the domain is paired with an element in the range.

cartografía Una ilustración que muestra cómo cada elemento del dominio está emparejado con un elemento del rango.

marginal frequencies In a two-way frequency table, the frequencies in the totals row and column; The totals of each subcategory in a two-way frequency table.

frecuencias marginales En una tabla de frecuencias de dos vías, las frecuencias en los totales de fila y columna; Los totales de cada subcategoría en una tabla de frecuencia bidireccional.

maximum The highest point on the graph of a function.

máximo El punto más alto en la gráfica de una función.

maximum error of the estimate The maximum difference between the estimate of the population mean and its actual value.

error máximo de la estimación La diferencia máxima entre la estimación de la media de la población y su valor real.

measurement data Data that have units and can be measured.

medicion de datos Datos que tienen unidades y que pueden medirse.

measures of center Measures of what is average.

medidas del centro Medidas de lo que es promedio.

measures of spread Measures of how spread out the data are.

medidas de propagación Medidas de cómo se extienden los datos son.

median The beginning of the second quartile that separates the data into upper and lower halves.

mediana El comienzo del segundo cuartil que separa los datos en mitades superior e inferior.

median of a triangle A line segment with endpoints that are a vertex of the triangle and the midpoint of the side opposite the vertex.

mediana de un triángulo Un segmento de línea con extremos que son un vértice del triángulo y el punto medio del lado opuesto al vértice.

metric A rule for assigning a number to some characteristic or attribute.

métrico Una regla para asignar un número a alguna caracteristica o atribuye.

midline The line about which the graph of a function oscillates.

linea media La línea sobre la cual oscila la gráfica de una función periódica.

midpoint The point on a line segment halfway between the endpoints of the segment.

punto medio El punto en un segmento de línea a medio camino entre los extremos del segmento.

midsegment of a trapezoid The segment that connects the midpoints of the legs of a trapezoid.

segment medio de un trapecio El segmento que conecta los puntos medios de las patas de un trapecio.

midsegment of a triangle The segment that connects the midpoints of the legs of a triangle.

segment medio de un triángulo El segmento que conecta los puntos medios de las patas de un triángulo.

minimum The lowest point on the graph of a function.

mínimo El punto más bajo en la gráfica de una función.

minor arc An arc with measure less than 180°.

mixture problems Problems that involve creating a mixture of two or more kinds of things and then determining some quantity of the resulting mixture.

monomial A number, a variable, or a product of a number and one or more variables.

monomial function A function of the form $f(x) = ax^n$, for which a is a nonzero real number and n is a positive integer.

multi-step equation An equation that uses more than one operation to solve it.

multiplicative identity Because the product of any number a and 1 is equal to a, 1 is the multiplicative identity.

multiplicative inverses Two numbers with a product of 1.

multiplicity The number of times a number is a zero for a given polynomial.

mutually exclusive Events that cannot occur at the same time.

arco menor Un arco con una medida inferior a 180°.

problemas de mezcla Problemas que implican crear una mezcla de dos o más tipos de cosas y luego determinar una cierta cantidad de la mezcla resultante.

monomio Un número, una variable, o un producto de un número y una o más variables.

función monomial Una función de la forma $f(x) = ax^n$, para la cual a es un número real no nulo y n es un entero positivo.

ecuaciones de varios pasos Una ecuación que utiliza más de una operación para resolverla.

identidad multiplicativa Dado que el producto de cualquier número a y 1 es igual a, 1 es la identidad multiplicativa.

inversos multiplicativos Dos números con un producto es igual a 1.

multiplicidad El número de veces que un número es cero para un polinomio dado.

mutuamente exclusivos Eventos que no pueden ocurrir al mismo tiempo.

N

natural base exponential function An exponential function with base e, written as $y = e^x$.

natural logarithm The inverse of the natural base exponential function, most often abbreviated as ln x.

negation A statement that has the opposite meaning, as well as the opposite truth value, of an original statement.

negative Where the graph of a function lies below the x-axis.

negative correlation Bivariate data in which y decreases as x increases.

negative exponent An exponent that is a negative number.

función exponencial de base natural Una función exponencial con base e, escrita como $y = e^x$.

logaritmo natural La inversa de la función exponencial de base natural, más a menudo abreviada como ln x.

negación Una declaración que tiene el significado opuesto, así como el valor de verdad opuesto, de una declaración original.

negativo Donde la gráfica de una función se encuentra debajo del eje x.

correlación negativa Datos bivariate en el cual y disminuye a x aumenta.

exponente negativo Un exponente que es un número negativo.

negatively skewed distribution A distribution that typically has a median greater than the mean and less data on the left side of the graph.

net A two-dimensional figure that forms the surfaces of a three-dimensional object when folded.

no correlation Bivariate data in which x and y are not related.

nonlinear function A function in which a set of points cannot all lie on the same line

nonrigid motion A transformation that changes the dimensions of a given figure.

normal distribution A continuous, symmetric, bell-shaped distribution of a random variable.

***n*th root** If $a^n = b$ for a positive integer n, then a is the nth root of b.

***n*th term of an arithmetic sequence** The nth term of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$, where n is a positive integer.

numerical expression A mathematical phrase involving only numbers and mathematical operations.

distribución negativamente sesgada Una distribución que típicamente tiene una mediana mayor que la media y menos datos en el lado izquierdo del gráfico.

red Una figura bidimensional que forma las superficies de un objeto tridimensional cuando se dobla.

sin correlación Datos bivariados en los que x e y no están relacionados.

función no lineal Una función en la que un conjunto de puntos no puede estar en la misma línea

movimiento no rígida Una transformación que cambia las dimensiones de una figura dada.

distribución normal Distribución con forma de campana, simétrica y continua de una variable aleatoria.

raíz enésima Si $a^n = b$ para cualquier entero positive n, entonces a se llama una raíz enésima de b.

enésimo término de una secuencia aritmética El enésimo término de una secuencia aritmética con el primer término a_1 y la diferencia común d viene dado por $a_n = a_1 + (n - 1)d$, donde n es un número entero positivo.

expresión numérica Una frase matemática que implica sólo números y operaciones matemáticas.

O

oblique asymptote An asymptote that is neither horizontal nor vertical.

observational study Members of a sample are measured or observed without being affected by the study.

octant One of the eight divisions of three-dimensional space.

odd functions Functions that are symmetric in the origin.

one-to-one function A function for which each element of the range is paired with exactly one element of the domain.

onto function A function for which the codomain is the same as the range.

asíntota oblicua Una asíntota que no es ni horizontal ni vertical.

estudio de observación Los miembros de una muestra son medidos o observados sin ser afectados por el estudio.

octante Una de las ocho divisiones del espacio tridimensional.

funciones extrañas Funciones que son simétricas en el origen.

función biunívoca Función para la cual cada elemento del rango está emparejado con exactamente un elemento del dominio.

sobre la función Función para la cual el codomain es el mismo que el rango.

Glossary · Glosario

open half-plane The solution of a linear inequality that does not include the boundary line.

medio plano abierto La solución de una desigualdad linear que no incluye la línea de límite.

opposite rays Two collinear rays with a common endpoint.

rayos opuestos Dos rayos colineales con un punto final común.

optimization The process of seeking the optimal value of a function subject to given constraints.

optimización El proceso de buscar el valor óptimo de una función sujeto a restricciones dadas.

order of symmetry The number of times a figure maps onto itself.

orden de la simetría El número de veces que una figura se asigna a sí misma.

ordered triple Three numbers given in a specific order used to locate points in space.

triple ordenado Tres números dados en un orden específico usado para localizar puntos en el espacio.

orthocenter The point of concurrency of the altitudes of a triangle.

ortocentro El punto de concurrencia de las altitudes de un triángulo.

orthographic drawing The two-dimensional views of the top, left, front, and right sides of an object.

dibujo ortográfico Las vistas bidimensionales de los lados superior, izquierdo, frontal y derecho de un objeto.

oscillation How much the graph of a function varies between its extreme values as it approaches positive or negative infinity.

oscilación Cuánto la gráfica de una función varía entre sus valores extremos cuando se acerca al infinito positivo o negativo.

outcome The result of a single event; The result of a single performance or trial of an experiment.

resultado El resultado de un solo evento; El resultado de un solo rendimiento o ensayo de un experimento.

outlier A value that is more than 1.5 times the interquartile range above the third quartile or below the first quartile.

parte aislada Un valor que es más de 1,5 veces el rango intercuartílico por encima del tercer cuartil o por debajo del primer cuartil.

P

parabola A curved shape that results when a cone is cut at an angle by a plane that intersects the base; The graph of a quadratic function.

parábola Forma curvada que resulta cuando un cono es cortado en un ángulo por un plano que interseca la base; La gráfica de una función cuadrática.

paragraph proof A paragraph that explains why the conjecture for a given situation is true.

prueba de párrafo Un párrafo que explica por qué la conjetura para una situación dada es verdadera.

parallel lines Coplanar lines that do not intersect; Nonvertical lines in the same plane that have the same slope.

líneas paralelas Líneas coplanares que no se intersecan; Líneas no verticales en el mismo plano que tienen pendientes iguales.

parallel planes Planes that do not intersect.

planos paralelas Planos que no se intersecan.

parallelogram A quadrilateral with both pairs of opposite sides parallel.

paralelogramo Un cuadrilátero con ambos pares de lados opuestos paralelos.

parameter A measure that describes a characteristic of a population; A value in the equation of a function that can be varied to yield a family of functions.

parámetro Una medida que describe una característica de una población; Un valor en la ecuación de una función que se puede variar para producir una familia de funciones.

parent function The simplest of functions in a family.

función basica La función más fundamental de un familia de funciones.

Pascal's triangle A triangle of numbers in which a row represents the coefficients of an expanded binomial $(a + b)^n$.

triángulo de Pascal Un triángulo de números en el que una fila representa los coeficientes de un binomio expandido $(a + b)^n$.

percent rate of change The percent of increase per time period.

por ciento tasa de cambio El porcentaje de aumento por período de tiempo.

percentile A measure that tells what percent of the total scores were below a given score.

percentil Una medida que indica qué porcentaje de las puntuaciones totales estaban por debajo de una puntuación determinada.

perfect cube A rational number with a cube root that is a rational number.

cubo perfecto Un número racional con un raíz cúbica que es un número racional.

perfect square A rational number with a square root that is a rational number.

cuadrado perfecto Un número racional con un raíz cuadrada que es un número racional.

perfect square trinomials Squares of binomials.

trinomio cuadrado perfecto Cuadrados de los binomios.

perimeter The sum of the lengths of the sides of a polygon.

perímetro La suma de las longitudes de los lados de un polígono.

period The horizontal length of one cycle.

período La longitud horizontal de un ciclo.

periodic function A function with y-values that repeat at regular intervals.

función periódica Una función con y-valores aquella repetición con regularidad.

permutation An arrangement of objects in which order is important.

permutación Un arreglo de objetos en el que el orden es importante.

perpendicular Intersecting at right angles.

perpendicular Intersección en ángulo recto.

perpendicular bisector Any line, segment, or ray that passes through the midpoint of a segment and is perpendicular to that segment.

mediatriz Cualquier línea, segmento o rayo que pasa por el punto medio de un segmento y es perpendicular a ese segmento.

perpendicular lines Nonvertical lines in the same plane for which the product of the slopes is –1.

líneas perpendiculares Líneas no verticales en el mismo plano para las que el producto de las pendientes es −1.

phase shift A horizontal translation of the graph of a trigonometric function.

cambio de fase Una traducción horizontal de la gráfica de una función trigonométrica.

pi The ratio $\frac{\text{cricumference}}{\text{diameter}}$.

pi Relación $\frac{\text{circunferencia}}{\text{diámetro}}$.

piecewise-defined function A function defined by at least two subfunctions, each of which is defined differently depending on the interval of the domain.

función definida por piezas Una función definida por al menos dos subfunciones, cada una de las cuales se define de manera diferente dependiendo del intervalo del dominio.

piecewise-linear function A function defined by at least two linear subfunctions, each of which is defined differently depending on the interval of the domain.

función lineal por piezas Una función definida por al menos dos subfunciones lineal, cada una de las cuales se define de manera diferente dependiendo del intervalo del dominio.

plane A flat surface made up of points that has no depth and extends indefinitely in all directions.

plano Una superficie plana compuesta de puntos que no tiene profundidad y se extiende indefinidamente en todas las direcciones.

plane symmetry When a plane intersects a three-dimensional figure so one half is the reflected image of the other half.

simetría plana Cuando un plano cruza una figura tridimensional, una mitad es la imagen reflejada de la otra mitad.

Platonic solid One of five regular polyhedra.

sólido platónico Uno de cinco poliedros regulares.

point A location with no size, only position.

punto Una ubicación sin tamaño, solo posición.

point discontinuity An area that appears to be a hole in a graph.

discontinuidad de punto Un área que parece ser un agujero en un gráfico.

point of concurrency The point of intersection of concurrent lines.

punto de concurrencia El punto de intersección de líneas concurrentes.

point of symmetry The point about which a figure is rotated.

punto de simetría El punto sobre el que se gira una figura.

point of tangency For a line that intersects a circle in one point, the point at which they intersect.

punto de tangencia Para una línea que cruza un círculo en un punto, el punto en el que se cruzan.

point symmetry A figure or graph has this when a figure is rotated 180° about a point and maps exactly onto the other part.

simetría de punto Una figura o gráfica tiene esto cuando una figura se gira 180° alrededor de un punto y se mapea exactamente sobre la otra parte.

polygon A closed plane figure with at least three straight sides.

polígono Una figura plana cerrada con al menos tres lados rectos.

polyhedron A closed three-dimensional figure made up of flat polygonal regions.

poliedros Una figura tridimensional cerrada formada por regiones poligonales planas.

polynomial A monomial or the sum of two or more monomials.

polinomio Un monomio o la suma de dos o más monomios.

polynomial function A continuous function that can be described by a polynomial equation in one variable.

función polinómica Función continua que puede describirse mediante una ecuación polinómica en una variable.

polynomial identity A polynomial equation that is true for any values that are substituted for the variables.

identidad polinomial Una ecuación polinómica que es verdadera para cualquier valor que se sustituya por las variables.

population All of the members of a group of interest about which data will be collected.

población Todos los miembros de un grupo de interés sobre cuáles datos serán recopilados.

population proportion The number of members in the population with a particular characteristic divided by the total number of members in the population.

proporción de la población El número de miembros en la población con una característica particular dividida por el número total de miembros en la población.

positive Where the graph of a function lies above the *x*-axis.

positiva Donde la gráfica de una función se encuentra por encima del eje *x*.

positive correlation Bivariate data in which *y* increases as *x* increases.

correlación positiva Datos bivariate en el cual *y* aumenta a *x* disminuye.

positively skewed distribution A distribution that typically has a mean greater than the median.

distribución positivamente sesgada Una distribución que típicamente tiene una media mayor que la mediana.

postulate A statement that is accepted as true without proof.

postulado Una declaración que se acepta como verdadera sin prueba.

power function A function of the form $f(x) = ax^n$, where a and n are nonzero real numbers.

función de potencia Una ecuación polinomial que es verdadera para una función de la forma $f(x) = ax^n$, donde a y n son números reales no nulos.

precision The repeatability, or reproducibility, of a measurement.

precisión La repetibilidad, o reproducibilidad, de una medida.

preimage The original figure in a transformation.

preimagen La figura original en una transformación.

prime polynomial A polynomial that cannot be written as a product of two polynomials with integer coefficients.

polinomio primo Un polinomio que no puede escribirse como producto de dos polinomios con coeficientes enteros.

principal root The nonnegative root of a number.

raíz principal La raíz no negativa de un número.

principal square root The nonnegative square root of a number.

raíz cuadrada principal La raíz cuadrada no negativa de un número.

principal values The values in the restricted domains of trigonometric functions.

valores principales Valores de los dominios restringidos de las funciones trigonométricas.

principle of superposition Two figures are congruent if and only if there is a rigid motion or series of rigid motions that maps one figure exactly onto the other.

principio de superposición Dos figuras son congruentes si y sólo si hay un movimiento rígido o una serie de movimientos rígidos que traza una figura exactamente sobre la otra.

prism A polyhedron with two parallel congruent bases connected by parallelogram faces.

prisma Un poliedro con dos bases congruentes paralelas conectadas por caras de paralelogramo.

probability The number of outcomes in which a specified event occurs to the total number of trials.

probabilidad El número de resultados en los que se produce un evento especificado al número total de ensayos.

probability distribution A function that maps the sample space to the probabilities of the outcomes in the sample space for a particular random variable.

distribución de probabilidad Una función que mapea el espacio de muestra a las probabilidades de los resultados en el espacio de muestra para una variable aleatoria particular.

probability model A mathematical representation of a random event that consists of the sample space and the probability of each outcome.

modelo de probabilidad Una representación matemática de un evento aleatorio que consiste en el espacio muestral y la probabilidad de cada resultado.

projectile motion problems Problems that involve objects being thrown or dropped.

problemas de movimiento del proyectil Problemas que involucran objetos que se lanzan o caen.

proof A logical argument in which each statement is supported by a statement that is accepted as true.

prueba Un argumento lógico en el que cada sentencia está respaldada por una sentencia aceptada como verdadera.

proof by contradiction One assumes that the statement to be proven is false and then uses logical reasoning to deduce that a statement contradicts a postulate, theorem, or one of the assumptions.

prueba por contradicción Se supone que la afirmación a ser probada es falsa y luego utiliza el razonamiento lógico para deducir que una afirmación contradice un postulado, teorema o uno de los supuestos.

proportion A statement that two ratios are equivalent.

proporción Una declaración de que dos proporciones son equivalentes.

pure imaginary number A number of the form bi, where b is a real number and i is the imaginary unit.

número imaginario puro Un número de la forma bi, donde b es un número real e i es la unidad imaginaria.

pyramid A polyhedron with a polygonal base and three or more triangular faces that meet at a common vertex.

pirámide Poliedro con una base poligonal y tres o más caras triangulares que se encuentran en un vértice común.

Pythagorean identities Identities that express the Pythagorean Theorem in terms of the trigonometric functions.

identidades pitagóricas Identidades que expresan el Teorema de Pitágoras en términos de las funciones trigonométricas.

Pythagorean triple A set of three nonzero whole numbers that make the Pythagorean Theorem true.

triplete Pitágorico Un conjunto de tres números enteros distintos de cero que hacen que el Teorema de Pitágoras sea verdadero.

Q

quadrantal angle An angle in standard position with a terminal side that coincides with one of the axes.

ángulo de cuadrante Un ángulo en posición estándar con un lado terminal que coincide con uno de los ejes.

quadratic equation An equation that includes a quadratic expression.

ecuación cuadrática Una ecuación que incluye una expresión cuadrática.

quadratic expression An expression in one variable with a degree of 2.

expresión cuadrática Una expresión en una variable con un grado de 2.

quadratic form A form of polynomial equation, $au^2 + bu + c$, where u is an algebraic expression in x.

forma cuadrática Una forma de ecuación polinomial, $au^2 + bu + c$, donde u es una expresión algebraica en x.

quadratic function A function with an equation of the form $y = ax^2 + bx + c$, where $a \neq 0$.

función cuadrática Una función con una ecuación de la forma $y = ax^2 + bx + c$, donde $a \neq 0$.

quadratic inequality An inequality that includes a quadratic expression.

desigualdad cuadrática Una desigualdad que incluye una expresión cuadrática.

quadratic relations Equations of parabolas with horizontal axes of symmetry that are not functions.

relaciones cuadráticas Ecuaciones de parábolas con ejes horizontales de simetría que no son funciones.

quartic function A fourth-degree function.

función cuartica Una función de cuarto grado.

quartiles Measures of position that divide a data set arranged in ascending order into four groups, each containing about one fourth or 25% of the data.

cuartiles Medidas de posición que dividen un conjunto de datos dispuestos en orden ascendente en cuatro grupos, cada uno de los cuales contiene aproximadamente un cuarto o el 25% de los datos.

quintic function A fifth-degree function.

función quíntica Una función de quinto grado.

R

radian A unit of angular measurement equal to $\frac{180°}{\pi}$ or about 57.296°.

radián Una unidad de medida angular igual o $\frac{180°}{\pi}$ alrededor de 57.296°.

radical equation An equation with a variable in a radicand.

ecuación radical Una ecuación con una variable en un radicand.

radical expression An expression that contains a radical symbol, such as a square root.

expresión radicales Una expresión que contiene un símbolo radical, tal como una raíz cuadrada.

radical form When an expression contains a radical symbol.

forma radical Cuando una expresión contiene un símbolo radical.

radical function A function that contains radicals with variables in the radicand.

función radical Función que contiene radicales con variables en el radicand.

radicand The expression under a radical sign.

radicando La expresión debajo del signo radical.

radius of a circle or sphere A line segment from the center to a point on a circle or sphere.

radio de un círculo o esfera Un segmento de línea desde el centro hasta un punto en un círculo o esfera.

radius of a regular polygon The radius of the circle circumscribed about a regular polygon.

radio de un polígono regular El radio del círculo circunscrito alrededor de un polígono regular.

range The difference between the greatest and least values in a set of data; The set of second numbers of the ordered pairs in a relation; The set of y-values that actually result from the evaluation of the function.

rango La diferencia entre los valores de datos más grande or menos en un sistema de datos; El conjunto de los segundos números de los pares ordenados de una relación; El conjunto de valores y que realmente resultan de la evaluación de la función.

rate of change How a quantity is changing with respect to a change in another quantity.

tasa de cambio Cómo cambia una cantidad con respecto a un cambio en otra cantidad.

rational equation An equation that contains at least one rational expression.

ecuación racional Una ecuación que contiene al menos una expresión racional.

rational exponent An exponent that is expressed as a fraction.

exponente racional Un exponente que se expresa como una fracción.

rational expression A ratio of two polynomial expressions.

expresión racional Una relación de dos expresiones polinomiales.

rational function An equation of the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial expressions and $b(x) \neq 0$.

función racional Una ecuación de la forma $f(x) = \frac{a(x)}{b(x)}$, donde $a(x)$ y $b(x)$ son expresiones polinomiales y $b(x) \neq 0$.

rational inequality An inequality that contains at least one rational expression.

desigualdad racional Una desigualdad que contiene al menos una expresión racional.

rationalizing the denominator A method used to eliminate radicals from the denominator of a fraction or fractions from a radicand.

racionalizando el denominador Método utilizado para eliminar radicales del denominador de una fracción o fracciones de una radicand.

ray Part of a line that starts at a point and extends to infinity.

rayo Parte de una línea que comienza en un punto y se extiende hasta el infinito.

reciprocal function An equation of the form $f(x) = \frac{n}{b(x)}$, where n is a real number and $b(x)$ is a linear expression that cannot equal 0.

función recíproca Una ecuación de la forma $f(x) = \frac{n}{b(x)}$, donde n es un número real y $b(x)$ es una expresión lineal que no puede ser igual a 0.

reciprocal trigonometric functions Trigonometric functions that are reciprocals of each other.

funciones trigonométricas recíprocas Funciones trigonométricas que son reciprocales entre sí.

reciprocals Two numbers with a product of 1.

recíprocos Dos números con un producto de 1.

rectangle A parallelogram with four right angles.

rectángulo Un paralelogramo con cuatro ángulos rectos.

recursive formula A formula that gives the value of the first term in the sequence and then defines the next term by using the preceding term.

formula recursiva Una fórmula que da el valor del primer término en la secuencia y luego define el siguiente término usando el término anterior.

reduction A dilation with a scale factor between 0 and 1.

reducción Una dilatación con un factor de escala entre 0 y 1.

reference angle The acute angle formed by the terminal side of an angle and the x-axis.

ángulo de referencia El ángulo agudo formado por el lado terminal de un ángulo en posición estándar y el eje x.

reflection A function in which the preimage is reflected in the line of reflection; A transformation in which a figure, line, or curve is flipped across a line.

reflexión Función en la que la preimagen se refleja en la línea de reflexión; Una transformación en la que una figura, línea o curva se voltea a través de una línea.

regression function A function generated by an algorithm to find a line or curve that fits a set of data.

función de regresión Función generada por un algoritmo para encontrar una línea o curva que se ajuste a un conjunto de datos.

regular polygon A convex polygon that is both equilateral and equiangular.

polígono regular Un polígono convexo que es a la vez equilátero y equiangular.

regular polyhedron A polyhedron in which all of its faces are regular congruent polygons and all of the edges are congruent.

poliedro regular Un poliedro en el que todas sus caras son polígonos congruentes regulares y todos los bordes son congruentes.

regular pyramid A pyramid with a base that is a regular polygon.

pirámide regular Una pirámide con una base que es un polígono regular.

regular tessellation A tessellation formed by only one type of regular polygon.

teselado regular Un teselado formado por un solo tipo de polígono regular.

relation A set of ordered pairs.

relación Un conjunto de pares ordenados.

relative frequency In a two-way frequency table, the ratios of the number of observations in a category to the total number of observations; The ratio of the number of observations in a category to the total number of observations.

frecuencia relativa En una tabla de frecuencia bidireccional, las relaciones entre el número de observaciones en una categoría y el número total de observaciones; La relación entre el número de observaciones en una categoría y el número total de observaciones.

relative maximum A point on the graph of a function where no other nearby points have a greater y-coordinate.

máximo relativo Un punto en la gráfica de una función donde ningún otro punto cercano tiene una coordenada y mayor.

relative minimum A point on the graph of a function where no other nearby points have a lesser y-coordinate.

mínimo relativo Un punto en la gráfica de una función donde ningún otro punto cercano tiene una coordenada y menor.

remote interior angles Interior angles of a triangle that are not adjacent to an exterior angle.

ángulos internos no adyacentes Ángulos interiores de un triángulo que no están adyacentes a un ángulo exterior.

residual The difference between an observed y-value and its predicted y-value on a regression line.

residual La diferencia entre un valor de y observado y su valor de y predicho en una línea de regresión.

rhombus A parallelogram with all four sides congruent.

rombo Un paralelogramo con los cuatro lados congruentes.

rigid motion A transformation that preserves distance and angle measure.

movimiento rígido Una transformación que preserva la distancia y la medida del ángulo.

Glossary · Glosario

English	Español
root A solution of an equation.	**raíz** Una solución de una ecuación.
rotation A function that moves every point of a preimage through a specified angle and direction about a fixed point.	**rotación** Función que mueve cada punto de una preimagen a través de un ángulo y una dirección especificados alrededor de un punto fijo.
rotational symmetry A figure can be rotated less than 360° about a point so that the image and the preimage are indistinguishable.	**simetría rotacional** Una figura puede girar menos de 360° alrededor de un punto para que la imagen y la preimagen sean indistinguibles.

S

English	Español
sample A subset of a population.	**muestra** Un subconjunto de una población.
sample space The set of all possible outcomes.	**espacio muestral** El conjunto de todos los resultados posibles.
sampling error The variation between samples taken from the same population.	**error de muestreo** La variación entre muestras tomadas de la misma población.
scale The distance between tick marks on the *x*- and *y*-axes.	**escala** La distancia entre las marcas en los ejes *x* e *y*.
scale factor of a dilation The ratio of a length on an image to a corresponding length on the preimage.	**factor de escala de una dilatación** Relación de una longitud en una imagen con una longitud correspondiente en la preimagen.
scatter plot A graph of bivariate data that consists of ordered pairs on a coordinate plane.	**gráfica de dispersión** Una gráfica de datos bivariados que consiste en pares ordenados en un plano de coordenadas.
secant Any line or ray that intersects a circle in exactly two points; The ratio of the length of the hypotenuse to the length of the leg adjacent to the angle.	**secante** Cualquier línea o rayo que cruce un círculo en exactamente dos puntos; Relación entre la longitud de la hipotenusa y la longitud de la pierna adyacente al ángulo.
sector A region of a circle bounded by a central angle and its intercepted arc.	**sector** Una región de un círculo delimitada por un ángulo central y su arco interceptado.
segment bisector Any segment, line, plane, or point that intersects a line segment at its midpoint.	**bisectriz del segmento** Cualquier segmento, línea, plano o punto que interseca un segmento de línea en su punto medio.
self-selected sample Members volunteer to be included in the sample.	**muestra auto-seleccionada** Los miembros se ofrecen como voluntarios para ser incluidos en la muestra.
semicircle An arc that measures exactly 180°.	**semicírculo** Un arco que mide exactamente 180°.
semiregular tessellation A tessellation formed by two or more regular polygons.	**teselado semiregular** Un teselado formado por dos o más polígonos regulares.
sequence A list of numbers in a specific order.	**secuencia** Una lista de números en un orden específico.

English	Español
series The indicated sum of the terms in a sequence.	**serie** La suma indicada de los términos en una secuencia.
set-builder notation Mathematical notation that describes a set by stating the properties that its members must satisfy.	**notación de construción de conjuntos** Notación matemática que describe un conjunto al declarar las propiedades que sus miembros deben satisfacer.
sides of an angle The rays that form an angle.	**lados de un ángulo** Los rayos que forman un ángulo.
sigma notation A notation that uses the Greek uppercase letter S to indicate that a sum should be found.	**notación de sigma** Una notación que utiliza la letra mayúscula griega S para indicar que debe encontrarse una suma.
significant figures The digits of a number that are used to express a measure to an appropriate degree of accuracy.	**dígitos significantes** Los dígitos de un número que se utilizan para expresar una medida con un grado apropiado de precisión.
similar polygons Two figures are similar polygons if one can be obtained from the other by a dilation or a dilation with one or more rigid motions.	**polígonos similares** Dos figuras son polígonos similares si uno puede ser obtenido del otro por una dilatación o una dilatación con uno o más movimientos rígidos.
similar solids Solid figures with the same shape but not necessarily the same size.	**sólidos similares** Figuras sólidas con la misma forma pero no necesariamente del mismo tamaño.
similar triangles Triangles in which all of the corresponding angles are congruent and all of the corresponding sides are proportional.	**triángulos similares** Triángulos en los cuales todos los ángulos correspondientes son congruentes y todos los lados correspondientes son proporcionales.
similarity ratio The scale factor between two similar polygons.	**relación de similitud** El factor de escala entre dos polígonos similares.
similarity transformation A transformation composed of a dilation or a dilation and one or more rigid motions.	**transformación de similitud** Una transformación compuesto por una dilatación o una dilatación y uno o más movimientos rígidos.
simple random sample Each member of the population has an equal chance of being selected as part of the sample.	**muestra aleatoria simple** Cada miembro de la población tiene la misma posibilidad de ser seleccionado como parte de la muestra.
simplest form An expression is in simplest form when it is replaced by an equivalent expression having no like terms or parentheses.	**forma reducida** Una expresión está reducida cuando se puede sustituir por una expresión equivalente que no tiene ni términos semejantes ni paréntesis.
simulation The use of a probability model to imitate a process or situation so it can be studied.	**simulación** El uso de un modelo de probabilidad para imitar un proceso o situación para que pueda ser estudiado.
sine The ratio of the length of the leg opposite an angle to the length of the hypotenuse.	**seno** La relación entre la longitud de la pierna opuesta a un ángulo y la longitud de la hipotenusa.

sinusoidal function A function that can be produced by translating, reflecting, or dilating the sine function.	**función sinusoidal** Función que puede producirse traduciendo, reflejando o dilatando la función sinusoidal.
skew lines Noncoplanar lines that do not intersect.	**líneas alabeadas** Líneas no coplanares que no se cruzan.
slant height of a pyramid or right cone The length of a segment with one endpoint on the base edge of the figure and the other at the vertex.	**altura inclinada de una pirámide o cono derecho** La longitud de un segmento con un punto final en el borde base de la figura y el otro en el vértice.
slope The rate of change in the y-coordinates (rise) to the corresponding change in the x-coordinates (run) for points on a line.	**pendiente** La tasa de cambio en las coordenadas y (subida) al cambio correspondiente en las coordenadas x (carrera) para puntos en una línea.
slope criteria Outlines a method for proving the relationship between lines based on a comparison of the slopes of the lines.	**criterios de pendiente** Describe un método para probar la relación entre líneas basado en una comparación de las pendientes de las líneas.
solid of revolution A solid figure obtained by rotating a shape around an axis.	**sólido de revolución** Una figura sólida obtenida girando una forma alrededor de un eje.
solution A value that makes an equation true.	**solución** Un valor que hace que una ecuación sea verdadera.
solve an equation The process of finding all values of the variable that make the equation a true statement.	**resolver una ecuación** El proceso en que se hallan todos los valores de la variable que hacen verdadera la ecuación.
solving a triangle When you are given measurements to find the unknown angle and side measures of a triangle.	**resolver un triángulo** Cuando se le dan mediciones para encontrar el ángulo desconocido y las medidas laterales de un triángulo.
space A boundless three-dimensional set of all points.	**espacio** Un conjunto tridimensional ilimitado de todos los puntos.
sphere A set of all points in space equidistant from a given point called the center of the sphere.	**esfera** Un conjunto de todos los puntos del espacio equidistantes de un punto dado llamado centro de la esfera.
square A parallelogram with all four sides and all four angles congruent.	**cuadrado** Un paralelogramo con los cuatro lados y los cuatro ángulos congruentes.
square root One of two equal factors of a number.	**raíz cuadrada** Uno de dos factores iguales de un número.
square root function A radical function that contains the square root of a variable expression.	**función raíz cuadrada** Función radical que contiene la raíz cuadrada de una expresión variable.
square root inequality An inequality that contains the square root of a variable expression.	**square root inequality** Una desigualdad que contiene la raíz cuadrada de una expresión variable.
standard deviation A measure that shows how data deviate from the mean.	**desviación tipica** Una medida que muestra cómo los datos se desvían de la media.

standard error of the mean The standard deviation of the distribution of sample means taken from a population.	**error estandar de la media** La desviación estándar de la distribución de los medios de muestra se toma de una población.
standard form of a linear equation Any linear equation can be written in this form, $Ax + By = C$, where $A \geq 0$, A and B are not both 0, and A, B, and C are integers with a greatest common factor of 1.	**forma estándar de una ecuación lineal** Cualquier ecuación lineal se puede escribir de esta forma, $Ax + By = C$, donde $A \geq 0$, A y B no son ambos 0, y A, B y C son enteros con el mayor factor común de 1.
standard form of a polynomial A polynomial that is written with the terms in order from greatest degree to least degree.	**forma estándar de un polinomio** Un polinomio que se escribe con los términos en orden del grado más grande a menos grado.
standard form of a quadratic equation A quadratic equation can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$ and a, b, and c are integers.	**forma estándar de una ecuación cuadrática** Una ecuación cuadrática puede escribirse en la forma $ax^2 + bx + c = 0$, donde $a \neq 0$ y a, b, y c son enteros.
standard normal distribution A normal distribution with a mean of 0 and a standard deviation of 1.	**distribución normal estándar** Distribución normal con una media de 0 y una desviación estándar de 1.
standard position An angle positioned so that the vertex is at the origin and the initial side is on the positive x-axis.	**posición estándar** Un ángulo colocado de manera que el vértice está en el origen y el lado inicial está en el eje x positivo.
statement Any sentence that is either true or false, but not both.	**enunciado** Cualquier oración que sea verdadera o falsa, pero no ambas.
statistic A measure that describes a characteristic of a sample.	**estadística** Una medida que describe una característica de una muestra.
statistics An area of mathematics that deals with collecting, analyzing, and interpreting data.	**estadísticas** El proceso de recolección, análisis e interpretación de datos.
step function A type of piecewise-linear function with a graph that is a series of horizontal line segments.	**función escalonada** Un tipo de función lineal por piezas con un gráfico que es una serie de segmentos de línea horizontal.
straight angle An angle that measures 180°.	**ángulo recto** Un ángulo que mide 180°.
stratified sample The population is first divided into similar, nonoverlapping groups. Then members are randomly selected from each group.	**muestra estratificada** La población se divide primero en grupos similares, sin superposición. A continuación, los miembros se seleccionan aleatoriamente de cada grupo.
substitution A process of solving a system of equations in which one equation is solved for one variable in terms of the other.	**sustitución** Un proceso de resolución de un sistema de ecuaciones en el que una ecuación se resuelve para una variable en términos de la otra.
supplementary angles Two angles with measures that have a sum of 180°.	**ángulos suplementarios** Dos ángulos con medidas que tienen una suma de 180°.

surface area The sum of the areas of all faces and side surfaces of a three-dimensional figure.

área de superficie La suma de las áreas de todas las caras y superficies laterales de una figura tridimensional.

survey Data are collected from responses given by members of a group regarding their characteristics, behaviors, or opinions.

encuesta Los datos se recogen de las respuestas dadas por los miembros de un grupo con respecto a sus características, comportamientos u opiniones.

symmetric distribution A distribution in which the mean and median are approximately equal.

distribución simétrica Un distribución en la que la media y la mediana son aproximadamente iguales.

symmetry A figure has this if there exists a rigid motion—reflection, translation, rotation, or glide reflection—that maps the figure onto itself.

simetría Una figura tiene esto si existe una reflexión-reflexión, una traducción, una rotación o una reflexión de deslizamiento rígida-que mapea la figura sobre sí misma.

synthetic division An alternate method used to divide a polynomial by a binomial of degree 1.

división sintética Un método alternativo utilizado para dividir un polinomio por un binomio de grado 1.

synthetic geometry The study of geometric figures without the use of coordinates.

geometría sintética El estudio de figuras geométricas sin el uso de coordenadas.

synthetic substitution The process of using synthetic division to find a value of a polynomial function.

sustitución sintética El proceso de utilizar la división sintética para encontrar un valor de una función polynomial.

system of equations A set of two or more equations with the same variables.

sistema de ecuaciones Un conjunto de dos o más ecuaciones con las mismas variables.

system of inequalities A set of two or more inequalities with the same variables.

sistema de desigualdades Un conjunto de dos o más desigualdades con las mismas variables.

systematic sample Members are selected according to a specified interval from a random starting point.

muestra sistemática Los miembros se seleccionan de acuerdo con un intervalo especificado desde un punto de partida aleatorio.

T

tangent The ratio of the length of the leg opposite an angle to the length of the leg adjacent to the angle.

tangente La relación entre la longitud de la pata opuesta a un ángulo y la longitud de la pata adyacente al ángulo.

tangent to a circle A line or segment in the plane of a circle that intersects the circle in exactly one point and does not contain any points in the interior of the circle.

tangente a un círculo Una línea o segmento en el plano de un círculo que interseca el círculo en exactamente un punto y no contiene ningún punto en el interior del círculo.

tangent to a sphere A line that intersects the sphere in exactly one point.

tangente a una esfera Una línea que interseca la esfera exactamente en un punto.

term A number, a variable, or a product or quotient of numbers and variables.

término Un número, una variable, o un producto o cociente de números y variables.

term of a sequence A number in a sequence.

término de una sucesión Un número en una secuencia.

terminal side The part of an angle that rotates about the center.

lado terminal La partede un ángulo que gira alrededor de un centro.

tessellation A repeating pattern of one or more figures that covers a plane with no overlapping or empty spaces.

teselado Patrón repetitivo de una o más figuras que cubre un plano sin espacios superpuestos o vacíos.

theorem A statement that can be proven true using undefined terms, definitions, and postulates.

teorema Una afirmación o conjetura que se puede probar verdad utilizando términos, definiciones y postulados indefinidos.

theoretical probability Probability based on what is expected to happen.

probabilidad teórica Probabilidad basada en lo que se espera que suceda.

transformation A function that takes points in the plane as inputs and gives other points as outputs. The movement of a graph on the coordinate plane.

transformación Función que toma puntos en el plano como entradas y da otros puntos como salidas. El movimiento de un gráfico en el plano de coordenadas.

translation A function in which all of the points of a figure move the same distance in the same direction; A transformation in which a figure is slid from one position to another without being turned.

traslación Función en la que todos los puntos de una figura se mueven en la misma dirección; El movimiento de un gráfico en el plano de coordenadas.

translation vector A directed line segment that describes both the magnitude and direction of the slide if the magnitude is the length of the vector from its initial point to its terminal point.

vector de traslación Un segmento de línea dirigido que describe tanto la magnitud como la dirección de la diapositiva si la magnitud es la longitud del vector desde su punto inicial hasta su punto terminal.

transversal A line that intersects two or more lines in a plane at different points.

transversal Una línea que interseca dos o más líneas en un plano en diferentes puntos.

trapezoid A quadrilateral with exactly one pair of parallel sides.

trapecio Un cuadrilátero con exactamente un par de lados paralelos.

trend A general pattern in the data.

tendencia Un patrón general en los datos.

trigonometric equation An equation that includes at least one trigonometric function.

ecuación trigonométrica Una ecuación que incluye al menos una función trigonométrica.

trigonometric function A function that relates the measure of one nonright angle of a right triangle to the ratios of the lengths of any two sides of the triangle.

función trigonométrica Función que relaciona la medida de un ángulo no recto de un triángulo rectángulo con las relaciones de las longitudes de cualquiera de los dos lados del triángulo.

trigonometric identity An equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

identidad trigonométrica Una ecuación que implica funciones trigonométricas que es verdadera para todos los valores para los cuales se define cada expresión en la ecuación.

English	Español
trigonometric ratio A ratio of the lengths of two sides of a right triangle.	**relación trigonométrica** Una relación de las longitudes de dos lados de un triángulo rectángulo.
trigonometry The study of the relationships between the sides and angles of triangles.	**trigonometría** El estudio de las relaciones entre los lados y los ángulos de los triángulos.
trinomial The sum of three monomials.	**trinomio** La suma de tres monomios.
truth value The truth or falsity of a statement.	**valor de verdad** La verdad o la falsedad de una declaración.
two-column proof A proof that contains statements and reasons organized in a two-column format.	**prueba de dos columnas** Una prueba que contiene declaraciones y razones organizadas en un formato de dos columnas.
two-way frequency table A table used to show frequencies of data classified according to two categories, with the rows indicating one category and the columns indicating the other.	**tabla de frecuencia bidireccional** Una tabla utilizada para mostrar las frecuencias de los datos clasificados de acuerdo con dos categorías, con las filas que indican una categoría y las columnas que indican la otra.
two-way relative frequency table A table used to show frequencies of data based on a percentage of the total number of observations.	**tabla de frecuencia relativa bidireccional** Una tabla usada para mostrar las frecuencias de datos basadas en un porcentaje del número total de observaciones.

U

English	Español
unbounded When the graph of a system of constraints is open.	**no acotado** Cuando la gráfica de un sistema de restricciones está abierta.
undefined terms Words that are not formally explained by means of more basic words and concepts.	**términos indefinidos** Palabras que no se explican formalmente mediante palabras y conceptos más básicos.
uniform motion problems Problems that use the formula $d = rt$, where d is the distance, r is the rate, and t is the time.	**problemas de movimiento uniforme** Problemas que utilizan la fórmula $d = rt$, donde d es la distancia, r es la velocidad y t es el tiempo.
uniform tessellation A tessellation that contains the same arrangement of shapes and angles at each vertex.	**teselado uniforme** Un teselado que contiene la misma disposición de formas y ángulos en cada vértice.
union The graph of a compound inequality containing *or*.	**unión** La gráfica de una desigualdad compuesta que contiene la palabra *o*.
union of A and B The set of all outcomes in the sample space of event A combined with all outcomes in the sample space of event B.	**unión de A y B** El conjunto de todos los resultados en el espacio muestral del evento A combinado con todos los resultados en el espacio muestral del evento B.
unit circle A circle with a radius of 1 unit centered at the origin on the coordinate plane.	**círculo unitario** Un círculo con un radio de 1 unidad centrado en el origen en el plano de coordenadas.
univariate data Measurement data in one variable.	**datos univariate** Datos de medición en una variable.

English	Español
upper quartile The median of the upper half of a set of data.	**cuartil superior** La mediana de la mitad superior de un conjunto de datos.

V

English	Español
valid argument An argument is valid if it is impossible for all of the premises, or supporting statements, of the argument to be true and its conclusion false.	**argumento válido** Un argumento es válido si es imposible que todas las premisas o argumentos de apoyo del argumento sean verdaderos y su conclusión sea falsa.
variable A letter used to represent an unspecified number or value; Any characteristic, number, or quantity that can be counted or measured.	**variable** Una letra utilizada para representar un número o valor no especificado; Cualquier característica, número, o cantidad que pueda ser contada o medida.
variable term A term that contains a variable.	**término variable** Un término que contiene una variable.
variance The square of the standard deviation.	**varianza** El cuadrado de la desviación estándar.
vertex Either the lowest point or the highest point of a function.	**vértice** El punto más bajo o el punto más alto en una función.
vertex angle of an isosceles triangle The angle between the sides that are the legs of an isosceles triangle.	**ángulo del vértice de un triángulo isósceles** El ángulo entre los lados que son las patas de un triángulo isósceles.
vertex form A quadratic function written in the form $f(x) = a(x - h)^2 + k$.	**forma de vértice** Una función cuadrática escribirse de la forma $f(x) = a(x - h)^2 + k$.
vertex of a polyhedron The intersection of three edges of a polyhedron.	**vértice de un polígono** La intersección de tres bordes de un poliedro.
vertex of an angle The common endpoint of the two rays that form an angle.	**vértice de un ángulo** El punto final común de los dos rayos que forman un ángulo.
vertical angles Two nonadjacent angles formed by two intersecting lines.	**ángulos verticales** Dos ángulos no adyacentes formados por dos líneas de intersección.
vertical asymptote A vertical line that a graph approaches.	**asíntota vertical** Una línea vertical que se aproxima a un gráfico.
vertical shift A vertical translation of the graph of a trigonometric function.	**cambio vertical** Una traducción vertical de la gráfica de una función trigonométrica.
volume The measure of the amount of space enclosed by a three-dimensional figure.	**volumen** La medida de la cantidad de espacio encerrada por una figura tridimensional.

W

English	Español
work problems Problems that involve two people working at different rates who are trying to complete a single job.	**problemas de trabajo** Problemas que involucran a dos personas trabajando a diferentes ritmos que están tratando de completar un solo trabajo.

X

x-intercept The *x*-coordinate of a point where a graph crosses the *x*-axis.

intercepción *x* La coordenada *x* de un punto donde la gráfica corte al eje de *x*.

Y

y-intercept The *y*-coordinate of a point where a graph crosses the *y*-axis.

intercepción *y* La coordenada *y* de un punto donde la gráfica corte al eje de *y*.

Z

z-value The number of standard deviations that a given data value is from the mean.

valor *z* El número de variaciones estándar que separa un valor dado de la media.

zero An *x*-intercept of the graph of a function; a value of *x* for which $f(x) = 0$.

cero Una intercepción *x* de la gráfica de una función; un punto *x* para los que $f(x) = 0$.

G42 Glossary

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